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SUPERMEMBRANES – A FOND FAREWELL? †

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Supermembranes have been proposed as models for elementary particles. One of the main reasons for the recent interest in these theories is that supermembranes, unlike string theories, can live in an eleven-dimensional supergravity background [1,2]. Thus they can possibly serve as a basis for a unified theory of elementary particles different from string theory, which could have maximally extended supergravity as its low-energy approximation. In this contribution, we will review some of the recent developments with special emphasis on the supermembrane in eleven dimensions, and on the topics that have been the focus of our own recent work [3,4,5]. Since the theory of supermembranes now appears to have entered a more tranquil stage of its development, it is perhaps also appropriate to offer some comments concerning their future.

Supermembranes, as they are presently known, are based on Green-Schwarz type actions. In their original form the Green-Schwarz action provided just another description of superstring theory. Some time ago, it was shown that such actions can also be formulated for supersymmetric p-branes, where $p = 0, 1, \ldots$ defines the spatial dimension of the "brane" [1,6,7,8]. Hence, for $p = 0$ we have a superparticle, for $p = 1$ the superstring, for $p = 2$ the supermembrane, and so on. The dimension d of space-time in which the superbranes can live is very restricted [9]. While d can take all positive integer values for $p = 0$, the only other possible values for (p, d) are (1,3), (1,4), (1,6), (1,10), (2,4) (2,5), (2,7), (2,11), (3,6), (3,8), (4,9) and (5,10). These restrictions arise from the fact that the action contains a Wess-Zumino-Witten term [10], whose supersymmetry depends sensitively on the dimension of space-time. If the coefficient of this term takes a particular value then the action has an

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additional fermionic gauge symmetry, the so-called κ-symmetry. This symmetry is necessary to ensure the matching of the bosonic and fermionic degrees of freedom on the mass shell. As the number of spinor components grows much more rapidly with the dimension than the number of vector (or tensor) components, it is also very plausible that there should be a maximal dimension in which super-p-branes can exist. In the remainder we will be mostly concerned with the maximally extended theory which corresponds to $(p, d) = (2, 11)$.

Much of our work is based on the observation that supermembranes can be regarded as the $N \to \infty$ limit of certain matrix models in supersymmetric quantum mechanics with $SU(N)$ invariance [3]. In this way, it becomes possible to regularize the theory at a non-perturbative level and to study its properties in a rigorous fashion. Already at this point, we would like to emphasize that these matrix models provide the only supersymmetric regularization which is presently available. Notwithstanding the numerous open problems that remain, especially concerning the limit where the "cutoff" is removed, we believe that this approach has led to new and important information about the supermembrane. In particular, using this approach, we have shown that the spectrum of the regulated supermembrane is continuous [4]. Furthermore, we have studied the (classical) supermembrane Lorentz algebra in the finite-N approximation [5], paving the way for a rigorous quantum-mechanical treatment.

As there already several extensive reviews of supermembrane theory [2], we here only sketch some of the basic results. In the notation and conventions of [3] the supermembrane Lagrangian reads

$$\mathcal{L} = -\sqrt{-g(X,\theta)} - \epsilon^{ijk}\left[\tfrac{1}{2}\,\partial_i X^\mu(\partial_j X^\nu + \bar\theta\Gamma^\nu\partial_j\theta) + \tfrac{1}{6}\,\bar\theta\Gamma^\mu\partial_i\theta\,\bar\theta\Gamma^\nu\partial_j\theta\right]\bar\theta\Gamma_{\mu\nu}\partial_k\theta, \quad (1)$$

where $X^\mu(\zeta)$ and $\theta(\zeta)$ denote the superspace coordinates of the membrane parametrized in terms of world-tube parameters ζ^i $(i = 0, 1, 2)$. The fermionic coordinates θ transform as $d = 11$ spinors and have thus 32 components. The gamma matrices are denoted by Γ^μ; gamma matrices with more than one index denote antisymmetrized products of gamma matrices in the usual fashion. The metric $g_{ij}(X,\theta)$ is the induced metric on the world tube,

$$g_{ij} = \left(\partial_i X^\mu + \bar\theta\Gamma^\mu\partial_i\theta\right)\,\left(\partial_j X^\nu + \bar\theta\Gamma^\nu\partial_j\theta\right)\,\eta_{\mu\nu}, \quad (2)$$

where $\eta_{\mu\nu}$ is the flat $d = 11$ Minkowski metric. It is easy to see that g_{ij}, and therefore the first term in (1), is invariant under space-time supersymmetry. If $d = 4, 5, 7$ or 11 the second term proportional to ϵ^{ijk} is also supersymmetric (up to a total divergence). Furthermore, the action is invariant under κ-symmetry for these values of d.

In order to study the quantum-mechanical properties of the supermembrane we pass to the light-cone gauge. First we define light-cone coordinates

$$X^\mu = \begin{cases} X^\pm = \dfrac{1}{\sqrt{2}}(X^{10} \pm X^0) \\ X^a \qquad (a = 1, \ldots, 9) \end{cases} \quad (3)$$

In the light-cone gauge we are left with the transverse coordinates $\vec{X}$ and corresponding momenta $\vec{P}$, which transform as vectors under the $SO(9)$ group of transverse rotations. Only sixteen spinor components θ remain, which transform as $SO(9)$ spinors. Furthermore we have the centre-of-mass (CM) momentum P_0^+ in the direction associated with the CM coordinate X_0^- (the remaining modes in X^- are dependent), while the CM momentum P_0^- is equal to minus the supermembrane Hamiltonian.

The CM transverse coordinates and momenta and the CM spinors are defined by

$$\vec{P}_0 = \int \mathrm{d}^2\sigma\, \vec{P}, \qquad \vec{X}_0 = \int \mathrm{d}^2\sigma \sqrt{w(\sigma)}\, X(\sigma), \qquad \theta_0 = \int \mathrm{d}^2\sigma \sqrt{w(\sigma)}\, \theta(\sigma). \tag{4}$$

where σ^r $(r = 1, 2)$ are (space-like) coordinateswhich parametrize the membrane, and we employ a density $\sqrt{w(\sigma)}$ which is normalized according to

$$\int \mathrm{d}^2\sigma\, \sqrt{w(\sigma)} = 1. \tag{5}$$

The supermembrane theory is now expressed in terms of the various CM coordinates and momenta and the "oscillatory" modes contained in $\vec{X}$, $\vec{P}$ and θ. The Hamiltonian takes the following form

$$H = \frac{\vec{P}_0^{\,2}}{2P_0^+} + \frac{\mathcal{M}^2}{2P_0^+}, \tag{6}$$

where $\mathcal{M}$ is the supermembrane mass operator, which does *not* depend on any of the CM coordinates or momenta. An explicit expression for $\mathcal{M}^2$ will be given shortly.

The gauge conditions adopted above leave a residual reparametrization invariance consisting of so-called area-preserving transformations. They are defined by

$$\sigma^r \to \sigma^r + \xi^r(\sigma) \qquad \text{with} \qquad \partial_r \left(\sqrt{w(\sigma)}\, \xi^r(\sigma)\right) = 0. \tag{7}$$

The general solution of (7) can be decomposed into co-exact and harmonic vector fields. For a membrane of genus g there are precisely $2g$ independent harmonic vectors. Furthermore, there can be homotopically nontrivial transformations; these will not be considered in what follows. The co-exact components are parametrized in terms of globally defined functions $\xi(\sigma)$,

$$\xi^r(\sigma) = \frac{\epsilon^{rs}}{\sqrt{w(\sigma)}}\, \partial_s \xi(\sigma). \tag{8}$$

and generate an *invariant* subgroup, which we denote by G. In the following, we will restrict our attention to this invariant subgroup when referring to area-preserving transformations. The commutator of two infinitesimal G-transformations characterized by functions ξ_1 and ξ_2 yields a similar transformation characterized by

$$\xi_3 = \{\xi_2, \xi_1\}, \tag{9}$$

where the bracket $\{A,B\}$ for two functions $A(\sigma)$ and $B(\sigma)$ is defined by [11,12]

$$\{A,B\}(\sigma) \equiv \frac{1}{\sqrt{w(\sigma)}}\epsilon^{rs}\partial_r A(\sigma)\partial_s B(\sigma). \tag{10}$$

It is straightforward to show that (10) satisfies the Jacobi identity. According to (9) the bracket is related to the structure constants of the group G. This relationship will be made more precise in a moment.

As it turns out, the supermembrane moving in a d-dimensional space-time can be regarded as a limiting case of certain models in supersymmetric quantum mechanics in $d-2$ dimensions. As explained above, this observation plays a central role in what follows [3]. In order to derive this result one expands all the coordinates and momenta into a complete orthonormal basis of functions consisting of the constant function 1 and functions $Y^A(\sigma)$ (where $A = 1, 2, \ldots, \infty$). The coefficient of 1 represents the CM value, so we have

$$\vec{X}(\sigma) = \vec{X}_0 + \sum_{A=1}^{\infty} \vec{X}_A Y^A(\sigma), \tag{11}$$

and similar expansions for all other quantities of interest such as the momenta or the fermionic coordinates.

The bracket $\{Y^A, Y^B\}$ is again expressible in terms of the basis functions, so we may write

$$\{Y^A, Y^B\} = f^{AB}{}_C Y^C, \tag{12}$$

where the constants $f^{AB}{}_C$ can be regarded as the structure constants of the infinite-dimensional group G. Other tensors related to the diffeomorphisms generated by harmonic vectors and tensors needed for the Lorentz algebra generators were defined in [5]. After these decompositions, the supermembrane mass operator takes the form

$$\mathcal{M}^2 = \left(P_a^A\right)^2 + \tfrac{1}{2}\left(f_{ABC}X_a^B X_b^C\right)^2 - i f_{ABC}\,\theta^A\gamma^a X_a^B\theta^C. \tag{13}$$

We stress once more that this expression is independent of the CM coordinates and momenta. Furthermore we have adopted a somewhat different notation for the spinor coordinates, which are regarded as real $SO(9)$ spinors. Corresponding to the original $d=11$ spinors, they have have been rescaled with a factor proportional to $(P_0^+)^{1/2}$. The gamma matrices are chosen accordingly and indices a, b run from 1 to 9.

For completeness we give the supersymmetry charge associated with (13)

$$Q = \left(P_a^A\gamma^a + \tfrac{1}{2}f_{ABC}X_a^B X_b^C\,\gamma^{ab}\right)\theta^A. \tag{14}$$

Observe that this charge is a sixteen-component spinor. It satisfies the supersymmetry algebra,

$$\left[Q,\mathcal{M}^2\right] = \text{constraints}, \qquad \{Q_\alpha, Q_\beta\} = \delta_{\alpha\beta}\,\mathcal{M}^2 + \text{constraints}, \tag{15}$$

where the constraints are of the form $\varphi_A \approx 0$, with

$$\varphi_A = f_{ABC}\left(\vec{X}^B.\vec{P}^C - \tfrac{i}{2}\theta^B\theta^C\right). \tag{16}$$

These constraints, which imply the invariance of the wave function under the action of the group G, follow from the requirement that the light-cone coordinate X^- be a globally defined function on the membrane. Apart from its CM value X^- is then determined in terms of the other coordinates and momenta, which is only possible if $\varphi^A \approx 0$.

If we replace G by a finite group, then (13) defines the Hamiltonian of a supersymmetric quantum-mechanical system. These models have been discussed in the literature [14] and follow from dimensional reduction of pure supersymmetric Yang-Mils theories. In the limit to the infinite-dimensional group G we thus recover the supermembrane. This is the result quoted above. It enables us to regularize the supermembrane in a supersymmetric way by considering a limiting procedure based on a sequence of groups whose limit yields the group G. Since the precise meaning of this "limit" is somewhat subtle (see below), let us first note the following result. For spherical and toroidal membranes, there exists a basis of functions $Y^A(\sigma)$ such that the associated structure constants f_{ABC} are the $N \to \infty$ limit of $SU(N)$ structure constants (for toroidal membranes, there are, of course, additional area-preserving diffeomorphisms generated by the two harmonic vectors). To be specific, the following relations exist for the $SU(N)$ structure constants [11,12,13,5],

$$\begin{aligned} f^{ABC}_{\mathrm{SU(N)}} &= f^{ABC}_{\mathrm{sphere}} + \mathrm{O}\left(\frac{1}{N^2}\right), \\ f^{ABC}_{\mathrm{SU(N)}} &= f^{ABC}_{\mathrm{torus}} + \mathrm{O}\left(\frac{1}{N^2}\right), \end{aligned} \tag{17}$$

where it is important that the $SU(N)$ structure constants on the left-hand sides of these equations are defined in a different $SU(N)$ basis, and therefore not the same. Although these bases are related by a similarity transformation for any finite N, this is no longer true for $N = \infty$ (as is also suggested by the non-equivalence of area-preserving diffeomorphisms on the sphere and the torus).

There has been some discussion in the recent literature on the precise meaning of the $N \to \infty$ limit, especially in connection with spherical and toroidal membranes. Although this limit can be taken "pointwise" in (17), one can by no means infer that $SU(\infty)$ really coincides with the group of area-preserving diffeomorphisms on membranes of a particular topology. The important point here is that these diffeomorphisms, and hence the vector fields $\xi^r(\sigma)$ that generate them, are by definition C^∞-functions. Now, it is not difficult to construct sequences of area-preserving maps corresponding to elements of $SU(N)$ which are C^∞ for any finite N and which, in the limit $N \to \infty$, become singular (i.e. discontinuous or non-differentiable). Hence the group $SU(\infty)$ also contains singular area-preserving maps.

Such singular maps do not even have to preserve the topology of the membrane, so that it is quite meaningless to associate $SU(\infty)$ with any particular topology (see also [15]). Rather, the limit $N \to \infty$ must be interpreted as a "weak" limit: it only makes sense for matrix elements of ($SU(N)$-invariant) operators between two fixed states (the existence of such limits may require non-trivial renormalizations as $N \to \infty$). One can even go farther and adopt the point of view that the $SU(N)$ approximation *defines* the membrane theory, and therefore takes care of *all* topologies. Then there would actually be no need to sum over different topologies, just as in lattice gauge theories, where there is also no need to sum over "instanton sectors" of different topological charge as these are automatically included. On the other hand, this point of view ignores the fact that, for membranes of nontrival topolgy, there are $2g$ additional constraints which require the wave function to be invariant under area-preserving diffeomorphisms associated with the harmonic vectors on the membrane surface. It is not clear how to incorporate such constraints into the finite-N approximation (see appendix A of [5] for a further discussion of this). Of course, one may certainly prefer other approaches to make sense of the theory, but we remind the reader that so far all other proposals (e.g. those based on semi-classical approximations) are essentially perturbative and therefore prone to miss important qualitative features.

The structure of the Hamiltonian (6) shows that the wave functions for the supermembrane now factorize into a wave function pertaining to the CM modes and a wave function of the supersymmetric quantum-mechanical system that describes the other modes. It follows also from (6) that the fermionic CM modes, which are the generators of a sixteen-dimensional Clifford algebra, should be realized on the CM wave functions. Therefore the possible wave functions pertaining to the CM modes constitute a *massless* $d = 11$ supermultiplet. In terms of its $SO(9)$ representation content, this multiplet consists of $\mathbf{44} \oplus \mathbf{84}$ bosonic and $\mathbf{128}$ fermionic states.

Whether or not the supermembrane states can be massless is determined by the second part of the wave function, which must be an eigenfunction of the operator (13). This shows that it is not possible to obtain nontrivial information regarding the mass spectrum by studying a collapsed membrane, an approximation that is sometimes used in order to eliminate the nonlinearities of the supermembrane Hamiltonian (see, for instance, [16]). Furthermore, in view of the supersymmetry algebra (15), the mass can only be zero if the state is annihilated by the supersymmetry charge (14), and vice versa. Therefore it follows that, as for their dependence on the modes other than the CM modes, massless supermembrane states correspond to supersymmetric singlets. After combination with the CM modes we thus have a massless representation of $d = 11$ supersymmetry. If the states are not massless they will, after being combined with the CM part of the wave function, constitute massive representations of $d = 11$ supersymmetry.

For $d = 11$ one would hope that there is precisely one supersymmetric vacuum, so that the corresponding massless $d = 11$ supermultiplet will be the supergravity multiplet. For other dimensionalities, the smallest supermultiplet does not contain spin-2. In order to have a massless supermultiplet that contains the graviton, there must be a further degeneracy of the groundstate associated with nonzero angular momentum. It is not known how this additional degeneracy would effect the analysis of [16].

The nonlinear structure of the mass operator (13) makes it very difficult to prove or disprove the existence of massless states for the supermembrane, despite the fact that supersymmetry allows us to take the "square root" of the mass-operator and thereby reduce the second-order (functional) differential equation to a first order one. Below, we will demonstrate by means of a toy model introduced in [4], that even under most simplified circumstances, it is very difficult to settle this issue. Therefore, contrary to some claims in the literature, the question of massless states remains open. Among the few results that can be stated with some certainty, is a theorem that follows from the positivity of the bosonic part of the mass operator (13) [3]. According to this theorem there is no massless state whose wave function factorizes into a bosonic and a fermionic wave function such that one of these, or both, is invariant under either G or under the $SO(9)$ group of transverse rotations. This result shows that the structure of these groundstate wave functions (if they exist) must be very complicated (see [3] for a further discussion).

Having stressed that the problem of finding massless states is more difficult than one might have anticipated, we would like to add some comments on alternative proposals to study the nature of the groundstate. One is based on the Witten index [17],

$$\Delta \equiv \mathrm{Tr}\left[(-)^{\mathrm{F}}\, \mathrm{e}^{-\frac{1}{2}\beta\mathcal{M}^2}\right]. \tag{18}$$

and motivated by the hope that, even though it may not be possible to derive the groundstate wave functions explicitly, there may still be indirect methods to count their number. As is well-known, if $\Delta \neq 0$ there must be an unequal number of massless fermionic or bosonic states. At least some of them must be annihilated by the supersymmetry charge (14). Motivated by the connection of the supersymmetric quantum mechanics with supersymmetric Yang-Mills theories, the Witten index has been determined for $G = SU(2)$ in the so-called ultralocal limit [18]. Usually one finds that $\Delta \neq 0$, with the exception of the two-dimensional models (corresponding to a supermembrane moving in a four-dimensional space-time), where $\Delta = 0$ for odd-dimensional gauge groups G. Furthermore, in [19] a "twisted" version of the Witten index was proposed which was argued to be strictly positive. If this were correct massless supermembrane states should exist. On the other hand, the validity of the ultralocal approximation is rather questionable. Moreover, we have already pointed out that the spectrum is continuous in the finite-N approximation without a gap. In such a situation it is known that the very notion of the Witten index will sensitively depend on the regularization that one employs [20]. In any case, the issue cannot just be resolved by formal manipulations of certain functional integrals. Rather than being positive the result is likely to be ill-defined!

Another method to investigate the problem of massless states relies on semi-classical approximations [21] and arguments based on collapsed membranes [16]. However, the results do not always agree. The reason is probably that the various approximations involved here are difficult to control.

An important feature of the quantum-mechanical models based on (13) is that the potential vanishes if the coordinates $\vec{X}^A$ take values in some abelian subalgebra. These valleys in the potential are also familiar from supersymmetric gauge theories; obviously in the small-volume limit, where one drops all the nonconstant modes, these valleys are still present although they remain compact as long as the volume is not strictly equal to zero. Also the (super)membrane has these zero-energy configurations; they correspond to stringlike configurations of arbitrary length. The same feature exists for general p-branes. Classical (super)p-branes are unstable: the zero-energy configurations correspond to collapsed branes of lower dimensionality $p-1$ (obviously, this observation is only relevant for $p > 1$).

It is evident that these classical degeneracies will also bear upon the quantized theories and the nature of the mass spectrum. In quantum mechanics, the spectrum of the Hamiltonian is usually discrete if the potential confines the wave function to a finite volume in configuration space. In the presence of zero-energy valleys, there is a latent danger that the wave functions will no longer be confined. There is then no obvious reason why the spectrum should be discrete. However, quantum-mechanical effects may still prevent the wave function from escaping through the zero-energy valley. In the valley the transverse width of the wave function is reduced, and because of the uncertainty principle the kinetic energy must be finite and positive. Another way to see this is by noting that the oscillations perpendicular to the valley direction give rise to a zero-point energy. This effect thus induces an effective potential barrier, which tends to confine the wave function. To make this more explicit, consider the following two-dimensional Hamiltonian,

$$H_B = p_x^2 + p_y^2 + x^2 y^2. \tag{19}$$

Obviously the potential in (19) has zero-energy valleys along the x- and the y-axis. Nevertheless the eigenfunctions of (19) are confined and cannot escape through these valleys. This follows from decomposing (19) as

$$H_B = H_1 + H_2, \tag{20}$$

where $H_1 = p_x^2 + \frac{1}{2}x^2y^2$ and $H_2 = p_y^2 + \frac{1}{2}x^2y^2$. Since H_1 and H_2 take the form of harmonic oscillator Hamiltonians in x and y, respectively, with frequencies proportional to $|x|$ or $|y|$, we immediately derive the operator inequality

$$H_B \geq \frac{|x| + |y|}{\sqrt{2}}. \tag{21}$$

Therefore the wave function will be confined and the spectrum of H_B is discrete [22].

It is now obvious why the introduction of supersymmetry could drastically change the situation described above. As is well-known, supersymmetric harmonic oscillators have no zero-point energy, so that the confining effective potential may vanish. Whether or not the potential in the valley will vanish completely can also be investigated in the two-dimensional model. We first introduce a supersymmetry charge by

$$Q = Q^{\dagger} = \begin{pmatrix} -xy & p_x + ip_y \\ p_x - ip_y & xy \end{pmatrix}, \tag{22}$$

which acts in a two-dimensional fermionic Fock space. The Hamiltonian then follows in the usual fashion,

$$H = Q^2 = \begin{pmatrix} H_B & x - iy \\ x + iy & H_B \end{pmatrix}. \tag{23}$$

In order to establish the absence of an effective potential barrier that may prevent the wave function from escaping through the valley, we consider a set of normalized trial wave functions

$$\psi_\lambda(x,y) = \chi(x-\lambda)\, \varphi_0(x,y)\, \xi_F, \tag{24}$$

characterized by some parameter λ. Here χ is a one-dimensional free particle wave packet satisfying $\int \mathrm{d}x\, |\chi|^2 = 1$, which has compact support so that (24) is only different from zero for $x \approx \lambda$, φ_0 is the normalized groundstate wave function for a one-dimensional harmonic oscillator,

$$\varphi_0(x,y) = \pi^{-\frac{1}{4}}\, |x|^{\frac{1}{4}} \exp\left(-\tfrac{1}{2}|x|\, y^2\right), \tag{25}$$

and ξ_F is a normalized two-dimensional spinor. When the parameter λ is large, the wave function (24) thus has its support in a narrow region along the x-axis.

What we now intend to show is that, by making λ large, ψ_λ will tend to a supersymmetric wave function, which, by virtue of the supersymmetry algebra, must have zero energy. To see how this works, let us consider $Q\,\psi_\lambda$,

$$Q\,\psi_\lambda = \left\{ xy \begin{pmatrix} -1 & -\operatorname{sgn} x \\ \operatorname{sgn} x & 1 \end{pmatrix} + \begin{pmatrix} 0 & p_x \\ p_x & 0 \end{pmatrix} \right\} \psi_\lambda. \tag{26}$$

If we now choose ξ_F equal to $\xi_F = \frac{1}{2}\sqrt{2}(1,-1)$, then the first term cancels (without loss of generality, we can assume $x \approx \lambda > 0$) and we are left with

$$Q\,\psi_\lambda = -p_x\,\psi_\lambda. \tag{27}$$

It is now straightforward to derive the following result for the norm of $Q\,\psi_\lambda$,

$$\|\, Q\,\psi_\lambda \,\|^2 = |p_x \chi|^2 + \mathrm{O}(\lambda^{-1}), \tag{28}$$

where the first term is obvious and represents the norm of the one-dimensional wave function $p_x\,\chi$, which is equal to the energy of the wave packet since χ is normalized to unity. The second term originates from the operator p_x acting on φ_0. This introduces a factor $|x|^{-1}$ or a factor y^2; however, the latter becomes also proportional to $|x|^{-1}$ after integrating over y. As $x \approx \lambda$, the contributions from $p_x\,\varphi_0$ are thus of order λ^{-1}.

Obviously the right-hand side of (28) can be made arbitrarily small by choosing a wave packet χ of sufficiently low energy and by making λ sufficiently large. The latter implies that the wave function will extend further and further into the valley, so that there is apparently no confining force, as ψ_λ approaches a supersymmetric wave function with zero energy. Although suggestive, the above result is by itself not yet sufficient to conclude that the spectrum of H is continuous, but it nicely illustrates the main ingredients of the proof. Namely, along the same lines, one shows that for any E

$$\| (H-E)\psi_\lambda \|^2 = |(p_x^2 - E)\chi|^2 + \mathrm{O}(\lambda^{-1}). \tag{29}$$

For any positive E and ε we can then choose a wave packet χ such that

$$|(p_x^2 - E)\chi|^2 \leq \varepsilon/2. \tag{30}$$

By making λ sufficiently large we can make the $\mathrm{O}(\lambda^{-1})$ corrections in (29) smaller than $\varepsilon/2$. Combining (29) and (30) then shows that for $\| \psi_\lambda \| = 1$,

$$\| (H-E)\psi_\lambda \|^2 \leq \varepsilon \tag{31}$$

for any positive E and ε. This proves that any *non-negative* E is a spectral value of the Hamiltonian H, so that the spectrum is continuous. It is clear that the reason for the continuity of the spectrum is that wave functions can excape to infinity along the valleys that have zero classical energy.

The above example exhibits all the qualitative features of the supersymmetric quantum-mechanical models that we are interested in. For the supersymmetric $SU(N)$ matrix models, the proof is technically more involved, but the only essential ingredients are the existence of the potential valleys extending to infinity and supersymmetry. We thus conclude that the $\mathcal{M}^2$ operator of the supersymmetric quantum-mechanical models that are related to the supermembrane has a continuous spectrum starting at zero for any finite (compact) gauge group [4]. By contrast the spectrum of the corresponding bosonic theory is discrete [22].

At this point we would like to return once more to the issue of finding discrete and possibly zero-mass eigenstates. Discrete eigenstates *within* the continuum are possible, although such examples are not easy to construct and always suffer from instabilities (as the discrete states can "decay" into the nearby continuum). For our toy model, it is not known whether such states actually occur. The problem is somewhat simpler to address for the zero-eigenvalues. In this case, the problem boils down to the question of whether there exist *normalizable* solutions of

$$Q\psi = 0. \tag{31}$$

Writing out the two components of ψ and assuming for simplicity that $\psi_1 = i\psi_2^*$, we must check whether there are square-integrable solutions of

$$(i\partial_x + \partial_y)\,\psi_1 = -ixy\psi_1^*. \tag{32}$$

Now, although solutions certainly exist, it is not known even in this utterly simple example whether these are square-integrable or not (see e.g. [23]). The case we are ultimately interested in, namely solving (31) with (14), is, of course, far more complicated as the spinorial part of the wave function now has $2^{8(N^2-1)}$ components!

We stress once more that the continuity of the spectrum has so far only been established for supersymmetric matrix models based on a *finite* Lie group. Furthermore, it is not known whether this result remains valid for supermembranes moving in a curved background, and whether the proof can be suitably modified in this case. Apart from this important open problem, the crucial question is what will happen when we take the limit to the infinite-dimensional group of area-preserving diffeomorphisms. The regularization procedure that we employ emphasizes supersymmetry, and indeed supersymmetry is crucial for our conclusions. Of course, it is not unlikely that alternative regularization methods exist. However, irrespective of these technical considerations, it should be noted that there is no *physical* reason for the supermembrane to behave any differently from the finite-dimensional matrix model, because it exhibits the very same features that led to the continuity of the spectrum in the finite-dimensional case, namely flat directions in the classical potential and supersymmetry. At this point, one might raise the problem of whether or not the supermembrane is renormalizable (or even finite). In the former case, various quantities may diverge in the $N \to \infty$ limit, but these infinities can be controlled by making a finite number of renormalizations. As the spectrum is continuous for finite N, one would first define a spectral density and then study its deformations as N is varied. For $N \to \infty$, certain renormalizations may again be necessary to keep the spectral density well defined. It would take more than a fortuitous accident for the spectral density to develop discontinuities (i.e. δ-function-like spikes) in this limit, quite apart from the formidable technical problem of proving this! If, on the other hand, the supermembrane should turn out not to be renormalizable after all, little can be said, but it is quite obvious that the theory would then be plagued by the same diseases which afflict any theory of gravity in more than two dimensions. One may also ask whether the above result only applies to a particular membrane topology. In view of our discussion concerning the relation between $SU(\infty)$ and area-preserving transformations, we emphasize once more that the topologies are automatically accounted for, at least in the sense that $SU(\infty)$ contains more than just the invariant subgroup G of area-preserving diffeomorphisms associated with one particular membrane topology. So it seems that there is very little chance for the supermembrane to evade the dilemma of a continuous spectrum: it is a theory without an intrinsic mass scale! We emphasize that, of all supersymmetric p-branes, only the superparticle and the superstring can evade the above conclusion. The particle simply because it has no internal structure at all, and the string because, apart from the centre-of-mass motion, all modes are confined by harmonic oscillator potentials.

In conclusion, we believe that, in its present form, the supermembrane cannot be a viable candidate as a model for elementary particles, at least not in the same relatively simple context in which superstrings are ordinarily considered. In addition to the difficulties concerning its physical interpretation which we have described above, there remain the enormous technical complications of membrane theory that derive from its non-linear structure (to name but a few: little progress has been made with a Polyakov-type approach, no membrane analog of vertex operators suitable for the computation of scattering amplitudes is known, etc.). To overcome all these problems, one would need some major miracles (like hidden integrability of supermembrane theory, for instance). On the other hand, there is a beautiful and unique algebraic structure in these theories. Perhaps supermembrane theory - like string theory - will make a comeback in an entirely different context from that originally envisaged.

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