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Abstract

The length spectrum of closed geodesics on a compact Riemann surface corresponding to a regular octagon on the Poincaré disc is investigated. The general form of the elements of the "octagon group", a discrete subgroup of SU(1,1), in terms of 2×2 matrices is derived, and the Aurich-Steiner law for the length of periodic orbits is proved analytically. An algorithm for the multiplicity of geodesics with a given length is developed, which leads to an efficient enumeration of the periodic orbits of this strongly chaotic system.

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I Introduction

The free motion on a compact two-dimensional surface of constant negative curvature is one of the simplest and best investigated ergodic models of classical mechanics (see e. g. Ref. [1] and references therein). In Ref. [2] some properties of periodic trajectories were investigated for one of such surfaces which corresponds to a regular octagon on the Poincaré disc with opposite sides being identified.

It is known [1,2] that for such a system the periodic trajectories are in one-to-one correspondence with the conjugacy classes of fundamental group matrices. For the problem considered, i. e. the "octagon group", the latter can be represented as a product of an arbitrary number of the following 8 generators [1,2]:

$$b_k = \begin{pmatrix} 1+\sqrt{2} & e^{i\frac{k\pi}{4}}\sqrt{2}\sqrt{\sqrt{2}+1} \\ e^{-i\frac{k\pi}{4}}\sqrt{2}\sqrt{\sqrt{2}+1} & 1+\sqrt{2} \end{pmatrix}, \tag{1}$$

where $k = 0, 1, \ldots, 7$.

In Ref. [2] all products up to 11 generators were found and, using a particular algorithm for separating the conjugacy classes, the length spectrum of 206 796 242 primitive periodic trajectories was calculated. The numerical results strongly suggest that there exists an exact formula for the lengths of primitive periodic trajectories [2]:

$$\cosh \frac{l_n}{2} = m + n\sqrt{2} , \tag{2}$$

where l_n is the length of a periodic trajectory with n being a natural number and m an odd natural number, which is uniquely defined by the condition that the modulus of the difference

$$\Delta := |m - n\sqrt{2}| \tag{3}$$

has a minimum value at given n.

The existence of such arithmetic relations in terms of algebraic numbers for this ergodic system was not expected before. In particular, from these relations it could be concluded [2] that the mean multiplicity $\bar{g}(l)$ of periodic trajectories with a fixed length l is unexpectedly large, i. e. $\bar{g}(l) \sim 8\sqrt{2}e^{l/2}/l$, $l \to \infty$.

In Section 2 of this note we shall study the fundamental group matrices for the regular octagon and shall find their general form from which we shall prove the Aurich-Steiner law (2), (3) analytically.

In Section 3 we present a few important symmetry relations for such matrices.

In Section 4 we develop an algorithm for the calculation of the multiplicity of periodic trajectories with a given length. The usual method of constructing the fundamental group matrices as products of a finite number of generators (which was used in Ref. [2]) suffers from the drawback that products consisting of a large number of generators can give a periodic trajectory with a small period. This means that the multiplicity of periodic trajectories (even for small lengths) obtained by such a method, in general, will be underestimated due to the contribution from products with a larger number of generators. This fact restricts the applicability of such calculations, especially for checking the Selberg trace formula (periodic-orbit theory) for the system considered [3].

The method proposed in Section 4 permits us to find the exact multiplicity of periodic trajectories with a given length independent of the number of generators taken into account.

II General form of fundamental group matrices for the regular octagon

An arbitrary fundamental group matrix of the "octagon group" corresponding to the regular octagon shown in Fig. 1 represented as a product of generators (1) can be written as [2]:

$$M = \begin{pmatrix} A_1 + iA_2 & \sqrt{\sqrt{2}-1}(B_1 + iB_2) \\ \sqrt{\sqrt{2}-1}(B_1 - iB_2) & A_1 - iA_2 \end{pmatrix} , \tag{4}$$

where A_1, A_2, B_1, B_2 are algebraic numbers of the form

$$m + n\sqrt{2} \tag{5}$$

with integers m, n.

(Note that we choose in the off-diagonal elements the factor $\sqrt{\sqrt{2}-1}$ instead of $\sqrt{\sqrt{2}+1}$ as in [2]. The reason for it will become clear below).

The obvious property of (4) is that its determinant must be equal to 1:

$$A_1^2 + A_2^2 - (\sqrt{2}-1)(B_1^2 + B_2^2) = 1 . \tag{6}$$

This condition seems to be trivial, but we shall show in a moment that it gives a lot of information about the matrix elements.

First of all, we introduce a few definitions. Let us call the algebraic numbers of form (5) even or odd depending on the parity of m. (Here the parity of an algebraic number $m + n\sqrt{2}$ is defined by $p(m + n\sqrt{2}) \equiv m(\mathrm{mod}2)$). It is easy to show that A_1 must be odd, A_2 even, and B_1, B_2 must have the same parity (both even or odd). Among the general algebraic numbers (5) which are defined by two independent integers m and n we shall be interested in particular subsets of these numbers for which m is uniquely connected with n by the requirement that the quantity

$$\Delta = |m - n\sqrt{2}| \tag{7}$$

acquires its minimum value for fixed n and for a given parity of m. We shall call the numbers with this property minimal numbers. There are two types of minimal numbers: even and odd depending on whether m is allowed to be even or odd in the minimization of (7). The necessary and sufficient condition that an algebraic number $C = m + n\sqrt{2}$ ($n \neq 0$) belongs to the set of minimal numbers can be expressed in form of the inequality

$$|\tilde{C}| := |m - n\sqrt{2}| < 1 . \tag{8}$$

In Table 1 we present the first 20 minimal numbers for the case $n > 0$. Minimal numbers have the interesting property that each class of minimal numbers is closed under multiplication. This means that if one multiplies two arbitrary minimal numbers with the same parity, the result will be again a minimal number with the same parity.

Let us consider condition (6) in detail. It is an algebraic relation for the numbers(5). It is clear that it will remain true if one changes the sign of $\sqrt{2}$ in all terms. This implies that if A_1, A_2, B_1, B_2 of form (5) obey (6), then their conjugated partners $\tilde{A}_1$ etc. will obey the following relation:

$$\tilde{A}_1^2 + \tilde{A}_2^2 + (\sqrt{2}+1)(\tilde{B}_1^2 + \tilde{B}_2^2) = 1 , \tag{9}$$

n	1	2	3	4	5	6	7	8	9	10
m even	2	2	4	6	8	8	10	12	12	14
m odd	1	3	5	5	7	9	9	11	13	15
n	11	12	13	14	15	16	17	18	19	20
m even	16	16	18	20	22	22	24	26	26	28
m odd	15	17	19	19	21	23	25	25	27	29

Table 1: First positive minimal numbers

where $\tilde{A}_1 := m_1 - n_1\sqrt{2}$ etc.. But all terms in (9) are positive numbers, and therefore they are restricted by the following values

$$|\tilde{A}_i| < 1 , \ |\tilde{B}_i| < \sqrt{\sqrt{2}-1} < 1 , i = 1,2 . \tag{10}$$

These inequalities mean that all A_i and B_i belong to minimal numbers.

Taking into account the above-mentioned parity properties, we conclude that all fundamental group matrices for the regular octagon must have form (4) where:

$$\begin{array}{ll} A_1 & \text{is an odd minimal number} \\ A_2 & \text{is an even minimal number} \\ B_1 & \text{and } B_2 \text{ are minimal numbers of the same parity .} \end{array} \tag{11}$$

The length l of a periodic trajectory corresponding to a fundamental group matrix M can be calculated from the relation [1,2]:

$$\cosh\frac{l}{2} = \frac{1}{2}|\mathrm{Tr}M| = |A_1| . \tag{12}$$

Combining this with (11), one obtains formulae (2), (3) which were proposed before in Ref. [2].

Now we prove the reverse statement, i. e. that any matrix of form (4) with unit determinant and with A_i, B_i obeying (11) belongs to the fundamental group of the regular octagon. Our proof will be based on the theorem proven in Ref. [4] and cited in Ref. [1]. (Actually, Ref. [4] was not available to us).

According to this theorem the group of all matrices of form (4) with unit determinant differs from the considered "octagon group" by the existence of an additional generator

$$R_\pi = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{13}$$

with the properties

$$R_\pi b_k R_\pi^{-1} = b_{k+4} = b_k^{-1} , \ R_\pi^2 = -1 . \tag{14}$$

The theorem states that an arbitrary matrix (4) can be represented as a word constructed from the generators b_k and the additional matrix R_π. But according to (14) matrices with an even number of R_π's can be reduced to fundamental group matrices (without any R_π), and matrices with an odd number of R_π's can be reduced to matrices with one R_π. Therefore, an arbitrary matrix (4) with algebraic elements A_1, A_2, B_1, B_2 belongs either to a fundamental group matrix or to a product of a fundamental group matrix with one R_π. It is not difficult to find a criterium which distinguishes these two cases. As was indicated above, A_1 must be an odd algebraic number for any fundamental group matrix and A_2 must be an even one. The application of R_π to a matrix (4) results in the following substitution: $A_1 \to -A_2$, $A_2 \to A_1$ and $B_1 \to -B_2$, $B_2 \to B_1$. Hence, for a product of R_π and a fundamental group matrix the A_1 element will be an even algebraic number and A_2 will be an odd one. (B_1 and B_2 will be, as before, numbers of the same parity). This means that the parity of A_1 uniquely discriminates between these two cases. If A_1 is an odd number, the matrix (4) belongs to the fundamental group, and if A_1 is an even number, the matrix (4) is a product of R_π and a fundamental group matrix.

Thus we have proved the following Theorem: The necessary and sufficient condition that a matrix (4) with unit determinant belongs to the fundamental group of the regular octagon is that the A_1 element is an even minimal number (and all other elements obey (11)).

From this theorem it follows that to construct a fundamental group matrix it is enough to sort out minimal algebraic numbers obeying conditions (11) and select from them those obeing (6). Let us emphasize that the minimality conditions (7) (or(8)), which in the end is a simple consequence of the unit determinant condition (6), is of very importance. As it will be shown below, due to this condition it will be enough to sort out only a finite number of minimal numbers in order to find all periodic trajectories with a fixed length.

III Symmetry transformations

Let

$$\begin{aligned} A_1 &= m_1 + n_1\sqrt{2}, \quad A_2 = m_2 + n_2\sqrt{2}, \\ B_1 &= l_1 + k_1\sqrt{2}, \quad B_2 = l_2 + k_2\sqrt{2} \end{aligned} \tag{15}$$

be the representation of the matrix elements in terms of integers. As was noted above, m_1 is an odd integer, m_2 is an even integer, whereas l_1 and l_2 can be either even or odd, but must have the same parity.

We present here a few more parity properties. From Eq.(6) one can show the following:

1) if n_1 is even, then n_2, l_1 and l_2 are even and k_1 and k_2 are of the same parity;

2) if n_1 is odd and l_1, l_2 are even, then n_2 is even and k_1 and k_2 are of different parity;

3) if n_1 is odd and l_1, l_2 are odd, then n_2 is even or odd depending on whether k_1 and k_2 have the same or opposite parity.

From Fig.1 it is clear that the simplest symmetry of the octagon is a rotation over $\pi/4$. If A_1, A_2, B_1, B_2 define an admissible matrix (4), then A_1, A_2, B_1', B_2' with

$$\begin{aligned} B_1' &= (B_1 - B_2)/\sqrt{2} \\ B_2' &= (B_1 + B_2)/\sqrt{2} \end{aligned} \tag{16}$$

also give an admissible fundamental group matrix. Note that an inverse matrix corresponds to A_1, $-A_2$, $-B_1$, $-B_2$. The reflection over the coordinate axis is equivalent to the inversion with respect to the line which has the angle $\pi/8$ with the abscissa. (It is denoted by 1 in Fig 1). This inversion corresponds to the following transformation:

$$T_1 \; : \; \begin{cases} A_1' = A_1 \\ A_2' = -A_2 \\ B_1' = (B_1 + B_2)/\sqrt{2} \\ B_2' = (B_1 - B_2)/\sqrt{2} \end{cases} , \; T_1^2 = 1 \, . \tag{17}$$

The inversion over circle 2 in Fig.1 gives the transformation

$$T_2 \; : \; \begin{cases} A_1' = A_1 \\ A_2' = -(\sqrt{2}+1)A_2 - \sqrt{2}B_2 \\ B_2' = (2+\sqrt{2})A_2 + (\sqrt{2}+1)B_2 \\ B_1' = -B_1 \end{cases} , \; T_2^2 = 1 \, . \tag{18}$$

Analogously, the inversion over circle 3 in Fig.1 corresponds to the transformation

$$T_3 \; : \; \begin{cases} A_1' = A_1 \\ A_2' = -(1+\sqrt{2})A_2 + B_1 - B_2 \\ B_1' = B_1/\sqrt{2} - (1+\sqrt{2})/\sqrt{2}B_2 - (1+\sqrt{2})A_2 \\ B_2' = -(1+\sqrt{2})/\sqrt{2}B_1 + B_2/\sqrt{2} + (1+\sqrt{2})A_2 \end{cases} , \; T_3^2 = 1 \, . \tag{19}$$

Note that

$$T_2T_1 = T_3T_2 \tag{20}$$

as it must be for the reflections over 3 lines having an angle of 60° with each other. And, finally, the inversion over circle 4 in Fig.1 gives

$$T_4 \; : \; \begin{cases} A_1' = A_1 \\ A_2' = -(3+2\sqrt{2})A_2 + \sqrt{2}B_2 - (2+\sqrt{2})B_2 \\ B_1' = -(1+\sqrt{2})B_2 - (2+\sqrt{2})A_2 \\ B_2' = (4+3\sqrt{2})A_2 - (1+\sqrt{2})B_1 + (2+2\sqrt{2})B_2 \end{cases} , \; T_4^2 = 1 \, . \tag{21}$$

Two important properties of the transformations (17)-(21) are:

i) If A_1, A_2, B_1, B_2 obey Eq.(6), then A_1, A_2', B_1', B_2' also obey this relation.

ii) If A_1, A_2, B_1, B_2 are integer algebraic numbers of type (15) obeying (11), then A_1, A_2', B_1', B_2' will be also integer algebraic numbers obeying (11).

Any sequence of these transformations will be an admissible transformation. So, knowing a fundamental group matrix, one can construct another one with the same trace by the above symmetry transformations.

IV Separation of conjugacy classes

In this Section we discuss the connection between fundamental group matrices and periodic trajectories. It is known (see e.g. Ref.[1]) that if M is a certain fundamental group matrix, then all conjugated matrices

$$M' = SMS^{-1} \, . \tag{22}$$

where S is an arbitrary fundamental group matrix, correspond to the same periodic trajectory. Therefore, to enumerate the periodic trajectories it is necessary to know which fundamental group matrices are conjugated to each other. If the matrices are given as products of fundamental group generators, then a pure algebraic algorithm exists [2], which solves this problem within a finite number of steps.

In the approach developed in this paper, we can construct any fundamental group matrix directly, but, a priori, we do not know its representation as a generator product, and the question of the separation of conjugacy classes has to be considered in detail.

Let us recall a few general facts [1]. Any geodesic on the Poincaré disc is a circle which is perpendicular to the boundary circle $|z| = 1$. Inside the fundamental region a closed geodesic (i.e. a periodic trajectory) is a set of segments of such circles connected with each other by the identification of the boundary arcs via the generators (1). An arbitrary fundamental group matrix of the form

$$M = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \tag{23}$$

with unit determinant defines the linear fractional transformation ($z = x + iy$)

$$z' = \frac{\alpha z + \beta}{\beta^* z + \alpha^*} , \tag{24}$$

which leaves the circle $|z| = 1$ invariant.

Simultaneously, a matrix (23) defines a unique geodesic on the Poincaré disc which is not changed by the transformation (24). In Cartesian coordinates this invariant geodesic is given by the equation

$$x^2 + y^2 - \frac{2}{\alpha_2}(\beta_1 y - \beta_2 x) + 1 = 0 , \tag{25}$$

where β_1, β_2, α_1, α_2 are real and imaginary parts, respectively, of β and α:

$$\beta = \beta_1 + i\beta_2 , \ \alpha = \alpha_1 + i\alpha_2 . \tag{26}$$

If $\alpha_2 = 0$, then the invariant geodesic is the straight line

$$\beta_1 y = \beta_2 x . \tag{27}$$

It is not difficult to show that geometrically the conjugated matrix (22) is the result of the translation of the circle (25) (corresponding to the matrix M) under a transformation of type (24) defined by the matrix S.

Let us assume that we know a fundamental group matrix and we want to construct the corresponding circle (25) on the Poincaré disc. Two variants are possible. Either the circle (25) goes through the fundamental domain or it entirely lies outside of it. Only the first case corresponds to an arc of a periodic trajectory of the free motion on the surface considered. The second case has to be considered as the result of a transformation of a geodesic under the action of a fundamental group matrix. This means that we have not to consider matrices for which the invariant circle (25) lies outside the fundamental region.

The necessary and sufficient condition that the circle (25) goes through the fundamental octagon is that the distance between the centre of the circle and a certain corner of the

octagon is smaller or equal to the radius of (25). If the matrix M is written in the form (4), this condition is equivalent to the following inequality

$$|A_2| \le (2 - \sqrt{2})(|B_1| + (\sqrt{2} - 1)|B_2|) , \tag{28}$$

where we assume that $|B_1| \ge |B_2|$ (this can always be achieved by rotations over $\pi/4$ as in Eq. (17)).

Using Eq. (6) one obtains the following condition on B_1, B_2 (assuming $B_1 \ge B_2 \ge 0$)

$$B_1^2 + 5B_2^2 - 4B_1B_2 \le (1 + \sqrt{2})^3(A_1^2 - 1) . \tag{29}$$

Therefore, if the length of the geodesic is fixed (i.e. A_1 is fixed), there exists only a finite number of matrices (4) which we have to consider. These and only these matrices correspond to invariant geodesics (25) which go through the fundamental region.

To numerically construct such matrices, we shall use the following algorithm.

Fix the matrix trace, i.e. choose the number A_1. We repeat that A_1 must be an odd minimal number of the form $m_1 + n_1\sqrt{2}$ which is uniquely defined by the integer n_1. The numbers B_1 and B_2 have to be minimal numbers with equal parity, so at fixed k_1, k_2 there exist two sets of B_1, B_2 with even and odd l_1, l_2. Let us sort out all k_1, k_2 such that B_1, B_2 obey inequality (29). As the left-hand side of it is the equation of an ellipse there are only a finite number of pairs k_1, k_2 obeying it. For each of such pair we find A_2 from Eq. (6) and keep only such cases where A_2 has the form $m_2 + n_2\sqrt{2}$ with m_2, n_2 being integers (and m_2 is even). This can be achieved, e.g. by a direct solution of Eq. (6) over variables m_2, n_2 or by sorting out all n_2 obeying (28). It is these cases that give the fundamental group matrices (4) with unit determinant.

Let us consider such a matrix. By construction, i.e. by Eq. (28) an arc of the geodesic corresponding to this matrix is located inside the fundamantal octagon. In general, this arc is restricted from both sides by two boundary arcs of our octagon. Since the opposite sides of the octagon are identified, the part of the geodesic which goes out of the octagon has to be brought back with the aid of a proper generator (1). As was mentioned above, this corresponds to the conjugation (22) where S is the matrix of this generator. This means that after the conjugation of the initial matrix with a certain generator (1) one obtains a matrix whose geodesic goes through the fundamental region (and which obeys (28)).

It is clear geometrically that in general (when the geodesic does not go through a corner of the fundamental region) there exist only two generators which have this property. They correspond to the two boundary arcs which are crossed by the geodesic considered.

This suggests the following scheme for the construction of periodic trajectories. Let us assume that we know a fundamental group matrix (4) which obeys (28), i.e. its geodesic (25) goes through the fundamental octagon. Do the conjugation (22) with all 8 generators (1). For all matrices obtained by this procedure check condition (28). As was noted, in general this condition will be satisfied only for two generators, hence only for two matrices the corresponding geodesic goes through the fundamental octagon. Choose one of these two matrices and repeat the conjugation with the 8 generators once more. As before only two generators give matrices whose geodesics lie inside the fundamental region. One of them is the inverse to the generator used in the first step. Therefore only one gives a new matrix to be considered. Repeat this process until the new matrix does coincide with the initial one. (As our matrices belong to the fundamental group, this procedure will stop after a finite number of steps.)

Let $k_1, k_2, \ldots, k_n$ be the labels of the generators obtained by the described procedure and

$$\begin{aligned} M_{k_1} &= b_{k_1} M b_{k_1}^{-1} \\ M_{k_2} &= b_{k_2} M_{k_1} b_{k_2}^{-1} \\ &\vdots \\ M_{k_n} &= b_{k_n} M_{k_{n-1}} b_{k_n}^{-1} \end{aligned} \tag{30}$$

be the sequence of the corresponding matrices. By construction, $\mathrm{tr} M_{k_m} = \mathrm{tr} M$, and each M_{k_m} obeys the condition (28), where the last matrix satisfies

$$M_{k_n} = M \ . \tag{31}$$

This condition gives a representation of the initial matrix M as a product of generators:

$$M = b_{k_n} b_{k_{n-1}} \ldots b_{k_1} \ . \tag{32}$$

The periodic trajectory corresponding to the initial matrix M consists exactly of those segments of geodesics (25) which correspond to the matrices M_{k_j} $(j = 1, 2, \ldots, n)$. Simultaneously, the proposed method gives, as a by-product, the canonical representation of the matrix as a product of generators.

In cases where the geodisic goes through a corner of the fundamental region the arguments have to be slightly modified but we don't dwell on it here.

Actually, the initial matrix M (or, strictly speaking, the geodesic corresponding to it by Eq. (25)) defines the position and the momentum of the point particle in the initial moment. The subsequent motion is uniquely defined by these values. Between boundaries the particle moves on a geodesic - an arc of a circle of the form (25). When the particle collides with a boundary arc, it is transformed to another boundary arc according to the identification used. It is this process that is described by the sequence of matrices (30). It is not difficult to construct an explicit algorithm which describes the motion inside the fundamental region directly. The above method with the sorting out of all generators seems to be more algebraic and more suitable for numerical calculations.

The method outlined above was used to write a program which permits to find all periodic trajectories for a given length. The detailed description of this program and results of the calculations will be published elsewhere. Here we mention the results only for the first 80 lengths presented in Table 1 of Ref. [2] where products of up to 11 generators were calculated. We find that the length spectrum given in this Table is almost complete; there are only two lengths for which the multiplicity is too low, since products of 12 generators give a contribution. They correspond to $A_1 = 97 + 68\sqrt{2}$ and $A_4 = 97 + 69\sqrt{2}$ for which the multiplicities have to be equal to 48 and 576 instead of 40 and 560 respectively.

V Summary

In this note it is shown that the fundamental group matrices for the regular octagon can be expressed in terms of minimal algebraic numbers with well defined parity (see Eq. (11)). This proves the geodesic-length law conjectured in Ref. [2] on the ground of numerical calculations. A geometrical method for the separation of conjugacy classes is proposed. The method permits to construct a simple algorithm for calculating all periodic trajectories with a given length.

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Figure caption

Fig. 1: Regular octagon on the Poincaré disc. Numbers denote circles of inversion associated with the symmetries $T_1, \ldots, T_4$.

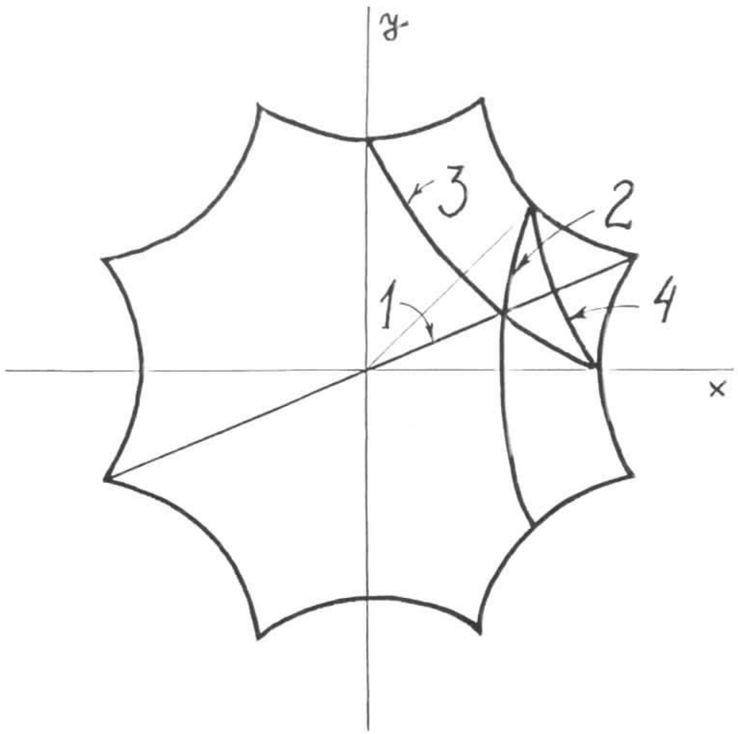

Fig. 1