DEUTSCHES ELEKTRONEN – SYNCHROTRON DESY

DESY 90-001
January 1990

On the Impact of Linear Coupling on Nonlinear Dynamics

G. Ripken, F. Willeke

Deutsches Elektronen-Synchrotron DESY, Hamburg

ISSN 0418-9833

NOTKESTRASSE 85 · 2 HAMBURG 52

To be sure that your preprints are promptly included in the
HIGH ENERGY PHYSICS INDEX,
send them to the following address (if possible by air mail):

DESY
Bibliothek
Notkestrasse 85
2 Hamburg 52
Germany

DESY 90-001
January 1990

ISSN 0418-9833

On the Impact of Linear Coupling on Nonlinear Dynamics

G. Ripken, F. Willeke

July, 1989

Abstract

The impact of the interference of linear coupling with nonlinear fields on the dynamics of accelerator beams is studied. A systematic canonical perturbation theory for linearly coupled systems has been developed. The analysis shows that linear coupling causes as a new feature the excitation of skew resonances ($nQ_x + mQ_y = integer$; m odd) by nonlinear fields with midplane symmetry. These effects occur in the vicinity of the coupling resonance $Q_x - Q_y = integer$. Broadening of the width of nonlinear resonance clusters is a consequence of the split in the eigenfrequencies. One has to expect from these effects that the dynamic aperture limit in storage rings is reduced along the main tune diagonal. The reduction depends on the size of the tune split and, in the case of randomly excited coupling, on the random distortions of the lattice functions which disturb the intrinsic cancellation of nonlinear effects in large regular lattices. All these effects can easily be controlled by compensating the coupling resonance using skew quadrupoles. Our conclusion is that when operating on the coupling resonance, full linear coupling does not significantly enhance the nonlinear effects of sextupole fields. The formalism has been applied to the HERA proton ring which suffers from large coupling effects coming from strong skew quadrupole components of the superconducting magnets at low excitation. Predictions derived from the theory agree well with the results of particle tracking.

Contents

1 Introduction **1**

2 Variation of Constants in the Coupled Case **2**
2.1 The Hamiltonian 2
2.2 The Unperturbed System. 3
2.2.1 The Equations of Motion for the Unperturbed System. 3
2.2.2 Eigenvectors for the Particle Motion ; Floquet-Theorem. 5
2.3 The Perturbed System. 9

3 Nonlinear Perturbation Theory **14**

4 Estimation of Coupled Lattice Functions **19**

5 Analytical Estimation of Skew Resonances Near the Coupling Resonances **24**

6 Expectation Value of Driving Terms of Nonlinear Resonances excited by Random Field Errors **28**

7 Application to HERA and Comparison with Simulation **29**

8 Conclusion **33**

Acknowledgements **34**

Appendix A: Lattice Functions for Linear Coupled Motion **34**

Appendix B: Perturbation Theory for Linear Coupled Motion **37**
B.1 Calculation for the Perturbed Part of the Revolution Matrix 38
B.2 Perturbation Theory 39
B.2.1 Outside the Resonances 39
B.2.2 Resonances 43

References **49**

1 Introduction

Usually, linear coupling between horizontal and vertical betatron oscillations is undesirable in circular accelerators because it complicates the motion of the beam and this causes operational problems. Very little is known about the interference of linear coupling and the nonlinearities in an accelerator. In the HERA electron-proton collider this question is of special interest for two reasons. The main dipole field generated by the superconducting magnets of the proton ring exhibits a considerable relative random skew quadrupole component ($a_2^{rms} = 3 \cdot 10^{-4}$ for $r = 25mm$ at injection fields [1]) which excites the coupling resonance with an rms width of $\kappa = 0.04$. Only a single pair of skew quadrupoles was planned for HERA and there is some concern that residual effects of the skew quadrupole will further reduce the dynamic aperture which is already limited by strong persistent current dominated nonlinear field errors. The second reason why it is interesting to investigate this question for HERA is that a naturally flat electron beam will collide with a round proton beam. In $\bar{p}-p$ collisions it has been observed that when $\bar{p}$ and p beam sizes at the collision point are different, the large amplitude particles in the larger beam are lost quickly[2]. Therefore in HERA we expect that the protons with vertical oscillation amplitudes exceeding the electron beam size will be lost. So, in great contrast to $e^+ - e^-$-colliders, a round beam is desirable in the HERA electron ring for colliding beam operation. A convenient way of generating a large vertical emittance is linear coupling. The interference of linear coupling with the nonlinearities of the machine has therefore to be investigated. We start our investigation by the formulation of nonlinear accelerator theory for coupled betatron oscillations. In particular, explicit expressions for the widths of nonlinear resonances will be derived for linearly coupled motion. In the case that coupling is generated by random errors, we are able to give expectation values for the widths of nonlinear resonances. Using these expressions, the motion of the particles in a beam in the vicinity of resonances can be described. In parallel, numerical particle tracking has been performed for HERA in order to check the results obtained by analytical treatment.

2 Variation of Constants in the Coupled Case

2.1 The Hamiltonian

We use the notation of Ref. [3]. Then the Hamiltonian for coupled betatron motion is given by:

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}\cdot[p_x + H\cdot y]^2 + \frac{1}{2}\cdot[p_y - H\cdot x]^2 + \frac{1}{2}\cdot[K_x^2(s) + k(s)]\cdot x^2 \\ &+ \frac{1}{2}\cdot[K_y^2(s) - k(s)]\cdot y^2 + N(s)\cdot xy + \sum_{n+m\geq 3} c_{nm}x^n y^m \ . \end{aligned} \tag{2.1}$$

The corresponding canonical equations read as:

$$x' = p_x + H\cdot y \ ; \tag{2.2a}$$

$$p_x' = -\left[\left(K_x^2 + k\right) + H^2\right]\cdot x - N\cdot y + H\cdot p_y + \sum_{n,m} n\cdot c_{nm}x^{n-1}y^m \ ; \tag{2.2b}$$

$$y' = p_y - H\cdot x \ ; \tag{2.2c}$$

$$p_y' = -\left[\left(K_y^2 - k\right) + H^2\right]\cdot y - N\cdot x - H\cdot p_x + \sum_{n,m} m\cdot c_{nm}x^n y^{m-1} \ . \tag{2.2d}$$

In detail, one has:

a)	$k \neq 0;$	$N = H = K_x = K_y = c_{nm} = 0:$	quadrupole;
b)	$N \neq 0;$	$k = H = K_x = K_y = c_{nm} = 0:$	skew quadrupole;
c)	$K_x^2 + K_y^2 \neq 0;$	$k = N = H = c_{nm} = 0:$	bending magnet;
d)	$H \neq 0;$	$k = N = K_x = K_y = c_{nm} = 0:$	solenoid;
e)	$c_{nm} \neq 0;$	$k = N = H = K_x = K_y = 0:$	multipole.

Eliminating p_x and p_y, we obtain from eqn. (2.2):

$$x'' + \left[(K_x^2 + k) + H^2\right]\cdot x + (N - H')\cdot y - 2H\cdot y' + \sum_{n,m} n\cdot c_{nm}x^{n-1}y^m = 0 \ ; \tag{2.3a}$$

$$y'' + \left[(K_y^2 - k) + H^2\right]\cdot y + (N + H')\cdot x + 2H\cdot x' + \sum_{n,m} m\cdot c_{nm}x^n y^{m-1} = 0 \ . \tag{2.3b}$$

These equations represent a periodic canonical system described by a Hamiltonian

$$\mathcal{H}(x, y, p_x, p_y, s) = \mathcal{H}^{(0)} + \mathcal{H}^{(1)} \tag{2.4}$$

with an unperturbed part

$$\mathcal{H}^{(0)} = \sum_{\mu_1+\mu_2+\mu_3+\mu_4=2} c_{\mu_1\mu_2\mu_3\mu_4}(s)\cdot x^{\mu_1}p_x^{\mu_2}y^{\mu_3}p_y^{\mu_4} \tag{2.5a}$$

and a perturbative part

$$\mathcal{H}^{(1)} = \sum_{\nu=3}^{\infty}\sum_{n+m=\nu} c_{nm}(s)\cdot x^n y^m \tag{2.5b}$$

where s designates the distance around the ring and is the independent variable of the system of equations and the $c(s)$ coefficients are periodic functions.

If we designate the periodicity length of the system by L (length of one turn) the periodicity condition reads as:

$$\mathcal{H}(x, y, p_x, p_y, s + L) = \mathcal{H}(x, y, p_x, p_y, s) \ . \tag{2.6}$$

Combining the coordinates x, y and momenta p_x, p_y into the vector

$$\vec{z}(s) \ = \ \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \tag{2.7}$$

the canonical equations of motion may be written in the form

$$\frac{d}{ds} \vec{z} \ = \ \underline{S} \cdot \frac{\partial \mathcal{H}}{\partial \vec{z}} \tag{2.8}$$

where the matrix $\underline{S}$ is given by

$$\underline{S} = \begin{pmatrix} \underline{S}_2 & \underline{0} \\ \underline{0} & \underline{S}_2 \end{pmatrix} \ ; \ \ \underline{S}_2 = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \ . \tag{2.9}$$

2.2 The Unperturbed System.

2.2.1 The Equations of Motion for the Unperturbed System.

Taking into account only the first component, $\mathcal{H}^{(0)}$, of the Hamiltonian (2.4) we obtain from (2.8) the equations of motion for the unperturbed system:

$$\frac{d}{ds} \vec{z}^{\,(0)} \ = \ \underline{S} \cdot \frac{\partial \mathcal{H}^{(0)}}{\partial \vec{z}^{\,(0)}}$$

or

$$\frac{d}{ds} \vec{z}^{\,(0)} = \underline{A} \cdot \vec{z}^{\,(0)} \tag{2.10a}$$

with

$$\underline{A} \cdot \vec{z}^{\,(0)} = \underline{S} \cdot \frac{\partial \mathcal{H}^{(0)}}{\partial \vec{z}^{\,(0)}} \tag{2.10b}$$

and

$$\vec{z}^{\,(0)} = \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} \ .$$

In the special case (2.1) one has:

$$\begin{aligned} \mathcal{H}^{(0)} \ &= \ \frac{1}{2} \cdot [p_x + H(s) \cdot y]^2 + \frac{1}{2} \cdot [p_y - H(s) \cdot x]^2 + \frac{1}{2} \cdot [K_x^2(s) + k(s)] \cdot x^2 \\ &+ \ \frac{1}{2} \cdot [K_y^2(s) - k(s)] \cdot y^2 + N(s) \cdot xy \end{aligned} \tag{2.11a}$$

and

$$\underline{A}(s) \;=\; \begin{pmatrix} 0 & 1 & H & 0 \\ -[(K_x^2+k)+H^2] & 0 & -N & H \\ -H & 0 & 0 & 1 \\ -N & -H & -[(K_y^2-k)+H^2] & 0 \end{pmatrix} . \qquad (2.11b)$$

Because the equations of motion (2.10) are linear, the solution can be written in the form:

$$\vec{z}^{\,(0)}(s) = \underline{M}(s,s_0)\; \vec{z}^{\,(0)}(s_0) \qquad (2.12)$$

which defines the transfer matrix $\underline{M}(s,s_0)$.

From (2.10), $\underline{M}(s,s_0)$ is determined by the differential equations

$$\begin{aligned} \frac{d}{ds}\,\underline{M}(s,s_0) &= \underline{A}(s)\cdot\underline{M}(s,s_0)\;; \qquad (2.13a) \\ \underline{M}(s_0,s_0) &= \underline{1}\;. \qquad (2.13b) \end{aligned}$$

Since the variables $x,\ p_x,\ y,\ p_y$ are canonical, the transfer matrix is symplectic [3] :

$$\underline{M}^T(s,s_0)\cdot\underline{S}\cdot\underline{M}(s,s_0) = \underline{S}\;. \qquad (2.14)$$

The symplecticity condition (2.14) ensures that the transfer matrix , $\underline{M}(s,s_0)$, contains complete information about the stability of the (linear) betatron motion.

Finally, we mention for later considerations that the coefficient matrix $\underline{A}(s)$ of eqn. (2.10) satisfies the condition

$$\underline{A}^T(s)\cdot\underline{S}+\underline{S}\cdot\underline{A}(s) = \underline{0}\;. \qquad (2.15)$$

This can be obtained by differentiating eqn (2.14) with respect to s and using eqn. (2.13).

2.2.2 Eigenvectors for the Particle Motion ; Floquet-Theorem.

In order to prepare the investigation of the perturbed system we first investigate the eigenmotion of the particles for the unperturbed problem.

To begin, we note that from the symplecticity condition of eqn. (2.14) and with the aid of arbitrary solution vectors $\vec{z_1}$ and $\vec{z_2}$ of eqn. (2.10), one can construct a constant of motion for the betatron oscillation, the so called Lagrange invariant [3,4]

$$W[\vec{z_1}(s),\vec{z_2}(s)] = \vec{z_2}^{\,T}(s)\cdot\underline{S}\cdot\vec{z_1}(s) = \text{ const }. \qquad (2.16)$$

With the help of this invariant one is in a position to study the eigenvalue spectrum of the one turn transfer matrix $\underline{M}(s_0+L,s_0)$:

$$\begin{aligned} \underline{M}(s_0+L,s_0)\;\vec{v}_\mu(s_0) &= \lambda_\mu\cdot\vec{v}_\mu(s_0)\;; \qquad (2.17) \\ (\mu &= 1,2,3,4)\;. \end{aligned}$$

As is shown in Ref. [3] the following statements are then valid:

1) The eigenvectors of $\underline{M}$ can be separated into two groups

$$(\vec{v}_k,\ \vec{v}_{-k})\ ;\ k = I,\ II$$

with the properties

$$\underline{M}\ \vec{v}_k = \lambda_k \cdot \vec{v}_k\ ;\ \ \underline{M}\ \vec{v}_{-k} = \lambda_{-k} \cdot \vec{v}_{-k}\ ;\ \ \lambda_k \cdot \lambda_{-k} = 1\ ; \tag{2.18a}$$

$$\left\{ \begin{array}{l} (\vec{v}_{-k})^T \cdot \underline{S} \cdot \vec{v}_k = -(\vec{v}_k)^T \cdot \underline{S} \cdot \vec{v}_{-k} \neq 0\ ; \\ (\vec{v}_\mu)^T \cdot \underline{S} \cdot \vec{v}_\nu = 0 \ \text{ otherwise}\ ; \end{array} \right. \tag{2.18b}$$

$$(k = I, II)\ .$$

In the following we put :

$$\left\{ \begin{array}{l} \lambda_k = e^{i \cdot 2\pi Q_k}\ ; \\ \lambda_{-k} = e^{i \cdot 2\pi Q_{-k}}\ ; \end{array} \right. \tag{2.19}$$

$$(k = I, II)\ .$$

Then according to eqn. (2.18a) :

$$Q_{-k} = -Q_k\ , \tag{2.20}$$

where Q_k can be either real or complex.

2) Eqn. (2.18a) shows that the eigenvalues of $\underline{M}(s_0 + L, s_0)$ always appear in reciprocal pairs

$$(\lambda_k,\ \lambda_{-k} = 1/\lambda_k)\ ; \tag{2.21}$$

$$(k = I, II)\ .$$

Since $\underline{M}(s_0 + L, s_0)$ is real, then λ^* as well as λ is an eigenvalue.

For the eigenvalue spectrum of $\underline{M}(s_0 + L, s_0)$ there are then various possibilities [5] as illustrated in Fig. 1.

In the following we assume that the optical conditions have been set up so that the storage ring orbits are stable, i.e. all 4 eigenvalues are complex with unit absolute value [3] and therefore lie on a unit circle in the complex plane (case a)) :

$$|\lambda_k| = |\lambda_{-k}| = 1\ ;\ \ (k = I,\ II)\ ; \tag{2.22}$$

(stability criterion) .

Then :

$$Q_k \ \text{ real}\ ; \tag{2.23}$$

$$\lambda_k = \lambda^*_{-k}\ ;\ \ \vec{v}_{-k} = (\vec{v}_k)^*\ .$$

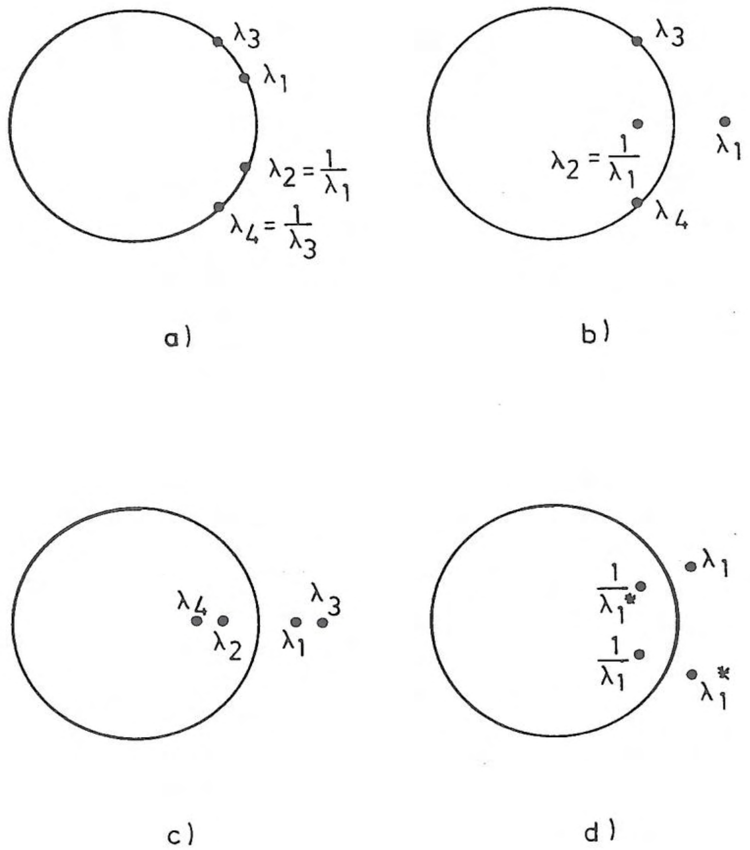

Figure 1: Location of the eigenvalues.

3) We define :

$$\vec{v}_\mu(s) = \underline{M}(s, s_0)\ \vec{v}_\mu(s_0)\ . \tag{2.24}$$

Then the vector $\vec{v}_\mu(s)$ is an eigenvector of the matrix $\underline{M}(s+L, s)$ with the eigenvalue λ_μ:

$$\underline{M}(s+L, s)\ \vec{v}_\mu(s) \ = \ \lambda_\mu \cdot \vec{v}_\mu(s)\ . \tag{2.25}$$

The eigenvector $\vec{v}_\mu(s)$ thus has the same eigenvalue as $\vec{v}_\mu(s_0)$: The eigenvalue is therefore independent of s.

4) We put

$$\vec{v}_\mu(s) = \vec{u}_\mu(s) \cdot e^{i \cdot 2\pi Q_\mu \cdot (s/L)}\ . \tag{2.26a}$$

Then :

$$\vec{u}_\mu(s+L) = \vec{u}_\mu(s)\ . \tag{2.26b}$$

Eqn. (2.26) is a statement of the Floquet theorem : Vectors $\vec{v}_\mu(s)$ are special solutions of the equations of motion (2.10) which can be expressed as the product of a periodic function $\vec{u}_\mu(s)$ and a harmonic function

$$e^{i \cdot 2\pi Q_\mu \cdot (s/L)}\ .$$

5) The general solution of the equation of motion (2.10) is a linear combination of the special solutions (2.26a) and can be therefore written in the form

$$\vec{z}(s) = \sum_{k=I,II} \left\{ A_k \cdot \vec{u}_k(s) \cdot e^{i \cdot 2\pi Q_k \cdot (s/L)} + A_{-k} \cdot \vec{u}_{-k}(s) \cdot e^{i \cdot 2\pi Q_{-k} \cdot (s/L)} \right\} . \tag{2.27}$$

6) Taking into account the stability condition (2.22) we get from eqn. (2.23):

$$\vec{v}_{-k} = (\vec{v}_k)^* \; ; \quad (k = I,\ II) \; , \tag{2.28}$$

and (2.18b) simplifies to $\left(\vec{v}^{+} = (\vec{v}^{\,T})^*\right)$:

$$\left\{ \begin{array}{l} \vec{v}_k^{\,+}(s_0) \cdot \underline{S} \cdot \vec{v}_k(s_0) = -\vec{v}_{-k}^{\,+}(s_0) \cdot \underline{S} \cdot \vec{v}_{-k}(s_0) \neq 0 \; ; \\ \vec{v}_\mu^{\,+}(s_0) \cdot \underline{S} \cdot \vec{v}_\nu(s_0) = 0 \;\; \text{for } \mu \neq \nu \; ; \end{array} \right. \qquad (k = I, II) \; . \tag{2.29}$$

Furthermore the terms $\vec{v}_\mu^{\,+}(s_0) \cdot \underline{S} \cdot \vec{v}_\mu(s_0)$ in eqn. (2.29) are pure imaginary :

$$\begin{aligned} \left[\vec{v}_\mu^{\,+}(s_0) \cdot \underline{S} \cdot \vec{v}_\mu(s_0)\right]^+ &= \vec{v}_\mu^{\,+}(s_0) \cdot \underline{S}^+ \cdot \vec{v}_\mu(s_0) \\ &= -\left[\vec{v}_\mu^{\,+}(s_0) \cdot \underline{S} \cdot \vec{v}_\mu(s_0)\right] \end{aligned}$$

(since $\underline{S}^+ = -\underline{S}$), so that in future the vectors $\vec{v}_k(s_0)$ and $\vec{v}_{-k}(s_0)$ $(k = I,\ II)$ will be normalised according to :

$$\vec{v}_k^{\,+}(s_0) \cdot \underline{S} \cdot \vec{v}_k(s_0) = -\vec{v}_{-k}^{\,+}(s_0) \cdot \underline{S} \cdot \vec{v}_{-k}(s_0) = i \; ; \qquad (k = I, II) \; . \tag{2.30}$$

From the validity of the symplecticity condition (2.14) it then follows that the vectors $\vec{v}_k(s)$ und $\vec{v}_{-k}(s)$ (k=I, II) satisfy the conditions (2.29), (2.30) also at position s :

$$\left\{ \begin{array}{l} \vec{v}_k^{\,+}(s) \cdot \underline{S} \cdot \vec{v}_k(s) = -\vec{v}_{-k}^{\,+}(s) \cdot \underline{S} \cdot \vec{v}_{-k}(s) = i \; ; \\ \vec{v}_\mu^{\,+}(s) \cdot \underline{S} \cdot \vec{v}_\nu(s) = 0 \;\; \text{for } \mu \neq \nu \; . \end{array} \right. \tag{2.31a}$$

Note that the Floquet-vectors

$$\vec{u}_\mu(s) = \vec{v}_\mu(s) \cdot e^{-i \cdot 2\pi Q_\mu \cdot (s/L)}$$

then fullfill the same relationships:

$$\left\{ \begin{array}{l} \vec{u}_k^{\,+}(s) \cdot \underline{S} \cdot \vec{u}_k(s) = -\vec{u}_{-k}^{\,+}(s) \cdot \underline{S} \cdot \vec{u}_{-k}(s) = i \; ; \\ \vec{u}_\mu^{\,+}(s) \cdot \underline{S} \cdot \vec{u}_\nu(s) = 0 \;\; \text{for } \mu \neq \nu \; . \end{array} \right. \tag{2.31b}$$

Remark:

For the special case of a vanishing coupling between the the betatron oscillations (no skew quadrupoles and no solenoids) the revolution-matrix $\underline{M}(s+L,s)$ takes the form:

$$\underline{M}(s+L,s) = \begin{pmatrix} \underline{m_x}(s+L,s) & \underline{0}_2 \\ \underline{0}_2 & \underline{m_y}(s+L,s) \end{pmatrix} . \tag{2.32}$$

The symplecticity condition (2.14) now reads:

$$\underline{m_z}^T \cdot \underline{S}_2 \cdot \underline{m_z} = \underline{S}_2 \tag{2.33}$$

$(z = x,\ y)$ or

$$\text{Det } (\underline{m_z}) = 1 . \tag{2.34}$$

According to Courant and Snyder [5] we now can write for the revolution matrix $\underline{m_z}$:

$$\underline{m_z}(s+L,s) \quad = \quad \begin{pmatrix} \cos 2\pi Q_z + \alpha_z(s) \cdot \sin 2\pi Q_z & \beta_z(s) \cdot \sin 2\pi Q_z \\ -\gamma_z(s) \cdot \sin 2\pi Q_z & \cos 2\pi Q_z - \alpha_z(s) \cdot \sin 2\pi Q_z \end{pmatrix} \tag{2.35}$$

with

$$\beta_z \cdot \gamma_z - \alpha_z^2 = 1 \tag{2.36}$$

where in addition we require:

$$\beta_z \geq 0 . \tag{2.37}$$

Using this representation of $\underline{m_z}(s+L,s)$ we may calculate the normalized eigenvectors of the revolution matrix (2.32):

$$\vec{v}_I \quad = \quad \begin{pmatrix} \vec{w}_x \\ \vec{0}_2 \end{pmatrix} ; \tag{2.38a}$$

$$\vec{v}_{II} \quad = \quad \begin{pmatrix} \vec{0}_2 \\ \vec{w}_y \end{pmatrix} \tag{2.38b}$$

with the eigenvalues:

$$\begin{aligned} \lambda_I &= e^{i \cdot 2\pi Q_x} ; \\ \lambda_{II} &= e^{i \cdot 2\pi Q_y} \end{aligned} \tag{2.39}$$

and where the vector $\vec{w}_z$ $(z = x,\ y)$ is given by

$$\vec{w}_z(s) = \frac{1}{\sqrt{2\beta_z(s)}} \cdot \begin{pmatrix} \beta_z(s) \\ -[\alpha_z(s) - i] \end{pmatrix} \cdot e^{i \cdot \phi_z(s)} . \tag{2.40}$$

Comparing (2.39) and (2.19) we can make the following identifications for the decoupled case:

$$\begin{aligned} Q_I &\longleftrightarrow Q_x ; \\ Q_{II} &\longleftrightarrow Q_y . \end{aligned}$$

The stability condition (2.22) then reads:

$$Q_x, Q_y \quad \text{real}$$

or using (2.35):

$$-2 \leq \mathrm{Sp}\,(\underline{m}_z) \;\leq +2 \;. \tag{2.41}$$

In this way we rediscover the notation and the results of Courant-Snyder.

For more details see Refs. [7] and [8].

2.3 The Perturbed System.

Using these results we are now able to introduce a new set of canonical variables which will be important for further investigations.

We first remark that the general solution of the unperturbed equation of motion (2.10) may be written in the form (see (2.25a) and (2.26)):

$$\vec{z}(s) = \sum_{k=I,II} \{A_k \cdot \vec{v}_k(s) + A_{-k} \cdot \vec{v}_{-k}(s)\} \tag{2.42}$$

with A_k, A_{-k} being constants of integration ($k = I,\ II$).

In order to solve the perturbed problem (2.8) we now make the following "ansatz" (variation of constants) :

$$\vec{z}(s) = \sum_{k=I,II} \{A_k(s) \cdot \vec{v}_k(s) + A_{-k}(s) \cdot \vec{v}_{-k}(s)\} \;. \tag{2.43}$$

Writing then for the coefficients A_k and A_{-k} $(k = I, II)$

$$A_k = \sqrt{J_k(s)} \cdot e^{+i\psi_k(s)} \;; \tag{2.44a}$$

$$A_{-k} = \sqrt{J_k(s)} \cdot e^{-i\psi_k(s)} \tag{2.44b}$$

eqn. (2.43) takes the form:

$$\vec{z} = \sum_{k=I,II} \sqrt{J_k(s)} \cdot \left\{\vec{v}_k(s) \cdot e^{+i\psi_k(s)} + \vec{v}_{-k}(s) \cdot e^{-i\psi_k(s)}\right\} \equiv \vec{z}(s, \psi_k, J_k) \;. \tag{2.45}$$

Note that in eqns. (2.44) and (2.45) the s dependence in J_k, ψ_k is to be understood as implicit, not as explicit. We intend to treat the J_k, ψ_k as dynamical variables. The explicit s dependence of $\vec{z}(s, \psi_k, J_k)$ is incorporated in the eigenvectors $\vec{v}_{\pm k}(s)$ which obey the unperturbed equations of motion (2.10).

From (2.45) we now get:

$$\frac{d\vec{z}}{ds} = \frac{\partial \vec{z}}{\partial s} + \sum_k \frac{\partial \vec{z}}{\partial \psi_k} \cdot \psi_k' + \sum_k \frac{\partial \vec{z}}{\partial J_k} \cdot J_k' = \underline{S} \cdot \frac{\partial \mathcal{H}}{\partial \vec{z}} \;. \tag{2.46}$$

With

$$\frac{\partial \vec{z}}{\partial s} = \underline{S} \cdot \frac{\partial \mathcal{H}^{(0)}}{\partial \vec{z}} \tag{2.47}$$

we obtain:

$$\sum_k \frac{\partial \vec{z}}{\partial \psi_k} \cdot \psi'_k + \sum_k \frac{\partial \vec{z}}{\partial J_k} \cdot J'_k = \underline{S} \cdot \frac{\partial \mathcal{H}^{(1)}}{\partial \vec{z}} \ . \tag{2.48}$$

Furthermore we have from (2.45):

$$\frac{\partial \vec{z}}{\partial \psi_k} = +i \cdot \sqrt{J_k(s)} \cdot \left\{ \vec{v}_k(s) \cdot e^{+i\psi_k} - \vec{v}_{-k}(s) \cdot e^{-i\psi_k} \right\} \ ; \tag{2.49a}$$

$$\frac{\partial \vec{z}}{\partial J_k} = \frac{1}{2\sqrt{J_k(s)}} \cdot \left\{ \vec{v}_k(s) \cdot e^{+i\psi_k} + \vec{v}_{-k}(s) \cdot e^{-i\psi_k} \right\} \ . \tag{2.49b}$$

Taking into account the relations (2.31) one obtains the equations :

$$\left\{ \vec{v}_k^{\,+} \cdot e^{-i\psi_k} + \vec{v}_{-k}^{\,+} \cdot e^{+i\psi_k} \right\} \cdot \underline{S} \cdot \frac{\partial \vec{z}}{\partial \psi_l} = -2 \cdot \sqrt{J_k} \cdot \delta_{kl} \ ; \tag{2.50a}$$

$$\left\{ \vec{v}_k^{\,+} \cdot e^{-i\psi_k} - \vec{v}_{-k}^{\,+} \cdot e^{+i\psi_k} \right\} \cdot \underline{S} \cdot \frac{\partial \vec{z}}{\partial \psi_l} = 0 \ ; \tag{2.50b}$$

$$\left\{ \vec{v}_k^{\,+} \cdot e^{-i\psi_k} + \vec{v}_{-k}^{\,+} \cdot e^{+i\psi_k} \right\} \cdot \underline{S} \cdot \frac{\partial \vec{z}}{\partial J_l} = 0 \ ; \tag{2.50c}$$

$$\left\{ \vec{v}_k^{\,+} \cdot e^{-i\psi_k} - \vec{v}_{-k}^{\,+} \cdot e^{+i\psi_k} \right\} \cdot \underline{S} \cdot \frac{\partial \vec{z}}{\partial J_l} = 2i \cdot \frac{1}{2 \cdot \sqrt{J_k}} \cdot \delta_{kl} \tag{2.50d}$$

and it follows from (2.48) with the help of (2.28) and (2.50) :

$$\begin{aligned} -2 \cdot \sqrt{J_k} \cdot \psi'_k &= \left\{ \vec{v}_k^{\,+} \cdot e^{-i\psi_k} + \vec{v}_{-k}^{\,+} \cdot e^{+i\psi_k} \right\} \cdot \underline{S} \cdot \left[\underline{S} \cdot \frac{\partial \mathcal{H}^{(1)}}{\partial \vec{z}} \right] \\ &= -\left\{ (\vec{v}_{-k})^T \cdot e^{-i\psi_k} + (\vec{v}_k)^T \cdot e^{+i\psi_k} \right\} \cdot \frac{\partial \mathcal{H}^{(1)}}{\partial \vec{z}} \end{aligned}$$

or

$$\begin{aligned} \psi'_k &= \frac{1}{2 \cdot \sqrt{J_k}} \cdot \left\{ \vec{v}_{-k} \cdot e^{-i\psi_k} + \vec{v}_k \cdot e^{+i\psi_k} \right\}^T \cdot \frac{\partial \mathcal{H}^{(1)}}{\partial \vec{z}} \\ &= \frac{\partial \vec{z}^{\,T}}{\partial J_k} \cdot \frac{\partial \mathcal{H}^{(1)}}{\partial \vec{z}} \\ &= \frac{\partial \mathcal{H}^{(1)}}{\partial J_k} \end{aligned}$$

and

$$\begin{aligned} 2i \cdot \frac{1}{2 \cdot \sqrt{J_k}} \cdot J'_k &= \left\{ \vec{v}_k^{\,+} \cdot e^{-i\psi_k} - \vec{v}_{-k}^{\,+} \cdot e^{+i\psi_k} \right\} \cdot \underline{S} \cdot \left[\underline{S} \cdot \frac{\partial \mathcal{H}^{(1)}}{\partial \vec{z}} \right] \\ &= -\left\{ (\vec{v}_{-k})^T \cdot e^{-i\psi_k} - (\vec{v}_k)^T \cdot e^{+i\psi_k} \right\} \cdot \frac{\partial \mathcal{H}^{(1)}}{\partial \vec{z}} \end{aligned}$$

or

$$\begin{aligned} J'_k &= i \cdot \sqrt{J_k} \cdot \left\{ \vec{v}_{-k} \cdot e^{-i\psi_k} - \vec{v}_k \cdot e^{+i\psi_k} \right\}^T \cdot \frac{\partial \mathcal{H}^{(1)}}{\partial \vec{z}} \\ &= -\frac{\partial \vec{z}^{\,T}}{\partial \psi_k} \cdot \frac{\partial \mathcal{H}^{(1)}}{\partial \vec{z}} \\ &= -\frac{\partial \mathcal{H}^{(1)}}{\partial \psi_k} \ . \end{aligned}$$

So, by this ansatz, the motion of J and ψ can be attributed entirely to the perturbative part $\mathcal{H}_1$ and the unperturbed motion is embodied in the motion of the eigenvectors $\vec{v}_k(s)$ (eigenmotion).

Finally we mention that it is convenient to change the independent variable from s to the machine azimuth Θ. The Hamiltonian then has to be multiplied with the scale factor

$$R = \frac{L}{2\pi} \tag{2.51}$$

linking both variables:

$$\hat{\mathcal{H}}^{(1)} = R \cdot \mathcal{H}^{(1)} \; . \tag{2.52}$$

The resulting equations

$$\frac{d\psi_k}{d\Theta} = +\frac{\partial \hat{\mathcal{H}}^{(1)}}{\partial J_k} \; ; \tag{2.53a}$$

$$\frac{dJ_k}{d\Theta} = -\frac{\partial \hat{\mathcal{H}}^{(1)}}{\partial \psi_k} \tag{2.53b}$$

which are similar to those of the well known uncoupled case [9,10,11] can now be the starting point for detailed investigations of specific cases.

Remarks:

1) From (2.53a,b) it follows that the quantities J_k and ψ_k ($k = I, II$) defined by eqn. (2.44a,b) are canonical variables and that eqn. (2.45) represents a canonical transformation

$$x,\ p_x,\ y,\ p_y \longrightarrow \psi_I,\ J_I,\ \psi_{II},\ J_{II} \; . \tag{2.54}$$

2) From eqn. (2.43) and the relations (2.31a) we obtain

$$A_k = -i \cdot \vec{v}_k^{\,+}(s) \cdot \underline{S} \cdot \vec{z}(s) \tag{2.55}$$

and from (2.44a,b) we have:

$$J_k(s) = |\vec{v}_k^{\,+}(s) \cdot \underline{S} \cdot \vec{z}(s)|^2 \; . \tag{2.56}$$

In the special case, (2.40), of vanishing coupling we thus may write:

$$J_x(s) = \frac{1}{2\beta_x(s)} \cdot \left\{ [\alpha_x \cdot x + \beta_x \cdot p_x]^2 + x^2 \right\} \; ; \tag{2.57a}$$

$$J_y(s) = \frac{1}{2\beta_y(s)} \cdot \left\{ [\alpha_y \cdot y + \beta_y \cdot p_y]^2 + y^2 \right\} \; . \tag{2.57b}$$

The terms on the r.h.s. of (2.57) just represent the well known Courant-Snyder invariants for the linear uncoupled case. Therefore the term on the r.h.s. of (2.56) may be interpreted as the generalized Courant-Snyder invariant for the linear coupled case.

3) Introducing the Jacobian matrix

$$\underline{\mathcal{J}} = \left(\frac{\partial \vec{z}}{\partial \psi_I}, \frac{\partial \vec{z}}{\partial J_I}, \frac{\partial \vec{z}}{\partial \psi_{II}}, \frac{\partial \vec{z}}{\partial J_{II}} \right) \tag{2.58}$$

as 4×4-matrix just written as a row of column vectors $(\partial \vec{z}/\partial \psi_I)$ etc. and taking into account (2.49), the relations (2.50) may be combined into the matrix form [12]

$$\underline{\mathcal{J}}^T \cdot \underline{S} \cdot \underline{\mathcal{J}} = \underline{S} \ . \tag{2.59}$$

This matrix equation may also be written as:

$$\underline{\mathcal{J}} \cdot \underline{S} \cdot \underline{\mathcal{J}}^T = \underline{S} \tag{2.60a}$$

or

$$\left(\frac{\partial \vec{z}}{\partial \psi_I}, \frac{\partial \vec{z}}{\partial J_I}, \frac{\partial \vec{z}}{\partial \psi_{II}}, \frac{\partial \vec{z}}{\partial J_{II}}\right) \cdot \begin{pmatrix} +\left(\frac{\partial \vec{z}}{\partial J_I}\right)^T \\ -\left(\frac{\partial \vec{z}}{\partial \psi_I}\right)^T \\ +\left(\frac{\partial \vec{z}}{\partial J_{II}}\right)^T \\ -\left(\frac{\partial \vec{z}}{\partial \psi_{II}}\right)^T \end{pmatrix} = \underline{S} \tag{2.60b}$$

since

$$\begin{aligned} \underline{\mathcal{J}}^T \cdot \underline{S} \cdot \underline{\mathcal{J}} = \underline{S} \quad &\Longrightarrow \quad [\underline{S}^T \cdot \underline{\mathcal{J}}^T] \cdot [\underline{S} \cdot \underline{\mathcal{J}}] = \underline{1} \\ &\Longrightarrow \quad [\underline{S}^T \cdot \underline{\mathcal{J}}^T] = [\underline{S} \cdot \underline{\mathcal{J}}]^{-1} \\ &\Longrightarrow \quad [\underline{S} \cdot \underline{\mathcal{J}}] \cdot [\underline{S}^T \cdot \underline{\mathcal{J}}^T] = \underline{1} \\ &\Longrightarrow \underline{S}^2 \cdot \underline{\mathcal{J}} \cdot [\underline{S}^T \cdot \underline{\mathcal{J}}^T] = \underline{S} \\ &\Longrightarrow \quad \underline{\mathcal{J}} \cdot \underline{S} \cdot \underline{\mathcal{J}}^T = \underline{S} \ . \end{aligned}$$

Using components, one obtains from (2.60b):

$$\begin{cases} [x, p_x]_{(I,\psi)} = [y, p_y]_{(I,\psi)} = 1 \ ; \\ [x, y]_{(I,\psi)} = [p_x, p_y]_{(I,\psi)} = [x, p_y]_{(I,\psi)} = [y, p_x]_{(I,\psi)} = 0 \end{cases}$$

where $[f, g]_{(I,\psi)}$ represents the Poisson-bracket defined by

$$[f, g]_{(I,\psi)} = \left[\frac{\partial f}{\partial \psi_I} \cdot \frac{\partial g}{\partial J_I} - \frac{\partial f}{\partial J_I} \cdot \frac{\partial g}{\partial \psi_I}\right] + \left[\frac{\partial f}{\partial \psi_{II}} \cdot \frac{\partial g}{\partial J_{II}} - \frac{\partial f}{\partial J_{II}} \cdot \frac{\partial g}{\partial \psi_I}\right] \ .$$

These relations demonstrate again that (2.54) represents a canonical transformation [13]. The new Hamiltonian in terms of the variables J_k, ψ_k is just $\mathcal{H}_1$ as may be seen from (2.53).

4) Starting from the Floquet-form (2.26) of $\vec{z}(s)$ and using (2.44):

$$\vec{z}(s) = \sum_{k=I,II} \sqrt{J_k} \cdot \left\{\vec{u}_k(s) \cdot e^{+i\Phi_k} + \vec{u}_{-k}(s) \cdot e^{-i\Phi_k}\right\} \tag{2.61}$$

with

$$\Phi_k = \psi_k + 2\pi Q_k \cdot \frac{s}{L} \tag{2.62}$$

we may define another Jacobian matrix

$$\underline{\tilde{\mathcal{J}}} = \left(\frac{\partial \vec{z}}{\partial \Phi_I}, \frac{\partial \vec{z}}{\partial J_I}, \frac{\partial \vec{z}}{\partial \Phi_{II}}, \frac{\partial \vec{z}}{\partial J_{II}}\right) \tag{2.63}$$

in terms of the variables J_k, Φ_k which obeys the same relation as $\underline{\mathcal{J}}$:

$$\underline{\tilde{\mathcal{J}}}^T \cdot \underline{S} \cdot \underline{\tilde{\mathcal{J}}} = \underline{S} \tag{2.64}$$

as may be seen by using eqn. (2.31b). Therefore Φ_k, J_k are again canonical variables.

For the unperturbed case

$$\mathcal{H}_1 = 0 \implies (J_k = const,\ \psi_k = const)$$

and

$$\frac{dJ_k}{ds} = 0\ ; \tag{2.65a}$$

$$\frac{d\Phi_k}{ds} = \frac{2\pi}{L} Q_k\ . \tag{2.65b}$$

So in that case, the quantities J, Φ appearing in the Floquet form (2.61) are standard action angle variables.

The transition

$$\psi_k,\ J_k \longrightarrow \Phi_k,\ \tilde{J}_k = J_k$$

may be affected by a canonical transformation using a generating function of the form

$$F_3(J_k, \Phi_k; s) = \sum_{k=I,II} \left\{-J_k \cdot \Phi_k + J_k \cdot \frac{2\pi}{L} Q_k \cdot s\right\}\ . \tag{2.66}$$

The corresponding transformation equations:

$$\psi_k = -\frac{\partial F_3}{\partial J_k} = \Phi_k - \frac{2\pi}{L} Q_k \cdot s\ ; \tag{2.67a}$$

$$\tilde{J}_k = -\frac{\partial F_3}{\partial \Phi_k} = J_k \tag{2.67b}$$

are indeed identical with the defining equations for Φ_k and J_k (see eqn. (2.62)).

The new Hamiltonian $\tilde{\mathcal{H}}$ in terms of $\tilde{J}_k = J_k$ and Φ_k then reads as:

$$\begin{aligned}\tilde{\mathcal{H}} &= \mathcal{H}_1 + \frac{\partial F_3}{\partial s} \\ &= \mathcal{H}_1 + J_I \cdot \frac{2\pi}{L} Q_I + J_{II} \cdot \frac{2\pi}{L} Q_{II}\ .\end{aligned} \tag{2.68}$$

This form of the Hamiltonian is useful for calculating the detuning terms [8,14].

5) We have restricted our investigation to canonical systems which contain only two oscillation modes (four-dimensional phase space). But it is easy to generalize these considerations to an arbitrary number of modes. Thus one is able to treat also coupled synchro-betatron oscillations [6,16,17] in the framework of this formalism (six-dimensional phase space).

3 Nonlinear Perturbation Theory

The Hamiltonian $\mathcal{H}^{(1)}$ defined in (2.5b) has now to be expressed using the new canonical variables J_k and ψ_k. This can be achieved by using eqn. (2.45):

$$\vec{z} = \sum_{k=I,II} \sqrt{J_k(s)} \cdot \left\{ \vec{v}_k(s) \cdot e^{+i\psi_k(s)} + \vec{v}_k^{\,*}(s) \cdot e^{-i\psi_k(s)} \right\} \equiv \vec{z}(s, \psi_k, J_k) \ . \tag{3.1}$$

From (3.1) one has:

$$x = \sum_{k=I,II} \sqrt{J_k} \left\{ v_{k1} \cdot e^{+i\psi_k} + v_{k1}^* \cdot e^{-i\psi_k} \right\} \ ; \tag{3.2a}$$

$$y = \sum_{k=I,II} \sqrt{J_k} \left\{ v_{k3} \cdot e^{+i\psi_k} + v_{k3}^* \cdot e^{-i\psi_k} \right\} \ . \tag{3.2b}$$

Thus we get:

$$\begin{aligned} x^n &= \sum_{p=0}^{n} \begin{pmatrix} n \\ p \end{pmatrix} \cdot J_I^{(p/2)} \cdot \left\{ v_{I1} \cdot e^{+i\psi_I} + v_{I1}^* \cdot e^{-i\psi_{II}} \right\}^p \\ &\quad \times J_{II}^{(n-p)/2} \cdot \left\{ v_{II1} \cdot e^{+i\psi_{II}} + v_{II1}^* \cdot e^{-i\psi_{II}} \right\}^{n-p} \\ &= \sum_{p=0}^{n} \begin{pmatrix} n \\ p \end{pmatrix} \cdot J_I^{(p/2)} \cdot J_{II}^{(n-p)/2} \cdot \sum_{\nu_1} \sum_{\nu_2} \begin{pmatrix} p \\ \frac{p-\nu_1}{2} \end{pmatrix} \cdot \begin{pmatrix} n-p \\ \frac{n-p-\nu_2}{2} \end{pmatrix} \\ &\quad \times \left[v_{I1}^{(p+\nu_1)/2} \cdot (v_{I1}^*)^{(p-\nu_1)/2} \right] \cdot \left[v_{II1}^{(n-p+\nu_2)/2} \cdot (v_{II1}^*)^{(n-p-\nu_2)/2} \right] \\ &\quad \times e^{i[\nu_1 \cdot \psi_I + \nu_2 \cdot \psi_{II}]} \ ; \end{aligned} \tag{3.3a}$$

$$\begin{aligned} y^m &= \sum_{q=o}^{m} \begin{pmatrix} m \\ q \end{pmatrix} \cdot J_I^{(q/2)} \cdot \left\{ v_{I3} \cdot e^{+i\psi_I} + v_{I3}^* \cdot e^{-i\psi_I} \right\}^q \\ &\quad \times J_{II}^{(m-q)/2} \cdot \left\{ v_{II3} \cdot e^{+i\psi_{II}} + v_{II3}^* \cdot e^{-i\psi_{II}} \right\}^{m-q} \\ &= \sum_{q=0}^{m} \begin{pmatrix} m \\ q \end{pmatrix} \cdot J_I^{(q/2)} \cdot J_{II}^{(m-q)/2} \cdot \sum_{\mu_1} \sum_{\mu_2} \begin{pmatrix} q \\ \frac{q-\mu_1}{2} \end{pmatrix} \cdot \begin{pmatrix} m-q \\ \frac{m-q-\mu_2}{2} \end{pmatrix} \\ &\quad \times \left[v_{I3}^{(q+\mu_1)/2} \cdot (v_{I3}^*)^{(q-\mu_1)/2} \right] \cdot \left[v_{II3}^{(m-q+\mu_2)/2} \cdot (v_{II3}^*)^{(m-q-\mu_2)/2} \right] \\ &\quad \times e^{i[\mu_1 \cdot \psi_I + \mu_2 \cdot \psi_{II}]} \end{aligned} \tag{3.3b}$$

with

$$\begin{aligned} &\nu_1 \in \{-p, -p+2, ..., +p\} \ ; \\ &\nu_2 \in \{-(n-p), -(n-p)+2, ..., +(n+p)\} \ ; \\ &\mu_1 \in \{-q, -q+2, ..., +q\} \ ; \\ &\mu_2 \in \{-(m-q), -(m-q)+2, ..., +(m+q)\} \end{aligned} \tag{3.4}$$

and therefore:

$$\begin{aligned}
\mathcal{H}_1 &= \sum_{n,m} \sum_{p=o}^{n} \sum_{q=o}^{m} c_{nm}(s) \cdot \begin{pmatrix} n \\ p \end{pmatrix} \cdot \begin{pmatrix} m \\ q \end{pmatrix} \cdot J_I^{(p+q)/2} \cdot J_{II}^{[n+m-(p+q)]/2} \\
&\times \sum_{\nu_1} \sum_{\mu_1} \begin{pmatrix} p \\ \frac{p-\nu_1}{2} \end{pmatrix} \cdot \begin{pmatrix} q \\ \frac{q-\mu_1}{2} \end{pmatrix} \cdot \sum_{\nu_2} \sum_{\mu_2} \begin{pmatrix} n-p \\ \frac{n-p-\nu_2}{2} \end{pmatrix} \cdot \begin{pmatrix} m-q \\ \frac{m-q-\mu_2}{2} \end{pmatrix} \cdot \\
&\times \left[v_{I1}^{(p+\nu_1)/2} \cdot (v_{I1}^*)^{(p-\nu_1)/2} \right] \cdot \left[v_{II1}^{(n-p+\nu_2)/2} \cdot (v_{II1}^*)^{(n-p-\nu_2)/2} \right] \\
&\times \left[v_{I3}^{(q+\mu_1)/2} \cdot (v_{I3}^*)^{(q-\mu_1)/2} \right] \cdot \left[v_{II3}^{(m-q+\mu_2)/2} \cdot (v_{II3}^*)^{(m-q-\mu_2)/2} \right] \\
&\times e^{i \cdot \{[\nu_1 + \mu_1] \cdot \psi_I + [\nu_2 + \mu_2] \cdot \psi_{II}\}} \; .
\end{aligned} \tag{3.5}$$

In order to factorize the terms of $\mathcal{H}_1$ into a periodic and an harmonic function we use the components

$$\begin{aligned}
u_{I1}(s) &= v_{I1}(s) \cdot e^{-i \cdot 2\pi Q_I \cdot (s/L)} \; ; \\
u_{II1}(s) &= v_{II1}(s) \cdot e^{-i \cdot 2\pi Q_{II} \cdot (s/L)}
\end{aligned}$$

and

$$\begin{aligned}
u_{I3}(s) &= v_{I3}(s) \cdot e^{-i \cdot 2\pi Q_I \cdot (s/L)} \; ; \\
u_{II3}(s) &= v_{II3}(s) \cdot e^{-i \cdot 2\pi Q_{II} \cdot (s/L)}
\end{aligned}$$

of the Floquet-vectors $\vec{u}_k(s)$ (see eqn. (2.26)).

Eqn. (3.5) then reads as:

$$\begin{aligned}
\mathcal{H}_1 &= \sum_{n,m} \sum_{p=o}^{n} \sum_{q=o}^{m} c_{nm}(s) \cdot \begin{pmatrix} n \\ p \end{pmatrix} \cdot \begin{pmatrix} m \\ q \end{pmatrix} \cdot J_I^{(p+q)/2} \cdot J_{II}^{[n+m-(p+q)]/2} \\
&\times \sum_{\nu_1} \sum_{\mu_1} \begin{pmatrix} p \\ \frac{p-\nu_1}{2} \end{pmatrix} \cdot \begin{pmatrix} q \\ \frac{q-\mu_1}{2} \end{pmatrix} \cdot \sum_{\nu_2} \sum_{\mu_2} \begin{pmatrix} n-p \\ \frac{n-p-\nu_2}{2} \end{pmatrix} \cdot \begin{pmatrix} m-q \\ \frac{m-q-\mu_2}{2} \end{pmatrix} \cdot \\
&\times \left[u_{I1}^{(p+\nu_1)/2} \cdot (u_{I1}^*)^{(p-\nu_1)/2} \right] \cdot \left[u_{II1}^{(n-p+\nu_2)/2} \cdot (u_{II1}^*)^{(n-p-\nu_2)/2} \right] \\
&\times \left[u_{I3}^{(q+\mu_1)/2} \cdot (u_{I3}^*)^{(q-\mu_1)/2} \right] \cdot \left[u_{II3}^{(m-q+\mu_2)/2} \cdot (u_{II3}^*)^{(m-q-\mu_2)/2} \right] \\
&\times e^{i \cdot \{(\nu_1 + \mu_1) \cdot \psi_I + (\nu_2 + \mu_2) \cdot \psi_{II} + 2\pi \cdot [(\nu_1 + \mu_1) \cdot Q_I + (\nu_2 + \mu_2) \cdot Q_{II}] \cdot s/L\}} \\
&= \sum_{\lambda_1} \sum_{\lambda_2} A_{(\lambda_1,\lambda_2)}(s) \cdot e^{i \cdot \{\lambda_1 \cdot \psi_I + \lambda_2 \cdot \psi_{II} + 2\pi \cdot [\lambda_1 \cdot Q_I + \lambda_2 \cdot Q_{II}] \cdot s/L\}}
\end{aligned} \tag{3.6}$$

with

$$\begin{aligned}
A_{(\lambda_1,\lambda_2)}(s) &= \sum_{n,m} \sum_{p=o}^{n} \sum_{q=o}^{m} c_{nm}(s) \cdot \binom{n}{p} \cdot \binom{m}{q} \cdot J_I^{(p+q)/2} \cdot J_{II}^{[n+m-(p+q)]/2} \\
&\times \sum_{\substack{\nu_1,\mu_1 \\ \nu_1+\mu_1=\lambda_1}} \binom{p}{\frac{p-\nu_1}{2}} \cdot \binom{q}{\frac{q-\mu_1}{2}} \cdot \sum_{\substack{\nu_2,\mu_2 \\ \nu_2+\mu_2=\lambda_2}} \binom{n-p}{\frac{n-p-\nu_2}{2}} \cdot \binom{m-q}{\frac{m-q-\mu_2}{2}} \\
&\times \left[u_{I1}^{(p+\nu_1)/2} \cdot (u_{I1}^*)^{(p-\nu_1)/2}\right] \cdot \left[u_{II1}^{(n-p+\nu_2)/2} \cdot (u_{II1}^*)^{(n-p-\nu_2)/2}\right] \\
&\times \left[u_{I3}^{(q+\mu_1)/2} \cdot (u_{I3}^*)^{(q-\mu_1)/2}\right] \cdot \left[u_{II3}^{(m-q+\mu_2)/2} \cdot (u_{II3}^*)^{(m-q-\mu_2)/2}\right] \\
&= A_{(\lambda_1,\lambda_2)}(s+L) \ .
\end{aligned} \tag{3.7}$$

Writing

$$\begin{aligned}
\hat{n} &= p+q && \in \{0,1,2,...,n+m\} \ ; \\
\hat{m} &= (n+m)-(p+q) && \in \{0,1,2,...,n+m\}
\end{aligned}$$

we get:

$$\begin{aligned}
A_{(\lambda_1,\lambda_2)}(s) &= \sum_{\hat{n}} \sum_{\hat{m}} J_I^{\hat{n}/2} \cdot J_{II}^{\hat{m}/2} \\
&\times \sum_{\substack{n,m \\ n+m=\hat{n}+\hat{m}}} c_{nm}(s) \cdot \sum_{\substack{p,q \\ p+q=\hat{n}}} \binom{n}{p} \cdot \binom{m}{q} \\
&\times \sum_{\substack{\nu_1,\mu_1 \\ \nu_1+\mu_1=\lambda_1}} \binom{p}{\frac{p-\nu_1}{2}} \cdot \binom{q}{\frac{q-\mu_1}{2}} \cdot \sum_{\substack{\nu_2,\mu_2 \\ \nu_2+\mu_2=\lambda_2}} \binom{n-p}{\frac{n-p-\nu_2}{2}} \cdot \binom{m-q}{\frac{m-q-\mu_2}{2}} \\
&\times \left[u_{I1}^{(p+\nu_1)/2} \cdot (u_{I1}^*)^{(p-\nu_1)/2}\right] \cdot \left[u_{II1}^{(n-p+\nu_2)/2} \cdot (u_{II1}^*)^{(n-p-\nu_2)/2}\right] \\
&\times \left[u_{I3}^{(q+\mu_1)/2} \cdot (u_{I3}^*)^{(q-\mu_1)/2}\right] \cdot \left[u_{II3}^{(m-q+\mu_2)/2} \cdot (u_{II3}^*)^{(m-q-\mu_2)/2}\right] \\
&= \sum_{\hat{n}} \sum_{\hat{m}} h_{\hat{n}\hat{m}\lambda_1\lambda_2} \cdot J_I^{\hat{n}/2} \cdot J_{II}^{\hat{m}/2}
\end{aligned} \tag{3.8}$$

where the coefficients $h_{\hat{n}\hat{m}\lambda_1\lambda_2}$ are given by:

$$\begin{aligned}
h_{\hat{n}\hat{m}\lambda_1\lambda_2} &= \sum_{\substack{n,m \\ n+m=\hat{n}+\hat{m}}} c_{nm}(s) \cdot \sum_{\substack{p,q \\ p+q=\hat{n}}} \binom{n}{p} \cdot \binom{m}{q} \\
&\times \sum_{\substack{\nu_1,\mu_1 \\ \nu_1+\mu_1=\lambda_1}} \binom{p}{\frac{p-\nu_1}{2}} \cdot \binom{q}{\frac{p-\mu_1}{2}} \cdot \sum_{\substack{\nu_2,\mu_2 \\ \nu_2+\mu_2=\lambda_2}} \binom{n-p}{\frac{n-p-\nu_2}{2}} \cdot \binom{m-q}{\frac{m-q-\mu_2}{2}} \\
&\times \left[u_{I1}^{(p+\nu_1)/2} \cdot (u_{I1}^*)^{(p-\nu_1)/2}\right] \cdot \left[u_{II1}^{(n-p+\nu_2)/2} \cdot (u_{II1}^*)^{(n-p-\nu_2)/2}\right] \\
&\times \left[u_{I3}^{(q+\mu_1)/2} \cdot (u_{I3}^*)^{(q-\mu_1)/2}\right] \cdot \left[u_{II3}^{(m-q+\mu_2)/2} \cdot (u_{II3}^*)^{(m-q-\mu_2)/2}\right] \ .
\end{aligned} \tag{3.9}$$

Thus we have:

$$\mathcal{H}_1 = \sum_{\hat{n}} \sum_{\hat{m}} \sum_{\lambda_1} \sum_{\lambda_2} h_{\hat{n}\hat{m}\lambda_1\lambda_2} \cdot J_I^{\hat{n}/2} \cdot J_{II}^{\hat{m}/2} \times e^{i \cdot \{\lambda_1 \cdot \psi_I + \lambda_2 \cdot \psi_{II} + 2\pi \cdot (\lambda_1 \cdot Q_I + \lambda_2 \cdot Q_{II}) \cdot s/L\}} . \tag{3.10}$$

For the Hamiltonian $\hat{\mathcal{H}}_1$ defined by eqns. (2.52) and (2.53) we finally obtain:

$$\hat{\mathcal{H}}_1 = \sum_{\hat{n}} \sum_{\hat{m}} \sum_{\lambda_1} \sum_{\lambda_2} \hat{h}_{\hat{n}\hat{m}\lambda_1\lambda_2} \cdot J_I^{\hat{n}/2} \cdot J_{II}^{\hat{m}/2} \times e^{i \cdot \{\lambda_1 \cdot \psi_I + \lambda_2 \cdot \psi_{II} + 2\pi \cdot (\lambda_1 \cdot Q_I + \lambda_2 \cdot Q_{II}) \cdot s/L\}} \tag{3.11}$$

with the coefficients

$$\hat{h}_{\hat{n}\hat{m}\lambda_1\lambda_2} = R \cdot h_{\hat{n}\hat{m}\lambda_1\lambda_2} \tag{3.12}$$

which are periodic functions of Θ:

$$\hat{h}_{\hat{n}\hat{m}\lambda_1\lambda_2}(\Theta + 2\pi) = \hat{h}_{\hat{n}\hat{m}\lambda_1\lambda_2}(\Theta)$$

$$\Longrightarrow \quad \hat{h}_{\hat{n}\hat{m}\lambda_1\lambda_2}(\Theta) = \sum_{q=-\infty}^{+\infty} \hat{h}_{\hat{n}\hat{m}\lambda_1\lambda_2 q} \cdot e^{i \cdot q\Theta} = R \cdot \sum_{q=-\infty}^{+\infty} h_{\hat{n}\hat{m}\lambda_1\lambda_2 q} \cdot e^{i \cdot q\Theta} . \tag{3.13}$$

With eqns. (3.11) and (3.13) we have established the connection with the canonical perturbation theory described in Ref. [11].

Note that the complex periodic functions u_{k1} and u_{k3} ($k = I,\ II$) appearing in (3.9) are determined by eqns. (2.17), (2.30), (2.24) and (2.26a). They can be conveniently directly calculated using computer programs (for example PETROS [15]). A description of a method to determine the eigenvectors of the transfer matrix may be found in Refs. [4,7].

So far, the unperturbed but coupled system has been represented by the eigenvectors of the revolution matrix. In analogy to uncoupled systems we can also introduce real generalized lattice functions and can thus describe the unperturbed system using amplitude and phase functions for each of the two eigenmodes [3,4].

If one considers randomly distributed perturbation in the lattice, the formulation using generalized amplitude functions, instead of the periodic functions u_{k1} and u_{k3} is advantageous because the average values over phase functions for one turn around the ring are easier to estimate (see chap. 6, eqn. (6.2)) and one obtains compact expressions for the expectation values of the strengths of nonlinear effects. Moreover, generalized amplitude functions can be expressed in terms of the familiar uncoupled lattice functions if the coupling fields are weak and distributed around the ring. Corresponding expressions will be derived in the next chapter using a perturbation treatment similar to that used for solving Schrödinger equation problems.

Remarks:

1) The investigations of this chapter can easily be extended to a Hamiltonian of the most general form:

$$\mathcal{H}^{(1)} = \sum_{\mu_1+\mu_2+\mu_3+\mu_4=3}^{\infty} c_{\mu_1\mu_2\mu_3\mu_4} \cdot x^{\mu_1} p_x^{\mu_2} y^{\mu_3} p_y^{\mu_4} \tag{3.14}$$

which includes the Hamiltonian (2.5b) as a special case.

In this generalization, also the second and the fourth component of the Floquet-vectors $\vec{u}_k(s)$ appear in the periodic perturbation-functions $\hat{h}_{\hat{n}\hat{m}\lambda_I\lambda_{II}}$. The equations so derived may find many applications in machine physics which involve coupled motion (coupled synchro-betatron oscillations; beam-beam interactions etc.). In this report we only consider the impact of linear coupling on nonlinear betatron motion.

2) In terms of the variables Φ_k, J_k which are defined by eqn. (2.61) the Hamiltonian takes the form (see eqn. (2.68)):

$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_0 + \tilde{V}(\Phi_I, \Phi_{II}, J_I, J_{II}, s) \tag{3.15}$$

with

$$\tilde{\mathcal{H}}_0 = +J_I \cdot \frac{2\pi}{L} Q_I + J_{II} \cdot \frac{2\pi}{L} Q_{II}$$

and

$$\tilde{V}(\Phi_I, \Phi_{II}, J_I, J_{II}, s) \equiv \mathcal{H}_1 \ .$$

Here the term $\tilde{V}(\Phi_I, \Phi_{II}, J_I, J_{II}, s)$ describes the perturbation and is periodic in s and Φ_I, Φ_{II}:

$$\begin{aligned} \tilde{V}(\Phi_I, \Phi_{II}, J_I, J_{II}, s) &= \tilde{V}(\Phi_I, \Phi_{II}, J_I, J_{II}, s + L) \\ &= \tilde{V}(\Phi_I + 2\pi, \Phi_{II}, J_I, J_{II}, s) \\ &= \tilde{V}(\Phi_I, \Phi_{II} + 2\pi, J_I, J_{II}, s) \ . \end{aligned} \tag{3.16}$$

The corresponding canonical equations read as:

$$\frac{d\Phi_k}{ds} = +\frac{\partial \tilde{\mathcal{H}}}{\partial J_k} = \frac{2\pi}{L} Q_k + \frac{\partial \tilde{V}}{\partial J_k} \ ; \tag{3.17a}$$

$$\frac{dJ_k}{ds} = -\frac{\partial \tilde{\mathcal{H}}}{\partial \Phi_k} = -\frac{\partial \tilde{V}}{\partial \Phi_k} \ . \tag{3.17b}$$

In this form the Hamiltonian can be used for a version of perturbation theory given by Courant, Ruth and Weng [8,14].

4 Estimation of Coupled Lattice Functions

For simplicity we only investigate the case where coupling is produced by skew quadrupoles. Then one has:

$$p_x = x' \ ; \quad p_y = y' \ . \tag{4.1}$$

In Appendix A we show how coupled eigenvectors can be written in terms of periodic, linear coupled lattice functions, i.e. in terms of generalized amplitude functions.

The procedure is to form the real and the imaginary parts of the eigenvectors, the so called generating vectors $\vec{z}_{1,3},\ \vec{z}_{2,4}$:

$$\begin{aligned}\vec{v}_I &= \frac{1}{\sqrt{2}} \cdot (\vec{z}_1 + i \cdot \vec{z}_2) \ ; \\ \vec{v}_{II} &= \frac{1}{\sqrt{2}} \cdot (\vec{z}_3 + i \cdot \vec{z}_4) \ .\end{aligned}$$

The lattice functions are then given by:

$$\begin{pmatrix} \beta_{xI} = x_1^2 + x_2^2 & \gamma_{xI} = x_1'^2 + x_2'^2 & \alpha_{xI} = -(x_1 x_1' + x_2 x_2') & \phi_{xI} = \arctan(x_2/x_1) \\ \beta_{xII} = x_3^2 + x_4^2 & \gamma_{xII} = x_3'^2 + x_4'^2 & \alpha_{xII} = -(x_3 x_3' + x_4 x_4') & \phi_{xII} = \arctan(x_4/x_3) \\ \beta_{yI} = y_1^2 + y_2^2 & \gamma_{yI} = y_1'^2 + y_2'^2 & \alpha_{yI} = -(y_1 y_1' + y_2 y_2') & \phi_{yI} = \arctan(y_2/y_1) \\ \beta_{yII} = y_3^2 + y_4^2 & \gamma_{IIy} = y_3'^2 + y_4'^2 & \alpha_{yII} = -(y_3 y_3' + y_4 y_4') & \phi_{yII} = \arctan(y_4/y_3) \end{pmatrix} \qquad (4.2)$$

where x_i, x_i', y_i, y_i' are the components of the generating vectors $\vec{z}_i$:

$$\vec{z}_i = \begin{pmatrix} x_i \\ x_i' \\ y_i \\ y_i' \end{pmatrix} . \qquad (4.3)$$

In terms of these functions the the motion of a particle $\{x(s), x'(s), y(s), y'(s)\}$ can be written as [3,4]:

$$x(s) = \sqrt{2J_I}\sqrt{\beta_{xI}} \cos(\phi_{xI} + \phi_I) + \sqrt{2J_{II}}\sqrt{\beta_{xII}} \cos(\phi_{xII} + \phi_{II}) \ ; \qquad (4.4)$$

$$x'(s) = \sqrt{2J_I}\sqrt{\gamma_{xI}} \cos(\tilde{\phi}_{xI} + \phi_I) + \sqrt{2J_{II}}\sqrt{\gamma_{xII}} \cos(\tilde{\phi}_{xII} + \phi_{II}) \ ; \qquad (4.5)$$

$$y(s) = \sqrt{2J_I}\sqrt{\beta_{yI}} \cos(\phi_{yI} + \phi_I) + \sqrt{2J_{II}}\sqrt{\beta_{yII}} \cos(\phi_{yII} + \phi_{II}) \ ; \qquad (4.6)$$

$$y'(s) = \sqrt{2J_I}\sqrt{\gamma_{yI}} \cos(\tilde{\phi}_{yI} + \phi_I) + \sqrt{2J_{II}}\sqrt{\gamma_{yII}} \cos(\tilde{\phi}_{yII} + \phi_{II}) \ . \qquad (4.7)$$

Thus in order to obtain the lattice functions for the coupled lattice, one only has to calculate the eigenvectors of the latter.

The expressions for β_{xII}, β_{yI} allow one to estimate the strength of the interference of linear coupling and nonlinearities.

As we have already mentioned, the coupled eigenvectors (and hence the lattice functions) can be calculated directly using computer programs e.g. "PETROS" [15] to solve the eigenproblem. However, more insight can be obtained using analytic estimates based on a perturbation procedure.

The technique for estimating the coupled eigenvectors depends on how close the tunes lie to a linear resonance: We define the linear resonance condition as $(Q_x \pm Q_y = integer)$.

a) Eigenvectors outside the resonances $(Q_x \pm Q_y \neq integer)$.

As is shown in Appendix B, outside a resonance we may write the modification to the uncoupled eigenvectors as:

$$\begin{aligned}\delta\vec{v}_I &= a_{12} \cdot [\vec{v}_I]^* + a_{13} \cdot \vec{v}_{II} + a_{14} \cdot [\vec{v}_{II}]^* \ ; \\ \delta\vec{v}_{II} &= a_{31} \cdot \vec{v}_I + a_{32} \cdot [\vec{v}_I]^* + a_{34} \cdot [\vec{v}_{II}]^*\end{aligned}$$

with coefficients a_{mn} given by eqn. (B.16)

Using the relations:

$$\vec{v}_1(s) \equiv \vec{v}_I = \frac{1}{\sqrt{2\beta_x(s)}} \cdot \begin{pmatrix} \beta_x(s) \\ -[\alpha_x(s) - i] \\ 0 \\ 0 \end{pmatrix} \cdot e^{i\phi_x(s)} \; ; \quad \vec{v}_2(s) = [\vec{v}_I(s)]^* \; ; \tag{4.8a}$$

$$\vec{v}_3(s) \equiv \vec{v}_{II}(s) = \frac{1}{\sqrt{2\beta_y(s)}} \cdot \begin{pmatrix} 0 \\ 0 \\ \beta_y(s) \\ -[\alpha_y(s) - i] \end{pmatrix} \cdot e^{i\phi_y(s)} \; ; \quad \vec{v}_4(s) = [\vec{v}_{II}(s)]^* \tag{4.8b}$$

for the uncoupled eigenvectors and the coupling perturbation due to skew quadrupoles,

$$\delta \underline{A}(s) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -N & 0 \\ 0 & 0 & 0 & 0 \\ -N & 0 & 0 & 0 \end{pmatrix} , \tag{4.9}$$

we thus obtain the modifications to the eigenvectors:

$$\begin{aligned} \delta\vec{v}_I(s) &= \vec{v}_{II}(s) \cdot \frac{1}{4} \cdot \frac{e^{-i\pi[Q_x - Q_y]}}{\sin\pi[Q_x - Q_y]} \cdot N_{13}(s) \\ &- [\vec{v}_{II}(s)]^* \cdot \frac{1}{4} \cdot \frac{e^{-i\pi[Q_x + Q_y]}}{\sin\pi[Q_x + Q_y]} \cdot N_{14}(s) \; ; \end{aligned} \tag{4.10a}$$

$$\begin{aligned} \delta\vec{v}_{II}(s) &= \vec{v}_I(s) \cdot \frac{1}{4} \cdot \frac{e^{-i\pi[Q_y - Q_x]}}{\sin\pi[Q_y - Q_x]} \cdot N_{31}(s) \\ &- [\vec{v}_I(s)]^* \cdot \frac{1}{4} \cdot \frac{e^{-i\pi[Q_y + Q_x]}}{\sin\pi[Q_y + Q_x]} \cdot N_{32}(s) \end{aligned} \tag{4.10b}$$

with

$$\begin{aligned} N_{13} &= \int_s^{s+L} d\tilde{s} \cdot N(\tilde{s}) \cdot \sqrt{\beta_x(\tilde{s}) \cdot \beta_y(\tilde{s})} \cdot e^{i[\phi_x(\tilde{s}) - \phi_y(\tilde{s})]} \; ; \\ N_{14} &= \int_s^{s+L} d\tilde{s} \cdot N(\tilde{s}) \cdot \sqrt{\beta_x(\tilde{s}) \cdot \beta_y(\tilde{s})} \cdot e^{i[\phi_x(\tilde{s}) + \phi_y(\tilde{s})]} \; ; \\ N_{31} &= \int_s^{s+L} d\tilde{s} \cdot N(\tilde{s}) \cdot \sqrt{\beta_x(\tilde{s}) \cdot \beta_y(\tilde{s})} \cdot e^{i[\phi_y(\tilde{s}) - \phi_x(\tilde{s})]} \; ; \\ N_{32} &= \int_s^{s+L} d\tilde{s} \cdot N(\tilde{s}) \cdot \sqrt{\beta_x(\tilde{s}) \cdot \beta_y(\tilde{s})} \cdot e^{i[\phi_y(\tilde{s}) + \phi_x(\tilde{s})]} \; . \end{aligned} \tag{4.11}$$

Eqns. (4.10a,b) finally lead to:

$$\left\{ \begin{aligned} \beta_{xI}(s) &= \beta_x(s) \; ; \\ \beta_{yI}(s) &= \left| \frac{e^{-i\pi[Q_x - Q_y]}}{4\sin\pi[Q_x - Q_y]} \cdot N_{13}(s) - \frac{e^{-i\pi[Q_x + Q_y]}}{4\sin\pi[Q_x + Q_y]} \cdot N_{14}(s) \right|^2 \cdot \beta_x(s) \; ; \end{aligned} \right. \tag{4.12a}$$

$$\left\{ \begin{array}{l} \beta_{yII}(s) = \beta_y(s) \; ; \\ \beta_{xII}(s) = \left| \dfrac{e^{-i\pi[Q_y - Q_x]}}{4 \sin \pi [Q_y - Q_x]} \cdot N_{31}(s) - \dfrac{e^{-i\pi[Q_y + Q_x]}}{4 \sin \pi [Q_y + Q_x]} \cdot N_{32}(s) \right|^2 \cdot \beta_y(s) \; . \end{array} \right. \qquad (4.12b)$$

These relations are not valid in the resonance case ($Q_x \pm Q_y \approx integer$) where the functions $\beta_{yI}(s)$ and $\beta_{xII}(s)$ would be undefined.

b) Eigenvectors in the neighbourhood of a difference resonance ($Q_x - Q_y \approx integer$).

As is also demonstrated in Appendix B one may approximate the eigenvectors in the neighbourhood of a difference resonance by (see eqn. (B.49)):

$$\vec{v}_I = \frac{1}{\sqrt{[\delta Q_I - \delta Q_x]^2 + \kappa^2/4}} \cdot \left\{ \tilde{B}_{12} \cdot \vec{v}_I^{\,(0)} + [\delta Q_I - \delta Q_x] \cdot \vec{v}_{II}^{\,(0)} \right\} ; \; \vec{v}_{-I} = [\vec{v}_I]^* \; ; \qquad (4.13a)$$

$$\vec{v}_{II} = \frac{1}{\sqrt{[\delta Q_{II} - \delta Q_x]^2 + \kappa^2/4}} \cdot \left\{ \tilde{B}_{12} \cdot \vec{v}_I^{\,(0)} + [\delta Q_{II} - \delta Q_x] \cdot \vec{v}_{II}^{\,(0)} \right\} ; \; \vec{v}_{-II} = [\vec{v}_{II}]^* \qquad (4.13b)$$

where δQ_I and δQ_{II} and the quantity κ^2 with (B.44), (B.45) and (B.46) are given by:

$$\left\{ \begin{array}{l} \delta Q_I \; = \frac{1}{2} \cdot [\delta Q_x + \delta Q_y] + \frac{1}{2}\sqrt{[\delta Q_x - \delta Q_y]^2 + \kappa^2} \; ; \\ \delta Q_{II} = \frac{1}{2} \cdot [\delta Q_x + \delta Q_y] - \frac{1}{2}\sqrt{[\delta Q_x - \delta Q_y]^2 + \kappa^2} \; ; \end{array} \right. \qquad (4.14)$$

$$\begin{aligned} \kappa^2 &= \left| \frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot N(\tilde{s}) \cdot \sqrt{\beta_x(\tilde{s}) \cdot \beta_y(\tilde{s})} \cdot e^{i[\phi_y(\tilde{s}) - \phi_x(\tilde{s})]} \right|^2 \\ &= \left| \frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot N(\tilde{s}) \cdot \sqrt{\beta_x(\tilde{s}) \cdot \beta_y(\tilde{s})} \cdot \cos[\phi_y(\tilde{s}) - \phi_x(\tilde{s})] \right|^2 + \\ & \quad \left| \frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot N(\tilde{s}) \cdot \sqrt{\beta_x(\tilde{s}) \cdot \beta_y(\tilde{s})} \cdot \sin[\phi_y(\tilde{s}) - \phi_x(\tilde{s})] \right|^2 . \end{aligned} \qquad (4.15)$$

Note that κ is just the driving term in the linearly coupled case (Ref. [3]; eqn. (3.97b)). It also measures the resonance width. A more general definition for driving terms will be given at eqn. (5.11).

With

$$Q_x - Q_y = n + \Lambda \; ; \quad \delta Q_x = -\delta Q_y = \frac{1}{2}\Lambda \; ;$$
$$\delta Q = +\frac{1}{2}\sqrt{\Lambda^2 + \kappa^2} \equiv \delta Q_I$$

and taking into account (B.22) one then obtains:

$$\vec{v}_I = \frac{1}{\sqrt{[\delta Q - \Lambda/2]^2 + \kappa^2/4}} \cdot \left\{ \tilde{B}_{12} \cdot \frac{1}{\sqrt{2\beta_x(s)}} \cdot \begin{pmatrix} \beta_x(s) \\ -[\alpha_x(s) - i] \\ 0 \\ 0 \end{pmatrix} \cdot e^{i \cdot \phi_x(s)} \right.$$

$$+[\delta Q - \Lambda/2] \cdot \frac{1}{\sqrt{2\beta_y(s)}} \cdot \left. \begin{pmatrix} 0 \\ 0 \\ \beta_y(s) \\ -[\alpha_y(s) - i] \end{pmatrix} \cdot e^{i \cdot \phi_y(s)} \right\} ; \tag{4.16a}$$

$$\vec{v}_{II} = \frac{1}{\sqrt{[\delta Q + \Lambda/2]^2 + \kappa^2/4}} \cdot \left\{ \tilde{B}_{12} \cdot \frac{1}{\sqrt{2\beta_x(s)}} \cdot \begin{pmatrix} \beta_x(s) \\ -[\alpha_x(s) - i] \\ 0 \\ 0 \end{pmatrix} \cdot e^{i \cdot \phi_x(s)} \right.$$

$$\left. -[\delta Q + \Lambda/2] \cdot \frac{1}{\sqrt{2\beta_y(s)}} \cdot \begin{pmatrix} 0 \\ 0 \\ \beta_y(s) \\ -[\alpha_y(s) - i] \end{pmatrix} \cdot e^{i \cdot \phi_y(s)} \right\} \tag{4.16b}$$

and it follows that:

$$\left\{ \begin{aligned} \beta_{xI} &= \frac{\kappa^2}{[\sqrt{\Lambda^2 + \kappa^2} - \Lambda]^2 + \kappa^2} \cdot \beta_x ; \\ \beta_{yI} &= \frac{[\sqrt{\Lambda^2 + \kappa^2} - \Lambda]^2}{[\sqrt{\Lambda^2 + \kappa^2} - \Lambda]^2 + \kappa^2} \cdot \beta_y ; \end{aligned} \right. \tag{4.17a}$$

$$\left\{ \begin{aligned} \beta_{xII} &= \frac{\kappa^2}{[\sqrt{\Lambda^2 + \kappa^2} + \Lambda]^2 + \kappa^2} \cdot \beta_x ; \\ \beta_{yII} &= \frac{[\sqrt{\Lambda^2 + \kappa^2} + \Lambda]^2}{[\sqrt{\Lambda^2 + \kappa^2} + \Lambda]^2 + \kappa^2} \cdot \beta_y . \end{aligned} \right. \tag{4.17b}$$

On the coupling resonance, $\Lambda = 0$, the mode I and mode II β–functions have the same value, namely, one half of the original unperturbed β–function:

$$\beta_{xI} = \beta_{xII} = \frac{1}{2} \cdot \beta_x ; \tag{4.18a}$$

$$\beta_{yI} = \beta_{yII} = \frac{1}{2} \cdot \beta_y . \tag{4.18b}$$

This demostrates that there is a large exchange of energy between both degrees of freedom but that no instability appears.

Moving away from the resonance, these formulae give the same result as those in the previous section (see eqn. (4.12a,b)) if the contribution from the sum resonance term is neglected.

For the nonlinear effects is it important to notice that on the coupling resonance, the coupled beta functions become independent of the strengths of the coupling.

We do not treat the case of the sum resonance since, as is shown in Appendix B, in this case the motion becomes unstable, exhibiting resonant blow up.

5 Analytical Estimation of Skew Resonances Near the Coupling Resonances

Using the relations

$$v_{k1} = \sqrt{\frac{\beta_{xk}}{2}} \cdot e^{i\phi_{xk}} = u_{k1} \cdot e^{i \cdot 2\pi Q_k \cdot (s/L)} \; ; \tag{5.1a}$$

$$v_{k3} = \sqrt{\frac{\beta_{yk}}{2}} \cdot e^{i\phi_{yk}} = u_{k3} \cdot e^{i \cdot 2\pi Q_k \cdot (s/L)} \tag{5.1b}$$

resulting from (A.2) and (2.25a) the Hamiltonian $\hat{\mathcal{H}}_1$ in (3.11) may now be written in terms of the generalized amplitude and phase functions. Using (3.10) and (3.13) we then obtain:

$$\begin{aligned}\hat{\mathcal{H}}_1 &= R \cdot \sum_{\hat{n}} \sum_{\hat{m}} \sum_{\lambda_1} \sum_{\lambda_2} \sum_{q} h_{\hat{n}\hat{m}\lambda_1\lambda_2 q} \cdot J_I^{\hat{n}/2} \cdot J_{II}^{\hat{m}/2} \\ &\quad \times e^{i \cdot \{\lambda_1 \cdot \psi_I + \lambda_2 \cdot \psi_{II} + [\lambda_1 \cdot Q_I + \lambda_2 \cdot Q_{II} + q] \cdot \Theta\}}\end{aligned} \tag{5.2}$$

with

$$h_{\hat{n}\hat{m}\lambda_1\lambda_2 q} = \frac{1}{2\pi} \cdot \int_0^{2\pi} d\Theta \cdot h_{\hat{n}\hat{m}\lambda_1\lambda_2}(\Theta) \cdot e^{-iq\Theta} \tag{5.3}$$

and

$$\begin{aligned} h_{\hat{n}\hat{m}\lambda_1\lambda_2} &= \sum_{\substack{n,m \\ n+m=\hat{n}+\hat{m}}} c_{nm}(s) \cdot \sum_{\substack{p,q \\ p+q=\hat{n}}} \binom{n}{p} \cdot \binom{m}{q} \\ &\quad \times \left(\frac{\beta_{Ix}}{2}\right)^{p/2} \cdot \left(\frac{\beta_{IIx}}{2}\right)^{(n-p)/2} \cdot \left(\frac{\beta_{Iy}}{2}\right)^{q/2} \cdot \left(\frac{\beta_{IIy}}{2}\right)^{(m-q)/2} \\ &\quad \times \sum_{\substack{\nu_1,\mu_1 \\ \nu_1+\mu_1=\lambda_1}} \binom{p}{\frac{p-\nu_1}{2}} \cdot \binom{q}{\frac{p-\mu_1}{2}} \cdot \sum_{\substack{\nu_2,\mu_2 \\ \nu_2+\mu_2=\lambda_2}} \binom{n-p}{\frac{n-p-\nu_2}{2}} \cdot \binom{m-q}{\frac{m-q-\mu_2}{2}} \\ &\quad \times e^{i \cdot \{\nu_1 \cdot \tilde{\phi}_{xI} + \mu_1 \cdot \tilde{\phi}_{yI} + \nu_2 \cdot \tilde{\phi}_{xII} + \mu_2 \cdot \tilde{\phi}_{yII}\}}\end{aligned} \tag{5.4}$$

where the (periodic) functions $\tilde{\phi}_{xk}$ and $\tilde{\phi}_{yk}$ $(k = I, II)$ are defined by:

$$\tilde{\phi}_{xk}(s) = \phi_{xk}(s) - 2\pi Q_k \cdot \frac{s}{L} \; ; \tag{5.5a}$$

$$\tilde{\phi}_{yk}(s) = \phi_{yk}(s) - 2\pi Q_k \cdot \frac{s}{L} \; ; \tag{5.5b}$$

$$(k = I, \; II) \; .$$

Note that if the linear motion is decoupled one has:

$$\beta_{1x} \equiv \beta_x \; ; \quad \beta_{2y} \equiv \beta y \; ; \quad \beta_{1y} = \beta_{2x} = 0 \; ;$$
$$\phi_{1x} \equiv \phi_x \; ; \quad \phi_{2y} \equiv \phi y \; ; \quad \phi_{1y} = \phi_{2x} = 0 \; ;$$

$$\Longrightarrow \quad \begin{cases} p = n \; , \quad q = 0 \; ; \\ \hat{n} = n \; , \quad \hat{m} = m \; ; \\ \nu_1 = n_1 \; , \quad \nu_2 = 0 \; ; \\ \mu_1 = 0 \; , \quad \mu_2 = n_2 \; . \end{cases}$$

In this case eqn. (5.4) simplifies to the formula:

$$h_{nmn_1n_2} = c_{nm}(s) \cdot \left(\frac{\beta_x}{2}\right)^{n/2} \cdot \left(\frac{\beta_y}{2}\right)^{m/2} \cdot \begin{pmatrix} n \\ \frac{n-n_1}{2} \end{pmatrix} \cdot \begin{pmatrix} m \\ \frac{m-n_2}{2} \end{pmatrix} \times e^{i \cdot \{\cdot \tilde{\phi}_x + n_2 \cdot \tilde{\phi}_y\}} \tag{5.6}$$

with

$$\tilde{\phi}_x(s) = \phi_x(s) - 2\pi Q_x \cdot \frac{s}{L} ; \tag{5.7a}$$

$$\tilde{\phi}_y(s) = \phi_y(s) - 2\pi Q_y \cdot \frac{s}{L} \tag{5.7b}$$

which was used in Ref. [11].

Furthermore, if we restrict our investigations to resonances excited by sextupoles we can write for the coefficients c_{mn} in (5.4):

$$\begin{cases} c_{30} = +\frac{1}{6} \cdot B_3 ; \\ c_{12} = -\frac{1}{2} \cdot B_3 ; \\ c_{mn} = 0 \quad \text{otherwise} \end{cases} \tag{5.8}$$

with

$$B_3 = \frac{e}{p_0} \left(\frac{\partial^2 B_y}{\partial x^2}\right)_{x=y=0} \tag{5.9}$$

(B_3=sextupole strength).

In order to study the impact of nonlinear resonances, one now restricts the series (5.2) to the slowly varying terms after having set the tunes close to a single resonance $n_1 Q_I + n_2 Q_{II} + p = 0$. Then the Hamiltonian $\hat{\mathcal{H}}_1$ simplifies to the form:

$$\hat{\mathcal{H}}_1 = \sum_{\hat{n}} \sum_{\hat{m}} J_I^{\hat{n}/2} \cdot J_{II}^{\hat{m}/2} \cdot \kappa_{\bar{n}} \cdot \cos\{n_1 \cdot \psi_I + n_2 \cdot \psi_{II} + \Delta \cdot \Theta + \varphi_{\bar{n}}\} \tag{5.10}$$

where we have written for abreviation:

$$R \cdot h_{\hat{n}\hat{m}n_1n_2p} = \frac{1}{2}\kappa_{\bar{n}} \cdot e^{i\varphi_{\bar{n}}} \tag{5.11}$$

($\bar{n} \equiv \hat{n}\hat{m}n_1n_2p$). Here the coefficients $\kappa_{\bar{n}}$ are called the driving or detuning terms.

A canonical transformation

$$(\psi_I, \psi_{II}, J_I, J_{II}) \longrightarrow (\varphi_I, \varphi_{II}, J_I, J_{II})$$

using the generating function

$$F_2(\psi_I, \psi_{II}, J_I, J_{II}, \Theta) = \psi_I \cdot J_I + \psi_{II} \cdot J_{II} + \Delta \cdot \left[\frac{J_I}{n_1} \cdot \frac{n_1^2}{n_1^2 + n_2^2} + \frac{J_{II}}{n_2} \cdot \frac{n_2^2}{n_1^2 + n_2^2}\right] \cdot \Theta \tag{5.12}$$

then gives

$$\varphi_I = \frac{\partial F_2}{\partial J_I} = \psi_I + \Delta \cdot \Theta \cdot \frac{1}{n_1} \cdot \frac{n_1^2}{n_1^2 + n_2^2} ; \tag{5.13a}$$

$$\varphi_{II} = \frac{\partial F_2}{\partial J_{II}} = \psi_{II} + \Delta \cdot \Theta \cdot \frac{1}{n_2} \cdot \frac{n_2^2}{n_1^2 + n_2^2} \tag{5.13b}$$

and

$$
\begin{aligned}
\hat{\mathcal{H}}_1 \longrightarrow \mathcal{K} &= \hat{\mathcal{H}}_1 + \frac{\partial F_2}{\partial \Theta} \\
&= \sum_{\hat{n}} \sum_{\hat{m}} J_I^{\hat{n}/2} \cdot J_{II}^{\hat{m}/2} \cdot \kappa_{\bar{n}} \cdot \cos\{n_1 \cdot \varphi_I + n_2 \cdot \varphi_{II} + \varphi_{\bar{n}}\} \\
&\quad + [\Delta_I \cdot J_I + \Delta_{II} \cdot J_{II}] \qquad (5.14)
\end{aligned}
$$

with

$$\Delta_I = \Delta \cdot \frac{n_1}{n_1^2 + n_2^2} \; ; \qquad (5.15a)$$

$$\Delta_{II} = \Delta \cdot \frac{n_2}{n_1^2 + n_2^2} \qquad (5.15b)$$

whereby the canonical equations in $\mathcal{K}$ read as:

$$
\begin{aligned}
\frac{d\varphi_I}{d\Theta} &= \frac{\partial \mathcal{K}}{\partial J_I} \\
&= \sum_{\hat{n}} \sum_{\hat{m}} \frac{\hat{n}}{2} \cdot J_I^{(\hat{n}/2-1)} J_{II}^{\hat{m}/2} \cdot \kappa_{\bar{n}} \cdot \cos\{n_1 \cdot \varphi_I + n_2 \cdot \varphi_{II} + \varphi_{\bar{n}}\} + \Delta_I \; ; \qquad (5.16a) \\
\frac{d\varphi_{II}}{d\Theta} &= \frac{\partial \mathcal{K}}{\partial J_{II}} \\
&= \sum_{\hat{n}} \sum_{\hat{m}} \frac{\hat{m}}{2} \cdot J_I^{\hat{n}/2} J_{II}^{(\hat{m}/2-1)} \cdot \kappa_{\bar{n}} \cdot \cos\{n_1 \cdot \varphi_I + n_2 \cdot \varphi_{II} + \varphi_{\bar{n}}\} + \Delta_{II} \; ; \qquad (5.16b) \\
\frac{dJ_I}{d\Theta} &= -\frac{\partial \mathcal{K}}{\partial \varphi_I} \\
&= n_I \cdot \sum_{\hat{n}} \sum_{\hat{m}} J_I^{\hat{n}/2} J_{II}^{\hat{m}/2} \cdot \kappa_{\bar{n}} \cdot \sin\{n_1 \cdot \varphi_I + n_2 \cdot \varphi_{II} + \varphi_{\bar{n}}\} \; ; \qquad (5.16c) \\
\frac{dJ_{II}}{d\Theta} &= -\frac{\partial \mathcal{K}}{\partial \varphi_{II}} \\
&= n_{II} \cdot \sum_{\hat{n}} \sum_{\hat{m}} J_I^{\hat{n}/2} J_{II}^{\hat{m}/2} \cdot \kappa_{\bar{n}} \cdot \sin\{n_1 \cdot \varphi_I + n_2 \cdot \varphi_{II} + \varphi_{\bar{n}}\} \; . \qquad (5.16d)
\end{aligned}
$$

Since $\mathcal{K}$ is not explicitly Θ dependent, one has:

$$\mathcal{K}(\varphi_I, \varphi_{II}, J_I, J_{II}) = const. \qquad (5.17)$$

From eqns. (5.16c) and (5.16d) it follows also that

$$\frac{J_I}{n_1} - \frac{J_{II}}{n_2} = const. \qquad (5.18)$$

Then we see that for a "difference resonance" for which

$$\mathrm{sgn}(n_1) = -\,\mathrm{sgn}(n_2)$$

the motion remains stable whereas for a "sum resonance" for which

$$\mathrm{sgn}(n_1) = \mathrm{sgn}(n_2)$$

the variables J_I and J_{II} may in principle become arbitrarily large so that the stability of the particle motion is no longer guaranteed.

Thus in the most naive model, one assumes that the particle dynamics is completely determined by the single resonant harmonics as described above. In the case of a sum resonance, the motion of the particles suffers from an unlimited growth of the oscillation amplitude if the initial amplitudes are chosen to be larger than a certain limit. This limit depends on the distance of the tunes from the resonance lines defined by $n_1 Q_I + n_2 Q_{II} = integer$. On the contrary, if the initial amplitudes are fixed, there is a certain distance of the working point from the resonance line for which the motion takes place just on the separatrix which separates stable and unstable motion. This particular distance in tune space is then called "width of the resonance". A situation where the particle dynamics is completely determined by just one resonant term is quite artificial and can in practice easily be avoided by appropriate choice of the tunes. In addition, most of the potentially resonant terms will not lead to instability since the presence of constant terms $\kappa_{\bar{n}}$ with $n_1 = 0,\ n_2 = 0,\ p = 0$ will cause an amplitude dependence of the tune which stabilizes weak resonances. Therefore one should consider the width of a resonance primarily as a number which characterizes the strength of the nonlinearities. Resonance widths are well suited for the comparison of the impact of various multipole components. Only in the rare case of the occurrance of a strong single resonance, does the resonance width describe also a stability limit in tune space.

The amplitude dependent width of the resonance is calculated from the fixed point condition

$$\partial\mathcal{K}/\partial J_I = \partial\mathcal{K}/\partial J_{II} = \partial\mathcal{K}/\partial\varphi_I = \partial\mathcal{K}/\partial\varphi_{II} = 0 \tag{5.19}$$

under the condition that $c = n_2 J_I - n_1 J_{II}$ is a constant of motion (see eqn. (5.18)), yielding Δ, the width of the nonlinear resonance for given amplitudes $\sqrt{J_I}, \sqrt{J_{II}}$ (for more details see Ref. [10]). For the $\frac{1}{3}$ integer resonance driven by sextupoles as described by the Hamiltonian

$$\mathcal{K} \;=\; \Delta_x \cdot J_x + \kappa_{3030p} \cdot J_x^{3/2} \cdot \cos 3\varphi \tag{5.20}$$

(no linear coupling) one obtains for example for the resonance width

$$Q_x/3 - p = \Delta_x = \frac{3}{2} \cdot \kappa_{3030p} \cdot J_x^{1/2} \; . \tag{5.21}$$

For two dimensional resonances and for high order resonances, the algebra to evaluate the fixed point equation may be quite cumbersome or even impossible to handle analytically.

The formulae (5.4) – (5.16) show that, for fields with midplane symmetry ($m = even$), normal ($n_2 = even$) and skew ($n_2 = odd$) type resonances are excited. In addition the number of terms which drive these resonances increases drastically (compare eqns. (5.4) and (5.6)). For normal sextupole fields, we find 28 different driving terms which drive 8 resonances. Without coupling, only 4 resonances are driven by 5 driving terms.

6 Expectation Value of Driving Terms of Nonlinear Resonances excited by Random Field Errors

According to eqns. (5.3), (5.4) and (5.11), a driving term of a nonlinear resonance is of the form

$$\kappa_{\bar{n}} = \left| \int_s^{s+L} d\tilde{s} \cdot \sum_\alpha b_k(s) \cdot F_\alpha(s) \cdot \exp i\phi_\alpha \right|$$

which involves a one turn integration around the accelerator lattice. Here $b_k(s)$ is the $k-th$ coefficient of the nonlinear disturbing fields, $F_\alpha(s)$ contains the amplitude functions and numerical factors, ϕ_α is the phase function in eqn. (5.4) and α denotes the family of terms which drive the same resonance. For the $\frac{1}{3}$ integer resonance driven by normal sextupoles one obtains for example (no coupling)

$$\kappa_{3030p} = \left| \frac{1}{2\pi} \int_s^{s+L} d\tilde{s} \cdot \frac{b_3}{3r^2\rho} \left(\frac{\beta_x(s)}{2} \right)^{3/2} \cdot \exp 3i\phi_x \right|$$

where b_3 is the multipole coefficient for a normal sextupole, $\frac{1}{\rho}$ is the reference bending strength, r is the reference radius and β_x is the horizontal β-function. Since the betatron phase advance over a magnet is usually small, it is convenient to introduce thin lenses so that the integral transforms into a sum over discrete elements:

$$\kappa_{\bar{n}} = \left| \sum_j \sum_\alpha b_{kj} \cdot F_{\alpha j} \exp i\phi_{\alpha j} \right| .$$

The rms value of $\kappa_{\bar{n}}$ is defined by

$$\kappa_{\bar{n}}^{rms} = \sqrt{< \kappa_{\bar{n}}^2 >}$$

where the brackets mean averaging over different configurations (seeds) of random errors of the multipole coefficients b_k:

$$\kappa_{\bar{n}}^{rms} = \sqrt{< [\sum_j \sum_\alpha b_{kj} \cdot F_{\alpha j} \cos(\phi_{\alpha j})]^2 + [\sum_j \sum_\alpha b_{kj} \cdot F_{\alpha j} \sin(\phi_{\alpha j})]^2 >} .$$

A "random" sequence b_{kj} is defined here to obey:

$$< b_{ki} b_{kj} >= \delta_{ij} \cdot b_{krms}^2$$

so that the expression for $\kappa_{\bar{n}}^{rms}$ simplifies to

$$\kappa_{\bar{n}}^{rms} = b_k^{rms} \sqrt{\sum_j \left([\sum_\alpha F_{\alpha j} \cos(\phi_{\alpha j})]^2 + [\sum_\alpha F_{\alpha j} \sin(\phi_{\alpha j})]^2 \right)} .$$

Expressions for $F_{\alpha j} \cos \phi_{\alpha j}$ and $F_{\alpha j} \sin \phi_{\alpha j}$ can be calculated exactly using linear beam optics codes like PETROS.

For long regular structures, where a certain pattern of F_α values repeats many times (for example $\check{F}, \hat{F}, ...$) we finally obtain as a conservative estimate

$$\kappa_{\bar{n}}^{rms} = b_k^{rms} \sqrt{\frac{N}{2}} \sqrt{(\sum_\alpha \hat{F}_\alpha)^2 + (\sum_\alpha \check{F}_\alpha)^2} \tag{6.1}$$

where we have assumed that the squares of cosine and sine functions average to $\frac{1}{2}$ (but not to $\frac{1}{2}\delta_{\alpha\alpha'}$!):

$$\overline{\cos\phi_\alpha \cdot \cos\phi_{\alpha'}} \approx \overline{\sin\phi_\alpha \cdot \sin\phi_{\alpha'}} \approx \frac{1}{2} \,. \tag{6.2}$$

(The bar means one turn average). N is the number of disturbed elements.

7 Application to HERA and Comparison with Simulation

The results have been applied to the HERA proton ring where the nonlinear field errors are dominated by random sextupolar field errors from the superconducting magnets. The following is a brief summary of the results. According to the results of field measurements on the superconducting dipole magnets [1], we assume for this study Gaussianly distributed sextupolar field errors with a standard deviation of $b_3^{rms} = 3 \cdot 10^{-4}$ at $r = 25mm$ referred to the dipole strength (bend angle) of $\theta = 15mrad$. Higher order multipoles which are also present have been omitted. Random and systematic coupling is introduced in the system by tilts of the quadrupole magnets in the arc. For example, an rms tilt of $1mrad$ would cause an rms tune split of $\kappa = |Q_I - Q_{II}| = 0.01$ due to linear coupling between horizontal and vertical betatron oscillations (see eqn. (B.43) and take $\delta \underline{A}_1 = 0$. Recall also that κ is a measure of resonance width.) We first calculate the width of the resonances excited by the random sextupoles in the third order resonance cluster (with fractional tunes around 1/3) near the main diagonal in the tune space (Q_I, Q_{II}) for the case with and without coupling and compare it with numerical tracking calculations. For analytical estimates in the coupled case we use the formulae in the previous section: $\beta_{xI} = \beta_{xII} = \frac{1}{2}\beta_x$ and $\beta_{yI} = \beta_{yII} = \frac{1}{2}\beta_y$ (see eqn. (4.18a,b)). For the width of the $\frac{1}{3}$-integer resonance for randomly distributed sextupole errors one obtains from (5.20) and (6.1) without coupling

$$3Q_I = p: \quad \Delta = \frac{1}{8\pi\sqrt{2}} \frac{b_3^{rms}\theta}{r^2} \sqrt{\frac{N}{2}} \sqrt{\hat{\beta}_x^3 + \check{\beta}_x^3} \sqrt{\varepsilon_I} \,; \tag{7.1a}$$

$$3Q_{II} = p: \quad \Delta = 0 \tag{7.1b}$$

(ε_I, mode 1, horizontal beam emittance; $\varepsilon_I = 2J_I$) and with coupling

$$\left.\begin{array}{l} 3Q_I = p \\ 3Q_{II} = p \end{array}\right\} \quad \Delta = \frac{1}{32\pi} \frac{b_3^{rms}\theta}{r^2} \sqrt{\frac{N}{2}} \sqrt{(\hat{\beta}_x^{3/2} + 3\hat{\beta}_x^{1/2}\check{\beta}_y)^2 + (\check{\beta}_x^{3/2} + 3\check{\beta}_x^{1/2}\hat{\beta}_y)^2} \sqrt{\varepsilon_k} \tag{7.2}$$

(N is the number of sextupoles). Skew and normal resonances have the same width as may be seen from eqns. (3.9) and (4.18a,b). Table I summarizes the analytical estimates of the widths of the $3rd$-order resonances on the diagonal together with those of the linear coupling, $Q_I - Q_{II}$, resonance for comparison. Equation 5.19 (or 5.21) which gives the resonance widths, determines the stability limit (Acceptance $= J_{I,II}^{max}$ [1]) for a fixed distance of the tunes from

[1] The acceptance is proportional to the square of the dynamic aperture.

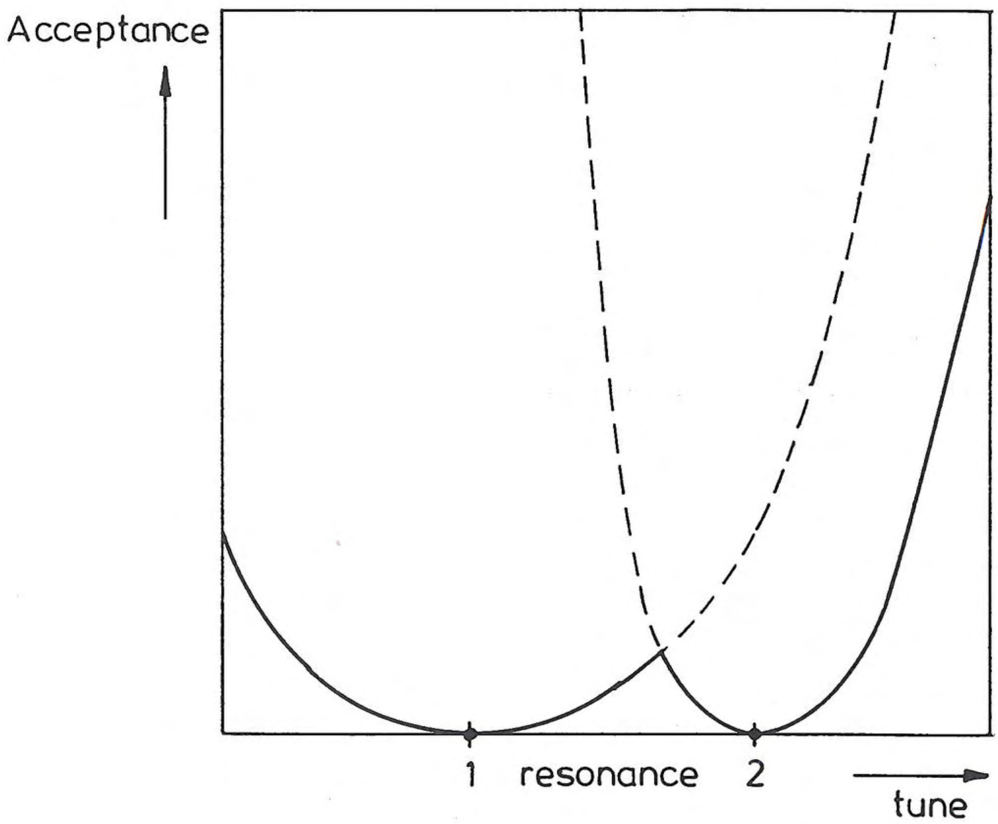

Figure 2: Analytical estimate of dynamic aperture in the vicinity of a resonance cluster by superposing single resonance stability limits

the resonance. This equation can be used to calculate approximately the dynamic aperture over a wide range of tune in the vicinity of the 1/3-order resonance cluster by superposing the dynamic aperture curves obtained for the different "isolated" resonances. This procedure is sketched in Fig 2.

Table I: Comparison of 3rd order Resonance Widths for $b_3^{rms}\theta/r^2 = 0.02m^{-2}$ and $\varepsilon_I = \varepsilon_{II} = 1\mu m$

	resonance	Δ (without coupling)	Δ (with coupling)
	$Q_I - Q_{II}$	0.0000	0.0300
normal	$3Q_I$	0.0162	0.0098
normal	$Q_I + 2Q_{II}$	0.0323	0.0316
skew	$3Q_{II}$	0.0000	0.0098
skew	$Q_{II} + 2Q_I$	0.0000	0.0316

Tracking calculations have been performed for 5 different seeds of random numbers. The simulations and extraction of the acceptance or dynamic aperture are made in the same way as described in references [18]. The width of the linear coupling resonance has been adjusted to $\kappa = 0.03$.

Table I shows that the widths of the $3Q_I$ and $Q_I + 2Q_{II}$ resonance turn out to be comparable with and without coupling. The reason is on one hand, the fact that $\beta_{xI} = \beta_{xII} = \frac{1}{2}\beta_x$, $\beta_{yI} = \beta_{yII} = \frac{1}{2}\beta_y$ reduces the resonance driving terms, but that on the other hand, the

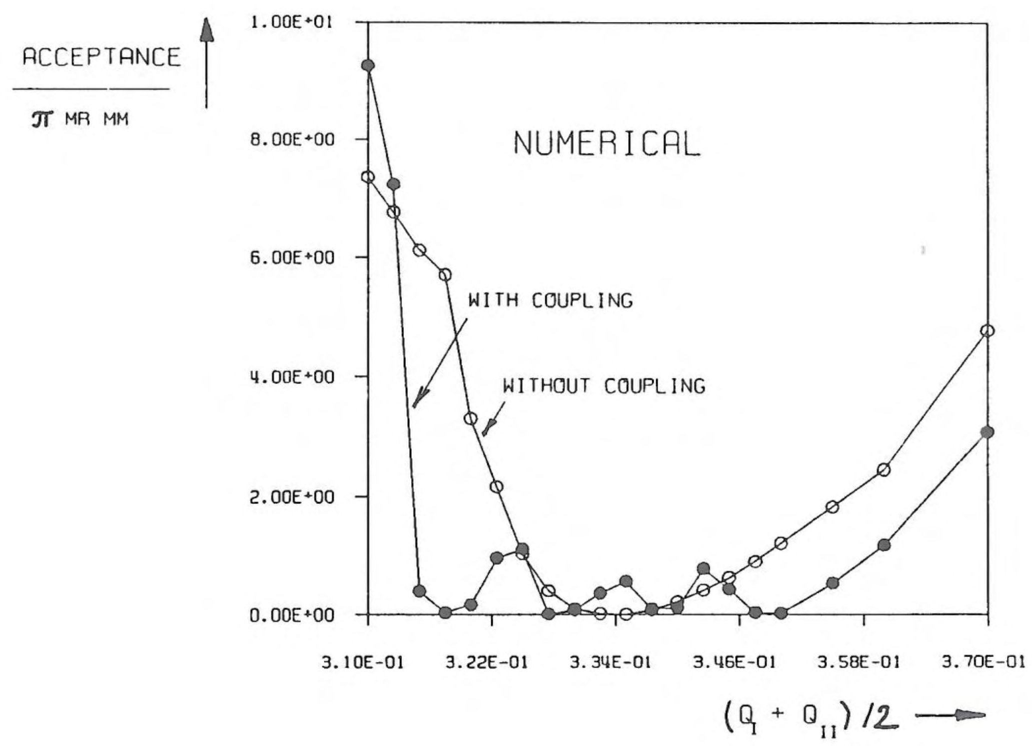

Figure 3: Acceptance of HERA-P for $b_3^{rms}\theta/r^2 = 0.02$ near $Q_I = Q_{II} \simeq 0.033$ by particle tracking. Open circles: without coupling, full circles: with coupling, resonance width $\kappa = 0.03$ as function of uncoupled $Q_x = Q_y$.

increased number of terms which contribute to the driving term (see eqn. (3.9)) enhances the resonance driving terms.

Fig. 3 shows the tracking results for one of the random seeds (a typical case) and Fig. 4 the analytical results. Since one compares an analytical expectation value with the result from a particular random seed, the widths of the resonance do not agree absolutely. Moreover higher order detuning effects cause a small asymmetry in the resonance. These have not been taken into account in the formulae (7.1) and (7.2). Nonetheless, a certain broadening of the resonance cluster due to coupling shows up well in both the analytic and numerical results.

This broadening is due to the additional skew type resonances which do not completely overlap with the normal type resonances because the tunes Q_I and Q_{II} are split due to the linear coupling (see sketch in figure 5). Tracking calculations around $\frac{1}{4}$-integer and $\frac{2}{7}$-integer resonances show that the broadening occurs also at higher order resonances. Note that the octupolar detuning which can be extracted from the formalism of the previous section [14], increases by about 50% due to coupling (for HERA-p). The simulations show that the second order sextupolar detuning is not changed significantly. Besides the broadening of major resonances, the tracking calculations for HERA also show a small but significant overall reduction of the dynamic aperture of about 20 – 30% along the diagonal away from the resonances. A large part of this reduction is produced by the random distortion of the lattice functions due to the random skew quadrupoles. A similar reduction would occur as a consequence of optical distortions by random normal gradient errors (beta beat effects)

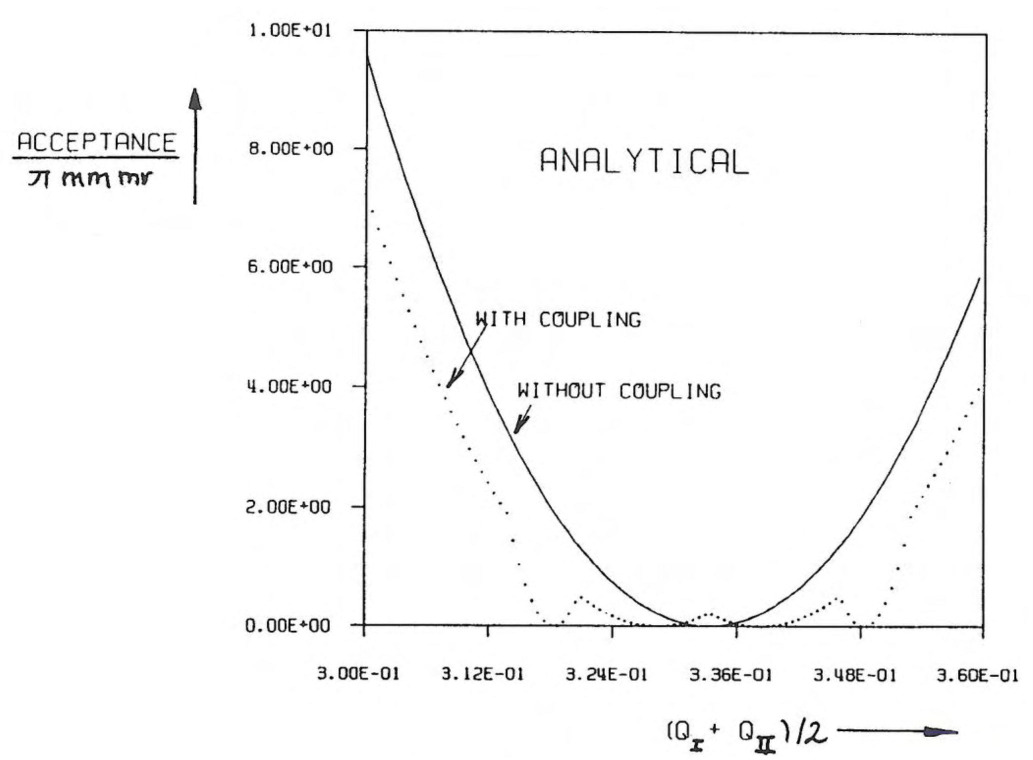

Figure 4: Analytical calculation of the acceptance for the conditions as in figure 3. The lines are without coupling, the dots are with a coupling resonance of strength $\kappa = 0.03$.

and is not an effect of coupling. Consequently this perturbational part vanishes and the dynamic aperture improves if the random quadrupole tilts are replaced by systematic ones which produce the same tune split but which preserve the regularity of the lattice.

What remains to explain is still a residual $10 - 20\%$ reduction of dynamic aperture due to coupling. One can assume that this is the effect of the higher density of resonance lines in the vicinity of the main tune diagonal due to the occurrence of the skew type resonances when linear coupling is present. If the strength of the coupling is reduced, the tunes can be adjusted closer to the diagonal. In this situation, although the motion is still fully coupled, the dynamic aperture improves monotonically with decreasing width of the coupling resonance.

If the randomly generated coupling is compensated with two families of systematically distributed skew quadrupoles, the dynamic aperture is more or less restored apart from the effects of the symmetry breaking optical distortions. If only one pair of skew quadrupoles is used, as planned in HERA-p, however, the reduction of dynamic aperture is found to be even larger due to the lumped compensation, which may, while compensating the coupling resonance, drive the sum resonance $Q_I + Q_{II} = integer$. This would increase the width of the corresponding stopband, and would lead to considerable distortions of the linear optics. This is the explanation for why we find for HERA a drastic reduction of the dynamic aperture if coupling compensation is turned on. Well off the tune diagonal, the coupling which would produce a tune split of $\kappa = 0.03$, does not have an important effect on the beam dynamics in HERA-p. However, at some tunes, not too far from the diagonal, the influence of additional skew resonances is noticeable.

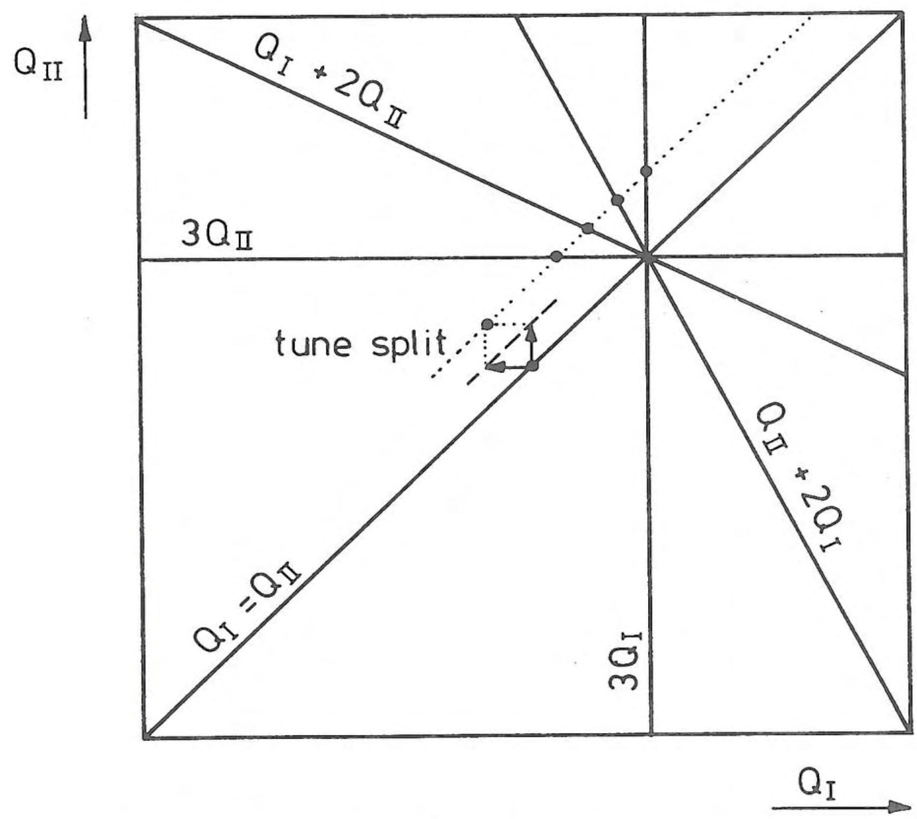

Figure 5: Tune Diagram. The dotted line gives the values of the tunes in presence of coupling while the unperturbed tunes are varied along the main tune diagonal

8 Conclusion

In the preceding sections we have presented canonical perturbation theory and a treatment of nonlinear resonances for coupled linear systems with weak nonlinear perturbations. A special application is the betatron coupling between the horizontal and vertical plane. In this case, nonlinear coupling introduces as a new feature the excitation of skew resonances by fields with midplane symmetry. Far away from the the main tune diagonal this is only of academic interest since the coupling fields in real accelerators are too weak to produce a significant effect. On the main diagonal, coupling causes a split of the tunes. Therefore, the normal resonances and the additional skew resonances do not completely overlap and this leads to effective broadening of a resonance cluster near the main diagonal. As far as the strength of the resonances is concerned, the reduction of β-functions by a factor of two is balanced by the fact that there are more driving terms present, so that the single resonance widths are only insignificantly reduced. Two orthogonal families of skew quadrupoles are sufficient to control this effect. Due to the strong skew quadrupole component of the HERA dipole magnet, it might be difficult to compensate the coupling effects with only two single skew quadrupoles in the HERA proton ring.

If one wishes to use the coupling to increase the vertical emittance of an electron beam in order to produce a round beam cross section and hence minimise beam-beam interaction effects, it is essential that the width of the coupling resonance be carefully minimised.

Therefore, if one wants to make use of coupling, installation of symmetrically distributed skew quadrupoles is necessary. Under optimized conditions, when operating on the coupling resonance and with sextupoles turned on, linear coupling will not lead to a disadvantageous situation for the beam dynamics (e.g. the resonance broadening) provided that the linear tune split is minimised so that the "non-overlap" phenomenon is avoided.

Acknowledgements

We wish to thank Dr. D. P. Barber for helpful suggestions, stimulating discussions and guidance and for careful reading of the manuscript and Prof. Dr. B. Wiik for continued encouragement.

Appendix A: Lattice Functions for Linear Coupled Motion

From eqns. (2.28) and (2.45), the orbit vector

$$\vec{z}(s) = \begin{pmatrix} x(s) \\ p_x(s) \\ y(s) \\ p_y(s) \end{pmatrix}$$

for coupled betatron motion may be written as $(2 \cdot J_k = \epsilon_k)$:

$$\vec{z} = \sqrt{\frac{\epsilon_I}{2}} \cdot \vec{v}_I \cdot e^{i\psi_I} + \sqrt{\frac{\epsilon_{II}}{2}} \cdot \vec{v}_{II} \cdot e^{i\psi_{II}} + c.c. \tag{A.1}$$

We now represent the eigenvectors $\vec{v}_k$ in the form

$$\vec{v}_k \equiv \begin{pmatrix} v_{k1} \\ v_{k2} \\ v_{k3} \\ v_{k4} \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} \sqrt{\beta_{xk}} \cdot e^{i\phi_{xk}} \\ \sqrt{\gamma_{xk}} \cdot e^{i\tilde{\Phi}_{xk}} \\ \sqrt{\beta_{yk}} \cdot e^{i\phi_{yk}} \\ \sqrt{\gamma_{yk}} \cdot e^{i\tilde{\Phi}_{yk}} \end{pmatrix} \tag{A.2}$$

with

$$\beta_{xk} = 2v_{k1} \cdot (v_{k1})^* \; ; \quad \gamma_{xk} = 2v_{k2} \cdot (v_{k2})^* \; ; \tag{A.3a}$$

$$\beta_{yk} = 2v_{k3} \cdot (v_{k3})^* \; ; \quad \gamma_{yk} = 2v_{k4} \cdot (v_{k4})^* \; . \tag{A.3b}$$

Additionally, we introduce the functions :

$$\alpha_{xk} = -[v_{k1} \cdot (v_{k2})^* + v_{k2} \cdot (v_{k1})^*] \; ; \quad \alpha_{yk} = -[v_{k3} \cdot (v_{k4})^* + v_{k4} \cdot (v_{k3})^*] \; ; \qquad \text{(A.3b)}$$

$(k = I, \ II)$.

As can be seen by eqn. (2.25), α_{xk}, β_{xk}, γ_{xk} and α_{yk}, β_{yk}, γ_{yk} are periodic functions:

$$\alpha_{xk}(s+L) = \alpha_{xk}(s) \; ; \quad \beta_{xk}(s+L) = \beta_{xk}(s) \; ; \quad \gamma_{xk}(s+L) = \gamma_{xk}(s) \; ; \tag{A.4a}$$

$$\alpha_{yk}(s+L) = \alpha_{yk}(s) \; ; \quad \beta_{yk}(s+L) = \beta_{yk}(s) \; ; \quad \gamma_{yk}(s+L) = \gamma_{yk}(s) \; . \tag{A.4b}$$

For the phase functions $\phi_{xk}(s)$, $\tilde{\phi}_{xk}(s)$, $\phi_{yk}(s)$, $\tilde{\phi}_{yk}(s)$ one has from (2.17) and (2.19) :

$$\begin{aligned} \phi_{xk}(s+L) - \phi_{xk}(s) &= \phi_{yk}(s+L) - \phi_{yk}(s) = 2\pi Q_k \ ; &\text{(A.5a)} \\ \tilde{\phi}_{xk}(s+L) - \tilde{\phi}_{xk}(s) &= \tilde{\phi}_{yk}(s+L) - \tilde{\phi}_{yk}(s) = 2\pi Q_k \ ; &\text{(A.5b)} \end{aligned}$$

$(k = I,\ II)$.

For the uncoupled case (see eqns. (2.38) and (2.40)) one obtains:

$$\begin{aligned} \alpha_{xI} &= \alpha_x \ ; \\ \beta_{xI} &= \beta_x \ ; \\ \gamma_{xI} &= \gamma_x \ ; \\ \alpha_{yI} &= 0 \ ; \\ \beta_{yI} &= 0 \ ; \\ \gamma_{yI} &= 0 \end{aligned} \qquad \text{(A.6a)}$$

and

$$\begin{aligned} \alpha_{xII} &= 0 \ ; \\ \beta_{xII} &= 0 \ ; \\ \gamma_{xII} &= 0 \ ; \\ \alpha_{yII} &= \alpha_y \ ; \\ \beta_{yII} &= \beta_y \ ; \\ \gamma_{yII} &= \gamma_y \ . \end{aligned} \qquad \text{(A.6b)}$$

Decomposing the eigenvectors $\vec{v}_k$ into a real and an imaginary part:

$$\vec{v}_I = \frac{1}{\sqrt{2}} \cdot (\vec{z}_1 + i \cdot \vec{z}_2) \ ; \quad \vec{v}_{II} = \frac{1}{\sqrt{2}} \cdot (\vec{z}_3 + i \cdot \vec{z}_4) \qquad \text{(A.7)}$$

with

$$\vec{z}_i = \begin{pmatrix} x_i \\ p_{xi} \\ y_i \\ p_{yi} \end{pmatrix} \ ; \quad (i = 1, 2, 3, 4) \qquad \text{(A.8)}$$

one has:

$$\begin{aligned} \alpha_{xI} &= -[x_1 \cdot p_{x1} + x_2 \cdot p_{x2}] \ ; \\ \beta_{xI} &= x_1^2 + x_2^2 \ ; \\ \gamma_{xI} &= p_{x1}^2 + p_{x2}^2 \ ; \\ \alpha_{yI} &= -[y_1 \cdot p_{y1} + y_2 \cdot p_{y2}] \ ; \\ \beta_{yI} &= y_1^2 + y_2^2 \ ; \\ \gamma_{xI} &= p_{y1}^2 + p_{y2}^2 \end{aligned} \qquad \text{(A.9a)}$$

and

$$\begin{aligned} \alpha_{xII} &= -[x_3 \cdot p_{x3} + x_4 \cdot p_{x4}] \; ; \\ \beta_{xII} &= x_3^2 + x_4^2 \; ; \\ \gamma_{xII} &= p_{x3}^2 + p_{x4}^2 \; ; \\ \alpha_{yII} &= -[y_3 \cdot p_{y3} + y_4 \cdot p_{y4}] \; ; \\ \beta_{yII} &= y_3^2 + y_4^2 \; ; \\ \gamma_{xII} &= p_{y3}^2 + p_{y4}^2 \; . \end{aligned} \tag{A.9b}$$

The trajectory $\vec{z}(s)$ is expressed in terms of these lattice functions as:

$$\vec{z} = \begin{pmatrix} \sqrt{\epsilon_I}\sqrt{\beta_{xI}}\cos(\phi_{xI}+\psi_I) + \sqrt{\epsilon_{II}}\sqrt{\beta_{xII}}\cos(\phi_{xII}+\psi_{II}) \\ \sqrt{\epsilon_I}\sqrt{\gamma_{xI}}\cos(\bar{\phi}_{xI}+\psi_I) + \sqrt{\epsilon_{II}}\sqrt{\gamma_{xII}}\cos(\bar{\phi}_{xII}+\psi_{II}) \\ \sqrt{\epsilon_I}\sqrt{\beta_{yI}}\cos(\phi_{yI}+\psi_I) + \sqrt{\epsilon_{II}}\sqrt{\beta_{yII}}\cos(\phi_{yII}+\psi_{II}) \\ \sqrt{\epsilon_I}\sqrt{\gamma_{yI}}\cos(\bar{\phi}_{yI}+\psi_I) + \sqrt{\epsilon_{II}}\sqrt{\gamma_{yII}}\cos(\bar{\phi}_{yII}+\psi_{II}) \end{pmatrix} . \tag{A.10}$$

For the calculation of coupled linear beam optics in a circular machine we now follow the following recipe:

- Find the eigenvectors and the eigenvalues of the revolution matrix at s_0. The eigenvalues must be complex and must have the absolute value 1.
- Normalize these eigenvectors so that $(\vec{v}_k)^+ \underline{S}\vec{v}_k = i$;   (k=I, II) .
- Form the generating vectors: $\vec{z}_{1,3} = \frac{1}{\sqrt{2}}[\vec{v}_{I,II} + (\vec{v}_{I,II})^*]$; $\vec{z}_{2,4} = -\frac{i}{\sqrt{2}}[\vec{v}_{I,II} - (\vec{v}_{I,II})^*]$.
- Calculate the initial lattice functions at s_0 :

$$\begin{pmatrix} \alpha_{xI} = -[x_1 p_{x1} + x_2 p_{x2}] & \beta_{xI} = x_1^2 + x_2^2 & \gamma_{xI} = p_{x1}^2 + p_{x2}^2 & \phi_{xI} = \arctan x_2/x_1 \\ \alpha_{xII} = -[x_3 p_{x3} + x_4 p_{x4}] & \beta_{xII} = x_3^2 + x_4^2 & \gamma_{xII} = p_{x3}^2 + p_{x4}^2 & \phi_{xII} = \arctan x_4/x_3 \\ \alpha_{yI} = -[y_1 p_{y1} + y_2 p_{y2}] & \beta_{yI} = y_1^2 + y_2^2 & \gamma_{yI} = p_{y1}^2 + p_{y2}^2 & \phi_{yI} = \arctan y_2/y_1 \\ \alpha_{yII} = -[y_3 p_{y3} + y_4 p_{y4}] & \beta_{yII} = y_3^2 + y_4^2 & \gamma_{yII} = p_{y3}^2 + p_{y4}^2 & \phi_{yII} = \arctan y_4/y_3 \end{pmatrix} \tag{A.11}$$

The lattice functions elsewhere in the ring are obtained the same way after transporting the generating vectors $\vec{z}_i$ through the lattice :

$$\vec{z}_i(s) = \underline{M}(s, s_0) \cdot \vec{z}_i(s_0) \; . \tag{A.12}$$

What is the meaning of these four sets of lattice functions? If the four dimensional torus on which our particle, represented by the vector $\vec{z}$, is moving is projected onto the $x - p_x$-plane we get an elliptical band filled with possible coordinates x, p_x. This region is obtained by the superposition of the two ellipses defined by the two sets of lattice functions:

$$E_{xI} = \sqrt{\epsilon_I \beta_{xI}} \; ; \quad A_{xI} = \sqrt{\epsilon_I \gamma_{xI}} \; ; \quad G_{xI} = -\sqrt{\epsilon_I}\frac{\alpha_{xI}}{\sqrt{\beta_{xI}}} \; ; \tag{A.13a}$$

$$E_{xII} = \sqrt{\epsilon_I \beta_{xII}} \; ; \quad A_{xII} = \sqrt{\epsilon_{II} \gamma_{xII}} \; ; \quad G_{xII} = -\sqrt{\epsilon_{II}}\frac{\alpha_{xII}}{\sqrt{\beta_{xII}}} \; . \tag{A.13b}$$

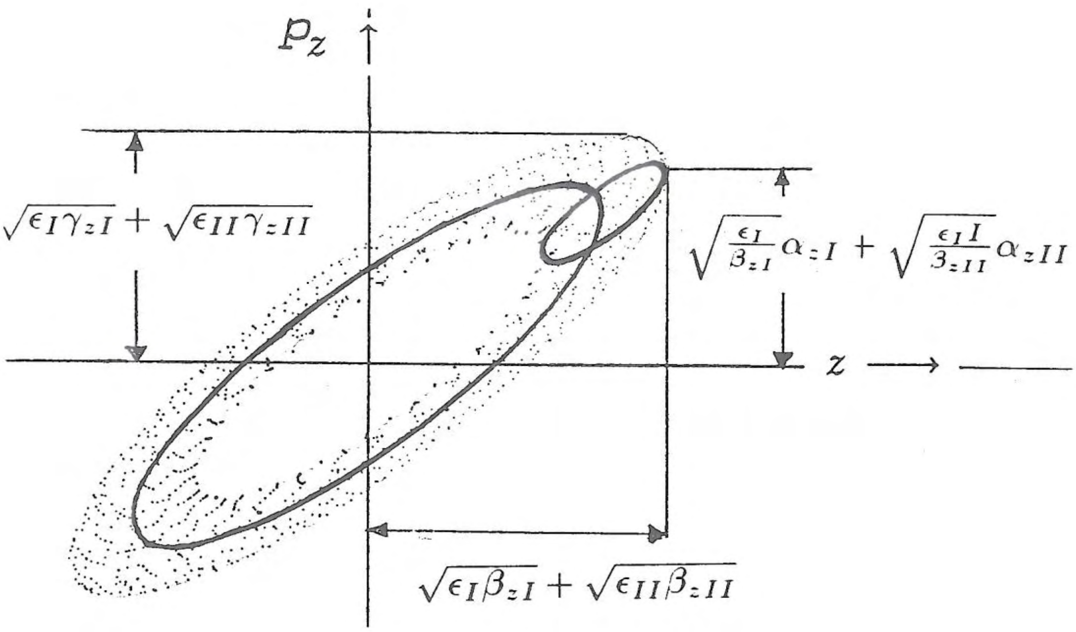

Figure 6: Two sets of lattice functions for each plane describe the projection of a four dimensional torus on the $x - x'$ and $y - y'$-planes.

The same thing happens for the projection onto the $y - p_y$-plane. This is illustrated in Fig. 6. We see from eqn. (A.11) that, as in the uncoupled case, the lattice functions describe the focussing properties of the lattice: Mode I and mode II lattice functions tell us where the the particle oscillation amplitudes or the trajectory slopes are large or small just in the same way as for the uncoupled case.

The areas of the two ellipses

$$\Gamma_{xI} = \pi \cdot |x_1 \cdot p_{x2} - x_2 \cdot p_{x1}| = \pi \cdot E_{xI}\sqrt{A_{xI}^2 - G_{xI}^2} \; ; \qquad \text{(A.14a)}$$

$$\Gamma_{xII} = \pi \cdot |x_3 \cdot p_{x4} - x_4 \cdot p_{x3}| = \pi \cdot E_{xII}\sqrt{A_{xII}^2 - G_{xII}^2} \qquad \text{(A.14b)}$$

are not preserved during the motion through the lattice but the sum or difference (depending if $(x_1 \cdot p_{x2} - x_2 \cdot p_{x1})$ and $(y_1 \cdot p_{y2} - y_2 \cdot p_{y1})$ resp. $(x_3 \cdot p_{x4} - x_4 \cdot p_{x3})$ and $(y_3 \cdot p_{y4} - y_4 \cdot p_{y3})$ have the same or opposite sign) of the horizontal and vertical ellipse areas of mode I and II respectively :

$$\pi\{[x_1 \cdot p_{x2} - x_2 \cdot p_{x1}] + [y_1 \cdot p_{y2} - y_2 \cdot p_{y1}]\} = \pi\epsilon_I \; ; \qquad \text{(A.15a)}$$

$$\pi\{[x_3 \cdot p_{x4} - x_4 \cdot p_{x3}] + [y_3 \cdot p_{y4} - y_4 \cdot p_{y3}]\} = \pi\epsilon_{II} \qquad \text{(A.15b)}$$

are constants of motion because of the normalization of the generating vectors. Thus in the coupled case the invariants $\epsilon_I, \epsilon_{II}$ replace the horizontal and vertical invariants which we obtain in the uncoupled case.

For more details see Ref. [3].

Appendix B: Perturbation Theory for Linear Coupled Motion

We decompose the Hamiltonian (2.11a) into two parts:

$$\mathcal{H}^{(0)} = \mathcal{H}_{00} + \mathcal{H}_{01} \tag{B.1}$$

with

$$\begin{aligned} \mathcal{H}_{00} &= \frac{1}{2}\cdot p_x^2 + \frac{1}{2}\cdot p_y^2 + \frac{1}{2}\cdot [K_x^2(s) + k(s) + H^2(s)]\cdot x^2 \\ &+ \frac{1}{2}\cdot [K_y^2(s) - k(s) + H^2(s)]\cdot y^2\ ; \end{aligned} \tag{B.2a}$$

$$\begin{aligned} \mathcal{H}_{01} &= H(s)\cdot p_x\cdot y - H(s)\cdot p_y\cdot x \\ &+ N(s)\cdot xy \end{aligned} \tag{B.2b}$$

where in $\mathcal{H}_{01}$ we have gathered the (linear) coupling terms of (2.11a).

For the uncoupled problem we now have:

$$\frac{d}{ds}\vec{z} = \underline{S}\cdot\frac{\partial\mathcal{H}_{00}}{\partial\vec{z}} \tag{B.3a}$$

or

$$\frac{d}{ds}\vec{z} = \underline{A}_0\cdot\vec{z} \tag{B.3b}$$

and the linear coupled problem reads as:

$$\begin{aligned} \frac{d}{ds}\vec{z} &= \underline{S}\cdot\frac{\partial\mathcal{H}^{(0)}}{\partial\vec{z}} \\ &= \underline{S}\cdot\frac{\partial\mathcal{H}_{00}}{\partial\vec{z}} + \underline{S}\cdot\frac{\partial\mathcal{H}_{01}}{\partial\vec{z}} \end{aligned} \tag{B.4a}$$

or

$$\frac{d}{ds}\vec{z} = (\underline{A}_0 + \delta\underline{A})\cdot\vec{z}\ . \tag{B.4b}$$

Comparing (B.3) and (B.4) with (2.10a,b) we then get from eqn. (2.15) the relation:

$$\delta\underline{A}^T(s)\cdot\underline{S} + \underline{S}\cdot\delta\underline{A}(s) = \underline{0}\ . \tag{B.5}$$

For the revolution matrix $\underline{M}(s+L,s)$ we may write:

$$\underline{M}(s+L,s) = \underline{M}_0(s+L,s) + \delta\underline{M}(s+L,s)\ .$$

B.1 Calculation for the Perturbed Part of the Revolution Matrix

In order to determine the perturbation part $\delta\underline{M}(s+L,s)$ of the revolution matrix of the linear coupled problem we first remark that according to eqn. (B.4) the transfer matrix

$$\underline{M}_0(s,s_0) + \delta\underline{M}(s,s_0)$$

obeys the equation:

$$\frac{d}{ds}[\underline{M}_0(s,s_0) + \delta\underline{M}(s,s_0)] = [\underline{A}_0(s) + \delta\underline{A}(s)] \cdot [\underline{M}_0(s,s_0) + \delta\underline{M}(s,s_0)] \; ; \qquad \text{(B.6a)}$$

$$\underline{M}_0(s_0,s_0) + \delta\underline{M}(s_0,s_0) = \underline{1} \; . \qquad \text{(B.6b)}$$

Taking into account the corresponding equations for the unperturbed transfer matrix $\underline{M}_0(s,s_0)$:

$$\frac{d}{ds}\underline{M}_0(s,s_0) = \underline{A}_0(s) \cdot \underline{M}_0(s,s_0) \; ;$$
$$\underline{M}_0(s_0,s_0) = \underline{1}$$

we obtain from (B.6) in first order the differential equation for $\delta\underline{M}(s,s_0)$ [12] :

$$\frac{d}{ds}\; \delta\underline{M}(s,s_0)] = \underline{A}_0(s) \cdot \delta\underline{M}(s,s_0) + \delta\underline{A}(s) \cdot \underline{M}_0(s,s_0)$$

with the initial condition:

$$\delta\underline{M}(s_0,s_0) = \underline{0} \; .$$

The solution of this equation (and thus the first order solution of eqn. (B.6)) reads as:

$$\delta\underline{M}(s,s_0) = \int_{s_0}^{s} d\tilde{s} \cdot \underline{M}_0(s,\tilde{s}) \cdot \delta\underline{A}(\tilde{s}) \cdot \underline{M}_0(\tilde{s},s_0)$$
$$= \underline{M}_0(s,s_0) \cdot \int_{s_0}^{s} d\tilde{s} \cdot \underline{M}_0^{-1}(\tilde{s},s_0) \cdot \delta\underline{A}(\tilde{s}) \cdot \underline{M}_0(\tilde{s},s_0) \; .$$

For the perturbative part $\delta\underline{M}(s+L,s)$ of the revolution matrix $\underline{M}(s+L,s)$ one therefore gets in first order the expression:

$$\delta\underline{M}(s+L,s) = \int_{s}^{s+L} d\tilde{s} \cdot \underline{M}_0(s+L,\tilde{s}) \cdot \delta\underline{A}(\tilde{s}) \cdot \underline{M}_0(\tilde{s},s)$$
$$= \underline{M}_0(s+L,s) \cdot \int_{s}^{s+L} d\tilde{s} \cdot \underline{M}_0^{-1}(\tilde{s},s) \cdot \delta\underline{A}(\tilde{s}) \cdot \underline{M}_0(\tilde{s},s) \; . \qquad \text{(B.7)}$$

Another way to derive eqn. (B.7) is represented in Ref. [3].

B.2 Perturbation Theory

B.2.1 Outside the Resonances

With the help of eqn. (B.7) we are now in the position to calculate the Q-shift

$$\delta Q_\kappa = -\frac{i}{2\pi \cdot \lambda_\kappa} \cdot \delta\lambda_\kappa \qquad \text{(B.8)}$$

caused by $\delta\underline{M}$ [4,12].

For that purpose we write the eigenvalue equation as:

$$(\underline{M}_0 + \delta\underline{M}) \cdot (\vec{v}_\mu + \delta\vec{v}_\mu) = (\lambda_\mu + \delta\lambda_\mu) \cdot (\vec{v}_\mu + \delta\vec{v}_\mu) \; ; \qquad \text{(B.9a)}$$
$$(\mu = \pm I, \; \pm II, \; \pm III)$$

where $\delta\vec{v}$ and $\delta\lambda$ are the modifications to the uncoupled eigenvectors and eigenvalues.

Since

$$\underline{M}_0\vec{v}_\mu = \lambda_\mu\vec{v}_\mu$$

we have in first order:

$$\underline{M}_0 \cdot \delta\vec{v}_\mu + \delta\underline{M} \cdot \vec{v}_\mu = \lambda_\mu \cdot \delta\vec{v}_\mu + \delta\lambda_k \cdot \vec{v}_\mu \,. \tag{B.9b}$$

By expanding $\delta\vec{v}_\mu$ in terms of the eigenvectors $\vec{v}_\mu$ of the unperturbed problem:

$$\delta\vec{v}_\mu = \sum_\nu a_{\mu\nu} \cdot \vec{v}_\nu \tag{B.10}$$

and by inserting (B.10) into (B.9b) we then get:

$$\sum_\mu a_{\mu\nu} \cdot \lambda_\nu\vec{v}_\nu + \delta\underline{M} \cdot \vec{v}_\mu = \lambda_\mu \cdot \sum_\mu a_{\mu\nu}\vec{v}_\nu + \delta\lambda_\mu \cdot \vec{v}_\mu \,. \tag{B.11}$$

Multiplying this equation from the left hand side with

$$\frac{1}{i} \cdot \vec{v}_\kappa^{\,+}\underline{S}$$

and taking into account eqn. (2.31a) we obtain

$$a_{\mu\kappa} \cdot \lambda_\kappa \cdot \left(\frac{1}{i}\vec{v}_\kappa^{\,+}\underline{S}\vec{v}_\kappa\right) + \frac{1}{i}\vec{v}_\kappa^{\,+}\underline{S} \cdot \delta\underline{M} \cdot \vec{v}_\mu = \lambda_\mu \cdot a_{\mu\kappa}\left(\frac{1}{i}\vec{v}_\kappa^{\,+}\underline{S}\vec{v}_\kappa\right) + \delta\lambda_\kappa\left(\frac{1}{i}\vec{v}_\kappa^{\,+}\underline{S}\vec{v}_\kappa\right) \cdot \delta_{\mu\kappa} \tag{B.12}$$

with

$$\left(\frac{1}{i}\vec{v}_\kappa^{\,+}\underline{S}\vec{v}_\kappa\right) = \begin{cases} +1 & \text{for } \kappa = I,\ II\ ; \\ -1 & \text{for } \kappa = -I, -II\ . \end{cases}$$

For $\kappa \neq \mu$ the expansion coefficients are given by (see eqn. (B.7))

$$\begin{aligned} a_{\mu\kappa} &= \left(\frac{1}{i}\vec{v}_\kappa^{\,+}\underline{S}\vec{v}_\kappa\right) \cdot \frac{1}{\lambda_\mu - \lambda_\kappa} \cdot \frac{1}{i}\vec{v}_\kappa^{\,+}\underline{S} \cdot \delta\underline{M}(s+L,s) \cdot \vec{v}_\mu(s) \\ &= \left(\frac{1}{i}\vec{v}_\kappa^{\,+}\underline{S}\vec{v}_\kappa\right) \cdot \frac{1}{\lambda_\mu - \lambda_\kappa} \cdot \frac{1}{i}\vec{v}_\kappa^{\,+}\underline{S} \cdot \underline{M}_0(s+L,s) \times \\ &\quad \int_s^{s+L} d\tilde{s} \cdot \underline{M}_0^{-1}(\tilde{s},s) \cdot \delta\underline{A}(\tilde{s}) \cdot \underline{M}_0(\tilde{s},s) \cdot \vec{v}_\mu(s) \end{aligned} \tag{B.13a}$$

and for $\kappa = \mu$ the normalization condition (2.31) leads to:

$$\text{Re}\,\{a_{\mu\mu}\} = 0 \implies a_{\mu\mu} = i \cdot \varphi_\mu$$

where φ_μ is an arbitrary real number. This is in agreement with the fact that one can multiply an eigenvector $\vec{v}_\mu$ with an arbitrary phase factor $e^{i\varphi_\mu}$ without disturbing the normalization.

In the following we will set:

$$\varphi_\mu = 0 \implies a_{\mu\mu} = 0\,. \tag{B.13b}$$

Using the symplecticity condition of the transfer matrix $\underline{M}_0(s_1, s_2)$:

$$\underline{M}_0^T(s_1, s_2) \cdot \underline{S} \cdot \underline{M}_0(s_1, s_2) = \underline{S}$$

and the equation

$$\begin{aligned}
\vec{v}_\kappa^{\,+}(s) \cdot \underline{S} \cdot \underline{M}_0(s+L, s) &= \vec{v}_\kappa^{\,+}(s) \cdot \left[\underline{M}_0^{-1}(s+L, s)\right]^T \cdot \underline{S} \\
&= \left[\underline{M}_0^{-1}(s+L, s) \cdot \vec{v}_\kappa(s)\right]^+ \cdot \underline{S} \\
&= \left[\lambda_\kappa^{-1} \cdot \vec{v}_\kappa(s)\right]^+ \cdot \underline{S} \\
&= \lambda_\kappa \cdot \vec{v}_\kappa^{\,+}(s) \cdot \underline{M}_0^T(\bar{s}, s) \cdot \underline{S} \cdot \underline{M}_0(\bar{s}, s) \\
&\quad \left(\text{since } \ (\lambda_\kappa^{-1})^* = \lambda_\kappa \ \text{ and } \ \underline{S} = \underline{M}_0^T \cdot \underline{S} \cdot \underline{M}_0\right) \\
&= \lambda_\kappa \cdot [\underline{M}_0(\bar{s}, s) \cdot \vec{v}_\kappa(s)]^+ \cdot \underline{S} \cdot \underline{M}_0(\bar{s}, s) \\
&= \lambda_\kappa \cdot \vec{v}_\kappa^{\,+}(\bar{s}) \cdot \underline{S} \cdot \underline{M}_0(\bar{s}, s)
\end{aligned} \tag{B.14}$$

$a_{\mu\kappa}$ can be rewritten as

$$\begin{aligned}
a_{\mu\kappa} &= \left(\frac{1}{i}\vec{v}_\kappa^{\,+}\underline{S}\vec{v}_\kappa\right) \frac{\lambda_\kappa}{\lambda_\mu - \lambda_\kappa} \times \\
&\quad \frac{1}{i} \cdot \int_s^{s+L} d\bar{s} \cdot \vec{v}_\kappa^{\,+}(\bar{s}) \cdot \underline{S} \cdot \delta\underline{A}(\bar{s}) \cdot \vec{v}_\mu(\bar{s}) \ .
\end{aligned} \tag{B.15}$$

With the notation

$$\begin{aligned}
&\vec{v}_1(s) \equiv \vec{v}_I(s) \ ; \\
&\vec{v}_2(s) = \vec{v}_{-I}(s) \ ; \\
&\vec{v}_3(s) = \vec{v}_{II}(s) \ ; \\
&\vec{v}_4(s) = \vec{v}_{-II}(s)
\end{aligned} \tag{B.16}$$

$$\Longrightarrow \quad \lambda_2 = \lambda_1^* \ ; \quad \lambda_4 = \lambda_3^*$$

and using the relation

$$\frac{1}{i} \cdot \frac{\lambda_\mu}{\lambda_\mu - \lambda_\kappa} = -\frac{1}{2} \cdot \frac{e^{-i\pi[Q_\mu - Q_\kappa]}}{\sin\pi[Q_\mu - Q_\kappa]}$$

we then obtain:

$$\delta\vec{v}_I = a_{12} \cdot [\vec{v}_I]^* + a_{13} \cdot \vec{v}_{II} + a_{14} \cdot [\vec{v}_{II}]^* \ ; \quad \delta\vec{v}_{-I} = [\delta\vec{v}_I]^* \tag{B.16a}$$

with

$$\begin{aligned}
a_{12} &= +\frac{1}{2} \cdot \frac{e^{-i \cdot 2\pi Q_I}}{\sin 2\pi Q_I} \cdot \int_s^{s+L} d\bar{s} \cdot \vec{v}_2^{\,+}(\bar{s}) \cdot \underline{S} \cdot \delta\underline{A}(\bar{s}) \cdot \vec{v}_1(\bar{s}) \ ; \\
a_{13} &= -\frac{1}{2} \cdot \frac{e^{-i\pi \cdot [Q_I - Q_{II}]}}{\sin\pi[Q_I - Q_{II}]} \cdot \int_s^{s+L} d\bar{s} \cdot \vec{v}_3^{\,+}(\bar{s}) \cdot \underline{S} \cdot \delta\underline{A}(\bar{s}) \cdot \vec{v}_1(\bar{s}) \ ; \\
a_{14} &= +\frac{1}{2} \cdot \frac{e^{+i\pi \cdot [Q_I + Q_{II}]}}{\sin\pi[Q_I + Q_{II}]} \cdot \int_s^{s+L} d\bar{s} \cdot \vec{v}_4^{\,+}(\bar{s}) \cdot \underline{S} \cdot \delta\underline{A}(\bar{s}) \cdot \vec{v}_1(\bar{s}) \ .
\end{aligned}$$

Furthermore

$$\delta\vec{v}_{II} = a_{31}\cdot\vec{v}_I + a_{32}\cdot[\vec{v}_I]^* + a_{34}\cdot[\vec{v}_{II}]^* \ ; \quad \delta\vec{v}_{-I} = [\delta\vec{v}_I]^* \tag{B.16b}$$

with

$$\begin{aligned}
a_{31} &= -\frac{1}{2}\cdot\frac{e^{-i\pi\cdot[Q_{II}-Q_I]}}{\sin\pi[Q_{II}-Q_{II}]}\cdot\int_s^{s+L} d\tilde{s}\cdot\vec{v}_1^{+}(\tilde{s})\cdot\underline{S}\cdot\delta\underline{A}(\tilde{s})\cdot\vec{v}_3(\tilde{s}) \ ; \\
a_{32} &= +\frac{1}{2}\cdot\frac{e^{-i\pi\cdot[Q_{II}+Q_I]}}{\sin\pi[Q_{II}+Q_I]}\cdot\int_s^{s+L} d\tilde{s}\cdot\vec{v}_2^{+}(\tilde{s})\cdot\underline{S}\cdot\delta\underline{A}(\tilde{s})\cdot\vec{v}_3(\tilde{s}) \ ; \\
a_{34} &= +\frac{1}{2}\cdot\frac{e^{-i2\pi\cdot Q_{II}}}{\sin 2\pi Q_{II}}\cdot\int_s^{s+L} d\tilde{s}\cdot\vec{v}_4^{+}(\tilde{s})\cdot\underline{S}\cdot\delta\underline{A}(\tilde{s})\cdot\vec{v}_3(\tilde{s}) \ .
\end{aligned}$$

For $\mu = \kappa$ the first terms on both sides of eqn. (B.12) cancel and one obtains with (B.7), (B.10) and (B.14) the following approximate expression for the Q-shift δQ_κ in linear order:

$$\begin{aligned}
\delta Q_\kappa &= -\left(\frac{1}{i}\cdot\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa\right)\cdot\frac{1}{2\pi\cdot\lambda_\kappa}\cdot\vec{v}_\kappa^{+}\underline{S}\cdot\delta\underline{M}(s+L,s)\cdot\vec{v}_\kappa \\
&= -\left(\frac{1}{i}\cdot\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa\right)\cdot\frac{1}{2\pi\cdot\lambda_\kappa}\cdot\vec{v}_\kappa^{+}\underline{S}\cdot\underline{M}(s+L,s)\times \\
&\quad \int_s^{s+L} d\tilde{s}\cdot\underline{M}^{-1}(\tilde{s},s)\cdot\delta\underline{A}(\tilde{s})\cdot\underline{M}_0(\tilde{s},s)\vec{v}_\kappa(s) \\
&= -\left(\frac{1}{i}\cdot\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa\right)\cdot\frac{1}{2\pi}\cdot\int_s^{s+L} d\tilde{s}\cdot\vec{v}_\kappa^{+}(\tilde{s})\cdot\underline{S}\cdot\delta\underline{A}(\tilde{s})\cdot\vec{v}_\kappa(\tilde{s}) \\
&= -\left(\frac{1}{i}\cdot\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa\right)\cdot\frac{1}{2\pi}\cdot\int_{s_0}^{s_0+L} d\tilde{s}\cdot\vec{v}_\kappa^{+}(\tilde{s})\cdot\underline{S}\cdot\delta\underline{A}(\tilde{s})\cdot\vec{v}_\kappa(\tilde{s})
\end{aligned}$$

(in the last step we have used the fact that the integrand is a periodic function of period L). Thus for $\kappa = k$ and $\kappa = -k \quad (k = \pm I, \pm II)$:

$$\delta Q_k = -\frac{1}{2\pi}\cdot\int_{s_0}^{s_0+L} d\tilde{s}\cdot\vec{v}_k^{+}(\tilde{s})\cdot\underline{S}\cdot\delta\underline{A}(\tilde{s})\cdot\vec{v}_k(\tilde{s}) \ ; \tag{B.17a}$$

$$\delta Q_{-k} = +\frac{1}{2\pi}\cdot\int_{s_0}^{s_0+L} d\tilde{s}\cdot\vec{v}_{-k}^{+}(\tilde{s})\cdot\underline{S}\cdot\delta\underline{A}(\tilde{s})\cdot\vec{v}_{-k}(\tilde{s}) \ . \tag{B.17b}$$

Taking into account that :

$$\begin{aligned}
\delta Q_\kappa^* &= -\left(\frac{1}{i}\cdot\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa\right)^{+}\cdot\frac{1}{2\pi}\int_{s_0}^{s_0+L} d\tilde{s}\cdot\left[\vec{v}_\kappa^{+}(\tilde{s})\cdot\underline{S}\cdot\delta\underline{A}(\tilde{s})\cdot\vec{v}_\kappa(\tilde{s})\right]^{+} \\
&= -\left(\frac{1}{i}\cdot\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa\right)\cdot\frac{1}{2\pi}\int_{s_0}^{s_0+L} d\tilde{s}\cdot\left[-\vec{v}_\kappa^{+}(\tilde{s})\cdot\delta\underline{A}^T(\tilde{s})\cdot\underline{S}\cdot\vec{v}_\kappa(\tilde{s})\right]
\end{aligned}$$

as well as

$$\vec{v}_{-\kappa} = (\vec{v}_\kappa)^*$$

the following relations can be derived from (B.17a,b):

$$\Re e\{\delta Q_k\} = -\frac{1}{4\pi}\cdot\int_{s_0}^{s_0+L} d\tilde{s}\cdot\vec{v}_k^{+}(\tilde{s})\cdot\left[\underline{S}\cdot\delta\underline{A}(\tilde{s}) - \delta\underline{A}^T(\tilde{s})\cdot\underline{S}\right]\cdot\vec{v}_k(\tilde{s})$$

$$= \quad -\Re e\{\delta Q_{-k}\} \; ;$$

$$\Im m\{\delta Q_k\} \quad = \quad +\frac{i}{4\pi} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \vec{v}_k^{+}(\tilde{s}) \cdot \left[\underline{S} \cdot \delta \underline{A}(\tilde{s}) + \delta \underline{A}^T(\tilde{s}) \cdot \underline{S}\right] \cdot \vec{v}_k(\tilde{s})$$
$$= \quad +\Im m\{\delta Q_{-k}\} \; .$$

Using (B.5) we now obtain:

$$\Im m\{\delta Q_k\} = \Im m\{\delta Q_{-k}\} = 0$$

and therefore

$$\delta Q_k \quad = \quad -\delta Q_{-k} \quad \text{real} \; . \tag{B.18}$$

This relation is in agreement with eqn. (2.20).

Finally we point out that the perturbation theory used here is only valid if $\delta\lambda_\mu$ and $\delta\vec{v}_\mu$ are small compared with the unperturbed quantities λ_μ and $\vec{v}_\mu$:

$$|\delta\lambda_\mu| \quad \ll \quad |\lambda_\mu| \; ;$$
$$\|\delta\vec{v}_\mu\| \quad \ll \quad \|\vec{v}_\mu\| \; .$$

Therefore, in order to apply this kind of perturbation theory the following condition must hold (see eqn. (B.15)):

$$\left| \int_s^{s+L} d\tilde{s} \cdot \left[\vec{v}_\kappa^{+}(\tilde{s}) \cdot \underline{S} \cdot \delta\underline{A}(\tilde{s}) \cdot \vec{v}_\mu(\tilde{s})\right] \right| \ll |\lambda_\mu - \lambda_\kappa| \; .$$

This condition is well satisfied if the values for different $\lambda_\mu, \lambda_\kappa$ are far apart. However the calculation breaks down at resonance i.e. if two eigenvalues coincide:

$$\lambda_k = e^{-i \cdot 2\pi Q_k} \approx \lambda_{k'} = e^{-i \cdot 2\pi Q_{k'}} \Longleftrightarrow Q_k - Q_{k'} = n \tag{B.19a}$$

or

$$\lambda_k = e^{-i \cdot 2\pi Q_k} \approx \lambda_{-k'} = e^{+i \cdot 2\pi Q_{k'}} \Longleftrightarrow Q_k + Q_{k'} = n \tag{B.19b}$$

where n is an integer.

These Q-resonances can lead to instabilities of the particle motion. This will be discussed in the next section.

B.2.2 Resonances

We now consider the case where the eigenvalue spectrum of the (uncoupled) revolution matrix $\underline{M}_0(s+L,s)$ is degenerate:

$$\begin{cases} \lambda_{II} = \lambda_I & \Longrightarrow \quad Q_x^{(0)} - Q_y^{(0)} = n \quad (\text{ difference resonance}) \; ; \\ \lambda_{II} = \lambda_I^* & \Longrightarrow \quad Q_x^{(0)} + Q_y^{(0)} = n \quad (\text{ sum resonance}) \end{cases} \tag{B.20}$$

with

$$\lambda_I \quad = \quad e^{i \cdot 2\pi Q_x^{(0)}} \; ;$$
$$\lambda_{II} \quad = \quad e^{i \cdot 2\pi Q_y^{(0)}} \; .$$

In the absence of coupling

$$\underline{A}_0(s) \;=\; \begin{pmatrix} 0 & 1 & 0 & 0 \\ -[K_x^2+k] & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -[K_y^2-k] & 0 \end{pmatrix} \qquad \text{(B.21)}$$

and the eigenvectors of $\underline{M}_0(s+L,s)$ then take the form:

$$\vec{v}_I^{\,(0)}(s) \;=\; \frac{1}{\sqrt{2\beta_x(s)}} \cdot \begin{pmatrix} \beta_x(s) \\ -[\alpha_x(s)-i] \\ 0 \\ 0 \end{pmatrix} \cdot e^{i\phi_x(s)} \;;\quad \vec{v}_{-I}^{\,(0)}(s) = [\vec{v}_I^{\,(0)}(s)]^* \;; \qquad \text{(B.22a)}$$

$$\vec{v}_{II}^{\,(0)}(s) \;=\; \frac{1}{\sqrt{2\beta_y(s)}} \cdot \begin{pmatrix} 0 \\ 0 \\ \beta_y(s) \\ -[\alpha_y(s)-i] \end{pmatrix} \cdot e^{i\phi_y(s)} \;;\quad \vec{v}_{-II}^{\,(0)}(s) = [\vec{v}_{II}^{\,(0)}(s)]^* \;. \qquad \text{(B.22b)}$$

For the difference resonance the uncoupled eigenvalue problem reads as:

$$\begin{aligned} \underline{M}_0(s+L,s)\;\vec{v}_1(s) &= \lambda_1 \cdot \vec{v}_1(s) \equiv \lambda \cdot \vec{v}_1(s) \;; \\ \underline{M}_0(s+L,s)\;\vec{v}_2(s) &= \lambda_2 \cdot \vec{v}_2(s) \equiv \lambda \cdot \vec{v}_2(s) \;; \\ \underline{M}_0(s+L,s)\;\vec{v}_3(s) &= \lambda_3 \cdot \vec{v}_3(s) \equiv \lambda^* \cdot \vec{v}_3(s) \;; \\ \underline{M}_0(s+L,s)\;\vec{v}_3(s) &= \lambda_4 \cdot \vec{v}_4(s) \equiv \lambda^* \cdot \vec{v}_4(s) \end{aligned} \qquad \text{(B.23)}$$

where

$$\lambda \equiv \lambda_I = e^{i \cdot 2\pi Q_x^{(0)}} \qquad \text{(B.24)}$$

and

$$\begin{aligned} \vec{v}_1(s) &\equiv \vec{v}_I^{\,(0)}(s) \;; \\ \vec{v}_2(s) &= \vec{v}_{II}^{\,(0)}(s) \;; \\ \vec{v}_3(s) &= [\vec{v}_1(s)]^* \;; \\ \vec{v}_4(s) &= [\vec{v}_2(s)]^* \;. \end{aligned} \qquad \text{(B.25a)}$$

For a sum resonance the vectors in (B.23) are replaced by:

$$\begin{aligned} \vec{v}_1(s) &\equiv \vec{v}_I^{\,(0)}(s) \;; \\ \vec{v}_2(s) &= [\vec{v}_{II}^{\,(0)}(s)]^* \;; \\ \vec{v}_3(s) &= [\vec{v}_1(s)]^* \;; \\ \vec{v}_4(s) &= [\vec{v}_2(s)]^* \;. \end{aligned} \qquad \text{(B.25b)}$$

Furthermore we assume that the coupling is weak and is distributed around the circumference of the machine and we also allow for small quadrupole focussing errors Δk:

$$\delta\underline{A}(s) \;=\; \begin{pmatrix} 0 & 0 & H & 0 \\ -[\Delta k + H^2] & 0 & -N & H \\ -H & 0 & 0 & 0 \\ -N & -H & -[-\Delta k + H^2] & 0 \end{pmatrix} . \qquad \text{(B.26)}$$

Here the perturbation matrix $\delta \underline{A}(s)$ is the sum of two parts:

$$\delta \underline{A} = \delta \underline{A}_1 + \delta \underline{A}_2 \tag{B.27}$$

with

$$\delta \underline{A}_1(s) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -[\Delta k + H^2] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -[-\Delta k + H^2] & 0 \end{pmatrix} \tag{B.28a}$$

and

$$\delta \underline{A}_2(s) = \begin{pmatrix} 0 & 0 & H & 0 \\ 0 & 0 & -N & H \\ -H & 0 & 0 & 0 \\ -N & -H & 0 & 0 \end{pmatrix} \tag{B.28b}$$

where we have gathered the uncoupled perturbative terms in $\delta \underline{A}_1(s)$ and all coupling terms in $\delta \underline{A}_2(s)$.

Note that the matrix $\delta \underline{A}_1(s)$ produces a Q-shift of the uncoupled motion which is given by

$$\delta Q_x = -\frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} d\bar{s} \cdot [\vec{v}_I^{\,(0)}(\bar{s})]^+ \cdot \underline{S} \cdot \delta \underline{A}_1(\bar{s}) \cdot \vec{v}_I^{\,(0)}(\bar{s}) \; ; \tag{B.29a}$$

$$\delta Q_y = -\frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} d\bar{s} \cdot [\vec{v}_{II}^{\,(0)}(\bar{s})]^+ \cdot \underline{S} \cdot \delta \underline{A}_1(\bar{s}) \cdot \vec{v}_{II}^{\,(0)}(\bar{s}) \tag{B.29b}$$

(see eqn. (B.17)).

Returning to the treatment of the full matrix $\delta \underline{A}$: In contrast to (B.9), now that the system is degenerate, it is no longer sensible to attempt to treat the modes and their perturbations separately. Instead we try to construct orthogonal eigenvectors from linear combinations of degenerate eigenvectors by methods similar to those used in quantum mechanics. Thus we write:

$$(\underline{M}_0 + \delta \underline{M}) \, (c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + \delta \vec{v}) = (\lambda + \delta\lambda) \cdot (c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + \delta \vec{v})$$

(with $\lambda = \lambda_1 = \lambda_2$; using the choices (25a,b) the condition $\lambda_1 = \lambda_2$ is valid both for the sum and difference resonance) or, using (B.23):

$$\underline{M}_0 \, \delta \vec{v} + \delta \underline{M} \, (c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) = \delta\lambda \cdot (c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) + \lambda \cdot \delta \vec{v} \; . \tag{B.30}$$

Multiplying this equation from the left side by

$$\frac{1}{i} \vec{v}_k^{\,+} \underline{S}$$

and using the relation

$$\begin{aligned} \frac{1}{i} \cdot \vec{v}_k^{\,+} \cdot \underline{S} \cdot \underline{M}_0 \cdot \delta \vec{v} &= \frac{1}{i} \cdot \vec{v}_k^{\,+} \cdot \left[\underline{M}_0^{-1}\right]^T \cdot \underline{S} \cdot \delta \vec{v} \\ &= \frac{1}{i} \cdot \left[\underline{M}_0^{-1} \cdot \vec{v}_k\right]^+ \cdot \underline{S} \cdot \delta \vec{v} \\ &= \frac{1}{i} \cdot [\lambda_k^* \cdot \vec{v}_k]^+ \cdot \underline{S} \cdot \delta \vec{v} \\ &= \lambda_k \cdot \frac{1}{i} \cdot \vec{v}_k^{\,+} \cdot \underline{S} \cdot \delta \vec{v} \end{aligned} \tag{B.31}$$

(see eqns. (2.14) and (2.22)) we obtain the equation:

$$\frac{1}{i} \cdot \vec{v}_k^{\,+} \cdot \underline{S} \cdot \delta \underline{M}\ (c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) = \delta\lambda \cdot \frac{1}{i} \cdot \vec{v}_k^{\,+} \underline{S} \vec{v}_k \cdot c_k \ ; \qquad (k = 1, 2) \tag{B.32}$$

or

$$\left[\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} - \frac{1}{i} \cdot \begin{pmatrix} \vec{v}_1^{\,+} \underline{S} \vec{v}_1 & 0 \\ 0 & \vec{v}_2^{\,+} \underline{S} \vec{v}_2 \end{pmatrix} \cdot \delta\lambda \right] \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{B.33}$$

with

$$\begin{aligned} B_{kl} &= \frac{1}{i} \cdot \vec{v}_k^{\,+}(s) \cdot \underline{S} \cdot \delta \underline{M}(s+L, s) \cdot \vec{v}_l(s_0) \\ &= \frac{1}{i} \cdot \vec{v}_k^{\,+}(s) \cdot \underline{S} \cdot \underline{M}_0(s+L, s) \cdot \int_s^{s+L} d\bar{s} \cdot \underline{M}_0^{-1}(\bar{s}, s) \cdot \delta \underline{A}(\bar{s}) \cdot \underline{M}_0(\bar{s}, s) \cdot \vec{v}_l(s) \\ &= \lambda \cdot \frac{1}{i} \cdot \int_s^{s+L} d\bar{s} \cdot \vec{v}_k^{\,+}(\bar{s}) \cdot \underline{S} \cdot \delta \underline{A}(\bar{s}) \cdot \vec{v}_l(\bar{s}) \end{aligned} \tag{B.34}$$

where we have used the relations (B.7) and (B.14).

From (B.24) we have:

$$\delta\lambda = \lambda \cdot i \cdot 2\pi \delta Q \ ; \quad \delta Q = \frac{1}{i} \cdot \frac{\delta\lambda}{2\pi \cdot \lambda} \ , \tag{B.35}$$

where δQ represents the tune shift induced by the perturbation $\delta \underline{A}$ in (B.26).

Thus eqn. (B.33) can also be written in the form:

$$\left[\begin{pmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \end{pmatrix} - \frac{1}{i} \cdot \begin{pmatrix} \vec{v}_1^{\,+} \underline{S} \vec{v}_1 & 0 \\ 0 & \vec{v}_2^{\,+} \underline{S} \vec{v}_2 \end{pmatrix} \cdot \delta Q \right] \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{B.36}$$

with

$$\tilde{B}_{kl} = \frac{1}{i} \cdot \frac{1}{2\pi \cdot \lambda} \cdot B_{kl} = -\frac{1}{2\pi} \cdot \int_s^{s+L} d\bar{s} \cdot \vec{v}_k^{\,+}(\bar{s}) \cdot \underline{S} \cdot \delta \underline{A}(\bar{s}) \cdot \vec{v}_l(\bar{s}) \ . \tag{B.37}$$

Note that

$$\begin{aligned} \tilde{B}_{kl}^* &= -\frac{1}{2\pi} \cdot \int_s^{s+L} d\bar{s} \cdot \left[\vec{v}_k^{\,+}(\bar{s}) \cdot \underline{S} \cdot \delta \underline{A}(\bar{s}) \cdot \vec{v}_l(\bar{s}) \right]^+ \\ &= -\frac{1}{2\pi} \cdot \int_s^{s+L} d\bar{s} \cdot \vec{v}_l^{\,+}(\bar{s}) \cdot \delta \underline{A}^T(\bar{s}) \cdot \underline{S}^T \cdot \vec{v}_k(\bar{s}) \\ &= -\frac{1}{2\pi} \cdot \int_s^{s+L} d\bar{s} \cdot \vec{v}_l^{\,+}(\bar{s}) \cdot \underline{S} \cdot \delta \underline{A}(\bar{s}) \cdot \vec{v}_k(\bar{s}) \\ &= \tilde{B}_{lk} \end{aligned} \tag{B.38}$$

since

$$\begin{aligned} \delta \underline{A}^T(\bar{s}) \cdot \underline{S}^T &= -\delta \underline{A}^T(\bar{s}) \cdot \underline{S} \\ &= \underline{S} \cdot \delta \underline{A}(\bar{s}) \end{aligned} \tag{B.39}$$

as may be seen from (B.5).

A solution

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

of (B.36) only exists if

$$\text{Det} \left[\begin{pmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \end{pmatrix} - \frac{1}{i} \cdot \begin{pmatrix} \vec{v}_1^{+} \underline{S} \vec{v}_1 & 0 \\ 0 & \vec{v}_2^{+} \underline{S} \vec{v}_2 \end{pmatrix} \cdot \delta Q \right] = 0 \tag{B.40}$$

(characteristic determinant) or

$$\left[\tilde{B}_{11} - \frac{1}{i} \vec{v}_1^{+} \underline{S} \vec{v}_1 \cdot \delta Q\right] \cdot \left[\tilde{B}_{22} - \frac{1}{i} \vec{v}_2^{+} \underline{S} \vec{v}_2 \cdot \delta Q\right] - \tilde{B}_{12} \cdot \tilde{B}_{21} = 0$$

$$\Longrightarrow \quad \delta Q_{I,II} = \frac{1}{2} \cdot \left[\frac{1}{i} \vec{v}_1^{+} \underline{S} \vec{v}_1 \cdot \tilde{B}_{11} + \frac{1}{i} \vec{v}_2^{+} \underline{S} \vec{v}_2 \cdot \tilde{B}_{22}\right]$$
$$\pm \frac{1}{2} \sqrt{\left[\frac{1}{i} \vec{v}_1^{+} \underline{S} \vec{v}_1 \cdot \tilde{B}_{11} - \frac{1}{i} \vec{v}_2^{+} \underline{S} \vec{v}_2 \cdot \tilde{B}_{22}\right]^2 + \frac{1}{i} \vec{v}_1^{+} \underline{S} \vec{v}_1 \cdot \frac{1}{i} \vec{v}_2^{+} \underline{S} \vec{v}_2 \cdot \kappa^2} \tag{B.41}$$

with

$$\begin{aligned} \kappa^2 &= 4 \cdot \tilde{B}_{12} \cdot \tilde{B}_{21} = 4 \cdot |\tilde{B}_{12}|^2 \\ &= \left| \frac{1}{\pi} \cdot \int_s^{s+L} d\tilde{s} \cdot [\vec{v}_1^{(0)}(\tilde{s})]^{+} \cdot \underline{S} \cdot \delta \underline{A}(\tilde{s}) \cdot \vec{v}_2^{(0)}(\tilde{s}) \right|^2 \\ &= \left| \frac{1}{\pi} \cdot \int_s^{s+L} d\tilde{s} \cdot [\vec{u}_1^{(0)}(\tilde{s})]^{+} \cdot \underline{S} \cdot \delta \underline{A}(\tilde{s}) \cdot \vec{u}_2^{(0)}(\tilde{s}) \cdot e^{i \cdot 2\pi n \cdot \frac{s}{L}} \right|^2 \\ &= \left| \frac{1}{\pi} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot [\vec{v}_1^{(0)}(\tilde{s})]^{+} \cdot \underline{S} \cdot \delta \underline{A}(\tilde{s}) \cdot \vec{v}_2^{(0)}(\tilde{s}) \right|^2 \end{aligned} \tag{B.42}$$

where we have used eqns. (B.38), (B.20) and (2.26).

In order to discuss this relation for the tune shifts, δQ, we have to investigate the two possible cases:

a) difference resonance;

b) sum resonance

separately.

a) Difference resonance.

In this case by eqn. (B.25a) we have:

$$\begin{aligned} \frac{1}{i} \vec{v}_1^{+} \underline{S} \vec{v}_1 &\equiv \frac{1}{i} [\vec{v}_I^{(0)}]^{+} \underline{S} \vec{v}_I^{(0)} = 1 \; ; \\ \frac{1}{i} \vec{v}_2^{+} \underline{S} \vec{v}_2 &\equiv \frac{1}{i} [\vec{v}_{II}^{(0)}]^{+} \underline{S} \vec{v}_{II}^{(0)} = 1 \end{aligned}$$

and therefore:

$$\left\{ \begin{aligned} \delta Q_I &= \tfrac{1}{2} \cdot \left[\tilde{B}_{11} + \tilde{B}_{22}\right] + \tfrac{1}{2} \sqrt{\left[\tilde{B}_{11} - \tilde{B}_{22}\right]^2 + \kappa^2} \; ; \\ \delta Q_{II} &= \tfrac{1}{2} \cdot \left[\tilde{B}_{11} + \tilde{B}_{22}\right] - \tfrac{1}{2} \sqrt{\left[\tilde{B}_{11} - \tilde{B}_{22}\right]^2 + \kappa^2} \; . \end{aligned} \right. \tag{B.43}$$

From (B.37) and (B.29), for the terms $\tilde{B}_{11}$ and $\tilde{B}_{22}$ we obtain:

$$\begin{aligned}
\tilde{B}_{11} &= -\frac{1}{2\pi}\cdot\int_{s_0}^{s_0+L} d\bar{s}\cdot[\vec{v}_I^{\,(0)}(\bar{s})]^+\cdot\underline{S}\cdot\delta\underline{A}(\bar{s})\cdot\vec{v}_I^{\,(0)}(\bar{s}) \\
&= -\frac{1}{2\pi}\cdot\int_{s_0}^{s_0+L} d\bar{s}\cdot[\vec{v}_I^{\,(0)}(\bar{s})]^+\cdot\underline{S}\cdot\delta\underline{A}_1(\bar{s})\cdot\vec{v}_I^{\,(0)}(\bar{s}) = \delta Q_x \;; \qquad \text{(B.44a)}\\
\tilde{B}_{22} &= -\frac{1}{2\pi}\cdot\int_{s_0}^{s_0+L} d\bar{s}\cdot[\vec{v}_{II}^{\,(0)}(\bar{s})]^+\cdot\underline{S}\cdot\delta\underline{A}(\bar{s})\cdot\vec{v}_{II}^{\,(0)}(\bar{s}) \\
&= -\frac{1}{2\pi}\cdot\int_{s_0}^{s_0+L} d\bar{s}\cdot[\vec{v}_{II}^{\,(0)}(\bar{s})]^+\cdot\underline{S}\cdot\delta\underline{A}_1(\bar{s})\cdot\vec{v}_{II}^{\,(0)}(\bar{s}) = \delta Q_y \;. \qquad \text{(B.44b)}
\end{aligned}$$

It follows from eqns. (B.43) and (B.44) that the motion remains stable in the neighbourhood of a difference resonance since δQ in (B.43) is a real number and the $|\lambda|'s$ are still unity.

For the coefficients c_1 and c_2 in (B.36) one gets:

$$\begin{aligned}
c_{k1} &= d_k\cdot\tilde{B}_{12} \;; \qquad \text{(B.45a)}\\
c_{k2} &= d_k\cdot[\delta Q_k-\delta Q_x] \;; \qquad \text{(B.45b)}
\end{aligned}$$

$$(k = I, II)$$

and the normalization condition (2.31a) leads to:

$$d_k = \frac{1}{\sqrt{[\delta Q_k-\delta Q_x]^2+\kappa^2/4}} \;. \qquad \text{(B.46)}$$

The eigenvectors of the coupled problem are thus given approximately by:

$$\begin{aligned}
\vec{v}_I &= \frac{1}{\sqrt{[\delta Q_I-\delta Q_x]^2+\kappa^2/4}}\cdot\left\{\tilde{B}_{12}\cdot\vec{v}_I^{\,(0)}+[\delta Q_I-\delta Q_x]\cdot\vec{v}_{II}^{\,(0)}\right\} \;; \qquad \text{(B.47a)}\\
\vec{v}_{II} &= \frac{1}{\sqrt{[\delta Q_{II}-\delta Q_x]^2+\kappa^2/4}}\cdot\left\{\tilde{B}_{12}\cdot\vec{v}_I^{\,(0)}+[\delta Q_{II}-\delta Q_x]\cdot\vec{v}_{II}^{\,(0)}\right\} \;. \qquad \text{(B.47b)}
\end{aligned}$$

In order to calculate the vector $\delta\vec{v}$ appearing in (B.33) one needs a higher step of perturbation theory.

b) Sum resonance.

For a sum resonance eqn. (B.25b) gives:

$$\begin{aligned}
\frac{1}{i}\vec{v}_1^{\,+}\underline{S}\vec{v}_1 &\equiv \frac{1}{i}[\vec{v}_I^{\,(0)}]^+\underline{S}\vec{v}_I^{\,(0)} = 1 \;;\\
\frac{1}{i}\vec{v}_2^{\,+}\underline{S}\vec{v}_2 &\equiv \frac{1}{i}[\vec{v}_{-II}^{\,(0)}]^+\underline{S}\vec{v}_{-II}^{\,(0)} = -1
\end{aligned}$$

and thus:

$$\delta Q_{I,II} = \frac{1}{2}\cdot\left[\tilde{B}_{11}-\tilde{B}_{22}\right]\pm\frac{1}{2}\sqrt{\left[\tilde{B}_{11}+\tilde{B}_{22}\right]^2-\kappa^2} \qquad \text{(B.48)}$$

where the terms $\tilde{B}_{11}$ and $\tilde{B}_{22}$ as obtained from (B.29) and (B.37) are again given by:

$$\tilde{B}_{11} = -\frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot [\vec{v}_I^{(0)}(\tilde{s})]^+ \cdot \underline{S} \cdot \delta \underline{A}(\tilde{s}) \cdot \vec{v}_I^{(0)}(\tilde{s}) = \delta Q_x ; \tag{B.49a}$$

$$\tilde{B}_{22} = \tilde{B}_{22}^* = -\frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot [\vec{v}_{II}^{(0)}(\tilde{s})]^+ \cdot \underline{S} \cdot \delta \underline{A}(\tilde{s}) \cdot \vec{v}_{II}^{(0)}(\tilde{s}) = \delta Q_y . \tag{B.49b}$$

In this case δQ is a complex number in the neighbourhod of such a resonance and the motion becomes unstable. Therefore sum resonances must be avoided for reasons of stability.

Another treatment of the behaviour of coupled motion (without solenoids) in the neighbourhood of a linear resonance can be found in Ref. [5].

References

[1] H. Brück, M. Meinke and P. Schmueser, DESY HERA-25-89 (1989)

[2] L. Evans (CERN SPS Division), private communication

[3] F. Willeke, G. Ripken: "Methods of Beam Optics"; DESY 88-114 and Cornell Summer School 1987 (American Institute of Physics Conference Proceedings 184).

[4] G. Ripken: "Untersuchungen zur Strahlführung und Stabilität der Teilchenbewegung in Beschleunigern und Storage-Ringen unter strenger Berücksichtigung einer Kopplung der Betatronschwingungen"; DESY R1-70/04.

[5] E. Courant and H. Snyder: "Theory of the Alternating-Gradient Synchrotron"; Annals of Physics 3,1-48 (1958)

[6] H. Mais and G. Ripken: "Theory of Spin-Orbit Motion in Electron-Positron Storage Rings. Summary of Results"; DESY 83-62.

[7] H. Mais and G. Ripken: "Theory of Coupled Synchro-Betatron Oscillations (I)"; DESY M-82-05.

[8] D.P. Barber, H. Mais, G. Ripken, F. Willeke: "Nonlinear Theory of Coupled Synchro-Betatron Motion"; DESY 86-147 .

[9] A. Schoch: "Theory of Linear and Nonlinear Perturbations of Betatron Oscillations in Alternating-Gradient Synchrotrons"; Cern 57-21, 1958.

[10] G. Guignard: "A General Treatment of Resonances in Accelerators"; Cern 78-11, 1978.

[11] F. Willeke: "Analytical Study of the Tevatron Nonlinear Dynamics"; Fermi National Accelerator Laboratory, FN-422(July 1985)

[12] D.P. Barber, K. Heinemann, H. Mais, G. Ripken: "A Fokker-Planck-Treatment of Stochastic Particle Motion within the Framework of a fully coupled 6-dimensional Formalism for Electron-Positron Storage Rings including Classical Spin Motion in linear Approximation"; (in preparation).

[13] Herbert Goldstein: "Classical Mechanics" ; Addison-Wesley(1980).

[14] E.D. Courant, R.D. Ruth, W.T. Weng: "A Stability in Dynamical Systems I"; SLAC-PUB-3415 (August 1984) and BNL/SUNY Summer School 1983 (American Institute of Physics Conference Proceedings 127).

[15] J. Kewisch, Diplomarbeit, University of Hamburg (1976), unpublished

[16] G. Ripken: "Non-linear canonical equations of coupled synchro- betatron motion and their solution within the framework of a non-linear 6-dimensional (symplectic) tracking program for ultra-relativistic protons"; DESY 85-84.

[17] D.P. Barber, G. Ripken, F. Schmidt: "A non-linear canonical formalism for the coupled synchro-betatron motion of protons with arbitrary energy"; DESY 87-36.

[18] R. Brinkmann, F. Willeke, DESY 86/79 (1985) and DESY 88/77(1988)