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NUMERICAL INVESTIGATION OF FOUR-DIMENSIONAL FIELD THEORIES

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INTRODUCTION

The work discussed below has been done in collaboration with Ch. Frick, K. Jansen, J. Jersák, I. Montvay, P. Seuferling, T. Trappenberg, and U. Wolff in different combinations and is presented in more detail in refs. [1–5]. The general framework of our investigation is a non-perturbative study of ϕ^4-theory and the Ising model in four dimensions. In the continuum the Lagrange density of euclidean ϕ^4-theory reads

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}(\phi^2)^2, \tag{4}$$

where ϕ represents a real N-component scalar field $\phi^\alpha(x)$. The Ising model, on the other hand, describes a field which only assumes values $\phi(x) = \pm 1$. It is a standard test-ground for many ideas in field theory and statistical mechanics.

The physical relevance of ϕ^4-theory with a four-component scalar field is based on the fact that it describes the Higgs-sector of the standard model of electro-weak interactions. The Glashow-Salam-Weinberg model of electro-weak interactions contains gauge fields coupled to a four-component Higgs-field. Due to the smallness of the gauge coupling questions about the Higgs mechanism can be studied in the context of the pure ϕ^4-theory [6]. One of the questions relevant for phenomenology concerns the mass m_H of the Higgs particle. As a simplification one may reduce the number of components to one, which leads to the ordinary ϕ^4-theory. Another special case is obtained by sending the bare quartic self-coupling g_0 to infinity in a such a way that the field ϕ is constrained to unit length:

$$\phi \cdot \phi = 1.$$

The theory is then called the O(4)-symmetric non-linear sigma model in the case of four components, and the Ising model in the case of one component. It is in this limit that we studied the models. In this limit accurate Monte Carlo calculations are feasible. On the other hand this limit is relevant for upper bounds on the Higgs-mass as will be discussed below. The mass of the Higgs-particle is related to the value of the renormalized quartic coupling g_R and the field expectation value $v \approx 250$ GeV through

$$m_H = \sqrt{\frac{g_R}{3}}\, v.$$

Thus non-perturbative upper bounds on the coupling g_R in the phase with broken symmetry yield upper bounds on the Higgs-mass.

Lecture given at the NATO Advanced Research Workshop: "Probabilistic Methods in Quantum Field Theory and Quantum Gravity", Cargèse, France, August 1989

ϕ^4-THEORY

For simplicity of notation I begin with a consideration of the one-component ϕ^4-theory. In order to regularize the model it is defined on a hypercubical lattice $\mathbb{Z}^4$. The euclidean action

$$S = \sum_x \left\{ \frac{1}{2}(\partial_\mu \phi_0)^2 + \frac{m_0^2}{2}\phi_0^2 + \frac{g_0}{4!}\phi_0^4 \right\} \tag{2}$$

is parametrized as

$$S = \sum_x \left\{ -2\kappa \sum_{\mu=1}^{4} \phi(x)\phi(x+\hat{\mu}) + \phi(x)^2 + \lambda[\phi(x)^2 - 1]^2 \right\} , \tag{3}$$

where the lattice spacing a is set to 1 and $\hat{\mu}$ denotes the unit vector in the positive μ-direction. The parameters κ and λ are related to the bare mass m_0 and the bare coupling g_0 through

$$m_0^2 = \frac{1-2\lambda}{\kappa} - 8 , \quad g_0 = \frac{6\lambda}{\kappa^2}. \tag{4}$$

The phase diagram in terms of κ and λ is shown in figure 1 and results from numerous studies in recent years. (See e.g. ref. [7].) For values of κ below a certain $\kappa_c(\lambda)$ all expectation values respect the symmetry $\phi \to -\phi$ of the action, whereas for values of κ above κ_c the symmetry is broken spontaneously. For $\lambda = 0$ we have a free field theory, whereas in the limit $\lambda \to \infty$ the Ising model is obtained.

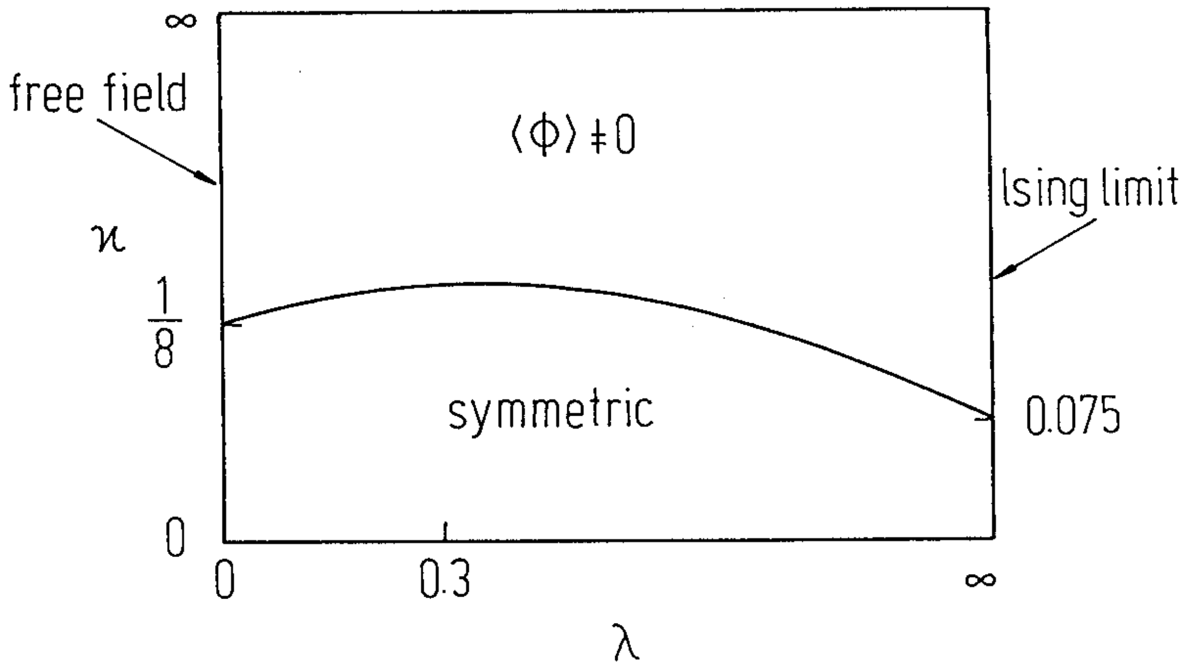

Fig. 1. Phase diagram of ϕ^4-theory in four dimensions.

The physical quantities of central interest are the renormalized mass m_R and the renormalized coupling g_R. The renormalized mass is defined together with the wave function renormalization Z_R through the small momentum behaviour of the propagator:

$$G(p)^{-1} = \frac{1}{Z_R}\{m_R^2 + p^2 + \mathcal{O}(p^4)\}. \tag{5}$$

m_R is numerically close to the physical mass $m = m_H$ given by the pole of the propagator. The renormalized coupling g_R is defined in terms of the 4-point vertex function by

$$\Gamma^{(4)}(0,0,0,0) = -\frac{1}{Z_R^2}\, g_R\, . \tag{6}$$

The renormalized coupling and the bare coupling are numerically quite different. In particular g_R remains finite even in the limit where λ goes to infinity. g_R and m_R are functions of κ and λ and for an understanding of the model one has to know this dependence. Of particular interest is the scaling region near κ_c where the mass m_R (in lattice units) vanishes as $\kappa \to \kappa_c$.

Most important for these considerations is the fact that ϕ^4-theory is almost certainly trivial, as work by various authors in recent years indicates, including ref. [7,8]. This means that in the continuum limit, where the cutoff $\Lambda/m_R = 1/am_R$ goes to infinity while $\kappa \to \kappa_c$, the renormalized coupling g_R vanishes and free field theory is obtained. Therefore ϕ^4-theory can only be used as a low-energy effective field theory with some finite cutoff Λ. The coupling g_R will then also be finite but will decrease with increasing cutoff. Since this concept is only meaningful if the cutoff is larger than, say, twice the renormalized mass, upper bounds on g_R result. The determination of numerical values for these upper bounds is the aim of recent nonperturbative investigations. Lüscher and Weisz [7,8] addressed this problem by means of a combination of high-temperature expansions in the symmetric phase and renormalization group methods, which allowed them to get control over m_R and g_R in the whole symmetric phase. These functions could then be continued to the scaling region in the phase with broken symmetry. The scaling region can be identified approximately with the strip around the critical line, in which $\Lambda/m_R > 2$. The upper bounds on g_R mentioned above are obtained on the boundaries of the scaling region. Here g_R assumes its maximum in the Ising limit which is therefore the relevant limit.

OUR AIMS

The aims of our investigation are

1. A high precision calculation of m, g_R, Z_R and other quantities.
   The results are to be used to check the scaling behaviour of physical quantities and to make a comparison with the results of Lüscher and Weisz. Triviality bounds on g_R are to be derived from them. The calculations are done in the Ising limit by means of the Monte Carlo method.

2. A study of finite size effects.
   The calculations are done for a system in a finite volume $L^3 \cdot T$. This implies finite size effects for all quantities under consideration. The L-dependence of m, g_R etc. can be measured accurately in the Monte Carlo calculation. On the other hand, if the coupling g_R is small enough, these finite size effects can be calculated in renormalized lattice perturbation theory, which we did in the one-loop approximation. This then allows an extrapolation of the Monte Carlo results to $L = \infty$.
   Furthermore a precise determination of the volume dependence of two-particle masses allows information to be obtained about scattering lengths [9].

CLUSTER ALGORITHM

One aspect of the numerical simulations deserves special note. For the Ising model we have made use of the so-called cluster algorithm of Swendsen and Wang [10]. This algorithm has been developed further by Wolff [11] in order to be applicable to the case of continuous spin models. We have employed his version in our studies of the O(4) non-linear sigma model.

Cluster algorithms are highly efficient updating algorithms and their use was essential for achieving high precision. The basic idea of cluster algorithms is to enforce spin-flips for large domains on the lattice. As a result the problem of critical slowing down is avoided. There is, however, also another advantage which turned out to be even more important in our calculations. Namely, the cluster algorithms allow the measurement of observables through estimators with significantly reduced variance.

For the case of the Ising model the cluster algorithm works as follows. In the Monte Carlo simulation an alternating sequence of spin-configurations and bond-configurations is generated. The spin-configurations are the usual configurations of Ising spins. Bond configurations are represented by the values 0 or 1 on the links which connect neighbouring lattice sites. The mapping between these types of configurations is probabilistic. Given a spin-configuration $\{\phi(x)\}$, a bond with value 1 is created on the link between x and y with probability $p = 1 - \exp(-4\kappa)$ if $\phi(x) = \phi(y)$. If the spins $\phi(x)$ and $\phi(y)$ are unequal no bond is created. A cluster is a maximal set of points connected by bonds with value 1. It may consist of a single site. From a bond-configuration a new spin-configuration is now obtained by identifying the clusters and assigning random values $r_i = \pm 1$ to all spins contained in the same cluster C_i. The efficiency of the cluster updating algorithm in fighting critical slowing-down is due to the fact that in the step from the bond-configuration to the spin-configuration whole clusters are statistically assigned a new spin value. Since there are also large clusters, this can imply a non-local change of spins. The generated sequence of bond-configurations and their cluster structure can also be used to measure physical quantities. In this way one obtains the same expectation values as in the spin representation, but the fluctuations, and therefore the statistical errors, are smaller [11]. This is the variance reduction mentioned above.

To illustrate the efficiency of the Swendsen-Wang algorithm let me compare it to the usual local Metropolis algorithm. On a 12^4-lattice at a value of the coupling where $m \approx 0.5$ the cluster algorithm is ten times slower than the Metropolis algorithm. But the relative errors of various physical quantities are smaller by factors 5–20. Thus the effective gain in speed is a factor 3–50 for the cluster algorithm. The situation is even better on a 24^4-lattice where the gain in speed is 20–200.

RESULTS

Ising Model in the Symmetric Phase [1]

In the symmetric phase of the Ising model we have performed simulations on lattices with a spatial extent of $L = 12, 16, 20$ at points where $\Lambda/m_R = 2.0, 2.6$ and 3.3. Very precise results could be obtained. As an example I quote the final numbers (extrapolated to $L = \infty$) at $\kappa = 0.0732$:

$$m_R = 0.3078(3), \quad g_R = 32.9(13), \quad Z_R = 0.9707(8). \tag{7}$$

The analysis of the data leads to the following conclusions:

1. We observe a very good agreement with the analytical results of Lüscher and Weisz. The precision could be improved.

2. Scaling according to the perturbative β-function (in the three loop-approximation) is confirmed.

3. Finite size effects are under control. The observed effects are in agreement with the theoretical results. The finite size dependence of two-particle masses allows a determination of S-wave scattering lengths to an accuracy of 10–15% [12].

Ising Model in the Phase with Broken Symmetry [2,3]

The results in the broken symmetry phase of the Ising model are of a quality comparable to those in the symmetric phase. Therefore I will not discuss them here in more detail but merely quote the triviality bound

$$g_R \leq 34 \tag{8}$$

which can be derived from them. A novel aspect is the appearance of a new type of finite size effects due to tunneling. In the phase under consideration the effective potential which governs the dynamics of the long wave-length modes has a double-well shape. If the volume L^3 is finite this leads to tunneling phenomena as in the case of an anharmonic oscillator. In particular the spectrum of the Hamiltonian exhibits a small splitting of all levels, which vanishes for large volumes. For the splitting ΔE between the two lowest masses the large volume behaviour is

$$\Delta E \simeq C \cdot L^{1/2} \exp\left\{-\sigma L^3\right\} \tag{9}$$

where a "surface tension" σ appears. The prefactor C and the surface tension σ can be calculated in an instanton-type semi-classical calculation. Including one-loop corrections one gets [4]

$$C = 1.65058 \sqrt{2 \frac{m_R^3}{g_R}} \tag{10}$$

and an L-dependent surface tension

$$\sigma(L) = \sigma_\infty \left(1 - \frac{g_R}{16\pi^2} \frac{3\sqrt{3}\pi}{(m_R L)^2} \exp(-\frac{\sqrt{3}}{2} m_R L) + \mathcal{O}(e^{-m_R L}) + \mathcal{O}(g_R^2)\right) \tag{11}$$

$$\sigma_\infty = 2 \frac{m_R^3}{g_R} \left(1 - \frac{g_R}{16\pi^2} \left(\frac{1}{8} + \frac{\pi}{4\sqrt{3}}\right) + \mathcal{O}(g_R^2)\right). \tag{12}$$

The mass splitting was also calculated numerically for various L in our Monte Carlo simulation. The observed L-dependence is as predicted in (9), see fig. 2. The values of the surface tension σ and the constant C have been determined from a fit of ΔE up to $L = 10$ in ref. [2]. Combined with the Monte Carlo value $m_R = 0.395(1)$ in lattice units the results are

$$\sigma/m_R^3 = 0.0581(5), \qquad C = 0.101(4)\,. \tag{13}$$

The measurements were done at a point where the coupling is $g_R = 30.2(4)$. On the other hand for this value of g_R the theoretical predictions are

$$\sigma/m_R^3 = 0.0585(8) \qquad \text{for } L = 10\,, \qquad C = 0.105(1)\,. \tag{14}$$

The agreement with the numbers above is remarkably good. This shows that the semiclassical one-loop approximation is reliable for the value of g_R above. Furthermore it supports the evidence that at this point the model is in the scaling region, which was also found from a study of the scaling behaviour of g_R and m_R.

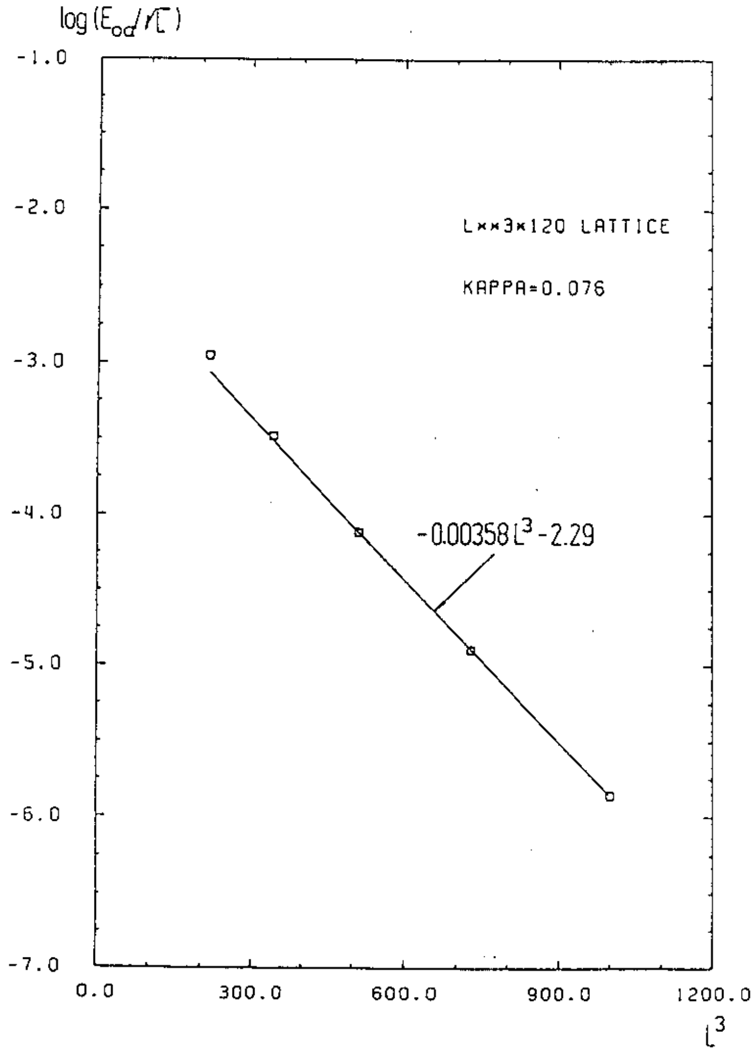

Fig. 2. Volume dependence of the mass splitting $\Delta E \equiv E_{0a}$ for the four-dimensional Ising model in the phase with broken symmetry.

O(4)-symmetric Non-linear Sigma Model in the Symmetric Phase [5]

Using both Wolff's cluster updating algorithm [11] and the conventional Metropolis algorithm we performed numerical simulations of the non-linear sigma model in the symmetric phase for two values of κ and various lattice sizes, namely

$$\kappa = 0.290 \qquad (m \approx 0.45)$$

$$L = 4, 6, 8, 10, 12, \qquad T = 12$$

and

$$\kappa = 0.297 \qquad (m \approx 0.3)$$

$$L = 8, 10, 12, 14, 16, \qquad T = 16.$$

With the cluster algorithm accurate numerical results could be obtained. A comparison of the measured values of m and g_R for different lattice sizes L showed that finite size effects are well reproduced by one-loop renormalized lattice perturbation theory. As a consequence the extrapolation to $L = \infty$ with the help of the perturbative formulae is reliable.

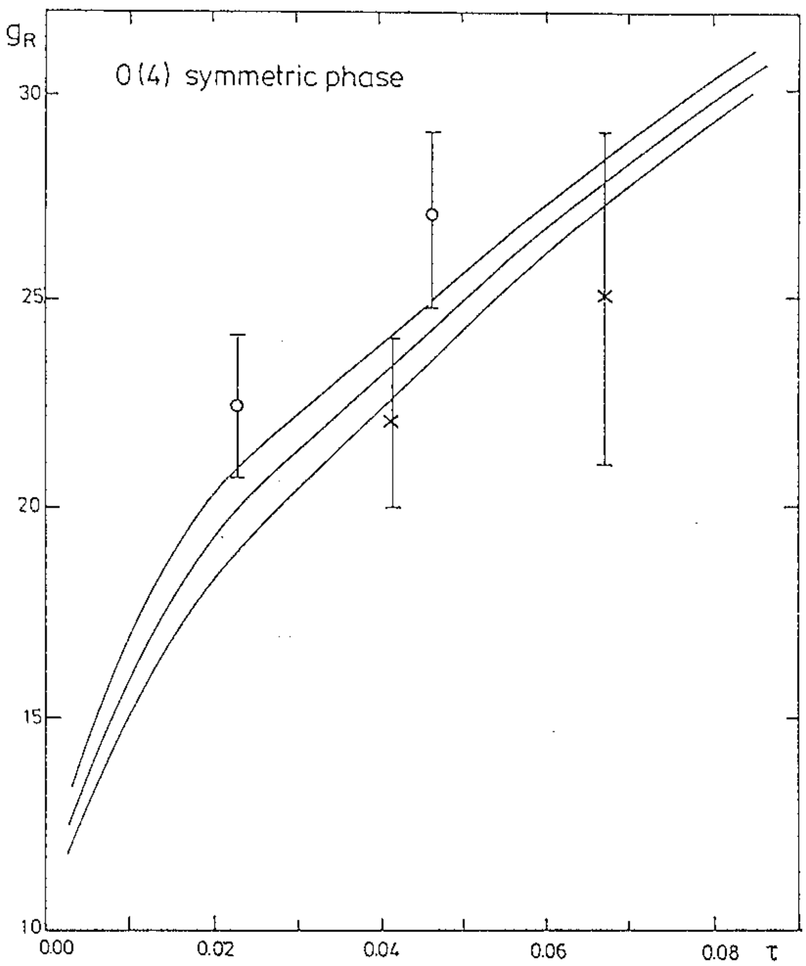

Fig. 3. Comparison of our results (open circles) to the analytical work in ref. [8] (strip given by the three lines) and to the numerical results of ref. [13]. The renormalized coupling g_R is shown as a function of $\tau = 1 - \kappa/\kappa_c$ with $\kappa_c \equiv 0.30411$.

The extrapolated numbers are

$$\kappa = 0.290 \tag{15}$$

$$m_R = 0.4500(6), \quad g_R = 26.9(2.1), \quad Z_R = 0.988(2)$$

and

$$\kappa = 0.297 \tag{16}$$

$$m_R = 0.3044(4), \quad g_R = 22.4(1.7), \quad Z_R = 0.981(2).$$

The extrapolated infinite volume results for m_R and g_R agree well with the analytical results of Lüscher and Weisz [8]. In fig. 3 this is shown for the case of the renormalized coupling. The figure also includes the results of a previous numerical simulation by Kuti et al.[13] which also agrees within errors with ref. [8] and with us. The estimated relative errors for our values for m_R are up to a factor of 5–10 smaller than those of ref. [8], but the errors of the renormalized coupling are somewhat worse here, as is shown by the figure. In fig. 3 the slope of the solid curve below our data points reflects the scaling prediction from the renormalization group. As can be seen from the figure our data are consistent with scaling behaviour.

The finite-size effects on two-particle masses again permitted a determination of scattering lengths, similarly to the case of a single-component ϕ^4-theory.

CONCLUSIONS

- ϕ^4-theory can be fully understood by means of present day methods.
- Finite size effects are under control.
- Presently available results confirm the existing theoretical picture based on triviality and scaling. The qualitative behaviour of the four-component model in the symmetric phase is very similar to the one-component case.
- Our calculations agree with the work of Lüscher and Weisz [7,8], leading to the bound $m_H < 650$ GeV on the Higgs mass.

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THE CRITICAL BEHAVIOUR OF A

NONTRIVIAL RANDOM SURFACE MODEL

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INTRODUCTION

Since Jan Ambjørn has given an introduction and review on random surfaces at this workshop I need not explain the motivations and basic concepts of random surface theory. Of the two different approaches to this subject he has mainly discussed the one which deals with triangulated random surfaces. The other one, which will be considered in the following, considers a lattice regularization of surfaces. The surfaces are then embedded in the d-dimensional lattice $\mathbb{Z}^d$. In this type of regularization the surfaces are specified geometrically without any parametrization and the problems related to reparametrization invariance do not arise.

A model of random surfaces is specified through

1. The set of allowed surfaces, which may be restricted through conditions on topology, geometry etc..
2. The action S.

Most commonly used is the Nambu action

$$S = \beta A \,, \qquad\qquad A = Area = \#(plaquettes).$$

The basic quantities of a model are the partition function

$$Z = \sum_S e^{-\beta A(S)}, \tag{1}$$

where the sum goes over all closed surfaces S (modulo translations), and the correlation functions

$$G(\gamma_1, ..., \gamma_n) = \sum_{S \in \mathcal{S}(\gamma_1,...,\gamma_n)} e^{-\beta A(S)}. \tag{2}$$

Here the sum is over those surfaces which have the loops γ_1 to γ_n as boundaries. The increase in the number of surfaces as the area A increases is characterized by the entropy $n_\gamma(A)$. It is defined through

$$G(\gamma) = \sum_{A=1}^{\infty} n_\gamma(A) e^{-\beta A} \tag{3}$$

Lecture given at the NATO Advanced Research Workshop: "Probabilistic Methods in Quantum Field Theory and Quantum Gravity", Cargèse, France, August 1989 for some fixed loop γ. For unrestricted random surfaces it has been shown [1] that

$$n_\gamma(A) \geq A! \tag{4}$$

and the correlation functions do not exist. On the other hand for fixed topology (Euler number) we have

$$n_\gamma(A) \leq e^{b_0 A} \tag{5}$$

for some positive constant b_0, independent of γ. This motivated the introduction of *Planar Random Surfaces (PRS)*, where only surfaces of the simplest topology without handles are allowed. Other models of physical interest include the so-called *Self-avoiding Random Surfaces (SARS)* and the *Solid-on-Solid Model (SOS)*. In these latter two models, however, the surfaces obey nonlocal constraints and therefore they are not of much interest in the framework of string theory, where one would like to consider models which are defined in a local way.

In the following some important physical observables and critical exponents of random surface models, which have already been introduced in Jan's lectures, will be recalled briefly. The susceptibility is defined by

$$\chi(\beta) = \sum_{p'} G(\partial p, \partial p') \tag{6}$$

where ∂p denotes the boundary of a plaquette p. The mass gap $m(\beta)$ is given as usual by the asymptotic exponential decay of the two-plaquette correlation function:

$$G(\partial p, \partial p') \sim e^{-mx}, \qquad x = dist.(p, p'). \tag{7}$$

The Wilson loop

$$W_{L,M}(\beta) = G(\gamma_{L,M})$$

is the correlation function for a single loop $\gamma_{L,M}$ of side-lengths L and M. Its asymptotic decay determines the string tension $\tau(\beta)$:

$$W_{L,M} \sim e^{-\tau LM}. \tag{8}$$

Another quantity of interest is the Hausdorff or fractal dimension [2]. In the context of random surface models a convenient definition is

$$d_H = 2 \lim_{A\to\infty} \frac{\log A}{\log < x^2 >_A} \tag{9}$$

where $< x^2 >_A$ is the mean squared distance of a surface to its center of gravity, averaged over all surfaces of area A. Its asymptotic behaviour for large A is thus given by

$$< x^2 >_A \sim A^{\frac{2}{d_H}}. \tag{10}$$

On the basis of considerations for random walks Parisi [3] conjectured the value

$$d_H = 4$$

for random surfaces. Furthermore he suggests that d_H is related to the upper critical dimension d_c^u through

$$d_c^u = 2d_H = 8.$$

The upper critical dimension is the number of euclidean space-time dimensions above which trivial (mean field) behaviour sets in. The status of these conjectures will be discussed later.

Numerical and analytical results for different models indicate that the entropy in general behaves like

$$n_\gamma(A) \sim A^c e^{\beta_0 A} \qquad \text{as} \qquad A \to \infty \tag{11}$$

for some positive constant β_0 and some real index ε. In this case the correlation functions exist for $\beta > \beta_0$. If β_0 is a critical point, where the susceptibility diverges and the mass gap vanishes, we define critical exponents through

$$\begin{aligned} \chi(\beta) &\sim (\beta - \beta_0)^{-\gamma} & (12) \\ m(\beta) &\sim (\beta - \beta_0)^{\nu} & (13) \\ \frac{d\tau}{d\beta} &\sim (\beta - \beta_0)^{\mu - 1} & (14) \\ G(\partial p, \partial p') &\sim |x|^{-(d-2+\eta)}, \qquad \text{for} \qquad 1 \ll |x| \ll \frac{1}{m} \; . & (15) \end{aligned}$$

These exponents and the Hausdorff dimension are not independent of each other. As in the case of other models of statistical mechanics, scaling relations can be derived under some standard scaling hypotheses. If γ is positive they are

$$\begin{aligned} \gamma &= 2 + \varepsilon & (16) \\ \gamma &= \nu(2 - \eta) & (17) \\ \nu &= \frac{1}{2}\,\mu = \frac{1}{d_H} \; . & (18) \end{aligned}$$

Therefore there are only two independent exponents. We take ε and d_H as basic critical exponents because they are most easily determined in numerical calculations.

Ultimately we are interested in the existence of a continuum limit. If a sensible continuum limit exists as $\beta \to \beta_0$ such that the mass gap and string tension have a finite physical limit, the corresponding quantities in lattice units should vanish:

$$m(\beta) \to 0, \qquad \tau(\beta) \to 0 \; .$$

PLANAR RANDOM SURFACES

The simplest ansatz for a regularized string theory is the model of planar random surfaces (PRS) which was mentioned above. Due to the large entropy of planar random surfaces a Monte Carlo simulation did not appear to be feasible [4]. An extensive study of this model by numerical methods was, however, made unnecessary by results due to Durhuus, Fröhlich and Jonsson [5]. Assuming that the susceptibility $\chi(\beta)$ diverges as $\beta \to \beta_0$ and that a certain self-similarity property holds they were able to establish the following

Triviality Theorem :

> For the model of planar random surfaces, mean-field theory is exact and the exponents assume their classical values
>
> $$\nu = \frac{1}{4}, \; \mu = \frac{1}{2}, \; d_H = 4 \qquad (19)$$
>
> $$\eta = 0, \; \gamma = \frac{1}{2}, \; \varepsilon = -1.5 \; .$$
>
> Furthermore, the string tension does not go to zero,
>
> $$\tau(\beta_0) > 0,$$
>
> the correlation function $G(\partial p, \partial p')$ has the free massive scalar propagator as its continuum limit, and the higher correlation functions of n loops for n ≥ 3 do not possess continuum limits if $d < 6$.

Monte Carlo calculations [6] for $d = 2$ and 3 yield $\varepsilon = -1.5 \pm 0.2$ which implies the divergence of $\chi(\beta)$. On the other hand mean field theory is known to apply for large enough d. These results indicate that the triviality theorem applies for all $d \geq 2$. This disease of PRS can be traced back to the fact that the surfaces behave like non-interacting branched polymers ("cacti") near the critical point. This means that a typical surface consists mainly of long thin tubes. A corresponding behaviour in the case of triangulated random surfaces, namely the abundance of spikes, has been discussed by Jan Ambjørn in his lectures.

PLANAR RANDOM SURFACES WITHOUT SPIKES

How can one cure this disease? Two possibilities are

a) to change the action, or as a special case
b) to restrict the class of allowed surfaces in order to prevent the outgrowth of "fingers".

Such an attempt is made in the model of planar random surfaces without spikes (PRSWS). Spikes are 180° wedges, where two plaquettes which occupy the same place are attached to each other. Forbidding the occurrence of spikes represents a local constraint. The model has been introduced by Berg, Billoire and Förster [7] as an analogue to fermionic random walks which contribute to the random walk representation of the Dirac propagator. The hope is that the constraint is strong enough to suppress "fingers" in low dimensions.

The PRSWS model is related to a string model with extrinsic curvature [8] studied recently by several authors. A regularized model on a lattice including extrinsic curvature terms has been formulated by Durhuus and Jonsson [9]. Its action reads symbolically

$$S = \beta_1 \cdot \#(\mathit{spikes}) + \beta_2 \cdot \#(90^\circ \mathit{wedges}) + \beta_3 \cdot \#(\mathit{flat\ links}).$$

It reduces to the Planar Random Surface model for $\beta_1 = \beta_2 = \beta_3$. On the other hand, our PRSWS model is obtained as a special case containing extrinsic curvature terms in the limit $\beta_1 \to \infty$ with $\beta_2 = \beta_3$. Durhuus and Jonsson extended the triviality theorem for PRS to the model above for the case that the couplings β_i are finite.

For PRSWS some general results have been derived [10,11] including Osterwalder-Schrader-positivity and scaling relations (16,17,18). Most important is the observation [12,10] that the triviality proof as given by Durhuus, Fröhlich and Jonsson for PRS does not apply to the PRSWS model. This allows the hope for a non-trivial behaviour of this surface model in low dimensions.

In large numbers of dimensions d, mean field theory applies as usual and can be used to calculate various quantities [10,11]. The critical exponents take their classical values (19) in mean field theory. Furthermore the string tension $\tau(\beta_0)$ at the critical point assumes a finite value, which can be calculated in mean field theory. For large d the model is thus trivial like the planar random surface model.

Monte Carlo Calculations

Using an algorithm [13,7,4] developed particularly for the study of such models Baumann and Berg [14] obtained the first Monte Carlo results for PRSWS. They measured the exponent ε and the Hausdorff dimension d_H in $d = 4$ dimensions. The results are

$$d_H = 4.2 \pm 0.3 \tag{20}$$

in accordance with the Parisi conjecture, and

$$\varepsilon = -1.74 \pm 0.03 \tag{21}$$

which implies critical exponents

$$\nu \approx \frac{1}{4}, \;\; \gamma \approx \frac{1}{4}, \;\; \eta \approx 1, \tag{22}$$

that differ from the mean field values (19). These results were then the main motivation for further studies of PRSWS. One of the questions to be answered is about the upper critical dimension, which in view of the results above is conjectured to be

$$d_c^u = 8. \tag{23}$$

The Monte Carlo calculations have been extended since then to other dimensions d and the statistics in 4 dimensions has been improved [10,11]. A new Monte Carlo algorithm due to Baumann [15] was very advantageous for this purpose. First of all an accurate determination of β_0 was achieved for d=4,6,8,10,12 and 26, which is necessary for a determination of critical exponents. The results are shown in fig. 1 together with the predictions from mean field theory (upper curve). For large d the agreement is quite good whereas for $d < 8$ deviations show up.

The Hausdorff dimension and the exponent ε have then been calculated in extensive runs in eight and ten dimensions. For the Hausdorff dimension the results are

$$\begin{aligned} d_H &= 4.0 \pm 0.2\,, \quad \text{for } d = 8 & (24)\\ &= 3.8 \pm 0.1\,, \quad \text{for } d = 10. & (25) \end{aligned}$$

The number in $d = 10$ is obtained with somewhat less statistics than in $d = 8$. Together with the value in $d = 4$ the results indicate the validity of the

Assumption: $d_H = 4$ for $d \geq 4$.

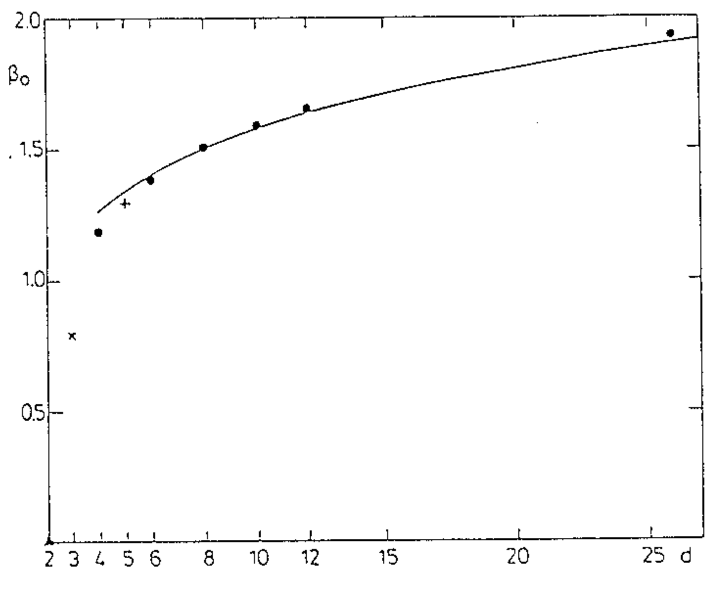

Fig. 1. The critical coupling β_0 versus the number of dimensions d for PRSWS (from [15]). The dots and crosses represent Monte Carlo data. The curve shows the prediction from mean field theory.

The exponent ε comes out to be

$$\begin{aligned} \varepsilon &= -1.58 \pm 0.03\,, \quad \text{for } d = 8 \qquad (26) \\ &= -1.55 \pm 0.05\,, \quad \text{for } d = 10. \qquad (27) \end{aligned}$$

The numbers may contain some systematic errors due to the finite maximal area occuring in the simulations. But they are consistent with the classical value of $\varepsilon = -1.5$, in contrast to the case of $d = 4$ dimensions. Thus the calculations support the hypothesis $d_c^u = 8$ and allowed the hope that PRSWS yield a non-trivial regularized string model for d < 8.

The String Tension

Concerning the continuum limit a crucial question is whether the physical string tension has a finite value. This amounts to $\tau(\beta_0) = 0$ as has been mentioned earlier. In large dimensions d mean field theory shows that this is not the case. To decide the case of low dimensions is a difficult task for Monte Carlo calculations. Therefore we have studied the string tension by means of a strong coupling expansion [16]. Strong coupling means large β in this case. From diagrams involving up to 12 plaquettes we derived the expansion up to the 5th term. The result is

$$\begin{aligned} \tau &= -\frac{1}{2}\log t - 2(d-2)t^2 - 8(d-1)(d-2)t^3 - 2(d-2)(9d-20)t^4 \\ &\quad - 8(d-2)(18d^2 - 48d + 19)t^5\,, \qquad (28) \end{aligned}$$

$$\text{where} \qquad t = e^{-2\beta}.$$

In fig. 2 the leading logarithmic term and the successive partial sums of this series are displayed for the case $d = 4$. Also indicated is the value of the critical coupling t_0.

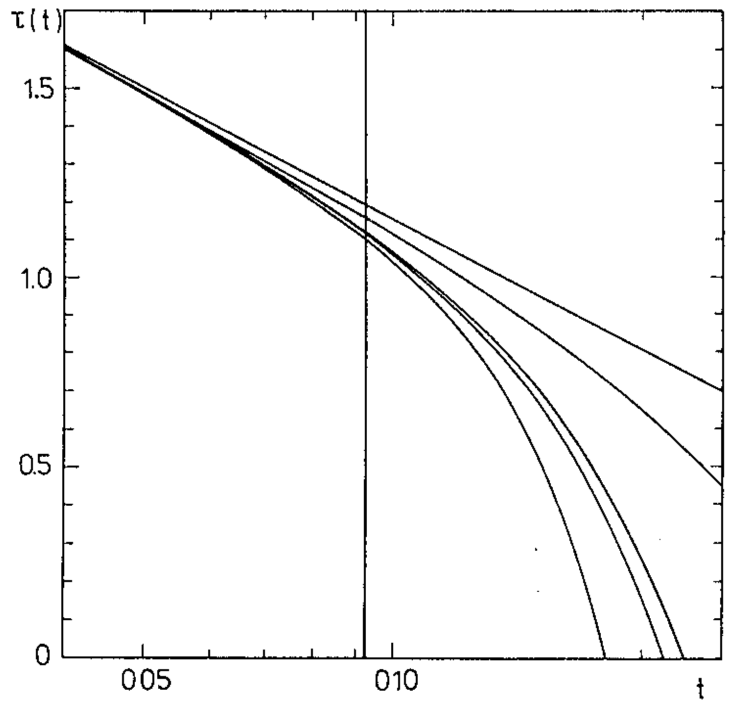

Fig. 2. The string tension as a function of $t = e^{-2\beta}$ for PRSWS in $d = 4$ dimensions. The uppermost curve represents the leading order of the strong coupling series; the curves below show the successive partial sums. The vertical line indicates the critical point.

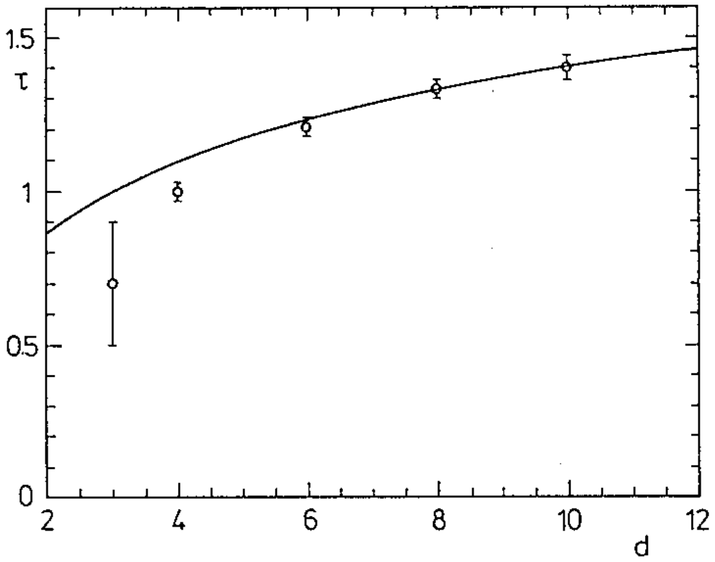

Fig. 3. The string tension at the critical point versus d for PRSWS. The circles represent the results of the strong coupling analysis, the curve shows the prediction from mean field theory.

From fig. 2 one gets the impression that at t_0 the series is still well convergent to some non-zero value. However, in order to find out whether some significance can be attributed to this observation, we analyzed the series by means of various extrapolation methods assuming a critical behaviour of the type

$$\tau \sim \tau(t_0) + A\,|t - t_0|^{\mu}. \tag{29}$$

For the exponent μ the outcome is consistent with the classical value of $\mu = 1/2$, which also results from the Monte Carlo calculations of d_H in combination with the scaling relations. On the other hand the values obtained for $\tau(\beta_0)$ always deviate significantly from 0. In fig. 3 the strong coupling results for $\tau(\beta_0)$ are shown as a function of the number of dimensions d, together with the prediction from mean field theory. Again we see a deviation from mean field behaviour for low dimensions. However it appears not to be big enough to bring the points down to zero.

As a check on the method we have also applied it to the case of the exponent ε, and the results are consistent with the Monte Carlo data. Despite the limitations of the strong coupling method, in particular the shortness of the series, we see that it gives strong evidence for a non-vanishing $\tau(\beta_0)$. It would of course be desirable to get more information about this question by means of the Monte Carlo or some other method.

CONCLUSIONS

The picture which emerges from the discussion above is the following. The simplest lattice model for regularized strings, namely planar random surfaces PRS, is trivial in all dimensions. The modification introduced in the PRSWS model appears to be able to produce non-trivial critical behaviour for $d < 8$, but the physical string tension diverges in the continuum limit as in the case of PRS. On the other hand there are models with stronger, nonlocal constraints, like self-avoiding random surfaces SARS, which show non-trivial behaviour in low dimensions. However, they are not interesting for string theory.

d	PRS	PRSWS	?	SARS
12	⋮	⋮		
10	⋮	⋮		
8	⋮	⋮ trivial		
6	⋮	| non-trivial		?
4	⋮	|		|
2	⋮	|		|
	trivial			
	$\tau(\beta_0) > 0$	$\tau(\beta_0) > 0$	$\tau(\beta_0) = 0$	

It is presently unknown what a regularized string theory would have to look like. But, as has been mentioned in Jan's lectures, there are some indications that it should contain extrinsic curvature terms amongst other modifications of the simplest ansatz.

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