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CONVERGENT PERTURBATION EXPANSIONS IN IRRELEVANT INTERACTIONS FOR THE RENORMALIZATION GROUP FLOW OF A FOUR-DIMENSIONAL HIERARCHICAL SU(2) LATTICE GAUGE FIELD MODEL

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Abstract

New perturbation expansions in irrelevant interactions are proposed for coupling constant recursion relations in renormalization group studies. These expansions are formulated and investigated in detail for a four-dimensional hierarchical $SU(2)$ lattice gauge field model. It is found that they are absolutely convergent for all values of the running gauge coupling constant. Only the leading term in the effective action which involves the (marginal) running gauge coupling constant needs to be treated by nonperturbative means. Low order approximations suffice to describe the renormalization group flow very precisely. The expansions make it possible to link the ultraviolet (continuum) properties to the infrared (confinement) properties of the model. Effective Boltzmannians and expectation values of local gauge invariant operators can be calculated.

CONTENTS

1. INTRODUCTION

Renormalization group techniques have become an important tool in investigations of critical or nearly critical systems in statistical mechanics and Euclidean quantum field theory [1,2]. They provide a method to tackle the problem of calculation of partition functions for systems with infinite (or finite but very large) correlation length. This can be done, at least in principle, by performing the functional integration in steps. In every step, one integrates out the high frequency parts within the momentum slice between the old and the new ultraviolet cutoff. The new cutoff is lowered by some scale factor $L > 1$. All degrees of freedom with momenta below the new cutoff are kept fixed in such an integration step. In this way, one obtains a sequence of effective Boltzmannians (or actions). They describe the theory for different but decreasing ultraviolet cutoffs, or, equivalently, at and above different but increasing length scales. The recursion relations between the effective Boltzmannians (or actions) are called renormalization group transformations, or, in the real space renormalization group approach, block spin transformations. In a lattice theory, the lattice spacing a determines the ultraviolet cutoff while the size of the lattice serves as infrared cutoff. A block spin transformation maps a lattice theory onto an effective theory living on a "block lattice" with lattice spacing La. The effective Boltzmannians are given as partition functions of "auxiliary systems" with short correlation lengths (measured in units of the associated length scale or lattice spacing). Such systems can be investigated by expansion methods of classical statistical mechanics, e.g. by polymer expansions.

In general, it is not possible to calculate even a single effective action[1] explicitly, and certainly not the whole sequence of effective actions. Nevertheless, there is a chance to achieve control over the flow of effective actions. This becomes possible when renormalization group transformations can be *iterated*. For this purpose, it is necessary to describe and to estimate the effective Boltzmannians (or actions) in an *inductive* manner. The explicitly known parts are characterized by a set of running coupling constants, whereas the unknown parts are estimated by appropriate bounds. The running couplings are then calculated by recursion relations (also called "recursive flow equations" in the following). It suffices to know these recursion relations approximately if the errors are bounded and sufficiently small. Usually, the recursion relations are given by perturbation expansions up to a given order in the running coupling constant. To estimate the perturbative and nonperturbative remainders, the bounds on the effective actions or Boltzmannians are needed. These estimates make the low order perturbative analysis rigorous and permit its application in exact renormalization group studies of weakly coupled models. However, perturbation theory fails when the running coupling constant does not remain small enough during the iteration. This explains why one can, up to now, only handle weakly coupled asymptotically free models by iterated renormalization group transformations in a rigorous mathematical way. For example, Gawędzki and Kupiainen have proven the infrared asymptotic freedom of the ϕ^4 lattice model in four dimensions by block spin renormalization group techniques [3]. Balaban has treated four-dimensional non-Abelian lattice gauge field theories by the renormalization group approach. In a sequence of papers, he was able to prove the ultraviolet stability of these models [4]. Mack and Göpfert have applied iterated Mayer expansions to prove permanent confinement of static quarks in the three-dimensional $U(1)$ lattice gauge theory [5,6]. These expansions have further been developed by Mack and Pordt. They obtained convergent and computable

[1] In statistical mechanics it is called Hamiltonian instead of action.

multigrid expansions [7,8], which are candidates for future computer simulations of continuum field theories without ultraviolet cutoff [9,10].

In this paper, an attempt is made to derive series expansions for coupling constant recursion relations which do not require a small running coupling constant. Nevertheless, the series should be "perturbation series" in the sense that their leading orders yield the leading contributions. This would imply that the expansions provide good approximations even in low order calculations. Finally, the series expansions should converge absolutely.

With regard to these demands, I propose to organize series expansions for coupling constant recursion relations in the following way. First, the irrelevant interactions are split off from the relevant and marginal parts of the effective Boltzmannians[1]. The explicit form of the split is taken from the discussion by Mack [10]. It replaces the split of actions in the work of Gawędzki and Kupiainen[2]. The irrelevant interactions are parametrized by conveniently chosen coupling constants, called irrelevant couplings. The idea is to treat the irrelevant interactions as "perturbations" of the relevant and marginal ones. The latter are handled by nonperturbative methods. Then the running coupling constants, which are given by the recursive flow equations, are expanded into the contributions from these irrelevant "perturbations". One obtains thus expansions in powers and products of the (running) irrelevant couplings. The coefficients depend on the (running) relevant and marginal coupling constants. The individual orders of the expansions are defined by a "degree of irrelevance". It describes the (canonical) scaling behaviour of the irrelevant interactions. I will denote such perturbation expansions in irrelevant interactions as "irrelevant perturbation expansions" for short.

This paper is designed to explain, to analyze, and to illustrate this idea for a simple but nontrivial model, namely a four-dimensional hierarchical $SU(2)$ lattice gauge model. Following Ito [11,12], this hierarchical lattice model can be understood as a full gauge model on a lattice with some hierarchical structure. The hierarchical structure is defined in such a way that renormalization group transformations are *exactly* described by Migdal recursion formulae [13][3]. The hierarchical structure destroys not only the translation invariance, but also the reflection (Osterwalder-Schrader) positivity of the lattice model. On the other hand, it leads to the absense of any nonlocal interaction term in the effective actions. (In general theories, effective actions in the sense of Wilson contain nonlocal terms.) This simplifying feature of hierarchical models is advantageous for rigorous and nonperturbative studies. This is because essential ideas and mechanisms are not obscured by tedious technicalities. Thus the studies become easier and more transparent than in complete models.

In the hierarchical $SU(2)$ model, there is no relevant and one marginal coupling constant, namely the gauge coupling g, that needs to be renormalized. Accordingly, one introduces a *running* gauge coupling constant g_j. The index j denotes the length or cutoff scale, respectively, of the corresponding effective action.

[1] The characterization of relevant, marginal, and irrelevant interactions and coupling constants, respectively, is done by studying the renormalization group transformation in the vicinity of an ultraviolet fixed point. One determines the eigenvectors and -values of the linearized renormalization group transformation. Interactions and their couplings are called relevant, marginal, or irrelevant with respect to this fixed point when the corresponding eigenvalues are larger than 1, equal to 1, or smaller than 1.

[2] See [3] and earlier papers cited therein.

[3] As a consequence, the definition of the hierarchical structure – and hence of the model itself – depends on the scale factor L of the renormalization group transformations.

Let us briefly recall what is known about this hierarchical model. Ito proved permanent confinement of static quarks [12]. The continuum limit of this hierarchical model, which is asymptotically free in the ultraviolet, was shown to exist by V.F. Müller and J. Schiemann [14][1]. They used a large field - small field split in order to attain control over the renormalization group flow. The large field - small field technique was originally proposed by Gallavotti and collaborators [16]. It starts by splitting the space into large field domains ("islands") and their complement, called small field region. In the small field region, the lattice fields stay bounded in a suitable way and standard perturbation theory converges. The nonperturbative large field contributions to the effective Boltzmannians are exponentially small. Müller and Schiemann calculated the recursion relations for the leading running couplings perturbatively in powers of $\beta_j^{-1} = g_j^2/4$ for large enough β_j (weak coupling expansion)[2]. The perturbative and nonperturbative remainders were estimated via a separate treatment of small and large fields. The recursive flow equations were reproduced by the present author in [18] by a conceptionally much simpler procedure. It is based on bounds for "reduced activities", which consist of the irrelevant parts of the effective Boltzmannians [10]. These bounds are valid for all fields. While they have the form of stability bounds for large fields, they are at the same time strong enough for small fields. The former property follows from the fact that the reduced activities parametrize the effective Boltzmannians and *not* the effective actions. The latter result comes about because the reduced activities contain only irrelevant interactions.

For the hierarchical model, the proposed perturbation theory in irrelevant interactions is explicitly formulated. In the recursive flow equations, the running coupling constants are represented by irrelevant perturbation series. The expansion coefficients of these series can be obtained from a generating function. They depend on the running coupling constant β_j. For small values of the gauge coupling constant (large β_j), the expansion coefficients may be further expanded. One then obtains not only the asymptotic weak coupling expansions in powers of β_j^{-1}, but also nonperturbative contributions. Therefore, the "perturbation" expansions in irrelevant interactions are generally nonperturbative with respect to the running gauge coupling constant. In addition, the model allows to numerically compute and iterate renormalization group steps. This can be done with great precision and provides "exact" results. These "exact" results can serve as reference data in testing perturbative approximations. It turns out that low order approximations in the degree of irrelevance are sufficient in order to reliably predict the renormalization group flow.

It will be shown in this paper, that the irrelevant perturbation series for the recursive calculation of the running coupling constants converge absolutely for all values of the bare coupling constant and for all iteration steps. For an individual renormalization group step, there exists a single condition of convergence. This condition is satisfied when the reduced activity is sufficiently small in a certain norm. The norm depends on the running coupling constant β_j. In order to prove the convergence for all iteration steps, it is thus necessary to control the flow of the running coupling constant and to iterate appropriate bounds on the reduced activities. This is done by a rigorous renormalization group analysis for all steps with a large enough running coupling constant β_j (weak coupling regime). For sufficiently small running coupling constants β_j (strong coupling regime), the condition of convergence is always satisfied. This is a consequence of the fact that all running couplings converge to zero under iterated renormalization group transformations, as has been proven by Ito [12]. Hence the norms of the effective reduced activities converge to zero too. Between the weak and strong coupling regimes, the renormalization group flow is studied by numerical means. The norms are calculated within very small errors. They are small enough to guarantee that the condition of convergence is always fulfilled. In this way, the main convergence result of this paper is partly proven by a rigorous analysis and partly established by numerical investigations.

[1] In addition, the uniqueness and universality of the continuum limit of effective actions were proven by Schiemann [15].

[2] The running coupling constant β_j parametrizes the Wilson action [17] as a part of the effective action at scale j.

In conclusion, the irrelevant perturbation expansions do not need a small parameter like β_j^{-1} or β_j – neither for convergence nor for reliable low order approximations. Thus iterated perturbation expansions in irrelevant interactions make it possible to link the ultraviolet and the infrared regime. They close the gap between weak coupling and high temperature expansions.

Organization of the Paper

In Sect. 2, the four-dimensional hierarchical $SU(2)$ lattice gauge model is introduced. The hierarchical structure is explained and the partition function is given. Some known properties of the model are rewieved, like the convergence of effective monomer activities g_j to the stable fixed point $g \equiv 1$, and the estimation of the string tension from below (implying confinement). The notations are introduced. In particular, the reduced activities are defined. They are parametrized by irrelevant couplings. A single renormalization group step is then completely described by a set of coupled recursion relations for the running coupling constant β_j and the infinitely many running irrelevant couplings. (Some subsections are only included for the convenience of the reader. Since they are not needed later on, they may be omitted.)

The first renormalization group step is performed in closed form in Sect. 3. The analyticity properties of the effective action are discussed. The effective couplings are represented by convergent series expansions. From these expansions, the perturbative and nonperturbative contributions with respect to the gauge coupling constant are obtained.

In Sect. 4, the main idea of the paper is introduced and discussed. The recursion relations for the renormalization group flow of the hierarchical $SU(2)$ model are represented in irrelevant perturbation theory. The running coupling constants are obtained as sums of contributions from irrelevant interactions. To all contributions, a "degree of irrelevance" is assigned. The properties of these irrelevant perturbation series are summarized together with the essential results of this paper.

It is shown in Sect. 5 that the irrelevant perturbation series converge for all running coupling constants if and only if a single condition is fulfilled. This condition will be called "condition of convergence". In order to derive it, interpolating activities are introduced. Some sufficient criteria for convergence are also given. They imply convergence if the reduced activities are small enough with respect to certain norms. That this will indeed be the case, is shown in the remaining sections.

The proof of convergence is stated in Sect. 6 for all iteration steps with a large or small enough running coupling constant β_j. For large values of β_j, convenient bounds on the reduced activities are iterated (Sect. 6.1 - Sect. 6.5). For small values of β_j, the convergence of the irrelevant perturbation theory turns out to be a simple consequence of the convergence of the effective monomer activities to the stable fixed point $g = 1$ (Sect. 6.6).

Finally, in Sect. 7, the renormalization group flow is studied by numerical means. It is compared with the low order predictions of the irrelevant perturbation theory. It is found that the flow can be approximated very well by a fourth-order perturbative calculation in the degree of irrelevance. The approach of individual renormalization group trajectories to the renormalized trajectory is displayed. The renormalized trajectory governs the renormalization group flow. It is parametrized by the renormalized coupling constant β. Under a single renormalization group step (with scale factor $L=\sqrt{2}$), this renormalized coupling constant runs according to the renormalization group (Callan-Symanzik) equation $\beta' = \beta + \Delta\beta(\beta)$. Here β' denotes the effective renormalized coupling constant, and $\Delta\beta(\beta)$ is the (discrete) renormalization group function. It is calculated up to large values of the renormalized coupling constant. The perturbative "two loop" result $-\frac{1}{3} - \frac{2}{27}\beta^{-1}$ for $\Delta\beta(\beta)$ is shown to be a good approximation for $\beta \geq 3$. Norms of reduced activities are calculated in a range of running couplings constants β_j around unity. It is found that the norms are always small enough to fulfil the criterium for convergence. Together with the rigorous analysis of Sect. 6, this establishes the convergence of the irrelevant perturbation series for the running coupling constants. The result holds for all values of the bare coupling constant and for all iteration steps.

In Sect. 8, the results of this paper are briefly summarized. The section ends with an outlook. App. A and App. B complete Sect. 6.2, and App. C is associated with Sect. 6.3. In App. D, some expansion coefficients are given explicitly. They are needed for perturbative computations up to the fourth order in degree of irrelevance.

2. THE MODEL

The 4-dimensional hierarchical $SU(2)$ lattice gauge theory model is introduced. It can either be understood as a full lattice gauge theory whose renormalization group transformations are simplified by omitting nonlocal effective interactions, or as a lattice gauge theory living on a hierarchically organized lattice with lattice spacing a in which Migdal-Kadanoff recursion formulae describe exact renormalization group transformations. Some known properties of this model are reviewed.

The effective Boltzmannian at length scale $a_j = L^{N-j}a_N$ can be written as partition function of a polymer system. The polymer system consists only of monomers as a consequence of the hierarchical structure of the model. The monomers are plaquettes p of the (hierarchical) lattice. Their activities $g_j^c(u)$ are class functions of the parallel transporters $u = U(\partial p) \in SU(2)$ around the plaquettes p. The partition function of the hierarchical model is expressed in terms of the monomer activities by

$$Z_j(\Lambda_j) = \int d\mu_j(u_j) \prod_{p\in\Lambda_j} g_j^c(u_j(p))$$

where the measure is hierarchically organized. For a scale factor $L=\sqrt{2}$, the Migdal recursion relations read

$$g_{j-1}^c(u) = \left(\mathcal{N}_j^{-1}\left[g_j * g_j\right](u)\right)^2 .$$

Here $*$ is the convolution on the group $G = SU(2)$, and the normalization factor $\mathcal{N}_j$ ensures $g_{j-1}^c(1) = 1$. The monomer activity is split into two parts, a Wilson activity $g_W^c(\beta_j, u)$ which depends on the running coupling constant β_j (inverse of the gauge coupling squared),

$$\beta_j = -\frac{d^2}{d\theta^2} g_j(\theta)\Big|_{\theta=0} = -g_j''(0) ,$$

and a remainder which sums irrelevant contributions

$$g_j^c(u) = g_W^c(\beta_j, u)\left[1 + r_j^c(u)\right] \equiv g_W(\beta_j,\theta)\left[1 + r_j(\theta)\right] = g_j(\theta)$$

$$g_W^c(\beta, u) \equiv \exp\left[-\frac{1}{2}\beta\,\mathrm{tr}\,(1-u)\right] \equiv \exp\left[-2\beta \sin^2\frac{\theta}{2}\right] = g_W(\beta,\theta) .$$

The angle θ parametrizes classes of $SU(2)$ elements v via $u = v\,e^{i\theta\sigma_3}v^{-1}$ (σ_3 = Pauli matrix) so that $\mathrm{tr}\,u = 2\cos\theta$. (The index c denotes class functions on the gauge group.) The reduced activities $r_j(\theta)$ fulfil the renormalization conditions $r_j(0) = 0$, $r_j'(0) = 0$ by definition. The iteration starts with a Wilson activity $g_N = g_W(\beta_N)$ so that $r_N = 0$. For all $j \leq N$, the remainders r_j, called reduced activities, are expanded into absolutely convergent series of irrelevant interactions

$$r_j(\theta) = \sum_{n\geq 2} \rho_n^{(j)} \left(\sin^2\frac{\theta}{2}\right)^n .$$

It is essential that the split is done at the level of the Boltzmannian, or rather polymer activities, and not in the Hamiltonian. This evades expansion of a logarithm which might diverge because of zeroes of the partition function for complex θ.

A renormalization group transformation induces a map $\left\{\beta_j, \{\rho_{\geq 2}^{(j)}\}\right\} \to \left\{\beta_{j-1}, \{\rho_{\geq 2}^{(j-1)}\}\right\}$.

MONOMER ACTIVITIES AND THEIR RECURSION RELATIONS

Consider a pure $SU(2)$ lattice gauge field theory on a lattice Λ_N with lattice spacing a_N. Let the Boltzmannian be represented as the partition function of a polymer system which consists only of monomers. The monomers are plaquettes p of the lattice. To each of them, a monomer activity $A_N(p|u)$ is assigned. It depends on the lattice gauge field U through the parallel transporter u around the plaquette p, defined by the path-ordered product

$$u = \prod_{b\in\partial p} U(b) . \tag{2.1}$$

The Boltzmannian is then simply given as the product $\prod_p A_N(p|u)$. For the case of a Wilson action [17] one has

$$A_N(p|u) = \exp\Big(-\beta_N\big[1 - \frac{1}{2}\,\mathrm{Re}\,\mathrm{tr}\,u\big]\Big) = \exp\Big(-\frac{1}{2}\beta_N\,\mathrm{tr}\,(1-u)\Big) \equiv g_N^c(u) \tag{2.2}$$

where use has been made of the fact that $SU(2)$ matrices have real trace. The (bare) coupling constant β_N is related to the (bare) gauge coupling g_N by

$$\beta_N = \frac{4}{g_N^2} . \tag{2.3}$$

This can be seen by replacing the gauge field $U(b)$ by $\exp(i g_N a_N \mathcal{A}_\mu^N)$ for a vector potential $\mathcal{A}_\mu^N$ with Lorentz index μ in direction of the bond b, and taking the naive continuum limit $a_N \to 0$.

The hierarchical model is defined in such a way that renormalization group transformations are completely described by recursion relations between monomer activities. More precisely, the monomer activities g_{j-1}^c of length scale $a_{j-1} = L^{N-(j-1)} a_N$ should be recursively defined by the Migdal recursion formulae [13]

$$g_{j-1}^c(u) = \left[\mathcal{N}_j^{-1} \int_G \prod_{i=1}^{L^2-1} dv_i \;\; g_j^c(uv_1^{-1}) \prod_{k=1}^{L^2-1} g_j^c(v_k v_{k+1}^{-1})\right]^{L^{D-2}} , \tag{2.4a}$$

where v_k with $k = L^2$ denotes the unit element 1 of the group $G = SU(2)$. dv_i denotes the normalized Haar measure on G. The scale factor $L > 1$ determines the lattice spacing a_{j-1} of the block lattice Λ_{j-1} in units of the lattice spacing a_j of Λ_j : $a_{j-1} = La_j$. $D \geq 2$ denotes the number of (Euclidean) space-time dimensions. The normalization factor $\mathcal{N}_j$ is chosen to maintain the normalization condition

$$g_{j-1}^c(1) = 1 . \tag{2.4b}$$

PARTITION FUNCTION OF THE HIERARCHICAL MODEL

Equation (2.4a) can be interpreted as an exact block spin transformation on a D-dimensional lattice which is hierarchically organized [11,12]. This will now be described. The reader may omit this and also the next subsection and proceed to the subsection "Elementary properties of effective activities".

Take $D = 4$ and start with a four-dimensional hypercubic lattice with lattice spacing a_N. Let the lattice be finite with a side length $a_k = L^{N-k} a_N$ for some integer $k \leq N-1$. Consider the corresponding hierarchical lattice $\Lambda_N^{[k]}$ as a stack of parallel two-dimensional lattice planes λ_N (called layers), which are connected by bonds $b_\perp$ on which the gauge degrees of freedom are eliminated by setting $U(b_\perp) = 1$. Note that this is a simplification in order to construct the hierarchical model; it should not be confused with "gauge fixing". Now set all couplings $\beta_{p_\perp}$ for the plaquettes $p_\perp$ perpendicular to the lattice planes λ_N equal to zero. This decouples all two-dimensional layers. The partition functions for the individual layers λ_N then have the form

$$Z_N(\lambda_N) = \int \prod_{b\in\lambda_N} dU_N(b) \prod_{p\in\lambda_N} g_N^c(u_N(p)) .$$

By a suitable gauge fixing, e.g. the axial gauge $U(b) = 1$ for all links b of some chosen maximal tree within λ_N, these integrals factorize into

$$Z_N(\lambda_N) = \prod_{p\in\lambda_N} \int du_N(p)\; g_N^c(u_N(p)) .$$

Thus, as long as the lattice $\Lambda_N^{[k]}$ consists only of decoupled two-dimensional layers λ_N, the total partition function is simply

$$Z_N(\Lambda_N^{[k]}) = \prod_{\lambda_N \subset \Lambda_N^{[k]}} Z_N(\lambda_N) .$$

In the hierarchical model, the decoupling between different layers is partially removed. To prepare for this, one introduces a hierarchical structure as follows. In a first step, one divides the lattice $\Lambda_N^{[k]}$ into blocks $\Delta_N^{N-1} \subset \Lambda_N^{[k]}$ with side length $a_{N-1} = La_N$. Then one considers for each block Δ_N^{N-1} the sets $\Delta_N^{N-1} \cap \lambda_N$ which are either empty or contain L^2 plaquettes p_N of λ_N. In the latter case, abbreviate $\Delta_N^{N-1}(\lambda_N) \equiv \Delta_N^{N-1} \cap \lambda_N$. Then, for each block Δ_N^{N-1}, there are L^2 subsets $\Delta_N^{N-1}(\lambda_N^i)$. Now define parallel transporters around these "layer blocks"

$$u_N(\Delta_N^{N-1}(\lambda_N)) = \prod_{b\in\partial\Delta_N^{N-1}(\lambda_N)} U_N(b) . \tag{2.5}$$

In order to couple distinct layers, all parallel transporters $u_N(\Delta_N^{N-1}(\lambda_N^i))$ around layer blocks belonging to the same block Δ_N^{N-1} are identified. This couples each layer λ_N exactly with (L^2-1) other layers. All links in the interior of the layer blocks $\Delta_N^{N-1}(\lambda_N)$ remain thus decoupled from the links of other layers. Consider now a coarser lattice $\Lambda_{N-1}^{[k]}$ with lattice spacing $a_{N-1} = La_N$ that may be imposed upon $\Lambda_N^{[k]}$ in such a way that its layers λ_{N-1} are also layers of $\Lambda_N^{[k]}$. The layers λ_{N-1} consist of block plaquettes p_{N-1} with side length La_N. For a given block Δ_N^{N-1}, one has thus exactly one block layer λ_{N-1} with nonempty

$$\Delta_N^{N-1} \cap \lambda_{N-1} = \Delta_N^{N-1}(\lambda_{N-1}) = p_{N-1} . \tag{2.6}$$

Note that the block lattice $\Lambda_{N-1}^{[k]}$ consists for $k = N-1$ only of a single block plaquette, i.e. $p_{N-1} = \lambda_{N-1} = \Lambda_{N-1}^{[k]}$. Technically, the couplings are introduced by inserting the following terms into the expression for the partition function

$$\prod_{\Delta_N^{N-1}\subset\Lambda_N^{[k]}} \prod_{\substack{\Delta_N^{N-1}(\lambda_N)\subset\Delta_N^{N-1}\\ \Delta_N^{N-1}(\lambda_N)\neq\Delta_N^{N-1}(\lambda_{N-1})}} \delta(u_N(\Delta_N^{N-1}(\lambda_N))\,u_N(\Delta_N^{N-1}(\lambda_{N-1}))^{-1})\,,$$

with the δ-function on the group. To complete the hierarchical structure of the lattice, one has to iterate this procedure because in the first step each layer has only been coupled to (L^2-1) other layers. One proceeds by inductively superimposing new block lattice structures $\Lambda_{N-2}^{[k]}, \Lambda_{N-3}^{[k]}, \ldots, \Lambda_k^{[k]}$ onto the previous ones, which amounts to the introduction of blocks of blocks of blocks etc. The largest block lattice $\Lambda_k^{[k]}$ is a single plaquette p_k with the side length of the whole lattice $\Lambda_k^{[k]}$. Thus one gets finally the following partition function for the hierarchical model with scale parameter L in $D=4$ dimensions

$$Z_N(\Lambda_N^{[k]}) = \int \prod_{p_N\in\lambda_N\subset\Lambda_N^{[k]}} du_N(p_N)\; g_N^c(u_N(p_N))$$
$$\times \prod_{j=N-1}^{k} \prod_{\Delta_{j+1}^j\subset\Lambda_{j+1}^{[k]}} \prod_{\substack{\Delta_{j+1}^j(\lambda_{j+1})\subset\Delta_{j+1}^j\\ \Delta_{j+1}^j(\lambda_{j+1})\neq\Delta_{j+1}^j(\lambda_j)}} \delta(u_N(\Delta_{j+1}^j(\lambda_{j+1}))\,u_N(\Delta_{j+1}^j(\lambda_j))^{-1})\,. \quad (2.7)$$

Exact Block Spin Transformations

Now consider the first block spin transformation $N\to N-1$ with block spins

$$U_{N-1}(b_{N-1}) = \prod_{b_N\in b_{N-1}} U_N(b_N) \quad (2.8)$$

for links b_{N-1} within the layers λ_{N-1} of the block lattice $\Lambda_N^{[k]}$. This fixes the parallel transporters $u_N(p_{N-1})$ around block plaquettes p_{N-1} to equal $u_{N-1}(p_{N-1})$. Hence, the block spin transformation can be described by inserting the following representation of unity into the integrand on the right-hand side of the partition function (2.7)

$$1 = \int \prod_{p_{N-1}\in\lambda_{N-1}\subset\Lambda_{N-1}^{[k]}} du_{N-1}(p_{N-1})\; \delta(u_{N-1}(p_{N-1})\,u_N(p_{N-1})^{-1})\,, \quad (2.9)$$

and by performing the u_N-integrations afterwards. To do this, it is useful to rewrite

$$\prod_{p_N\in\lambda_N\subset\Lambda_N^{[k]}} du_N(p_N)\; g_N^c(u_N(p_N)) =$$
$$= \prod_{\Delta_N^{N-1}\subset\Lambda_N^{[k]}} \prod_{\Delta_N^{N-1}(\lambda_N)\subset\Delta_N^{N-1}} \prod_{p_N\in\Delta_N^{N-1}(\lambda_N)} du_N(p_N)\; g_N^c(u_N(p_N))$$
$$= \prod_{\Delta_N^{N-1}\subset\Lambda_N^{[k]}} \prod_{\Delta_N^{N-1}(\lambda_N)\subset\Delta_N^{N-1}} \Big[du_N(\Delta_N^{N-1}(\lambda_N)) \prod_{i=1}^{L^2-1} dv_i$$
$$\times\; g_N^c(u_N(\Delta_N^{N-1}(\lambda_N))\,v_1^{-1}) \prod_{k=1}^{L^2-1} g_N^c(v_k v_{k+1}^{-1})\Big]\,.$$

Now the integrations over the variables $u_N(\Delta_N^{N-1}(\lambda_N))$ can easily be done for each block

Δ_N^{N-1} with the help of the (L^2-1) δ-functions $\delta(u_N(\Delta_{j+1}^j(\lambda_{j+1}))\,u_N(\Delta_{j+1}^j(\lambda_j))^{-1})$ for $j=N-1$ of Eq. (2.7) and the δ-function $\delta(u_{N-1}(p_{N-1})\,u_N(p_{N-1})^{-1})$ of the block spin transformation (2.9). Before integrating out the u_N parallel transporters, however, one has to replace

$$u_N(\Delta_{j+1}^j(\lambda_l)) = \prod_{p_{N-1}\in\Delta_{j+1}^j(\lambda_l)} u_N(p_{N-1}) \qquad \text{for} \quad l=j \text{ or } j+1 \quad \text{with} \quad j\le N-1$$

in the arguments of the remaining δ-functions with $j\le N-2$. Then the result becomes

$$Z_N(\Lambda_N^{[k]}) = \int \prod_{p_{N-1}\in\lambda_{N-1}\subset\Lambda_{N-1}^{[k]}} du_{N-1}(p_{N-1}) \left[\mathcal{N}_N^{L^2}\, g_{N-1}^c(u_{N-1}(p_{N-1}))\right]$$
$$\times \prod_{j=N-2}^{k} \prod_{\Delta_{j+1}^j\subset\Lambda_{j+1}^{[k]}} \prod_{\substack{\Delta_{j+1}^j(\lambda_{j+1})\subset\Delta_{j+1}^j\\ \Delta_{j+1}^j(\lambda_{j+1})\neq\Delta_{j+1}^j(\lambda_j)}} \delta(u_{N-1}(\Delta_{j+1}^j(\lambda_{j+1}))\,u_{N-1}(\Delta_{j+1}^j(\lambda_j))^{-1})\,, \quad (2.10)$$

where the new (effective) activity g_{N-1}^c is defined by the Migdal formula (2.4), which describes the fact that the block spin transformation factorizes within each block Δ_N^{N-1} into $L^{D-2}=L^2$ identical block spin transformations for the two-dimensional layer blocks. A graphical representation of the block spin definition and the blocking procedure is given in [12]. By comparison with (2.7), one gets

$$Z_N(\Lambda_N^{[k]}) = \text{const}\; Z_{N-1}(\Lambda_{N-1}^{[k]}) \quad (2.11a)$$

with an additional constant

$$\text{const} = \prod_{p_{N-1}\in\Lambda_{N-1}^{[k]}} \mathcal{N}_N^{L^2} = \left(\mathcal{N}_N^{L^2}\right)^{L^{4(N-k-1)}} \quad (2.11b)$$

that occurs as a consequence of the normalization of the new activities g_{N-1}. This constant is of no importance because it will be cancelled in calculating expectation values of physical (local gauge invariant) observables.

Finally, one gets the simple formula

$$Z_N(\Lambda_N^{[k]}) = \text{const}'\; Z_k(\Lambda_k^{[k]}) = \text{const}' \int du_k(p_k)\; g_k^c(u_k(p_k)) \quad (2.12)$$

for partition functions on finite hierarchical lattices.

Elementary Properties of Effective Activities

In the following, I will consider Migdal recursion relations with $L=\sqrt{2}$ and $D=4$, i.e.

$$g_{j-1}^c(u) = (\mathcal{N}_j^{-1}\,[g_j^c * g_j^c](u))^2 = \left[\mathcal{N}_j^{-1}\int_G dv\; g_j^c(v)\,g_j^c(uv^{-1})\right]^2 \quad (2.13)$$

where $*$ denotes convolution with respect to the Haar measure on $SU(2)$.[1]

The iteration starts with the Wilson activity (or single plaquette Wilson Boltzmannian) $g_N^c(u)$ of Eq. (2.2). Effective (single plaquette) Boltzmannians $g_j^c(u)$ with $j < N$ are recursively defined by Eq. (2.13). Then the following properties hold for all $j \leq N$:

(i) The $g_j^c(u)$ are real and positive for $u \in G$ with

$$0 < g_j^c(u) \leq 1 . \tag{2.14}$$

(ii) They are normalized

$$g_j^c(1) = 1 . \tag{2.15}$$

(iii) They are continuous class functions on the group G

$$g_j^c(vuv^{-1}) = g_j^c(u) \qquad \forall u, v \in G . \tag{2.16}$$

(iv) They are invariant under inversion of the argument

$$g_j^c(u^{-1}) = g_j^c(u) \qquad \forall u \in G . \tag{2.17}$$

Using the invariance properties of the Haar measure on $SU(2)$, the properties (i),...,(iv) – which are obviously fulfilled for $g_N^c(u)$ – can be proven inductively. For example, the estimate

$$\begin{aligned} g_{j-1}^c(u) = \left[\mathcal{N}_j^{-1} \int_G dv\, g_j^c(uv^{-1}) g_j^c(v)\right]^2 &\leq \left[\mathcal{N}_j^{-1} \frac{1}{2} \int_G dv \left(g_j^c(uv^{-1})^2 + g_j^c(v)^2\right)\right]^2 \\ &= \left[\mathcal{N}_j^{-1} \frac{1}{2} (\mathcal{N}_j + \mathcal{N}_j)\right]^2 = 1 , \end{aligned}$$

turns out to be a simple consequence of the inequality $ab \leq (a^2 + b^2)/2$ and of the invariance of the Haar measure under (left and right) translations. The translation invariance of the measure implies also the class function property (iii),

$$\begin{aligned} g_{j-1}^c(wuw^{-1}) &= \left[\mathcal{N}_j^{-1} \int_G dv\, g_j^c(wuw^{-1}v^{-1}) g_j^c(v)\right]^2 \\ &= \left[\mathcal{N}_j^{-1} \int_G dv\, g_j^c(uw^{-1}v^{-1}w) g_j^c(v)\right]^2 \\ &= \left[\mathcal{N}_j^{-1} \int_G dv\, g_j^c(uv^{-1}) g_j^c(wvw^{-1})\right]^2 = g_{j-1}^c(u) . \end{aligned}$$

Furthermore, one gets

$$\begin{aligned} g_{j-1}^c(u^{-1}) &= \left[\mathcal{N}_j^{-1} \int_G dv\, g_j^c(u^{-1}v^{-1}) g_j^c(v)\right]^2 = \left[\mathcal{N}_j^{-1} \int_G dv\, g_j^c(v^{-1}) g_j^c(uv)\right]^2 \\ &= \left[\mathcal{N}_j^{-1} \int_G dv\, g_j^c(uv^{-1}) g_j^c(v)\right]^2 = g_{j-1}^c(u) , \end{aligned}$$

[1] The same recursion relation (block spin transformation) holds for the $O(4)$-symmetric hierarchical Heisenberg ferromagnet in two dimensions with scale factor 2 [19]. However, the normalized Haar measure dv is then on the group $SO(4)$, and the monomer activities $g_j(v)$, $v \in SO(4)$, are no longer class functions because of lack of gauge invariance.

since the Haar measure is also inversion invariant.

Parametrization

It will be convenient to parametrize classes of $SU(2)$-elements by a rotation angle θ. Let the class of group elements for a given value of θ be defined as the set of all elements u with

$$u = v\, e^{i\theta\sigma_3} v^{-1} \qquad \text{for some} \quad v \in SU(2) , \tag{2.18}$$

where σ_3 is the Pauli matrix. Conversely, the class of a given $u \in SU(2)$ is determined by the trace

$$\operatorname{tr} u = 2 \cos\theta . \tag{2.19}$$

Hence the parameter θ is uniquely defined on the interval $[0, \pi]$. Now it is possible to consider the class functions $g_j^c(u)$ on the group $SU(2)$ as functions of the real variable θ :

$$g_j^c(u) = g_j^c(v\, e^{i\theta\sigma_3} v^{-1}) = g_j^c(e^{i\theta\sigma_3}) .$$

Introduce the short-hand notations

$$g_j(\theta) \equiv g_j^c(e^{i\theta\sigma_3}) \qquad \text{and} \qquad [g_j * g_j](\theta) \equiv [g_j^c * g_j^c](e^{i\theta\sigma_3}) . \tag{2.20}$$

By means of this equation, the properties of the functions g_j can easily be deduced from those of the class functions g_j^c. Starting again with

$$g_N(\theta) = g_N^c(e^{i\theta\sigma_3}) = \exp\left(-\frac{1}{2}\beta_N(2 - 2\cos\theta)\right) = \exp\left(-2\beta_N \sin^2\frac{\theta}{2}\right) , \tag{2.21}$$

one gets for all $j \leq N$ and for all real θ :

(i) The functions $g_j(\theta)$ are real and positive with

$$0 < g_j(\theta) \leq 1 . \tag{2.22}$$

(ii) They are normalized

$$g_j(0) = 1 . \tag{2.23}$$

(iii) They are continuous and 2π-periodic

$$g_j(\theta \pm 2\pi k) = g_j(\theta) . \tag{2.24}$$

(iv) They are even

$$g_j(-\theta) = g_j(\theta) . \tag{2.25}$$

Moreover, the $g_j(\theta)$ are monotonously decreasing functions on $[0, \pi]$:

$$g_j(\theta_1) \geq g_j(\theta_2) \qquad \text{for} \qquad 0 \leq \theta_1 \leq \theta_2 \leq \pi . \tag{2.26}$$

This has been shown by Schiemann [15, App. D], utilizing the fact that the g_j^c are functions of positive type (i.e. all coefficients in the character expansion, Eq. (2.38), are positive).

ANALYTIC CONTINUATION

The Boltzmannian $g_N(\theta)$ is of the Wilson type given by Eq. (2.21). By analytic continuation, it becomes an entire function of the complex variable θ. One can also analytically continue the effective Boltzmannians $g_j(\theta)$ for $j < N$ by defining them recursively. The result is [20]

Proposition 2.1. *The analytically continued effective Boltzmannians $g_j(\theta)$ are entire functions for all $j \le N$.*

PROOF. ⌈ Consider the auxiliary functions $g_j^c(e^{i\theta\sigma_3}u)$ for complex θ and $u \in SU(2)$. Using the recursion relation, it is shown by induction on $j \le N$ that they are entire in θ for any fixed $u \in G$ and continuous in u for fixed complex θ. Note that this is true for the case $j = N$ with

$$g_N^c(e^{i\theta\sigma_3}u) = \exp\Big[-\frac{1}{2}\beta_N \operatorname{tr}(1 - e^{i\theta\sigma_3}u)\Big]$$
$$= \exp\Big[-\frac{1}{2}\beta_N(2 - \cos\theta \operatorname{tr} u - i\sin\theta \operatorname{tr}(\sigma_3 u))\Big].$$

Suppose that the assertion is true for some $j \le N$ and consider the function

$$g_{j-1}^c(e^{i\theta\sigma_3}u)^{1/2} \equiv \mathcal{N}_j^{-1}\int_G dv\, g_j^c(e^{i\theta\sigma_3}uv^{-1})g_j^c(v)$$

which is continuous in u for fixed complex θ because the integrand is continuous by assumption and the integration domain is compact.
Now consider the function for fixed $u \in G$ and let γ be an arbitrary closed curve in the complex θ-plane. Then

$$\oint_\gamma d\theta\, g_{j-1}^c(e^{i\theta\sigma_3}u)^{1/2} = \mathcal{N}_j^{-1}\oint_\gamma d\theta\int_G dv\, g_j^c(e^{i\theta\sigma_3}uv^{-1})g_j^c(v)$$
$$= \mathcal{N}_j^{-1}\int_G dv\, g_j^c(v)\underbrace{\oint_\gamma d\theta\, g_j^c(e^{i\theta\sigma_3}uv^{-1})}_{=0}$$

by interchanging the integrations of the continuous integrand over compact domains and by using Cauchy's theorem. Since this result holds for all closed curves γ, Morera's theorem implies that $g_{j-1}^c(e^{i\theta\sigma_3}u)^{1/2}$ is entire in θ. This is of course also true for its square. This proves the induction step $j \to j-1$ and establishes the assertion for the auxiliary functions $g_j^c(e^{i\theta\sigma_3}u)$, $j \le N$. The proposition follows by setting u equal to unity. ⌋

Remark. Making use of the translation invariance of the Haar measure dv, one can replace v by $\exp(i(\theta/2)\sigma_3)v$ and write

$$g_{j-1}^c(e^{i\theta\sigma_3}u) = \Big[\mathcal{N}_j^{-1}\int_G dv\, g_j^c(e^{i(\theta/2)\sigma_3}uv^{-1})g_j^c(e^{i(\theta/2)\sigma_3}v)\Big]^2 \qquad (2.27)$$

Thus analyticity of $g_j^c(e^{i\theta\sigma_3}w)$, $w \in G$, on the complex strip $|\operatorname{Im}\theta| < \eta$ will imply the analyticity of $g_{j-1}^c(e^{i\theta\sigma_3}u)$, $u \in G$, on the wider strip $|\operatorname{Im}\theta| < 2\eta$.

CONVERGENCE OF EFFECTIVE BOLTZMANNIANS TO THE FIXED POINT $g^c = 1$

Consider again the class functions $g_j^c(u)$ for $u \in SU(2)$. Starting with a g_N^c of the Wilson type, the iteration of the Migdal transformation (2.13) generates a sequence $\{g_j^c\}_{j\le N}$ of effective Boltzmannians. For $D = 4$ the following proposition holds.

Proposition 2.2. *Let g_N^c be given by (2.2) with an arbitrary (bare) coupling constant $\beta_N > 0$. Then the sequence $\{g_j^c\}_{j\le N}$ converges uniformly on $G = SU(2)$ to the unit function 1, i.e.*

$$\lim_{j\to-\infty} g_j^c = 1. \qquad (2.28)$$

The proposition follows from a theorem first proven by Ito [12]. Ito showed that iterated Migdal-Kadanoff recursions drive the effective Boltzmannians towards the high temperature (strong coupling) fixed point 1. This holds for hierarchical lattice gauge field models, where the recursion formulae realize exact renormalization group transformations [11], with gauge groups $G = SU(N)$ or $U(N)$ and in case of the critical dimension $D = 4$. For dimensions D less than 4, the convergence had been established previously by Ito for $G = U(1)$ [11] and by Müller and Schiemann for $G = SU(N), U(N)$ [20]. In [21], Müller and Schiemann have combined and generalized these results on the convergence of Migdal-Kadanoff iterations with $D \le 4$ for compact connected Lie groups and for a large class of functions. They proposed a simple group-theoretical method which yields also estimates of the rate of convergence. I would like to apply their method in order to prove Proposition 2.2 and to give an impression how it works.

PROOF OF PROPOSITION 2.2. ⌈ Introduce plaquette effective actions S_j^c by

$$S_j^c(u) \equiv -\log g_j^c(u) \qquad \text{for} \qquad u \in G = SU(2). \qquad (2.29)$$

Due to the relations (2.14)-(2.17), they are class functions with $S_j^c(u) \ge 0$, $S_j^c(1) = 0$, $S_j^c(vuv^{-1}) = S_j^c(u)$, and $S_j^c(u^{-1}) = S_j^c(u)$ for all $u, v \in G$. Furthermore, by (2.26), the $S_j^c(e^{i\theta\sigma_3})$ are monotonously increasing functions for $\theta \in [0,\pi]$. Thus

$$\max_{u\in G} S_j^c(u) = \max_{\theta\in[0,\pi]} S_j^c(e^{i\theta\sigma_3}) = S_j^c(e^{i\pi\sigma_3}) \ge 0.$$

Note that the activities $g_j^c(e^{i\theta\sigma_3})$ and the corresponding actions $S_j^c(e^{i\theta\sigma_3})$ are real analytic for real θ and continuous in $u \in G$ (see the proof of Proposition 2.1). Now consider

$$\int_G du\, \frac{\partial^2}{\partial\theta^2}S_j^c(e^{i\theta\sigma_3}u) = \frac{\partial^2}{\partial\theta^2}\int_G du\, S_j^c(e^{i\theta\sigma_3}u) = \frac{\partial^2}{\partial\theta^2}\int_G du\, S_j^c(u) = 0.$$

which implies

$$\max_{u\in G}\frac{\partial^2}{\partial\theta^2}S_j^c(e^{i\theta\sigma_3}u) \ge 0,$$

where the equality holds if and only if $(\partial^2/\partial\theta^2)S_j^c(e^{i\theta\sigma_3}u)$ is identically zero. Then define the norm

$$\|S_j^c\| \equiv \max_{\theta\in[0,\pi]}\max_{u\in G}\frac{\partial^2}{\partial\theta^2}S_j^c(e^{i\theta\sigma_3}u) = \max_{\theta\in\mathbf{R}}\max_{u\in G}\frac{\partial^2}{\partial\theta^2}S_j^c(e^{i\theta\sigma_3}u) \qquad (2.30)$$

on the space of the plaquette effective actions S_j^c over G. The Migdal recursion for the plaquette effective actions is

$$S_{j-1}^c(e^{i\theta\sigma_3}u) = 2\log\mathcal{N}_j - 2\log\int_G dv\,\exp\Big\{-S_j^c(e^{i(\theta/2)\sigma_3}uv^{-1}) - S_j^c(e^{i(\theta/2)\sigma_3}v)\Big\}.$$

With respect to the expectation value

$$\langle\cdot\rangle \equiv \frac{\int_G dv\,\exp\Big\{-S_j^c(e^{i(\theta/2)\sigma_3}uv^{-1}) - S_j^c(e^{i(\theta/2)\sigma_3}v)\Big\}(\cdot)}{\int_G dv\,\exp\Big\{-S_j^c(e^{i(\theta/2)\sigma_3}uv^{-1}) - S_j^c(e^{i(\theta/2)\sigma_3}v)\Big\}},$$

one gets

$$\begin{aligned}\frac{\partial^2}{\partial\theta^2}S_{j-1}^c(e^{i\theta\sigma_3}u) &= -2\left\langle\frac{\partial^2}{\partial\theta^2}\Big[-S_j^c(e^{i(\theta/2)\sigma_3}uv^{-1}) - S_j^c(e^{i(\theta/2)\sigma_3}v)\Big]\right\rangle\\ &\quad -2\left\langle\left(\frac{\partial}{\partial\theta}\Big[-S_j^c(e^{i(\theta/2)\sigma_3}uv^{-1}) - S_j^c(e^{i(\theta/2)\sigma_3}v)\Big]\right)^{;2}\right\rangle.\end{aligned}$$

$\langle A;B\rangle$ denotes the truncated expectation value $\langle AB\rangle - \langle A\rangle\langle B\rangle$. Because of $\langle A^{;2}\rangle \equiv \langle A;A\rangle = \langle A^2\rangle - \langle A\rangle^2 \geq 0$ it follows that

$$\begin{aligned}\frac{\partial^2}{\partial\theta^2}S_{j-1}^c(e^{i\theta\sigma_3}u) &\leq 2\left\langle\frac{\partial^2}{\partial\theta^2}\Big[S_j^c(e^{i(\theta/2)\sigma_3}uv^{-1}) + S_j^c(e^{i(\theta/2)\sigma_3}v)\Big]\right\rangle\\ &= \frac{1}{2}\left\langle\frac{\partial^2}{\partial\omega^2}\Big[S_j^c(e^{i\omega\sigma_3}uv^{-1}) + S_j^c(e^{i\omega\sigma_3}v)\Big]\Big|_{\omega=\theta/2}\right\rangle\\ &= \frac{1}{2}\left\langle 2\|S_j^c\| - \left(2\|S_j^c\| - \frac{\partial^2}{\partial\omega^2}\Big[S_j^c(e^{i\omega\sigma_3}uv^{-1}) + S_j^c(e^{i\omega\sigma_3}v)\Big]\Big|_{\omega=\theta/2}\right)\right\rangle\\ &= \|S_j^c\| - \left\langle\|S_j^c\| - \frac{1}{2}\frac{\partial^2}{\partial\omega^2}\Big[S_j^c(e^{i\omega\sigma_3}uv^{-1}) + S_j^c(e^{i\omega\sigma_3}v)\Big]\Big|_{\omega=\theta/2}\right\rangle\\ &\equiv \|S_j^c\| - \langle X_j^c\rangle\end{aligned}$$

Due to the definition of the norm $\|S_j^c\|$, Eq. (2.30), the quantity X_j^c is obviously positive. Thus one gets $\|S_{j-1}^c\| \leq \|S_j^c\|$, which is, however, insufficient to prove the convergence. Therefore an improved lower bound on the expectation value $\langle X_j^c\rangle$ is needed. By taking advantage of $X_j^c \geq 0$, one gets

$$\begin{aligned}\langle X_j^c\rangle &\geq \frac{\int_G dv\,\exp\Big\{-2\max_{u\in G}S_j^c(u)\Big\}\Big(\|S_j^c\| - \frac{1}{2}\frac{\partial^2}{\partial\omega^2}\Big[S_j^c(e^{i\omega\sigma_3}uv^{-1}) + S_j^c(e^{i\omega\sigma_3}v)\Big]\Big)}{\int_G dv\,\exp\Big\{-2\min_{u\in G}S_j^c(u)\Big\}}\\ &= \exp\Big\{2\min_{u\in G}S_j^c(u) - 2\max_{u\in G}S_j^c(u)\Big\}\|S_j^c\|\end{aligned}$$

where use has been made of (cp. above)

$$\int_G dv\,\frac{\partial^2}{\partial\omega^2}S_j^c(e^{i\omega\sigma_3}uv^{-1}) = 0 = \int_G dv\,\frac{\partial^2}{\partial\omega^2}S_j^c(e^{i\omega\sigma_3}v).$$

Because of the monotony and the symmetry of $S_j^c(e^{i\theta\sigma_3})$, a Taylor expansion with remainder yields

$$\begin{aligned}\max_{u\in G}S_j^c(u) - \min_{u\in G}S_j^c(u) &= S_j^c(e^{i\pi\sigma_3}) - S_j^c(1)\\ &= \underbrace{\pi\,\frac{\partial}{\partial\theta}S_j^c(e^{i\theta\sigma_3})\Big|_{\theta=0}}_{=0} + \frac{1}{2!}\pi^2\,\frac{\partial^2}{\partial\theta^2}S_j^c(e^{i\theta\sigma_3})\Big|_{\theta\in(0,\pi)}\\ &\leq \frac{1}{2}\pi^2\|S_j^c\|.\end{aligned}$$

One gets the result[1]

$$\|S_{j-1}^c\| \leq \|S_j^c\|\Big[1 - \exp\big(-\pi^2\|S_j^c\|\big)\Big]. \tag{2.31}$$

The norm of the Wilson action S_N^c can be calculated explicitly. Write $\mathrm{tr}(e^{i\theta\sigma_3}u) = \cos\theta\,\mathrm{tr}\,u + i\sin\theta\,\mathrm{tr}(\sigma_3 u)$ in order to carry out the necessary differentiations with respect to θ. Then

$$\|S_N^c\| = \frac{1}{2}\beta_N\max_\theta\max_{u\in G}\Big[\cos\theta\,\mathrm{tr}\,u + i\sin\theta\,\mathrm{tr}(\sigma_3 u)\Big].$$

Parametrize $u = a_0 1 + i\sum_{k=1}^3 a_k\sigma_k$ with the Pauli matrices σ_k and the constraint $\sum_{\mu=0}^3 a_\mu^2 = 1$. Proceeding via

$$\frac{1}{2}\Big(\cos\theta\,\mathrm{tr}\,u + i\sin\theta\,\mathrm{tr}(\sigma_3 u)\Big) = a_0\cos\theta - a_3\sin\theta = \mathrm{Re}\Big[(a_0 + ia_3)e^{i\theta}\Big] = \mathrm{Re}\Big[r\,e^{i(\omega+\theta)}\Big]$$

for $a_0 + ia_3 = r\,e^{i\omega}$ with real ω and $r \in [0,1]$, one gets

$$\|S_N^c\| = \beta_N\max_{\theta,\omega}\max_{r\in[0,1]} r\cos(\omega+\theta) = \beta_N. \tag{2.32}$$

The proof of the proposition is completed by

$$\|S_j^c\| \leq \beta_N\Big[\underbrace{1 - \exp(-\pi^2\beta_N)}_{<1}\Big]^{N-j} \xrightarrow[N\geq j\to-\infty]{} 0. \qquad \rfloor \tag{2.33}$$

Remarks. (i) The convergence of effective Boltzmannians under iterated Migdal transformations (2.13), stated in Proposition 2.2 only for starting cases $g_N^c = g_W^c(\beta_N)$ of the Wilson type, holds more generally for a wider class of Boltzmannians g_N^c, cp. [21]. All that is really needed for the proof is that $g_N^c(u)$ fulfils the conditions (2.14)-(2.17) and that $g_N^c(e^{i\theta\sigma_3}u)$ can be continuously differentiated twice with respect to θ. On the space of Boltzmannians with these properties, $g^c \equiv 1$ is thus a stable fixed point under Migdal renormalization group transformations (2.13).

[1] If one considered more generally the recursion formula (2.4) instead of (2.13), one would obtain the result [21]

$$\|S_{j-1}^c\| \leq L^{D-4}\|S_j^c\|\Big[1 - \exp\big(-\pi^2\|S_j^c\|\big)\Big].$$

(ii) Besides this stable fixed point $g = 1$, which governs the infrared behaviour of the model, there "exists" an ultraviolet fixed point (for $\beta \to \infty$), which is responsible for the existence of continuum limits of renormalized effective Boltzmannians [14]. I will come back to these topics in Sect. 4 and Sect. 6.2.

CONFINEMENT

This subsection is not needed in the following and may be omitted by the reader.

The uniform convergence of the effective Boltzmannians g_j^c to the fixed point $g \equiv 1$, as stated in Proposition 2.2 for the group $SU(2)$, implies confinement of static quarks (see again [11,20], and [12]). This can be shown by proving that the expectation values $\langle W_{1/2}(C)\rangle_{\Lambda_N, g_N}$ of the Wilson loop operators $W_{1/2}(C) = \chi_{1/2}(u(C)) = \mathrm{tr}\,(u(C))$ obey an area law

$$\langle W_{1/2}(C)\rangle_{\Lambda_N, g_N} \sim \exp(-K_{1/2}\,\mathrm{area}\,(C)) \tag{2.34}$$

for large Wilson loops C. Here $u(C)$ denotes the parallel transporter along the closed loop C, i.e. $u(C)$ is the path-ordered product of link variables $U(b)$ along C, and $\chi_{1/2}$ is the character of the fundamental representation of $SU(2)$. The expectation values can be obtained as thermodynamical limits (infinite volume limits)

$$\langle W_{1/2}(C)\rangle_{\Lambda_N, g_N} = \lim_{k \to -\infty} \langle W_{1/2}(C)\rangle_{\Lambda_N^{[k]}, g_N} \tag{2.35}$$

of expectation values on finite hierarchical lattices $\Lambda_N^{[k]}$ (which have been previously introduced for integer L). The latter are defined as

$$\langle W_{1/2}(C)\rangle_{\Lambda_N^{[k]}, g_N} = Z_N(\Lambda_N^{[k]})^{-1} \int d\mu_{\Lambda_N^{[k]}}(u_N) \prod_{p_N \in \lambda_N \subset \Lambda_N^{[k]}} g_N^c(u_N(p_N))\, \chi_{1/2}(u_N(C)) \;, \tag{2.36}$$

where the "hierarchical measure" $d\mu_{\Lambda_N^{[k]}}(u_N)$ can be read off from Eq. (2.7).

For any fixed loop C, the thermodynamical limit of these expectation values for decreasing integers k (increasing lattice sizes) exists. More generally, Schiemann has proven that thermodynamical limits exist for the expectation values of arbitrary local gauge invariant observables within the four-dimensional hierarchical $SU(2)$ model [15].

I will now briefly explain why the convergence of the effective Boltzmannians g_j to the high-temperature fixed point – and especially the rate of this convergence near the fixed point – leads to confinement.

Suppose that the Wilson loop C is chosen to be the boundary ∂p_l of some large block p_l of the block lattice $\Lambda_l \subset \Lambda_N$ for $l \ll N$. Then consider finite hierarchical lattices $\Lambda_N^{[k]}$ with $k \le l$ such that $p_l \in \Lambda_l^{[k]} \subset \Lambda_N^{[k]}$. In the case $k = l$, the block plaquette p_l equals the whole block lattice $\Lambda_l^{[l]}$, and the expectation value $\langle W_{1/2}(\partial p_l)\rangle_{\Lambda_N^{[l]}, g_N}$ thus becomes, after applying $(N-l)$ exact block spin transformations on the right-hand side of Eq. (2.36), the simple form (cp. Eq. (2.12))

$$\langle W_{1/2}(\partial p_l)\rangle_{\Lambda_N^{[l]}, g_N} = Z_l(\Lambda_l^{[l]})^{-1} \int du_l(p_l)\; g_l^c(u_l(p_l))\; \chi_{1/2}(u_l(p_l)) \;. \tag{2.37}$$

The right-hand side of this equation can be computed most easily by taking advantage of the convergent character expansion for the effective monomer activity g_l^c, namely

$$g_l^c(u) = \sum_\tau d_\tau\, c_\tau^{(l)}\, \chi_\tau(u) \;, \tag{2.38}$$

where τ runs over a complete set of irreducible, continuous, and unitary representations $T^{(\tau)}$ of $SU(2)$. These representations $T^{(\tau)}$ are labelled by nonnegative half-integers $\tau \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ and have dimensions $d_\tau = 2\tau + 1$. Their characters χ_τ are class functions, i.e. $\chi_\tau(vuv^{-1}) = \chi_\tau(u)$ for all $u, v \in SU(2)$, and fulfil

$$\int du\; \chi_\tau(u)\, \overline{\chi_{\tau'}(u)} = \delta_{\tau\tau'} \tag{2.39}$$

$$\chi_\tau(1) = d_\tau \;. \tag{2.40}$$

Explicitly, they are given by [22]

$$\chi_\tau(e^{i\theta\sigma_3}) = \frac{\sin[(2\tau+1)\theta]}{\sin\theta} \;. \tag{2.41}$$

Note that the expansion coefficients $c_\tau^{(l)}$ are positive for all τ and all $l \le N$.

Thus, returning to Eq. (2.37), one gets

$$\langle W_{1/2}(\partial p_l)\rangle_{\Lambda_N^{[l]}, g_N} = \frac{d_{1/2}\, c_{1/2}^{(l)}}{d_0\, c_0^{(l)}} \;. \tag{2.42}$$

The thermodynamical limit can then be written as

$$\langle W_{1/2}(\partial p_l)\rangle_{\Lambda_N, g_N} = \lim_{k \to -\infty} \langle W_{1/2}(\partial p_l)\rangle_{\Lambda_N^{[k]}, g_N} = \frac{2\, c_{1/2}^{(l)}}{c_0^{(l)}}\,(1 + \Delta_l) \tag{2.43}$$

with a correction term Δ_l that vanishes for $l \to -\infty$, i.e. for increasing loops ∂p_l with $\mathrm{area}\,(\partial p_l) \to \infty$ [20].

Now consider the string tension $K_{1/2}$ which describes, in physical terms, the coefficient of the long-range potential between static quarks. It is given by

$$K_{1/2} = -\lim_{l \to -\infty} \frac{1}{\mathrm{area}\,(\partial p_l)} \log \langle W_{1/2}(\partial p_l)\rangle_{\Lambda_N, g_N} \;. \tag{2.44}$$

Using $\mathrm{area}\,(\partial p_l) = a_l^2 = (L^{(N-l)} a_N)^2$, one obtains

$$a_N^2 K_{1/2} = -\lim_{l \to -\infty} \frac{1}{L^{2(N-l)}} \log \langle W_{1/2}(\partial p_l)\rangle_{\Lambda_N, g_N} \tag{2.45}$$

for the dimensionless quantity $a_N^2 K_{1/2}$, the string tension in lattice units. Since g_N is a Wilson Boltzmannian and depends only on the bare coupling constant β_N, $a_N^2 K_{1/2}$ is a function of β_N. Inserting the result (2.43), one has

$$a_N^2 K_{1/2} = -\lim_{l \to -\infty} L^{2(l-N)} \log\left(\frac{2\, c_{1/2}^{(l)}}{c_0^{(l)}}\,(1 + \Delta_l)\right) = \lim_{l \to -\infty} L^{2(l-N)}\left(\log c_0^{(l)} - \log c_{1/2}^{(l)}\right) \;. \tag{2.46}$$

The convergence of the effective monomer activities g_l^c to the fixed point $g \equiv 1$ implies that $\lim_{l\to-\infty} c_0^{(l)} = 1$ whereas $\lim_{l\to-\infty} c_{1/2}^{(l)} = 0$ for all $\tau \neq 0$, because the fixed point activity has the character expansion $1 = \chi_0(u)$.

In order to derive a lower bound for the string tension, one needs upper bounds $\delta^{(l)}$ on the coefficients $c_{1/2}^{(l)}$ which hold for large l,

$$a_N^2 K_{1/2} = \lim_{l\to-\infty} L^{2(l-N)} \log \frac{1}{c_{1/2}^{(l)}} \geq \lim_{l\to-\infty} L^{2(l-N)} \log \frac{1}{\delta^{(l)}} . \tag{2.47}$$

Following [11,20], I define

$$\delta^{(l)} \equiv \sum_{\tau\neq 0} d_\tau^2 \, c_\tau^{(l)} \tag{2.48}$$

with $\delta^{(l)} \geq d_{1/2}^2 c_{1/2}^{(l)} = 4\, c_{1/2}^{(l)}$ due to the positivity of the coefficients $c_\tau^{(l)}$. The quantity $\delta^{(l)}$ is very useful for estimating the "rate of convergence" when the effective activities g_j^c are in close proximity to the fixed point.

Consider the renormalization group step $l \to l-1$. Then $\delta^{(l-1)}$ is given by $1 - c_0^{(l-1)}$ with

$$c_0^{(l-1)} = \frac{1}{d_0} \int du \; g_{l-1}^c(u)\, \overline{\chi_0(u)} = \int du \; g_{l-1}^c(u) \, .$$

The convolution integral in the Migdal recursion formula (2.4) for $g_{l-1}^c(u)$ can easily be evaluated by means of the character expansion (2.38) of g_l^c. One obtains

$$\int \prod_{i=1}^{L^2-1} dv_i \;\; g_l^c(uv_1^{-1}) \prod_{k=1}^{L^2-1} g_l^c(v_k v_{k+1}^{-1}) = \sum_\tau d_\tau \left[c_\tau^{(l)}\right]^{L^2} \chi_\tau(u) \tag{2.49}$$

together with the normalization factor $\mathcal{N}_l = \sum_\tau d_\tau^2 \left[c_\tau^{(l)}\right]^{L^2}$. Thus

$$\begin{aligned} c_0^{(l-1)} &= \mathcal{N}_l^{-L^2} \int du \left[\sum_\tau d_\tau \left[c_\tau^{(l)}\right]^{L^2} \chi_\tau(u)\right]^{L^2} \\ &= \mathcal{N}_l^{-L^2} \int du \; \sum_\tau d_\tau \, \epsilon_\tau^{(l)} \chi_\tau(u) = \mathcal{N}_l^{-L^2} \epsilon_0^{(l)} \, , \end{aligned}$$

where all coefficients $\epsilon_\tau^{(l)}$ are positive. This is obvious by recalling $c_\tau^{(l)} \geq 0$ and reducing

$$\prod_{i=1}^{L^2} \chi_{\tau_i}(u) = \sum_\tau n_\tau(\{\tau_i\}) \chi_\tau(u)$$

with positive integers $n_\tau(\{\tau_i\})$ that can be determined by repeated use of

$$\chi_\tau(u)\,\chi_{\tau'}(u) = \sum_{\nu=|\tau'-\tau|}^{\tau'+\tau} \chi_\nu(u) \, . \tag{2.50}$$

In particular, due to $\chi_0(u) = 1$, one gets $\epsilon_0^{(l)} \geq \left[c_0^{(l)}\right]^{L^4} = (1-\delta^{(l)})^{L^4}$. Now estimate the normalization factor from above

$$\begin{aligned} \mathcal{N}_l = \left[c_0^{(l)}\right]^{L^2} + \sum_{\tau\neq 0} d_\tau^2 \left[c_\tau^{(l)}\right]^{L^2} &\leq \left[c_0^{(l)}\right]^{L^2} + \sum_{\tau\neq 0} \left[d_\tau^2 c_\tau^{(l)}\right]^{L^2} \\ &\leq \left[c_0^{(l)}\right]^{L^2} + \left[\sum_{\tau\neq 0} d_\tau^2 c_\tau^{(l)}\right]^{L^2} = (1-\delta^{(l)})^{L^2} + (\delta^{(l)})^{L^2} \, . \end{aligned}$$

One obtains [20]

$$\begin{aligned} \delta^{(l-1)} &\leq 1 - \left[\frac{(1-\delta^{(l)})^{L^2}}{(1-\delta^{(l)})^{L^2} + (\delta^{(l)})^{L^2}}\right]^{L^2} = 1 - \left[1 + \left(\frac{\delta^{(l)}}{1-\delta^{(l)}}\right)^{L^2}\right]^{-L^2} \\ &\leq 1 - \left[1 - L^2 \left(\frac{\delta^{(l)}}{1-\delta^{(l)}}\right)^{L^2}\right] \leq \left(t\,\delta^{(l)}\right)^{L^2} \end{aligned} \tag{2.51}$$

for $\delta^{(l)} \leq 1/2$ and with the abbreviation $t = 2(L^2)^{1/L^2}$. By iteration, this becomes finally

$$\delta^{(l-m)} \leq \left[\exp\left(\sum_{k=0}^{m} L^{-2m} \log t\right) \delta^{(l)}\right]^{L^{2m}} \leq \left[\exp\left(\frac{1}{1-L^{-2}} \log t\right) \delta^{(l)}\right]^{L^{2m}} = \left(t'\delta^{(l)}\right)^{L^{2m}} \tag{2.52}$$

with $t' = t^{1/(1-L^{-2})} = 2(L^2)^{1/(L^2-1)}$.

In conclusion, the string tension is bounded from below by

$$a_N^2 K_{1/2} \geq \lim_{l\to-\infty} L^{2(l-N)} \log\left[\left(t'\delta^{(l_0)}\right)^{-L^{2(l_0-l)}}\right] = -L^{2(l_0-N)} \log\left(t'\delta^{(l_0)}\right) > 0 \tag{2.53}$$

where l_0 may be fixed e.g. by $l_0 \equiv \max\{l \,|\, t'\delta^{(l)} \leq 1/2\}$ due to the convergence $\delta^{(l)} \to 0$ for $l \to -\infty$ ensured by Proposition 2.2.

Definition of Reduced Activities r_j

Introduce the notation

$$g_W^c(\beta, u) \equiv \exp\left(-\frac{1}{2}\beta \operatorname{tr}(1-u)\right) \tag{2.54}$$

for monomer activities (plaquette Boltzmannians) of the Wilson type, i.e. for $g_W^c(\beta,u) = \exp(-S_W^c(\beta,u))$ with the Wilson plaquette action $S_W^c(\beta,u) = (\beta/2)\operatorname{tr}(1-u)$. Then Eq. (2.2) reads

$$g_N^c(u) = g_W^c(\beta_N, u) \, . \tag{2.55}$$

In the following, I will consider exclusively the activities $g_j(\theta)$ defined by Eq. (2.20), and analytically continued to the whole complex θ-plane. Rewriting Eq. (2.21) yields

$$g_W(\beta, \theta) = \exp\left(-2\beta \sin^2 \frac{\theta}{2}\right) \tag{2.56}$$

and

$$g_N(\theta) = g_W(\beta_N, \theta) \tag{2.57}$$

with the bare coupling constant β_N for the cutoff scale N.

Now I split the effective monomer activities g_j into a product of a Wilson activity and a correction factor[1]

$$g_j(\theta) = g_W(\beta_j, \theta)[1 + r_j(\theta)]\,. \tag{2.58}$$

This split will be made unique by fixing the parameter β_j according to

$$\beta_j = -\frac{d^2}{d\theta^2} g_j(\theta)\Big|_{\theta=0} = -g_j''(0)\,. \tag{2.59}$$

β_j is called the ***running coupling constant.***

Equivalently, the correction terms or ***reduced activities*** r_{j-1} can be obtained by means of the recursion relations ($j \leq N$)

$$r_{j-1}(\theta) = \Big[\mathcal{N}_j^{-1}\big[g_W(\beta_j)(1+r_j) * g_W(\beta_j)(1+r_j)\big](\theta)\Big]^2 g_W(\beta_{j-1}, \theta)^{-1} - 1 \tag{2.60}$$

together with the ***renormalization conditions***

$$r_{j-1}(0) = 0 \tag{2.61a}$$
$$r_{j-1}''(0) = 0 \tag{2.61b}$$

and the ***initial data***

$$\beta_N > 0 \quad \text{and} \quad r_N = 0\,. \tag{2.62}$$

The choice of Eq. (2.62) implies that the iteration starts with the Wilson activity given by Eq. (2.57). The first renormalization condition, (2.61a), can be written explicitly as

$$\mathcal{N}_j = \big[g_W(\beta_j)(1+r_j) * g_W(\beta_j)(1+r_j)\big](0)$$

and replaces the normalization condition (2.4b) for the full monomer activity g_{j-1}. The second renormalization condition, (2.61b), fixes the running coupling constant β_{j-1} via

$$\begin{aligned} r_{j-1}''(0) &= g_{j-1}''(0) g_W(-\beta_{j-1}, 0) + 2 g_{j-1}'(0) g_W'(-\beta_{j-1}, 0) + g_{j-1}(0) g_W''(-\beta_{j-1}, 0) \\ &= g_{j-1}''(0) + \beta_{j-1} = 0\,. \end{aligned}$$

Thus both renormalization constants $\mathcal{N}_j$, β_{j-1} included in the recursion relation (2.60) for $j \to j-1$ are determined. Note that the reduced activities $r_j(\theta)$, $j \leq N$, are 2π-periodic, symmetric, and entire functions of θ. For real arguments, they are real valued and, by inequality (2.22), bounded from below by -1.

Remark. With the help of the plaquette effective actions S_j^c, Eq. (2.29), and the norm (2.30), it is possible to bound the running coupling constant β_j by estimates

$$\beta_j \leq \|S_j^c\| \qquad \text{for all} \qquad j \leq N\,. \tag{2.63}$$

[1] Splits of this kind were introduced by Mack within the context of polymer representations for effective Boltzmannians on a multigrid [10].

Note that the running coupling constant β_j can be written as

$$\begin{aligned} \beta_j &= -\frac{d^2}{d\theta^2} g_j^c(e^{i\theta\sigma_3})\Big|_{\theta=0} = -\frac{d^2}{d\theta^2}\exp\Big(-S_j^c(e^{i\theta\sigma_3})\Big)\Big|_{\theta=0} \\ &= \frac{d^2}{d\theta^2} S_j^c(e^{i\theta\sigma_3})\Big|_{\theta=0} + \underbrace{\Big[\frac{d}{d\theta} S_j^c(e^{i\theta\sigma_3})\Big|_{\theta=0}\Big]^2}_{=0} \\ &\leq \max_{\theta\in[0,\pi]} \max_{u\in G} \frac{\partial^2}{\partial\theta^2} S_j^c(e^{i\theta\sigma_3} u) = \|S_j^c\| \end{aligned}$$

Then (2.33) implies ($j \leq N$)

$$\beta_j \leq \beta_N \Big[1 - \exp(-\pi^2 \beta_N)\Big]^{N-j}\,. \tag{2.64}$$

This estimate can be improved considerably near the fixed point $g = 1$. Let j_0 with $N \geq j_0 > -\infty$ be so small that $\|S_{j_0}^c\| \leq \alpha/\pi^2$ with $0 < \alpha < 1$. Then one obtains

$$\|S_{j-1}^c\| \leq \Big(\pi \|S_j^c\|\Big)^2 \qquad \text{for all} \qquad j \leq j_0 \tag{2.65}$$

and finally, by iteration,

$$\beta_j \leq \|S_j^c\| \leq \frac{1}{\pi^2}\alpha^{2^{j_0-j}} \qquad \text{for all} \qquad j \leq j_0\,. \tag{2.66}$$

Renormalization Group Transformations as Mappings in Coupling Constant Space

In order to parametrize the r_j, I use for all $j \leq N$ series expansions of the form [18]

$$r_j(\theta) = \sum_{n=2}^{\infty} \rho_n^{(j)} \Big(\sin^2\frac{\theta}{2}\Big)^n \tag{2.67}$$

which display in every order n not only the symmetry but also the periodicity of the reduced activities. Note that the absence of $\sin^{2k}(\theta/2)$-terms with $k = 0$ and $k = 1$ is ensured by the renormalization conditions. As a consequence, the reduced activities r_j include only *irrelevant interactions*. This will be discussed in more detail in Sect. 4. It will also be shown, as a conclusion from Lemma 4.1, that the series (2.67) converge absolutely for all $j \leq N$, all choices of β_N, and for all complex θ.

A renormalization group transformation $j \to j-1$ can now be described by a mapping $\mathcal{T}$ of the set of j-dependent couplings $\{\beta_j; \rho_n^{(j)}, n \geq 2\}$ to the set of effective couplings $\{\beta_{j-1}; \rho_n^{(j-1)}, n \geq 2\}$. I would like to describe these mappings by the following set of equations

$$\beta_{j-1} = \mathcal{T}_1(\beta_j|\{\rho_m^{(j)}\}) \tag{2.68a}$$
$$\rho_n^{(j-1)} = \mathcal{T}_n(\beta_j|\{\rho_m^{(j)}\}) \qquad \text{for} \qquad n \geq 2\,. \tag{2.68b}$$

Remark. An effective Boltzmannian $g_j(\theta)$ may have – and has in general – zeroes for complex fields. These complex zeroes correspond to singularities of the effective action $S_j(\theta) = \log g_j(\theta)$. Thus, in order to define the effective action as a single-valued and analytic branch of the (negative) logarithm of the effective Boltzmannian, one has to specify a simply connected region on which the effective Boltzmannian is different from zero. Such a region includes the real θ-axis. Selecting the right branch, the effective action $S_j(\theta)$ then becomes the analytic continuation of $-\log g_j(\theta)$ (where the principal branch of log is taken) onto the specified region.
Now consider the following expansion for the "reduced" effective action

$$S_j(\theta) - 2\beta_j \sin^2\frac{\theta}{2} = -\log\left[1 + r_j(\theta)\right] = \sum_{n=2}^{\infty} \gamma_n^{(j)} \left(\sin^2\frac{\theta}{2}\right)^n . \tag{2.69}$$

The irrelevant couplings $\gamma_n^{(j)}$ are uniquely defined and can be expressed through $\rho_m^{(j)}$-couplings with $m \leq n$. However, according to the analyticity properties of the effective action $S_j(\theta)$, the series (2.69) converges only for complex fields θ with an appropriately bounded modulus of $\sin^2(\theta/2)$. For the case $j = N-1$, a detailed discussion of these points will be presented in the next section.

3. THE FIRST RENORMALIZATION GROUP STEP

Starting from a Wilson Boltzmannian, $g_N(\theta) = g_W(\beta_N, \theta)$, the first renormalization group step $N \to N-1$ can be done in closed form. The result will be used to obtain the zeroth order approximation in later renormalization group steps.

Computation of $g_{N-1}(\theta)$

The effective Boltzmannian $g_{N-1}(\theta)$ which results from the first renormalization group step $N \to N-1$ is given by (see (2.20), (2.13) and (2.57))

$$\begin{aligned} g_{N-1}(\theta) = g^c_{N-1}(e^{i\theta\sigma_3}) &= \left(\mathcal{N}_N^{-1}\left[g^c_N * g^c_N\right](e^{i\theta\sigma_3})\right)^2 \\ &= \left(\mathcal{N}_N^{-1}\left[g^c_W(\beta_N) * g^c_W(\beta_N)\right](e^{i\theta\sigma_3})\right)^2 . \end{aligned} \tag{3.1}$$

Now consider the convolution of Wilson Boltzmannians

$$\begin{aligned} \left[g^c_W(\beta) * g^c_W(\beta)\right](e^{i\theta\sigma_3}) &= \int_G dv\, g^c_W(\beta, v) g^c_W(\beta, e^{i\theta\sigma_3} v^{-1}) \\ &= \int_G dv\, g^c_W(\beta, e^{i(\theta/2)\sigma_3} v) g^c_W(\beta, e^{i(\theta/2)\sigma_3} v^{-1}) \end{aligned} \tag{3.2}$$

where use has been made of the translation invariance of the Haar measure (cp. Eq. (2.27)). Using the definition (2.54), one gets

$$\left[g^c_W(\beta) * g^c_W(\beta)\right](e^{i\theta\sigma_3}) = \int_G dv\, \exp\left(-\frac{1}{2}\beta \sum_{l=\pm 1} \operatorname{tr}\left(1 - e^{i(\theta/2)\sigma_3} v^l\right)\right) \tag{3.3}$$

Parametrize the $SU(2)$ elements v, v^{-1} as points on the surface of the four-dimensional unit sphere S^3, i.e.

$$v^{\pm 1} = a_0 1 \pm i \sum_{k=1}^{3} a_k \sigma_k \quad \in SU(2) \quad \text{with} \quad |a|^2 \equiv \sum_{\mu=0}^{3} a_\mu^2 = 1 , \tag{3.4}$$

where σ_k denote the Pauli matrices. In this parametrization, the normalized Haar measure is simply $dv = d^4a\, \delta(|a|^2 - 1)/\pi^2 = d^4a\, \delta(|a| - 1)/(2\pi^2)$. It is useful to introduce four-dimensional polar coordinates [23] on S^3

$$\begin{aligned} a_1 &= \sin\phi \sin\chi \cos\psi \\ a_2 &= \sin\phi \sin\chi \sin\psi \\ a_3 &= \sin\phi \cos\chi \\ a_0 &= \cos\phi \end{aligned} \tag{3.5}$$

with angles $\psi \in [0, 2\pi]$ and $\chi, \phi \in [0, \pi]$. Using $\int_0^\infty d|a|\, |a|^3\, \delta(|a| - 1) = 1$, the Haar measure then becomes

$$dv = \frac{1}{2\pi^2} d\psi\, d\chi \sin\chi\, d\phi \sin^2\phi . \tag{3.6}$$

Now the calculation of the convolution is straightforward. With

$$\operatorname{tr}\left(e^{i(\theta/2)\sigma_3} v^{\pm 1}\right) = \cos\frac{\theta}{2}\operatorname{tr}(v^{\pm 1}) + i\sin\frac{\theta}{2}\operatorname{tr}(\sigma_3 v^{\pm 1}) = 2a_0\cos\frac{\theta}{2} \pm 2i^2 a_3 \sin\frac{\theta}{2} \tag{3.7}$$

and

$$\sum_{l=\pm 1} \operatorname{tr}\left(e^{i(\theta/2)\sigma_3} v^{l}\right) = 4a_0\cos\frac{\theta}{2} = 4\cos\frac{\theta}{2}\cos\phi \tag{3.8}$$

one has

$$\begin{aligned}\left[g^c_W(\beta) * g^c_W(\beta)\right](e^{i\theta\sigma_3}) &= \frac{2}{\pi}\int_0^\pi d\phi \sin^2\phi \exp\left(-2\beta + 2\beta\cos\frac{\theta}{2}\cos\phi\right) \\ &= 2\,e^{-2\beta}\frac{I_1(2\beta\cos\frac{\theta}{2})}{2\beta\cos\frac{\theta}{2}}\,. \end{aligned}\tag{3.9}$$

with the modified Bessel function I_1 [24]. The normalization factor $\mathcal{N}_N$ follows for $\theta = 0$:

$$\mathcal{N}_N = \left[g^c_W(\beta_N) * g^c_W(\beta_N)\right](e^{i\theta\sigma_3})\big|_{\theta=0} = 2\,e^{-2\beta_N}\frac{I_1(2\beta_N)}{2\beta_N}\,. \tag{3.10}$$

Hence one obtains the result

$$g_{N-1}(\theta) = \left[\frac{I_1(2\beta_N\cos\frac{\theta}{2})}{\cos\frac{\theta}{2}\, I_1(2\beta_N)}\right]^2 . \tag{3.11}$$

Using the formula [24]

$$I_\nu(\lambda\zeta) = \lambda^\nu \sum_{k=0}^{\infty}\frac{(\lambda^2-1)^k(\zeta/2)^k}{k!}\, I_{\nu+k}(\zeta)$$

for $\nu = 1$, $\zeta = 2\beta_N$ and $\lambda = \cos(\theta/2)$, one gets the series expansion

$$\frac{I_1(2\beta_N\cos\frac{\theta}{2})}{\cos\frac{\theta}{2}\, I_1(2\beta_N)} = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\beta_N^k\,\frac{I_{1+k}(2\beta_N)}{I_1(2\beta_N)}\left(\sin^2\frac{\theta}{2}\right)^k . \tag{3.12}$$

Note that $I_{1+k}(2\beta_N) \le I_1(2\beta_N)$ for all $\beta_N \ge 0$ and all integers $k \ge 0$.[1] Therefore, one can estimate

$$\sum_{k=0}^{\infty}\left|\frac{(-1)^k}{k!}\beta_N^k\,\frac{I_{1+k}(2\beta_N)}{I_1(2\beta_N)}\, z^k\right| \le \sum_{k=0}^{\infty}\frac{1}{k!}(\beta_N|z|)^k = e^{\beta_N|z|}\,,$$

[1] The modified Bessel functions $I_\nu(z)$, given by [24]

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1/2)\,\Gamma(\nu+1/2)}\int_0^\pi d\phi\,\sin^{2\nu}\phi\,\exp(z\cos\phi) \qquad \text{for} \qquad \operatorname{Re}\nu > -1/2\,,$$

are obviously positive for real positive arguments z and integer $\nu \ge 0$. They fulfil recursion relations $I_{\nu-1}(z) - I_{\nu+1}(z) = (2\nu/z)I_\nu(z)$, cp. [24], which immediately imply $I_1(z) \ge I_{2n+1}(z)$ and $I_2(z) \ge I_{2n+2}(z)$ for all $n \ge 1$ and $z \ge 0$. It remains to show that $I_1(z) \ge I_2(z)$ for all $z \ge 0$. After a partial integration in the formula for $I_2(z)$, one gets

$$I_1(z) - I_2(z) = \frac{z}{\pi}\int_0^\pi d\phi\,[1-\cos\phi]\sin^2\phi\,\exp(z\cos\phi) \ge 0\,,$$

which proves the assertion.

which proves the absolute convergence of the series (3.12) for all values of the complex variable $z = \sin^2(\theta/2)$. This implies that normalized convolutions of Wilson Boltzmannians are entire functions of the variable z. Hence the auxiliary function

$$G_{N-1}(z) \equiv \left[\frac{I_1(2\beta_N\sqrt{1-z})}{\sqrt{1-z}\, I_1(2\beta_N)}\right]^2 , \tag{3.13}$$

where the principal branch of the square root (with $\operatorname{Re}\sqrt{\zeta} \ge 0$) is taken, is entire. The monomer activity $g_{N-1}(\theta)$ can now be expressed in the form

$$g_{N-1}(\theta) = G_{N-1}(\sin^2\frac{\theta}{2})\,. \tag{3.14}$$

As a consequence, it is easy to see that $g_{N-1}(\theta)$ is indeed 2π-periodic, even, and entire in θ.

Zeroes of the Effective Boltzmannian

The zeroes of the effective plaquette Boltzmannian $g_{N-1}(\theta)$ can be obtained from Eq. (3.11). For this purpose, one needs the nontrivial zeroes of the modified Bessel function $I_1(\zeta)$. The trivial zero, $\zeta = 0$, can be excluded because the modified Bessel function I_1 enters g_{N-1} in the combination $\zeta^{-1}I_1(\zeta)$ which tends to $1/2$ for $\zeta \to 0$. All nontrivial zeros of I_1 are purely imaginary, simple, and of course isolated. By means of the relation

$$I_1(iy) = i\,J_1(y) \qquad \text{for real } y\,,$$

they are related to the real zeroes $j_{1,n}$ (n integer) of the Bessel function J_1 [24]. Note that $j_{1,-n} = -j_{1,n}$ since J_1 is odd. The nontrivial zeroes of I_1 are thus given by $ij_{1,n}$ with $n \neq 0$; their smallest modulus is $|j_{1,\pm 1}| = j_{1,1} = 3.83171\ldots$ Now consider the corresponding zeroes θ_n of $g_{N-1}(\theta)$ in the strip $\operatorname{Re}\theta \in [0,\pi]$. Let them be defined by

$$2\beta_N\cos\frac{\theta_n}{2} \equiv -i\,j_{1,n}\,,$$

where the extra minus sign ensures that $\operatorname{Im}\theta_n$ is positive for positive indices n. The equation is solved by

$$\theta_n = \pi + 2i\,\operatorname{arcsinh}\left(\frac{j_{1,n}}{2\beta_N}\right) \qquad \text{for} \qquad n \neq 0\,. \tag{3.15}$$

The zeroes z_n of $G_{n-1}(z)$ are real and positive ($n \ge 1$)

$$z_n = 1 + \left(\frac{j_{1,n}}{2\beta_N}\right)^2 . \tag{3.16}$$

Definition and Analyticity Properties of the Effective Action

For real angles θ, the effective plaquette action $S_{N-1}(\theta)$ is defined as the negative real logarithm of the effective plaquette Boltzmannian $g_{N-1}(\theta)$. The normalization $g_{N-1}(0) = 1$

implies $S_{N-1}(0) = 0$. An analytic continuation for complex angles θ within some small complex region around the real axis is then given by

$$S_{N-1}(\theta) = S^c_{N-1}(e^{i\theta\sigma_3}) = -\log g^c_{N-1}(e^{i\theta\sigma_3}) = -\log g_{N-1}(\theta)\,, \tag{3.17}$$

where log denotes the principal branch of the logarithm. However, it is possible to analytically continue $\log g_{N-1}(\theta)$ to almost all values of θ. Consider the logarithm $L_{N-1}(z)$ of $G_{N-1}(z)$ with the property $G_{N-1}(z) = \exp L_{N-1}(z)$. The logarithmic derivative $G'_{N-1}(z)/G_{N-1}(z)$ of $G_{N-1}(z)$ is meromorphic in the whole z-plane with poles z_n ($n \geq 1$) of order 1. Now cut the z-plane along the positive real axis for $z \geq z_1$ with $z_1 > 1$ given by Eq. (3.16). This cut z-plane provides a simply connected region on which the logarithmic derivative of $G_{N-1}(z)$ is holomorphic. Hence one is able to define single-valued analytic branches for the logarithm of $G_{N-1}(z)$ on this cut z-plane by

$$\int_0^z \frac{G'_{N-1}(\zeta)}{G_{N-1}(\zeta)}\, d\zeta + 2\pi i k\,,$$

where the path of integration can be chosen arbitrarily within the cut z-plane. The integer k denotes the branch. $L_{N-1}(z)$ is chosen to be the principal branch ($k = 0$) characterized by its reduction to the real logarithm $\log G_{N-1}(z)$ for $z \in [0,1]$. Thus define

$$L_{N-1}(z) \equiv \int_0^z \frac{G'_{N-1}(\zeta)}{G_{N-1}(\zeta)}\, d\zeta\,. \tag{3.18}$$

Introduce a cut θ-plane with cuts along the lines $\operatorname{Re}\theta = \pi + 2\pi l$, $|\operatorname{Im}\theta| \geq \operatorname{Im}\theta_1 > 0$ for integer l. Then the analytic continuation of the effective plaquette action (3.17) to this cut θ-plane is obtained by means of

$$S_{N-1}(\theta) \equiv -L_{N-1}(\sin^2\frac{\theta}{2})\,. \tag{3.19}$$

Utilizing the fact that the power series expansion[1]

$$-L_{N-1}(z) = 2\beta_{N-1} z + \sum_{m=2}^{\infty} \gamma_m^{(N-1)} z^m$$

has a radius of convergence equal to z_1, it follows immediately that the corresponding expansion for the effective action

$$S_{N-1}(\theta) = 2\beta_{N-1}\sin^2\frac{\theta}{2} + \sum_{m=2}^{\infty}\gamma_m^{(N-1)}\Big(\sin^2\frac{\theta}{2}\Big)^m \tag{3.20}$$

converges absolutely and locally uniformly in the region $|\sin^2(\theta/2)| = \sin^2\frac{\operatorname{Re}\theta}{2} + \sinh^2\frac{\operatorname{Im}\theta}{2} < z_1$. There one has the relation

$$r_{N-1}(\theta) = \sum_{n=2}^{\infty}\rho_n^{(N-1)}\Big(\sin^2\frac{\theta}{2}\Big)^n = \exp\Big[\sum_{m=2}^{\infty}\gamma_m^{(N-1)}\Big(\sin^2\frac{\theta}{2}\Big)^m\Big] - 1\,. \tag{3.21}$$

[1] This power series can – equivalently and without explicit construction of $L_{N-1}(z)$ – also be obtained by integration of the power series expansion for $G'_{N-1}(\zeta)/G_{N-1}(\zeta)$ around $\zeta = 0$, which converges for $|\zeta| < z_1$.

Note that the series expansion for $r_{N-1}(\theta)$ converges in the whole θ-plane because

$$R_{N-1}(z) \equiv G_{N-1}(z)\exp(+2\beta_{N-1}z) - 1 = \sum_{n=2}^{\infty}\rho_n^{(N-1)} z^n \tag{3.22}$$

is an entire function with $R_{N-1}(\sin^2(\theta/2)) = r_{N-1}(\theta)$.

Explicit Formulae for the Effective Couplings

For the effective (running) coupling constants β_{N-1}, $\rho_n^{(N-1)}$, and $\gamma_n^{(N-1)}$, respectively, the following equations hold

$$\beta_{N-1} = -\frac{1}{2}\frac{d}{dz}G_{N-1}(z)\Big|_{z=0} = -\frac{1}{2}G'_{N-1}(0) \tag{3.23}$$

$$\rho_n^{(N-1)} = \frac{1}{n!}\frac{d^n}{dz^n}R_{N-1}(z)\Big|_{z=0} = \frac{1}{n!}\frac{d^n}{dz^n}\Big[G_{N-1}(z)\exp(+2\beta_{N-1}z)\Big]\Big|_{z=0} \tag{3.24}$$

$$\gamma_n^{(N-1)} = -\frac{1}{n!}\frac{d^n}{dz^n}L_{N-1}(z)\Big|_{z=0} = -\frac{1}{n!}\frac{d^{n-1}}{dz^{n-1}}\Big[\frac{G'_{N-1}(z)}{G_{N-1}(z)}\Big]\Big|_{z=0}\,. \tag{3.25}$$

More explicitly, they become

$$\beta_{N-1} = \beta_N\frac{I_2(2\beta_N)}{I_1(2\beta_N)} < \beta_N\,, \tag{3.26}$$

$$\rho_n^{(N-1)} = \frac{1}{n!}(-\beta_N)^n\sum_{k=0}^{n}\binom{n}{k}\Big[(-2)\frac{I_2(2\beta_N)}{I_1(2\beta_N)}\Big]^k \sum_{\substack{i_1,i_2\geq 0\\ i_1+i_2=n-k}}\binom{n-k}{i_1,i_2}\prod_{m=1}^{2}\frac{I_{1+i_m}(2\beta_N)}{I_1(2\beta_N)}\,, \tag{3.27}$$

and

$$\gamma_n^{(N-1)} = -\frac{2}{n!}(-\beta_N)^n\sum_{k=1}^{n}(-1)^{k-1}(k-1)!\sum_{\substack{i_1,\ldots,i_k\geq 1\\ i_1+\ldots+i_k=n}}\frac{1}{k!}\binom{n}{i_1,i_2,\ldots,i_k}\prod_{m=1}^{k}\frac{I_{1+i_m}(2\beta_N)}{I_1(2\beta_N)}\,, \tag{3.28}$$

where I have used the multinomial coefficients

$$\binom{n}{i_1,i_2,\ldots,i_k} \equiv \frac{n!}{i_1!\,i_2!\ldots i_k!} \qquad \text{for} \qquad i_1+i_2+\ldots+i_k = n\,.$$

Series Expansions for the Effective Couplings

All effective couplings β_{N-1}, $\rho_n^{(N-1)}$, $\gamma_n^{(N-1)}$ are meromorphic functions of the complex variable β_N. The poles of these functions lie at $\beta_N = i\,j_{1,m}/2$ for integer $m \neq 0$. Thus the effective couplings can be represented by power series expansions in the (real and positive) bare coupling constant β_N which converge absolutely for $\beta_N < j_{1,1}/2 = 1.9153\ldots$ Besides such strong coupling (high temperature) expansions, one is also interested in weak coupling expansions, where the inverse coupling β_N^{-1} serves as small parameter. This will lead to divergent asymptotic expansions. However, as will be shown in the following, it is possible to modify the expansions by inclusion of contributions which are nonperturbative in β_N^{-1} and restore the convergence.

Set $\beta_N = \beta$ and consider the functions $I_\nu(2\beta)/I_1(2\beta)$. According to the formulae (3.26) - (3.28), they are the building blocks for the effective coupling constants. Write

$$\frac{I_\nu(2\beta)}{I_1(2\beta)} = \frac{e^{-2\beta} I_\nu(2\beta)}{e^{-2\beta} I_1(2\beta)} \tag{3.29}$$

and consider for $\nu \geq 1$ and $\beta > 0$ the series expansions

$$e^{-2\beta} I_\nu(2\beta) = \frac{1}{\sqrt{4\pi\beta}} \sum_{\mu=0}^{\infty} \frac{(-1)^\mu}{\mu!} \frac{\gamma(\nu+\mu+1/2;4\beta)}{\Gamma(\nu-\mu+1/2)} \left(\frac{1}{4\beta}\right)^\mu \tag{3.30}$$

which are due to Hadamard [25, Chap. 7.2.5] and converge absolutely. $\gamma(\alpha;x)$ denotes the incomplete Gamma function of Legendre [24]

$$\gamma(\alpha;x) = \int_0^x e^{-t} t^{\alpha-1} dt \qquad \text{for} \qquad \mathrm{Re}\,\alpha > 0\,. \tag{3.31}$$

For $\beta > 0$, the double inequality

$$0 < e^{-2\beta} I_1(2\beta) < \frac{1}{\sqrt{4\pi\beta}} \frac{\gamma(3/2;4\beta)}{\Gamma(3/2)} \tag{3.32}$$

holds. Both bounds can be obtained with the help of the integral representation for I_1

$$e^{-2\beta} I_1(2\beta) = \frac{\beta}{\Gamma(1/2)\Gamma(3/2)} \int_0^\pi e^{2\beta(\cos\omega - 1)} \sin^2\omega \, d\omega\,. \tag{3.33}$$

The upper bound follows by substituting $t = 4\beta \sin^2(\omega/2)$ and estimating

$$\begin{aligned} e^{-2\beta} I_1(2\beta) &= \frac{1}{\Gamma(1/2)\Gamma(3/2)} \int_0^{4\beta} e^{-t} \left(\frac{t}{4\beta}\right)^{1/2} \left(1 - \frac{t}{4\beta}\right)^{1/2} dt \\ &\leq \frac{1}{\Gamma(1/2)\Gamma(3/2)} \left(\frac{1}{4\beta}\right)^{1/2} \int_0^{4\beta} e^{-t} t^{1/2} dt = \frac{1}{\sqrt{4\pi\beta}} \frac{\gamma(3/2;4\beta)}{\Gamma(3/2)}\,. \end{aligned}$$

Now split

$$e^{-2\beta} I_1(2\beta) = \frac{1}{\sqrt{4\pi\beta}} \frac{\gamma(3/2;4\beta)}{\Gamma(3/2)} [1 - X_1(\beta)] \tag{3.34}$$

with $0 < X_1(\beta) < 1$ for $\beta > 0$ in order to develop

$$\frac{1}{e^{-2\beta} I_1(2\beta)} = \sqrt{4\pi\beta} \frac{\Gamma(3/2)}{\gamma(3/2;4\beta)} \sum_{n=0}^{\infty} X_1(\beta)^n\,. \tag{3.35}$$

This series converges absolutely. Insert

$$X_1(\beta) = \sum_{\mu=1}^{\infty} X_{1,\mu}(\beta) \tag{3.36}$$

with

$$\begin{aligned} X_{1,\mu}(\beta) &= -\frac{(-1)^\mu}{\mu!} \frac{\Gamma(3/2)}{\gamma(3/2;4\beta)} \frac{\gamma(3/2+\mu;4\beta)}{\Gamma(3/2-\mu)} \left(\frac{1}{4\beta}\right)^\mu \\ &= \frac{1}{\mu!} \frac{\Gamma(3/2)}{\gamma(3/2;4\beta)} \frac{\gamma(3/2+\mu;4\beta)}{|\Gamma(3/2-\mu)|} \left(\frac{1}{4\beta}\right)^\mu = |X_{1,\mu}(\beta)| > 0\,. \end{aligned} \tag{3.37}$$

Then

$$X_1(\beta)^n = \sum_{\mu=1}^{\infty} \sum_{\substack{\mu_1,\ldots,\mu_n \geq 1 \\ \mu_1+\ldots+\mu_n=\mu}} \prod_{i=1}^{n} X_{1,\mu_i}(\beta) \qquad \text{for} \qquad n \geq 1\,.$$

Note that

$$0 < \sum_{\mu=1}^{\infty} |X_{1,\mu}(\beta)| = \sum_{\mu=1}^{\infty} X_{1,\mu}(\beta) = X_1(\beta) < 1$$

since all coefficients $X_{1,\mu}(\beta)$ are positive (cp. Eq. (3.37)). Thus the series (3.35) still converges absolutely when the summation is over products $\prod_i X_{1,\mu_i}(\beta)$. Abbreviate for $\nu \geq 2$

$$e^{-2\beta} I_\nu(2\beta) = \frac{1}{\sqrt{4\pi\beta}} \frac{\gamma(\nu+1/2;4\beta)}{\Gamma(\nu+1/2)} \sum_{\mu=0}^{\infty} Y_{\nu,\mu}(\beta) \tag{3.38}$$

with

$$Y_{\nu,\mu}(\beta) = \frac{(-1)^\mu}{\mu!} \frac{\Gamma(\nu+1/2)}{\gamma(\nu+1/2;4\beta)} \frac{\gamma(\nu+\mu+1/2;4\beta)}{\Gamma(\nu-\mu+1/2)} \left(\frac{1}{4\beta}\right)^\mu \tag{3.39}$$

to get finally the series expansions

$$\begin{aligned} \frac{I_\nu(2\beta)}{I_1(2\beta)} &= \frac{\Gamma(3/2)}{\gamma(3/2;4\beta)} \frac{\gamma(\nu+1/2;4\beta)}{\Gamma(\nu+1/2)} \\ &\times \left\{ 1 + \sum_{\mu=1}^{\infty} \left[Y_{\nu,\mu}(\beta) + \sum_{\substack{\mu_X \geq 1, \mu_Y \geq 0 \\ \mu_X+\mu_Y=\mu}} \sum_{n=1}^{\mu_X} \sum_{\substack{\mu_1,\ldots,\mu_n \geq 1 \\ \mu_1+\ldots+\mu_n=\mu_X}} Y_{\nu,\mu_Y}(\beta) \prod_{i=1}^{n} X_{1,\mu_i}(\beta) \right] \right\}. \end{aligned} \tag{3.40}$$

which are valid for all nonzero positive values of β and converge absolutely (i.e. with $Y_{\nu,m}(\beta)$ replaced by $|Y_{\nu,m}(\beta)|$). Then, by inspection of Eqs. (3.26) - (3.28), it is easy to see that similar expansions hold for the effective couplings β_{N-1}, $\rho_n^{(N-1)}$, and $\gamma_n^{(N-1)}$.

Consider now asymptotic expansions[1] for the functions $I_\nu(2\beta)/I_1(2\beta)$. According to McMahon's expansion [24], the poles $i\,j_{1,n}/2$ of $I_\nu(2\beta)/I_1(2\beta)$ behave for large n like $i\,\pi(4n+1)/8+$

[1] An asymptotic expansion $g(z) + \sum_{n=1}^{\infty} a_n z^{-n}$ of the function $f(z)$ for *some part* of the neighbourhood of the point $z = \infty$, e.g. a *region* within the complex plane which reaches the point at infinity, is characterized by fulfilling the condition

$$\left| z^m \left(f(z) - g(z) - \sum_{n=1}^{m} \frac{a_n}{z^n} \right) \right| < \epsilon$$

for every $\epsilon > 0$ and every order m within the *considered part* of a conveniently chosen neighbourhood $|z| > \mathrm{const}(\epsilon, m)$ of $z = \infty$. Asymptotic expansions are thus useful when z becomes large while the order m is fixed. Note that the series $\sum_{n=1}^{\infty} a_n z^{-n}$ usually diverge, and write $f(z) \sim g(z) + \sum_{n=1}^{\infty} a_n z^{-n}$ in this

$O(n^{-1})$. As a consequence, there does not exist any large but finite constant R such that the $I_\nu(2\beta)/I_1(2\beta)$ are holomorphic in all finite points β with $|\beta| > R$. Thus the asymptotic expansions must diverge. They can be obtained by expanding

$$\gamma(\alpha;4\beta) = \Gamma(\alpha) - \int_{4\beta}^{\infty} e^{-t}\, t^{\alpha-1}\, dt = \Gamma(\alpha) - \Gamma(\alpha;4\beta). \tag{3.41}$$

The $\Gamma(\alpha;4\beta)$-part is purely nonperturbative in β^{-1} and can be represented by $\Gamma(\alpha;4\beta) = e^{-4\beta}(4\beta)^{\alpha-1}[1+O(\beta^{-1})]$ for large values of β [24]. Hence one gets the asymptotic expansion

$$\gamma(\alpha;4\beta) \sim \Gamma(\alpha) \tag{3.42}$$

which in combination with (3.37), (3.39), and (3.40) leads to

$$\frac{I_\nu(2\beta)}{I_1(2\beta)} \sim 1 + \sum_{\mu=1}^{\infty} \frac{(-1)^\mu}{\mu!} \left\{ \frac{\Gamma(\nu+1/2+\mu)}{\Gamma(\nu+1/2-\mu)} + \sum_{\substack{\mu_X \geq 1, \mu_Y \geq 0 \\ \mu_X+\mu_Y=\mu}} \sum_{n=1}^{\mu_X} \sum_{\substack{\mu_1,\ldots,\mu_n \geq 1 \\ \mu_1+\ldots+\mu_n=\mu_X}} \binom{\mu}{\mu_Y,\mu_1,\ldots,\mu_n} \times \frac{\Gamma(\nu+1/2+\mu_Y)}{\Gamma(\nu+1/2-\mu_Y)} \prod_{i=1}^{n} \left[-\frac{\Gamma(3/2+\mu_i)}{\Gamma(3/2-\mu_i)} \right] \right\} \left(\frac{1}{4\beta}\right)^{-1} . \tag{3.43}$$

As a result, one gets for $\beta > 0$

$$\frac{I_\nu(2\beta)}{I_1(2\beta)} = \text{asymptotic (weak coupling) expansion (r.h.s. of (3.43))} + \text{nonperturbative contributions (with respect to } \beta^{-1})$$

The decomposition for the effective couplings β_{N-1}, $\rho_n^{(N-1)}$, and $\gamma_n^{(N-1)}$ is obtained by inserting the result (3.43) into Eqs. (3.26), (3.27), and (3.28).

case. Consider as example the divergent asymptotic expansions for the modified Bessel functions [24]

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \sum_{\mu=0}^{\infty} \frac{(-1)^\mu}{\mu!} \frac{\Gamma(\nu+\mu+1/2)}{\Gamma(\nu-\mu+1/2)} \left(\frac{1}{2z}\right)^\mu$$

which hold for large $|z|$ in the *sector* $|\arg z| < \pi/2$. If an asymptotic expansion exists for a given function, it will be uniquely determined by this function. But note that different functions can have the same asymptotic expansion. For a more detailed discussion, the reader is referred to [26].

4. PERTURBATION THEORY IN IRRELEVANT INTERACTIONS

A perturbation theory in irrelevant interactions is introduced and explicitly formulated for the hierarchical $SU(2)$ model. It allows to compute the discrete renormalization group flow for all values of the bare gauge coupling and for all renormalization group steps by absolutely convergent series expansions of the form

$$\beta_{j-1} = \mathcal{T}_1(\beta_j|\rho_{\geq 2}^{(j)} = 0) + \sum_{d=2}^{\infty} \mathcal{T}_{1;d}^{irr}(\beta_j|\rho_{\leq d}^{(j)})$$
$$\rho_n^{(j-1)} = \mathcal{T}_n(\beta_j|\rho_{\geq 2}^{(j)} = 0) + \sum_{d=2}^{\infty} \mathcal{T}_{n;d}^{irr}(\beta_j|\rho_{\leq d}^{(j)})$$

for all running (effective) coupling constants. The terms of these expansions are labelled by a "degree of irrelevance" $\delta = d-1$. To compute the terms up to order δ, only the set $\{\rho_{\leq \delta+1}^{(j)}\}$ of couplings $\rho_n^{(j)}$ with degree of irrelevance $(n-1) \leq \delta$ is needed.

Fixed Point Structure of the Hierarchical Model

Consider the space of Boltzmannians $g(\theta)$. It can be parametrized by the coupling constants β and ρ_n, $n \geq 2$, which have been introduced in Sect. 2. The parametrization is obtained by the split (2.58) of the Boltzmannian $g(\theta)$ into its Wilson part $g_W(\beta,\theta)$ and a correction factor $[1+r(\theta)]$. Equation (2.59) determines the coupling constant β to be $-(d^2g/d\theta^2)(0)$. This implies that the reduced activity $r(\theta)$ fulfils the renormalization conditions (2.61). The reduced activity is expanded into irrelevant interactions $\rho_n \sin^{2n}(\theta/2)$, see Eq. (2.67). A point $(\beta,\rho_2,\rho_3,\ldots)$ represents thus the Boltzmannian

$$g(\theta) = \exp\Big(-2\beta \sin^2\frac{\theta}{2}\Big) \left[1 + \sum_{n\geq 2} \rho_n \left(\sin^2\frac{\theta}{2}\right)^n\right] . \tag{4.1}$$

Renormalization group transformations in this space of coupling constants are denoted by $\mathcal{T}$. They map points $(\beta,\rho_2,\rho_3,\ldots)$ into new points $(\beta',\rho_2',\rho_3',\ldots)$ with coordinates

$$\beta' = \mathcal{T}_1(\beta|\rho_2,\rho_3,\ldots) \tag{4.2a}$$
$$\rho_n' = \mathcal{T}_n(\beta|\rho_2,\rho_3,\ldots) . \tag{4.2b}$$

The functions $\mathcal{T}_\nu$, $\nu \geq 1$, can be calculated by means of the Migdal recursion formula (2.13) and the decomposition (4.1). All points in the space of coupling constants that can be generated from a given point by one or more renormalization group transformations $\mathcal{T}$ lie on a (discrete) *renormalization group trajectory*. The Boltzmannians (4.1) describe unit lattice theories, which can be classified according to their (dimensionless) correlation lengths.[1] Boltzmannians with finite but different correlation lengths represent the same physical system at different length scales if they lie on a common renormalization group trajectory. The subspace with correlation length zero, $\xi = 0$, is given by a single point, namely by the

[1] The correlation length ξ is defined via the string tension $K_{1/2}$, i.e. $\xi^{-1} = \sqrt{a K_{1/2}}$.

stable fixed point $(0,0,0,...)$ or $g(\theta) \equiv 1$, respectively[1]. Its domain of attraction consists of all Boltzmannians with finite correlation lengths. For example, Wilson Boltzmannians $g_W(\beta,\theta) = \exp(-2\beta \sin^2(\theta/2))$ with positive and finite values of β belong to this domain (see Proposition 2.2). The subspace with infinite correlation length $\xi = \infty$ is called the ***critical surface***. The line $(\beta,0,0,...)$ of Wilson Boltzmannians $g_W(\beta)$ with $\beta \geq 0$ starts for $\beta = 0$ at the stable fixed point $(0,0,0,...)$ and ends in the limit $\beta \to \infty$ on the critical surface in the space of coupling constants. This line will be called *canonical line*. Note that the canonical line is *not* a renormalization group trajectory. If the iteration of renormalization group transformations starts on the canonical line near – but off – the critical surface, i.e. at $(\beta_N,0,0,...)$ for a large but finite value of the *bare coupling constant* β_N, then the renormalization group flow is governed by a ***renormalized trajectory***. The renormalized trajectory $(\beta,\rho_2(\beta),\rho_3(\beta),...)$ connects the stable fixed point (for $\beta = 0$) with an *unstable fixed point* on the critical surface (for $\beta = \infty$). The domain of attraction of this unstable fixed point (which is an ultraviolet fixed point) is a subspace of the critical surface. This subspace includes the "end point" $(\infty,0,0,...)$ of the canonical line. Within this domain of attraction, the renormalized trajectory is the only trajectory which "leaves" the critical surface. The renormalized trajectory governs renormalization and permits via scaling limits (of unit lattice theories with increasing correlation lengths) the construction of effective Boltzmannians for the (hierarchical) $SU(2)$ continuum gauge model. As one-dimensional subspace of all continuum effective Boltzmannians, the renormalized trajectory can be parametrized by a single variable, called ***renormalized coupling constant***. This explains universality, proven together with the existence of the continuum limit in [14,15].

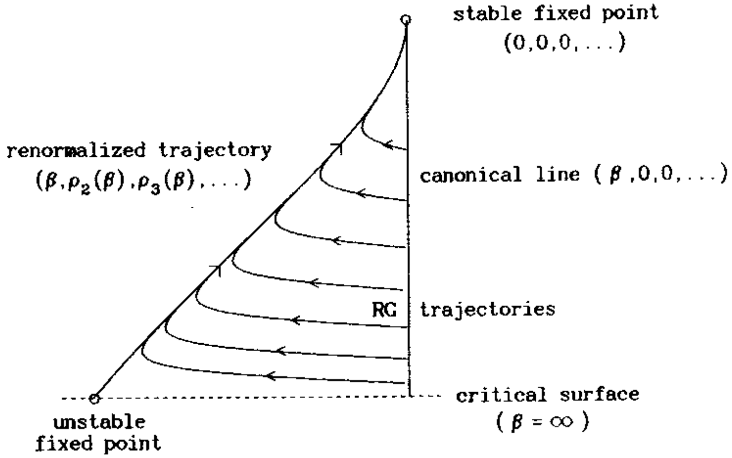


Fig. 4.1. The canonical line $(\beta,0,0,...)$ of Wilson Boltzmannians $g_W(\beta)$ and the renormalized trajectory. Besides the renormalized trajectory, some other renormalization group (RG) trajectories are shown. They start on the canonical line. The renormalization group flow is governed by the renormalized trajectory. All trajectories end in the stable fixed point $g \equiv 1$.

[1] A fixed point is called *stable* if it is approached by all trajectories in its neighbourhood, see the discussion in [2].

Marginal and Irrelevant Interactions

The coupling constants β and ρ_n can be characterized by their scaling properties under renormalization group transformations in a regime off the critical surface but near the unstable fixed point. This behaviour can be studied by linearizing the renormalization group transformation $\mathcal{T}$ around the fixed point, followed by a determination of the eigenvectors ("eigencouplings"). It is then possible to classify the distinct eigenvectors (or eigendirections in the space of coupling constants) according to their eigenvalues. Since all eigenvalues are smaller than or equal to one, there are no *relevant* couplings present. The eigenvector with eigenvalue one represents the *marginal* coupling that corresponds to the unstable direction of the fixed point. All other eigenvalues are smaller than one and belong to *irrelevant* couplings. The same classification holds for the coupling constants β and ρ_n which can be obtained as linear combinations of the eigencouplings. In conclusion, the coupling constant β is marginal whereas the couplings ρ_n are irrelevant with respect to the unstable fixed point.

Perturbation Theory in Irrelevant Interactions

Using the decomposition (4.1), which is based on the split (2.58), the Migdal recursion relations have been expressed through the functions $\mathcal{T}_\nu$ for $\nu \geq 1$, see Eqs. (4.2). The idea is now to expand these functions $\mathcal{T}_\nu(\beta|\rho_2,\rho_3,...)$, $\nu \geq 1$, for fixed β with respect to all remaining variables (irrelevant couplings) ρ_2, ρ_3, ... The expansions are around the point $(\beta,0,0,...)$ on the canonical line. As a result, one obtains (formal) Taylor series expansions in irrelevant couplings. In a next step, the terms of these expansions are reordered according to a "degree of irrelevance". It is assigned to all irrelevant couplings and describes their (canonical) scaling behaviour under (linearized) renormalization group transformations. The resulting expansions are called perturbation expansions in irrelevant interactions or "irrelevant perturbation expansions" for short.

The expansion coefficients of the irrelevant perturbation series depend on the running marginal coupling constant β. For the hierarchical model, the expansion coefficients can be explicitly calculated as functions of β. In case of small or large values of the running coupling constant, these β-dependent coefficients may be further expanded to yield the usual weak or strong coupling coefficients (together with bounded remainders). Moreover, the expansion coefficients can be decomposed into perturbative and nonperturbative parts with respect to the running gauge coupling.

Consider the functions $\mathcal{T}_\nu(\beta_j|\rho_2,\rho_3,...)$, $\nu \geq 1$, and their series expansions in the irrelevant variables ρ_n around the point $(\beta_j,0,0,...)$, where β_j is the running coupling constant. When the (running) irrelevant couplings $\rho_n^{(j)}$ are inserted for the variables ρ_n, one gets – by means of the recursion relations (4.2) – perturbation expansions for the effective (running) coupling constants β_{j-1} and $\rho_n^{(j-1)}$. These irrelevant perturbation expansions converge for a single renormalization group step $j \to j-1$ if the "distance" between the point $(\beta_j,\rho_2^{(j)},\rho_3^{(j)},...)$ and the associated expansion point $(\beta_j,0,0,...)$ is sufficiently small (with respect to a certain metric). In the model under consideration, this "condition of convergence" is fulfilled for *all* renormalization group steps since the "distance" between the renormalized trajectory and the (chosen) canonical line is ***always*** "small enough". Therefore the irrelevant perturbation expansions allow to track the coupling constant flow under iterated renormalization group transformations. In principle, one can thus calculate the ***whole*** renormalized trajectory – from the ultraviolet fixed point for $\beta = \infty$ to the infrared fixed point $(0,0,0,...)$. Note that the expansion point $(\beta_j, 0, 0, ...)$ moves along the canonical line during the iteration. Its "flow" is dictated by the flow of the running coupling constant β_j.

The irrelevant perturbation expansions are expansions in powers and products of irrelevant coupling constants. They are organized with respect to the degree of irrelevance which describes the "influence" of the irrelevant couplings on the renormalization group flow. This is because the irrelevant couplings ρ_n contribute to the coupling constant flow the less, the higher their degree of irrelevance is. In particular, the leading irrelevant couplings, i.e. those with a small degree of irrelevance, give the most important corrections to the zero-order approximations $\mathcal{T}_\nu(\beta|0,0,...)$ for $\mathcal{T}_\nu(\beta|\rho_2,\rho_3,...)$ in Eqs. (4.2). It is remarkable that this feature does not depend on the value of the (marginal) running coupling constant β_j. Note, in addition, that the irrelevant couplings themselves may become (and become indeed) very large for large values of β_j. As a result, series expansions in the degree of irrelevance provide *perturbation* expansions in the sense that *low order* approximations suffice to reliably predict the renormalization group flow of the model. This holds for all values of the running gauge coupling constant.

Taylor Expansions in Irrelevant Interactions

The mappings $\mathcal{T}$ in the space of coupling constants β and ρ_m are considered more explicitly now. In the following, I use the shorthand notations $\mathcal{T}_\nu(\beta|\rho_{\geq 2})$ for $\mathcal{T}_\nu(\beta|\rho_2,\rho_3,...)$. Write

$$\beta_{j-1} = \mathcal{T}_1(\beta_j|\rho^{(j)}_{\geq 2}) \tag{4.3a}$$

$$\rho^{(j-1)}_n = \mathcal{T}_n(\beta_j|\rho^{(j)}_{\geq 2}), \tag{4.3b}$$

where β_{j-1} and $\rho^{(j-1)}_n$ denote the new (effective) running coupling constants after the renormalization group step $j \to j-1$. Note that the first renormalization group step $N \to N-1$ is given by

$$\beta_{N-1} = \mathcal{T}_1(\beta_N|\rho^{(N)}_{\geq 2} = 0)$$

$$\rho^{(N-1)}_n = \mathcal{T}_n(\beta_N|\rho^{(N)}_{\geq 2} = 0) .$$

It is explicitly calculable, as has been discussed in the previous section. Following the idea of a perturbation theory in irrelevant interactions, one begins with formal Taylor expansions of the functions $\mathcal{T}_\nu(\beta_j|\rho_{\geq 2})$ in the irrelevant couplings ρ_n. The expansion point is $(\beta_j, 0, 0, ...)$. After inserting the values $\rho^{(j)}_n$ for ρ_n, one obtains $(\nu \geq 1)$

$$\mathcal{T}_\nu(\beta_j|\rho^{(j)}_{\geq 2}) = \mathcal{T}_\nu(\beta_j|\rho_{\geq 2} = 0) + \sum_{k=2}^{\infty} \frac{\partial \mathcal{T}_\nu(\beta_j|\rho_{\geq 2})}{\partial \rho_k}\bigg|_{\rho_{\geq 2}=0} \rho^{(j)}_k$$
$$+ \sum_{k_1,k_2=2}^{\infty} \frac{1}{2!} \frac{\partial^2 \mathcal{T}_\nu(\beta_j|\rho_{\geq 2})}{\partial \rho_{k_1} \partial \rho_{k_2}}\bigg|_{\rho_{\geq 2}=0} \rho^{(j)}_{k_1} \rho^{(j)}_{k_2} + \text{higher orders} \tag{4.4}$$

All expansion coefficients

$$\frac{\partial^n \mathcal{T}_\nu(\beta_j|\rho_{\geq 2})}{\partial \rho_{k_1} ... \partial \rho_{k_n}}\bigg|_{\rho_{\geq 2}=0} ,$$

which depend on the running coupling constant β_j, can be calculated explicitly within the hierarchical model.

Reordering According to the Degree of Irrelevance

Now I reorder the expansion (4.4) according to a conveniently chosen *degree of irrelevance*. Instead of linearizing the mapping $\mathcal{T}$ around the ultraviolet fixed point and determining its eigenvectors and their eigenvalues, it is more useful to consider the linear term of the above expansion (4.4) near the critical surface, i.e. for $\beta \to \infty$ where the canonical line meets the critical surface. The diagonal elements of the corresponding matrix are given by the limit

$$\lim_{\beta \to \infty} \frac{\partial \mathcal{T}_n(\beta|\rho_{\geq 2})}{\partial \rho_n}\bigg|_{\rho_{\geq 2}=0} .$$

Define the degree of irrelevance $\operatorname{degr} \rho_n$ of the irrelevant coupling ρ_n (and the corresponding interaction) for $n \geq 2$ through these diagonal elements via

$$\operatorname{degr} \rho_n \equiv -\log_{L^D} \left[\lim_{\beta \to \infty} \frac{\partial \mathcal{T}_n(\beta|\rho_{\geq 2})}{\partial \rho_n}\bigg|_{\rho_{\geq 2}=0} \right] \tag{4.5}$$

with $\log_{L^D} x = (D \log L)^{-1} \log x$. One obtains $(D = 4)$

$$\frac{\partial \mathcal{T}_n(\beta|\rho_{\geq 2})}{\partial \rho_n}\bigg|_{\rho_{\geq 2}=0} = L^{-4(n-1)} + O(\beta^{-1}) , \tag{4.6}$$

and hence

$$\operatorname{degr} \rho_n = n - 1 . \tag{4.7}$$

Remarks. (i) This definition of the degree of irrelevance avoids the necessity of calculating and diagonalizing the matrix $\partial \mathcal{T}_n(\beta|\rho_{\geq 2})/\partial \rho_m$ at the fixed point, which is not known a priori.

(ii) The diagonal elements $\partial \mathcal{T}_n/\partial \rho_n(\beta|0)$ display for $\beta \to \infty$ the canonical scaling behaviour of the dimensionless irrelevant couplings ρ_n. To see this, note that the irrelevant interactions $\sin^{2n}(\theta/2)$ correspond for small lattice spacings (i.e. for large dimensionless correlation lengths near the critical surface) to terms $(\operatorname{tr} F^{\mu\nu} F_{\mu\nu})^n$ with canonical dimension $(\text{length})^{-Dn}$. Thus, if the length scale is increased by a factor $L > 1$ due to a block spin transformation, the canonical scaling of the irrelevant couplings is described by $\rho'_n = L^{D(1-n)} \rho_n = L^{-D \operatorname{degr} \rho_n} \rho_n$.

Note that products of couplings occur always in combination with the correspondingly multiplied interactions. Hence, one gets for products of couplings the following degree of irrelevance

$$\operatorname{degr} \prod_i \rho_{n_i} = \sum_i (\operatorname{degr} \rho_{n_i} + 1) - 1 \tag{4.8}$$

Introduce multiindices $X = (n_2, n_3, n_4, ...)$ with $X(k) = n_k$ for $k \geq 2$ and integer numbers $n_k \geq 0$. Set

$$|X| \equiv \sum_{k \geq 2} k\, X(k) \quad , \quad X! \equiv \prod_{k \geq 2} X(k)! , \tag{4.9}$$

and define for finite X (with $|X| < \infty$)

$$\rho^{(j)}_X \equiv \prod_{k \geq 2} {\rho^{(j)}_k}^{X(k)} \quad \text{and} \quad \frac{\partial}{\partial \rho^{(j)}_X} \equiv \prod_{k \geq 2} \frac{\partial^{X(k)}}{\partial {\rho^{(j)}_k}^{X(k)}} . \tag{4.10}$$

Now consider the finite sums ($\nu \geq 1,\ d \geq 2$)

$$\mathcal{T}^{irr}_{\nu;d}(\beta_j|\rho^{(j)}_{\leq d}) \equiv \sum_{X:|X|=d} \frac{1}{X!}\left[\left.\frac{\partial \mathcal{T}_\nu(\beta_j|\rho_{\geq 2})}{\partial \rho_X}\right|_{\rho_{\geq 2}=0}\right]\rho^{(j)}_X$$

$$= \sum_{X:|X|=d} \frac{1}{X!}\left[\left.\frac{\partial \mathcal{T}_\nu(\beta_j|\rho_{\geq 2})}{\partial \rho_X}\right|_{\rho_{\geq 2}=0}\right]\prod_{k=2}^{d} \rho_k^{(j)\,X(k)} \tag{4.11}$$

which consist of all contributions from irrelevant interactions with degree of irrelevance $\mathrm{degr}\,\rho_X = |X|-1 = d-1$ to $\mathcal{T}_\nu(\beta_j|\rho^{(j)}_{\geq 2})$. Then reorder the Taylor series (4.4) according to these contributions with increasing degree of irrelevance. By Eqs. (4.3), this leads to the following set of *recursive flow equations*:

$$\beta_{j-1} = \mathcal{T}_1(\beta_j|\rho^{(j)}_{\geq 2}=0) + \sum_{d=2}^{\infty} \mathcal{T}^{irr}_{1;d}(\beta_j|\rho^{(j)}_{\leq d}) \tag{4.12a}$$

$$\rho_n^{(j-1)} = \mathcal{T}_n(\beta_j|\rho^{(j)}_{\geq 2}=0) + \sum_{d=2}^{\infty} \mathcal{T}^{irr}_{n;d}(\beta_j|\rho^{(j)}_{\leq d}) \ . \tag{4.12b}$$

These flow equations lie at the heart of the analysis to be presented. They express the running couplings β_{j-1}, $\rho_n^{(j-1)}$ of the scale $j-1$ by expanding them with respect to irrelevant couplings $\rho_m^{(j)}$ of the preceeding scale j. The expansions are expansions in products of irrelevant coupling constants with coefficients that depend on the running coupling constant.

Remark. The expansion (2.35) for the reduced activity, $r_j(\theta) = \sum_{n\geq 2} \rho_n^{(j)} \sin^{2n}(\theta/2)$, can also be considered as an expansion into irrelevant interactions with increasing degree of irrelevance, but it is an expansion into irrelevant interactions of the *same* scale j.

A discussion of the leading contributions $\mathcal{T}_\nu(\beta_j|0)$ to the couplings β_{j-1}, $\rho_n^{(j-1)}$ in (4.12) has already been given in Sect. 3. Since there were no irrelevant couplings present in the bare Wilson Boltzmannian $g_N = g_W(\beta_N)$, from which the iteration started, the effective couplings β_{N-1}, $\rho_n^{(N-1)}$ could be fully described by Eqs. (3.26) and (3.27). For a general iteration step $j \to j-1$ with $j < N$, however, they provide only the parts

$$\mathcal{T}_1(\beta_j|0) = \beta_j \, \frac{I_2(2\beta_j)}{I_1(2\beta_j)} \ , \tag{4.13}$$

$$\mathcal{T}_n(\beta_j|0) = \frac{1}{n!}(-\beta_j)^n \sum_{k=0}^{n} \binom{n}{k}\left[(-2)\frac{I_2(2\beta_j)}{I_1(2\beta_j)}\right]^k \sum_{\substack{i_1,i_2\geq 0\\ i_1+i_2=n-k}} \binom{n-k}{i_1,i_2} \prod_{m=1}^{2} \frac{I_{1+i_m}(2\beta_j)}{I_1(2\beta_j)} \ . \tag{4.14}$$

In Sect. 5, I will describe how the other expansion coefficients $\partial\mathcal{T}_\nu(\beta_j|\rho_{\geq 2})/\partial\rho_X|_{\rho_{\geq 2}=0}$, which occur in the contributions $\mathcal{T}^{irr}_{\nu;d}(\beta_j|\rho^{(j)}_{\leq d})$ from the irrelevant interactions, can be computed.

SUMMARY OF RESULTS

Introduce the complex variable

$$z = \sin^2\frac{\theta}{2} \ . \tag{4.15}$$

Proposition 4.1. *For all $j \leq N$, there exist unique functions $G_j(z)$, $R_j(z)$, called "auxiliary activities", such that*

$$G_j(\sin^2\frac{\theta}{2}) = g_j(\theta) \quad , \qquad R_j(\sin^2\frac{\theta}{2}) = r_j(\theta) \ . \tag{4.16}$$

The auxiliary activities G_j and R_j are entire functions of z. They obey the relations

$$G_j(z) = G_W(\beta_j, z)\left[1 + R_j(z)\right] \qquad \textit{with} \qquad G_W(\beta_j, z) = \exp(-2\beta_j z) \tag{4.17}$$

and fulfil the renormalization conditions

$$R_j(0) = 0 \quad , \qquad R_j'(0) = \left.\frac{d}{dz}R_j(z)\right|_{z=0} = 0 \ . \tag{4.18}$$

The proof of this proposition will be given in the next section. Note that (4.17) and (4.18) can be obtained by (4.16) from Eqs. (2.58), (2.56), and from the renormalization conditions (2.61).

Example. The auxiliary activities $G_{N-1}(z)$ and $R_{N-1}(z)$ for the scale $N-1$ are given explicitly in Eqs. (3.13) and (3.22).

As an immediate consequence of Proposition 4.1, the absolute and locally uniform convergence of the series (2.67),

$$r_j(\theta) = \sum_{n=2}^{\infty} \rho_n^{(j)}\left(\sin^2\frac{\theta}{2}\right)^n \ ,$$

follows for all complex θ and each $j \leq N$. This ensures that the use of the parametrization (2.67) does not require any restrictions.

Let $\mathcal{G}_\beta$ denote the space of all entire Boltzmannians $G(z)$ with $G(\bar{z}) = \overline{G(z)}$, $0 < G(z) \leq 1$ for real $z \in [0,1]$, $G(0) = 1$, and $G'(0) = -2\beta$. Each Boltzmannian $G \in \mathcal{G}_\beta$ can be parametrized through its reduced activity R by means of $G = G_W(\beta)[1 + R]$. Denote the space of all $R = G\,G_W(\beta)^{-1} - 1$ for $G \in \mathcal{G}_\beta$ by $\mathcal{G}^{irr}_\beta$.

For finite $\beta \geq 0$, a norm on $\mathcal{G}^{irr}_\beta$ can be defined by (see Sect. 5)

$$\|R\|_{2,\beta} \equiv \left(\max_{\varphi\in[0,2\pi]} \int_0^1 d\mu_\beta(s)\,|R(s\,e^{i\varphi})|^2\right)^{1/2} \quad , \qquad R \in \mathcal{G}^{irr}_\beta \ , \tag{4.19}$$

where $d\mu_\beta(s)$ is the normalized β-dependend measure

$$d\mu_\beta(s) \equiv \frac{ds\,\sqrt{s(1-s)}\,\exp(-4\beta s)}{\int_0^1 ds\,\sqrt{s(1-s)}\,\exp(-4\beta s)} \ . \tag{4.20}$$

Proposition 4.2. (A sufficient criterion for convergence) *Suppose that for fixed $j \leq N$ the estimate*

$$\|R_j\|_{2,\beta_j} < \sqrt{2} - 1 \tag{4.21}$$

holds. Then the irrelevant perturbation series in the recursive flow equations (4.12) *for the effective running couplings β_{j-1}, $\rho_n^{(j-1)}$ converge absolutely for the step $j \to j-1$.*

This proposition will also be proven, together with an improved criterion, in Sect. 5.

Note that the norm $\|R_j\|_{2,\beta_j}$ depends not only on the irrelevant interactions included in the reduced activity R_j, but rather on the *proportion* (or balance) between the reduced activity R_j and the corresponding Wilson part parametrized by the marginal β_j-coupling. One may therefore consider $\|R_j\|_{2,\beta_j}$ as a measure for the "magnitude" of the irrelevant perturbation R_j, where the β_j-dependence of the norm refers to the expansion point, namely the unperturbed Wilson activity $G_W(\beta_j)$.

The norm $\|\cdot\|_{2,\beta_j}$ induces a metric on the space $\mathcal{G}_\beta^{irr}$. Thus $\|R_j\|_{2,\beta_j} = \|R_j - 0\|_{2,\beta_j}$ measures the "distance" between the Wilson Boltzmannian $G_W(\beta_j)$, characterized by $R_W \equiv 0$ in $\mathcal{G}_{\beta_j}^{irr}$, and the effective Boltzmannian G_j, characterized by R_j.

Proposition 4.3. *The reduced activities $R_j \in \mathcal{G}_{\beta_j}^{irr}$ obey for all $j \leq N$ with a sufficiently large running coupling constant β_j the uniform estimates*

$$\|R_j\|_{2,\beta_j} \leq \frac{15\sqrt{105}}{256}\,\beta_j^{-1} + \text{const}\,\beta_j^{-3/2} \;, \tag{4.22}$$

where the constant is of order 1.

Proposition 4.4. *Let the bare coupling constant β_N be chosen arbitrarily. Then*

$$\|R_j\|_{2,\beta_j} \to 0 \qquad \textit{for} \qquad N \geq j \to -\infty \;. \tag{4.23}$$

By Proposition 4.3, Proposition 4.4, and the criterion of convergence, Proposition 4.2, one gets

Theorem 4.5. *Let the bare coupling β_N be chosen large enough. Then the irrelevant perturbation series in the recursive flow equations* (4.12) *converge absolutely for all iteration steps $j \to j-1$ with a sufficiently large or a sufficiently small running coupling constant β_j, i.e. for the weak coupling regime ($N \geq j \geq j_1$) as well as for the strong coupling (high temperature) regime ($j \leq j_2$ with some $j_2 \leq j_1$).*

For the hierarchical $SU(2)$ model, the renormalization group flow can be obtained with great precision by a numerical iteration of the renormalization group transformation. This offers the possibility to study the flow of the running coupling constants. When one starts from a Wilson action with finite bare coupling constant β_N, the resulting (discrete) renormalization group trajectory approaches the renormalized trajectory fairly quickly. The number of iteration steps needed to "reach" the renormalized trajectory depends only logarithmically on β_N. For $\beta_N = 10$, five steps are necessary. The norms $\|R_j\|_{2,\beta_j}$ can be calculated within small errors. The numerical analysis of Sect. 7 shows that they will always stay small enough in order to fulfil the criterion of convergence. This guarantees the convergence of the irrelevant perturbation series for the running coupling constants (4.12).

Let me summarize the essential properties of the proposed perturbation expansion in irrelevant interactions for the example of the hierarchical $SU(2)$ lattice gauge theory model.

Properties of Irrelevant Perturbation Expansions.

(i) Computability. *Consider the recursive flow equations* (4.12). *It is possible to explicitly compute the contributions $T_{1;d}^{irr}(\beta_j|\rho_{\leq d}^{(j)})$, $T_{n;d}^{irr}(\beta_j|\rho_{\leq d}^{(j)})$ from irrelevant interactions of the scale j with degree of irrelevance $(d-1)$ to the effective couplings β_{j-1}, $\rho_n^{(j-1)}$ on the scale $(j-1)$. This holds for all $j \leq N$, for all $n \geq 2$, and for all degrees of irrelevance $(d-1) \geq 1$. Note that the $T_{\nu;d}^{irr}(\beta_j|\rho_{\geq 2}^{(j)})$ are obtained as finite sums, see Eq.* (4.11).

(ii) Convergence. *The irrelevant perturbation series in the recursive flow equations* (4.12) *converge absolutely for all choices of the bare coupling constant $\beta_N > 0$ and for all iteration steps $j \to j-1$. This statement is rigorously proven by Theorem 4.5 both for the weak and the strong coupling regime. Between these regimes, a precise tracking of the renormalized trajectory is possible by numerical iteration of the Migdal recursion formula* (2.13). *Proceeding this way, the norms $\|R_j\|_{2,\beta_j}$ can be calculated accurately. They are bounded by some constant less than $(\sqrt{2}-1)$ and imply thus the convergence due to the criterion of Proposition 4.2. The values for the norms have only a very small β_N-dependence (because the effective Boltzmannians G_j lie almost on the renormalized trajectory). Hence one can deduce that the irrelevant perturbation theory never leaves the "domain of convergence" since the renormalization group flow is governed by the renormalized trajectory.*

(iii) Perturbative vs. Nonperturbative Contributions. *The perturbative expansions in irrelevant interactions could be further expanded and reorganized in powers of β_j^{-1} or β_j, respectively, to yield the conventional weak coupling or high temperature expansions. But note that the weak coupling series for the coupling constant recursion relations are only asymptotic and do not converge, whereas the irrelevant perturbation series in* (4.12) *converge absolutely. This is due to the correct inclusion of nonperturbative contributions with respect to the running gauge coupling constant $\propto \beta_j^{-1/2}$, as has been discussed in some detail in Sect. 3.*

Recursion Relations for Finite Order Approximations

For the purpose of doing calculations up to some finite degree of irrelevance δ, it is useful to study in place of the flow equations (4.12) the following set of *truncated* recursion equations

$$\beta_{j-1,\delta} = T_1(\beta_{j,\delta}|0) + \sum_{d=2}^{\delta+1} T_{1;d}^{irr}(\beta_{j,\delta}|\rho_{Y,\delta}^{(j)}\ {}_{\text{with}\,|Y|=d}) \tag{4.24a}$$

$$\rho_{X,\delta}^{(j-1)} = T_X(\beta_{j,\delta}|0) + \sum_{d=2}^{\delta+1} T_{X;d}^{irr}(\beta_{j,\delta}|\rho_{Y,\delta}^{(j)}\ {}_{\text{with}\,|Y|=d}) \tag{4.24b}$$

with initial values

$$\rho_{X,\delta}^{(N)} = \prod_{k\geq 2} \rho_k^{(N)X(k)} = 0 \qquad \text{for} \qquad r_N = 0 \;. \tag{4.25}$$

The contributions $T^{irr}_{X,\delta}$ are defined as

$$T^{irr}_{X;d}(\beta_{j,\delta}|\rho^{(j)}_{Y,\delta} \text{ with } |Y|=d) \equiv \sum_{Y:|Y|=d} \frac{1}{Y!} \left[\frac{\partial T_X(\beta_{j,\delta}|\rho_{\geq 2})}{\partial \rho_Y} \bigg|_{\rho_{\geq 2}=0} \right] \rho^{(j)}_{Y,\delta} \quad \text{for} \quad d \geq 2 \tag{4.26}$$

with

$$T_X(\beta_j|\rho^{(j)}_{\geq 2}) \equiv \prod_{k\geq 2} T_k(\beta_j|\rho^{(j)}_{\geq 2})^{X(k)} = \prod_{k\geq 2} \rho_k^{(j-1)\,X(k)} = \rho_X^{(j-1)} . \tag{4.27}$$

The same equation holds for $T^{irr}_{1;d}$ when X is replaced by 1. For fixed δ, the coupling $\beta_{j,\delta}$ and the *generalized irrelevant couplings* $\rho^{(j)}_{X,\delta}$ are recursively defined by Eqs. (4.24) and (4.25). Note that

$$\rho^{(j)}_{X,\delta} \neq \prod_{k\geq 2} \rho^{(j)\,X(k)}_{k,\delta} \qquad \text{for} \qquad j < N \tag{4.28}$$

in contrast to Eq. (4.10) for the exact couplings $\rho^{(j)}_X$ and $\rho^{(j)}_k$. The product of couplings $\rho^{(j)}_{k,\delta}$ on the right-hand side of (4.28) contains contributions from irrelevant interactions of the scale $j+1$ with degrees of irrelevance higher than δ. Those contributions are *not* contained in the generalized irrelevant couplings $\rho^{(j)}_{X,\delta}$. By induction, the same holds for the contributions to $\rho^{(j)}_{X,\delta}$ from irrelevant interactions of former scales $j+2$, $j+3$, etc.

In Sect. 7, a numerical investigation of Eqs. (4.24) is given up to the order $\delta = 4$.

5. THE CONDITION OF CONVERGENCE

In this section, the preliminaries are given for the proof of convergence of the irrelevant perturbation series in the recursive flow equations (4.12). After relating the quantities of interest, namely the $T^{irr}_{\nu;d}(\beta_j|\rho^{(j)}_{\leq d})$, to the activities g_j, r_j, the convergence properties of (4.12) are studied. Simple criteria which are sufficient but not necessary in order to prove the convergence will be given.

After the introduction of auxiliary activities $G_j(z)$, $R_j(z)$ and the following proof of Proposition 4.1, interpolating activities $\check{G}_{j-1}(z|\tau_j)$, $\check{R}_{j-1}(z|\tau_j)$ are defined. They depend on a parameter τ_j that modifies the irrelevant interactions of the preceeding scale j for $\tau_j \neq 1$ and switches them off for $\tau_j \to 0$. In addition, one gets corresponding interpolating renormalization constants $\check{N}_j(\tau_j)$ and $\check{\beta}_{j-1}(\tau_j)$. One obtains ($n \geq 2$)

$$T^{irr}_{1;d}(\beta_j|\rho^{(j)}_{\leq d}) = \frac{1}{d!}\frac{\partial^d}{\partial \tau_j^d}\check{\beta}_{j-1}(\tau_j)\bigg|_{\tau_j=0}$$

$$T^{irr}_{n;d}(\beta_j|\rho^{(j)}_{\leq d}) = \frac{1}{d!}\frac{\partial^d}{\partial \tau_j^d}\frac{1}{n!}\frac{\partial^n}{\partial z^n}\check{R}_{j-1}(z|\tau_j)\bigg|_{z=0=\tau_j} ,$$

i.e. the orders of the irrelevant perturbation expansions in the recursive flow equations (4.12) can be classified by powers of the parameter τ_j. By studying the analyticity properties of the interpolating quantities, it turns out that the expansions (4.12) for a single renormalization group step $j \to j-1$ converge if and only if the interpolating normalization factor $\check{N}_j(\tau_j)$ is free from zeroes on the closed unit disc $|\tau_j| \leq 1$ (condition of convergence). Finally, more convenient sufficient criteria for convergence, like $\|R_j\|_{2,\beta_j} < \sqrt{2}-1$ of Proposition 4.2, are formulated. They measure the auxiliary reduced activities $R_j(z)$ with respect to suitably defined norms depending on the running coupling constant β_j.

As an explicit example, the flow equation (4.12a) for β_{j-1} is computed up to the fourth order in the degree of irrelevance.

INTRODUCTION OF AUXILIARY ACTIVITIES G_j, R_j

The monomer activities or effective plaquette Boltzmannians $g_j(\theta)$ are even, 2π-periodic, and entire functions of the complex variable θ (cf. Eqs. (2.25), (2.24), and Proposition 2.1). Of course, the same is true for the reduced activities $r_j(\theta)$. It is useful to introduce "auxiliary" activities $G_j(z)$ and $R_j(z)$ depending on a complex variable z such that the original activities $g_j(\theta)$, $r_j(\theta)$ can be obtained by a replacement of z by $\sin^2(\theta/2)$, i.e.

$$G_j(\sin^2\frac{\theta}{2}) = g_j(\theta) \quad , \quad R_j(\sin^2\frac{\theta}{2}) = r_j(\theta) . \tag{5.1}$$

These functions $G_j(z)$, $R_j(z)$ are not only uniquely defined through the relations (5.1) for arbitrary $j \leq N$, but in addition also entire in z. This is the assertion of Proposition 4.1. The proposition is a simple consequence of

Proposition 5.1. *Let f be an entire function. Suppose that $f(\theta)$ is even and 2π-periodic in θ. Then there exist uniquely determined entire functions $\widetilde{F}(z)$, $F(z)$ which fulfil $\widetilde{F}(\cos\theta) = f(\theta) = F(\sin^2(\theta/2))$.*

PROOF. ⌈ First, consider the function $f(\arccos z)$. Obviously, it is unique and analytic in the complex z plane with branch cuts on the real axis for $|\mathrm{Re}\, z| \geq 1$. Those arguments z correspond via $z = \cos\theta$ to angles θ in the interior of the complex strip $0 \leq \mathrm{Re}\,\theta \leq \pi$. It suffices to consider this strip because of the symmetry and periodicity of f. It is also due to these properties that $f(\arccos z)$ is continuous on the cuts of the arccos.

Now construct the functions $\widetilde{F}$, F of the proposition. To this end, one introduces two functions f_1, f_2 by

$$f_1(z) = f(\arccos z) \quad \text{for} \quad \mathrm{Im}\, z > 0$$
$$f_2(z) = f(\arccos z) \quad \text{for} \quad \mathrm{Im}\, z < 0 ,$$

which are analytic in the upper and lower half plane, respectively. In addition, they are continuous on the real axis, where they are equal valued: $f_1(z) = f_2(z)$. Now choose a region[1] Ω which is devided by the real axis into a region Ω_1 in the upper half plane ($\mathrm{Im}\, z > 0$) and a second region Ω_2 in the lower half plane ($\mathrm{Im}\, z < 0$). The regions are enclosed by positively oriented boundary curves $\partial\Omega_1 = \gamma_1 + \gamma$ and $\partial\Omega_2 = \gamma_2 - \gamma$, respectively. $\gamma = \partial\Omega_1 \cap \partial\Omega_2$ denotes the curve on the real axis. Define the functions

$$F_1(z) = \frac{1}{2\pi i} \oint_{\gamma_1+\gamma} \frac{f_1(\zeta)}{\zeta - z} d\zeta$$
$$F_2(z) = \frac{1}{2\pi i} \oint_{\gamma_2-\gamma} \frac{f_2(\zeta)}{\zeta - z} d\zeta .$$

Note that both functions $F_j(z)$ are analytic in z for such z which lie either in the interior of their closed integration curves or in the outside. Due to Cauchy's integral formula and Cauchy's theorem, one has $F_1(z) = f_1(z)$ for $z \in \Omega_1$ and $F_1(z) = 0$ outside of $\overline{\Omega_1} = \Omega_1 \cup \partial\Omega_1$, whereas $F_2(z) = f_2(z)$ for $z \in \Omega_2$ and $F_2(z) = 0$ in the complement of $\overline{\Omega_2}$. Now consider the function $F_\Omega(z) = F_1(z) + F_2(z)$. It is analytic in Ω_1, Ω_2 and the complement of $\overline{\Omega}$. Note that

$$\int_\gamma \frac{f_1(\zeta)}{\zeta - z} d\zeta + \int_{-\gamma} \frac{f_2(\zeta)}{\zeta - z} d\zeta = \int_\gamma \frac{f_1(\zeta) - f_2(\zeta)}{\zeta - z} d\zeta = 0$$

because $f_1(z) = f_2(z)$ for real z and hence for $z \in \gamma$. Thus one gets

$$F_\Omega(z) = \frac{1}{2\pi i}\left[\int_{\gamma_1} \frac{f_1(\zeta)}{\zeta - z} d\zeta + \int_{\gamma_2} \frac{f_2(\zeta)}{\zeta - z} d\zeta\right] ,$$

where both integrals define analytic functions for arguments beyond their integration curves. Hence, $F_\Omega(z)$ is analytic in the interior Ω of $\gamma_1 + \gamma_2 = \partial\Omega$. Because $f_1(z), f_2(z)$ and $F_\Omega(z)$ are continuous, the equation $F_\Omega(z) = f_1(z) = f_2(z)$ holds on γ.

Finally, inserting the definition of f_1, f_2 yields

$$F_\Omega(z) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f(\arccos \zeta)}{\zeta - z} d\zeta ,$$

where the principal branch of arccos can be taken. The function F_Ω has the desired properties on Ω. Then $\widetilde{F}$ is uniquely defined by analytic continuation of functions F_{Ω_j} for conveniently chosen regions Ω_j.

[1] A *region* is an open and connected subset of the complex plane.

The function F is now simply given by the equation $F(z) = \widetilde{F}(1 - 2z)$ and thus also entire. It obeys $F(\sin^2(\theta/2)) = f(\theta)$ because $1 - 2z = \cos\theta$ for $z = \sin^2(\theta/2)$. ⌋

CONVOLUTION AND RECURSION FORMULAE FOR AUXILIARY ACTIVITIES

As a next step, recursion relations are derived for the auxiliary activities. For this purpose, consider the Migdal recursion formula (2.13) for $L = \sqrt{2}$, namely

$$g^c_{j-1}(u) = \left(\mathcal{N}_j^{-1}\, [g^c_j * g^c_j](u)\right)^2 = \left[\mathcal{N}_j^{-1} \int_{SU(2)} dv\; g^c_j(v)\, g^c_j(uv^{-1})\right]^2 .$$

Then, by use of Eqs. (2.20) and (2.27),

$$g_{j-1}(\theta) = g^c_{j-1}(e^{i\theta\sigma_3}) = \left[\mathcal{N}_j^{-1} \int_{SU(2)} dv \prod_{l=\pm1} g^c_j(e^{i(\theta/2)\sigma_3} v^l)\right]^2 . \tag{5.2}$$

Now set

$$e^{i(\theta/2)\sigma_3} v^{\pm1} = w_\pm\, e^{i\theta_\pm\sigma_3} w_\pm^{-1} \qquad \text{with some} \quad w_\pm \in SU(2) \tag{5.3}$$

to obtain

$$\begin{aligned} g^c_j(e^{i(\theta/2)\sigma_3} v^{\pm1}) &= g^c_j(w_\pm e^{i\theta_\pm\sigma_3} w_\pm^{-1}) = g^c_j(e^{i\theta_\pm\sigma_3}) = g_j(\theta_\pm) = G_j(\sin^2\frac{\theta_\pm}{2}) \\ &= G_j(\frac{1}{4}\,\mathrm{tr}\,(1 - e^{i\theta_\pm\sigma_3})) = G_j(\frac{1}{4}\,\mathrm{tr}\,(1 - w_\pm e^{i\theta_\pm\sigma_3} w_\pm^{-1})) \\ &= G_j(\frac{1}{4}\,\mathrm{tr}\,(1 - e^{i(\theta/2)\sigma_3} v^{\pm1})) , \end{aligned} \tag{5.4}$$

where use has been made of

$$\mathrm{tr}\,(1 - e^{i\theta_\pm\sigma_3}) = 2 - 2\cos\theta_\pm = 4\sin^2\frac{\theta_\pm}{2} . \tag{5.5}$$

Employing $g_{j-1}(\theta) = G_j(\sin^2(\theta/2))$, one gets therefore

$$G_{j-1}(\sin^2\frac{\theta}{2}) = \left[\mathcal{N}_j^{-1} \int_{SU(2)} dv \prod_{l=\pm1} G_j(\frac{1}{4}\,\mathrm{tr}\,(1 - e^{i(\theta/2)\sigma_3} v^l))\right]^2 . \tag{5.6}$$

Finally, parametrize the $SU(2)$-elements v, v^{-1} by polar coordinates on the sphere S^3 like in Sect. 3. This yields, cp. Eqs. (3.7) and (3.5),

$$\mathrm{tr}\,(e^{i(\theta/2)\sigma_3} v^{\pm1}) = 2\left[\cos\frac{\theta}{2}\cos\phi \mp \sin\frac{\theta}{2}\sin\phi\cos\chi\right] , \tag{5.7}$$

whereas the Haar measure dv becomes

$$dv = \frac{1}{\pi} d\phi \sin^2\phi\, d\chi \sin\chi \tag{5.8}$$

after integration of the angle ψ in (3.6). Hence, the recursion relations (5.6) take the form

$$G_{j-1}(\sin^2\frac{\theta}{2}) = \left[\mathcal{N}_j^{-1}\frac{1}{\pi}\int_0^\pi d\phi\sin^2\phi\int_0^\pi d\chi\sin\chi\; G_j(\sin^2\frac{\theta_+}{2})\,G_j(\sin^2\frac{\theta_-}{2})\right]^2 \tag{5.9}$$

with arguments

$$\sin^2\frac{\theta_\pm}{2} = \frac{1}{2}\Big[1-\cos\frac{\theta}{2}\cos\phi \pm \sin\frac{\theta}{2}\sin\phi\cos\chi\Big] \tag{5.10}$$

depending on the integration variables ϕ, χ, and on the external (= "block spin") angle θ. Since the activities $g_{j-1}(\theta)$, $j \le N$, are even and 2π-periodic in θ, it is sufficient to consider only the range $\operatorname{Re}\theta \in [0,\pi]$ with

$$\operatorname{Re}\cos\frac{\theta}{2} = \cos(\frac{\operatorname{Re}\theta}{2})\cosh(\frac{\operatorname{Im}\theta}{2}) \ge 0 \quad , \quad \operatorname{Re}\sin\frac{\theta}{2} = \sin(\frac{\operatorname{Re}\theta}{2})\cosh(\frac{\operatorname{Im}\theta}{2}) \ge 0 \; .$$

Thus one can express $\cos(\theta/2)$ and $\sin(\theta/2)$ in (5.10) by $\sqrt{1-z}$ and $\sqrt{z}$, where the principal branch of the square root is taken, i.e.

$$\sqrt{1-z} = \sqrt{\cos^2\frac{\theta}{2}} = \cos\frac{\theta}{2} \quad \text{with} \quad \operatorname{Re}\cos\frac{\theta}{2} \ge 0$$
$$\sqrt{z} = \sqrt{\sin^2\frac{\theta}{2}} = \sin\frac{\theta}{2} \quad \text{with} \quad \operatorname{Re}\sin\frac{\theta}{2} \ge 0 \; .$$

Abbreviating

$$z = \sin^2\frac{\theta}{2} \quad , \quad z_\pm = \sin^2\frac{\theta_\pm}{2} \, , \tag{5.11}$$

relation (5.10) yields the arguments $z_\pm$ as functions of z

$$z_\pm = \frac{1}{2}\Big[1-\sqrt{1-z}\cos\phi \pm \sqrt{z}\sin\phi\cos\chi\Big] \; . \tag{5.12}$$

It is convenient to introduce the shorthand notation

$$G_{j-1}(z) = \left[\mathcal{N}_j^{-1}\int dv\; G_j(z_+(z))\,G_j(z_-(z))\right]^2 = \left(\mathcal{N}_j^{-1}[G_j * G_j](z)\right)^2 \, , \tag{5.13}$$

where $*$ denotes "convolution" of the z-dependent auxiliary activities $G_j(z)$.

Remarks. (i) Note that the "convolution" $F * H$ of two entire functions F, H is itself entire. This is *not* obvious from its definition,

$$\begin{aligned}[F*H](z) &\equiv \int dv\; F(z_+(z))\,H(z_-(z)) \\ &= \frac{1}{\pi}\int_0^\pi d\phi\sin^2\phi\int_0^\pi d\chi\sin\chi\; F(\tfrac{1}{2}[1-\sqrt{1-z}\cos\phi+\sqrt{z}\sin\phi\cos\chi]) \\ &\quad\times H(\tfrac{1}{2}[1-\sqrt{1-z}\cos\phi-\sqrt{z}\sin\phi\cos\chi]) \; .\end{aligned} \tag{5.14}$$

but becomes clear if, for example, one sets $z = \sin^2(\theta/2)$ and $z_\pm = \sin^2(\theta_\pm/2)$. Then the right-hand side of (5.14) is of course entire in θ. Moreover, one can easily show that it is an even and 2π-periodic function of θ.
For $\theta \to -\theta$, one has $\sin^2(\theta_\pm/2) \to \sin^2(\theta_\mp/2)$. A transformation $\chi \to \pi-\chi$, however, interchanges $\sin^2(\theta_+/2)$ and $\sin^2(\theta_-/2)$ without any further change of the measure or the range of integration. Hence the right-hand side of (5.14) is even in θ.
Now consider $\theta \to \theta + 2\pi k$ implying $\cos\theta_\pm \to (-1)^k\cos\theta_\pm$. Thus only the case of odd k needs to be discussed. Transforming the integration variable ϕ according to $\phi \to \pi-\phi$ yields $(-1)\cos\theta_\pm \to \cos\theta_\mp$. This becomes again $\cos\theta_\pm$ after an additional substitution $\chi \to \pi-\chi$. Therefore the 2π-periodicity of the right-hand side of (5.14) follows.
Proposition 5.1 then proves the analyticity of $[F*H](z)$ for all complex z.

(ii) It should also be mentioned that the convolution can as well be written in the form

$$[f*h](\theta) = \int dv\; f(\theta_+)\,h(\theta_-) \, , \tag{5.15}$$

which can be deduced by setting

$$f(\theta) = F(\sin^2\frac{\theta}{2}) \quad , \quad h(\theta) = H(\sin^2\frac{\theta}{2}) \quad , \quad [f*h](\theta) = [F*H](\sin^2\frac{\theta}{2}) \; .$$

Thus the convolution integral is well defined for arbitrary even and 2π-periodic functions $f(\theta), h(\theta)$. According to (5.10), the arguments $\theta_\pm$ are given by

$$\theta_\pm = \arccos\Big[\cos\frac{\theta}{2}\cos\phi \mp \sin\frac{\theta}{2}\sin\phi\cos\chi\Big] \, , \tag{5.16}$$

where one can generally use the principal branch of arccos. Although the arguments $\theta_\pm$ are affected by the branch cuts, the integrand is single valued, continuous, and holomorphic in $\theta_\pm$ or in θ, ϕ, and χ, respectively.

Introduction of Interpolating Activities

The reduced activities $R_j(z)$ are recursively defined through

$$\begin{aligned}R_{j-1}(z) &= \left(\mathcal{N}_j^{-1}[G_j*G_j](z)\right)^2 G_W(\beta_{j-1},z)^{-1} - 1 \\ &= \left(\mathcal{N}_j^{-1}[G_W(\beta_j)(1+R_j)*G_W(\beta_j)(1+R_j)](z)\right)^2 G_W(\beta_{j-1},z)^{-1} - 1 \\ &= \left[\mathcal{N}_j^{-1}\int dv\prod_{l=\pm} G_W(\beta_j,z_l(z))\,[1+R_j(z_l(z))]\right]^2 G_W(\beta_{j-1},z)^{-1} - 1\end{aligned} \tag{5.17}$$

with renormalization constants $\mathcal{N}_j$ and β_{j-1} fixed by the renormalization conditions

$$R_{j-1}(0) = 0 \quad , \quad \frac{d}{dz}R_{j-1}(z)\Big|_{z=0} = R'_{j-1}(0) = 0 \; . \tag{5.18}$$

In order to evaluate the quantities $\mathcal{T}^{irr}_{\nu;d}(\beta_j|\rho^{(j)}_{\leq d})$ in the recursive flow equations (4.12), it is advantageous to define *interpolating reduced activities* $\check{R}_{j-1}(z|\tau_j)$ for complex parameters τ_j by means of the relations $(j \leq N)$

$$\check{R}_{j-1}(z|\tau_j) = \left[\check{\mathcal{N}}_j(\tau_j)^{-1} \int dv \prod_{l=\pm} G_W(\beta_j, z_l(z))\,[1 + R_j(\tau_j z_l(z))]\right]^2 G_W(\check{\beta}_{j-1}(\tau_j), z)^{-1} - 1 \tag{5.19}$$

and the renormalization conditions $(j \leq N)$

$$\check{R}_{j-1}(0|\tau_j) = 0 \quad , \quad \frac{\partial}{\partial z}\check{R}_{j-1}(z|\tau_j)\Big|_{z=0} = 0 \qquad \text{for all} \quad \tau_j \ . \tag{5.20}$$

The renormalization conditions yield the following set of equations for the *interpolating renormalization constants* $\check{\mathcal{N}}_j(\tau_j)$ and $\check{\beta}_{j-1}(\tau_j)$

$$\check{\mathcal{N}}_j(\tau_j) = \int dv \prod_{l=\pm} G_W(\beta_j, z_l(0))\,[1 + R_j(\tau_j z_l(0))] \ , \tag{5.21}$$

$$\check{\beta}_{j-1}(\tau_j) = -\check{\mathcal{N}}_j(\tau_j)^{-1}\frac{\partial}{\partial z}\int dv \prod_{l=\pm} G_W(\beta_j, z_l(z))\,[1 + R_j(\tau_j z_l(z))]\Big|_{z=0} \ . \tag{5.22}$$

For every $j \leq N$, the complex parameter τ_j modifies the irrelevant interactions on the corresponding scale j. This happens in such a way that all irrelevant interactions on the scale j remain unchanged for $\tau_j = 1$ while they will be switched off when τ_j is set equal to zero. Thus one obtains for $\tau_j = 1$ the quantities of interest, namely

$$\check{\mathcal{N}}_j(1) = \mathcal{N}_j \tag{5.23}$$

$$\check{\beta}_{j-1}(1) = \beta_{j-1} \tag{5.24}$$

$$\check{R}_{j-1}(z|1) = R_{j-1}(z) \ . \tag{5.25}$$

In comparison, the computation of the step $j \to j-1$ with $\tau_j = 0$ is the same as the one for the first renormalization group step $N \to N-1$, except that the bare coupling constant β_N has to be replaced by the effective (or renormalized) coupling constant β_j. Equations (3.10), (3.26), and (3.22), (3.13) yield immediately

$$\check{\mathcal{N}}_j(0) = \frac{I_1(2\beta_j)}{\beta_j}\exp(-2\beta_j) \tag{5.26}$$

$$\check{\beta}_{j-1}(0) = \beta_j\,\frac{I_2(2\beta_j)}{I_1(2\beta_j)} \tag{5.27}$$

$$\check{R}_{j-1}(z|0) = \left[\frac{I_1(2\beta_j\sqrt{1-z})}{\sqrt{1-z}\,I_1(2\beta_j)}\right]^2 \exp(+2\check{\beta}_{j-1}(0)\,z) - 1 \ . \tag{5.28}$$

The main effect of the parameter τ_j, however, is that it allows the bookkeeping of the contributions of irrelevant interactions of the scale j to the activities and effective couplings on the next scale, $j-1$. I will explain this in more detail.

Parametrize the interpolating reduced activities

$$\check{R}_{j-1}(z|\tau_j) = \sum_{n=2}^{\infty} \check{\rho}^{(j-1)}_n(\tau_j)\, z^n \tag{5.29}$$

by *interpolating irrelevant couplings* $\check{\rho}^{(j-1)}_n(\tau_j)$. They are given by $(n \geq 2)$

$$\check{\rho}^{(j-1)}_n(\tau_j) = \frac{1}{n!}\frac{\partial^n}{\partial z^n}\check{R}_{j-1}(z|\tau_j)\Big|_{z=0} \tag{5.30}$$

$$= \frac{1}{n!}\frac{\partial^n}{\partial z^n}\left[\check{G}_{j-1}(z|\tau_j)\exp(+2\check{\beta}_{j-1}(\tau_j)\,z)\right]\Big|_{z=0} \tag{5.31}$$

where

$$\check{G}_{j-1}(z|\tau_j) = \left[\check{\mathcal{N}}_j(\tau_j)^{-1}\int dv \prod_{l=\pm} G_W(\beta_j, z_l(z))\,[1 + R_j(\tau_j z_l(z))]\right]^2 \ . \tag{5.32}$$

Of course, the interpolating irrelevant couplings satisfy

$$\check{\rho}^{(j-1)}_n(1) = \rho^{(j-1)}_n \qquad \text{for all} \qquad n \geq 2 \ . \tag{5.33}$$

The case $\tau_j = 0$ will be described by Eqs. (3.27) with β_N replaced by β_j.

Reformulation of the Recursive Flow Equations (4.12)

Note that the parameters τ_j modify the irrelevant couplings $\rho^{(j)}_n$, $n \geq 2$, according to

$$\rho^{(j)}_n \ \to \ \check{\rho}^{(j)}_n(\tau_j) = \tau_j^n\,\rho^{(j)}_n \ . \tag{5.34}$$

Hence one gets for $\tau_j \neq 1$ the following generalizations of Eqs. (4.3)

$$\check{\beta}_{j-1}(\tau_j) = \mathcal{T}_1(\beta_j|\check{\rho}^{(j)}_{\geq 2}(\tau_j)) \tag{5.35a}$$

$$\check{\rho}^{(j-1)}_n(\tau_j) = \mathcal{T}_n(\beta_j|\check{\rho}^{(j)}_{\geq 2}(\tau_j)) \qquad \text{for} \qquad n \geq 2 \ . \tag{5.35b}$$

The expansion of these functions into powers and products $\rho^{(j)}_X$ of irrelevant couplings $\rho^{(j)}_n$ according to their degrees of irrelevance, $\operatorname{degr}\rho^{(j)}_X = |X| - 1$, as introduced in Sect. 4, leads to

$$\check{\beta}_{j-1}(\tau_j) = \mathcal{T}_1(\beta_j|\check{\rho}^{(j)}_{\geq 2}(\tau_j) = 0) + \sum_{d=2}^{\infty}\mathcal{T}^{irr}_{1;d}(\beta_j|\check{\rho}^{(j)}_{\leq d}(\tau_j)) \tag{5.36a}$$

$$\check{\rho}^{(j-1)}_n(\tau_j) = \mathcal{T}_n(\beta_j|\check{\rho}^{(j)}_{\geq 2}(\tau_j) = 0) + \sum_{d=2}^{\infty}\mathcal{T}^{irr}_{n;d}(\beta_j|\check{\rho}^{(j)}_{\leq d}(\tau_j)) \ . \tag{5.36b}$$

with $(\nu \geq 1,\ d \geq 2)$

$$\mathcal{T}^{irr}_{\nu;d}(\beta_j|\check{\rho}^{(j)}_{\leq d}(\tau_j)) = \sum_{X:|X|=d}\frac{1}{X!}\left[\frac{\partial\mathcal{T}_\nu(\beta_j|\check{\rho}^{(j)}_{\geq 2}(\tau_j))}{\partial\rho^{(j)}_X}\Big|_{\rho^{(j)}_{\geq 2}=0}\right]\rho^{(j)}_X \ . \tag{5.37}$$

Making use of (4.10), (5.34), and (4.9), one gets the relations

$$\frac{\partial}{\partial \rho_X^{(j)}} = \prod_{k\geq 2} \frac{\partial^{X(k)}}{\partial \rho_k^{(j)\,X(k)}} = \prod_{k\geq 2} (\tau_j^k)^{X(k)} \frac{\partial^{X(k)}}{\partial \check{\rho}_k^{(j)\,X(k)}}$$
$$= \tau_j^{\sum_{k\geq 2} k\,X(k)} \prod_{m\geq 2} \frac{\partial^{X(m)}}{\partial \check{\rho}_m^{(j)\,X(m)}} = \tau_j^{|X|} \frac{\partial}{\partial \check{\rho}_X^{(j)}}\ , \qquad (5.38)$$

which, in combination with (5.37), lead to

$$\mathcal{T}^{irr}_{\nu;d}(\beta_j|\check{\rho}^{(j)}_{\leq d}(\tau_j)) = \tau_j^d \sum_{X:|X|=d} \frac{1}{X!} \left[\frac{\partial \mathcal{T}_\nu(\beta_j|\check{\rho}^{(j)}_{\geq 2})}{\partial \check{\rho}^{(j)}_X} \Bigg|_{\check{\rho}^{(j)}_{\geq 2}=0} \right] \rho^{(j)}_X$$
$$= \tau_j^d\ \mathcal{T}^{irr}_{\nu;d}(\beta_j|\rho^{(j)}_{\leq d})\ . \qquad (5.39)$$

Thus Eqs. (5.36) are actually Taylor expansions of $\mathcal{T}_\nu(\beta_j|\check{\rho}^{(j)}_{\geq 2}(\tau_j))$ with respect to the interpolating parameter τ_j. Therefore, one obtains

$$\mathcal{T}^{irr}_{\nu;d}(\beta_j|\rho^{(j)}_{\leq d}) = \frac{1}{d!}\frac{\partial^d}{\partial \tau_j^d}\ \mathcal{T}_\nu(\beta_j|\check{\rho}^{(j)}_{\geq 2}(\tau_j))\Bigg|_{\tau_j=0} \qquad \text{for} \qquad d\geq 2\ . \qquad (5.40)$$

Note that the (d=1)-Taylor coefficients vanish because $R_j(\tau_j z)$ is of order τ_j^2 for small τ_j as a consequence of the renormalization conditions (5.18). These conditions imply

$$\frac{\partial \check{\mathcal{N}}_j}{\partial \tau_j}(0) = 0 \quad \Rightarrow \quad \frac{\partial \check{\beta}_{j-1}}{\partial \tau_j}(0) = 0 \quad \Rightarrow \quad \frac{\partial}{\partial \tau_j}\ \check{R}_{j-1}(z|\tau_j)\Bigg|_{\tau_j=0} = 0\ , \qquad (5.41)$$

which can be read off from Eqs. (5.21), (5.22), and (5.19). Then, by Eq. (5.30),

$$\frac{\partial \check{\rho}_n^{(j-1)}}{\partial \tau_j}(0) = \frac{1}{n!}\frac{\partial^n}{\partial z^n}\left[\frac{\partial}{\partial \tau_j}\ \check{R}_{j-1}(z|\tau_j)\Bigg|_{\tau_j=0}\right]\Bigg|_{z=0} = 0\ . \qquad (5.42)$$

In conclusion, it is possible to reformulate the irrelevant perturbation series in the set of recursive flow equations (4.12) as Taylor series expansions for the interpolating effective couplings $\check{\beta}_{j-1}(\tau_j)$, $\check{\rho}_n^{(j-1)}(\tau_j)$ around $\tau_j = 0$. Setting $\tau_j = 1$ afterwards, one gets the equations (4.12) in the form

$$\beta_{j-1} = \check{\beta}_{j-1}(1) = \sum_{d=0}^{\infty} \frac{1}{d!}\frac{\partial^d}{\partial \tau_j^d}\,\check{\beta}_{j-1}(\tau_j)\Bigg|_{\tau_j=0} \qquad (5.43a)$$

$$\rho_n^{(j-1)} = \check{\rho}_n^{(j-1)}(1) = \sum_{d=0}^{\infty} \frac{1}{d!}\frac{\partial^d}{\partial \tau_j^d}\,\check{\rho}_n^{(j-1)}(\tau_j)\Bigg|_{\tau_j=0}\ . \qquad (5.43b)$$

These representations provide the key for the proof of convergence of the irrelevant perturbation expansions for the running coupling constants (4.12).

STUDY OF CONVERGENCE PROPERTIES

Proposition 5.2. (Condition of Convergence) *The irrelevant perturbation series in the recursive flow equations* (4.12) *for the renormalization group step* $j \to j-1$ *converge absolutely if and only if the interpolating normalization factor* $\check{\mathcal{N}}_j(\tau_j)$, *given by Eq.* (5.21), *is free from zeroes on the closed unit disc* $|\tau_j| \leq 1$.

PROOF. ⌈ Consider the step $j \to j-1$. The interpolating monomer activity $\check{G}_{j-1}(z|\tau_j)$ is given by

$$\check{G}_{j-1}(z|\tau_j) = \left(\check{\mathcal{N}}_j(\tau_j)^{-1}[G_j^{\tau_j} * G_j^{\tau_j}](z)\right)^2 \qquad (5.44)$$

where $G_j^{\tau_j}(z)$ denotes the τ_j-depending activity $G_W(\beta_j, z)[1 + R_j(\tau_j z)]$ and $\check{\mathcal{N}}_j(\tau_j)$ the interpolating normalization factor $[G_j^{\tau_j} * G_j^{\tau_j}](0)$. First, $G_j^{\tau_j}(z)$ is clearly entire in both variables z and τ_j. Then the same holds true for the convolution $[G_j^{\tau_j} * G_j^{\tau_j}](z)$. For fixed parameter τ_j, this can be seen by recalling that the convolution of two functions which are entire with respect to the "class variable" z is itself entire in this variable z, cp. the corresponding remark in Sect. 2. On the other hand, for fixed z, the analyticity of the convolution in the parameter τ_j is a simple consequence of the fact that the integration domain for the convolution integral is compact and that the integrand depends continuously on the integration variables. Now, $\check{\mathcal{N}}_j(\tau_j) = [G_j^{\tau_j} * G_j^{\tau_j}](0)$ is of course an entire function in τ_j. Hence, by the presumption $\check{\mathcal{N}}_j(\tau_j) \neq 0$ for $|\tau_j| \leq 1$, it follows moreover that $\check{\mathcal{N}}_j(\tau_j)$ is different from zero on some open disc $|\tau_j| < 1+\epsilon$ with $\epsilon > 0$. Therefore, the inverse interpolating normalization factor $\check{\mathcal{N}}_j(\tau_j)^{-1}$ is analytic in the region $\{\tau_j : |\tau_j| < 1+\epsilon\}$ at least.

By use of the explicit formulae for the interpolating renormalization constant $\check{\beta}_{j-1}(\tau_j)$, Eq. (5.22), the interpolating reduced activity $\check{R}_{j-1}(z|\tau_j)$, Eq. (5.19), and the interpolating irrelevant couplings $\check{\rho}_n^{(j-1)}(\tau_j)$, Eqs. (5.30), one concludes that they are also analytic on $|\tau_j| < 1+\epsilon$. This implies the existence of power series expansions for these interpolating couplings $\check{\beta}_{j-1}(\tau_j)$, $\check{\rho}_n^{(j-1)}(\tau_j)$ around $\tau_j = 0$ with radii of convergence $1+\epsilon$. Thus the power series converge absolutely and uniformly on the compact domain $|\tau_j| \leq 1 < 1+\epsilon$, for example. By setting $\tau_j = 1$, the assertion follows for the set of equations (5.43) which may be rewritten in the form (4.12). ⌋

Remark. The proof of Proposition 5.2 has shown that the analyticity properties of the interpolating couplings $\check{\beta}_{j-1}(\tau_j)$, $\check{\rho}_n^{(j-1)}(\tau_j)$ are completely determined by those of the corresponding inverse interpolating normalization factor $\check{\mathcal{N}}_j(\tau_j)^{-1}$.

Since $\check{\mathcal{N}}_j$ is entire, the function $1/\check{\mathcal{N}}_j$ is holomorphic in the whole τ_j-plane with exception of the zeroes of $\check{\mathcal{N}}_j$. These zeroes are of course isolated and lie outside of some neighbourhood of the origin because $\check{\mathcal{N}}_j$ is continuous and $\check{\mathcal{N}}_j(0)$ different from zero. The radius of convergence for the power series expansion

$$\check{\mathcal{N}}_j(\tau_j)^{-1} = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{d^k}{d\tau^k}\check{\mathcal{N}}_j(\tau)^{-1}\right]_{\tau=0} \tau_j^k \qquad (5.45)$$

around $\tau_j = 0$ is hence given by the smallest modulus of all zeroes. Now split

$$\check{\mathcal{N}}_j(\tau_j) = \check{\mathcal{N}}_j(0) + \Delta\check{\mathcal{N}}_j(\tau_j) = \check{\mathcal{N}}_j(0)\left[1 + \frac{\Delta\check{\mathcal{N}}_j(\tau_j)}{\check{\mathcal{N}}_j(0)}\right]. \qquad (5.46)$$

The function $\Delta\check{\mathcal{N}}_j(\tau_j)/\check{\mathcal{N}}_j(0)$ is entire in τ_j and fulfils $\Delta\check{\mathcal{N}}_j(0)/\check{\mathcal{N}}_j(0) = 0$. Now one can formulate the ***criterion for convergence***

$$\max_{|\tau_j|\le 1} \left|\frac{\Delta\check{\mathcal{N}}_j(\tau_j)}{\check{\mathcal{N}}_j(0)}\right| < 1\,, \tag{5.47}$$

which is equivalent to $\min_{|\tau_j|\le 1} |\check{\mathcal{N}}_j(\tau_j)| > 0$ and thus sufficient to imply $\check{\mathcal{N}}_j(\tau_j) \neq 0$ for $|\tau_j| \le 1$, i.e. the applicability of the condition of convergence, Proposition 5.2.

The next step will be to express this criterion for convergence in terms of ordinary reduced activities $R_j(z)$ instead of interpolating normalization factors $\check{\mathcal{N}}_j(\tau_j)$.

Introduce the normalized measure

$$d\mu_\beta(s) \equiv \frac{ds\,\sqrt{s(1-s)}\exp(-4\beta s)}{\int_0^1 ds\,\sqrt{s(1-s)}\exp(-4\beta s)} \tag{5.48}$$

for finite $\beta \ge 0$ and consider the spaces $\mathcal{G}_\beta^{irr}$ defined in Sect. 4. Then for every positive β, the mapping $\|\ \|_\beta : \mathcal{G}_\beta^{irr} \to \mathbf{R}$ with

$$\|F\|_\beta \equiv \max_{\tau:|\tau|\le 1}\left|\int_0^1 d\mu_\beta(s)\,F(\tau s)\right| = \max_{\tau:|\tau|=1}\left|\int_0^1 d\mu_\beta(s)\,F(\tau s)\right| \quad , \quad F\in\mathcal{G}_\beta^{irr} \tag{5.49}$$

defines a norm on $\mathcal{G}_\beta^{irr}$.

PROOF. ⌈ Let $F \in \mathcal{G}_\beta^{irr}$. Then $\|F\|_\beta \ge 0$. Furthermore, $F \equiv 0$ implies $\|F\|_\beta = 0$. Suppose, on the other hand, that $\|F\|_\beta = 0$. Thus $\int d\mu_\beta(s)\,F(\tau s)$ is identically zero on the closed unit disk and hence for all τ because it is an entire function. $\tau = 0$ implies $F(0) = 0$. By differentiation it follows that $\int d\mu_\beta(s)\,s^n F^{(n)}(\tau s) = 0$ for all $n \ge 1$. Setting $\tau = 0$ yields $F^{(n)}(0) = 0$ for all n. Hence $\|F\|_\beta = 0$ implies $F = 0$ on the space $\mathcal{G}_\beta^{irr}$.
The property $\|wF\|_\beta = |w|\cdot\|F\|_\beta$ for all complex w and $F \in \mathcal{G}_\beta^{irr}$, is obviously fulfilled. The same is true for the triangle inequality $\|F+H\|_\beta \le \|F\|_\beta + \|H\|_\beta$ for all $F, H \in \mathcal{G}_\beta^{irr}$. ⌋

In addition, norms $\|\ \|_{p,\beta} : \mathcal{G}_\beta^{irr} \to \mathbf{R}$ with $p = 1, 2$ and

$$\|F\|_{p,\beta} \equiv \left(\max_{\tau:|\tau|=1}\int_0^1 d\mu_\beta(s)\,|F(\tau s)|^p\right)^{1/p} \tag{5.50}$$

are introduced. That they satisfy the conditions of a norm is obvious with exception of the triangle inequality for $p = 2$. For a proof, the "Hölder inequality"

$$\|F\cdot H\|_{1,\beta} \le \|F\|_{2,\beta}\cdot\|H\|_{2,\beta} \qquad \text{for all} \quad F, H \in \mathcal{G}_\beta^{irr} \tag{5.51}$$

is needed. It is trivial for $F = 0$ or $H = 0$, and follows for $F \neq 0$ and $H \neq 0$ by

$$\begin{aligned}\frac{\|F\cdot H\|_{1,\beta}}{\|F\|_{2,\beta}\|H\|_{2,\beta}} &= \max_{\tau:|\tau|=1}\int_0^1 d\mu_\beta(s)\,\frac{|F(\tau s)|}{\|F\|_{2,\beta}}\,\frac{|H(\tau s)|}{\|H\|_{2,\beta}} \\ &\le \max_{\tau:|\tau|=1}\int_0^1 d\mu_\beta(s)\left\{\frac{1}{2}\frac{|F(\tau s)|^2}{\|F\|_{2,\beta}^2} + \frac{1}{2}\frac{|H(\tau s)|^2}{\|H\|_{2,\beta}^2}\right\} \\ &\le \frac{1}{2}\frac{1}{\|F\|_{2,\beta}^2}\max_{\tau:|\tau|=1}\int_0^1 d\mu_\beta(s)\,|F(\tau s)|^2 \\ &\quad + \frac{1}{2}\frac{1}{\|H\|_{2,\beta}^2}\max_{\tau:|\tau|=1}\int_0^1 d\mu_\beta(s)\,|H(\tau s)|^2 = \frac{1}{2}+\frac{1}{2} = 1\,,\end{aligned}$$

where use has been made of $|ab| \le (|a|^2+|b|^2)/2$. Now, in order to prove $\|F+H\|_{2,\beta} \le \|F\|_{2,\beta} + \|H\|_{2,\beta}$ for $\|F+H\|_{2,\beta} \neq 0$, one proceeds via

$$\begin{aligned}\|F+H\|_{2,\beta}^2 &= \max_{\tau:|\tau|=1}\int_0^1 d\mu_\beta(s)\,|F(\tau s)+H(\tau s)|^2 \\ &\le \max_{\tau:|\tau|=1}\int_0^1 d\mu_\beta(s)\,\left\{\left|[F(\tau s)+H(\tau s)]\,F(\tau s)\right| + \left|[F(\tau s)+H(\tau s)]\,H(\tau s)\right|\right\} \\ &\le \|[F+H]F\|_{1,\beta} + \|[F+H]H\|_{1,\beta} \le \|F+H\|_{2,\beta}\left(\|F\|_{2,\beta}+\|H\|_{2,\beta}\right).\end{aligned}$$

Proposition 5.3. (Sufficient Criteria for Convergence) *Each of the following inequalities is sufficient to prove the convergence of perturbation series in terms of irrelevant interactions of the scale j :*

(i) $\|(1+R_j)^2 - 1\|_{\beta_j} < 1$ (5.52)

(ii) $\|(1+R_j)^2 - 1\|_{1,\beta_j} < 1$ (5.53)

(iii) $\|R_j\|_{2,\beta_j} < \sqrt{2}-1$ (5.54)

Remark. Proposition 4.2 is obviously given by the criterion (iii) of this lemma.

PROOF. ⌈ It is shown that the first inequality is equivalent to the criterion for convergence (5.47). Then the inequalities (ii) and (iii) are reduced to (i).
Consider the normalization function $\check{\mathcal{N}}_j(\tau_j)$. By its definition and the identity of z_+ and z_- for zero block spin z, i.e. $z_+(0) = (1-\cos\phi)/2 = z_-(0)$, one gets

$$\check{\mathcal{N}}_j(\tau_j) = \int dv\; G_W(\beta_j, z_\pm(0))^2\left[1 + R_j(\tau_j z_\pm(0))\right]^2. \tag{5.55}$$

Setting $z_\pm(0) = \sin^2(\phi/2) \equiv s$, this leads to

$$\begin{aligned}\check{\mathcal{N}}_j(\tau_j) &= \frac{1}{\pi}\int_0^\pi d\phi\sin^2\phi\int_0^\pi d\chi\sin\chi\,\exp\left(-4\beta_j\sin^2\frac{\phi}{2}\right)\left[1+R_j(\tau_j\sin^2\frac{\phi}{2})\right]^2 \\ &= \frac{2}{\pi}\int_0^\pi d\phi\sin^2\phi\,\exp\left(-4\beta_j\sin^2\frac{\phi}{2}\right)\left[1+R_j(\tau_j\sin^2\frac{\phi}{2})\right]^2 \\ &= \frac{8}{\pi}\int_0^1 ds\,\sqrt{s(1-s)}\,\exp(-4\beta_j s)\left[1+R_j(\tau_j s)\right]^2.\end{aligned} \tag{5.56}$$

Thus the function $\Delta\check{\mathcal{N}}_j(\tau_j)/\check{\mathcal{N}}_j(0)$ can be written as an expectation value with respect to the measure $d\mu_{\beta_j}(s)$

$$\frac{\Delta\check{\mathcal{N}}_j(\tau_j)}{\check{\mathcal{N}}_j(0)} = \int_0^1 d\mu_{\beta_j}(s)\left\{[1+R_j(\tau_j s)]^2 - 1\right\}, \tag{5.57}$$

and hence the criterion for convergence takes the form

$$\max_{\tau_j:|\tau_j|\le 1}\left|\frac{\Delta\check{\mathcal{N}}_j(\tau_j)}{\check{\mathcal{N}}_j(0)}\right| \equiv \|(1+R_j)^2 - 1\|_{\beta_j} < 1\,, \tag{5.58}$$

which proves (i). Due to the estimates

$$\|F\|_\beta = \max_{\tau:|\tau|=1} \left|\int_0^1 d\mu_\beta(s)\, F(\tau s)\right| \le \max_{\tau:|\tau|=1} \int_0^1 d\mu_\beta(s)\, |F(\tau s)| = \|F\|_{1,\beta} \tag{5.59}$$

and

$$\begin{aligned}\|(1+R_j)^2 - 1\|_{1,\beta_j} &= \|2R_j + R_j^2\|_{1,\beta_j} \le 2\|R_j\|_{1,\beta_j} + \|R_j^2\|_{1,\beta_j} \\ &\le 2\|1\|_{2,\beta_j}\|R_j\|_{2,\beta_j} + \|R_j\|^2_{2,\beta_j} = \left(1 + \|R_j\|_{2,\beta_j}\right)^2 - 1\,,\end{aligned}$$

one gets the inequalities

$$\|(1+R_j)^2 - 1\|_{\beta_j} \le \|(1+R_j)^2 - 1\|_{1,\beta_j} \le \left(1 + \|R_j\|_{2,\beta_j}\right)^2 - 1\,.$$

Hence, the criterion (i) will imply the convergence if $\|R_j\|_{2,\beta_j} < \sqrt{2}-1$ or $\|(1+R_j)^2-1\|_{1,\beta_j} < 1$, respectively, are fulfilled. This proves (ii) and (iii). ⌋

Computation of the Quantities $\mathcal{T}^{irr}_{\nu;d}(\beta_j|\rho^{(j)}_{\le d})$

The contributions $\mathcal{T}^{irr}_{\nu;d}(\beta_j|\rho^{(j)}_{\le d})$ to the recursive flow equations (4.12) for $j \to j-1$ have been related to the interpolating reduced activity $\tilde{R}_{j-1}(z|\tau_j)$ by Eqs. (5.40), (5.35), and (5.30), e.g.

$$\begin{aligned}\mathcal{T}^{irr}_{\nu;d}(\beta_j|\rho^{(j)}_{\le d}) &= \frac{1}{d!}\frac{\partial^d}{\partial\tau_j^d}\, \mathcal{T}_\nu(\beta_j|\tilde\rho^{(j)}_{\ge 2}(\tau_j))\Big|_{\tau_j=0} = \frac{1}{d!}\frac{\partial^d}{\partial\tau_j^d}\, \tilde\rho^{(j-1)}_\nu(\tau_j)\Big|_{\tau_j=0} \\ &= \frac{1}{d!}\frac{\partial^d}{\partial\tau_j^d}\frac{1}{\nu!}\frac{\partial^\nu}{\partial z^\nu}\, \tilde{R}_{j-1}(z|\tau_j)\Big|_{z=0=\tau_j}\,.\end{aligned} \tag{5.60}$$

It is possible to compute these terms for all (finite) values of ν and d (i.e. for all couplings $\beta_{j-1}, \rho^{(j-1)}_\nu$, and for all degrees of irrelevance), but the resulting formulae will become very complicated. In order to get an impression, consider the irrelevant perturbation expansion for the (unnormalized) convolution $[G_j^{\tau_j} * G_j^{\tau_j}](z)$. It is convenient to set $\rho^{(j)}_0 = 1$ and $\rho^{(j)}_1 = 0$. Then

$$\begin{aligned}[G_j^{\tau_j} * G_j^{\tau_j}](z) &= \int dv \prod_{l=\pm} G_W(\beta_j, z_l(z))\,[1 + R_j(\tau_j z_l(z))] \\ &= \int dv \left[\prod_{l=\pm} G_W(\beta_j, z_l(z))\right]\left(\sum_{n=0}^\infty \rho^{(j)}_n \tau_j^n z_+(z)^n\right)\left(\sum_{m=0}^\infty \rho^{(j)}_m \tau_j^m z_+(z)^m\right) \\ &= \sum_{n,m=0}^\infty \tau_j^{n+m}\rho^{(j)}_n\rho^{(j)}_m\, F_{nm}(\beta_j, z)\end{aligned} \tag{5.61}$$

where the "moments"

$$F_{nm}(\beta_j, z) \equiv \int dv\; G_W(\beta_j, z_+(z))\, G_W(\beta_j, z_-(z))\; z_+(z)^n\, z_-(z)^m \tag{5.62}$$

are entire functions of z. Note that

$$\check{\mathcal{N}}_j(0) = [G_j^{\tau_j} * G_j^{\tau_j}](0)\big|_{\tau_j=0} = F_{00}(\beta_j, 0)\,. \tag{5.63}$$

Introduce β_j-dependent expansion coefficients

$$L^{(k)}_{nm}(\beta_j) \equiv (2 - \delta_{nm})\, \check{\mathcal{N}}_j(0)^{-1}\, \frac{1}{k!}\frac{\partial^k}{\partial z^k} F_{nm}(\beta_j, z)\Big|_{z=0}\,. \tag{5.64}$$

Then the "almost normalized" convolution $\check{\mathcal{N}}_j(0)^{-1}[G_j^{\tau_j} * G_j^{\tau_j}](z)$ becomes

$$\begin{aligned}\check{\mathcal{N}}_j(0)^{-1}[G_j^{\tau_j} * G_j^{\tau_j}](z) &= \sum_{k=0}^\infty \sum_{n,m=0}^\infty \tau_j^{n+m}\rho^{(j)}_n\rho^{(j)}_m\, \check{\mathcal{N}}_j(0)^{-1}\frac{1}{k!}\frac{\partial^k}{\partial z^k}F_{nm}(\beta_j, z)\Big|_{z=0} z^k \\ &= \sum_{k=0}^\infty \sum_{\substack{n,m=0\\ n\le m}}^\infty \tau_j^{n+m}\rho^{(j)}_n\rho^{(j)}_m\, L^{(k)}_{nm}(\beta_j)\, z^k\,.\end{aligned} \tag{5.65}$$

The normalization factor gets the form

$$\begin{aligned}\check{\mathcal{N}}_j(\tau_j)\check{\mathcal{N}}_j(0)^{-1} &= \sum_{\substack{n,m=0\\ n\le m}}^\infty \tau_j^{n+m}\rho^{(j)}_n\rho^{(j)}_m\, L^{(0)}_{nm}(\beta_j) \\ &= 1 + \sum_{\substack{n,m=0\\ n\le m \text{ with } n+m\ge 2}}^\infty \tau_j^{n+m}\rho^{(j)}_n\rho^{(j)}_m\, L^{(0)}_{nm}(\beta_j)\end{aligned} \tag{5.66}$$

because of $L^{(0)}_{00}(\beta_j) = 1$ and $\rho^{(j)}_0 = 1$, $\rho^{(j)}_1 = 0$. In App. A it will be shown that the expansion coefficients $L^{(k)}_{nm}(\beta)$ can be obtained as derivatives

$$L^{(k)}_{nm}(\beta) = F_{00}(\beta, 0)^{-1}(2 - \delta_{nm})\left(-\frac{1}{2}\right)^{n+m}\frac{\partial^n}{\partial\beta^n}\frac{\partial^m}{\partial\beta'^m}\frac{1}{k!}\frac{\partial^k}{\partial z^k}F(\beta, \beta'; z)\Big|_{z=0,\beta'=\beta} \tag{5.67}$$

of the "generating function"

$$F(\beta, \beta'; z) \equiv [G_W(\beta) * G_W(\beta')](z) = 2\, e^{-(\beta+\beta')}\, \frac{I_1\!\left(\sqrt{(\beta+\beta')^2 - 4\beta\beta' z}\right)}{\sqrt{(\beta+\beta')^2 - 4\beta\beta' z}}\,. \tag{5.68}$$

Explicit expressions for all expansion coefficients $L^{(k)}_{nm}(\beta)$ that are needed for the evaluation of the recursive flow equations (4.24) up to a degree of irrelevance $\delta = 4$ are summarized in App. D.

Example. Consider Eq. (5.22) for the interpolating running coupling constant $\tilde\beta_{j-1}(\tau_j)$. Using the expansions (5.65), (5.66) given above, one gets

$$\tilde\beta_{j-1}(\tau_j) = -\left[1 + \sum_{\substack{n,m=0\\ n\le m \text{ with } n+m\ge 2}}^\infty \tau_j^{n+m}\rho^{(j)}_n\rho^{(j)}_m\, L^{(0)}_{nm}(\beta_j)\right]^{-1} \sum_{\substack{n,m=0\\ n\le m}}^\infty \tau_j^{n+m}\rho^{(j)}_n\rho^{(j)}_m\, L^{(1)}_{nm}(\beta_j) \tag{5.69}$$

Further expansion with respect to τ_j yields

$$\tilde{\beta}_{j-1}(\tau_j) = \mathcal{T}_1(\beta_j|\rho^{(j)}_{\geq 2} = 0) + \sum_{d=2}^{\infty} \mathcal{T}^{irr}_{1;d}(\beta_j|\rho^{(j)}_{\leq d})\,\tau_j^d \tag{5.70}$$

with

$$\mathcal{T}_1(\beta_j|\rho^{(j)}_{\geq 2} = 0) = \tilde{\beta}_{j-1}(0) = -L^{(1)}_{00}(\beta_j) = \beta_j\,\frac{I_2(2\beta_j)}{I_1(2\beta_j)} \tag{5.71}$$

and

$$\mathcal{T}^{irr}_{1;d}(\beta_j|\rho^{(j)}_{\leq d}) = \frac{1}{d!}\,\frac{\partial^d}{\partial \tau_j^d}\,\tilde{\beta}_{j-1}(\tau_j)\Big|_{\tau_j=0} \ . \tag{5.72}$$

The contributions from irrelevant interactions with degrees of irrelevance $(d-1) = 1,2,3,4$ are

$$\mathcal{T}^{irr}_{1;2}(\beta_j|\rho^{(j)}_2) = -\rho^{(j)}_2\Big[L^{(1)}_{02}(\beta_j) - L^{(1)}_{00}(\beta_j)\,L^{(0)}_{02}(\beta_j)\Big] \tag{5.73a}$$

$$\mathcal{T}^{irr}_{1;3}(\beta_j|\rho^{(j)}_{\leq 3}) = -\rho^{(j)}_3\Big[L^{(1)}_{03}(\beta_j) - L^{(1)}_{00}(\beta_j)\,L^{(0)}_{03}(\beta_j)\Big] \tag{5.73b}$$

$$\begin{aligned}\mathcal{T}^{irr}_{1;4}(\beta_j|\rho^{(j)}_{\leq 4}) = {}& -\rho^{(j)}_4\Big[L^{(1)}_{04}(\beta_j) - L^{(1)}_{00}(\beta_j)\,L^{(0)}_{04}(\beta_j)\Big] \\ & - \left(\rho^{(j)}_2\right)^2\Big[L^{(1)}_{22}(\beta_j) - L^{(1)}_{02}(\beta_j)\,L^{(0)}_{02}(\beta_j) \\ & \qquad - L^{(1)}_{00}(\beta_j)\,L^{(0}_{22}(\beta_j) + L^{(1)}_{00}(\beta_j)\,L^{(0)}_{02}(\beta_j)^2\Big]\end{aligned} \tag{5.73c}$$

$$\begin{aligned}\mathcal{T}^{irr}_{1;5}(\beta_j|\rho^{(j)}_{\leq 5}) = {}& -\rho^{(j)}_5\Big[L^{(1)}_{05}(\beta_j) - L^{(1)}_{00}(\beta_j)\,L^{(0)}_{05}(\beta_j)\Big] \\ & - \rho^{(j)}_2\rho^{(j)}_3\Big[L^{(1)}_{23}(\beta_j) - L^{(1)}_{03}(\beta_j)\,L^{(0)}_{02}(\beta_j) - L^{(1)}_{02}(\beta_j)\,L^{(0)}_{03}(\beta_j) \\ & \qquad - L^{(1)}_{00}(\beta_j)\,L^{(0)}_{23}(\beta_j) + 2\,L^{(1)}_{00}(\beta_j)\,L^{(0)}_{02}(\beta_j)\,L^{(0)}_{03}(\beta_j)\Big] \ .\end{aligned} \tag{5.73d}$$

6. THE PROOF OF CONVERGENCE

The convergence proof for weak coupling is completed by establishing bounds for the reduced activities $R_j(z)$ on the closed unit disc $|z| \leq 1$. They allow to derive bounds on the norms $\|(1+R_j)^2 - 1\|_{1,\beta_j}$ and $\|R_j\|_{2,\beta_j}$, respectively. Proposition 4.3 is proven.

6.1. INTRODUCTION AND PRELIMINARY CONSIDERATIONS

In this section, the setup of the proof is given. The bounds, which are needed in order to estimate the interpolating normalization factors, must hold on the closed unit disc $|z| \leq 1$, but have to be iterated on so-called Ω-regions. The smallest Ω-region that includes the unit disc is $\Omega(\bar{\eta})$ with $\bar{\eta} = \log(3+\sqrt{8})$. This region corresponds to a strip $|\operatorname{Im}\theta| < \bar{\eta}$ in the complex θ-plane.

The criteria for convergence, Proposition 5.3, offer the possibility to prove the convergence of the irrelevant perturbation series by bounding the reduced activities $R_j(z)$ on the closed unit disc. Since the activities R_j are recursively defined, it is thus necessary to find a bound which can be iterated. Consider a single renormalization group step $j \to j-1$ for some fixed $j \leq N$. Suppose that the reduced activity R_j is given together with the running coupling constant β_j. Further assume that the modulus of R_j obeys a bound of the form

$$|R_j(z)| \leq C\,\beta_j|z|^2\exp(\sigma\beta_j h(z)) \tag{6.1}$$

on the closure $\overline{\Omega}$ of a given region Ω in the complex z-plane. The constants C and σ should be of the order 1, and independent of β_j and the scale j. Let the function h be real-valued and positive.

Now express the effective reduced activity R_{j-1} in terms of the preceeding one, R_j, by means of the recursion relation (5.17),

$$R_{j-1}(z) = \left[\mathcal{N}_j^{-1}\int dv \prod_{l=\pm} G_W(\beta_j, z_l(z))\,[1 + R_j(z_l(z))]\right]^2 G_W(\beta_{j-1}, z)^{-1} - 1\ . \tag{6.2}$$

This formula includes the renormalization constants $\mathcal{N}_j$ and β_{j-1} that have to be fixed according to the renormalization conditions (5.18). One obtains

$$\mathcal{N}_j = \int dv \prod_{l=\pm} G_W(\beta_j, z_l(0))\,[1 + R_j(z_l(0))]\ , \tag{6.3}$$

$$\beta_{j-1} = -\,\mathcal{N}_j^{-1}\,\frac{\partial}{\partial z}\int dv \prod_{l=\pm} G_W(\beta_j, z_l(z))\,[1 + R_j(z_l(z))]\Big|_{z=0}\ . \tag{6.4}$$

The bound (6.1) iterates if it is possible to show that it holds also for $|R_{j-1}(z)|$ when β_j is replaced by the new effective coupling β_{j-1}, e.g. when

$$|R_{j-1}(z)| \leq C\,\beta_{j-1}|z|^2\exp(\sigma\beta_{j-1}h(z)) \tag{6.5}$$

with the same constants C, σ and the same function h as in (6.1). Of course, the bound (6.5) must be valid on $\overline{\Omega}$ again.

The region Ω has to satisfy two requirements, namely

(i) it must allow the iteration of bounds, i.e.

$$\{z_\pm(z) | \, z \in \Omega\} \subset \Omega \, , \qquad (6.6)$$

(ii) and it should include the unit disc,

$$\{z | \, |z| < 1\} \subset \Omega \, . \qquad (6.7)$$

In the following, I define regions $\Omega(\eta)$ which fulfil the first condition, (6.6), for all values of the parameter $\eta \geq 0$. Furthermore, it will be shown that these regions also fulfil condition (ii), (6.7), provided that η is chosen larger than or equal to $\tilde\eta = \log(3 + \sqrt{8})$.

COMPLEX REGIONS $\Omega(\eta)$ IN THE z-PLANE

Define regions $\Omega(\eta)$ depending on a positive parameter η by

$$\Omega(\eta) \equiv \left\{ z \, \middle| \, \left(\frac{\operatorname{Re} z - \frac{1}{2}}{\frac{1}{2}\cosh\eta} \right)^2 + \left(\frac{\operatorname{Im} z}{\frac{1}{2}\sinh\eta} \right)^2 < 1 \right\} \qquad (6.8)$$

and call them *Ω-regions* in the following. Their boundaries $\partial\Omega(\eta)$ are ellipses with focal points $F_1 = 1$, $F_2 = 0$, and symmetry axes $\operatorname{Re} z = 1/2$ and $\operatorname{Im} z = 0$. The lengths of the half axes of $\partial\Omega(\eta)$ are given by $a_\eta = (\cosh\eta)/2$ and $b_\eta = (\sinh\eta)/2$.

Points $z \in \partial\Omega(\eta)$ correspond via $z = \sin^2(\theta/2)$ to angles θ with $|\operatorname{Im}\theta| = \eta$, i.e.

$$z \in \partial\Omega(\eta) \quad \Leftrightarrow \quad |\operatorname{Im}\theta| = \eta \quad \text{for} \quad z = \sin^2\frac{\theta}{2} \, . \qquad (6.9)$$

This can easily be seen by writing

$$z = \sin^2\frac{\theta}{2} = \frac{1}{2}(1 - \cos\theta) = \frac{1}{2}(1 - \cos\omega\cosh\eta) \pm i\frac{1}{2}\sin\omega\sinh\eta \quad \text{for} \quad \theta = \omega \pm i\eta \, .$$

Then

$$\left(\frac{\operatorname{Re} z - \frac{1}{2}}{\frac{1}{2}\cosh\eta} \right)^2 + \left(\frac{\operatorname{Im} z}{\frac{1}{2}\sinh\eta} \right)^2 = \cos^2\omega + \sin^2\omega = 1 \, .$$

As a consequence, strips $|\operatorname{Im}\theta| < \eta$ in the complex θ-plane correspond to regions $\Omega(\eta)$ in the complex z-plane. Note that all Ω-regions include the interval $[0, 1]$ which corresponds to real angles θ.

Lemma 6.1. *Consider the angles $\theta_\pm$ which are defined as functions of the "block spin" angle θ and the $SU(2)$ "fluctuation fields" $v^{\pm 1}$ by Eq. (5.3), i.e. by*

$$e^{i(\theta/2)\sigma_3} v^{\pm 1} = w_\pm e^{i\theta_\pm \sigma_3} w_\pm^{-1} \qquad \textit{with conveniently chosen} \quad w_\pm \in SU(2) \, .$$

Then

$$|\operatorname{Im}\theta_\pm| \leq \frac{1}{2}|\operatorname{Im}\theta| \qquad (6.10)$$

and

$$z_\pm = \sin^2\frac{\theta_\pm}{2} \in \overline{\Omega(\eta(z)/2)} \qquad \textit{for} \qquad z = \sin^2\frac{\theta}{2} \in \overline{\Omega(\eta(z))} \, . \qquad (6.11)$$

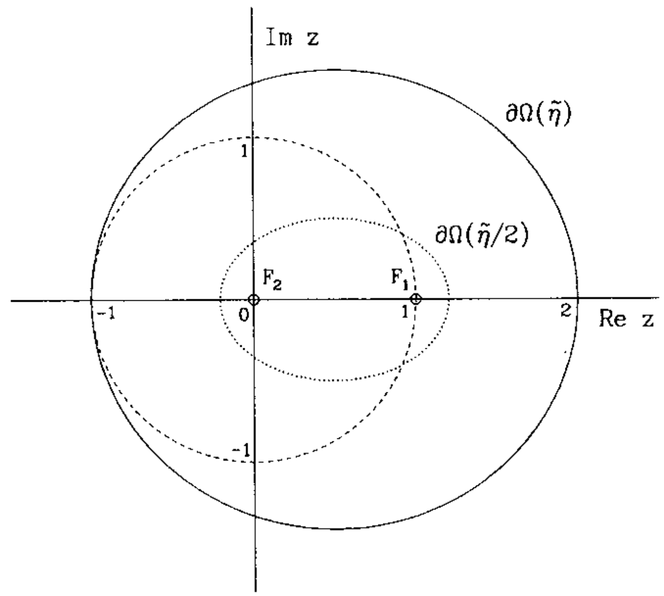


Fig. 6.1. The boundary curves $\partial\Omega(\tilde\eta)$, $\partial\Omega(\tilde\eta/2)$ of the two Ω-regions $\Omega(\tilde\eta)$, $\Omega(\tilde\eta/2)$ within the complex z-plane. F_1 and F_2 are the focal points of the ellipses $\partial\Omega(\eta)$ for $\eta > 0$. The parameters $\tilde\eta$ is given by arccosh 3.

PROOF. ⌈ Using Eq. (5.10),

$$z_\pm = \sin^2\frac{\theta_\pm}{2} = \frac{1}{2}\left[1 - \cos\frac{\theta}{2}\cos\phi \pm \sin\frac{\theta}{2}\sin\phi\cos\chi\right] ,$$

one obtains for $|\operatorname{Im}\theta| = \eta(z) > 0$

$$\begin{aligned} \left(\frac{\operatorname{Re} z_\pm - \frac{1}{2}}{\frac{1}{2}\cosh\frac{1}{2}\eta(z)} \right)^2 + \left(\frac{\operatorname{Im} z_\pm}{\frac{1}{2}\sinh\frac{1}{2}\eta(z)} \right)^2 &= \left(-\cos\frac{\operatorname{Re}\theta}{2}\cos\phi \pm \sin\frac{\operatorname{Re}\theta}{2}\sin\phi\cos\chi \right)^2 \\ &\quad + \left(\sin\frac{\operatorname{Re}\theta}{2}\cos\phi \pm \cos\frac{\operatorname{Re}\theta}{2}\sin\phi\cos\chi \right)^2 \\ &= \cos^2\phi + \sin^2\phi\cos^2\chi \\ &\leq \cos^2\phi + \sin^2\phi = 1 \, . \end{aligned}$$

This implies $z_\pm = \overline{\Omega(\eta(z)/2)}$ and $|\operatorname{Im}\theta_\pm| \leq \frac{1}{2}|\operatorname{Im}\theta|$, respectively. ⌋

Suppose that a given point z lies on the ellipse $\partial\Omega(\eta)$. Then the sum of the distances of z to the focal points F_1 and F_2 of the ellipse, $|1 - z| + |z|$, is obviously equal to the length of their great half axis multiplied by two. Thus

$$2a_\eta = \cosh\eta = |1 - z| + |z| \, .$$

Define therefore the function

$$\eta(z) \equiv \operatorname{arccosh}\left(|1-z|+|z|\right) \geq 0 \ . \tag{6.12}$$

Note that $\eta(\sin^2(\theta/2)) = \operatorname{arccosh}(\cosh(\operatorname{Im}\theta)) = |\operatorname{Im}\theta|$. Now consider the smallest closed Ω-domain that includes the closed unit disc $|z| \leq 1$. It is denoted $\overline{\Omega(\bar{\eta})}$. The parameter $\bar{\eta}$ is defined by

$$\bar{\eta} \equiv \min\{\eta | S^1 \subset \overline{\Omega(\eta)}\} \ ,$$

and can be calculated with help of (6.12)

$$\begin{aligned} \bar{\eta} &= \max_{z:|z|=1} \operatorname{arccosh}\left(|1-z|+|z|\right) = \operatorname{arccosh}\left(1 + \max_{z:|z|=1} |1-z|\right) \\ &= \operatorname{arccosh} 3 = \log\left(3+\sqrt{8}\right) \ . \end{aligned} \tag{6.13}$$

For further use, one has

$$\cosh\bar{\eta} = 3 \quad , \qquad \cosh\frac{\bar{\eta}}{2} = \left(\frac{1+\cosh\bar{\eta}}{2}\right)^{1/2} = \sqrt{2} \ . \tag{6.14}$$

As a preliminary result, one chooses the complex domain $\overline{\Omega(\bar{\eta})}$ for the iteration of the bound (6.1). In order to be precise, the bound (6.1) is completed by setting $h(z) = f(\eta(z))$, i.e.

$$|R_j(z)| \leq C\beta_j |z|^2 \exp(\sigma\beta_j f(\eta(z))) \tag{6.15}$$

where the function f is given by

$$f(\eta) \equiv 4\,\frac{\sinh^4(\eta/2)}{\cosh\eta} \qquad \text{for} \qquad 0 \leq \eta \leq \bar{\eta} \tag{6.16}$$

as will be shown later on. In the following, I will discuss why it should be possible to iterate the bound (6.15).

Role of the Renormalization Conditions

In order to establish the bound (6.5) for the reduced activity $R_{j-1}(z)$, one first has to show that $R_{j-1}(z)$ behaves like z^2 for small fields z with $|z| \leq O(\beta_{j-1}^{-1/2})$. This small field behaviour is a consequence of the renormalization conditions $R_{j-1}(0) = 0$ and $R'_{j-1}(0) = 0$ which guarantee that all relevant or marginal interactions included in the right-hand side of (6.2) cancel each other exactly. However, in general one is not able to calculate the renormalization constants (6.3), (6.4) precisely and in closed form, and these renormalization cancellations can not be exploited. Note that any approximative usage of the renormalization constants would destroy the desired small field behaviour of the effective reduced activity R_{j-1}. Instead, one has to take advantage of the renormalization conditions themselves. The key is given by the analyticity properties of the reduced activities. One has the simple but important

Lemma 6.2. *The functions $z^{-2}R_j(z)$ are entire for all $j \leq N$.*

Proof. ⌈ The assertion is a consequence of the fact that the reduced activities $R_j(z)$ are entire and satisfy the renormalization conditions $R_j(0) = 0$ and $R'_j(0) = 0$ for every $j \leq N$ (Proposition 4.1). ⌋

First estimate the reduced activity $R_{j-1}(z)$ outside some neighbourhood of the origin, e.g.

$$\begin{aligned} |z^{-2}R_{j-1}(z)| &\leq \max_{\zeta\in\partial\Omega(\eta(z))} |\zeta^{-2}R_{j-1}(\zeta)| \\ &\leq bound_{j-1}(\eta(z)) \qquad \text{for} \qquad \eta(z) \geq \eta_{min} = O(\beta_{j-1}^{-1/4}) \ . \end{aligned} \tag{6.17}$$

Here $bound_{j-1}(\eta)$ is some function that estimates the function $\max_{\zeta\in\partial\Omega(\eta)} |\zeta^{-2}R_{j-1}(\zeta)|$ from above. (It will be chosen later on in such a way that its scale dependence is only due to the running coupling constant β_{j-1}, cp. the bound (6.5).) Afterwards, in a second step, one uses Lemma 6.2 in order to apply the maximum principle and to extend the validity of the obtained bound onto the whole region $\Omega(\eta_{min})$

$$\begin{aligned} |z^{-2}R_{j-1}(z)| &\leq \max_{\zeta\in\partial\Omega(\eta_{min})} |\zeta^{-2}R_{j-1}(\zeta)| \\ &\leq bound_{j-1}(\eta_{min}) \qquad \text{for} \qquad z \in \Omega(\eta_{min}) \ . \end{aligned} \tag{6.18}$$

Therefore, combining the results,

$$|R_{j-1}(z)| \leq |z|^2\, bound_{j-1}\left(\max\{\eta_{min}, \eta(z)\}\right) \ . \tag{6.19}$$

Note that $bound_{j-1}(\eta)$ is always positive. Due to the maximum principle, which implies $\max_{\zeta\in\partial\Omega(\eta_1)} |\zeta^{-2}R_{j-1}(\zeta)| < \max_{\zeta\in\partial\Omega(\eta_2)} |\zeta^{-2}R_{j-1}(\zeta)|$ for $\eta_1 < \eta_2$, $bound_{j-1}(\eta)$ can be chosen to be a monotonuously increasing function of $\eta \geq 0$. Thus one may also write

$$|R_{j-1}(z)| \leq |z|^2 \left(bound_{j-1}(\eta_{min}) + \left[bound_{j-1}(\eta(z)) - bound_{j-1}(0)\right]\right) \ . \tag{6.20}$$

The estimates (6.19) and (6.20), respectively, prove the expected small field behaviour of the effective reduced activity $R_{j-1}(z)$.

Demanded Precision of Renormalization Constants

In order to estimate $\max_{\zeta\in\partial\Omega(\eta)} |\zeta^{-2}R_{j-1}(\zeta)|$ for $\eta \geq \eta_{min}$, cp. (6.17), one uses Eq. (6.2) together with (6.3) and (6.4). Since $\eta \geq \eta_{min}$ excludes very small fields z, it is sufficient to evaluate the renormalization constants $\mathcal{N}_j$ and β_j only approximately but with small and bounded errors. For example, instead of the normalization factor $\mathcal{N}_j$, one may insert the simple and very convenient approximation $\check{\mathcal{N}}_j(0) = \beta_j^{-1} I_1(2\beta_j)\exp(2\beta_j)$ into Eq. (6.2). The coupling constant β_{j-1}, on the other hand, does occur in the new bound (6.5) and has therefore to replace the old coupling β_j. Thus, by using (6.2) with $\mathcal{N}_j \to \check{\mathcal{N}}_j(0)$ and $\beta_j \to \beta_{j-1}$, one gets an "almost renormalized" reduced activity that is much better to handle and to bound than the "completely renormalized" one. Then of course, as a next step, one must take into account also the approximation errors. They are uniformly bounded in j by

$$\left|\frac{\mathcal{N}_j - \check{\mathcal{N}}_j(0)}{\check{\mathcal{N}}_j(0)}\right| \leq \text{const}\,\beta_j^{-1} \quad , \qquad \left|\frac{\beta_{j-1}-\beta_j}{\beta_j}\right| \leq \text{const}\,\beta_j^{-1} \tag{6.21}$$

provided that β_j is large enough. Hence, the renormalization constants can be written as

$$\mathcal{N}_j = \check{\mathcal{N}}_j(0)[1 + O(\beta_j^{-1})] \quad , \qquad \beta_{j-1} = \beta_j[1 + O(\beta_j^{-1})] \tag{6.22}$$

where the error terms $O(\beta_j^{-1})$ represent remainders bounded by const β_j^{-1} with some constant independent of j. Then, in order to get rigorous bounds for $|\zeta^{-2}R_{j-1}(\zeta)|$, one uses "almost renormalized" reduced activities given by (6.2) with renormalization constants (6.22) instead of $\mathcal{N}_j \to \mathcal{N}_j(0)$, $\beta_j \to \beta_{j-1}$.

To summarize, the renormalization constants may be calculated only approximately provided that the errors are bounded and sufficiently small. As will be discussed in Sect. 6.2, this information is achieved by controlling the renormalization group flow.

Iteration of Bounds on the Complex Region $\Omega(\bar{\eta})$ Including the Unit Disc

Assume that all β_j in the right-hand side of (6.20) have been replaced by β_{j-1} according to (6.22) and that a bound of the desired form

$$|R_{j-1}(z)| \le C'\beta_{j-1}|z|^2 \exp(\sigma'\beta_{j-1}f(\eta(z))) \qquad \text{for} \qquad z \in \overline{\Omega(\bar{\eta})} \tag{6.23}$$

results. I proceed by qualitatively relating the constants C' and σ' to the original ones C and σ. The constants C, C' are correlated with the small field behaviour of the reduced activities R_j, R_{j-1}. Recall the parametrization of R_j by irrelevant couplings $\rho_n^{(j)}$,

$$R_j(z) = \sum_{n=2}^{\infty} \rho_n^{(j)} z^n .$$

The canonical scaling behaviour of these couplings under a single renormalization group step with scale factor $L = \sqrt{2}$ is described by $\rho_n^{(j)} \to L^{-D\,\mathrm{degr}\,\rho_n^{(j)}}\rho_n^{(j)} = 4^{-(n-1)}\rho_n^{(j)}$, cp. Sect. 4. Now consider the reduced activity R_j for small fields $|z| \le O(\beta_j^{-1/2})$, where it is essentially given by the leading irrelevant part, i.e. $R_j(z) \sim \rho_2^{(j)}z^2$. Correspondingly, one obtains $R_{j-1}(z) \sim \rho_2^{(j-1)}z^2$. There are two kinds of contributions to the effective coupling $\rho_2^{(j-1)}$. First, one has a contribution that is generated by the marginal interaction of the Wilson activity $G_W(\beta_j)$. It may be written as (see Eq. (4.14))

$$\mathcal{T}_2(\beta_j|\rho_{\ge 2}^{(j)}) = \beta_j^2\left[\frac{I_3(2\beta_j)}{I_1(2\beta_j)} - \left(\frac{I_2(2\beta_j)}{I_1(2\beta_j)}\right)^2\right] = -\frac{1}{2}\beta_j + O(1) . \tag{6.24}$$

Contributions of a second type owe their existence to the irrelevant interactions of the reduced activity R_j. The main contribution comes from the leading irrelevant coupling $\rho_2^{(j)}$. According to its canonical scaling, it is given by

$$\mathcal{T}_{2;2}^{irr}(\beta_j|\rho_2^{(j)}) = \frac{1}{4}\rho_2^{(j)} + \rho_2^{(j)}O(\beta_j^{-1}) = \frac{1}{4}\rho_2^{(j)} + O(1) . \tag{6.25}$$

Altogether, one gets

$$\rho_2^{(j-1)} = -\frac{1}{2}\beta_j + \frac{1}{4}\rho_2^{(j)} + O(1) \tag{6.26}$$

where the contributions from the higher irrelevant couplings are also included in the remainder

term $O(1)$ (see the discussion in Sect. 6.2). Using $|\rho_2^{(j)}| \le C\beta_j$ and $|\rho_2^{(j-1)}| \le C'\beta_{j-1}$, it follows

$$|\rho_2^{(j-1)}| \le \frac{1}{2}\beta_j + \frac{1}{4}|\rho_2^{(j)}| + O(1) \le \frac{1}{2}\beta_j + \frac{1}{4}C\beta_j + O(1) .$$

This may be rewritten and further estimated by

$$|\rho_2^{(j-1)}| \le \left[\frac{1}{2} + \frac{1}{4}C + O(\beta_{j-1}^{-1})\right]\beta_{j-1} \le C'\beta_{j-1} .$$

Trying now to bound C' by C again, one gets the condition

$$\frac{1}{2} + \frac{1}{4}C + O(\beta_{j-1}^{-1}) \le C , \tag{6.27}$$

which has solutions for $C \ge 2/3 + O(\beta_{j-1}^{-1})$ or $C \ge 2/3 +$ const β_{j-1}^{-1}, respectively. Since const is independent of j, the parameter C can also be chosen independent of j provided that the running coupling constant stays large enough.

The renormalization procedure ensures that the reduced activities consist only of irrelevant interactions which will be scaled down by renormalization group transformations according to their canonical scaling behaviour. Although the resulting scaling effect on the small field bounds $C\beta_j|z|^2$ is reduced by the contributions from the marginal interaction, which drives the irrelevant terms in every step anew, a netto decrease of the small field bounds can be expected (and shown) for large enough C. This decrease is parametrized for a step $j \to j-1$ by replacing $C\beta_j$ by $C\beta_{j-1}$.

Remark. Since the recursion relations are not linear in the reduced activities R_j, a more realistic – but nevertheless still simplified – estimate would yield

$$|R_{j-1}(z)| \le \left[\frac{1}{2} + \frac{C}{4}\left\{1 + \sum_{d=1}^{3}\frac{1}{4}\left(\frac{4}{d+1}\right)\left(\frac{C}{16}\right)^d\right\} + O(\beta_{j-1}^{-1})\right]\beta_{j-1}|z|^2$$

for $|z| \le \beta_{j-1}^{-1/2}$ (cp. the rigorous calculation in Sect. 6.3). As a consequence, the lower bound on C will be increased while simultaneously an additional upper bound results

$$\frac{1}{2} + \frac{C}{4}\left\{1 + \sum_{d=1}^{3}\frac{1}{4}\left(\frac{4}{d+1}\right)\left(\frac{C}{16}\right)^d\right\} + O(\beta_{j-1}^{-1}) \le C . \tag{6.28}$$

The bounds will leave only a small window for C. Note that this range would increase considerably only if the scale factor L is chosen much larger than $\sqrt{2}$. For example, one would get

$$\frac{5}{8} + \frac{C}{16}\left\{1 + \sum_{d=1}^{3}\frac{1}{4}\left(\frac{4}{d+1}\right)\left(\frac{C}{256}\right)^d\right\} + O(\beta_{j-1}^{-1}) \le C$$

for the Migdal recursion (2.4) with $L = 2$.

Now consider the constants σ and σ'. Insert the bound (6.1) with $h(z) = f(\eta(z))$ into the following estimate for $|R_{j-1}|$

$$
\begin{aligned}
|R_{j-1}| \le & \left|\left(\mathcal{N}_j^{-1}[G_W(\beta_j) * G_W(\beta_j)]\right)^2 G_W(\beta_{j-1})^{-1} - 1\right| \\
& + \mathcal{N}_j^{-2}\,|G_W(\beta_{j-1})|^{-1}\Big\{2\big[|G_W(\beta_j)| * |G_W(\beta_j)|\big] \\
& \times \Big(2\big[|G_W(\beta_j)| * |G_W(\beta_j)R_j|\big] + \big[|G_W(\beta_j)R_j| * |G_W(\beta_j)R_j|\big]\Big) \\
& + \Big(2\big[|G_W(\beta_j)| * |G_W(\beta_j)R_j|\big] + \big[|G_W(\beta_j)R_j| * |G_W(\beta_j)R_j|\big]\Big)^2\Big\}
\end{aligned} \tag{6.29}
$$

which can be obtained by use of the triangle inequality after a convenient split of the recursion relation (6.2). Then use $\eta(z_\pm(z)) \le \eta(z)/2$ (cp. Lemma 6.1) to take an overall factor $\exp(4\sigma\beta_j f(\eta(z)/2))$ out of the right-hand side of this formula. Note that this factor follows from estimating the term

$$
\begin{aligned}
\Big(\big[|G_W(\beta_j)R_j| * |G_W(\beta_j)R_j|\big](z)\Big)^2 \le & \exp\Big(4\sigma\beta_j f(\tfrac{1}{2}\eta(z))\Big) \\
& \times \left(C^2\beta_j^2 \int dv\, |G_W(\beta_j, z_+(z))\, G_W(\beta_j, z_-(z))|\, |z_+(z)|^2 |z_-(z)|^2\right)^2 .
\end{aligned} \tag{6.30}
$$

By bounding the remaining products of convolution integrals with $|R_j(z)| \le C\beta_j|z|^2$ instead of (6.15), one finally gets the estimate

$$
|R_{j-1}(z)| \le C'\beta_{j-1}|z|^2 \exp(\sigma_{new}\beta_{j-1} f(\eta(z))) \exp(4\sigma\beta_j f(\tfrac{1}{2}\eta(z))) \tag{6.31}
$$

where the constant $\sigma_{new} > 1$ will be independent of j. Thus, in order to obtain the desired bound (6.23) with $\sigma' \le \sigma$, the following inequality must be satisfied

$$
\exp(\sigma_{new}\beta_{j-1} f(\eta(z))) \exp(4\sigma\beta_j f(\tfrac{1}{2}\eta(z))) \le \exp(\sigma\beta_{j-1} f(\eta(z))) \quad \text{for} \quad z \in \overline{\Omega(\bar\eta)} . \tag{6.32}
$$

This is equivalent to the condition

$$
\sigma \ge \sigma_{new}\left(1 - 4\frac{f(\frac{1}{2}\eta)}{f(\eta)}\right)^{-1} [1 + O(\beta_{j-1}^{-1})] \qquad \text{for all} \qquad z \in \overline{\Omega(\bar\eta)} . \tag{6.33}
$$

The function

$$
\frac{f(\frac{1}{2}\eta)}{f(\eta)} = \left(\frac{\sinh\frac{1}{4}\eta}{\sinh\frac{1}{2}\eta}\right)^4 \frac{\cosh\eta}{\cosh\frac{1}{2}\eta} = \frac{1}{4}\, \frac{2\cosh^2\frac{1}{2}\eta - 1}{[\cosh\frac{1}{2}\eta + 1]^2 \cosh\frac{1}{2}\eta} \tag{6.34}
$$

is monotonously increasing on $[0, \bar\eta]$. Thus one obtains

$$
\sigma \ge \sigma_{new}\left(1 - 4\frac{f(\frac{1}{2}\bar\eta)}{f(\bar\eta)}\right)^{-1} [1 + O(\beta_{j-1}^{-1})] \tag{6.35}
$$

with

$$
4\frac{f(\frac{1}{2}\bar\eta)}{f(\bar\eta)} = \frac{3}{\sqrt{2}(1+\sqrt{2})^2} = \frac{3}{4+3\sqrt{2}} < \frac{3}{8} ; \tag{6.36}
$$

providing for σ the lower bound

$$
\sigma > \left(1 - 4\frac{f(\frac{1}{2}\bar\eta)}{f(\bar\eta)}\right)^{-1} = \frac{4+3\sqrt{2}}{1+3\sqrt{2}} > 1.5722 . \tag{6.37}
$$

Need for Improved Bounds on the Unit Disc

It will be proven in Sect. 6.3 that the bounds (6.15) iterate on the domain $\overline{\Omega(\bar\eta)}$ when C is correctly adjusted and σ exceeds some lower bound. But, unfortunately, for the possible choices of σ these bounds are not good enough to yield reasonable estimates of the norms $\|(1+R_j)^2 - 1\|_{\beta_j}$. The explanation is the following. Consider the reduced activity $R_j(z)$ which is assumed to be bounded by (6.15) whose parameters C and σ should fulfil the conditions mentioned above. The bound holds on $\overline{\Omega(\bar\eta)}$ and hence especially on the closed unit disc $|z| \le 1$. Then estimate

$$
\begin{aligned}
\|(1+R_j)^2 - 1\|_{\beta_j} &= \max_{\tau:|\tau|=1}\left|\int_0^1 d\mu_{\beta_j}(s)\left[2R_j(\tau s) + R_j(\tau s)^2\right]\right| \\
&\le \max_{\tau:|\tau|=1}\int_0^1 d\mu_{\beta_j}(s)\left[2|R_j(\tau s)| + |R_j(\tau s)|^2\right] \\
&\le \int_0^1 d\mu_{\beta_j}(s)\Big[2C\beta_j s^2 \exp(\sigma\beta_j f(\eta(-s))) \\
&\qquad + C^2\beta_j^2 s^4 \exp(2\sigma\beta_j f(\eta(-s)))\Big] .
\end{aligned} \tag{6.38}
$$

The $d\mu_{\beta_j}(s)$-measure, Eq. (5.48), includes an exponential damping factor $\exp(-4\beta_j s)$ that has to be used in order to dominate the exponentially increasing bound. Thus one gets the condition

$$
\exp(2\sigma\beta_j f(\eta(-s))) \exp(-4\beta_j s) < 1 \qquad \text{for all} \qquad s \in [0,1] . \tag{6.39}
$$

For $s \to 1$, $f(\eta(-s))$ tends to $f(\bar\eta) = 4/3$. Hence one would need $\sigma < 3/2$ at least. But this is already in contradiction with the lower bound (6.37) for σ.

Generation of Improved Unit Disc Bounds

First, one could think of introducing much more complicated bounds and trying to iterate them. But there is another way. One can proceed as follows.

(i) Iterate the bound (6.1) with $h(z) = f(\eta(z))$ on $\overline{\Omega(\bar\eta/2)}$ instead of $\overline{\Omega(\bar\eta)}$. It will be called *Ω-bound* in the following.

(ii) Then use the Ω-bounds for the reduced activities $R_j(z)$ in order to generate *unit disc bounds* on $|z| \le 1$ whose form is accommodated to the requirements of estimating the norms $\|(1+R_j)^2 - 1\|_{\beta_j}$. This means that the unit disc bounds do not have to iterate by themselves and may therefore be chosen more complicated than the Ω-bounds

(6.1). The unit disc bound for a reduced activity $R_j(z)$ will be generated by first recurring to the preceeding scale, $j+1$, and then performing a single renormalization group step, $j+1 \to j$. As a consequence of Lemma 6.1, the activity $R_j(z)$ can now be bounded on $\overline{\Omega(\bar\eta)}$ and the closed unit disc, respectively, by use of the Ω-bound for $R_{j+1}(z)$ on $\overline{\Omega(\bar\eta/2)}$.

The results of the following sections 6.2 - 6.5 are summarized in

Proposition 6.3. *Assume that for some fixed scale $j \le N$ with a sufficiently large running coupling constant β_j the reduced activity $R_j(z)$ obeys the bound*

$$|R_j(z)| \le \beta_j |z|^2 \exp\Big(\frac{5}{2}\beta_j f(\eta(z))\Big) \qquad \textit{for} \qquad z \in \overline{\Omega(\bar\eta/2)}\ , \tag{6.40}$$

where the function $f(\eta)$ is given by Eq. (6.16). Furthermore, assume that the leading irrelevant couplings $\rho_2^{(j)}$, $\rho_3^{(j)}$, and $\rho_4^{(j)}$ fulfil the bounds

$$|\rho_2^{(j)}| \le \beta_j \ , \quad |\rho_3^{(j)}| \le O(\beta_j) \ , \quad |\rho_4^{(j)} - \frac{1}{2}(\rho_2^{(j)})^2| \le O(\beta_j)\ . \tag{6.41}$$

Then the following estimates hold:

(i) *renormalization constants*

$$\mathcal{N}_j = \mathcal{N}_{j0}[1 + O(\beta_j^{-1})] \qquad \textit{with} \qquad \mathcal{N}_{j0} = [G_W(\beta_j) * G_W(\beta_j)](0)$$

$$\beta_{j-1} < \beta_j \qquad \textit{with} \qquad \beta_j - \beta_{j-1} = O(1) \tag{6.42}$$

(ii) *leading irrelevant couplings*

$$|\rho_2^{(j-1)}| \le \beta_{j-1} \ , \quad |\rho_3^{(j-1)}| \le O(\beta_{j-1}) \ , \quad |\rho_4^{(j-1)} - \frac{1}{2}(\rho_2^{(j-1)})^2| \le O(\beta_{j-1}) \tag{6.43}$$

(iii) *inductive bounds*

$$|R_{j-1}(z)| \le \beta_{j-1} |z|^2 \exp\Big(\frac{5}{2}\beta_{j-1} f(\eta(z))\Big) \qquad \textit{for} \qquad z \in \overline{\Omega(\bar\eta/2)} \tag{6.44}$$

(iv) *norm estimates*

$$\|(1 + R_{j-1})^2 - 1\|_{1,\beta_{j-1}} \le \frac{75}{128}\beta_{j-1}^{-1} + O(\beta_{j-1}^{-3/2}) \tag{6.45a}$$

$$\|R_{j-1}\|_{2,\beta_{j-1}} \le \frac{15\sqrt{105}}{256}\beta_{j-1}^{-1} + O(\beta_{j-1}^{-3/2})\ . \tag{6.45b}$$

Outline of Proof. ⌈ The proof of (i) and (ii) is given in Sect. 6.2. The bound (iii) is proven in Sect. 6.3. After generating an additional bound for $|R_{j-1}(z)|$ on the closed unit disc, $|z| \le 1$, in Sect. 6.4 (Lemma 6.12), the norm estimates (iv) are calculated in Sect. 6.5. ⌋

6.2. CONTROL OF THE RENORMALIZATION GROUP FLOW

For sufficiently weak running gauge coupling, i.e. large enough β_j, the renormalization group flow is controlled by studying a system of recursion relations for the running coupling constant β_j and the leading irrelevant couplings $\rho_n^{(j)}$, $n \le 4$, with bounded remainders that depend on all other couplings.

Since the proof is restricted to small values of the running gauge coupling, i.e. large β_j, it is possible to use weak coupling expansions in order to establish the recursion relations for the running coupling constant and the leading irrelevant couplings. Although the weak coupling expansions do not converge, they are nevertheless useful in low order calculations because their remainders can be bounded rigorously.

Rigorous Perturbative Treatment by Low Order Asymptotic Expansions

In our case, perturbation expansions for the coupling constant recursion relations (2.68) or of the expansion coefficients $L_{nm}^{(k)}(\beta_j)$ in powers of the running gauge coupling constant $\beta_j^{-1/2}$ are actually *asymptotic expansions* with respect to the variable β_j. Although these asymptotic expansions do not converge, cp. Sect. 3, they are nevertheless usable for the desired rigorous treatment. This is because

(i) the perturbative or asymptotic approximation up to a fixed order l becomes arbitrarily good when β_j^{-1} tends to zero.

(ii) the remainders, which include both perturbative and nonperturbative parts with respect to β_j^{-1}, can be bounded by const$(l)\beta_j^{-(l+1)}$ with some constant that is independent of β_j and j, but depends on the order l.

Thus β_j^{-1} has to be the smaller, the higher the needed precision is.

As a next step, it has to be discussed to what order one must expand.

Discussion of the Renormalization Group Function

What one actually needs is the flow of the running coupling constant. It can be obtained approximately as solution of a finite system of recursion relations for the running coupling constant itself and for the leading irrelevant couplings. The solution describes the cutoff-dependence of the bare coupling constant, or, equivalently, the dependence of the renormalized coupling constant on the renormalization scale. The qualitative behaviour of the renormalization group flow, however, can easily be derived as follows. Consider the $SU(2)$ continuum gauge field theory. Its renormalization group function $\gamma(g)$ for the scale dependent gauge coupling $g = g(a)$ is given by

$$\gamma(g) = \frac{dg}{d\ln a} = \gamma_3 g^3 + \gamma_5 g^5 + O(g^7) \tag{6.46}$$

for small values of g. The coefficients γ_3, γ_5 are given by a two-loop calculation and do not depend on the renormalization scheme. Explicitly, they are

$$\gamma_3 = \frac{11}{24\pi^2} \quad , \quad \gamma_5 = \frac{17}{96\pi^4}\ . \tag{6.47}$$

After separating the variables g and a, one integrates

$$\int^a d\ln a = \int^{g(a)} \frac{dg}{\gamma_3 g^3 + \gamma_5 g^5 + O(g^7)} + \text{const}$$
$$= \int^{g(a)} \frac{1}{\gamma_3 g^3}\Big[1 - \frac{\gamma_5}{\gamma_3} g^2 + O(g^4)\Big] dg + \text{const} ,$$

where the integration constant must be determined by a convenient renormalization prescription which fixes the running gauge coupling on some renormalization scale (within the weak coupling regime). The result expresses the scale a as a nonperturbative function of the running gauge coupling constant g, i.e.

$$\ln a = -\frac{1}{2\gamma_3 g^2} - \frac{\gamma_5}{2\gamma_3^2} \ln g^2 + O(g^2) + \text{const} , \tag{6.48}$$

and implicitly defines the inverse function $g(a)$. Note that the inclusion of higher order terms in (6.46) would only give rise to small perturbative corrections to the leading nonperturbative behaviour. Now rewrite all these equations for the running coupling constant $\beta = 4\,g^{-2}$. Using $d\beta = -8\,g^{-3} dg$, one gets

$$\frac{d\beta}{d\ln a} = -8\gamma_3 - 32\gamma_5\beta^{-1} + O(\beta^{-2}) . \tag{6.49}$$

Especially, this yields the following formula for cutoff (or scale) changes $a \to a' = L^k a$ with $\beta \to \beta'$

$$\beta' - \beta = -8\gamma_3 \ln L^k + 4\frac{\gamma_5}{\gamma_3}(\ln\beta' - \ln\beta) + O(\beta^{-2}) . \tag{6.50}$$

Thus, by expanding with respect to powers of β^{-1}, one obtains for a single renormalization group step with scale factor L

$$\Delta\beta = \beta' - \beta = -(8\gamma_3 \ln L) - (32\gamma_5 \ln L)\beta^{-1} + O(\beta^{-2}) . \tag{6.51}$$

In conclusion, one must compute the coupling constant recursion relations up to the order β_j^{-1} at least.

Recursion Relations for Weak Coupling

For the following weak coupling analysis, only small Ω-regions $\Omega(\eta_j)$ with $\eta_j = \beta_j^{-1/4}$ are needed. They correspond to small strips $|\operatorname{Im}\theta| < \eta_j$ in the complex θ-plane.

Proposition 6.4.. *Assume that β_j is large enough and that the reduced activity R_j obeys the bound*

$$|R_j(z)| \leq 2\beta_j\, |z|^2 \qquad \textit{for} \qquad z \in \overline{\Omega(\eta_j)} \tag{6.52}$$

with $\eta_j \equiv \beta_j^{-1/4}$. Furthermore, assume that the leading irrelevant couplings fulfil the bounds

$$|\rho_2^{(j)}| \leq \beta_j \quad , \quad |\rho_3^{(j)}| \leq O(\beta_j) \quad , \quad |\rho_4^{(j)} - \tfrac{1}{2}(\rho_2^{(j)})^2| \leq O(\beta_j) . \tag{6.53}$$

Then the new coupling constants $\beta' = \beta_{j-1}$ and $\rho_k' = \rho_k^{(j-1)}$ are given in terms of $\beta = \beta_j$ and $\rho_k = \rho_k^{(j)}$ by the following formulae

$$\beta' = \beta - \frac{3}{4} - \frac{5}{8}\rho_2\beta^{-1} + \frac{3}{32}\beta^{-1} + \frac{15}{64}\rho_2\beta^{-2} - \frac{105}{128}\rho_3\beta^{-2} - \frac{25}{64}\rho_2^2\beta^{-3} + O(\beta^{-2}) \tag{6.54a}$$

$$\rho_2' = \frac{1}{4}\rho_2 - \frac{1}{2}\beta + \frac{3}{4} + \frac{7}{16}\rho_2\beta^{-1} + \frac{21}{32}\rho_3\beta^{-1} + \frac{11}{32}\rho_2^2\beta^{-2} + O(\beta^{-1}) \tag{6.54b}$$

$$\rho_3' = \frac{1}{16}\rho_3 - \frac{1}{4}\beta + \frac{1}{8}\rho_2 + O(\beta^0) . \tag{6.54c}$$

For the running coupling constant β_{j-1} one gets

$$\beta_{j-1} < \beta_j \qquad \text{with} \qquad \beta_j - \beta_{j-1} = O(1) . \tag{6.55}$$

The normalization factor $\mathcal{N}_j$ is given by

$$\mathcal{N}_j = \mathcal{N}_{j0}[1 + O(\beta_j^{-1})] \qquad \textit{with} \qquad \mathcal{N}_{j0} = [G_W(\beta_j) * G_W(\beta_j)](0) . \tag{6.56}$$

The error terms $O(\beta_j^{-n})$ represent remainders (including perturbative and nonperturbative contributions) that are bounded by const β_j^{-n} with some constant independent of j.

Note that the assumptions (6.52) and (6.53) of the proposition are consequences of (6.40) and (6.41) in Proposition 6.3.

Proof. [The proof uses low order perturbation expansions for β_{j-1} and for the leading irrelevant couplings. The expansions are organized in powers of the running gauge coupling constant $g_j = 2\beta_j^{-1/2}$. The remainders will be bounded rigorously.

The reduced activity $R_j(z)$ satisfy the renormalization conditions (5.18) and can therefore be represented by the Taylor expansion

$$R_j(z) = \sum_{n=2}^{4} \rho_n^{(j)} z^n + \tilde{R}_j^5(z) \tag{6.57a}$$

with remainder

$$\tilde{R}_j^5(z) = z^5 \int_0^1 dt\, \frac{(1-t)^4}{4!} R_j^{(5)}(zt) . \tag{6.57b}$$

The couplings $\rho_n^{(j)}$ are determined by Cauchy's integral formula

$$\rho_n^{(j)} = \frac{1}{2\pi i} \oint_{|\xi|=\kappa} \frac{R_j(\xi)}{\xi^{n+1}} d\xi$$

with some $\kappa < (\cosh\eta_j - 1)/2 = O(\beta_j^{-1/2})$, and can be estimated with help of the bound (6.52) for the reduced activity R_j

$$|\rho_n^{(j)}| \leq \frac{1}{\kappa^n} \max_{|\xi|=\kappa} |R_j(\xi)| \leq 2\beta_j\kappa^{2-n} = O(\beta_j^{n/2}) . \tag{6.58}$$

The function $z^{-5}\widetilde{R}_j^5(z)$ is entire and can hence be bounded on $\overline{\Omega(\eta_j)}$ by means of the maximum principle and the bound of $|R_j(z)|$

$$|z^{-5}\widetilde{R}_j^5(z)| = \left| z^{-5}\left\{ R_j(z) - \sum_{n=2}^{4} \rho_n^{(j)} z^n \right\}\right| \le \max_{\xi\in\partial\Omega(\eta_j)} \left| \xi^{-5}\left\{ R_j(\xi) - \sum_{n=2}^{4} \rho_n^{(j)} \xi^n \right\}\right|$$
$$\le \max_{\xi\in\partial\Omega(\eta_j)} \left\{ 2\beta_j|\xi|^{-3} + \sum_{n=2}^{4} |\rho_n^{(j)}|\, |\xi|^{n-5} \right\} = O(\beta_j^{5/2}) \ . \tag{6.59}$$

Thus the individual terms of the representation (6.57) satisfy

$$\rho_2^{(j)} = O(\beta_j) \quad , \quad \rho_3^{(j)} = O(\beta_j^{3/2}) \quad , \quad \rho_4^{(j)} = O(\beta_j^2) \quad , \quad \widetilde{R}_j^5(z) = O(\beta_j^{5/2}|z|^5) \ . \tag{6.60}$$

Now insert Eq. (6.57a) into the recursion relation for $G_{j-1}(z)$. Setting $\rho_0^{(j)} \equiv 1$, one gets

$$G_{j-1}(z) = \mathcal{N}_j^{-2}\Bigg(\sum_{\substack{n,m=0\\ n,m\neq 1}}^{4} \rho_n^{(j)}\rho_m^{(j)} F_{nm}(\beta_j,z) + \sum_{\substack{n=0\\ n\neq 1}}^{4} M_{n5}^{(j)}(z) + N_{55}^{(j)}(z)\Bigg)^2 \ . \tag{6.61}$$

The integrals $(0 \le n \le 4)$

$$M_{n5}^{(j)}(z) \equiv \int dv\ G_W(\beta_j,z_+)\,G_W(\beta_j,z_-)\ \rho_n^{(j)} \left[z_+^n\, \widetilde{R}_j^5(z_-) + \widetilde{R}_j^5(z_+)\, z_-^n \right] \tag{6.62}$$
$$N_{55}^{(j)}(z) \equiv \int dv\ G_W(\beta_j,z_+)\,G_W(\beta_j,z_-)\ \widetilde{R}_j^5(z_+)\,\widetilde{R}_j^5(z_-) \ . \tag{6.63}$$

contribute only to the remainders of the desired perturbation expansions for the effective couplings and will be estimated in App. B. Consider instead the "moments" (5.62),

$$F_{nm}(\beta_j,z) = \int dv\ G_W(\beta_j,z_+)\,G_W(\beta_j,z_-)\ z_+^n\, z_-^m \ , \tag{6.64}$$

with $n,m \ge 0$. Note that the indices $n=1$ or $m=1$ will not occur. The integrations can be done in closed form (see App. A). They yield modified Bessel functions. By Taylor expansion in z one obtains their (unnormalized) contributions to the z^k-terms of the new activities $G_{j-1}(z)$

$$\frac{1}{k!}\frac{\partial^k}{\partial z^k}F_{nm}(\beta,z)\bigg|_{z=0} = (-1)^{n+m+k}\left(\frac{1}{4}\right)^{n+m} \sum_{\nu=0}^{\min\{[\frac{n+m}{2}],k\}} (-1)^{\nu}\binom{k}{\nu}\frac{(2\nu)!}{k!}\,a_{2\nu}^{nm}\,\frac{\beta^{k-(n+m)-1}}{\sqrt{4\pi\beta}}$$
$$\times \left\{ \sum_{\mu=0}^{l} \frac{(-1)^{\mu}}{\mu!}\frac{\Gamma(k+\mu+3/2)}{\Gamma(k-\mu+3/2)}\frac{\Gamma(k-2\nu-\mu-1/2)}{\Gamma(k-n-m-\mu-1/2)}\left(\frac{1}{4\beta}\right)^{\mu} + O(\beta^{-l-1}) \right\} \tag{6.65}$$

with coefficients

$$a_{2\nu}^{nm} = \sum_{\substack{i=0\\ i+j=2\nu}}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j}(-1)^j \ . \tag{6.66}$$

Note that

$$F_{nm}^{(k)}(\beta_j) \equiv \frac{1}{k!}\frac{\partial^k}{\partial z^k}F_{nm}(\beta_j,z)\bigg|_{z=0} = O(\beta_j^{k-(n+m)-3/2}) \ . \tag{6.67}$$

Because of $F_{00}(\beta_j,0) = [G_W(\beta_j)*G_W(\beta_j)](0)$ it follows immediately that the normalization factor $\mathcal{N}_j$ obeys $\mathcal{N}_j = \mathcal{N}_{j0}[1+O(\beta_j^{-1})]$ with $\mathcal{N}_{j0} \equiv F_{00}(\beta_j,0)$. Consider also the irrelevant couplings $\rho_n^{(j)}$ multiplying the moments F_{nm} and the coefficients $F_{nm}^{(k)}$, respectively. With $\mathcal{N}_j^{-1} = O(\beta_j^{3/2})$ and the bounds of $|\rho_n^{(j)}|$, one gets

$$\rho_n^{(j)}\rho_m^{(j)}\,F_{nm}^{(k)}(\beta_j) = O(\beta_j^{k-\frac{1}{2}(n+m)-3/2}) \tag{6.68}$$

and

$$\rho_n^{(j)}\rho_m^{(j)}\,L_{nm}^{(k)}(\beta_j) = O(\beta_j^{k-\frac{1}{2}(n+m)}) \tag{6.69}$$

with the "almost normalized" expansion coefficients $L_{nm}^{(k)}(\beta_j)$ which were introduced in Sect. 5, Eq. (5.64). Therefore, to obtain the running coupling constant β_{j-1} up to terms of order β_j^{-1}, only moments $F_{nm}(\beta_j)$ with $n+m \le 4$ are needed. It can be shown (see App. B) that the contributions of the integrals $M_{n5}^{(j)}(z)$, $N_{55}^{(j)}(z)$ to the running coupling constant β_{j-1} are at most of order $\beta_j^{-3/2}$. More generally, one has the inequality

$$\mathcal{N}_j^{-1}\,\frac{1}{k!}\frac{\partial^k}{\partial z^k}\Bigg(\sum_{\substack{n=0\\ n\neq 1}}^{4} M_{n5}^{(j)}(z) + N_{55}^{(j)}(z)\Bigg)\Bigg|_{z=0} = O(\beta_j^{k-5/2}) \tag{6.70}$$

which corresponds to (6.68) and (6.69), respectively. Collecting all the contributions, one arrives at the following set of recursion relations for the marginal coupling β_j and for the irrelevant couplings $\rho_n^{(j)}$

$$\beta' = \beta - \frac{3}{4} - \frac{5}{8}\rho_2\beta^{-1} - \frac{105}{128}\rho_3\beta^{-2} + \frac{3}{32}\beta^{-1} +$$
$$+ \frac{15}{64}\rho_2\beta^{-2} + \frac{115}{512}\rho_2^2\beta^{-3} - \frac{315}{256}\rho_4\beta^{-3} + O(\beta^{-3/2}) \tag{6.71a}$$
$$\rho_2' = \frac{1}{4}\rho_2 - \frac{1}{2}\beta + \frac{21}{32}\rho_3\beta^{-1} + \frac{3}{4} + \frac{7}{16}\rho_2\beta^{-1} - \frac{101}{256}\rho_2^2\beta^{-2} + \frac{189}{128}\rho_4\beta^{-2} + O(\beta^{-1/2}) \tag{6.71b}$$
$$\rho_3' = \frac{1}{16}\rho_3 - \frac{1}{4}\beta + \frac{1}{8}\rho_2 - \frac{9}{64}\rho_2^2\beta^{-1} + \frac{9}{32}\rho_4\beta^{-1} + O(\beta^{1/2}) \tag{6.71c}$$
$$\rho_4' = \frac{1}{64}\rho_4 + \frac{1}{8}\beta^2 - \frac{1}{8}\rho_2\beta + \frac{3}{128}\rho_2^2 + O(\beta^{3/2}) \tag{6.71d}$$

Inserting the conditions (6.53), the system of equations (6.71) simplifies to (6.54). Because of the first relation of (6.53), β_{j-1} is smaller than β_j by an amount of order 1, provided that β_j is large enough. This is relation (6.55) and (6.42) respectively. Finally, it is easy to see that the conditions for $|\rho_2|$, etc. iterate under the recursion relations (6.71). ⌋

Remark. The recursion relations (6.54) were first rigorously derived by Müller and Schiemann [14] with the help of a small field - large field analysis. Afterwards, they have been rederived in [18] by use of reduced activities $R_j(z)$ with bounds (6.52) for all field configurations. It is remarkable that the bounds for the error terms in (6.54) are much better than the ones obtained by the small field - large field technique. The remainder in (6.54a), for example, is $O(\beta^{-2})$ in comparison with the result $O(\beta^{-2+2\alpha})$ of [14] for fixed α, $3/7 \le \alpha < 1/2$.

Solution of the Recursion Relations for Weak Coupling

It is possible to explicitly solve the recursion relations (6.54), cp. [14]. To this purpose, one introduces new "scaling" variables by

$$\tilde\beta \equiv \beta - \frac{5}{6}\rho_2\beta^{-1} + \frac{1}{108}\rho_2\beta^{-2} - \frac{35}{24}\rho_3\beta^{-2} - \frac{31}{36}\rho_2^2\beta^{-3} \tag{6.72a}$$

$$\tilde\rho_2 \equiv \rho_2 + \frac{2}{3}\beta + \frac{2}{9} + \frac{7}{2}\rho_3\beta^{-1} + \frac{11}{6}\rho_2^2\beta^{-2} \tag{6.72b}$$

$$\tilde\rho_3 \equiv \rho_3 - \frac{4}{45}\beta - \frac{2}{3}\rho_2 \tag{6.72c}$$

with corresponding inverse relations

$$\beta = \tilde\beta - \frac{5}{9} + \frac{5}{6}\tilde\rho_2\tilde\beta^{-1} + \frac{7}{162}\tilde\beta^{-1} + \frac{10}{27}\tilde\rho_2\tilde\beta^{-2} - \frac{35}{24}\tilde\rho_3\tilde\beta^{-2} - \frac{49}{36}\tilde\rho_2^2\tilde\beta^{-3} + O(\tilde\beta^{-2}) \tag{6.73a}$$

$$\rho_2 = \tilde\rho_2 - \frac{2}{3}\tilde\beta + \frac{26}{45} - \frac{4}{9}\tilde\rho_2\tilde\beta^{-1} - \frac{7}{2}\tilde\rho_3\tilde\beta^{-1} - \frac{11}{6}\tilde\rho_2^2\tilde\beta^{-2} + O(\tilde\beta^{-1}) \tag{6.73b}$$

$$\rho_3 = \tilde\rho_3 + \frac{2}{3}\tilde\rho_2 - \frac{16}{45}\tilde\beta + O(\tilde\beta^0)\ . \tag{6.73c}$$

The variables $\tilde\beta$, $\tilde\rho_2$ and $\tilde\rho_3$ decouple the recursion relations (6.54), i.e. lead to

$$\tilde\beta' = \tilde\beta - \frac{1}{3} - \frac{2}{27}\tilde\beta^{-1} + O(\tilde\beta^{-2}) \tag{6.74a}$$

$$\tilde\rho_2' = \frac{1}{4}\tilde\rho_2 + O(\tilde\beta^{-1}) \tag{6.74b}$$

$$\tilde\rho_3' = \frac{1}{16}\tilde\rho_3 + O(\tilde\beta^0)\ . \tag{6.74c}$$

These decoupled relations can easily be solved. For the initial data β_N, $\rho_2^{(N)} = 0$, and $\rho_3^{(N)} = 0$, one sets $\tilde\beta_N = \beta_N$, $\tilde\rho_2^{(N)} = \frac{2}{3}\beta_N + \frac{2}{9}$ and $\tilde\rho_3^{(N)} = -\frac{4}{45}\beta_N$. Note that the irrelevant scaling couplings $\tilde\rho_2, \tilde\rho_3$ decrease exponentially under renormalization group transformations. Hence, after a sufficiently large number of iteration steps, they become $\tilde\rho_2 = O(\tilde\beta^{-1})$ and $\tilde\rho_3 = O(\tilde\beta^0)$ according to the remainder terms in (6.74b) and (6.74c). This simplifies the inverse relations (6.73) to

$$\beta = \tilde\beta - \frac{5}{9} + \frac{7}{162}\tilde\beta^{-1} + O(\tilde\beta^{-2}) \tag{6.75a}$$

$$\rho_2 = -\frac{2}{3}\tilde\beta + \frac{26}{45} + O(\tilde\beta^{-1}) \tag{6.75b}$$

$$\rho_3 = -\frac{16}{45}\tilde\beta + O(\tilde\beta^0)\ . \tag{6.75c}$$

Consequently, the *renormalized* coupling constant β, e.g. the running coupling constant that parametrizes the continuum effective Boltzmannians on the renormalized trajectory, obeys the renormalization group (Callan-Symanzik) equation

$$\beta' = \beta + \Delta\beta(\beta) \tag{6.76a}$$

with the (discrete) renormalization group function

$$\Delta\beta(\beta) = -\frac{1}{3} - \frac{2}{27}\beta^{-1} + O(\beta^{-2})\ . \tag{6.76b}$$

This holds for single renormalization group steps with $L = \sqrt{2}$. Along the renormalized trajectory, all irrelevant couplings are functions of β alone. Equations (6.75) yield, for example,

$$\rho_2 = -\frac{2}{3}\beta + \frac{28}{135} + O(\beta^{-1}) \tag{6.77a}$$

$$\rho_3 = -\frac{16}{45}\beta + O(\beta^0)\ . \tag{6.77b}$$

Remarks. (i) In addition to the relation (6.76) between renormalized coupling constants, one can also obtain (by explicitly solving Eq. (6.74a)) a formula that connects the bare and the renormalized coupling constants, namely β_N and β_j for $j \ll N$. This formula is of crucial importance for the construction of the continuum limit.

In order to construct the continuum limit, one must define a sequence of bare Boltzmannians $G_N^{[N]} = G_W(\beta_N^{[N]})$. These bare Boltzmannians lie on the canonical line of Wilson Boltzmannians. They are parametrized by bare coupling constants $\beta_N^{[N]}$. Each of these Boltzmannians is the starting point of a discrete renormalization group trajectory (in the coupling constant space, cp. Sect. 4). The Boltzmannians $G_j^{[N]}$ along such a trajectory are generated from $G_N^{[N]}$ by

$$\mathrm{R}^{N-j}\, G_N^{[N]} \equiv G_j^{[N]}\ ,$$

where R denotes a single and R^n an n-fold renormalization group transformation. The running coupling constants of the effective Boltzmannians $G_j^{[N]}$ are denoted by $\beta_j^{[N]}$.

For a given renormalization scale j, for example $j = 0$, one fixes a renormalized coupling constant β_0^{ren}. Then the bare coupling constants $\beta_N^{[N]}$ are implicitly defined by the renormalization condition

$$\beta_0^{[N]} \equiv \beta_0^{ren}$$

on the running coupling constants $\beta_0^{[N]}$ at the renormalization scale. The length scales a_N of the bare (cutoff) Boltzmannians are related to the renormalization length scale a_0 by $a_N = L^{-N}a_0$. The continuum limit of effective Boltzmannians is then given by

$$\lim_{N\to\infty} \mathrm{R}^N G_N^{[N]} = \lim_{N\to\infty} G_0^{[N]} = G^{continuum}(\beta_0^{ren}) \quad \text{for} \quad \beta_0^{[N]} = \beta_0^{ren}\ .$$

Note that the continuum effective Boltzmannians are parametrized by a single variable, namely by the renormalized coupling constant. All continuum effective Boltzmannians lie on the renormalized trajectory, i.e.

$$\mathrm{R}\, G^{continuum}(\beta_j^{ren}) = G^{continuum}(\beta_{j-1}^{ren}) \qquad \text{etc.}$$

For the details of the construction and the proof of the existence of the continuum limit for the four-dimensional hierarchical $SU(2)$ lattice gauge model, the reader is referred to the work of Müller and Schiemann [14]. In [15], Schiemann has proven the uniqueness and the universality of the continuum limit of effective Boltzmannians. He considered different sequences of bare Boltzmannians (of Wilson, heat kernel and also more general type). All sequences must allow the contruction of continuum limits for the same prescribed renormalized coupling constant β_0^{ren} at the renormalization scale $j = 0$. That these limits are identical, follows from a stability analysis. Different bare Boltzmannians at the cutoff scale N lead to different renormalization group trajectories and hence to different effective Boltzmannians at the renormalization scale $j = 0$. The differences at the renormalization scale can be rigorously bounded by induction over $j \leq N$. Schiemann has shown that these bounds tend to zero in the limit $N \to \infty$, e.g. when the lattice cutoff is removed. His stability analysis is an improved version of the "method of iterated differences" of Gawędzki and Kupiainen [27].

(ii) The approach of the renormalization group trajectories towards the renormalized (continuum) trajectory can be considerably accelerated by using *improved* bare Boltzmannians of some "improved" canonical line which is "closer" to the renormalized trajectory than the line of Wilson Boltzmannians. The best choice for such an improved canonical line is, however, the renormalized trajectory itself. To demonstrate the effect, one sets – within the accuracy of the above analysis –

$$\rho_2^{(N)} = -\frac{2}{3}\beta_N + \frac{28}{135} + O(\beta_N^{-1}) \quad , \quad \rho_3^{(N)} = -\frac{16}{45}\beta_N + O(\beta_N^0)$$

implying

$$\bar{\rho}_2^{(N)} = O(\bar{\beta}_N^{-1}) \quad , \quad \bar{\rho}_3^{(N)} = O(\bar{\beta}_N^0)$$

with

$$\bar{\beta}_N = \beta_N + \frac{5}{9} - \frac{7}{162}\beta_N^{-1} + O(\beta_N^{-2}) \,.$$

Hence one can use Eqs. (6.75) instead of (6.73) even for the *first* renormalization group step. A general discussion of improved actions from the viewpoint of Wilson's renormalization group is given in [28] (for scalar field theories).

6.3. REPRODUCTION OF THE Ω-BOUNDS

It is shown that bounds of the following form iterate on complex domains $\overline{\Omega(\hat{\eta})}$ with $\hat{\eta} \in [0, \bar{\eta}]$ provided that the running coupling constant β_j is sufficiently large

$$|R_j(z)| \leq C\beta_j |z|^2 \exp\Big(\sigma\beta_j f(\eta(z))\Big) \,.$$

This corresponds to the iteration of bounds

$$|r_j(\theta)| \leq C\beta_j \left|\sin^2\frac{\theta}{2}\right|^2 \exp\Big(\sigma\beta_j f(\,\mathrm{Im}\,\theta\,)\Big)$$

on complex strips $|\mathrm{Im}\,\theta| \leq \hat{\eta}$. The constants C and σ are of order 1 and independent of the scale j. They have to satisfy conditions that replace (6.28) and (6.35). Especially, it is possible to choose $C = 1$ and $\sigma = 5/2$ for $\hat{\eta} = \bar{\eta}/2$ which proves (6.44) of Proposition 6.3.

Consider the renormalization group step $j \to j-1$ under the assumptions of Proposition 6.4. Let the activity $R_j(z)$ be bounded by

$$|R_j(z)| \leq C\beta_j |z|^2 \exp\Big(\sigma\beta_j f(\eta(z))\Big) \qquad \text{for} \qquad \eta(z) \leq \hat{\eta} \,. \tag{6.78}$$

Estimation of the Reduced Activity R_{j-1}

Split the recursion formula for the reduced activity $R_{j-1}(z)$ into two parts

$$\begin{aligned} R_{j-1} &= \Big(\mathcal{N}_j^{-1}[G_j * G_j]\Big)^2 G_W(\beta_{j-1})^{-1} - 1 \\ &= \Big(\mathcal{N}_j^{-1}[G_W(\beta_j) * G_W(\beta_j)]\Big)^2 G_W(\beta_{j-1})^{-1} - 1 \\ &\quad + \mathcal{N}_j^{-2}\, G_W(\beta_{j-1})^{-1} \int_0^1 ds \frac{d}{ds}\Big[G_W(\beta_j)(1 + sR_j) * G_W(\beta_j)(1 + sR_j)\Big]^2 \,. \end{aligned} \tag{6.79}$$

Then write for the first part

$$\mathcal{R}_{j-1}(z) \equiv \Big(\mathcal{N}_j^{-1}[G_W(\beta_j) * G_W(\beta_j)](z)\Big)^2 G_W(\beta_{j-1}, z)^{-1} - 1 \,. \tag{6.80}$$

Note that $\mathcal{R}_{N-1} = R_{N-1}$. The $\mathcal{R}_{j-1}$ activities do not fulfil the renormalization conditions for $j \neq N$. Nevertheless, they will yield very good bounds for $|z| \geq O(\beta_{j-1}^{-1/2})$. Now estimate R_{j-1} by use of the triangle inequality, i.e.

$$\begin{aligned} |R_{j-1}| \leq &\left|\Big(\mathcal{N}_j^{-1}[G_W(\beta_j) * G_W(\beta_j)]\Big)^2 G_W(\beta_{j-1})^{-1} - 1\right| \\ &+ \mathcal{N}_j^{-2} |G_W(\beta_{j-1})|^{-1} \int_0^1 ds \frac{d}{ds}\Big[|G_W(\beta_j)|(1 + s|R_j|) * |G_W(\beta_j)|(1 + s|R_j|)\Big]^2 \end{aligned}$$

Use (6.80) and rewrite

$$|R_{j-1}| \le |\mathcal{R}_{j-1}| + \left(\mathcal{N}_j^{-1}\left[|G_W(\beta_j)| * |G_W(\beta_j)|\right]\right)^2 |G_W(\beta_{j-1})|^{-1} \times \int_0^1 ds \frac{d}{ds} \left\langle \prod_{l=\pm} [1 + s|R_j(z_l(z))|] \right\rangle_{z,\beta_j}^2 \tag{6.81}$$

with the expectation value

$$\langle \cdot \rangle_{z,\beta_j} \equiv \frac{\int dv\, |G_W(\beta_j, z_+(z)) G_W(\beta_j, z_-(z))| (\,\cdot\,)}{\int dv\, |G_W(\beta_j, z_+(z)) G_W(\beta_j, z_-(z))|} = \int d\mu_{z,\beta_j}(\phi,\chi)\,(\,\cdot\,)\ . \tag{6.82}$$

The normalized measure $d\mu_{z,\beta}$ is explicitly given by

$$d\mu_{z,\beta}(\phi,\chi) = \left[2\, e^{-2\beta} \frac{I_1(2\beta \operatorname{Re}\sqrt{1-z})}{2\beta \operatorname{Re}\sqrt{1-z}}\right]^{-1} \frac{1}{\pi}\, d\phi\, \sin^2\phi\; d\chi\, \sin\chi\; e^{-2\beta(1-\operatorname{Re}\sqrt{1-z}\cos\phi)}\ . \tag{6.83}$$

Note that

$$\left\langle f(z_+(z)) h(z_-(z)) \right\rangle_{z,\beta_j} = \frac{[(|G_W(\beta_j)| f) * (|G_W(\beta_j)| h)](z)}{[|G_W(\beta_j)| * |G_W(\beta_j)|](z)} = \left\langle h(z_+(z)) f(z_-(z)) \right\rangle_{z,\beta_j}\ .$$

In order to estimate the expectation value

$$\left\langle \prod_{l=\pm} [1 + s|R_j(z_l(z))|] \right\rangle_{z,\beta_j} = 1 + s \left\langle \sum_{l=\pm} |R_j(z_l(z))| \right\rangle_{z,\beta_j} + s^2 \left\langle \prod_{l=\pm} |R_j(z_l(z))| \right\rangle_{z,\beta_j}, \tag{6.84}$$

the bound (6.78) of the reduced activity R_j, which holds by assumption, is needed. Then

$$\left\langle \sum_{l=\pm} |R_j(z_l(z))| \right\rangle_{z,\beta_j} \le C\beta_j \left\langle \sum_{l=\pm} |z_l(z)|^2 \exp\left(\sigma\beta_j f(\eta(z_l(z)))\right) \right\rangle_{z,\beta_j}$$

$$\le C\beta_j \exp\left(\sigma\beta_j f(\tfrac{1}{2}\eta(z))\right) \left\langle \sum_{l=\pm} |z_l(z)|^2 \right\rangle_{z,\beta_j}, \tag{6.85}$$

because of $\eta(z_\pm(z)) \le \eta(z)/2$ according to Lemma 6.1. In addition, one has

$$\left\langle \prod_{l=\pm} |R_j(z_l(z))| \right\rangle_{z,\beta_j} \le C^2\beta_j^2 \exp\left(2\sigma\beta_j f(\tfrac{1}{2}\eta(z))\right) \left\langle \prod_{l=\pm} |z_l(z)|^2 \right\rangle_{z,\beta_j}\ . \tag{6.86}$$

Abbreviate

$$\mathcal{E}_1(\beta_j, z) \equiv \left\langle \frac{1}{2} \sum_{l=\pm} |z_l(z)|^2 \right\rangle_{z,\beta_j} \tag{6.87a}$$

$$\mathcal{E}_2(\beta_j, z) \equiv \left\langle \prod_{l=\pm} |z_l(z)|^2 \right\rangle_{z,\beta_j}\ . \tag{6.87b}$$

Both expectation values $\mathcal{E}_1(\beta_j)$, $\mathcal{E}_2(\beta_j)$ can be explicitly calculated by means of a generating function (see App. C). The result is essentially

$$\mathcal{E}_k(\beta_j, z) = \sum_{l=1}^{2} \mathcal{E}_{kl}(\beta_j, z) \qquad \text{for} \qquad k = 1,2 \tag{6.88}$$

with leading terms

$$\mathcal{E}_{k1}(\beta_j, z) = \left(\frac{1}{16}|z|^2\right)^k\ . \tag{6.89}$$

Whereas $\mathcal{E}_1(\beta_j)$, $\mathcal{E}_2(\beta_j)$ are bounded by $\mathcal{E}_k(\beta_j,z) \le \cosh^{4k}(\eta(z)/4)$, their $\mathcal{E}_{k1}$-parts fulfil the estimates $\mathcal{E}_{k1}(\beta_j, z) \le (2^{-1}\cosh\eta(z))^{4k}$. Although the remainders $\mathcal{E}_{k2}(\beta_j, z)$ are also bounded by constants of order 1 on $\overline{\Omega(\tilde{\eta})}$, they provide in the important small field region only small corrections to the leading terms

$$\mathcal{E}_{k2}(\beta_j, z) = O(\beta_j^{-k-1/2}) \qquad \text{for} \qquad |z| \le O(\beta_j^{-1/2})\ . \tag{6.90}$$

It is convenient to introduce also $\mathcal{E}_0(\beta_j, z) \equiv 1$ with $\mathcal{E}_{01}(\beta_j, z) = 1$ and $\mathcal{E}_{02}(\beta_j, z) = 0$. Now continue the estimation of R_{j-1}. Based on (6.81), one obtains with (6.84) - (6.88)

$$|R_{j-1}(z)| \le |\mathcal{R}_{j-1}(z)| + \left(\mathcal{N}_j^{-1}\left[|G_W(\beta_j)| * |G_W(\beta_j)|\right](z)\right)^2 |G_W(\beta_{j-1}, z)|^{-1}$$

$$\times \int_0^1 ds \frac{d}{ds} \left(\sum_{k=0}^{2} \binom{2}{k} \left[sC\beta_j \exp\left(\sigma\beta_j f(\tfrac{1}{2}\eta(z))\right)\right]^k \mathcal{E}_k(\beta_j, z)\right)^2$$

$$\le |\mathcal{R}_{j-1}(z)| + \left(\mathcal{N}_j^{-1}\left[|G_W(\beta_j)| * |G_W(\beta_j)|\right](z)\right)^2 |G_W(\beta_{j-1}, z)|^{-1}$$

$$\times \exp\left(4\sigma\beta_j f(\tfrac{1}{2}\eta(z))\right) \int_0^1 ds \frac{d}{ds} \left(\sum_{k=0}^{2} \binom{2}{k} \left[sC\beta_j\right]^k \mathcal{E}_k(\beta_j, z)\right)^2\ . \tag{6.91}$$

Consider

$$\left(\sum_{k=0}^{2} \binom{2}{k} \left[sC\beta_j\right]^k \mathcal{E}_k(\beta_j)\right)^2 = \sum_{d=0}^{4} (sC\beta_j)^d \sum_{\substack{k_1,k_2=0,1,2\\ k_1+k_2=d}} \prod_{i=1,2} \left[\binom{2}{k_i} \mathcal{E}_{k_i}(\beta_j)\right]$$

$$= \sum_{d=0}^{4} (sC\beta_j)^d \sum_{\substack{k_1,k_2=0,1,2\\ k_1+k_2=d}} \sum_{l_1,l_2=1,2} \prod_{i=1,2} \left[\binom{2}{k_i} \mathcal{E}_{k_i l_i}(\beta_j)\right],$$

where the term for $l_1 = l_2 = 1$ is given by

$$\sum_{d=0}^{4} (sC\beta_j)^d \sum_{\substack{k_1,k_2=0,1,2\\ k_1+k_2=d}} \prod_{i=1,2} \left[\binom{2}{k_i} \mathcal{E}_{k_i 1}(\beta_j, z)\right] =$$

$$= \sum_{d=0}^{4} (sC\beta_j)^d \sum_{\substack{k_1,k_2=0,1,2\\ k_1+k_2=d}} \binom{2}{k_1}\binom{2}{k_2} \left(\frac{1}{16}|z|^2\right)^{k_1+k_2} = \sum_{d=0}^{4} \binom{4}{d} \left(s\,\frac{C\beta_j |z|^2}{16}\right)^d\ .$$

Then the inequality (6.91) becomes

$$
\begin{aligned}
|R_{j-1}(z)| \le |\mathcal{R}_{j-1}(z)| &\\
+ \exp\Big(4\sigma\beta_j f(\tfrac{1}{2}\eta(z))\Big) & \Big(\mathcal{N}_j^{-1}\big[|G_W(\beta_j)| * |G_W(\beta_j)|\big](z)\Big)^2 |G_W(\beta_{j-1},z)|^{-1} \\
&\times \left\{ \int_0^1 ds \frac{d}{ds} \sum_{d=0}^{4} \binom{4}{d} \left(s\,\frac{C\beta_j|z|^2}{16}\right)^d \right. \\
&\left. + \int_0^1 ds \frac{d}{ds} \sum_{d=0}^{4} (sC\beta_j)^d \sum_{\substack{k_1,k_2=0,1,2\\ k_1+k_2=d}} \sum_{\substack{l_1,l_2=1,2\\ l_1+l_2\ge 3}} \prod_{i=1,2} \left[\binom{2}{k_i} \mathcal{E}_{k_i l_i}(\beta_j,z) \right] \right\}. \qquad (6.92)
\end{aligned}
$$

Dividing both sides by $|z|^2$ for $|z| \ge O(\beta_{j-1}^{-1/2})$, this yields

$$
\begin{aligned}
|z^{-2}R_{j-1}(z)| \le |z^{-2}\mathcal{R}_{j-1}(z)| &\\
+ \exp\Big(4\sigma\beta_j f(\tfrac{1}{2}\eta(z))\Big) & \Big(\mathcal{N}_j^{-1}\big[|G_W(\beta_j)| * |G_W(\beta_j)|\big](z)\Big)^2 |G_W(\beta_{j-1},z)|^{-1} \\
&\times \left\{ \frac{1}{4}C\beta_j \sum_{d=0}^{3} \frac{1}{4} \binom{4}{d+1} \left(\frac{C\beta_j|z|^2}{16}\right)^d \right. \\
&\left. + \sum_{d=1}^{4} (C\beta_j)^d \sum_{\substack{k_1,k_2=0,1,2\\ k_1+k_2=d}} \sum_{\substack{l_1,l_2=1,2\\ l_1+l_2\ge 3}} \frac{1}{|z|^2} \prod_{i=1,2} \left[\binom{2}{k_i} \mathcal{E}_{k_i l_i}(\beta_j,z) \right] \right\}. \qquad (6.93)
\end{aligned}
$$

To proceed further, I need the following lemmata. The proofs of Lemma 6.5 and Lemma 6.6 will be postponed to the end of this section.

Lemma 6.5. *For large enough β_{j-1}, the following estimate holds*

$$
|z^{-2}\mathcal{R}_{j-1}(z)| \le \frac{1}{2}\beta_{j-1} \frac{1+\exp[\beta_{j-1} f(\eta(z))]}{2} [1 + \mathrm{const}\,\beta_{j-1}^{-1/2}] \quad \textit{for} \quad \eta(z) \in [\kappa\beta_{j-1}^{-1/4}, \tilde{\eta}] \qquad (6.94)
$$

where the constant is independent of the running coupling constant β and the value of j. For a positive parameter κ such that both κ and κ^{-1} are of order 1, the constant is also of order 1. (Note that the constant would diverge for $\kappa \to 0$.)

Lemma 6.6. *Let β_j be large enough and α such that $0 \le \alpha < 1/2$ with $(1-2\alpha)^{-1} = O(1)$. Then for $\eta \in [0, \tilde{\eta}]$*

$$
\begin{aligned}
\max_{z\in\partial\Omega(\eta)} &\left\{ \Big(\mathcal{N}_j^{-1}\big[|G_W(\beta_j)| * |G_W(\beta_j)|\big](z)\Big)^2 |G_W(\beta_{j-1},z)|^{-1} \exp(+\alpha\beta_{j-1}|z|^2) \right\} \\
&\le [1 + \mathrm{const}\,\beta_{j-1}^{-1/2}] \exp\left\{ \beta_{j-1}\left(\frac{1+2\alpha}{1-2\alpha} + \delta \right) f(\eta(z)) \right\}, \qquad (6.95)
\end{aligned}
$$

where the constant is of order 1 and δ is a small parameter, i.e. $\delta = 1/100$.

Lemma 6.7. *Let $\eta \in [0, \tilde{\eta}]$ and k be some natural number. Then*

$$
\begin{aligned}
\max_{z\in\partial\Omega(\eta)} &\left\{ \left(\beta_j|z|^2\right)^k \Big(\mathcal{N}_j^{-1}\big[|G_W(\beta_j)| * |G_W(\beta_j)|\big](z)\Big)^2 |G_W(\beta_{j-1},z)|^{-1} \right\} \\
&\le [1 + \mathrm{const}\,\beta_{j-1}^{-1/2}] \left(\frac{k}{\alpha e}\right)^k \exp\left\{ \beta_{j-1}\left(\frac{1+2\alpha}{1-2\alpha} + \delta \right) f(\eta(z)) \right\}, \qquad (6.96)
\end{aligned}
$$

with a constant of order 1.

PROOF. ⌈ Inserting $1 = \exp(-\alpha\beta_{j-1}|z|^2)\exp(+\alpha\beta_{j-1}|z|^2)$, one gets the estimate

$$
\begin{aligned}
\max_{z\in\partial\Omega(\eta)} &\left\{ \left(\beta_j|z|^2\right)^k \Big(\mathcal{N}_j^{-1}\big[|G_W(\beta_j)| * |G_W(\beta_j)|\big](z)\Big)^2 |G_W(\beta_{j-1},z)|^{-1} \right\} \\
&\le \max_{z\in\partial\Omega(\eta)} \left\{ \left(\beta_j|z|^2\right)^k \exp(-\alpha\beta_{j-1}|z|^2) \right\} \\
&\quad \times \max_{z\in\partial\Omega(\eta)} \left\{ \Big(\mathcal{N}_j^{-1}\big[|G_W(\beta_j)| * |G_W(\beta_j)|\big](z)\Big)^2 |G_W(\beta_{j-1},z)|^{-1} \exp(+\alpha\beta_{j-1}|z|^2) \right\}.
\end{aligned}
$$

The first factor is

$$
\begin{aligned}
\left(\frac{\beta_j}{\beta_{j-1}}\right)^k &\max_{z\in\partial\Omega(\eta)} \left\{ \left(\beta_{j-1}|z|^2\right)^k \exp(-\alpha\beta_{j-1}|z|^2) \right\} \\
&\le \left(\frac{\beta_j}{\beta_{j-1}}\right)^k \max_{y\ge 0} \left\{ y^k e^{-\alpha y} \right\} = [1 + O(\beta_{j-1}^{-1})] \left(\frac{k}{\alpha e}\right)^k.
\end{aligned}
$$

The second factor is bounded by use of Lemma 6.6. ⌋

Lemma 6.8. *Let $O(\beta_{j-1}^{-1/4}) \le \eta \le \tilde{\eta}$ and let k be 1 or 2. Then*

$$
\begin{aligned}
\max_{z\in\partial\Omega(\eta)} &\left\{ \mathcal{E}_{k2}(\beta_j,z) \Big(\mathcal{N}_j^{-1}\big[|G_W(\beta_j)| * |G_W(\beta_j)|\big](z)\Big)^2 |G_W(\beta_{j-1},z)|^{-1} \right\} \\
&\le \mathrm{const}\,\beta_{j-1}^{-k-1/2} \exp\left\{ \beta_{j-1}\left(\frac{1+2\alpha}{1-2\alpha} + \delta \right) f(\eta(z)) \right\}, \qquad (6.97)
\end{aligned}
$$

with a constant of order 1 independent of j and β_j, β_{j-1}.

PROOF. ⌈ As in the proof of Lemma 6.7, one begins with a bound

$$
\begin{aligned}
\max_{z\in\partial\Omega(\eta)} &\left\{ \mathcal{E}_{k2}(\beta_j,z) \Big(\mathcal{N}_j^{-1}\big[|G_W(\beta_j)| * |G_W(\beta_j)|\big](z)\Big)^2 |G_W(\beta_{j-1},z)|^{-1} \right\} \\
&\le \max_{z\in\Omega(\tilde{\eta})} \left\{ \mathcal{E}_{k2}(\beta_j,z) \exp(-\alpha\beta_{j-1}|z|^2) \right\} \\
&\quad \times [1 + O(\beta_{j-1}^{-1/2})] \exp\left\{ \beta_{j-1}\left(\frac{1+2\alpha}{1-2\alpha} + \delta \right) f(\eta(z)) \right\},
\end{aligned}
$$

where use has been made of Lemma 6.6 to get the second factor. In order to handle the first factor, it is necessary to expand the expectation values $\mathcal{E}_{k2}(\beta_j,z)$ up to terms of order β_j^{-2}.

To do this, one can either use the explicit formulae for $\mathcal{E}_k(\beta_j,z)$ given in App. C or get the desired terms of order 1 and also of order β_j^{-1} by an (asymptotic) expansion of the measure $d\mu_{z,\beta}(\phi,\chi)$, Eq. (6.83), in powers of β_j^{-1}. Note that the leading term will be a δ-function measure which causes the integration angle ϕ to be zero. Proceeding in this way, one would get immediately

$$\mathcal{E}_k(\beta,\sin^2\frac{\theta}{2}) = \left|\sin^4\frac{\theta}{4}\right|^k + O(\beta^{-1}) \quad \text{for} \quad k=1,2 \quad \text{and} \quad |\operatorname{Im}\theta| \le \tilde\eta .$$

For example, using (6.87a) for $\mathcal{E}_1(\beta,z)$ and Eq. (5.12) for $z_\pm(z)$,

$$\left\langle \frac{1}{2}\sum_{l=\pm}|z_l(z)|^2 \right\rangle_{z,\beta\to\infty} = \frac{1}{2}\sum_{l=\pm}\frac{1}{4}\left|1-\sqrt{1-z}\cos\phi + l\sqrt{z}\sin\phi\cos\chi\right|^2\Bigg|_{\phi=0} = \frac{1}{4}\left|1-\sqrt{1-z}\right|^2 .$$

This is easily rewritten as $|(1-\cos(\theta/2))/2|^2 = |\sin^2(\theta/4)|^2$ for $z=\sin^2(\theta/2)$. I restrict my attention in the following on $\mathcal{E}_1$. Developing also the next order, one gets

$$\mathcal{E}_1(\beta,\sin^2\frac{\theta}{2}) = \left|\sin^4\frac{\theta}{4}\right| + \beta^{-1}\left[\frac{3}{4}\operatorname{Re}\sin^2\frac{\theta}{4} + \frac{1}{8}\frac{|\sin^2\frac{\theta}{2}|}{\operatorname{Re}\cos\frac{\theta}{2}} - \frac{3}{8}\frac{\left(\operatorname{Im}\cos\frac{\theta}{2}\right)^2}{\operatorname{Re}\cos\frac{\theta}{2}}\right] + \text{terms of order } \beta^{-2} .$$

Now use $\sin(\theta/2) = 2\sin(\theta/4)\cos(\theta/4)$ and split off $\mathcal{E}_{11}(\beta,\sin^2(\theta/2)) = |\sin^4(\theta/2)|/16$ to obtain

$$\mathcal{E}_{12}(\beta,\sin^2\frac{\theta}{2}) = \beta^{-1}\left[\frac{3}{4}\operatorname{Re}\sin^2\frac{\theta}{4} + \frac{1}{8}\frac{|\sin^2\frac{\theta}{2}|}{\operatorname{Re}\cos\frac{\theta}{2}} - \frac{3}{8}\frac{\left(\operatorname{Im}\cos\frac{\theta}{2}\right)^2}{\operatorname{Re}\cos\frac{\theta}{2}}\right] + \left|\sin^4\frac{\theta}{4}\right|\left(1-\left|\cos^4\frac{\theta}{4}\right|\right) + \text{terms of order } \beta^{-2} .$$

Multiplication with the damping factor $\exp(-\alpha\beta_{j-1}|\sin^2(\theta/2)|^2)$ enforces the restriction to the small field region $|\theta| \le O(\beta_{j-1}^{-1/4})$ if one searches for the maximal value of the product. It turns out, however, that in this region the leading terms in the expansion of $\mathcal{E}_{12}(\beta_j)$, which can be at most of orders 1 or β_j^{-1}, respectively, are actually only of the order $\beta_{j-1}^{-3/2}$. A similar consideration yields the $O(\beta_{j-1}^{-5/2})$-bound in the case $k=2$. To summarize, one has

$$\max_{z\in\Omega(\tilde\eta)}\left\{\mathcal{E}_{k2}(\beta_j,z)\exp\left(-\alpha\beta_{j-1}|z|^2\right)\right\} \le O(\beta_{j-1}^{-k-1/2}) \quad \text{for} \quad k=1,2$$

which finally implies the bound (6.97). ⌋

By combination of Lemma 6.7 and Lemma 6.8 one gets

Lemma 6.9. *Let η be such that $\kappa\beta_{j-1}^{-1/4} \le \eta \le \tilde\eta$ with κ of Lemma 6.5. Then*

$$\begin{aligned}
&\max_{z\in\partial\Omega(\eta)}\left\{\frac{1}{|z|^2}\left[\prod_{i=1,2}\mathcal{E}_{k_il_i}(\beta_j,z)\right]\left(\mathcal{N}_j^{-1}\left[|Gw(\beta_j)| * |Gw(\beta_j)|\right](z)\right)^2|Gw(\beta_{j-1},z)|^{-1}\right\}\\
&\le \text{const}\,\beta_{j-1}^{-(k_1+k_2)-1/2(l_1+l_2-2)+1}\exp\left\{\beta_{j-1}\left(\frac{1+2\alpha}{1-2\alpha}+\delta\right)f(\eta(z))\right\}\\
&\le \text{const}\,\beta_{j-1}^{-(d-1)-1/2}\exp\left\{\beta_{j-1}\left(\frac{1+2\alpha}{1-2\alpha}+\delta\right)f(\eta(z))\right\}
\end{aligned} \tag{6.98}$$

for $k_1,k_2 = 0,1,2$ with $k_1+k_2 = d \in \{1,...,4\}$ and $l_1,l_2 = 1,2$ with $3 \le l_1+l_2 \le 4$. The constant is independent of j and the running coupling constants, but proportional to κ^{-4}. For $\kappa^{-1} = O(1)$, as presumed, it is of order 1.

PROOF. ⌈ The left-hand side of the assertion is bounded by

$$\left\{\max_{z\in\partial\Omega(\eta)}\frac{1}{|z|^2}\right\}\left[\prod_{i=1,2}\max_{z\in\Omega(\tilde\eta)}\left\{\mathcal{E}_{k_il_i}(\beta_j,z)\exp\left(-\frac{1}{2}\alpha\beta_{j-1}|z|^2\right)\right\}\right] \times \max_{z\in\partial\Omega(\eta)}\left\{\left(\mathcal{N}_j^{-1}\left[|Gw(\beta_j)| * |Gw(\beta_j)|\right](z)\right)^2|Gw(\beta_{j-1},z)|^{-1}\exp\left(+\alpha\beta_{j-1}|z|^2\right)\right\} .$$

The first factor is bounded by $O(\beta_{j-1})$ since $\eta \ge O(\beta_{j-1}^{-1/4})$. The treatment of the other factors including the expectation values $\mathcal{E}_{kl}$ has already been shown in the proofs of Lemma 6.7 and Lemma 6.8. The last factor is bounded by Lemma 6.6. ⌋

Inserting these individual estimates into (6.93) gives

$$\begin{aligned}
\max_{z\in\partial\omega(\eta)}|z^{-2}R_{j-1}(z)| &\le \frac{1}{2}\beta_{j-1}[1+O(\beta_{j-1}^{-1/2})]\frac{1+\exp(\beta_{j-1}f(\eta))}{2}\\
&+ [1+O(\beta_{j-1}^{-1/2})]\exp\left(4\sigma\beta_jf(\frac{1}{2}\eta)\right)\exp\left\{\beta_{j-1}\left(\frac{1+2\alpha}{1-2\alpha}+\delta\right)f(\eta)\right\}\\
&\times\left\{\frac{1}{4}C\beta_j\sum_{d=0}^{3}\frac{1}{4}\binom{4}{d+1}\left(\frac{Cd}{16\,\alpha\,e}\right)^d\right.\\
&\left.+\sum_{d=1}^{4}(C\beta_j)^d\sum_{\substack{k_1,k_2=0,1,2\\k_1+k_2=d}}\ \sum_{\substack{l_1,l_2=1,2\\l_1+l_2\ge3}}O(\beta_{j-1}^{-(d-1)-1/2})\right\} .
\end{aligned} \tag{6.99}$$

Now use

$$\begin{aligned}
\frac{1+\exp(\beta_{j-1}f(\eta))}{2} &= 1+\frac{1}{2}\left(\exp(\beta_{j-1}f(\eta))-1\right)\\
&\le 1+\left(\exp(\beta_{j-1}f(\eta))-1\right) = \exp(\beta_{j-1}f(\eta))\\
&\le \exp\left\{\beta_{j-1}\left(\frac{1+2\alpha}{1-2\alpha}+\delta\right)f(\eta)+4\sigma\beta_jf(\frac{1}{2}\eta)\right\}
\end{aligned}$$

and replace β_j by $\beta_{j-1}[1 + O(\beta_{j-1}^{-1})]$ in all expressions with the exception of the exponents. Thus the estimate (6.99) becomes

$$\max_{z\in\partial\omega(\eta)} |z^{-2}R_{j-1}(z)| \le C\beta_{j-1}\exp\left\{\beta_{j-1}\left(\frac{1+2\alpha}{1-2\alpha}+\delta\right)f(\eta) + 4\sigma\beta_j f(\tfrac{1}{2}\eta)\right\}$$
$$\times\left\{\frac{1}{2C}+\frac{1}{4}\sum_{d=0}^{3}\frac{1}{4}\left(\frac{4}{d+1}\right)\left(\frac{Cd}{16\,\alpha\,\mathrm{e}}\right)^d + O(\beta_{j-1}^{-1/2})\right\} \tag{6.100}$$

for $\eta \in [\kappa\beta_{j-1}^{-1/4}, \bar\eta]$. Consider now the inequality

$$\beta_{j-1}\left(\frac{1+2\alpha}{1-2\alpha}+\delta\right)f(\eta) + 4\sigma\beta_j f(\tfrac{1}{2}\eta) \le \sigma\beta_{j-1}f(\eta), \tag{6.101}$$

which holds for sufficiently large values of σ, i.e. for

$$\sigma \ge \left(\frac{1+2\alpha}{1-2\alpha}+\delta\right)\left(1-4\frac{\beta_j}{\beta_{j-1}}\frac{f(\frac{1}{2}\eta)}{f(\eta)}\right)^{-1} = \left(\frac{1+2\alpha}{1-2\alpha}+\delta\right)\left(1-4\frac{f(\frac{1}{2}\eta)}{f(\eta)}\right)^{-1}[1+O(\beta_{j-1}^{-1})]. \tag{6.102}$$

Note that $f(\eta/2)f(\eta)^{-1}$ is a monotonously increasing function on $[0,\bar\eta]$ (cp. Sect. 6.1). This implies that

$$\sigma_0(\eta) \equiv \left(1-4\frac{f(\frac{1}{2}\eta)}{f(\eta)}\right)^{-1} \ge \left(1-4\frac{1}{16}\right)^{-1} = \frac{4}{3} \tag{6.103}$$

also increases monotonously with η.

Hence, as far as the iteration of the bounds on $\overline{\Omega(\hat\eta)}$ for some fixed $\hat\eta \in [0,\bar\eta]$ is concerned, the parameter σ has to be chosen in such a way that the inequality (6.102) is fulfilled for $\eta = \hat\eta$. However, even for $\hat\eta = \bar\eta$ the bounds one obtains in this way can not be used to estimate the interpolating normalization factor on the closed unit disc. As discussed in Sect. 6.1, the reason is that σ must be larger than $\sigma_0(\bar\eta)$ which already exceeds the upper bound 3/2 for σ, cp. (6.37). Therefore I proceed by iterating bounds on $\overline{\Omega(\hat\eta)}$ with $\hat\eta = \bar\eta/2$. This has two advantages. First, σ can take smaller values because $\sigma_0(\bar\eta/2)$ is smaller than $\sigma_0(\bar\eta)$. But furthermore, it allows the computation of unit disc bounds which are adapted to the problem of estimating the norms $\|(1+R_j)^2 - 1\|_{\beta_j}$. Such bounds must be generated from the $\overline{\Omega(\bar\eta/2)}$-bounds by applying single renormalization group transformations in order to be valid on the closed unit disc. They do not iterate by themselves and are rather complicated.

Combining (6.100) with (6.101), we have for a suitably chosen σ (depending on the choice of the parameter $\hat\eta$)

$$\max_{z\in\partial\Omega(\eta)} |z^{-2}R_{j-1}(z)| \le C\beta_j\exp\big(\sigma\beta_{j-1}f(\eta)\big)$$
$$\times\left\{\frac{1}{2C}+\frac{1}{4}\sum_{d=0}^{3}\frac{1}{4}\left(\frac{4}{d+1}\right)\left(\frac{Cd}{16\,\alpha\,\mathrm{e}}\right)^d + O(\beta_{j-1}^{-1/2})\right\} \tag{6.104}$$

for $\eta \in [\kappa\beta_{j-1}^{-1/4}, \hat\eta]$ with $\hat\eta \le \bar\eta$.

Now continue the bound (6.104) to the whole η-interval $[0,\hat\eta]$. This is possible because the functions $z^{-2}R_{j-1}(z)$ are entire in z as a consequence of the renormalization conditions (Lemma 6.2). Thus one has for $0 \le \eta \le \eta_1$

$$\max_{z\in\partial\Omega(\eta)} |z^{-2}R_{j-1}(z)| \le \max_{z\in\overline{\Omega(\eta_1)}} |z^{-2}R_{j-1}(z)| = \max_{z\in\partial\Omega(\eta_1)} |z^{-2}R_{j-1}(z)|. \tag{6.105}$$

Hence, by setting $\eta_1 = \kappa\beta_{j-1}^{-1/4}$,

$$\max_{z\in\partial\Omega(\eta)} |z^{-2}R_{j-1}(z)| \le C\beta_j\exp\big(\sigma\beta_{j-1}f(\eta)\big)\exp\big(\sigma\beta_{j-1}f(\kappa\beta_{j-1}^{-1/4})\big)$$
$$\times\left\{\frac{1}{2C}+\frac{1}{4}\sum_{d=0}^{3}\frac{1}{4}\left(\frac{4}{d+1}\right)\left(\frac{Cd}{16\,\alpha\,\mathrm{e}}\right)^d + O(\beta_{j-1}^{-1/2})\right\}. \tag{6.106}$$

The bound (6.106) holds for all $\eta \in [0,\hat\eta] \subseteq [0,\bar\eta]$. It differs from (6.104) by the additional factor

$$\exp\big(\sigma\beta_{j-1}f(\kappa\beta_{j-1}^{-1/4})\big) = \exp\left(\sigma\beta_{j-1}\left[\frac{1}{4}\kappa^4\beta_{j-1}^{-1} + O(\beta_{j-1}^{-3/2})\right]\right) = [1+O(\beta_{j-1}^{-1/2})]\exp\left(\frac{1}{4}\sigma\kappa^4\right). \tag{6.107}$$

Conditions for the Iteration

I summarize the conditions for an iteration of bounds of the form (6.78) on $\overline{\Omega(\hat\eta)} \subseteq \overline{\Omega(\bar\eta)}$. For sufficiently large β_{j-1}, the following inequalities must be fulfilled simultanously:

(i) $\quad 0 < \alpha < \frac{1}{2} \quad$ with $\quad (1-2\alpha)^{-1} = O(1)$, (6.108a)

(ii) $\quad 0 < \kappa \le 1 \quad$ with $\quad \kappa^{-1} = O(1)$, (6.108b)

(iii) $\quad \sigma \ge \sigma_0(\hat\eta)\left(\frac{1+2\alpha}{1-2\alpha}+\delta\right)[1+O(\beta_{j-1}^{-1})]$, and (6.108c)

(iv) $\quad \exp\left(\frac{1}{4}\sigma\kappa^4\right)\left\{\frac{1}{2C}+\frac{1}{4}\sum_{d=0}^{3}\frac{1}{4}\left(\frac{4}{d+1}\right)\left(\frac{Cd}{16\,\alpha\,\mathrm{e}}\right)^d + O(\beta_{j-1}^{-1/2})\right\} \le 1$. (6.108d)

The function $\sigma_0(\eta)$ is defined in (6.103).

Choice of Parameters for the Iteration on $\overline{\Omega(\bar\eta/2)}$

Set $\hat\eta = \bar\eta/2$ and choose, for example,

$$\alpha = \frac{1}{8}, \quad \kappa = \frac{1}{2}, \quad \delta = \frac{1}{100}, \quad \sigma = \frac{5}{2}, \quad \text{and} \quad C = 1. \tag{6.109}$$

According to the conditions (6.108), this proves the bounds

$$|R_j(z)| \le \beta_j|z|^2\exp\left(\frac{5}{2}\beta_j f(\eta(z))\right) \quad \text{for} \quad \eta(z) \le \frac{1}{2}\bar\eta \tag{6.110}$$

inductively for all $j \le N$ with a sufficiently large running coupling constant β_j. Hence (6.44) of Proposition 6.3 follows.

I complete this section by giving the proofs of Lemma 6.5 and Lemma 6.6.

PROOF OF LEMMA 6.5. [Write η, β, β' instead of $\eta(z)$, β_j, β_{j-1}. Consider $\eta \geq O(\beta'^{-1/4})$. The $\mathcal{R}_{j-1}$-part of the activity R_{j-1} is given by Eq. (6.80). With

$$[G_{W'}(\beta) * G_{W'}(\beta)](z) = 2\, e^{-2\beta} \frac{I_1(2\beta\sqrt{1-z})}{2\beta\sqrt{1-z}}$$

from (5.68) and $\mathcal{N}_{j0} = [G_{W'}(\beta) * G_{W'}(\beta)](0)$, cp. (6.56), $\mathcal{R}_{j-1}$ takes the form

$$\mathcal{R}_{j-1} = \left(\frac{\mathcal{N}_{j0}}{\mathcal{N}_j} \frac{I_1(2\beta\sqrt{1-z})}{\sqrt{1-z}\, I_1(2\beta)}\right)^2 \exp(+2\beta' z) - 1 \,. \tag{6.111}$$

Because of $z \in \partial\Omega(\eta)$ with $\eta \geq O(\beta'^{-1/4})$, one can simply estimate

$$|z^{-2}\mathcal{R}_{j-1}(z)| \leq |z^{-2}(\mathcal{R}_{j-1}(z)+1)| + |z^{-2}| \tag{6.112}$$

As a consequence of $z \in \partial\Omega(\eta)$, one gets $|1-z| \geq (\cosh\eta - 1)/2 = \sinh^2(\eta/2)$. Hence the argument of the Bessel function can be estimated to be at least of the order $\beta^{3/4}$

$$|2\beta\sqrt{1-z}| \geq 2\beta \sinh\frac{\eta}{2} > \beta\eta \geq O(\beta^{3/4}) \,.$$

Therefore the Bessel function I_1 can be replaced by its leading asymptotic term within a small and bounded error, i.e.

$$I_1(2\beta) = \frac{\exp(2\beta)}{\sqrt{4\pi\beta}}[1 + O(\beta^{-1})] \tag{6.113a}$$

$$I_1(2\beta\sqrt{1-z}) = \frac{\exp(2\beta\sqrt{1-z})}{\sqrt{4\pi\beta\sqrt{1-z}}}[1 + O(\beta^{-3/4})] \,. \tag{6.113b}$$

Then, using $\mathcal{N}_{j0}\mathcal{N}_j^{-1} = 1 + O(\beta^{-1})$ from Proposition 6.4, Eq. (6.56),

$$\mathcal{R}_{j-1} + 1 = [1 + O(\beta^{-3/4})](1-z)^{-3/2} \exp(4\beta(\sqrt{1-z} - 1) + 2\beta' z) \,. \tag{6.114}$$

Consider first the exponent of (6.114). In the following, it will be shown that it fulfils the bound

$$4\beta(\operatorname{Re}\sqrt{1-z} - 1) + 2\beta' \operatorname{Re} z \leq \beta' f(\eta) \tag{6.115}$$

for $O(\beta'^{-1/4}) \leq \eta \leq \bar{\eta}$. Introduce the auxiliary function

$$H_\eta(x) \equiv 4\beta(x\cosh\frac{\eta}{2} - 1) + 2\beta'(\cosh^2\frac{\eta}{2} - x^2\cosh\eta) \quad \text{for} \quad x \in [0,1] \tag{6.116}$$

with the property

$$\begin{aligned} H_\eta(\cos\frac{\omega}{2}) &= 4\beta\left(\cos\frac{\omega}{2}\cosh\frac{\eta}{2} - 1\right) + 2\beta'\left(\sin^2\frac{\omega}{2}\cosh^2\frac{\eta}{2} - \cos^2\frac{\omega}{2}\sinh^2\frac{\eta}{2}\right) \\ &= 4\beta(\operatorname{Re}\sqrt{1-z} - 1) + 2\beta' \operatorname{Re} z \end{aligned} \tag{6.117}$$

for $z = \sin^2(\theta/2)$ with $\operatorname{Re}\theta = \omega$ and $|\operatorname{Im}\theta| = \eta(z) = \eta$. In order to determine $\max_x H_\eta(x)$ for given η, one has to compare $H_\eta(0)$, $H_\eta(1)$, and $H_\eta(x_{loc})$ for x_{loc} being the local extremum. One gets

$$H_\eta(0) = -4\beta + 2\beta'\cosh^2\frac{\eta}{2} \leq -4\beta + 2\beta'\sqrt{2}^2 = -4(\beta - \beta') < 0 \,,$$

where $\beta' < \beta$, (6.55), was used, and

$$\begin{aligned} H_\eta(1) &= 8\beta\sinh^2\frac{\eta}{4} - 2\beta'\sinh^2\frac{\eta}{2} = -8\beta\sinh^4\frac{\eta}{4} + 2(\beta-\beta')\sinh^2\frac{\eta}{2} \\ &= -8\beta\sinh^2\frac{\eta}{4}\left[\sinh^2\frac{\eta}{4} - \frac{\beta-\beta'}{\beta}\cosh^2\frac{\eta}{4}\right] \,. \end{aligned}$$

The value $H_\eta(1)$ is also negative because $\sinh^2(\eta/4)$ is larger than $\eta^2/16 \geq O(\beta'^{-1/2})$, whereas the second term in the bracket [...] is only of order β^{-1} due to $\beta - \beta' = O(1)$ by (6.55). The local extremum x_{loc} fulfils

$$H'_\eta(x_{loc}) = 4\beta\cosh\frac{\eta}{2} - 4\beta' x_{loc}\cosh\eta = 0 \qquad \text{and} \qquad H''_\eta(x_{loc}) < 0 \,,$$

and is hence a local maximum given by

$$x_{loc} = \frac{\beta}{\beta'}\frac{\cosh\frac{\eta}{2}}{\cosh\eta} \,. \tag{6.118}$$

It can be easily estimated from below

$$x_{loc}\cosh\frac{\eta}{2} > \frac{\cosh^2\frac{\eta}{2}}{\cosh\eta} \geq \frac{2}{3} \,.$$

To estimate it from above, one uses

$$(\beta-\beta')\cosh^2\frac{\eta}{2} \leq 2(\beta-\beta') = O(1) \qquad \text{and} \qquad \beta'\sinh^2\frac{\eta}{2} \geq \beta'\frac{\eta^2}{4} \geq O(\beta^{1/2}) \,.$$

These bounds imply

$$(\beta-\beta')\cosh^2\frac{\eta}{2} < \beta'\sinh^2\frac{\eta}{2} \quad \Rightarrow \quad \beta\cosh^2\frac{\eta}{2} < \beta'\cosh\eta$$

and finally

$$x_{loc}\cosh\frac{\eta}{2} = \frac{\beta}{\beta'}\frac{\cosh^2\frac{\eta}{2}}{\cosh\eta} < 1 \,. \tag{6.119}$$

This inequality is important, because it makes it possible to replace the coupling constant β by β' in the upper bound $H_\eta(x_{loc})$ of $H_\eta(x)$. For this purpose, define the function

$$\widetilde{H}_\eta(x) \equiv \beta'\{4(x\cosh\frac{\eta}{2} - 1) + 2\cosh^2\frac{\eta}{2} - 2x^2\cosh\eta\} \,. \tag{6.120}$$

Then, using $x_{loc}\cosh(\eta/2) - 1 < 0$ by (6.119),

$$H_\eta(x) \leq H_\eta(x_{loc}) < \widetilde{H}_\eta(x_{loc}) \leq \widetilde{H}_\eta(\widetilde{x}_{loc}) \,. \tag{6.121}$$

Obviously

$$\widetilde{x}_{loc} = \frac{\cosh\frac{\eta}{2}}{\cosh\eta} \in [\tfrac{1}{3}\sqrt{2}, 1] \qquad \text{for} \qquad \eta \in [0, \bar{\eta}]. \tag{6.122}$$

Thus one gets

$$\widetilde{H}_\eta(\widetilde{x}_{loc}) = \beta' \left\{ 4\left(\frac{\cosh^2\frac{\eta}{2}}{\cosh\eta} - 1\right) + 2\left(\cosh^2\frac{\eta}{2} - \frac{\cosh^2\frac{\eta}{2}}{\cosh\eta}\right)\right\} = \beta' \, 4\,\frac{\sinh^4\frac{\eta}{2}}{\cosh\eta}$$

and, finally,

$$4\beta(\operatorname{Re}\sqrt{1-z} - 1) + 2\beta' \operatorname{Re} z \le \beta' \, 4\,\frac{\sinh^4\frac{\eta}{2}}{\cosh\eta} = \beta' f(\eta), \tag{6.123}$$

which proves (6.115). Now estimate the factor $|z^{-2}(1-z)^{-3/2}|$ occuring in the estimation of $|z^{-2}(\mathcal{R}_{j-1}(z)+1)|$, cp. (6.112) and (6.114). Suppose that $0 < \eta \le \bar{\eta}$, then

$$\max_{z\in\partial\Omega(\eta)} \frac{1}{|z|^2|\sqrt{1-z}|^3} = \frac{1}{|z|^2|\sqrt{1-z}|^3}\bigg|_{z=-\sinh^2\frac{\eta}{2}} \le \frac{1}{\sinh^4\frac{\eta}{2}}. \tag{6.124}$$

The simple and straightforward proof of this inequality will be omitted. Since the right-hand side of (6.124) increases monotonously for decreasing η, let η be bounded from below by $\eta \ge 3\beta'^{-1/4}$. This yields $\sinh^{-4}(\eta/2) \le 16\,\eta^{-4} \le (2/3)^4\beta'$ and gives the result

$$|z^{-2}\mathcal{R}_{j-1}(z)| \le |z^{-2}| + |z^{-2}(\mathcal{R}_{j-1}(z)+1)| \le \left(\frac{2}{3}\right)^4 \beta'\left(1 + e^{\beta' f(\eta)}[1 + O(\beta^{-3/4})]\right)$$

by (6.114) and (6.123). It can be written as

$$|z^{-2}\mathcal{R}_{j-1}(z)| \le \frac{1}{2}\beta' \, \frac{1+\exp[\beta' f(\eta)]}{2}\,[1 + O(\beta'^{-3/4})] \qquad \text{for} \quad 3\beta'^{-1/4} \le \eta \le \bar{\eta}, \tag{6.125}$$

which already proves the proposition for $\kappa \ge 3$.

Small* η *analysis

To extend the proof for smaller κ of order 1 (which fulfil, however, also $\kappa^{-1} = O(1)$), I consider the case $\eta = O(\beta'^{-1/4})$ now. Although the functions $z^2\mathcal{R}_{j-1}(z)$ will diverge for $j \ne N$ and $z \to 0$ anyway, it is nevertheless necessary to take the (-1) in the definition of $\mathcal{R}_{j-1}$ into account; up to now I have simply used the split $\mathcal{R}_{j-1} = (\mathcal{R}_{j-1}+1) - 1$ in combination with the triangle inequality, see (6.112).

For $z \in \partial\Omega(\eta)$ with $\eta = O(\beta'^{-1/4})$, the maximum of $|z^{-2}\mathcal{R}_{j-1}(z)|$ lies in the small field region $|z| = O(\beta'^{-1/2})$. Outside this region, the real part of z clearly dominates the imaginary one, and this causes both factors $|z|^{-2}$ and $|\mathcal{R}_{j-1}(z)|$ to become small. Hence the analysis can be restricted to the small field region where $\mathcal{R}_{j-1}$ is easily expanded to be

$$\begin{aligned}\mathcal{R}_{j-1}(\sin^2\frac{\theta}{2}) &= [1 + O(\beta^{-1/2})]\,\frac{1}{\cos^3\frac{\theta}{2}}\,\exp\left(-8\beta\sin^4\frac{\theta}{4} - 2(\beta-\beta')\sin^2\frac{\theta}{2}\right) - 1 \\ &= [1 + O(\beta'^{-1/2})]\exp\left(-8\beta'\sin^4\frac{\theta}{4}\right) - 1.\end{aligned} \tag{6.126}$$

Therefore, by $z = \sin^2(\theta/2) = 4\sin^2(\theta/4)\cos^2(\theta/4)$,

$$\begin{aligned}\max_{z\in\partial\Omega(\eta)} |z^{-2}\mathcal{R}_{j-1}(z)| &\le \max_{\theta:|\operatorname{Im}\theta|=\eta}\left|\sin^{-4}\frac{\theta}{2}\left([1 + O(\beta'^{-1/2})]\exp\left(-8\beta'\sin^4\frac{\theta}{4}\right) - 1\right)\right| \\ &= \max_{\theta:|\operatorname{Im}\theta|=\eta} \frac{1}{|\sin^4\frac{\theta}{2}|}\left|[1 + O(\beta'^{-1/2})]\left(\exp\left(-8\beta'\sin^4\frac{\theta}{4}\right) - 1\right) + O(\beta'^{-1/2})\right| \\ &\le O(\beta'^{+1/2}) + \frac{1}{2}[1 + O(\beta'^{-1/2})]\max_{\theta:|\operatorname{Im}\theta|=\eta}\left|\frac{\exp(-8\beta'\sin^4\frac{\theta}{4}) - 1}{8\sin^4\frac{\theta}{4}}\right|.\end{aligned} \tag{6.127}$$

Now one can proceed by use of the fundamental theorem of calculus

$$\begin{aligned}\left|\frac{\exp(-8\beta'\sin^4\frac{\theta}{4}) - 1}{8\sin^4\frac{\theta}{4}}\right| &= \beta'\left|\int_0^1 dt\,\exp\left(-8\beta' t\sin^4\frac{\theta}{4}\right)\right| \le \beta'\int_0^1 dt\,\exp\left(-8\beta' t\operatorname{Re}\sin^4\frac{\theta}{4}\right) \\ &\le \beta'\int_0^1 dt\,\exp\left(-8\beta'[-\frac{1}{32}\eta^4 + O(\beta'^{-3/2})]\,t\right) \\ &= \beta'\,[1 + O(\beta'^{-1/2})]\int_0^1 dt\,\exp\left(\frac{1}{4}\beta'\eta^4 t\right) \\ &= \beta'\,\frac{\exp[\frac{1}{4}\beta'\eta^4] - 1}{\frac{1}{4}\beta'\eta^4}\,[1 + O(\beta'^{-1/2})].\end{aligned} \tag{6.128}$$

One gets the result

$$|z^{-2}\mathcal{R}_{j-1}(z)| \le \frac{1}{2}\beta'\,\frac{\exp[\frac{1}{4}\beta'\eta^4] - 1}{\frac{1}{4}\beta'\eta^4}\,[1 + O(\beta'^{-1/2})] \qquad \text{for} \qquad \eta = O(\beta'^{-1/4}). \tag{6.129}$$

Using $\sinh x \le x\cosh x$ for $x \ge 0$ and

$$f(\eta) = \frac{1}{4}\eta^4 + O(\beta'^{-3/2}) \qquad \text{for} \qquad \eta \le O(\beta'^{-1/4}), \tag{6.130}$$

one can further estimate

$$\begin{aligned}\frac{\exp[\frac{1}{4}\beta'\eta^4] - 1}{\frac{1}{4}\beta'\eta^4} &= \exp\left[\frac{1}{8}\beta'\eta^4\right]\frac{\sinh(\frac{1}{8}\beta'\eta^4)}{\frac{1}{8}\beta'\eta^4} \le \exp\left[\frac{1}{8}\beta'\eta^4\right]\cosh\left(\frac{1}{8}\beta'\eta^4\right) \\ &= \exp\left[\frac{1}{2}\beta' f(\eta) + O(\beta'^{-1/2})\right]\cosh\left(\frac{1}{2}\beta' f(\eta) + O(\beta'^{-1/2})\right) \\ &= \exp\left[\frac{1}{2}\beta' f(\eta)\right]\cosh\left(\frac{1}{2}\beta' f(\eta)\right)[1 + O(\beta'^{-1/2})] \\ &= \frac{\exp[\beta' f(\eta)] + 1}{2}\,[1 + O(\beta'^{-1/2})].\end{aligned} \tag{6.131}$$

The lemma follows by combining the estimate (6.125) for $\eta \in [3\beta'^{-1/4}, \bar{\eta}]$ with (6.129), (6.131) for $\eta = O(\beta'^{-1/4})$. ⌋

Proof of Lemma 6.6. ⌈ Set $\beta_j = \beta, \beta_{j-1} = \beta'$, and $z = \sin^2(\theta/2)$ with $\mathrm{Re}\,\theta = \omega$ and $|\mathrm{Im}\,\theta| = \eta(z) = \eta$. Abbreviate $\cos(\omega/2) = x$. Then

$$\begin{aligned}\left(\mathcal{N}_{j0}^{-1}\big[|G_W(\beta_j)| * |G_W(\beta_j)|\big](z)\right)^2 |G_W(\beta_{j-1},z)|^{-1}\exp(+\alpha\beta_{j-1}|z|^2)\\ = \left(\frac{I_1(2\beta\,\mathrm{Re}\cos\frac{\theta}{2})}{\mathrm{Re}\cos\frac{\theta}{2}\,I_1(2\beta)}\right)^2 \exp\left\{+2\beta'\,\mathrm{Re}\sin^2\frac{\theta}{2} + \alpha\beta'\left|\sin^2\frac{\theta}{2}\right|\right\}\\ = \left(\frac{I_1(2\beta x\cosh\frac{\eta}{2})}{x\cosh\frac{\eta}{2}\,I_1(2\beta)}\right)^2 \exp\left\{2\beta'(\cosh^2\frac{\eta}{2} - x^2\cosh\eta) + \alpha\beta'(\cosh^2\frac{\eta}{2} - x^2)^2\right\}, \quad (6.132)\end{aligned}$$

where

$$\big[|G_W(\beta)| * |G_W(\beta)|\big](z) = 2\,e^{-2\beta}\frac{I_1(2\beta\,\mathrm{Re}\sqrt{1-z})}{2\beta\,\mathrm{Re}\sqrt{1-z}}$$

has been used. Calculate $x_{max}(\eta) \in [0,1]$. Local extremums are determined by the equation

$$n(x) \equiv \frac{I_2(2\beta x\cosh\frac{\eta}{2})}{I_1(2\beta x\cosh\frac{\eta}{2})}2\beta\cosh\frac{\eta}{2} - 2\beta'\big[(2+\alpha)\cosh^2\frac{\eta}{2} - 1\big]x + 2\alpha\beta' x^3 = 0. \quad (6.133)$$

This function $n(x)$ has two zeroes in $[0,1]$; the first zero, $x = 0$, yields a local minimum, because the slope $n'(0)$ is positive. The second zero, x_{max}, is smaller than 1 but of order 1. Since $n'(x_{max})$ is negative, x_{max} is the local and global maximum on $[0,1]$. Consider thus the regime $x = O(1)$, where the asymptotic expansion of the modified Bessel functions can be used (see e.g. Sect. 3)

$$\frac{I_2(2\beta x\cosh\frac{\eta}{2})}{I_1(2\beta x\cosh\frac{\eta}{2})}2\beta\cosh\frac{\eta}{2} = 2\beta\cosh\frac{\eta}{2} - \frac{3}{2x} + \frac{3}{16\beta x^2\cosh\frac{\eta}{2}} + O(\beta^{-2}x^{-3}) \quad \text{for large } \beta x.$$

Expanding also

$$\frac{I_1(2\beta x\cosh\frac{\eta}{2})}{x\cosh\frac{\eta}{2}\,I_1(2\beta)} = \left(x\cosh\frac{\eta}{2}\right)^{-3/2}\exp\left\{2\beta(x\cosh\frac{\eta}{2} - 1)\right\}\big[1 + O(\beta^{-1})\big],$$

Eq. (6.132) implies

$$\begin{aligned}\max_{z\in\partial\Omega(\eta)}\left\{\left(\mathcal{N}_j^{-1}\big[|G_W(\beta_j)| * |G_W(\beta_j)|\big](z)\right)^2 |G_W(\beta_{j-1},z)|^{-1}\exp(+\alpha\beta_{j-1}|z|^2)\right\}\\ = \exp\left\{4\beta(x_{max}\cosh\frac{\eta}{2} - 1) + 2\beta'(\cosh^2\frac{\eta}{2} - x_{max}^2\cosh\eta) + \alpha\beta'(\cosh^2\frac{\eta}{2} - x_{max}^2)^2\right\}\\ \times\big[1 + O(\beta^{-1})\big](x_{max}\cosh\frac{\eta}{2})^{-3}. \quad (6.134)\end{aligned}$$

Upper and lower bounds of $x_{max}(\eta)$ for $O(\beta'^{-1/4}) \le \eta \le \tilde\eta$

First, one is interested in bounds on x_{max}. Substituting $n(x)$ by

$$\overline{n}(x) \equiv 2\beta\cosh\frac{\eta}{2} - 2\beta' x\cosh\eta,$$

the equation $\overline{n}(\overline{x}) = 0$ is easily solved

$$\overline{x} = \frac{\beta}{\beta'}\frac{\cosh\frac{\eta}{2}}{\cosh\eta} \in (\frac{\sqrt{2}}{3}, 1) \quad \text{for} \quad O(\beta'^{-1/4}) \le \eta \le \tilde\eta. \quad (6.135)$$

This value provides an upper bound $x_{max} < \overline{x}$ since

$$n(x) = \overline{n}(x) - 2\alpha\beta'(\cosh^2\frac{\eta}{2} - x^2)x + O(1) < \overline{n}(x) \quad \text{for} \quad x = O(1).$$

Note that the term proportional to α is at least of order $\beta'^{1/2}$ due to $\cosh^2(\eta/2) - x^2 \ge O(\beta'^{-1/2})$ for the η-values under consideration.
On the other hand, one can substitute $n(x)$ by

$$\underline{n}(x) \equiv 2\beta\cosh\frac{\eta}{2} - 2\beta'\big[(2+\alpha)\cosh^2\frac{\eta}{2} - 1\big]x$$

with the zero

$$\underline{x} = \frac{\beta}{\beta'}\frac{\cosh\frac{\eta}{2}}{(2+\alpha)\cosh^2\frac{\eta}{2} - 1} \in \left(\frac{\sqrt{2}}{3+2\alpha}, \frac{1}{1+\alpha}\right). \quad (6.136)$$

Because of

$$n(x) = \underline{n}(x) + 2\alpha\beta' x^3 + O(1) > \underline{n}(x) \quad \text{for} \quad x = O(1),$$

the lower bound $x_{max} > \underline{x}$ follows.

Now the factor $(x_{max}\cosh(\eta/2))^{-3}$ in (6.134) can be further estimated by

$$(x_{max}\cosh\frac{\eta}{2})^{-3} \le (\underline{x}\cosh\frac{\eta}{2})^{-3} = \left(\frac{\beta'}{\beta}\left[2 + \alpha - \frac{1}{\cosh^2\frac{\eta}{2}}\right]\right)^3 < \left(\frac{3+2\alpha}{2}\right)^3, \quad (6.137)$$

where $\beta' < \beta$ and $\cosh^2(\eta/2) \le \cosh^2(\tilde\eta/2) = 2$, cp. (6.14), has been used.

Estimation of the exponential factor for $O(\beta'^{-1/4}) \le \eta \le \tilde\eta$

By (6.135) and (6.119) one gets

$$x_{max}\cosh\frac{\eta}{2} < \overline{x}\cosh\frac{\eta}{2} < 1$$

and hence

$$\begin{aligned}&\exp\left\{4\beta(x_{max}\cosh\frac{\eta}{2} - 1) + 2\beta'(\cosh^2\frac{\eta}{2} - x_{max}^2\cosh\eta) + \alpha\beta'(\cosh^2\frac{\eta}{2} - x_{max}^2)^2\right\}\\ &\le \exp\left\{\beta'\left(4(x_{max}\cosh\frac{\eta}{2} - 1) + 2\cosh^2\frac{\eta}{2} - 2x_{max}^2\cosh\eta + \alpha(\cosh^2\frac{\eta}{2} - x_{max}^2)^2\right)\right\}\\ &\le \exp\left\{\beta'\max_{x\in[0,1]}\left(4(x\cosh\frac{\eta}{2} - 1) + 2\cosh^2\frac{\eta}{2} - 2x^2\cosh\eta + \alpha(\cosh^2\frac{\eta}{2} - x^2)^2\right)\right\}. \quad (6.138)\end{aligned}$$

Abbreviate the right-hand side of this inequality by $\exp\{\beta'\max_{y\in[0,1]}A_\eta(y)\}$ with the function

$$A_\eta(y) \equiv 4(y\cosh\frac{\eta}{2} - 1) + 2\cosh^2\frac{\eta}{2} - 2y^2\cosh\eta + \alpha(\cosh^2\frac{\eta}{2} - y^2)^2, \quad (6.139)$$

whose maximum y_{max} is determined by

$$\cosh\frac{\eta}{2} - y_{max}\cosh\eta - \alpha(\cosh^2\frac{\eta}{2} - y_{max}^2)y_{max} = 0$$

and fulfils the conditions

$$\underline{y} = \frac{\cosh\frac{\eta}{2}}{(2+\alpha)\cosh^2\frac{\eta}{2} - 1} < y_{max} < \frac{\cosh\frac{\eta}{2}}{\cosh\eta} = \overline{y}\,. \tag{6.140}$$

At the moment, we are only interested in a bound for the region $O(\beta'^{-1/4}) \le \eta \le \bar\eta$. For later use, however, one must be able to continue the bound to smaller values of η with the property that it has to go to zero for $\eta \to 0$. This gives an upper bound for $\alpha = O(1)$. Consider the case $\eta = 0$.

$$A_0(y) = 4(y-1) + 2 - 2y^2 + \alpha(1-y^2)^2 = -2(1-y)^2 + \alpha(1-y^2)^2$$
$$A_0(\cos\frac{\omega}{2}) = -2\Big[2\sin^2\frac{\omega}{4}\Big]^2 + \alpha\sin^4\frac{\omega}{2} = -8\sin^4\frac{\omega}{4}\Big[1 - 2\alpha\cos^4\frac{\omega}{4}\Big]\,.$$

Obviously, the function is less than or equal to zero if $1 - 2\alpha\cos^4(\omega/4) \ge 0$ for all $\omega \in [0,\pi]$. Therefore it follows that

$$y_{max} \underset{\eta\to 0}{\longrightarrow} 1 \qquad \text{for} \qquad \alpha \le \frac{1}{2}\,.$$

Now the function $A_\eta(x)$ has to be estimated by a bound which goes to zero for $\eta \to 0$. Using $(\cosh^2(\eta/2) - y^2)^2 = (\cosh(\eta/2)+y)^2(\cosh(\eta/2)-y)^2 \le 4\cosh^4(\eta/4)(\cosh(\eta/2)-y)^2$, one introduces

$$\widetilde{A}_\eta(y) \equiv 4(y\cosh\frac{\eta}{2} - 1) + 2\cosh^2\frac{\eta}{2} - 2y^2\cosh\eta + 4\alpha\cosh^4\frac{\eta}{4}(\cosh\frac{\eta}{2} - y)^2 \ge A_\eta(y)\,. \tag{6.141}$$

The maximum $\widetilde{y}_{max}$ of $\widetilde{A}_\eta(y)$ is now easily determined to be

$$\widetilde{y}_{max} = \frac{\cosh\frac{\eta}{2} - 2\alpha\cosh^4\frac{\eta}{4}\cosh\frac{\eta}{2}}{\cosh\eta - 2\alpha\cosh^4\frac{\eta}{4}} \qquad \text{for} \qquad 0 \le \alpha < \frac{1}{2}\,. \tag{6.142}$$

Thus

$$\max_{y\in[0,1]} A_\eta(y) \le \max_{y\in[0,1]} \widetilde{A}_\eta(y) = \widetilde{A}_\eta(\widetilde{y}_{max}) = \frac{1 + 2\alpha\cosh^4\frac{\eta}{4}}{\cosh\eta - 2\alpha\cosh^4\frac{\eta}{4}}\,4\sinh^4\frac{\eta}{2}\,. \tag{6.143}$$

This bound is well defined and positive for $\alpha < 1/2$.

Small η analysis

Now consider the region of small η, i.e. $\eta \le O(\beta'^{-1/4})$. In this region, it is useful to insert $x = \cos(\omega/2)$ into the function $n(x)$, Eq. (6.133), and to expand with respect to the variable ω, i.e.

$$\cos\frac{\omega}{2} = 1 - \frac{1}{2}\Big(\frac{\omega}{2}\Big)^2 + \frac{1}{4!}\Big(\frac{\omega}{2}\Big)^4 + O(\omega^6)\quad,\quad \cosh\frac{\eta}{2} = 1 + \frac{1}{2}\Big(\frac{\eta}{2}\Big)^2 + \frac{1}{4!}\Big(\frac{\eta}{2}\Big)^4 + O(\eta^6)\,.$$

Since $\omega_{max}(\eta)$ must be expected to be at most of order $\beta'^{-1/4}$ in this region, it follows that

$$\frac{n(\cos\frac{\omega}{2})}{\beta'} = 2\Big[\cosh\frac{\eta}{2} - \cos\frac{\omega}{2}\cosh\eta - \alpha(\cosh^2\frac{\eta}{2} - \cos^2\frac{\omega}{2})\cos\frac{\omega}{2}\Big] + O(\beta'^{-1})$$
$$= (1-2\alpha)\Big(\frac{\omega}{2}\Big)^2 - (3+2\alpha)\Big(\frac{\eta}{2}\Big)^2 + O(\beta'^{-1})\,.$$

Hence

$$\omega_{max}^2 = \frac{3+2\alpha}{1-2\alpha}\eta^2 + O(\beta'^{-1}) \qquad \text{for} \qquad \eta \le O(\beta'^{-1/4})\,. \tag{6.144}$$

Consider now the factor

$$(x_{max}\cosh\frac{\eta}{2})^{-3} = \Big(\cos\frac{\omega_{max}}{2}\cosh\frac{\eta}{2}\Big)^{-3} = \Big(1 - \frac{1+2\alpha}{1-2\alpha}\Big(\frac{\eta}{2}\Big)^2 + O(\beta'^{-1})\Big)^{-3}$$
$$= 1 + O(\beta'^{-1/2})\,, \tag{6.145}$$

provided that $\alpha < 1/2$ is chosen small enough in order to guarantee

$$(1-2\alpha)^{-1} = O(1)\,. \tag{6.146}$$

To complete the proof in the small η region, one estimates

$$\exp\Big\{\beta'\max_{x\in[0,1]}\Big(4(x\cosh\frac{\eta}{2} - 1) + 2\cosh^2\frac{\eta}{2} - 2x^2\cosh\eta + \alpha(\cosh^2\frac{\eta}{2} - x^2)^2\Big)\Big\}$$
$$= \exp\Big\{\beta'\Big(\frac{1+2\alpha}{1-2\alpha}4\Big(\frac{\eta}{2}\Big)^4 + O(\beta'^{-1/2})\Big)\Big\} = \exp\Big\{\beta'\frac{1+2\alpha}{1-2\alpha}f(\eta) + O(\beta'^{-1/2})\Big\}$$
$$= [1 + O(\beta'^{-1/2})]\exp\Big\{\beta'\frac{1+2\alpha}{1-2\alpha}f(\eta)\Big\}\,, \tag{6.147}$$

where use has been made of

$$4(\operatorname{Re}\cos\frac{\theta}{2} - 1) = 4\Big(\cos\frac{\omega}{2}\cosh\frac{\eta}{2} - 1\Big)$$
$$= -2\Big[\Big(\frac{\omega}{2}\Big)^2 - \Big(\frac{\eta}{2}\Big)^2\Big] + \frac{1}{3!}\Big[\Big(\frac{\omega}{2}\Big)^4 + \Big(\frac{\eta}{2}\Big)^4\Big] - \Big(\frac{\omega}{2}\Big)^2\Big(\frac{\eta}{2}\Big)^2\,,$$
$$2\operatorname{Re}\sin^2\frac{\theta}{2} = 2\Big[\sin^2\frac{\omega}{2}\cosh^2\frac{\eta}{2} - \cos^2\frac{\omega}{2}\sinh^2\frac{\eta}{2}\Big]$$
$$= 2\Big[\Big(\frac{\omega}{2}\Big)^2 - \Big(\frac{\eta}{2}\Big)^2\Big] - \frac{2}{3}\Big[\Big(\frac{\omega}{2}\Big)^4 + \Big(\frac{\eta}{2}\Big)^4\Big] + 4\Big(\frac{\omega}{2}\Big)^2\Big(\frac{\eta}{2}\Big)^2\,.$$

This proves the lemma for $\eta \le O(\beta'^{-1/4})$.

Completion of the proof for $O(\beta'^{-1/4}) \le \eta \le \bar\eta$

Let us return to the region $O(\beta'^{-1/4}) \le \eta \le \bar\eta$. Here, I first prove the assertion

$$\widetilde{A}_\eta(\widetilde{y}_{max}) \le \frac{1+2\alpha}{1-2\alpha}f(\eta)\,, \tag{6.148}$$

which reduces to

$$\frac{1+2\alpha\cosh^4\frac{\eta}{4}}{\cosh\eta-2\alpha\cosh^4\frac{\eta}{4}}\le\frac{1+2\alpha}{1-2\alpha}\,\frac{1}{\cosh\eta}$$

or, equivalently,

$$\Big[1+2\alpha\cosh^4\frac{\eta}{4}\Big](1-2\alpha)\cosh\eta\le(1+2\alpha)\Big[\cosh\eta-2\alpha\cosh^4\frac{\eta}{4}\Big].$$

After a short calculation, it turns out that this inequality is valid when the following relation holds

$$\cosh^2\frac{\eta}{2}\cosh^4\frac{\eta}{4}\le\cosh\eta\,.$$

But this is indeed the case for $\eta\in[0,\tilde\eta]$.

The remaining $(x_{max}\cosh(\eta/2))^{-3}$ factor can be absorbed for $\eta\ge a\beta'^{-1/4}$, $a=O(1)$, into $\exp(\delta\,\beta' f(\eta))$ with a small constant δ of order 1. This is a simple consequence of

$$\delta\,\beta' f(\eta)\ge\frac{1}{4}\delta a^4+O(\beta'^{-1/2})\qquad\text{for}\qquad\eta\ge a\beta'^{-1/4}$$

and the fact that $(x_{max}\cosh(\eta/2))^{-3}$ is smaller than $[(3+2\alpha)/2]^3$ for all $\eta\in[0,\tilde\eta]$. The following sufficient but very rough condition of domination,

$$\delta a^4=12\ln\left(\frac{3+2\alpha}{2}\right)<12\ln 2\,,$$

allows the choice of small parameters δ for $a=O(1)$. ⌋

6.4. GENERATION OF UNIT DISC BOUNDS

The reduced activities $R_j(z)$ are bounded on the closed unit disc $|z|\le 1$. The unit disc bounds do not iterate and must hence be generated from the Ω-bounds on the domain $\overline{\Omega(\tilde\eta/2)}$.

In order to derive a unit disc bound for $|R_{j-1}(z)|$, one starts again with the estimate (6.93) of Sect. 6.3, e.g.

$$\begin{aligned}|z^{-2}R_{j-1}(z)|\le{}&|z^{-2}\mathcal{R}_{j-1}(z)|\\&+\exp\Big(4\sigma\beta_j f(\tfrac{1}{2}\eta(z))\Big)\Big(\mathcal{N}_j^{-1}\big[|G_W(\beta_j)|*|G_W(\beta_j)|\big](z)\Big)^2|G_W(\beta_{j-1},z)|^{-1}\\&\times\left\{\frac{1}{4}C\beta_j\sum_{d=0}^{3}\frac{1}{4}\binom{4}{d+1}\left(\frac{C\beta_j|z|^2}{16}\right)^d\right.\\&\left.+\sum_{d=1}^{4}(C\beta_j)^d\sum_{\substack{k_1,k_2=0,1,2\\k_1+k_2=d}}\ \sum_{\substack{l_1,l_2=1,2\\l_1+l_2\ge3}}\frac{1}{|z|^2}\prod_{i=1,2}\left[\binom{2}{k_i}\mathcal{E}_{k_il_i}(\beta_j,z)\right]\right\}.\end{aligned}\tag{6.149}$$

To proceed further, however, the estimates of Lemma 6.6 and 6.5 have to be replaced by more convenient ones.

Lemma 6.10. *Let β_j be large enough and $|z|\in[0,1]$. Then*

$$\begin{aligned}&\Big(\mathcal{N}_j^{-1}\big[|G_W(\beta_j)|*|G_W(\beta_j)|\big](z)\Big)^2|G_W(\beta_{j-1},z)|^{-1}\\&\qquad\le[1+\text{const}\,\beta_{j-1}^{-1}]\left(1-\frac{1}{16}|z|^2\right)^{-3}\exp\Big\{\frac{1}{2}\beta_{j-1}|z|^2\Big\},\end{aligned}\tag{6.150}$$

with a constant of order 1.

Proof. ⌈ Set $z=\sin^2(\theta/2)$ with $\operatorname{Re}\theta=\omega$ and $|\operatorname{Im}\theta|=\eta(z)=\eta$. Abbreviate $\beta_j=\beta$, $\beta_{j-1}=\beta'$, and $\cos(\omega/2)=x$. Then consider

$$\begin{aligned}&\Big(\mathcal{N}_{j0}^{-1}\big[|G_W(\beta_j)|*|G_W(\beta_j)|\big](z)\Big)^2|G_W(\beta_{j-1},z)|^{-1}\\&=\left(\frac{I_1(2\beta x\cosh\frac{\eta}{2})}{x\cosh\frac{\eta}{2}\,I_1(2\beta)}\right)^2\exp\Big(2\beta'\big[\cosh^2\frac{\eta}{2}-x^2\cosh\eta\big]\Big)\\&\le\left[\frac{I_1(2\beta x\cosh\frac{\eta}{2})}{x\cosh\frac{\eta}{2}\,I_1(2\beta)}\exp\Big(\beta'\big[\cosh^2\frac{\eta}{2}-x^2\cosh\eta\big]\Big)\right.\\&\qquad\left.\times\exp\left\{\beta'\frac{(\cosh^2\frac{\eta}{2}-x^2)^4}{32}\ln\frac{1+\sqrt2}{x+\cosh\frac{\eta}{2}}\right\}\right]^2,\end{aligned}\tag{6.151}$$

with an additional factor that is larger than or equal to 1 due to

$$0\le\ln(x+\cosh\frac{\eta}{2})\le\ln(1+\sqrt2)\qquad\text{for}\qquad|z|\le1\,.$$

The task is to maximize the right-hand side of the inequality (6.151) under the constraint $\cosh^2(\eta/2) - x^2 = |z| \le 1$. Calculus yields the extremal values as zeroes of the function

$$n(x) \equiv (|z| + 2x^2)\frac{\beta}{\beta'}\frac{I_2(2\beta x\sqrt{|z| + x^2})}{I_1(2\beta x\sqrt{|z| + x^2})} + 2x\sqrt{|z| + x^2}\,(1 - |z| - 2x^2) - \frac{|z|^4}{64}. \qquad (6.152)$$

There a two zeroes within the interval $[0, 1]$. The first one, x_{min}, is of order β^{-1} and represents a local minimum since $n'(x_{min})$ is positive. Both facts follow from $n(0) = -(|z|^4)/64$ and $n'(0) = \frac{1}{2}\beta|z|^{3/2} + O(1)$. Now look at the second zero, called x_{max}. I show by establishing lower and upper bounds $\underline{x}$, $\overline{x}$ that it lies in a small neighbourhood of $1 - \frac{1}{4}|z|$.

A lower bound for x_{max}

Make use of

$$\frac{\beta}{\beta'}\frac{I_2(2\beta x\sqrt{|z| + x^2})}{I_1(2\beta x\sqrt{|z| + x^2})} > \frac{I_2(2\beta x\sqrt{|z| + x^2})}{I_1(2\beta x\sqrt{|z| + x^2})} > 1 - \frac{3}{4}\beta^{-1}\frac{1}{x\sqrt{|z| + x^2}},$$

where the first inequality holds due to $\beta' < \beta$ while the second is valid for $x = O(1)$. Introduce the auxiliary function

$$\underline{n}(x) \equiv (|z| + 2x^2)\left(1 - \frac{3}{4}\beta^{-1}\frac{1}{x\sqrt{|z| + x^2}}\right) + 2x\sqrt{|z| + x^2}\,(1 - |z| - 2x^2) - \frac{|z|^4}{64},$$

which fulfils obviously $\underline{n}(x) < n(x)$ for $x = O(1)$, or, more precisely, for $2\beta x \gg 1$. Now consider $\underline{n}$ and its first derivative $\underline{n}'$ at the point $x = 1 - \frac{1}{4}|z|$. One obtains

$$\underline{n}(1 - \frac{1}{4}|z|) = -\frac{3}{4}\beta^{-1}(1 - \frac{1}{16}|z|^2)^{-1}$$

$$\underline{n}'(1 - \frac{1}{4}|z|) = -\frac{8}{1 + \frac{1}{4}|z|}\left\{1 + \frac{|z|^4}{128} - \frac{3}{32}\beta^{-1}\left(\frac{|z|}{1 - \frac{1}{16}|z|^2}\right)^2\right\} < 0.$$

Thus one has

$$\underline{x} < 1 - \frac{1}{4}|z| \qquad \text{with} \qquad (1 - \frac{1}{4}|z|) - \underline{x} = O(\beta^{-1}). \qquad (6.153)$$

An upper bound for x_{max}

Define

$$\overline{n}(x) \equiv (|z| + 2x^2)\left(1 + \frac{\beta - \beta'}{\beta'}\right) + 2x\sqrt{|z| + x^2}\,(1 - |z| - 2x^2) - \frac{|z|^4}{64}$$

fulfilling $\overline{n}(x) > n(x)$ for all $x \in (0, 1]$ due to $I_2(2\beta x\sqrt{|z| + x^2}) \le I_1(2\beta x\sqrt{|z| + x^2})$, where the equality sign holds for $x = 0$. In a small vicinity of the point $x = 1 - \frac{1}{4}|z|$, the function $\overline{n}$ is determined by

$$\overline{n}(1 - \frac{1}{4}|z|) = \frac{\beta - \beta'}{\beta'}(2 + \frac{1}{8}|z|^2) = O(\beta'^{-1}) > 0$$

$$\overline{n}'(1 - \frac{1}{4}|z|) = -\frac{8}{1 + \frac{1}{4}|z|}\left(1 + \frac{|z|^4}{128}\right) + \frac{\beta - \beta'}{\beta'}4(1 - \frac{1}{4}|z|).$$

Thus

$$\overline{x} > 1 - \frac{1}{4}|z| \qquad \text{with} \qquad \overline{x} - (1 - \frac{1}{4}|z|) = O(\beta'^{-1}). \qquad (6.154)$$

As a result one gets $\underline{x} < x_{max} < \overline{x}$ and hence

$$\left|x_{max} - (1 - \frac{1}{4}|z|)\right| = O(\beta^{-1}) = O(\beta'^{-1}). \qquad (6.155)$$

Now use can be made of this information. Replace x by x_{max} in the right-hand side of (6.151) to get

$$\begin{aligned}
&\left(\mathcal{N}_{j0}^{-1}\left[|G_W(\beta_j)| * |G_W(\beta_j)|\right](z)\right)^2 |G_W(\beta_{j-1}, z)|^{-1} \\
&\le \left(\frac{I_1(2\beta x_{max}\sqrt{|z| + x^2_{max}})}{x_{max}\sqrt{|z| + x^2_{max}}\, I_1(2\beta)}\right)^2 \exp\left(2\beta'\left[|z| + x^2_{max} - x^2_{max}(2|z| + 2x^2_{max} - 1)\right]\right) \\
&\quad \times \exp\left(\beta'\frac{|z|^4}{16}\ln\frac{1 + \sqrt{2}}{x_{max} + \sqrt{|z| + x^2_{max}}}\right) \\
&= [1 + O(\beta^{-1})]\left(x_{max}\sqrt{|z| + x^2_{max}}\right)^{-3} \exp\left(4\beta\left[x_{max}\sqrt{|z| + x^2_{max}} - 1\right]\right) \\
&\quad \times \exp\left(2\beta'\left[|z| + x^2_{max} - x^2_{max}(2|z| + 2x^2_{max} - 1)\right]\right) \\
&\quad \times \exp\left(\beta'\frac{|z|^4}{16}\ln\frac{1 + \sqrt{2}}{x_{max} + \sqrt{|z| + x^2_{max}}}\right). \qquad (6.156)
\end{aligned}$$

Then

$$\begin{aligned}
\left(x_{max}\sqrt{|z| + x^2_{max}}\right)^{-3} &= \left[\left(1 - \frac{1}{4}|z| + O(\beta^{-1})\right)\left(1 + \frac{1}{4}|z| + O(\beta^{-1})\right)\right]^{-3} \\
&= [1 + O(\beta^{-1})]\left(1 - \frac{|z|^2}{16}\right)^{-3}. \qquad (6.157)
\end{aligned}$$

Estimation of the exponential factors in (6.156)

Begin with a look at the exponent

$$4\beta\left[x_{max}\sqrt{|z| + x^2_{max}} - 1\right] = 4\beta\left[\left(1 - \frac{1}{16}|z|^2 + O(\beta'^{-1})\right) - 1\right] \qquad (6.158)$$

which is obviously less than zero for sufficiently large values of $|z| \ge O(\beta'^{-1/2})$. Consider this "large field" case first.
Replacing β by $\beta' < \beta$, one estimates

$$\begin{aligned}
&\exp\left(4\beta\left[x_{max}\sqrt{|z| + x^2_{max}} - 1\right] + 2\beta'\left[|z| + x^2_{max} - x^2_{max}(2|z| + 2x^2_{max} - 1)\right]\right. \\
&\quad \left. + \beta'\frac{|z|^4}{16}\ln\frac{1 + \sqrt{2}}{x_{max} + \sqrt{|z| + x^2_{max}}}\right) \\
&\le \exp\{\beta'\,\Phi_{|z|}(x_{max})\} \le \exp\left\{\max_{y\in[0,1]}\beta'\,\Phi_{|z|}(y)\right\} \qquad (6.159a)
\end{aligned}$$

with the auxiliary function

$$\Phi_{|z|}(y) \equiv 4\left(y\sqrt{|z|+y^2}-1\right)+2|z|+2y^2-2y^2(2|z|+2y^2-1) + \frac{|z|^4}{16}\ln\frac{1+\sqrt{2}}{y+\sqrt{|z|+y^2}}\,. \tag{6.159b}$$

The maximum of $\Phi_{|z|}(y)$ for fixed $|z|$ is given by the zero y_{max} of

$$\frac{\partial}{\partial x}\Phi_{|z|}(y) = \frac{4}{\sqrt{|z|+y^2}}\left\{|z|+2y^2+2y\sqrt{|z|+y^2}\,(1-|z|-2y^2)-\frac{|z|^4}{64}\right\}.$$

Due to the term proportional to $|z|^4$, which is a consequence of the additional factor in (6.151), y_{max} comes out to be exactly $1-\frac{1}{4}|z|$. Explicitly, one has

$$\begin{aligned}\Phi_{|z|}(y_{max}) &= 4(1-\tfrac{1}{4}|z|)(1+\tfrac{1}{4}|z|)-4+2(1+\tfrac{1}{4}|z|)^2-4(1+\tfrac{1}{4}|z|)^2(1-\tfrac{1}{4}|z|)^2\\ &\quad+2(1-\tfrac{1}{4}|z|)^2+\frac{|z|^4}{16}\Big[\ln(1+\sqrt{2})-\ln\{(1-\tfrac{1}{4}|z|)+(1+\tfrac{1}{4}|z|)\}\Big]\\ &= \frac{1}{2}|z|^2-\frac{1}{64}|z|^4\Big[1-4\ln\frac{1+\sqrt{2}}{2}\Big] \le \frac{1}{2}|z|^2\,.\end{aligned} \tag{6.160}$$

The estimates (6.156), (6.157), (6.159), and (6.160) prove the assertion for the large field region $|z| \ge O(\beta'^{-1/2})$.

Next we study the remaining small field region $|z| \le O(\beta'^{-1/2})$. The replacement of β by β' in the exponent (6.158) yields a correction term proportional to $(\beta-\beta')$ which might become positive now. However, it is easy to show that this correction will be only of order β^{-1}. Therefore it is possible to split off its exponential,

$$\begin{aligned}\exp\Big(4(\beta-\beta')\big[x_{max}\sqrt{|z|+x_{max}^2}-1\big]\Big) &= \exp\Big(4(\beta-\beta')\Big[\Big(1-\frac{1}{16}|z|^2+O(\beta'^{-1})\Big)-1\Big]\Big)\\ &= \exp O(\beta'^{-1}) = [1+O(\beta'^{-1})]\,,\end{aligned}$$

and to absorb it into the $[1+O(\beta'^{-1})]$ factor of (6.156). Thus the lemma is proven for all $|z|\in[0,1]$.]

Next, one needs an estimate on the $\mathcal{R}_{j-1}$ term.

Lemma 6.11. *Let β_{j-1} be large enough and $|z|\in[\tilde\kappa\beta_{j-1}^{-1/2},1]$ for some fixed $\tilde\kappa<1$ with $\tilde\kappa^{-1}$ of order 1. Then*

$$|z^{-2}\mathcal{R}_{j-1}(z)| \le \frac{1}{2}\beta_{j-1}\,\frac{1+\exp[\frac{1}{2}\beta_{j-1}|z|^2]}{2}\,[1+\text{const}\,\beta_{j-1}^{-1/2}] \tag{6.161}$$

with a constant of order 1 independent of j.

PROOF. [Write again $\beta_j=\beta$ and $\beta_{j-1}=\beta'$. The proof started with a small field analysis. Suppose that $|z|$ is of order $\beta_{j-1}^{-1/2}$. Then, by (6.111) and (6.56),

$$|z^{-2}\mathcal{R}_{j-1}(z)| = \frac{1}{|z|^2}\left|[1+O(\beta'^{-1})]\left(\frac{I_1(2\beta\sqrt{1-z})}{\sqrt{1-z}\,I_1(2\beta)}\right)^2\exp(2\beta' z)-1\right|\,. \tag{6.162}$$

Now use

$$\begin{aligned}\left(\frac{I_1(2\beta\sqrt{1-z})}{\sqrt{1-z}\,I_1(2\beta)}\right)^2 &= [1+O(\beta'^{-1/2})]\exp(4\beta\sqrt{1-z}-4\beta)\\ &= [1+O(\beta'^{-1/2})]\exp\Big(4\beta\Big(-\frac{1}{2}z-\frac{1}{8}z^2\Big)+O(\beta'^{-1/2})\Big)\end{aligned}$$

to get

$$\begin{aligned}|z^{-2}\mathcal{R}_{j-1}(z)| &= \frac{1}{|z|^2}\left|[1+O(\beta'^{-1/2})]\exp\Big\{-\frac{1}{2}\beta' z^2+(\beta'-\beta)\Big[2z+\frac{1}{2}z^2\Big]\Big\}-1\right|\\ &= \frac{1}{|z|^2}\left|[1+O(\beta'^{-1/2})]\exp\Big(-\frac{1}{2}\beta' z^2\Big)-1\right|\\ &= \frac{1}{|z|^2}\left|[1+O(\beta'^{-1/2})]\Big(\exp\Big(-\frac{1}{2}\beta' z^2\Big)-1\Big)+O(\beta'^{-1/2})\right|\\ &\le O(\beta'^{+1/2})+[1+O(\beta'^{-1/2})]\left|\frac{\exp[-\frac{1}{2}\beta' z^2]-1}{z^2}\right|\,.\end{aligned} \tag{6.163}$$

Furthermore, using the fundamental theorem of calculus, one has

$$\begin{aligned}\left|\frac{\exp[-\frac{1}{2}\beta' z^2]-1}{z^2}\right| &= \left|-\frac{1}{2}\beta'\int_0^1 dt\exp\Big(-\frac{1}{2}\beta' z^2 t\Big)\right| \le \frac{1}{2}\beta'\int_0^1 dt\exp\Big(-\frac{1}{2}\beta'\,\mathrm{Re}(z^2)\,t\Big)\\ &\le \frac{1}{2}\beta'\int_0^1 dt\exp\Big(+\frac{1}{2}\beta'|z|^2 t\Big) = \frac{1}{2}\beta'\,\frac{\exp[\frac{1}{2}\beta'|z|^2]-1}{\frac{1}{2}\beta'|z|^2}\,.\end{aligned} \tag{6.164}$$

The inequalities (6.163) and (6.164) yield

$$|z^{-2}\mathcal{R}_{j-1}(z)| \le [1+O(\beta'^{-1/2})]\,\frac{1}{2}\beta'\,\frac{\exp[\frac{1}{2}\beta'|z|^2]-1}{\frac{1}{2}\beta'|z|^2} \qquad \text{for} \quad |z| = O(\beta'^{-1/2})\,, \tag{6.165}$$

where use has been made of

$$O(\beta'^{+1/2}) = O(\beta'^{-1/2})\,\frac{1}{2}\beta' \le O(\beta'^{-1/2})\,\frac{1}{2}\beta'\,\underbrace{\frac{\exp[\frac{1}{2}\beta'|z|^2]-1}{\frac{1}{2}\beta'|z|^2}}_{\ge 1}\,.$$

The following inequality completes the proof in the small field region

$$\begin{aligned}\frac{\exp[\frac{1}{2}\beta'|z|^2]-1}{\frac{1}{2}\beta'|z|^2} &= \exp\Big[\frac{1}{4}\beta'|z|^2\Big]\,\frac{\sinh(\frac{1}{4}\beta'|z|^2)}{\frac{1}{4}\beta'|z|^2}\\ &\le \exp\Big[\frac{1}{4}\beta'|z|^2\Big]\cosh\Big(\frac{1}{4}\beta'|z|^2\Big) = \frac{\exp[\frac{1}{2}\beta'|z|^2]+1}{2}\,.\end{aligned} \tag{6.166}$$

The remaining part of the proof is easily done with the help of Lemma 6.10. Equation (6.80) yields

$$|z^{-2}\mathcal{R}_{j-1}(z)| = \frac{1}{|z|^2}\left|\Big(\mathcal{N}_j^{-1}[G_{W}(\beta_j)*G_{W}(\beta_j)](z)\Big)^2 G_{W}(\beta_{j-1},z)^{-1}-1\right|$$

which can simply be bounded by use of the triangle inequality and (6.150)

$$|z^{-2}\mathcal{R}_{j-1}(z)| \le \frac{1}{|z|^2}\left\{\left(\mathcal{N}_j^{-1}\|G_W(\beta_j)\| * |G_W(\beta_j)|\|(z)\right)^2 |G_W(\beta_{j-1},z)|^{-1} + 1\right\}$$
$$\le \frac{1}{|z|^2}\left(1 + [1 + O(\beta_{j-1}^{-1})]\left(1 - \frac{1}{16}|z|^2\right)^{-3} \exp\left(\frac{1}{2}\beta_{j-1}|z|^2\right)\right) . \qquad (6.167)$$

By $(1 - \frac{1}{16}|z|^2)^{-3} \le (16/15)^3$ for $|z| \le 1$, it follows that

$$|z^{-2}\mathcal{R}_{j-1}(z)| \le 2\left(\frac{16}{15}\right)^3 \frac{1}{|z|^2}[1 + O(\beta'^{-1/2})]\frac{1 + \exp[\frac{1}{2}\beta'|z|^2]}{2} , \qquad (6.168)$$

which represents the assertion for $|z| \ge 2(16/15)^{3/2}\beta'^{-1/2}$. ∎

Returning to (6.149), one gets for $|z| \ge \bar\kappa\beta_{j-1}^{-1/2}$ by inserting Lemma 6.10 and Lemma 6.11 the estimate

$$|z^{-2}R_{j-1}(z)| \le \frac{1}{2}\beta_{j-1}[1 + O(\beta_{j-1}^{-1/2})]\frac{1 + \exp[\frac{1}{2}\beta_{j-1}|z|^2]}{2}$$
$$+ \frac{1}{4}C\beta_j[1 + O(\beta_{j-1}^{-1})]\left(1 - \frac{1}{16}|z|^2\right)^{-3}\exp\left(\frac{1}{2}\beta_{j-1}|z|^2 + 4\sigma\beta_j f(\frac{1}{2}\eta(z))\right)$$
$$\times \left\{\sum_{d=0}^{3}\frac{1}{4}\binom{4}{d+1}\left(\frac{C\beta_j|z|^2}{16}\right)^d\right.$$
$$\left. + \sum_{d=1}^{4} 4(C\beta_j)^{d-1}\sum_{\substack{k_1,k_2=0,1,2\\k_1+k_2=d}}\sum_{\substack{l_1,l_2=1,2\\l_1+l_2\ge 3}}\bar\kappa^{-2}\beta_{j-1}\prod_{i=1,2}\left[\binom{2}{k_i}\max_{\zeta:|\zeta|\le|z|}\mathcal{E}_{k_il_i}(\beta_j,\zeta)\right]\right\} . \qquad (6.169)$$

This is further simplified to

$$|z^{-2}R_{j-1}(z)| \le B_1^{(j-1)}(|z|) \qquad \text{for} \qquad |z| \ge \bar\kappa\beta_{j-1}^{-1/2} \qquad (6.170)$$

with

$$B_1^{(j-1)}(|z|) \equiv \beta_{j-1}[1 + O(\beta_{j-1}^{-1/2})]\exp\left(\frac{1}{2}\beta_{j-1}|z|^2\right)$$
$$\times \left\{\frac{1}{2} + \left[\frac{1}{4}\left(\frac{16}{15}\right)^3 C + b_1^{(j-1)}(|z|)\right]\exp\left(4\sigma\beta_j\frac{(\sqrt{1+|z|}-1)^2}{\sqrt{1+|z|}}\right)\right\} \qquad (6.171a)$$

and

$$b_1^{(j-1)}(|z|) \equiv \frac{1}{4}\left(\frac{16}{15}\right)^3 C\left[\sum_{d=1}^{3}\frac{1}{4}\binom{4}{d+1}\left(\frac{C\beta_j|z|^2}{16}\right)^d\right.$$
$$\left. + \sum_{d=1}^{4}\frac{4\beta_{j-1}(C\beta_j)^{d-1}}{\bar\kappa^2}\sum_{\substack{k_1,k_2=0,1,2\\k_1+k_2=d}}\sum_{\substack{l_1,l_2=1,2\\l_1+l_2\ge 3}}\prod_{i=1,2}\left[\binom{2}{k_i}\max_{\zeta:|\zeta|\le|z|}\mathcal{E}_{k_il_i}(\beta_j,\zeta)\right]\right] , \qquad (6.171b)$$

where use has been made of

$$f(\frac{1}{2}\eta(z)) \le f(\frac{1}{2}\max_{\zeta:|\zeta|=|z|}\eta(\zeta)) = f(\frac{1}{2}\operatorname{arccosh}(1 + 2|z|)) = \frac{(\sqrt{1+|z|}-1)^2}{\sqrt{1+|z|}} . \qquad (6.172)$$

Now the bound (6.170), which is valid only on $|z| \in [\bar\kappa\beta_{j-1}^{-1/2}, 1]$, must be continued to the whole interval $[0,1]$. This can be achieved again by means of the analyticity of $z^{-2}R_{j-1}(z)$, Lemma 6.2, and the maximum principle. Note that $\max_{\zeta:|\zeta|=|z|}|\zeta^{-2}R_{j-1}(\zeta)|$ is a monotonously increasing function in $|z|$. The same holds true for the bound $B_1^{(j-1)}(|z|)$. By writing

$$B_1^{(j-1)}(|z|) \le B_1^{(j-1)}(|z|) + \left\{\underbrace{B_1^{(j-1)}(\bar\kappa\beta_{j-1}^{-1/2}) - B_1^{(j-1)}(0)}_{>0}\right\}$$
$$= B_1^{(j-1)}(\bar\kappa\beta_{j-1}^{-1/2}) + \left\{\underbrace{B_1^{(j-1)}(|z|) - B_1^{(j-1)}(0)}_{\ge 0}\right\} \equiv B_2^{(j-1)}(|z|) , \qquad (6.173)$$

it is clear that the new bound $B_2^{(j-1)}(|z|)$ is indeed valid on $[0,1]$,

$$|z^{-2}R_{j-1}(z)| \le B_2^{(j-1)}(|z|) \qquad \text{for} \qquad z \in [0,1] . \qquad (6.174)$$

First calculate the z-independent part $B_1^{(j-1)}(\bar\kappa\beta_{j-1}^{-1/2})$. Use

$$\left.\frac{(\sqrt{1+|z|}-1)^2}{\sqrt{1+|z|}}\right|_{|z|=\bar\kappa\beta_{j-1}^{-1/2}} = \frac{1}{4}\bar\kappa^2\beta_{j-1}^{-1} + O(\beta_{j-1}^{-3/2})$$

and

$$\sum_{\substack{k_1,k_2=0,1,2\\k_1+k_2=d}}\sum_{\substack{l_1,l_2=1,2\\l_1+l_2\ge 3}}\prod_{i=1,2}\left[\binom{2}{k_i}\max_{\zeta:|\zeta|\le\bar\kappa\beta_{j-1}^{-1/2}}\mathcal{E}_{k_il_i}(\beta_j,\zeta)\right]$$
$$\le \sum_{\substack{k_1,k_2=0,1,2\\k_1+k_2=d}}\sum_{\substack{l_1,l_2=1,2\\l_1+l_2\ge 3}} O(\beta_{j-1}^{-(k_1+k_2)-(l_1+l_2)/2+1}) \le O(\beta_{j-1}^{-d-1/2}) ,$$

to get the result

$$B_1^{(j-1)}(\bar\kappa\beta_{j-1}^{-1/2}) = \beta_{j-1}[1 + O(\beta_{j-1}^{-1/2})]\exp\left(\frac{1}{2}\bar\kappa^2\right)$$
$$\times\left\{\frac{1}{2} + \left[\frac{1}{4}\left(\frac{16}{15}\right)^3 C + b_1^{(j-1)}(\bar\kappa\beta_{j-1}^{-1/2})\right]\exp\left(\sigma\bar\kappa^2 + O(\beta_{j-1}^{-1/2})\right)\right\} \qquad (6.175a)$$

with

$$b_1^{(j-1)}(\bar\kappa\beta_{j-1}^{-1/2}) = \frac{1}{4}\left(\frac{16}{15}\right)^3 C\left[\sum_{d=1}^{3}\frac{1}{4}\binom{4}{d+1}\left(\frac{C\bar\kappa^2}{16}\right)^d + O(\beta_{j-1}^{-1/2})\right] . \qquad (6.175b)$$

And similarly

$$B_1^{(j-1)}(0) = \beta_{j-1}\,[1+O(\beta_{j-1}^{-1/2})]\left\{\frac{1}{2}+\frac{1}{4}\left(\frac{16}{15}\right)^3 C + b_1^{(j-1)}(0)\right\} \tag{6.176}$$

with $b_1^{(j-1)}(0)$ of order β_{j-1}^{-1}. Now consider

$$\begin{aligned} B_1^{(j-1)}(|z|) - B_1^{(j-1)}(0) \le \beta_{j-1}\,[1+O(\beta_{j-1}^{-1/2})] &\left\{\frac{1}{2}\left[\exp\left(\frac{1}{2}\beta_{j-1}|z|^2\right)-1\right]\right. \\ &+ \frac{1}{4}\left(\frac{16}{15}\right)^3 C\left[\exp\left(\frac{1}{2}\beta_{j-1}|z|^2 + 4\sigma\beta_j\frac{(\sqrt{1+|z|}-1)^2}{\sqrt{1+|z|}}\right)-1\right] \\ &\left.+ b_1^{(j-1)}(|z|)\exp\left(\frac{1}{2}\beta_{j-1}|z|^2 + 4\sigma\beta_j\frac{(\sqrt{1+|z|}-1)^2}{\sqrt{1+|z|}}\right)\right\}. \end{aligned} \tag{6.177}$$

In order to exhibit the small field behaviour $\propto |z|^2$ of this bound, one uses the fundamental theorem of calculus together with the inequality $\sqrt{1+|z|} \le 1+\frac{1}{2}|z|$, e.g.

$$\begin{aligned} &\exp\left(\frac{1}{2}\beta_{j-1}|z|^2 + 4\sigma\beta_j\frac{(\sqrt{1+|z|}-1)^2}{\sqrt{1+|z|}}\right)-1 \\ &\le \left(\frac{1}{2}\beta_{j-1}|z|^2 + 4\sigma\beta_j\frac{(\sqrt{1+|z|}-1)^2}{\sqrt{1+|z|}}\right)\exp\left(\frac{1}{2}\beta_{j-1}|z|^2 + 4\sigma\beta_j\frac{(\sqrt{1+|z|}-1)^2}{\sqrt{1+|z|}}\right) \\ &\le |z|^2\left(\frac{1}{2}\beta_{j-1}+\sigma\beta_j\right)\exp\left(\frac{1}{2}\beta_{j-1}|z|^2 + 4\sigma\beta_j\frac{(\sqrt{1+|z|}-1)^2}{\sqrt{1+|z|}}\right). \end{aligned} \tag{6.178}$$

The result of this section is summarized in

Lemma 6.12 (Unit disc bound) *Let the assumptions of Proposition 6.3 be fulfilled for some fixed $j \le N$. Then the following estimate for the modulus $|R_{j-1}(z)|$ of the reduced activity R_{j-1} holds on the closed unit disc $|z| \le 1$*

$$\begin{aligned} |R_{j-1}(z)| &\le |z|^2 B_2^{(j-1)}(|z|) \\ &= |z|^2 B_1^{(j-1)}(\bar\kappa\beta_{j-1}^{-1/2}) + |z|^2\left[B_1^{(j-1)}(|z|) - B_1^{(j-1)}(0)\right], \end{aligned} \tag{6.179}$$

where the functions $B_1^{(j-1)}$, $B_2^{(j-1)}$ are defined in Eqs. (6.171) *and* (6.173). *The parameters σ and C have to be set $\sigma = 5/2$ and $C = 1$. Let $\bar\kappa$ be 1/2. Then, by* (6.175), *one has the estimate*

$$B_1^{(j-1)}(\bar\kappa\beta_{j-1}^{-1/2}) = B_1^{(j-1)}(\tfrac{1}{2}\beta_{j-1}^{-1/2}) \le \frac{5}{4}\beta_{j-1} + O(\beta_{j-1}^{+1/2}). \tag{6.180}$$

In addition, (6.177) *and* (6.178) *yield*

$$\begin{aligned} B_1^{(j-1)}(|z|) - B_1^{(j-1)}(0) &\le [1+O(\beta_{j-1}^{-1/2})]\,\beta_{j-1}^2|z|^2\exp\left(\frac{1}{2}\beta_{j-1}|z|^2\right) \\ &\times\left\{\frac{1}{4}+\left[\frac{1}{4}\left(\frac{16}{15}\right)^3 C\left(\frac{1}{2}+\sigma\frac{\beta_j}{\beta_{j-1}}\right)+b_1^{(j-1)}(|z|)\right]\exp\left(4\sigma\beta_j\frac{(\sqrt{1+|z|}-1)^2}{\sqrt{1+|z|}}\right)\right\} \end{aligned} \tag{6.181}$$

with $b_1^{(j-1)}(|z|)$ given by (6.171b).

6.5. NORM ESTIMATES

In this section, the unit disc bounds, given by Lemma 6.12, are used to show that the norm estimates (6.45a), (6.45b) of Proposition 6.3 are valid.

Consider the norm $\|(1+R_{j-1})^2-1\|_{1,\beta_{j-1}}$ and write

$$\|(1+R_{j-1})^2-1\|_{1,\beta_{j-1}} = \|2R_{j-1}+R_{j-1}^2\|_{1,\beta_{j-1}} \le 2\,\|R_{j-1}\|_{1,\beta_{j-1}} + \|R_{j-1}^2\|_{1,\beta_{j-1}}. \tag{6.182}$$

First, one estimates the leading term $\|R_{j-1}\|_{1,\beta_{j-1}}$ which will be of order β_{j-1}^{-1}.

$$\begin{aligned} \|R_{j-1}\|_{1,\beta_{j-1}} &= \max_{\tau:|\tau|=1}\int d\mu_{\beta_{j-1}}(s)\,|R_{j-1}(s\tau)| \\ &\le \max_{\tau:|\tau|=1}\int d\mu_{\beta_{j-1}}(s)\,(s|\tau|)^2 B_2^{(j-1)}(s|\tau|) = \int d\mu_{\beta_{j-1}}(s)\,s^2 B_2^{(j-1)}(s) \\ &= B_1^{(j-1)}(\tfrac{1}{2}\beta_{j-1}^{-1/2})\int d\mu_{\beta_{j-1}}(s)\,s^2 + \int d\mu_{\beta_{j-1}}(s)\,s^2\left[B_1^{(j-1)}(s) - B_1^{(j-1)}(0)\right] \end{aligned} \tag{6.184}$$

according to (5.50) and (6.179). Using

$$\int_0^1 d\mu_\beta(s)\,s^2 = \frac{1}{2}\left(1-\frac{I_2(2\beta)}{2\beta\,I_1(2\beta)}\left[2\beta+\frac{3}{2}\right]\right) = \frac{15}{64}\beta^{-2}+O(\beta^{-3}) \tag{6.184}$$

and the estimate (6.180) for the constant $B_1^{(j-1)}(\beta_{j-1}^{-1/2}/2)$, one gets for the first term on the right-hand side of (6.183)

$$B_1^{(j-1)}(\tfrac{1}{2}\beta_{j-1}^{-1/2})\int d\mu_{\beta_{j-1}}(s)\,s^2 \le \frac{75}{256}\beta_{j-1}^{-1}+O(\beta_{j-1}^{-3/2}). \tag{6.185}$$

The second term yields due to (6.181) and (6.171b)

$$\begin{aligned} \int d\mu_{\beta_{j-1}}(s)\,s^2\left[B_1^{(j-1)}(s)-B_1^{(j-1)}(0)\right] &= [1+O(\beta_{j-1}^{-1/2})]\,\frac{1}{4}\beta_{j-1}^2 \\ \times\Bigg\{\left(\frac{16}{15}\right)^3 C\left(\frac{1}{2}+\sigma\frac{\beta_j}{\beta_{j-1}}\right)&\int d\mu_{\beta_{j-1}}(s)\,s^4\exp\left(\frac{1}{2}\beta_{j-1}s^2+4\sigma\beta_j\frac{(\sqrt{1+s}-1)^2}{\sqrt{1+s}}\right) \\ +\int d\mu_{\beta_{j-1}}(s)\,s^4\exp\left(\frac{1}{2}\beta_{j-1}s^2\right) &+ \left(\frac{16}{15}\right)^3 C\sum_{d=1}^{3}\frac{1}{4}\binom{4}{d+1}\left(\frac{C\beta_j}{16}\right)^d \\ &\times\int d\mu_{\beta_{j-1}}(s)\,s^{4+2d}\exp\left(\frac{1}{2}\beta_{j-1}s^2+4\sigma\beta_j\frac{(\sqrt{1+s}-1)^2}{\sqrt{1+s}}\right) \\ +\left(\frac{16}{15}\right)^3 C\sum_{d=1}^{4}16\beta_{j-1}(C\beta_j)^{d-1} &\sum_{\substack{k_1,k_2=0,1,2\\k_1+k_2=d}}\ \sum_{\substack{l_1,l_2=1,2\\l_1+l_2\ge3}}\binom{2}{k_1}\binom{2}{k_2} \\ \times\int d\mu_{\beta_{j-1}}(s)\,s^4\exp\left(\frac{1}{2}\beta_{j-1}s^2+4\sigma\beta_j\frac{(\sqrt{1+s}-1)^2}{\sqrt{1+s}}\right)&\prod_{i=1,2}\max_{\zeta:|\zeta|\le s}\mathcal{E}_{k_il_i}(\beta_{j-1},\zeta)\Bigg\}. \end{aligned} \tag{6.186}$$

To proceed further, it is necessary to gain exponential damping factors from the $d\mu_\beta(s)$-measures. This can be achieved by use of the formula

$$d\mu_\beta(s) = \left[\frac{I_1(2t\beta)}{t\,I_1(2\beta)}\exp\big(2\beta(1-t)\big)\right] d\mu_{t\beta}(s)\, \exp\big(-4(1-t)\beta s\big) \tag{6.187}$$

for $t \in [0,1]$ which follows from the definition (5.48) of $d\mu_\beta(s)$ and from

$$\frac{\int_0^1 ds\,\sqrt{s(1-s)}\exp(-4t\beta s)}{\int_0^1 ds\,\sqrt{s(1-s)}\exp(-4\beta s)} = \frac{I_1(2t\beta)}{t\,I_1(2\beta)}\exp\big(2\beta(1-t)\big)\,. \tag{6.188}$$

For t of order unity and large β, one obtains

$$d\mu_\beta(s) = [1+O(\beta^{-1})]\,t^{-3/2}\,d\mu_{t\beta}(s)\,\exp\big(-4(1-t)\beta s\big)\,. \tag{6.189}$$

Thus one gets, for example,

$$\int_0^1 d\mu_{\beta_{j-1}}(s)\,s^4\exp\Big(\frac{1}{2}\beta_{j-1}s^2\Big) \le \int_0^1 d\mu_{\beta_{j-1}}(s)\,s^4\exp\Big(\frac{1}{2}\beta_{j-1}s\Big)$$
$$= [1+O(\beta_{j-1}^{-1})]\left(\frac{7}{8}\right)^{-3/2}\int_0^1 d\mu_{\beta_{j-1}7/8}(s)\,s^4 = O(\beta_{j-1}^{-4})\,. \tag{6.190}$$

Now use

$$\frac{(\sqrt{1+s}-1)^2}{\sqrt{1+s}} \le \frac{(\sqrt{2}-1)^2}{\sqrt{2}}\,s \qquad \text{for} \qquad s\in[0,1] \tag{6.191}$$

and $\sigma = 5/2$ to get

$$\exp\left(\frac{1}{2}\beta_{j-1}s^2 + 4\sigma\beta_j\frac{(\sqrt{1+s}-1)^2}{\sqrt{1+s}}\right) \le \exp\left(\left[\frac{1}{2}+10\frac{(\sqrt{2}-1)^2}{\sqrt{2}}[1+O(\beta_{j-1}^{-1})]\right]\beta_{j-1}s\right)$$
$$< \exp(2\beta_{j-1}s)\,. \tag{6.192}$$

The right-hand side of (6.192) is equal to $\exp(4(1-t)\beta_{j-1}s)$ for $t=1/2$. Therefore

$$\int d\mu_{\beta_{j-1}}(s)\,s^2\left[B_1^{(j-1)}(s)-B_1^{(j-1)}(0)\right] \le [1+O(\beta_{j-1}^{-1/2})]\,\frac{1}{4}\beta_{j-1}^2$$
$$\times\left\{\left(\frac{8}{7}\right)^{3/2}\int d\mu_{\beta_{j-1}7/8}(s)\,s^4 + \left(\frac{16}{15}\sqrt{2}\right)^3 C\left(\frac{1}{2}+\sigma\frac{\beta_j}{\beta_{j-1}}\right)\int d\mu_{\beta_{j-1}/2}(s)\,s^4\right.$$
$$+\left(\frac{16}{15}\sqrt{2}\right)^3 C\sum_{d=1}^{3}\frac{1}{4}\binom{4}{d+1}\left(\frac{C\beta_j}{16}\right)^d\int d\mu_{\beta_{j-1}/2}(s)\,s^{4+2d}$$
$$+\left(\frac{16}{15}\sqrt{2}\right)^3 C\sum_{d=1}^{4}16\beta_{j-1}(C\beta_j)^{d-1}\sum_{\substack{k_1,k_2=0,1,2\\k_1+k_2=d}}\;\sum_{\substack{l_1,l_2=1,2\\l_1+l_2\ge 3}}\binom{2}{k_1}\binom{2}{k_2}$$
$$\left.\times\int d\mu_{\beta_{j-1}/2}(s)\,s^4\left[\prod_{i=1,2}\max_{\zeta:|\zeta|\le s}\mathcal{E}_{k_il_i}(\beta_{j-1},\zeta)\right]\right\}. \tag{6.193}$$

By expanding the expectation values $\mathcal{E}_{kl}$ like in Sect. 6.3, one gets ($k_1+k_2=d$, $3\le l_1+l_2\le 4$)

$$\int_0^1 d\mu_{\beta_{j-1}/2}(s)\,s^4\left[\prod_{i=1,2}\max_{\zeta:|\zeta|\le s}\mathcal{E}_{k_il_i}(\beta_{j-1},\zeta)\right] = O(\beta_{j-1}^{-4-2(k_1+k_2)+\sum_i(k_i-1)(l_i-1)})$$
$$\le O(\beta_{j-1}^{-2-d-(l_1+l_2)}) \le O(\beta_{j-1}^{-5-d}) \tag{6.194}$$

Here it has been used that the measure $d\mu_{\beta_{j-1}/2}(s)$ is concentrated on the region $|s|\le O(\beta_{j-1}^{-1})$, where the expectation values $\mathcal{E}_{kl}$ fulfil estimates $\mathcal{E}_{1l}=O(\beta_{j-1}^{-2})$, $\mathcal{E}_{21}=O(\beta_{j-1}^{-4})$ and $\mathcal{E}_{22}=O(\beta_{j-1}^{-3})$. Thus one gets

$$\int d\mu_{\beta_{j-1}}(s)\,s^2\left[B_1^{(j-1)}(s)-B_1^{(j-1)}(0)\right] = O(\beta_{j-1}^{-2}) \tag{6.195}$$

and, finally,

$$\|R_{j-1}\|_{1,\beta_{j-1}} \le \frac{75}{256}\beta_{j-1}^{-1} + O(\beta_{j-1}^{-3/2})\,. \tag{6.196}$$

Now consider the norm $\|R_{j-1}^2\|_{1,\beta_{j-1}}$, cp. (6.182). Its leading contribution is given by

$$\int_0^1 d\mu_{\beta_{j-1}}(s)\left[B_1^{(j-1)}(\tfrac{1}{2}\beta_{j-1}^{-1/2})\,s^2\right]^2 \le \left[\frac{5}{4}\beta_{j-1}+O(\beta_{j-1}^{1/2})\right]^2\int_0^1 d\mu_{\beta_{j-1}}(s)\,s^4$$
$$= \frac{25}{16}\,\frac{3\cdot5\cdot7\cdot9}{8^4}\,\beta_{j-1}^{-2} + O(\beta_{j-1}^{-5/2}) < \frac{1}{e}\beta_{j-1}^{-2} + O(\beta_{j-1}^{-5/2})\,. \tag{6.197}$$

Although the other terms are of smaller order, it is sufficient to bound them all by const $\beta_{j-1}^{-3/2}$ with some constant independent of j. The explicit estimation of $\|R_{j-1}^2\|_{1,\beta_{j-1}}$ is straightforward and therefore omitted. However, it is crucial that the damping procedure of the in s exponentially increasing bounds can still be carried out. This is the case because

$$\exp\left(\beta_{j-1}s^2 + 8\sigma\beta_j\frac{(\sqrt{1+s}-1)^2}{\sqrt{1+s}}\right)\exp\big(-4(1-t)\beta_{j-1}s\big) \le 1 \qquad \forall s\in[0,1] \tag{6.198}$$

for $\sigma=5/2$, $t=1/8$ and sufficiently large β_{j-1}. Note that the inequality enforces small values of σ since it must hold for a small parameter $t\in(0,1)$ with t^{-1} of order unity.

In conclusion, one has for large enough values of the running coupling constant β_{j-1}

$$\|(1+R_{j-1})^2-1\|_{1,\beta_{j-1}} \le \frac{75}{128}\,\beta_{j-1}^{-1} + O(\beta_{j-1}^{-3/2})\,, \tag{6.199}$$

cp. (6.182) and (6.196). Analogously, one obtains

$$\|R_{j-1}\|_{2,\beta_{j-1}} \le \frac{15\sqrt{105}}{256}\,\beta_{j-1}^{-1} + O(\beta_{j-1}^{-3/2})\,, \tag{6.200}$$

where the leading term results from the square root of (6.197). Due to Proposition 5.3, the estimates (6.199) and (6.200), respectively, prove the convergence of the irrelevant perturbation series (4.12) for the running coupling constants.

6.6. CONVERGENCE NEAR THE STABLE FIXED POINT

The convergence of the norms $\|R_j\|_{2,\beta_j} \to 0$ for $j \to -\infty$ is proven (Proposition 4.4). This establishes the convergence of the irrelevant perturbation series for the running coupling constants, Eqs. (4.12), near the stable fixed point.

In order to derive (locally) uniform bounds for the effective Boltzmannians $g_j(\theta)$, $j \le N$, with $g_N(\theta) = g_W(\beta_N, \theta)$, one needs

Lemma 6.13. *Let $g_N^c(u)$ be given by the Wilson monomer activity $g_W^c(\beta_N, u)$ with a positive bare coupling constant β_N. Then*

$$\left|g_j^c(e^{i\theta\sigma_3}u)\right| \le g_j^c(e^{i\omega\sigma_3}u)\exp\left(2\beta_N \sinh^2\frac{\eta}{2}\right) \tag{6.201}$$

for all $j \le N$, $u \in SU(2)$, and for arbitrary complex $\theta = \omega + i\eta$.

PROOF. ⌈ (Induction on $j \le N$.) First, consider the case $j = N$. As in the proof of Proposition 2.2, one gets

$$\begin{aligned}\left|g_N^c(e^{i\theta\sigma_3}u)\right| = \left|g_W^c(\beta_N, e^{i\theta\sigma_3}u)\right| &= \exp\left[-\frac{1}{2}\beta_N \operatorname{Re}\operatorname{tr}(1 - e^{i\theta\sigma_3}u)\right] \\ &= \exp\left[-\frac{1}{2}\beta_N \operatorname{Re}(2 - \cos\theta \operatorname{tr} u - i\sin\theta \operatorname{tr}(\sigma_3 u))\right] \\ &= \exp\left[-\beta_N \operatorname{Re}(1 - a_0\cos\theta + a_3\sin\theta)\right] \\ &= \exp\left[-\beta_N (1 - a_0\cos\omega\cosh\eta + a_3\sin\omega\cosh\eta)\right]\end{aligned}$$

where use has been made of the parametrization $u = a_0 1 + i\sum_{k=1}^3 a_k\sigma_k$ with $\sum_{\mu=0}^3 a_\mu^2 = 1$, cp. Sect. 3. Now one splits

$$\begin{aligned}\left|g_N^c(e^{i\theta\sigma_3}u)\right| = &\exp\left[-\beta_N(1 - a_0\cos\omega + a_3\sin\omega)\right] \\ &\times \exp\left[-\beta_N(-a_0\cos\omega + a_3\sin\omega)(\cosh\eta - 1)\right] .\end{aligned}$$

The term $(-a_0\cos\omega + a_3\sin\omega)$ is bounded from below by -1, cp. again the proof of Proposition 2.2. Thus one gets the estimate

$$\begin{aligned}\left|g_N^c(e^{i\theta\sigma_3}u)\right| &\le \exp\left[-\frac{1}{2}\beta_N \operatorname{tr}(1 - e^{i\omega\sigma_3}u)\right]\exp\left[+\beta_N(\cosh\eta - 1)\right] \\ &= g_N^c(e^{i\omega\sigma_3}u)\exp\left[2\beta_N\sinh^2\frac{\eta}{2}\right] .\end{aligned}$$

Next, consider the inductive step from j to $j-1$. Assume that the assertion (6.201) holds for g_j^c. By Eq. (2.27), one has

$$g_{j-1}^c(e^{i\theta\sigma_3}u) = \left[\mathcal{N}_j^{-1}\int_G dv\, g_j^c(e^{i(\theta/2)\sigma_3}uv^{-1})\, g_j^c(e^{i(\theta/2)\sigma_3}v)\right]^2$$

where the normalization ensures $g_{j-1}^c(1) = 1$. Estimate

$$\begin{aligned}\left|g_{j-1}^c(e^{i\theta\sigma_3}u)\right| &\le \left[\mathcal{N}_j^{-1}\int_G dv\, \left|g_j^c(e^{i(\theta/2)\sigma_3}uv^{-1})\right|\,\left|g_j^c(e^{i(\theta/2)\sigma_3}v)\right|\right]^2 \\ &\le \left[\mathcal{N}_j^{-1}\int_G dv\, g_j^c(e^{i(\omega/2)\sigma_3}uv^{-1})\, g_j^c(e^{i(\omega/2)\sigma_3}v)\right]^2 \exp\left[8\beta_N\sinh^2\frac{\eta}{4}\right] \\ &= g_{j-1}^c(e^{i\omega\sigma_3}u)\exp\left[8\beta_N\sinh^2\frac{\eta}{4}\right] .\end{aligned}$$

Then

$$4\sinh^2\frac{\eta}{4} \le 4\sinh^2\frac{\eta}{4}\cosh^2\frac{\eta}{4} = \left[2\sinh\frac{\eta}{4}\cosh\frac{\eta}{4}\right]^2 = \sinh^2\frac{\eta}{2}$$

proves the assertion for $j-1$ and hence for all j by induction. ⌋

Setting $u = 1$ in (6.201), one obtains finally by Eq. (2.20)

$$|g_j(\theta)| \le \underbrace{g_j(\operatorname{Re}\theta)}_{\le 1}\exp\left[2\beta_N\sinh^2\frac{\operatorname{Im}\theta}{2}\right] . \tag{6.202}$$

The effective Boltzmannians $g_j(\theta)$ of the sequence $\{g_j\}_{j\le N}$ are entire functions of the complex variable θ (Proposition 2.1) and (locally) uniformly bounded by means of (6.202). Thus the set of all Boltzmannians g_j with $j \le N$ represents a *normal family* of entire functions. By Proposition 2.2, the sequence $\{g_j\}_{j\le N}$ converges (uniformly) for real variables θ. Vitali's theorem implies that the sequence $\{g_j\}$ converges (locally) uniformly to the limit function $g(\theta)$ which is entire. Because of Proposition 2.2, one has $g(\theta) = 1$ for all real θ and hence $g(\theta) \equiv 1$.

Since the running coupling constant β_j tends to zero in the limit $j \to -\infty$, this implies that the reduced activities $r_j(\theta)$ converge (locally) uniformly to the entire limit function $r(\theta) \equiv 0$. The same holds true for the "auxiliary" reduced activities $R_j(z)$. Hence

$$\|R_j\|_{2,0} \to 0 \qquad \text{for} \qquad j \to -\infty . \tag{6.203}$$

Using Eqs. (5.50) and (6.187),

$$\begin{aligned}\|R_j\|_{2,\beta_j} &\equiv \left(\max_{\tau:|\tau|=1}\int_0^1 d\mu_\beta(s)\,|R_j(\tau s)|^2\right)^{1/2} \\ &= \left(e^{2\beta_j}\left[\lim_{t\to 0}\frac{I_1(2\beta_j t)}{t\, I_1(2\beta_j)}\right]\max_{\tau:|\tau|=1}\int_0^1 d\mu_0(s)\, e^{-4\beta_j s}\,|R_j(\tau s)|^2\right)^{1/2} \\ &\le e^{\beta_j}\left(\max_{\tau:|\tau|=1}\int_0^1 d\mu_0(s)\,|R_j(\tau s)|^2\right)^{1/2} ,\end{aligned}$$

one obtains the estimate

$$\|R_j\|_{2,\beta_j} \le e^{\beta_j}\|R_j\|_{2,0} \le \exp(\|S_j^c\|)\,\|R_j\|_{2,0} \tag{6.204}$$

as a consequence of the bound $\beta_j \le \|S_j^c\|$, (2.63). The class functions $S_j^c(u)$ and their norms $\|S_j^c\|$ have been introduced in Eqs. (2.29) and (2.30). The norms $\|S_j^c\|$ converge to zero for $j \to -\infty$, see the result (2.33) in the proof of Proposition 2.2. Therefore one has

$$\|R_j\|_{2,\beta_j} \le \exp(\|S_j^c\|)\,\|R_j\|_{2,0} \underset{j\to-\infty}{\longrightarrow} 0 \tag{6.205}$$

which proves Proposition 4.4.

7. NUMERICAL STUDY OF THE RENORMALIZATION GROUP FLOW

To study the question of convergence of the irrelevant perturbation expansions in the recursive flow equations (4.12) for arbitrary values of the running coupling constant, the renormalization group flow $\{g_j\}_{j\leq N}$ is studied numerically, and the norms $\|(1+R_j)^2-1\|_{1,\beta_j}$ are computed. It is found that an approximation to g_j, in which all terms with degree of irrelevance $\delta > 4$ are neglected, remains extremely accurate for all steps and arbitrary couplings. It is also found that the sufficient criteria for convergence, Proposition 4.2 and Proposition 5.3, respectively, remains satisfied for all values of the running coupling constant.

Numerical Derivation of "Exact" Results

The Migdal recursion relation (5.13) can be solved by numerical integration. To do this, one uses the parametrizations (5.8) and (5.12) of the Haar measure and of the arguments $z_\pm(z)$, and approximates the effective Boltzmannians by

$$G_j(z) = \exp\Big(-\sum_{\mu=1}^{k} \gamma_\mu^{(j)} z^\mu\Big) \qquad \text{with} \qquad \gamma_1^{(j)} = 2\beta_j \tag{7.1}$$

for some (sufficiently large) k. I have used $k = 20$. The coefficients $\gamma_\mu^{(j)}$ can be calculated simultaneously in every renormalization group step by means of Cauchy's integral formula. Thus one needs three numerical integrations for a renormalization group step; two for the convolution integral and a third to determine the γ_μ coefficients. For the bare coupling constant $\beta_N = 10$, which will be used in the following, the contour of the Cauchy integration can be chosen as the unit circle $|z| = 1$, i.e.

$$\begin{aligned}\gamma_\mu^{(j-1)} &= -\frac{1}{\pi i}\oint_{|\xi|=1} \frac{d\xi}{\xi^{\mu+1}} \log \int dv\; G_j(z_+(\xi))\, G_j(z_-(\xi)) \\ &= -\frac{2}{\pi}\int_0^\pi d\omega\; \mathrm{Re}\Big\{ e^{-i\mu\omega} \log \int dv\, \exp\Big(-\sum_{\nu=1}^{k}\gamma_\nu^{(j)} \sum_{l=\pm} z_l(e^{i\omega})^\nu\Big)\Big\}\;,\end{aligned} \tag{7.2}$$

provided that the logarithm is analytically continued along the unit circle, cp. the discussion in Sect. 3. Use (5.8) for the Haar measure dv and (5.12) for the arguments $z_\pm$. In the above formula, use has been made of $G_j(\bar z) = \overline{G_j(z)}$, which is obviously true for $G_N(z) = G_W(\beta_N, z)$, and follows inductively for all $j < N$. The precision of the determination of the coefficients $\gamma_\mu^{(j)}$ is very good even for higher μ. I have compared the numerical results for $\gamma_\mu^{(N-1)}$ with the exact formulae (3.26) and (3.28). The relative errors are astonishingly small. For the stepsize that I have chosen for the numerical Cauchy integration, the relative error for $\gamma_1^{(N-1)}$ is smaller than 10^{-12} while the largest relative error, reached for $\gamma_k^{(N-1)} = \gamma_{20}^{(N-1)}$, is 10^{-2}. That the relative error grows with increasing μ is a consequence of the fact that all coefficients γ_μ, $\mu = 1, \ldots, k$, are calculated simultaneously, which makes the routine very fast. But it is not necessary to improve the accuracy of the higher coefficients because their influence decreases rapidly with μ. Besides, one should note that the approximation (7.1) is not able to represent zeroes of the effective activities. However, since (7.1) becomes exponentially small in such cases, the systematical error in evaluating the convolution integral is also exponentially small.

Numerical Investigation of the Finite Order Recursion Relations (4.24)

The "exact" renormalization group flow, obtained as discussed above, provides the reference data which are needed in order to test the irrelevant perturbation theory approximations. For this purpose, one uses the finite order recursion relations (4.24), which approximate the recursive flow equations (4.12) by taking into account only contributions up to some fixed degree of irrelevance δ.

Starting the iteration with a Wilson Boltzmannian $G_W(\beta_N)$, one may first calculate the flow of the running coupling constant β_j in lowest order approximation, e.g. for $\delta = 0$. The first iteration step $N \to N-1$ is of course treated correctly, whereas all further steps $j \to j-1$ with $j \leq N-1$ are affected by omitting the contributions from irrelevant interactions. The finite set of equations (4.24) is reduced for $\delta = 0$ to the single recursion relation

$$\beta_{j-1,0} = T_1(\beta_{j,0}|0) = \beta_{j,0}\,\frac{I_2(2\beta_{j,0})}{I_1(2\beta_{j,0})} \qquad \text{for} \qquad \rho_{n,0}^{(j)} \equiv 0\;, \tag{7.3}$$

which is indeed exact for the first step. But, for sufficiently large running coupling constant, the $\delta = 0$ approximation predicts the renormalization group flow

$$\beta_{j-1,0} = \beta_{j,0} - \frac{3}{4} + \frac{3}{32}\beta_{j,0} + O(\beta_{j,0}^{-2}) \qquad \text{for} \qquad \rho_{n,0}^{(j)} \equiv 0\;, \tag{7.4}$$

as can be seen by asymptotically expanding (7.3) according to (3.43), cp. also (6.71a). This result for $\delta = 0$ has to be compared with the exact flow (6.76), (6.77) along the renormalized trajectory.

It is instructive to consider also the $\delta = 1$ approximation in some detail. One obtains by (4.24)

$$\beta_{j-1,1} = T_1(\beta_{j,1}|0) + T_{1;2}^{irr}(\beta_{j,1}|\rho_{2,1}^{(j)}) \tag{7.5a}$$

$$\rho_{2,1}^{(j-1)} = T_2(\beta_{j,1}|0) + T_{2;2}^{irr}(\beta_{j,1}|\rho_{2,1}^{(j)}) \tag{7.5b}$$

with

$$T_2(\beta|0) = \beta^2\left[\frac{I_3(2\beta)}{I_1(2\beta)} - \left(\frac{I_2(2\beta)}{I_1(2\beta)}\right)^2\right]$$

and

$$\begin{aligned} T_{1;2}^{irr}(\beta|\rho_2) &= -\rho_2\Big[L_{02}^{(1)}(\beta) - L_{00}^{(1)}(\beta)\,L_{02}^{(0)}(\beta)\Big] \\ T_{2;2}^{irr}(\beta|\rho_2) &= 2\rho_2\Big[L_{02}^{(2)}(\beta) - L_{00}^{(2)}(\beta)\,L_{02}^{(0)}(\beta) \\ &\qquad - L_{00}^{(1)}(\beta)\,L_{02}^{(1)}(\beta) + L_{00}^{(1)}(\beta)^2\,L_{02}^{(0)}(\beta)\Big] \end{aligned}$$

where the expansion coefficients $L_{nm}^{(k)}(\beta)$ are given in App. D. Now expand Eqs. (7.5) for large $\beta_{j,1}$

$$\beta_{j-1,1} = \Big(\beta_{j,1} - \frac{3}{4} + \frac{3}{32}\beta_{j,1}^{-1} + O(\beta_{j,1}^{-2})\Big) + \rho_{2,1}^{(j)}\Big(-\frac{5}{8}\beta_{j,1}^{-1} + \frac{15}{64}\beta_{j,1}^{-2} + O(\beta_{j,1}^{-3})\Big) \tag{7.6a}$$

$$\rho_{2,1}^{(j-1)} = \Big(-\frac{1}{2}\beta_{j,1} + \frac{3}{4} + O(\beta_{j,1}^{-1})\Big) + \rho_{2,1}^{(j)}\Big(\frac{1}{4} + \frac{7}{16}\beta_{j,1}^{-1} + O(\beta_{j,1}^{-2})\Big)\;. \tag{7.6b}$$

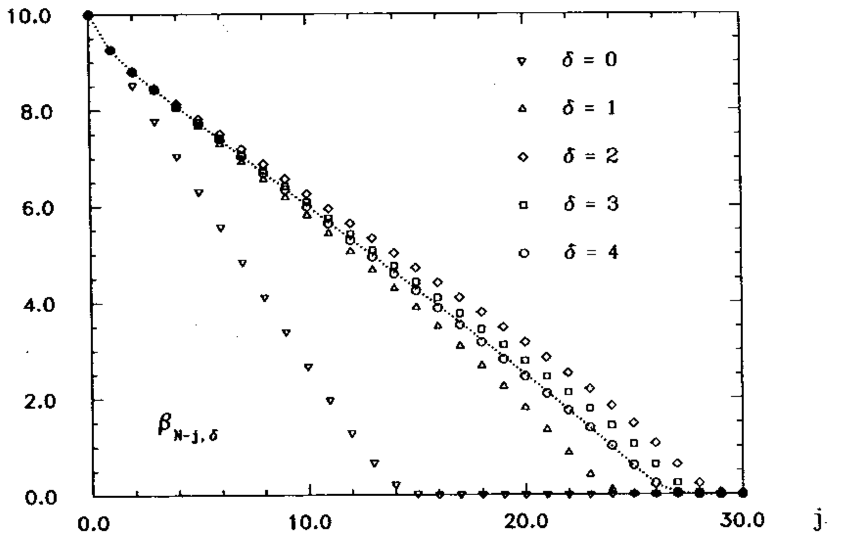
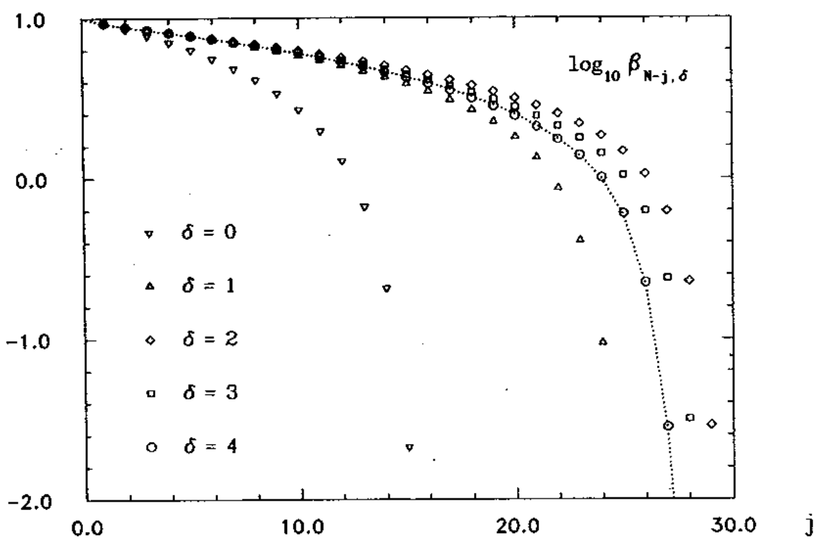


Fig. 7.1a+7.1b. Flow of the running coupling constant β_{N-j} for $\beta_N = 10$ in different orders δ of irrelevant perturbation theory. The dotted line indicates the exact result obtained by a numerical integration of the recursion relations. The fifth-order result differs only slightly from the fourth-order one and is therefore not given here. ($\log_{10}\beta = (\log 10)^{-1}\log\beta$)



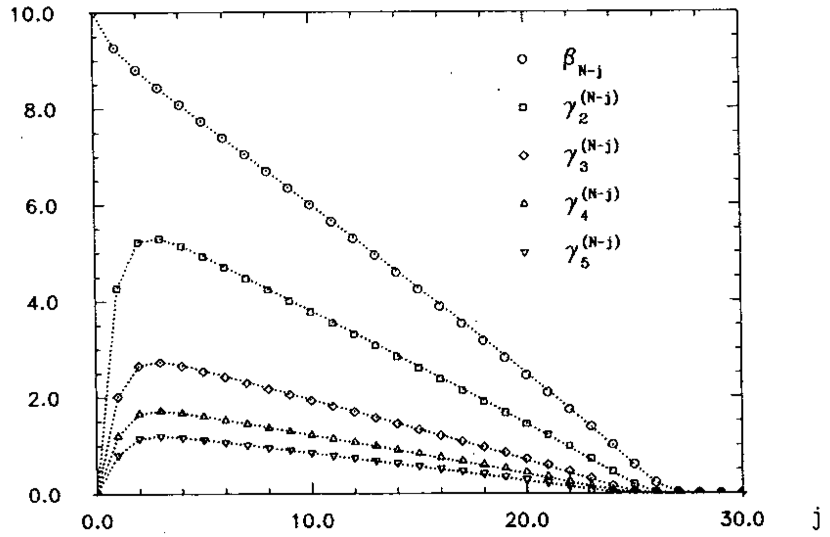
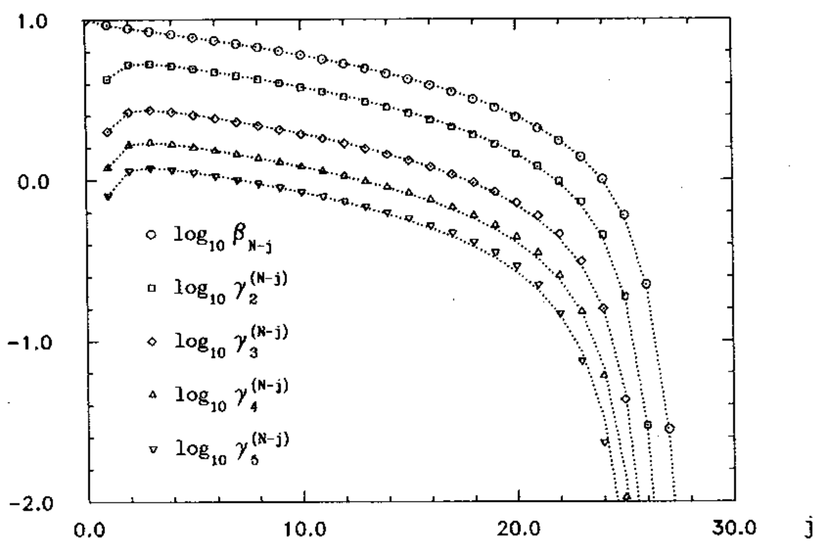


Fig. 7.2a+7.2b. Flow of the four leading irrelevant couplings $\gamma_2^{(N-j)}, \dots, \gamma_5^{(N-j)}$ calculated in fourth-order irrelevant perturbation theory ($\delta = 4$). For comparison, the exact data for the running couplings are connected by dotted lines.



Using the fact that $\rho_{2,1}^{(j)}$ is of order $\beta_{j,1}$, cp. Sect. 6.2, these equations can be rewritten in the form

$$\beta_{j-1,1} = \beta_{j,1} - \left(\frac{3}{4} + \frac{5}{8}\rho_{2,1}^{(j)}\beta_{j,1}^{-1}\right) + \left(\frac{3}{32}\beta_{j,1}^{-1} + \frac{15}{64}\rho_{2,1}^{(j)}\beta_{j,1}^{-2}\right) + O(\beta_{j,1}^{-2}) \tag{7.7a}$$

$$\rho_{2,1}^{(j-1)} = \left(-\frac{1}{2}\beta_{j,1} + \frac{1}{4}\rho_{2,1}^{(j)}\right) + \left(\frac{3}{4} + \frac{7}{16}\rho_{2,1}^{(j)}\beta_{j,1}^{-1}\right) + O(\beta_{j,1}^{-1}) \,. \tag{7.7b}$$

Thus, in case of a large running coupling constant, one concludes that the recursion relations for $\delta = 1$ yield the two leading orders of the weak coupling expansions Eqs. (6.71a,b) correctly. Hence one expects that the $\delta = 1$ approximation describes the running coupling constant flow along the renormalized trajectory correctly for large $\beta_{j,1}$ at the "one loop" level. Indeed, one obtains the result

$$\beta_{j-1,1} = \beta_{j,1} - \frac{1}{3} - \frac{7}{27}\beta_{j,1}^{-1} + O(\beta_{j,1}^{-2}) \tag{7.8a}$$

$$\rho_{2,1}^{(j-1)} = -\frac{2}{3}\beta_{j,1} + \frac{1}{3} + O(\beta_{j,1}^{-1}) \tag{7.8b}$$

by solving Eqs. (7.7).

Following the discussion of Sect. 6.2, one can easily see that it is necessary to choose at least $\delta = 3$ in order to reproduce the correct "two loop" result (6.76) for the running coupling constant flow.

Moreover, expanding up to contributions with a degree of irrelevance $\delta = 4$, one obtains not only an additional improvement of the approximations of the flow for large β_j, but also a very precise description for smaller values of the running coupling constant, see Fig. 7.1 for the choice $\beta_N = 10$.

It is instructive, to consider also the flow of the leading irrelevant couplings ρ_2, ρ_3, ρ_4 and ρ_5 for $\delta = 4$. In order to display the flow in a single plot, Fig. 7.2, one calculates the couplings γ_2, γ_3, γ_4 and γ_5 which parametrize the corresponding effective actions, see (2.69). Explicitly, one has the relations

$$\gamma_2 = -\rho_2 \,, \quad \gamma_3 = -\rho_3 \,, \quad \gamma_4 = -\rho_4 + \frac{1}{2}\rho_2^2 \,, \qquad \text{and} \qquad \gamma_5 = -\rho_5 + \rho_2\rho_3 \,. \tag{7.9}$$

Again, as Fig. 7.2 shows, the agreement with the "exact" data is very good. Note that after four or five initial iteration steps the flow of the irrelevant couplings is obviously "governed" by the flow of the running coupling constant. Such a behaviour is typical for the renormalized trajectory, cp. the discussion in Sect. 6.2.

Computation of the Renormalized Trajectory

To study the flow of individual renormalization group trajectories towards the renormalized trajectory more precisely, it is first of all necessary to calculate the renormalized trajectory itself. It will be obtained as the limit of a sequence of conveniently chosen renormalization group trajectories, cp. Sect. 6.2.

Along a renormalization group trajectory (originating in the point β_N, $\rho_n^{(N)} = 0$ on the canonical line), the effective activities are denoted by $G_j^{[N]}$ with $j \leq N$ and $G_N^{[N]} \equiv G_W(\beta_N)$.

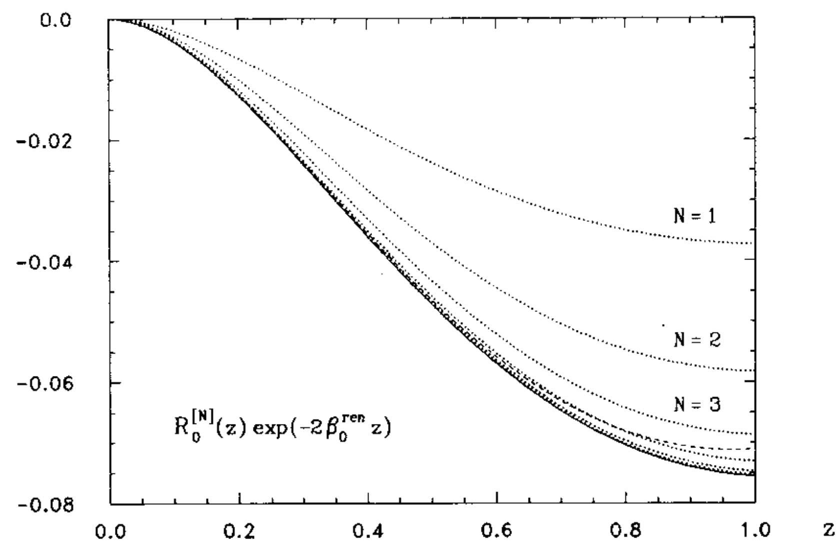


Fig. 7.3. The continuum effective monomer activity $R_0^{[\infty]}(z)\exp(-2\beta_0^{ren}z)$ on the unit lattice for a renormalized coupling constant $\beta_0^{ren} = 0.993$ (solid curve). A sequence of approximations $R_0^{[N]}(z)\exp(-2\beta_0^{ren}z)$ of this reduced activity is also shown for $N = 1, \ldots, 6$ (dotted curves), where $L^N = \sqrt{2}^N$ represents the underlying cutoff. All these approximations lie on renormalization group trajectories indexed by N which connect the canonical line of Wilson Boltzmannians to the hyperplane $\beta_0^{[N]} = \beta_0^{ren} = 0.993$ within exactly N iteration steps. The limit activity $R_0^{[\infty]}(z) \equiv \lim_{N\to\infty} R_0^{[N]}(z)$ lies on the renormalized trajectory and describes the continuum theory. All curves are calculated by irrelevant perturbation theory up to the fourth order $\delta = 4$. The corresponding "continuum" result obtained by the described "exact" (numerical) treatment of the Migdal recursion formula is represented by the dashed curve. The differences for $z \to 1$ are due to contributions of irrelevant interactions with higher degrees of irrelevance ($\delta > 4$), which are, however, of minor importance for the small field behaviour.

The running coupling constants are then denoted by $\beta_j^{[N]}$ with $\beta_N^{[N]} \equiv \beta_N$. One starts by choosing a renormalization scale j. For simplicity, set $j = 0$. Afterwards, one fixes a renormalized coupling constant β_0^{ren} on this scale (according to some "physical renormalization prescription"). Then one has to specify a sequence of bare coupling constants β_N, $N \geq 1$, which parametrize the bare Boltzmannians $G_W(\beta_N)$ on the canonical line and also the associated discrete renormalization group trajectories $\{G_j^{[N]}\}_{j\leq N}$. These bare couplings β_N depend on the choice of the renormalized coupling β_0^{ren} by enforcing the renormalization condition $\beta_0^{[N]} = \beta_0^{ren}$ for all running coupling constants $\beta_0^{[N]}$ with $N \geq 1$.

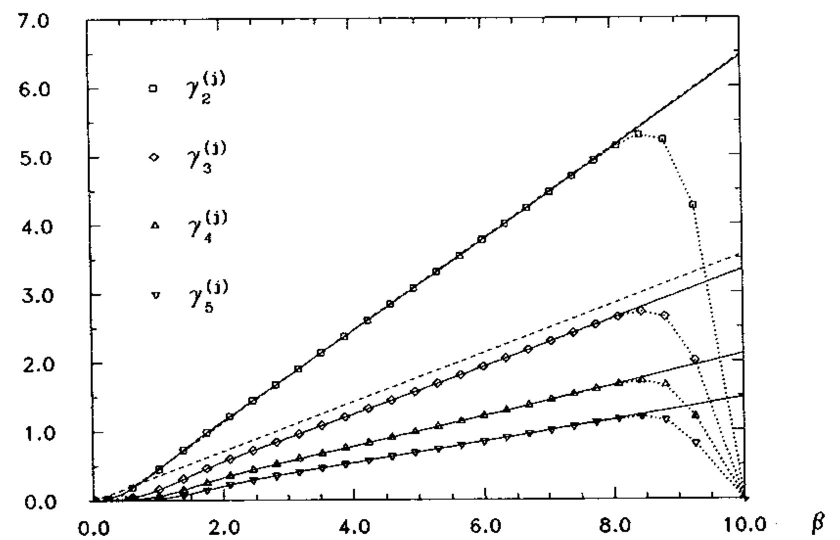


Fig. 7.4. Approach of the renormalization group trajectory with $\beta_N = 10$, $\rho_n^{(N)} = 0$ to the renormalized trajectory. The solid curves show the leading irrelevant couplings on the renormalized trajectory as functions of the renormalized coupling β, i.e. $\gamma_2(\beta)$, $\gamma_3(\beta)$, $\gamma_4(\beta)$, and $\gamma_5(\beta)$. For the discrete flow along the chosen renormalization group trajectory, the irrelevant couplings $\gamma_n^{(j)}$ are compared to the corresponding continuum values $\gamma_n(\beta_j)$. (The running couplings β_j, $\gamma_n^{(j)}$ have already been displayed in Fig. 7.2.) The renormalized trajectory is reached (within very small errors) after four or five iteration steps. The two dashed lines represent the approximations $\frac{2}{3}\beta - \frac{28}{135}$ for γ_2 and $\frac{16}{45}\beta$ for γ_3 of Eqs. (6.77). Whereas the agreement with γ_2 is very good even for small values $\beta > 1/2$, the approximation for γ_3 holds only up to a shift of order one, as expected by (6.77b).

All calculations can be done most easily with the help of the recursive flow equations (4.24) for $\delta = 4$, which describe the renormalization group trajectories almost exactly (besides the fact that only the leading irrelevant couplings are included in the approximation). Within this frame, the results of the above procedure show that the trajectories with $\beta_0^{[N]} = \beta_0^{ren}$ converge with increasing $N \geq 1$ to a limit trajectory that approximates the renormalized trajectory very well (as far as the leading couplings $\beta, \gamma_2, ..., \gamma_5$ are concerned), see Fig. 7.3.

Along the renormalized trajectory, all irrelevant couplings are functions of the renormalized coupling constant β, i.e. $\gamma_n = \gamma_n(\beta)$. These functions are plotted for $n = 2, ..., 5$ in Fig. 7.4 (solid curves). The flow of the renormalized coupling constant β itself is dictated by the renormalization group (Callan-Symanzik) equation

$$\beta' = \beta + \Delta\beta(\beta) . \tag{7.10}$$

Here β' denotes the effective renormalized coupling constant on a length scale enlarged by a

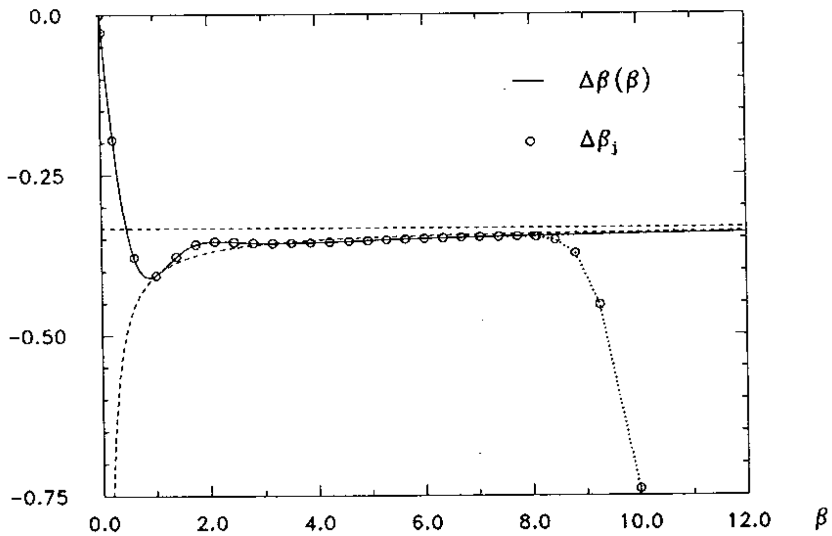


Fig. 7.5. The renormalization group function $\Delta\beta(\beta)$ for renormalization group steps with $L = \sqrt{2}$ as calculated in fourth-order irrelevant perturbation theory ($\delta = 4$, solid curve). The lower dashed line gives the "two loop" approximation $-\frac{1}{3} - \frac{2}{27}\beta^{-1}$ for large β, cp. Eq. (6.76b). The "one loop" result, $-\frac{1}{3}$, is represented by the upper dashed line. In addition, the values $\Delta\beta_j = \beta_{j-1} - \beta_j$ are given for the iteration starting at $\beta_N = 10$, $\rho_n^{(N)} = 0$, compare Fig. 7.2. After some initial steps, when the trajectory is very close to the renormalized trajectory, one gets $\Delta\beta_j = \Delta\beta(\beta_j)$ up to very small corrections.

factor $L = \sqrt{2}$. $\Delta\beta(\beta)$ is the (discrete) renormalization group function. It describes the change of the renormalized coupling constant under (discrete) renormalization group transformations. By Eq. (6.76b), one has the rigorous result

$$\Delta\beta(\beta) = -\frac{1}{3} - \frac{2}{27}\beta^{-1} + O(\beta^{-2}) , \tag{7.11}$$

which holds for sufficiently large values of β, where the remainder is bounded by const β^{-2} with a β-independent constant. However, by perturbation expansions in irrelevant interactions, the $\Delta\beta$-function can be calculated for *all* values of the coupling constant. The result is shown in Fig. 7.5 (solid curve) together with the asymptotic behaviour (7.11) (lower dashed curve).

Approach to the Renormalized Trajectory

In order to investigate the approach of individual renormalization group trajectories to the renormalized trajectory, one can plot the running couplings $\gamma_n^{(j)}$ as well as the differences

$\beta_{j-1} - \beta_j \equiv \Delta\beta_j$ against the running coupling constant β_j. Then one compares these data with the "continuum" values $\gamma_n(\beta_j)$ and $\Delta\beta(\beta_j)$. The results are shown in Fig. 7.4 and Fig. 7.5, where the trajectory starts at $\beta_N = 10$ with $\rho_n^{(j)} = 0$. For this choice of the bare coupling constant, the renormalized trajectory is reached after five iteration steps, i.e. $\gamma_n^{(j)}$, $\Delta\beta_j$ become $\gamma_n(\beta_j)$, $\Delta\beta(\beta_j)$.

The number of iteration steps that is needed in order to reach the renormalized trajectory from a starting point $G_W(\beta_N)$ on the canonical line depends of course on the size of the bare coupling constant β_N. However, as a consequence of the exponentially fast decay of the irrelevant scaling couplings $\tilde{\rho}_n$, this number grows only logarithmically with increasing β_N. The scaling couplings have been introduced and discussed with respect to their influence on the renormalization group flow in Sect. 6.2.

Calculation of the Norms $\|(1+R_j)^2 - 1\|_{1,\beta_j}$

The "exact" solutions (7.1) for the effective monomer activities G_j allow to calculate the norms introduced in Sect. 5. Consider for example the norm $\|(1+R_j)^2 - 1\|_{1,\beta_j}$. By the definition (5.50), one gets

$$\|(1+R_j)^2 - 1\|_{1,\beta_j} = \max_{|\tau|=1} \int d\mu_{\beta_j}(s) \left| \exp\Big(-2\sum_{\nu=2}^{k} \gamma_\nu^{(j)}(\tau s)^\nu\Big) - 1 \right| , \qquad (7.12)$$

where the measure $d\mu_\beta(s)$ is given in (5.48). The "exact" results for the iteration with $\beta_N = 10$ are shown in Fig. 7.6 together with the norms $\|(1+R)^2 - 1\|_{1,\beta}$ along the renormalized trajectory as predicted by the irrelevant perturbation theory with $\delta = 4$.

The "exact" (numerical) data for the norms $\|(1+R_j)^2 - 1\|_{1,\beta_j}$ along the (discrete) renormalization group trajectory with $\beta_N = 10$, $\rho_n^{(N)} = 0$ describe the norms along the renormalized trajectory very precisely, because the renormalized trajectory is reached after a few iteration steps only and because the computational and systematical errors of the numerical treatment based on Eqs. (7.1) and (7.2) are extremely small. Furthermore, the "exact" data are well approximated and inter- and extrapolated by the data from the perturbation expansion in irrelevant interactions up to the fourth order in degree of irrelevance. Of course, these finite order perturbative results can not and should not be used to decide wether the irrelevant perturbation expansions converge or not. But they show that the "exact" data given in Fig. 7.6 covers the interesting region between (very) small and large values of β.

Hence, due to Proposition (5.3.ii) and Theorem 4.5, the convergence of the irrelevant perturbation series in the recursive flow equations (4.12) is established for all values of the bare coupling constant and for all iteration steps.

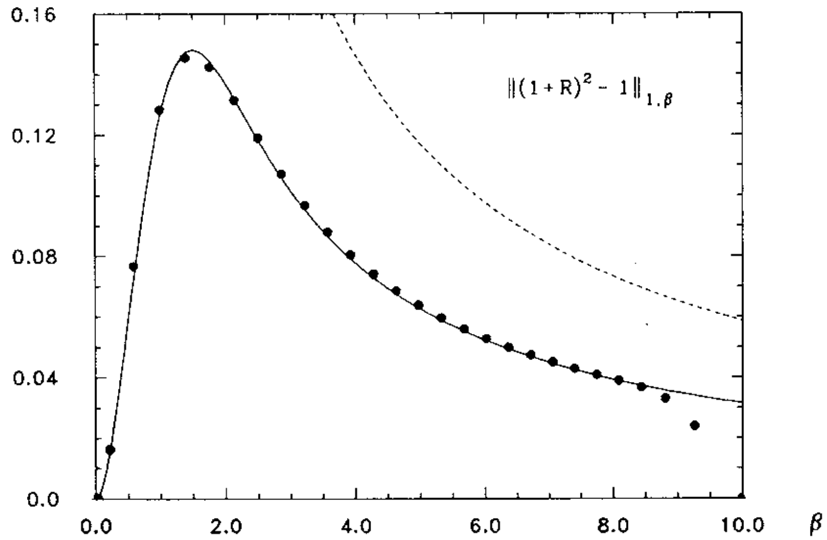


Fig. 7.6. The norms $\|(1+R)^2 - 1\|_{1,\beta}$. The solid curve gives the norms along the renormalized trajectory as calculated in irrelevant perturbation theory up to the degree of irrelevance $\delta = 4$. For comparison, the exact results $\|(1+R_j)^2 - 1\|_{1,\beta_j}$ along the renormalization group trajectory starting at $\beta_N = 10$, $\rho_n^{(N)} = 0$, are plotted against the running coupling constant β_j (black dots). The dashed line represents the leading order $\frac{75}{128}\beta^{-1}$ of the bound (6.45a) which is proven to be valid for sufficiently large β.

8. SUMMARY AND OUTLOOK

New perturbative expansions in irrelevant interactions have been proposed for exact renormalization group studies. They have been investigated in detail for a hierarchical $SU(2)$ lattice gauge field model in four Euclidean dimensions, which is renormalizable and ultraviolet asymptotically free.

Organization of the Expansions

The basic idea from the technical point of view is to decompose the effective Boltzmannians into a relevant Wilson part, parametrized by the running coupling constant β_j, and residual interactions which are irrelevant. The Wilson Boltzmannian, i.e. the exponential of the Wilson action, defines an unperturbed measure. This measure serves as the basis for a perturbative treatment of the residual interactions. For all field configurations, the residual interactions are parametrized by irrelevant couplings $\rho_n^{(j)}$. The resulting expansions of the effective Boltzmannians can be inserted into the recursive renormalization group flow equations. For the running coupling constants, expansions in powers and products of the irrelevant couplings $\rho_n^{(j)}$ result. They are organized according to a degree of irrelevance. The degree of irrelevance describes the (canonical) scaling behaviour of the irrelevant (dimensionless) couplings under linearized renormalization group transformations. All expansion coefficients are functions of the running (marginal) coupling constant β_j.

Results

- Every order δ of the expansion is given by a finite sum of terms with the same degree of irrelevance δ. The expansion coefficients are explicitly calculable as functions of the running coupling constant β_j.
- The perturbation expansions in irrelevant interactions converge absolutely for all values of the running coupling constant. This holds also in the weak coupling regime where standard perturbation theory in powers of β_j^{-1} yields only asymptotic expansions. The convergence is a consequence of the fact that the renormalized trajectory and the canonical line (of Wilson Boltzmannians) have only a small "distance" in the space of coupling constants.
- For small gauge coupling constants (large β_j), the β_j-dependent expansion coefficients may be represented as series expansions which yield not only the asymptotic weak coupling expansions in powers of β_j^{-1} for large β_j, but also nonperturbative contributions. Therefore, the "perturbation" expansions in irrelevant interactions are generally nonperturbative in the running gauge coupling constant.
- Low order approximations are very accurate. This holds even after many iteration steps when β_j runs over several orders of magnitude. It turns out that a fourth-order calculation, i.e. a calculation up to a degree of irrelevance $\delta = 4$, produces very precise results.
- High order calculations are possible because all expansion coefficients can be generated with the help of an algebraic computer program.

Application

- Since the irrelevant perturbation expansions converge for all values of the running coupling constant, they will allow to calculate the whole renormalized trajectory with arbitrary accuracy (in principle). The same holds for the continuum effective Boltzmannians (or actions).
- The iterated expansions make it possible to link ultraviolet and infrared properties of the model. Because no small parameter like β^{-1} or β is needed, this closes the gap between weak coupling and high temperature expansions.
- It is also possible to calculate expectation values of local gauge invariant operators within this setup of perturbation expansions in irrelevant interactions. The convergence of the series expansions for the recursive flow equations is governed by the interpolating normalization constants $\acute{N}_j(\tau_j)$. Thus recursive equations for expectation values will also converge in irrelevant perturbation theory.
- Expectation values can be calculated in the thermodynamical (infinite volume) limit by iterated renormalization group transformations.

Conclusion

Within the framework of the proposed perturbation theory in irrelevant interactions, it becomes possible to track the renormalization group flow along the *whole* renormalized trajectory – linking the ultraviolet to the infrared behaviour of the model – by taking into account only a small number of leading irrelevant interactions. Thus the perturbation theory provides not only a scheme for the calculation of *continuum* effective Boltzmannians, but also for the computation of expectation values of local gauge invariant operators in the continuum (hierarchical) $SU(2)$ gauge theory. The thermodynamical limit of the expectation values can also be calculated.

Future Developments

(i) *Hierarchical $SU(2)$ model for $L = \sqrt{2}$*

- A purely analytical proof of the convergence of the irrelevant perturbation theory between the weak and the strong coupling regime is desirable. But in order to do this, a new idea would be needed.
- *Multigrid formulation.* Following Mack and Pordt's program towards convergent and computable multigrid expansions for continuum field theories without ultraviolet cutoff [7,8,9,10], one can try to iterate the irrelevant perturbation expansions for the recursive flow equations. This would lead to a scheme where the cluster expansions of Mack and Pordt are partly replaced by simpler perturbation expansions.

(ii) *Hierarchical $SU(2)$ model for $L = 2$*

In this paper, I have only considered the hierarchical model which is defined by the Migdal recursion relation for a scale factor $L = \sqrt{2}$. The hierarchical model with $L = 2$, for which an appealing choice of the block spin definition and of the hierarchical structure of the model is available (cp. Sect. 2), can be derived easily from the $L = \sqrt{2}$ case. By inspection of the Migdal recursion formulae it follows that a renormalization group transformation with scale factor $L = 2$ can be performed as a sequence of two modified recursion steps,

$$g_{j-1/2}^{[L=2]} = \mathcal{N}_j^{-1} \left[g_j^{[L=2]} * g_j^{[L=2]} \right]$$
$$g_{j-1}^{[L=2]} = \left(\mathcal{N}_{j-1/2}^{-1} \left[g_{j-1/2}^{[L=2]} * g_{j-1/2}^{[L=2]} \right] \right)^4 .$$

Note that both recursion relations are again of the Migdal type, Eq. (2.4), for scale factors $L = \sqrt{2}$. They can be distinguished by the parameter D of Eq. (2.4a). For the first equation, one has $D = 2$. The second equation is obtained for $D = 6$. It is possible to explicitly introduce this parameter D in all the formulae and estimates of this paper. No additional insight or technique would be required. The set of recursive flow equations (4.12) has to be replaced by two sets of equations. The first determines the couplings $\beta_{j-1/2}$, $\rho_n^{(j-1/2)}$ for the "intermediate" Boltzmannian $g_{j-1/2}$. The second set then calculates the effective couplings β_{j-1}, $\rho_n^{(j-1)}$ from the "intermediate" ones.

(iii) *Treatment of complete (non-hierarchical) lattice gauge models*

The renormalization group treatment of non-hierarchical lattice gauge field theories is a complicated task. Roughly, one may proceed by the following major steps:

- Choose a block spin. Some proposals of different authors are summarized together with their common properties in [29]. The choice of a "good" block spin is of crucial importance, see [6, Remark C].
- The effective Boltzmannians are represented as partition functions of nontrivial polymer systems. Reduced polymer activities have to be defined in such a way that they consist only of irrelevant interactions and fulfil convenient renormalization conditions [10].
- The reduced polymer activities must then be parametrized by irrelevant coupling constants.
- The expansion coefficients of the recursive renormalization group flow equations will again depend on the (marginal) gauge coupling constant (when no irrelevant couplings are added in order to improve the irrelevant perturbation theory, cp. the remark at the end of Sect. 6.2). These coefficients can be calculated nonperturbatively by standard Monte Carlo renormalization group (MCRG) methods [30]. The Monte Carlo simulations would be done with Wilson actions and the running coupling constant β_j. One simulation is necessary for each renormalization group step.
- *Multigrid formulation.* As in the hierarchical case, one is finally interested in obtaining expansions for *iterated* renormalization group steps. Such expansions represent effective Boltzmannians as partition functions of polymer systems on a *multigrid*. They point the way to a computer simulation of continuum field theories without cutoff, as discussed by Mack [10]. Monte Carlo simulations of polymer systems were investigated by many authors [31]. A method which is suitable for nonlocal Hamiltonians was described by Mack and Pinn [32] and has been further investigated by [33].

Final Conclusion

Convergent perturbation expansions for running coupling constant recursion relations were proposed. They are organized in irrelevant interactions. These expansions promise to provide a useful scheme for the calculation of effective Boltzmannians for continuum field theories. There is *no* need for a "small parameter" like a small running gauge coupling (or a small inverse gauge coupling).

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APPENDICES

APPENDIX A. PERTURBATION EXPANSION FOR CONVOLUTION INTEGRALS

The generating function $F(\beta,\beta';z)$ of the moments $F_{nm}(\beta,z)$ is calculated. It leads to an expression for the expansion coefficients $F^{(k)}_{nm}(\beta)$ which is used to derive an explicit formula for all orders of the asymptotic (weak coupling) expansion of these functions $F^{(k)}_{nm}(\beta)$. For calculations up to a finite order l, an integral representation of the remainder is given. In addition, the remainder and its derivatives are estimated. The results of this appendix contribute to the rigorous perturbative treatment of Sect. 6.2 and provide the basis for the computation of the "almost normalized" expansion coefficients $L^{(k)}_{nm}(\beta)$ which are needed for perturbation expansions in irrelevant interactions (cp. Sect. 5 and App. D).

Consider the moments

$$F_{nm}(\beta,z) = \int_G dv\; G_W(\beta,z_+(z))\,G_W(\beta,z_-(z))\; z_+(z)^n\, z_-(z)^m \;, \tag{A.1}$$

that have been introduced in (5.62), and define the generating function, cp. (5.68),

$$F(\beta,\beta';z) = \int_G dv\; G_W(\beta,z_+(z))\,G_W(\beta',z_-(z)) = \int_G dv\; e^{-2\beta\, z_+(z) - 2\beta'\, z_-(z)}\,, \tag{A.2}$$

which is entire in all variables β,β', and z. Then obviously

$$F_{nm}(\beta,z) = \left(-\frac{1}{2}\frac{\partial}{\partial\beta}\right)^n \left(-\frac{1}{2}\frac{\partial}{\partial\beta'}\right)^m F(\beta,\beta';z)\Big|_{\beta=\beta'} \;. \tag{A.3}$$

By the explicit parametrization introduced in Sect. 3 and Sect. 5, respectively, one has $z_\pm(z) = [1-\sqrt{1-z}\cos\phi \pm \sqrt{z}\sin\phi\cos\chi]/2$, Eq. (5.12), and $dv = (d\phi\sin^2\phi\; d\chi\sin\chi)/\pi$ for the normalized Haar measure. Now the computation of F is straightforward (let $\beta_\pm = \beta \pm \beta'$)

$$\begin{aligned} F(\beta,\beta';z) &= \frac{1}{\pi}\int_0^\pi d\phi\sin^2\phi \int_0^\pi d\chi\sin\chi \\ &\qquad \times \exp\!\left(-\beta_+[1-\sqrt{1-z}\cos\phi] - \beta_-[\sqrt{z}\sin\phi\cos\chi]\right) \\ &= \frac{2}{\pi\beta_-\sqrt{z}}\int_0^\pi d\phi\sin\phi\,\sinh(\beta_-\sqrt{z}\sin\phi)\exp\!\left(-\beta_+[1-\sqrt{1-z}\cos\phi]\right) \;. \end{aligned} \tag{A.4}$$

Writing the sinh as a power series, one obtains by integration an expansion in terms of modified Bessel functions

$$\begin{aligned} F(\beta,\beta';z) &= \frac{2\,e^{-\beta_+}}{\pi\beta_-\sqrt{z}} \sum_{\nu=0}^\infty \frac{(\beta_-\sqrt{z})^{2\nu+1}}{(2\nu+1)!}\int_0^\pi d\phi\,\sin^{2(\nu+1)}\phi\; e^{+\beta_+\sqrt{1-z}\cos\phi} \\ &= e^{-\beta_+}\sum_{\nu=0}^\infty \frac{\Gamma(\nu+3/2)}{\Gamma(3/2)}\,\frac{2^{\nu+1}}{(2\nu+1)!}\,\beta_-^{2\nu}\, z^\nu\, \frac{I_{\nu+1}(\beta_+\sqrt{1-z})}{(\beta_+\sqrt{1-z})^{\nu+1}} \end{aligned} \tag{A.5}$$

Using the well known duplication formula $\Gamma(2\xi) = (2\pi)^{-1/2}\, 2^{2\xi-1/2}\,\Gamma(\xi)\,\Gamma(\xi+1/2)$ for $\xi = \nu+1$, this becomes

$$F(\beta,\beta';z) = \frac{2\,e^{-\beta_+}}{\beta_+\sqrt{1-z}}\sum_{\nu=0}^\infty \frac{1}{\nu!}\left(\frac{\beta_-^2 z}{2\beta_+\sqrt{1-z}}\right)^\nu I_{\nu+1}(\beta_+\sqrt{1-z}) \;. \tag{A.6}$$

For $\beta_+ \neq 0$ and $z \neq 1$, the series can be summed with help of the formula [24]

$$I_1(\lambda\zeta) = \lambda\sum_{k=0}^\infty \frac{(\lambda^2-1)^k(\zeta/2)^k}{k!}\, I_{1+k}(\zeta)$$

when one sets $\zeta = \beta_+\sqrt{1-z}$ and $\lambda = \zeta^{-1}\sqrt{\beta_+^2(1-z)+\beta_-^2 z}$. Thus, finally,

$$\begin{aligned} F(\beta,\beta';z) &= 2\,e^{-\beta_+}\,\frac{I_1\!\left(\sqrt{\beta_+^2(1-z)+\beta_-^2 z}\right)}{\sqrt{\beta_+^2(1-z)+\beta_-^2 z}}\Bigg|_{\beta_\pm=\beta\pm\beta'} \\ &= 2\,e^{-(\beta+\beta')}\,\frac{I_1\!\left(\sqrt{(\beta+\beta')^2-4\beta\beta' z}\right)}{\sqrt{(\beta+\beta')^2-4\beta\beta' z}} \;. \end{aligned} \tag{A.7}$$

The cases $\beta_+ = 0$ and $z = 1$, respectively, can be investigated by means of

$$\lim_{\zeta\to 0}\frac{I_n(\zeta)}{\zeta^n} = \frac{2^{-n}}{n!} \qquad \text{and} \qquad I_1(\xi) = \frac{\xi}{2}\sum_{k=0}^\infty \frac{(\xi/2)^{2k}}{k!\,(k+1)!} \;.$$

It turns out that they are treated correctly by (A.7).

Having calculated the generating function $F(\beta,\beta';z)$, the problem of integrating (A.1) is reduced to the evaluation of Eqs. (A.3). In order to compute the coefficients $F^{(k)}_{nm}(\beta)$ defined by Eq. (6.67), it is useful to split

$$\begin{aligned} F^{(k)}_{nm}(\beta) &= \frac{1}{k!}\frac{\partial^k}{\partial z^k}F_{nm}(\beta,z)\Big|_{z=0} = \frac{1}{k!}\left(-\frac{1}{2}\right)^{n+m}\frac{\partial^n}{\partial\beta^n}\frac{\partial^m}{\partial\beta'^m}\frac{\partial^k}{\partial z^k}F(\beta,\beta';z)\Big|_{z=0,\beta'=\beta} \\ &= \frac{1}{k!}\left(-\frac{1}{2}\right)^{n+m}\left[\frac{\partial}{\partial\beta_+}+\frac{\partial}{\partial\beta_-}\right]^n\left[\frac{\partial}{\partial\beta_+}-\frac{\partial}{\partial\beta_-}\right]^m \\ &\qquad \times \frac{\partial^k}{\partial z^k}F\!\left(\frac{\beta_++\beta_-}{2},\frac{\beta_+-\beta_-}{2};z\right)\Big|_{z=0,\beta_-=0,\beta_+=2\beta} \\ &= \frac{1}{k!}\left(-\frac{1}{2}\right)^{n+m}\sum_{\nu=0}^{[\frac{n+m}{2}]}\left\{\sum_{\substack{i=0 \\ i+j=2\nu}}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}(-1)^j\right\} \\ &\qquad \times \frac{\partial^{n+m-2\nu}}{\partial\beta_+^{n+m-2\nu}}\frac{\partial^{2\nu}}{\partial\beta_-^{2\nu}}\frac{\partial^k}{\partial z^k}F\!\left(\frac{\beta_++\beta_-}{2},\frac{\beta_+-\beta_-}{2};z\right)\Big|_{z=0,\beta_-=0,\beta_+=2\beta} \end{aligned} \tag{A.8}$$

and then to start by performing the derivations of F with respect to the variable z, which will lead to simple formulae. Abbreviate $Q(z) = \sqrt{\beta_+^2(1-z) + \beta_-^2 z}$. Then

$$\frac{\partial^k}{\partial z^k}\frac{I_{\nu+1}(Q(z))}{Q(z)^{\nu+1}} = \frac{\partial^{k-1}}{\partial z^{k-1}}\frac{\partial Q(z)}{\partial z}\frac{\partial}{\partial Q}\frac{I_{\nu+1}(Q)}{Q^{\nu+1}} = \frac{\partial^{k-1}}{\partial z^{k-1}}\frac{\beta_-^2-\beta_+^2}{2Q(z)}\frac{I_{\nu+2}(Q(z))}{Q(z)^{\nu+1}}$$
$$= \left(\frac{\beta_-^2-\beta_+^2}{2}\right)\frac{\partial^{k-1}}{\partial z^{k-1}}\frac{I_{\nu+2}(Q(z))}{Q(z)^{\nu+2}} = \left(\frac{\beta_-^2-\beta_+^2}{2}\right)^k\frac{I_{\nu+k+1}(Q(z))}{Q(z)^{\nu+k+1}}$$

and hence

$$\frac{\partial^k}{\partial z^k}F\left(\frac{\beta_++\beta_-}{2},\frac{\beta_+-\beta_-}{2};z\right)\bigg|_{z=0} = 2\,e^{-\beta_+}\frac{I_{k+1}(\beta_+)}{\beta_+^{k+1}}\left(\frac{\beta_-^2-\beta_+^2}{2}\right)^k. \tag{A.9}$$

Now proceed by differentiating with respect to β_-

$$\frac{\partial^{2\nu}}{\partial\beta_-^{2\nu}}(\beta_-^2-\beta_+^2)^k\bigg|_{\beta_-=0} = \frac{\partial^{2\nu}}{\partial\beta_-^{2\nu}}\sum_{i=0}^{k}\binom{k}{i}(-\beta_+^2)^{k-i}\beta_-^{2i}\bigg|_{\beta_-=0} = \binom{k}{\nu}(2\nu)!(-\beta_+^2)^{k-\nu}$$

for $0\le\nu\le k$. When $\nu>k$, the derivatives yield obviously zero. To summarize, one has

$$F_{nm}^{(k)}(\beta) = 2\left(-\frac{1}{2}\right)^{n+m+k}\sum_{\nu=0}^{\min\{[\frac{n+m}{2}],k\}}(-1)^\nu\binom{k}{\nu}\frac{(2\nu)!}{k!}a_{2\nu}^{nm}$$
$$\times\frac{\partial^{n+m-2\nu}}{\partial\beta_+^{n+m-2\nu}}e^{-\beta_+}\frac{I_{k+1}(\beta_+)}{\beta_+^{k+1}}\beta_+^{2(k-\nu)}\bigg|_{\beta_+=2\beta} \tag{A.10}$$

with the coefficients $a_{2\nu}^{nm}$ of (6.66).

In order to carry out the remaining differentiations in (A.10), one first replaces the modified Bessel function by the following finite order asymptotic expansion with remainder.

Lemma A.1. *Let z and ν be complex variables with $\mathrm{Re}\,\nu>-1/2$. Then the expansions*

$$e^{-z}I_\nu(z) = \frac{1}{\sqrt{2\pi z}}\sum_{\mu=0}^{l}\frac{(-1)^\mu}{\mu!}\frac{\gamma(\nu+\mu+1/2;2z)}{\Gamma(\nu-\mu+1/2)}\left(\frac{1}{2z}\right)^\mu + R_l(\nu,z) \tag{A.11}$$

hold for finite order l with remainders $R_l(\nu,z)$ given by

$$R_l(\nu,z) = \frac{(2z)^\nu(-1)^{l+1}}{\Gamma(1/2)\Gamma(\nu-l-1/2)}\int_0^1 dx\,e^{-2zx}x^{\nu+l+1/2}\int_0^1 dt\,\frac{(1-t)^l}{l!}(1-xt)^{\nu-l-3/2}. \tag{A.12}$$

For $\mathrm{Re}\,\nu>1/2$, the remainders and their derivatives with respect to z fulfil estimates

$$\left|\frac{\partial^r}{\partial z^r}\left[z^{-\nu}R_l(\nu,z)\right]\right| \le \frac{2^{\mathrm{Re}\,\nu+r}}{c(\mathrm{Re}\,\nu,l)}\frac{\gamma(\mathrm{Re}\,\nu+l+r+3/2;2\,\mathrm{Re}\,z)}{\sqrt{\pi}\;l!\;|\Gamma(\nu-l-1/2)|}\left(\frac{1}{2\,\mathrm{Re}\,z}\right)^{\mathrm{Re}\,\nu+l+r+3/2} \tag{A.13}$$

with $c(\mathrm{Re}\,\nu,l) = \min\{\mathrm{Re}\,\nu-1/2,\,l+1\}$.

PROOF. ⌈ Consider the integral representation [24]

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)}\int_0^\pi d\phi\;e^{z\cos\phi}\sin^{2\nu}\phi\,.$$

which is valid for modified Bessel functions $I_\nu(z)$ with $\mathrm{Re}\,\nu>-1/2$. Note that the modified Bessel functions $I_\nu(z)$ have branch cuts along the negative real axis for $\nu\ne\pm n$ due to $z^\nu=\exp(\nu\log z)$ with the principal branch of log. Substitute $x=\sin^2(\phi/2)$ to get

$$e^{-z}I_\nu(z) = \frac{(2z)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)}\int_0^1 dx\;e^{-2zx}x^{\nu-1/2}(1-x)^{\nu-1/2}\,.$$

By the Taylor expansion with remainder

$$(1-x)^{\nu-1/2} = \sum_{\mu=0}^{l}\frac{1}{\mu!}\left[\frac{\partial^\mu}{\partial y^\mu}(1-y)^{\nu-1/2}\bigg|_{y=0}\right]x^\mu$$
$$+\,x^{l+1}\int_0^1 dt\,\frac{(1-t)^l}{l!}\left[\frac{\partial^{l+1}}{\partial y^{l+1}}(1-y)^{\nu-1/2}\bigg|_{y=xt}\right]$$
$$= \sum_{\mu=0}^{l}\frac{\Gamma(\nu+1/2)}{\mu!\,\Gamma(\nu-\mu+1/2)}(-1)^\mu x^\mu$$
$$+\,x^{l+1}\int_0^1 dt\,\frac{(1-t)^l}{l!}(-1)^{l+1}\frac{\Gamma(\nu+1/2)}{\Gamma(\nu-l-1/2)}(1-xt)^{\nu-l-3/2}\,,$$

one obtains the result

$$e^{-z}I_\nu(z) = \frac{(2z)^\nu}{\Gamma(1/2)}\sum_{\mu=0}^{l}\frac{(-1)^\mu}{\mu!\,\Gamma(\nu-\mu+1/2)}\int_0^1 dx\;e^{-2zx}x^{\nu+\mu-1/2} + R_l(\nu,z)$$

where the remainder $R_l(\nu,z)$ is given by Eq. (A.12). Now substitute $w=2zx$,

$$e^{-z}I_\nu(z) = \frac{(2z)^{\nu-1}}{\Gamma(1/2)}\sum_{\mu=0}^{l}\frac{(-1)^\mu}{\mu!\,\Gamma(\nu-\mu+1/2)}\int_0^{2z}dw\;e^{-w}\left(\frac{w}{2z}\right)^{\nu+\mu-1/2} + R_l(\nu,z)$$
$$= \frac{1}{\sqrt{2\pi z}}\sum_{\mu=0}^{l}\frac{(-1)^\mu}{\mu!}\frac{\gamma(\nu+\mu+1/2;2z)}{\Gamma(\nu-\mu+1/2)}\left(\frac{1}{2z}\right)^\mu + R_l(\nu,z)$$

where the incomplete Gamma function of Legendre

$$\gamma(\alpha;\zeta) = \int_0^\zeta e^{-t}t^{\alpha-1}dt \qquad \text{for} \qquad \mathrm{Re}\,\alpha>0$$

has been used for complex arguments ζ.

The derivatives of the functions $z^{-\nu}R_l(\nu,z)$ can easily be obtained from the integral representation (A.12),

$$\frac{\partial^r}{\partial z^r}\left[z^{-\nu}R_l(\nu,z)\right] = \frac{2^{\nu+r}(-1)^{l+r+1}}{\Gamma(1/2)\Gamma(\nu-l-1/2)}$$
$$\times\int_0^1 dx\;e^{-2zx}x^{\nu+l+r+1/2}\int_0^1 dt\,\frac{(1-t)^l}{l!}(1-xt)^{\nu-l-3/2}\,.$$

Consequently,

$$\left|\frac{\partial^r}{\partial z^r}\left[z^{-\nu}R_l(\nu,z)\right]\right| \le \frac{2^{\mathrm{Re}\,\nu+r}}{\sqrt{\pi}\,|\Gamma(\nu-l-1/2)|} \times \int_0^1 dx\; e^{-2(\mathrm{Re}\,z)x}\, x^{\mathrm{Re}\,\nu+l+r+1/2} \int_0^1 dt\,\frac{(1-t)^l}{l!}\,(1-xt)^{\mathrm{Re}\,\nu-l-3/2}\;.$$

Now $(1-xt)^{\mathrm{Re}\,\nu-l-3/2}$ can be estimated by 1 for $\mathrm{Re}\,\nu-l-3/2\ge 0$, whereas it is bounded by $(1-t)^{\mathrm{Re}\,\nu-l-3/2}$ for $\mathrm{Re}\,\nu-l-3/2<0$. Therefore, the t-integration yields for $\mathrm{Re}\,\nu>1/2$

$$\int_0^1 dt\,(1-t)^l(1-xt)^{\mathrm{Re}\,\nu-l-3/2} \le \frac{1}{c(\mathrm{Re}\,\nu,l)}\;.$$

Finally, one rewrites

$$\begin{aligned}\int_0^1 dx\; e^{-2(\mathrm{Re}\,z)x}\,x^{\mathrm{Re}\,\nu+l+r+1/2} &= \frac{1}{2\,\mathrm{Re}\,z}\int_0^{2\,\mathrm{Re}\,z} dw\; e^{-w}\left(\frac{w}{2\,\mathrm{Re}\,z}\right)^{\mathrm{Re}\,\nu+l+r+1/2}\\ &= \left(\frac{1}{2\,\mathrm{Re}\,z}\right)^{\mathrm{Re}\,\nu+l+r+3/2}\gamma(\mathrm{Re}\,\nu+l+r+3/2;2\,\mathrm{Re}\,z)\\ &\le \left(\frac{1}{2\,\mathrm{Re}\,z}\right)^{\mathrm{Re}\,\nu+l+r+3/2}\Gamma(\mathrm{Re}\,\nu+l+r+3/2) \qquad \text{for} \qquad \mathrm{Re}\,z>0\;. \quad \rfloor\end{aligned}$$

The replacement of the incomplete Gamma functions of Legendre $\gamma(k+\mu+3/2;2\beta_+)$ in (A.11) by "complete" Gamma functions $\Gamma(k+\mu+3/2)$ gives an additional remainder term which is exponentially small in β_+, provided that $\mu\le l$ and $k\le k_{max}$ for fixed l and k_{max}, and that $\beta_+=2\beta_j$ is sufficiently large. Its derivatives are exponentially small too. Thus the β_+ differentiations of (A.10) can be performed easily

$$\begin{aligned}\frac{\partial^{n+m-2\nu}}{\partial\beta_+^{n+m-2\nu}}\left[\frac{e^{-\beta_+}I_{k+1}(\beta_+)}{\beta_+^{k+1}}\,\beta_+^{2(k-\nu)}\right] &= \sum_{\mu=0}^{l}\frac{(-1)^\mu}{\mu!}\,\frac{2^{-\mu}}{\sqrt{2\pi}}\,\frac{\Gamma(k+\mu+3/2)}{\Gamma(k-\mu+3/2)}\,\frac{\Gamma(k-2\nu-\mu-1/2)}{\Gamma(k-n-m-\mu-1/2)}\,\beta_+^{k-n-m-\mu-3/2}\\ &\quad + O(\beta_+^{k-n-m-l-5/2})\;. \end{aligned} \tag{A.14}$$

Here, in order to bound the derivatives of the rest term, use has been made of Lemma A.1.

APPENDIX B. BOUNDS FOR CONVOLUTION INTEGRALS

The rigorous treatment of the renormalization group flow for weak coupling of Sect. 6.2 is completed by the proof of the bound (6.70).

Consider the convolution integrals $M_{n5}^{(j)}(z)$, $N_{55}^{(j)}(z)$ defined in Eqs. (6.62) and (6.63). They contain the remainder term $\widetilde{R}_j^5$ of expansion (6.57). I will prove the estimate (6.70), namely

$$\mathcal{N}_j^{-1}\,\frac{1}{k!}\,\frac{\partial^k}{\partial z^k}\left(\sum_{\substack{n=0\\n\ne1}}^{4}M_{n5}^{(j)}(z)+N_{55}^{(j)}(z)\right)\Bigg|_{z=0} = O(\beta_j^{k-5/2})\;. \tag{B.1}$$

For technical reasons, I expand $\widetilde{R}_j^5(z)=\rho_5^{(j)}z^5+\widetilde{R}_j^6(z)$, which yields the following representation

$$\sum_{\substack{n=0\\n\ne1}}^{4}M_{n5}^{(j)}(z)+N_{55}^{(j)}(z) = \sum_{\substack{n=0\\n\ne1}}^{5}(2-\delta_{n5})\,\rho_n^{(j)}\rho_5^{(j)}\,F_{n5}(\beta_j,z)+\sum_{\substack{n=0\\n\ne1}}^{5}M_{n6}^{(j)}(z)+N_{66}^{(j)}(z)\;. \tag{B.2}$$

The contributions of the F_{nm} terms to (B.1), given by the coefficients $F_{nm}^{(k)}(\beta_j)$, have already been estimated in App. A. The remaining coefficients can be bounded now for $0\le n\le 5$ by

$$\mathcal{N}_j^{-1}\left|\frac{1}{k!}\frac{\partial^k M_{n6}^{(j)}}{\partial z^k}(0)\right| \le O(\beta_j^{k-3-[n/2]})\quad,\quad \mathcal{N}_j^{-1}\left|\frac{1}{k!}\frac{\partial^k N_{66}^{(j)}}{\partial z^k}(0)\right| \le O(\beta_j^{k-6})\;, \tag{B.3}$$

where $[n/2]$ equals $n/2$ for even n and $(n-1)/2$ for odd n. Note that the case $n=1$ is excluded. The functions $M_{n6}^{(j)}(z)$, $N_{66}^{(j)}(z)$ are defined by the integrals ($\rho_0^{(j)}\equiv 1$, $\rho_1^{(j)}\equiv 0$)

$$M_{n6}^{(j)}(z) = \int_G dv\;\exp(-2\beta_j z_+ - 2\beta_- z_-)\;\rho_n^{(j)}\left[z_+^n\,\widetilde{R}_j^6(z_-)+\widetilde{R}_j^6(z_+)\,z_-^n\right] \tag{B.4a}$$

$$N_{66}^{(j)}(z) = \int_G dv\;\exp(-2\beta_j z_+ - 2\beta_j z_-)\;\widetilde{R}_j^6(z_+)\,\widetilde{R}_j^6(z_-)\;, \tag{B.4b}$$

and are obviously entire in the variable z. Due to Cauchy's estimate (6.58), $|\rho_n^{(j)}|\le O(\beta_j^{n/2})$, and the bound $|\widetilde{R}_j^6(z)|\le O(\beta_j^3)\,|z|^6$ for $z\in\overline{\Omega(\eta_j)}$, cp. (6.59), one gets for $p=O(\beta_j^{-1})$

$$\left|\frac{1}{k!}\frac{\partial^k M_{n6}^{(j)}}{\partial z^k}(0)\right| \le \frac{1}{p^k}\max_{|z|=p}|M_{n6}^{(j)}(z)| \le \frac{1}{p^k}\,O(\beta_j^{3+n/2})\max_{|\sin^2(\theta/2)|=p}|\widetilde{F}_{n6}(\theta)+\widetilde{F}_{6n}(\theta)| \tag{B.5a}$$

$$\left|\frac{1}{k!}\frac{\partial^k N_{66}^{(j)}}{\partial z^k}(0)\right| \le \frac{1}{p^k}\max_{|z|=p}|N_{66}^{(j)}(z)| \le \frac{1}{p^k}\,O(\beta_j^{6})\max_{|\sin^2(\theta/2)|=p}\widetilde{F}_{66}(\theta) \tag{B.5b}$$

with moments of the type

$$\widetilde{F}_{nm}(\theta) = \int_G dv\;\exp\left(-2\beta\,\mathrm{Re}\left(\sin^2\frac{\theta_+}{2}+\sin^2\frac{\theta_-}{2}\right)\right)\left|\sin^2\frac{\theta_+}{2}\right|^n\left|\sin^2\frac{\theta_-}{2}\right|^m\;. \tag{B.6}$$

The expressions $\sin^2(\theta_\pm/2)$ are given in Eq. (5.10). Suppose that n, m are even. Then we can calculate these moments explicitly by introducing the generating function

$$\tilde{F}(\beta,\beta',\gamma,\gamma';\theta) = \int_G dv\, \exp\Big(-2\beta\,\mathrm{Re}\sin^2\frac{\theta_+}{2} - 2\beta'\,\mathrm{Re}\sin^2\frac{\theta_-}{2} - 2\gamma\,\mathrm{Im}\sin^2\frac{\theta_+}{2} - 2\gamma'\,\mathrm{Im}\sin^2\frac{\theta_-}{2}\Big) \quad \text{(B.7)}$$

which is entire in the variables β, β', γ, and γ' for arbitrarily fixed θ. Let $\beta_\pm = \beta \pm \beta'$ and $\gamma_\pm = \gamma \pm \gamma'$ to obtain (cp. App. A)

$$\tilde{F}(\beta,\beta',\gamma,\gamma';\theta) = e^{-\beta_+}\sum_{\nu=0}^{\infty}\frac{\Gamma(\nu+3/2)}{\Gamma(3/2)}\frac{2^{\nu+1}}{(2\nu+1)!}\,\alpha_-(\beta_-,\gamma_-)^{2\nu}\,\frac{I_{\nu+1}(\alpha_+(\beta_+,\gamma_+))}{\alpha_+(\beta_+,\gamma_+)^{\nu+1}}$$
$$= 2\,e^{-\beta_+}\frac{I_1\big(\sqrt{\alpha_+(\beta_+,\gamma_+)^2+\alpha_-(\beta_-,\gamma_-)^2}\big)}{\sqrt{\alpha_+(\beta_+,\gamma_+)^2+\alpha_-(\beta_-,\gamma_-)^2}} \quad \text{(B.8)}$$

with the abbreviations

$$\alpha_+(\beta_+,\gamma_+) = \beta_+\,\mathrm{Re}\cos\frac{\theta}{2} + \gamma_+\,\mathrm{Im}\cos\frac{\theta}{2} \quad \text{(B.9a)}$$
$$\alpha_-(\beta_-,\gamma_-) = \beta_-\,\mathrm{Re}\sin\frac{\theta}{2} + \gamma_-\,\mathrm{Im}\sin\frac{\theta}{2}\,. \quad \text{(B.9b)}$$

From the simple identities

$$\Big|\sin^2\frac{\theta_\pm}{2}\Big|^n = \sum_{i=0}^{n/2}\binom{n/2}{i}\Big(\mathrm{Re}\sin^2\frac{\theta_\pm}{2}\Big)^{2i}\Big(\mathrm{Im}\sin^2\frac{\theta_\pm}{2}\Big)^{n-2i}$$

and

$$\frac{\partial^l}{\partial\beta^l} = \Big(\frac{\partial}{\partial\beta_+}+\frac{\partial}{\partial\beta_-}\Big)^l = \sum_{i=0}^{l}\binom{l}{i}\frac{\partial^i}{\partial\beta_+^i}\frac{\partial^{l-i}}{\partial\beta_-^{l-i}} \qquad \text{etc.}$$

one gets immediately

$$\tilde{F}_{nm}(\theta) = \sum_{\substack{i_+,i_-,j_+,j_-\ge 0\\ i_++i_-+j_++j_-=n+m}}\{\text{combinatorial factors}\} \times \partial^{i_+}_{\beta_+}\partial^{i_-}_{\beta_-}\partial^{j_+}_{\gamma_+}\partial^{j_-}_{\gamma_-}\tilde{F}(\beta,\beta',\gamma,\gamma';\theta)\Big|_{\substack{\beta_-=\gamma_+=\gamma_-=0\\ \beta_+=2\beta}}\,. \quad \text{(B.10)}$$

It remains to show that every derivative of the generating function $\tilde{F}$ gives a small factor β_j^{-1}. Therefore I carry out the differentiations with respect to β_- and γ_- explicitly and use, on the other hand, analyticity of $\tilde{F}$ in β_+, γ_+ to apply Cauchy's estimate

$$\partial^{i_-}_{\beta_-}\partial^{j_-}_{\gamma_-}\tilde{F}(\beta,\beta',\gamma,\gamma';\theta)\Big|_{\beta_-=\gamma_-=0} = \Big(\mathrm{Re}\sin\frac{\theta}{2}\Big)^{i_-}\Big(\mathrm{Im}\sin\frac{\theta}{2}\Big)^{j_-} \times e^{-\beta_+}\frac{\Gamma(l+3/2)}{\Gamma(3/2)}\frac{2^{l+1}}{(2l+1)!}(2l)!\,\frac{I_{l+1}(\alpha_+(\beta_+,\gamma_+))}{\alpha_+(\beta_+,\gamma_+)^{l+1}}\Big|_{i_-+j_-=2l}\,. \quad \text{(B.11)}$$

Because of (B.8) and (B.9b), the sum $i_- + j_-$ has to be even. Note that

$$\max_{|\sin^2(\theta/2)|=p=O(\beta_j^{-1})}\Big|\Big(\mathrm{Re}\sin\frac{\theta}{2}\Big)^{i_-}\Big(\mathrm{Im}\sin\frac{\theta}{2}\Big)^{j_-}\Big| = O(\beta_j^{-(i_-+j_-)/2})\,. \quad \text{(B.12)}$$

Furthermore,

$$\Big|\partial^{i_+}_{\beta_+}\partial^{j_+}_{\gamma_+}\Big[e^{-\beta_+}\frac{I_{l+1}(\alpha_+(\beta_+,\gamma_+))}{\alpha_+(\beta_+,\gamma_+)^{l+1}}\Big]\Big|_{\gamma_+=0,\beta_+=2\beta}\Big| \le$$
$$\le \frac{i_+!}{q^{i_+}}\frac{j_+!}{s^{j_+}}\max_{|\xi-2\beta|=q}\max_{|\zeta|=s}\Big|e^{-\xi}\frac{I_{l+1}(\alpha_+(\xi,\zeta))}{\alpha_+(\xi,\zeta)^{l+1}}\Big|$$
$$\le \frac{i_+!}{q^{i_+}}\frac{j_+!}{s^{j_+}}\Big(\max_{|\xi-2\beta|=q}\max_{|\zeta|=s}\Big|e^{-\alpha_+(\xi,\zeta)}\frac{I_{l+1}(\alpha_+(\xi,\zeta))}{\alpha_+(\xi,\zeta)^{l+1}}\Big|\Big) \times\Big(\max_{|\xi-2\beta|=q}\max_{|\zeta|=s}\Big|e^{-\xi+\alpha_+(\xi,\zeta)}\Big|\Big)$$
$$\le \frac{i_+!}{q^{i_+}}\frac{j_+!}{s^{j_+}}\,O(\beta_j^{-(l+1)-1/2}) \quad \text{(B.13)}$$

with $p = O(\beta_j^{-1})$, $q = O(\beta_j)$, and $s = O(\beta_j)$ chosen in such a way that $\alpha_+(\xi,\zeta)$ fulfils $|\alpha_+(\xi,\zeta)| = O(\beta_j)$ and $\exp(\,\mathrm{Re}[\alpha_+(\xi,\zeta)-\xi]) = O(1)$ for all ξ,ζ,θ with $|\xi-2\beta| = q$, $|\zeta| = s$, $|\sin^2(\theta/2)| = p$. Note that

$$\Big|e^{-\alpha_+}\frac{I_{l+1}(\alpha_+)}{\alpha_+^{l+1}}\Big| \le O(\beta_j^{-(l+1)-1/2})$$

is then a consequence of Lemma A.1. Thus one gets $(n, m = 0, 2, 4, 6)$

$$\max_{|\sin^2(\theta/2)|=p}\tilde{F}_{nm}(\theta) \le O(\beta_j^{-(i_-+j_-)/2})\,O(\beta_j^{-(i_++j_+)})\,O(\beta_j^{-(i_-+j_-)/2-3/2}) = O(\beta_j^{-(n+m)-3/2})\,.$$

Finally, the estimates (B.5) give

$$\Big|\frac{1}{k!}\frac{\partial^k M^{(j)}_{n6}}{\partial z^k}(0)\Big| \le O(\beta_j^{k-3-n/2-3/2}) \qquad \text{for} \qquad n = 0,2,4 \quad \text{(B.14a)}$$
$$\Big|\frac{1}{k!}\frac{\partial^k N^{(j)}_{66}}{\partial z^k}(0)\Big| \le O(\beta_j^{k-6-3/2})\,. \quad \text{(B.14b)}$$

In order to establish (B.3), use $\mathcal{N}_j^{-1} = O(\beta_j^{3/2})$ and the rough estimates

$$\tilde{F}_{3,6}(\theta) \le O(1)\,\tilde{F}_{2,6}(\theta) \quad , \quad \tilde{F}_{5,6}(\theta) \le O(1)\,\tilde{F}_{4,6}(\theta)$$

following from the integral representations (B.6) of the moments $\tilde{F}_{nm}(\theta)$. They yield the missing bounds for odd n, e.g.

$$\Big|\frac{1}{k!}\frac{\partial^k M^{(j)}_{36}}{\partial z^k}(0)\Big| \le O(\beta_j^{k-5}) \quad , \quad \Big|\frac{1}{k!}\frac{\partial^k M^{(j)}_{56}}{\partial z^k}(0)\Big| \le O(\beta_j^{k-6})\,. \quad \text{(B.14c)}$$

APPENDIX C. COMPUTATION OF THE EXPECTATION VALUES $\mathcal{E}_1$, $\mathcal{E}_2$

The expectation values $\mathcal{E}_1(\beta,z)$ and $\mathcal{E}_2(\beta,z)$ of (6.87) are calculated.

First, express the functions $\mathcal{E}_1(\beta,z)$ and $\mathcal{E}_2(\beta,z)$, defined in Eqs. (6.87), by the moments $\widetilde{F}_{nm}(\theta)$, Eq. (B.6),

$$\mathcal{E}_1(\beta,\sin^2\frac{\theta}{2}) = \frac{\widetilde{F}_{20}(\theta)+\widetilde{F}_{02}(\theta)}{2\,\widetilde{F}_{00}(\theta)} \quad , \quad \mathcal{E}_2(\beta,\sin^2\frac{\theta}{2}) = \frac{\widetilde{F}_{22}(\theta)}{\widetilde{F}_{00}(\theta)} .$$

By means of the formula (B.10), the moments $\widetilde{F}_{nm}$ are related to the generating function $\widetilde{F}$, Eqs. (B.7) and (B.8). The combinatorial factors in (B.10) can be obtained from the binomial theorem. The derivatives of $\widetilde{F}$ with respect to β_- and γ_- can be read off from (B.11). In contrast to the discussion of App. B, the remaining differentiations with respect to β_+ and γ_+ must be performed explicitly too. It is useful to apply the well known formula $I'_\nu(x) = I_{\nu\pm1}(x) \pm \frac{\nu}{x} I_\nu(x)$ in order to get the following decompositions

$$\frac{\partial}{\partial x}\frac{I_\nu(x)}{x^\nu} = \frac{I_{\nu+1}(x)}{x^\nu} \quad , \quad \frac{\partial^2}{\partial x^2}\frac{I_\nu(x)}{x^\nu} = \frac{I_\nu(x)}{x^\nu} - (2\nu+1)\frac{I_{\nu+1}(x)}{x^{\nu+1}} ,$$

where each term on the right hand sides is nonsingular for $x \to 0$. Then, as a result, the expectation values get the form ($\beta = \beta_j$)

$$\mathcal{E}_1(\beta,\sin^2\frac{\theta}{2}) = \frac{1}{4}\left[1+a^2+b^2\right] + \frac{I_2(2\beta a)}{2\beta a\, I_1(2\beta a)}\left\{\frac{1}{4}\left[-4\beta a^2 - 3(a^2+b^2)+c^2+d^2\right]\right\} \quad \text{(C.1a)}$$

$$\begin{aligned}\mathcal{E}_2(\beta,\sin^2\frac{\theta}{2}) = {} & \frac{1}{16}\left[1+6a^2+2b^2+(a^2+b^2)^2\right] + \frac{1}{8}\beta^{-1}\left[3(a^2+b^2)+c^2-d^2\right] + \\ & + \frac{5}{64}\beta^{-2}\left[3a^2+6b^2+2(c^2-d^2)\right] + \\ & + \frac{I_2(2\beta a)}{2\beta a\, I_1(2\beta a)}\left\{-\frac{1}{2}\beta a^2(1+a^2+b^2) - \frac{1}{8}\left[9a^2+3b^2+3(a^2+b^2)^2\right] - \right. \\ & \quad - \frac{1}{8}(1+a^2-b^2)(c^2-d^2) - \frac{1}{2}\beta^{-1}\left[3(a^2+b^2)+c^2-d^2\right] - \\ & \quad \left. - \frac{5}{16}\beta^{-2}\left[3a^2+6b^2+2(c^2-d^2)\right]\right\} + \\ & + \frac{I_3(2\beta a)}{(2\beta a)^2\, I_1(2\beta a)}\left\{\frac{15}{16}b^4 - \frac{5}{8}b^2(c^2-d^2) + \frac{3}{16}(c^2+d^2)^2\right\}\end{aligned} \quad \text{(C.1b)}$$

with the abbreviations

$$a = \frac{\partial\alpha_+}{\partial\beta_+} = \mathrm{Re}\cos\frac{\theta}{2} \quad , \quad b = \frac{\partial\alpha_+}{\partial\gamma_+} = \mathrm{Im}\cos\frac{\theta}{2}$$
$$c = \frac{\partial\alpha_-}{\partial\beta_-} = \mathrm{Re}\sin\frac{\theta}{2} \quad , \quad d = \frac{\partial\alpha_-}{\partial\gamma_-} = \mathrm{Im}\sin\frac{\theta}{2} . \quad \text{(C.2)}$$

In the small field region $\theta = O(\beta^{-1/4})$ we have $1-a = O(\beta^{-1/2})$, $a^2 = O(1)$, $b^2 = O(\beta^{-1})$, $c^2 = O(\beta^{-1/2})$, and $d^2 = O(\beta^{-1/2})$. According to the expansions

$$\frac{I_2(2\beta a)}{2\beta a\, I_1(2\beta a)} = \frac{1}{2\beta a}\left[1 - \frac{3}{4}\frac{1}{a}\beta^{-1} + \frac{3}{32}\frac{1}{a^2}\beta^{-2} + O(a^{-3}\beta^{-3})\right]$$
$$\frac{I_3(2\beta a)}{(2\beta a)^2\, I_1(2\beta a)} = \left(\frac{1}{2\beta a}\right)^2\left[1 + O(a^{-1}\beta^{-1})\right] ,$$

$\mathcal{E}_1, \mathcal{E}_2$ can be expressed as

$$\mathcal{E}_i(\theta) = \left(\frac{1}{4}\left[(1-a)^2+b^2\right]\right)^i + O(\beta^{-1/2-i}) = \left(\frac{1}{16}\left|\sin^4\frac{\theta}{2}\right|\right)^i + O(\beta^{-1/2-i}) .$$

APPENDIX D. COMPUTATION OF EXPANSION COEFFICIENTS

The expansion coefficients $L^{(k)}_{nm}(\beta) = (2 - \delta_{nm})\, F^{(k)}_{nm}(\beta)/F^{(0)}_{nm}(\beta)$ *are needed in order to obtain the contributions from irrelevant interactions to the running coupling constant recursion relations* (4.12), *cp. Sect.* 5. *In this appendix, they are given explicitly for* $k = 0, \ldots, 5$ *and* $n + m = 0, \ldots, 5$ *with* $n, m \neq 1$, *allowing to compute the set of finite order recursion relations* (4.24) *up to contributions with a degree of irrelevance* $\delta = 4$.

As has been shown in App. A, the expansion coefficients $F^{(k)}_{nm}(\beta)$ and $L^{(k)}_{nm}(\beta)$, respectively, can be represented as derivatives of the generating function $F(\beta, \beta'; z)$. With the help of an analytic computer program, for example REDUCE [34], their explicit calculation is an easy task. In the following, I will state the coefficients needed for calculations up to the fourth order in the degree of irrelevance. The expressions include modified Bessel functions and are organized in such a way that all individual terms remain finite in the limit $\beta \to 0$. Unfortunately, one can not immediately read off the leading behaviour of the coefficients for large β-values because of cancellations between the different terms.

Note that it is possible to expand these expansion coefficients in absolutely convergent series (including incomplete Gamma functions of Legendre) for all $\beta > 0$. However, series expansions in powers of β will have only a radius of convergence given by $\frac{1}{2} j_{1,1} = 1.9155\ldots$ The reason for this is simply the fact that the expansion coefficients $L^{(k)}_{nm}(\beta)$ are analytic for all complex β with exception of the nontrivial zeroes of the modified Bessel function $I_1(2\beta)$. These zeros lie on the imaginary axis and the smallest one is $2\beta = i\, j_{1,1}$.
On the other hand, power series expansions in β^{-1} are only asymptotic.

Expansion Coefficients for $k = 0$

$$L^{(0)}_{00}(\beta) = 1$$

$$L^{(0)}_{02}(\beta) = \frac{1}{2}\left(-[4\beta + 3]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + 2\right)$$

$$L^{(0)}_{03}(\beta) = \frac{1}{4}\left(6\beta\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - [8\beta + 9]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + 4\right)$$

$$L^{(0)}_{04}(\beta) = \frac{1}{8}\left([24\beta + 15]\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - [16\beta + 24]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + 8\right)$$

$$L^{(0)}_{22}(\beta) = \frac{1}{16}\left([24\beta + 15]\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - [16\beta + 24]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + 8\right)$$

$$L^{(0)}_{05}(\beta) = \frac{1}{16}\left(-30\beta\frac{I_4(2\beta)}{I_1(2\beta)}(2\beta)^{-3} + [72\beta + 75]\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - \right.$$
$$\left. - [32\beta + 60]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + 16\right)$$

$$L^{(0)}_{23}(\beta) = \frac{1}{16}\left(-30\beta\frac{I_4(2\beta)}{I_1(2\beta)}(2\beta)^{-3} + [72\beta + 75]\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - \right.$$
$$\left. - [32\beta + 60]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + 16\right)$$

Expansion Coefficients for $k = 1$

$$L^{(1)}_{00}(\beta) = -2\beta^2\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1}$$

$$L^{(1)}_{02}(\beta) = \frac{1}{4}\left(-[8\beta^2 + 8\beta + 4]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + 4\beta + 1\right)$$

$$L^{(1)}_{03}(\beta) = \frac{1}{8}\left(-[16\beta^2 + 18\beta + 12]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + 8\beta + 3\right)$$

$$L^{(1)}_{04}(\beta) = \frac{1}{16}\left(-30\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - [32\beta^2 + 40\beta + 27]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + 16\beta + 8\right)$$

$$L^{(1)}_{22}(\beta) = \frac{1}{32}\left([64\beta + 50]\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - [32\beta^2 + 40\beta + 59]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + 16\beta + 8\right)$$

$$L^{(1)}_{05}(\beta) = \frac{1}{32}\left(140\beta\frac{I_4(2\beta)}{I_1(2\beta)}(2\beta)^{-3} - [10\beta + 150]\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - \right.$$
$$\left. - [64\beta^2 + 88\beta + 55]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + 32\beta + 20\right)$$

$$L^{(1)}_{23}(\beta) = \frac{1}{32}\left(-100\beta\frac{I_4(2\beta)}{I_1(2\beta)}(2\beta)^{-3} + [182\beta + 210]\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - \right.$$
$$\left. - [64\beta^2 + 88\beta + 151]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + 32\beta + 20\right)$$

Expansion Coefficients for $k = 2$

$$L^{(2)}_{00}(\beta) = \frac{1}{2}\left(-4\beta^2\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + \beta^2\right)$$

$$L^{(2)}_{02}(\beta) = \frac{1}{8}\left(-[8\beta^3 + 14\beta^2 + 16\beta + 4]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + 4\beta^2 + 4\beta + 1\right)$$

$$L^{(2)}_{03}(\beta) = \frac{1}{16}\left(-24\beta\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - [16\beta^3 + 26\beta^2 + 24\beta + 12]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + \right.$$
$$\left. + 8\beta^2 + 7\beta + 3\right)$$

$$L^{(2)}_{04}(\beta) = \frac{1}{64}\left(-[192\beta + 336]\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - [64\beta^3 + 96\beta^2 + 64\beta - 60]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + \right.$$
$$\left. + 32\beta^2 + 24\beta - 1\right)$$

$$L^{(2)}_{22}(\beta) = \frac{1}{128}\left([320\beta + 304]\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - [64\beta^3 + 96\beta^2 + 192\beta + 292]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + \right.$$
$$\left. + 32\beta^2 + 24\beta + 63\right)$$

$$L_{05}^{(2)}(\beta) = \frac{1}{128}\left(1120\beta\frac{I_4(2\beta)}{I_1(2\beta)}(2\beta)^{-3} - [920\beta + 1680]\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - \right.$$
$$\left. - [128\beta^3 + 176\beta^2 + 30\beta - 460]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + 64\beta^2 + 40\beta - 45\right)$$

$$L_{23}^{(2)}(\beta) = \frac{1}{128}\left(-608\beta\frac{I_4(2\beta)}{I_1(2\beta)}(2\beta)^{-3} + [904\beta + 1104]\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - \right.$$
$$\left. - [128\beta^3 + 176\beta^2 + 414\beta + 740]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + 64\beta^2 + 40\beta + 147\right)$$

Expansion Coefficients for $k = 3$

$$L_{00}^{(3)}(\beta) = \frac{1}{6}\left(-[2\beta^4 + 12\beta^2]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + 3\beta^2\right)$$

$$L_{02}^{(3)}(\beta) = \frac{1}{24}\left(-[8\beta^4 + 24\beta^3 + 36\beta^2 + 48\beta]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + 4\beta^3 + 9\beta^2 + 12\beta\right)$$

$$L_{03}^{(3)}(\beta) = \frac{1}{48}\left(-144\beta\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - [16\beta^4 + 42\beta^3 + 60\beta^2 + 36\beta]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + \right.$$
$$\left. + 8\beta^3 + 15\beta^2 + 15\beta\right)$$

$$L_{04}^{(3)}(\beta) = \frac{1}{96}\left(-[576\beta + 792]\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - \right.$$
$$- [32\beta^4 + 72\beta^3 + 75\beta^2 - 48\beta - 324]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} +$$
$$\left. + 16\beta^3 + 24\beta^2 + 12\beta - 48\right)$$

$$L_{22}^{(3)}(\beta) = \frac{1}{192}\left([576\beta + 648]\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - \right.$$
$$- [32\beta^4 + 72\beta^3 + 171\beta^2 + 336\beta + 540]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} +$$
$$\left. + 16\beta^3 + 24\beta^2 + 60\beta + 108\right)$$

$$L_{05}^{(3)}(\beta) = \frac{1}{384}\left(4320\beta\frac{I_4(2\beta)}{I_1(2\beta)}(2\beta)^{-3} - [5040\beta + 7920]\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - \right.$$
$$- [128\beta^4 + 240\beta^3 + 78\beta^2 - 930\beta - 3240]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} +$$
$$\left. + 64\beta^3 + 72\beta^2 - 45\beta - 480\right)$$

$$L_{23}^{(3)}(\beta) = \frac{1}{384}\left(-2592\beta\frac{I_4(2\beta)}{I_1(2\beta)}(2\beta)^{-3} + [3312\beta + 4176]\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - \right.$$
$$- [128\beta^4 + 240\beta^3 + 654\beta^2 + 1446\beta + 2664]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} +$$
$$\left. + 64\beta^3 + 72\beta^2 + 243\beta + 492\right)$$

Expansion Coefficients for $k = 4$

$$L_{00}^{(4)}(\beta) = \frac{1}{24}\left(-[12\beta^4 + 48\beta^2]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + \beta^4 + 12\beta^2\right)$$

$$L_{02}^{(4)}(\beta) = \frac{1}{48}\left(-[4\beta^5 + 19\beta^4 + 48\beta^3 + 60\beta^2 + 96\beta - 24]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + \right.$$
$$\left. + 2\beta^4 + 8\beta^3 + 16\beta^2 + 24\beta - 6\right)$$

$$L_{03}^{(4)}(\beta) = \frac{1}{192}\left(-864\beta\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - \right.$$
$$- [16\beta^5 + 66\beta^4 + 144\beta^3 + 168\beta^2 - 144]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} +$$
$$\left. + 8\beta^4 + 27\beta^3 + 48\beta^2 + 36\beta - 36\right)$$

$$L_{04}^{(4)}(\beta) = \frac{1}{768}\left(-[6912\beta + 8352]\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - \right.$$
$$- [64\beta^5 + 224\beta^4 + 384\beta^3 + 72\beta^2 - 1536\beta - 4320]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} +$$
$$\left. + 32\beta^4 + 88\beta^3 + 111\beta^2 - 96\beta - 732\right)$$

$$L_{22}^{(4)}(\beta) = \frac{1}{1536}\left([5376\beta + 7008]\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - \right.$$
$$- [64\beta^5 + 224\beta^4 + 640\beta^3 + 1544\beta^2 + 3072\beta + 5280]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} +$$
$$\left. + 32\beta^4 + 88\beta^3 + 239\beta^2 + 544\beta + 1028\right)$$

$$L_{05}^{(4)}(\beta) = \frac{1}{1536}\left(18240\beta\frac{I_4(2\beta)}{I_1(2\beta)}(2\beta)^{-3} - [27840\beta + 41760]\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - \right.$$
$$- [128\beta^5 + 368\beta^4 + 350\beta^3 - 1368\beta^2 - 7440\beta - 19680]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} +$$
$$\left. + 64\beta^4 + 136\beta^3 + 43\beta^2 - 795\beta - 3180\right)$$

$$L_{23}^{(4)}(\beta) = \frac{1}{1536}\left(-14016\beta\frac{I_4(2\beta)}{I_1(2\beta)}(2\beta)^{-3} + [15936\beta + 20448]\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - \right.$$
$$- [128\beta^5 + 368\beta^4 + 1118\beta^3 + 2952\beta^2 + 6672\beta + 12576]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} +$$
$$\left. + 64\beta^4 + 136\beta^3 + 427\beta^2 + 1077\beta + 2292\right)$$

EXPANSION COEFFICIENTS FOR $k = 5$

$$L^{(5)}_{00}(\beta) = \frac{1}{60}\Big(-\big[\beta^6 + 36\beta^4 + 120\beta^2\big]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + 4\beta^4 + 30\beta^2\Big)$$

$$L^{(5)}_{02}(\beta) = \frac{1}{480}\Big(-\big[8\beta^6 + 56\beta^5 + 200\beta^4 + 480\beta^3 + 480\beta^2 + 960\beta - 480\big]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + \\ + 4\beta^5 + 25\beta^4 + 80\beta^3 + 140\beta^2 + 240\beta - 120\Big)$$

$$L^{(5)}_{03}(\beta) = \frac{1}{960}\Big(-5760\beta\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - \\ - \big[16\beta^6 + 98\beta^5 + 312\beta^4 + 600\beta^3 + 480\beta^2 - 720\beta - 1440\big]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + \\ + 8\beta^5 + 43\beta^4 + 120\beta^3 + 180\beta^2 + 60\beta - 360\Big)$$

$$L^{(5)}_{04}(\beta) = \frac{1}{3840}\Big(-\big[46080\beta + 50400\big]\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - \big[64\beta^6 + 336\beta^5 + 886\beta^4 + \\ + 960\beta^3 - 2160\beta^2 - 13440\beta - 29520\big]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + \\ + 32\beta^5 + 144\beta^4 + 320\beta^3 + 165\beta^2 - 1440\beta - 5280\Big)$$

$$L^{(5)}_{22}(\beta) = \frac{1}{7680}\Big(\big[30720\beta + 45600\big]\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - \big[64\beta^6 + 336\beta^5 + 1206\beta^4 + \\ + 3520\beta^3 + 8720\beta^2 + 17280\beta + 31920\big]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + \\ + 32\beta^5 + 144\beta^4 + 480\beta^3 + 1325\beta^2 + 3040\beta + 6080\Big)$$

$$L^{(5)}_{05}(\beta) = \frac{1}{7680}\Big(81600\beta\frac{I_4(2\beta)}{I_1(2\beta)}(2\beta)^{-3} - \big[170400\beta + 252000\big]\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - \\ - \big[128\beta^6 + 560\beta^5 + 1038\beta^4 - 1150\beta^3 - \\ - 14640\beta^2 - 54000\beta - 128400\big]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + \\ + 64\beta^5 + 232\beta^4 + 315\beta^3 - 935\beta^2 - 6825\beta - 21600\Big)$$

$$L^{(5)}_{23}(\beta) = \frac{1}{7680}\Big(-91200\beta\frac{I_4(2\beta)}{I_1(2\beta)}(2\beta)^{-3} + \big[94560\beta + 122400\big]\frac{I_3(2\beta)}{I_1(2\beta)}(2\beta)^{-2} - \\ - \big[128\beta^6 + 560\beta^5 + 1998\beta^4 + 6170\beta^3 + \\ + 16560\beta^2 + 38160\beta + 73200\big]\frac{I_2(2\beta)}{I_1(2\beta)}(2\beta)^{-1} + \\ + 64\beta^5 + 232\beta^4 + 795\beta^3 + 2365\beta^2 + 6075\beta + 13200\Big)$$

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