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# Convergent Multigrid Polymer Expansions and Renormalization for Euclidean Field Theory

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# CONVERGENT MULTIGRID POLYMER EXPANSIONS AND RENORMALIZATION FOR EUCLIDEAN FIELD THEORY [1]

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## Abstract

The representation of Euclidean quantum field theories by multigrid polymer systems provides convergent perturbation expansions for sufficiently weak coupling constants. These expansions are computable in the sense that the n-th term is given by a O(n)-dimensional integral. Nonperturbative contributions from large field configurations can be controlled rigorously by recursive bounds on polymer activities. The expansions presented here incorporate phase space expansions and the renormalization group approach. Convergence of these expansions is proven at the example of weakly coupled critical lattice $\Phi^4$-theory in four dimensions in hierarchical approximation and for the complete model and for the ultraviolet problem of three-dimensional $\Phi^4$-theory in hierarchical approximation.

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## CONTENTS

1.INTRODUCTION 1

2.THE MULTIGRID 7

3.CONVERGENT POLYMER SYSTEMS FOR HIERARCHICAL MODELS 10

3.1. Hierarchical Renormalization Group Transformations 10
3.2. Hierarchical Multigrid Expansions 23
3.3. Hierarchical Ultraviolet Problem 35

4.MULTIGRID POLYMER SYSTEMS FOR EUCLIDEAN QUANTUM FIELD THEORY 47

4.1. Layerwise Polymer System 47
4.2. Cluster Expansion Formulas for Activities 59
4.3. Recursive Bounds on the Layerwise Polymer System 65
4.4. Stability Bound 68
4.5. Cluster Expansion Formulas for Vacuum Energy Counterterms and Running Coupling Constants 71
4.6. Coarsening Step : Bounds on $\delta R_j^c$ 74
4.7. Renormalization Subtractions : Bounds on $R_{j-1}^{div}$ 82
4.8. Integration Step (I) : Bounds on $M_{j-1}^n$ 95
4.9. Bounds on Vacuum Energy Counterterms 104
4.10. Bounds on Running Coupling Constants : Induction Step 106
4.11. Integration Step (II) : Bounds on $R_{j-1}$ 112
4.12. Bounds on Polymer-dependent Counterterms 121
4.13. Repolymerization Step : Bounds on $\delta R_{j-1}^{ren}$ 126

APPENDICES 132

A. Polymer Systems 132
B. Multigrid Operators 135
C. Tree Graph Formulas 143
D. Perturbation Expansions 146

References 148

## 1. INTRODUCTION

Quantum field theory describes the behavior of elementary particles in the microscopic world. One of the main tasks of constructive quantum field theory is to prove the existence of non-trivial models in quantum field theory and to present computable results. The scattering of particles or coupling of fields is related to the computation of Green functions or correlation functions.

In standard perturbation theory Green functions are expressed by a sum of terms. The n-th term is determined by Feynman diagrams with n vertices. The Feynman diagrams represents nd-dimensional integrals in d-space time dimensions.

For non-trivial bosonic theories standard perturbation expansions suffers from the lack of convergence. They are at least Borel-summable. This lack of convergence does not vanish by introduction of a space time lattice and for very small coupling constants (not equal to zero).

One of the aims of this article is to present series expansions which are computable in the sense as conventional perturbation expansions. These expansions are convergent for sufficiently small coupling constants and the n-th term can be expressed by a O(n)-dimensional integral.

Throughout this article, quantum fields are defined in terms of their analytic continuation to imaginary time, i.e. the Minkowski metric is changed into Euclidean metric. This is necessary for the use of statistical mechanical methods

For weakly coupled massive lattice models a convergent expansion which is very similar to standard perturbation theory were described by G. Mack and the author [34] (see also [36]). Two conditions are essential for convergence. The first one is the smallness of the coupling constants and the second condition is that the correlation length $\xi$ in units of lattice spacing $a$ is finite, i.e. the model has to be noncritical. A massless lattice field theory has infinite correlation length $\xi$ and a field theory on continuous space may be regarded as continuum limit $a \to 0$ of a lattice theory. Therefore in both cases the ration $\xi/a$ is not finite, i.e. the massless lattice field theory and field theory on continuous space are critical theories.

To deal with this problem, K. Wilson has invented the renormalization group approach [28, 41, 42]. The idea is to solve the critical or nearly critical theory by performing renormalization group steps. After each renormalization group step the (effective) theory is less critical. In a field theory, renormalization group steps are performed by integration over high frequency modes, keeping low frequency modes constant. In the block spin version of the renormalization group [27] the low frequency modes are given by averaging the fields over $d$-dimensional hypercubes (blocks) for fields which live on $d$-dimensional space. After each renormalization group step the block size is increased and the resulting effective theory is less critical.

K. Gawędzki and A. Kupiainen [18 – 20] used the block spin version of exact renormalization group for a rigorous control of critical field theories.

In the present work, a critical theory is transformed into a statistical mechanical system which lives on a multigrid $\Lambda$. Then the application of renormalization group methods is possible. For a continuum theory the multigrid consists of an infinite sequence of lattices



$\Lambda_k$, $k = 0, 1, 2, \ldots$ with decreasing lattice spacings $a_k = L^{-k} a_0$, where $L$ is some integer $\geq 2$. $\Lambda_j$ is called the $j$th multigrid layer of $\Lambda$. The largest lattice spacing $a_0$ is chosen to be of the order of the physical correlation length. The continuum space $\mathbf{R}^d$ ( or a finite volume of $\mathbf{R}^d$) is called base space. The base space can be covered by $d$-dimensional hypercubes of side length $a_j$. The points $y \in \Lambda_j$ of the $j$th multigrid layer may be identified with the hypercubes of side length $a_j$ in the base space. If $z$ is a point of this hypercube $y$ then we write $z \underline{\in} y$. There is a canonical partial ordering "$\leq$" on $\Lambda$. For $y, x \in \Lambda$ we write $y \leq x$ if $z \underline{\in} y$ implies $z \underline{\in} x$ for all $z$.

Similarly, a massless lattice theory is translated into a theory on the multigrid $\Lambda$ with layers $\Lambda_k$, $k = 0, -1, -2, \ldots$ of increasing lattice spacings $a_k = L^{-k} a_0$. The smallest lattice spacing $a_0$ is the lattice spacing of the original lattice on which the theory lives.

Multigrid methods can be used for numerical studies of lattice field theories. G. Mack and S. Meyer applied multigrid Monte Carlo techniques to $\Phi^4$-theory with spontaneous symmetry breaking [32].

Let us explain how the transformation of a field theory into a multigrid system is obtained (cp. [31, 33]). Consider for example a Euclidean quantum field theory on the continuum $\mathbf{R}^d$ (or a compact subset of $\mathbf{R}^d$, finite volume). The evaluation of an infinite dimensional functional integral (generating functional of Green functions) is required. The integration variables are fields $\Phi(z)$ on the continuum, $z \in \mathbf{R}^d$. This field is decomposed into a sum of fields $\Phi^k(z)$, one for each layer $\Lambda_k$ of the multigrid. $\Phi^k$ is determined by a field $\varphi^k$ on the lattice $\Lambda_k$

$$\Phi^k(z) = a_k^d \sum_{x \in \Lambda_k} \mathcal{A}^k(z, x) \varphi^k(x).$$

(The $\mathcal{A}^k$-kernels are the same as used by K. Gawędzki and A. Kupiainen [18 – 20] ). The main contribution of the Fourier transform of $\Phi^k$ comes from the momentum range $a_k^{-1} < \|p\| < a_{k+1}^{-1}$. Thus the fields $\varphi^k$ in this expansion are localized both in coordinate space and momentum space in a way consistent with the uncertainity principle. This expansion presents an example of socalled "phase space cell expansion".

This multigrid tranformation were used by G. Mack and the author [33] to provide computable convergent expansions of superrenormalizable $\lambda\Phi^4$-theory (on continuous three dimensional space).

Phase space cell expansions were firstly introduced by J. Glimm and A. Jaffe [21] discussing the positivity of $\Phi_3^4$-Hamiltonian. These investigations were further developped by J. Magnen and R. Sénéor [35], G. Battle and P. Federbush [3, 4] and J. Feldman, J. Magnen, V. Rivasseau, R. Sénéor [10 – 12].

Another example of a phase cell expansion is the iterated Mayer expansion. Iterated Mayer expansion with Pauli-Villars cutoff were applied by M. Göpfert and G. Mack [24, 25] to prove permanent confinement of static quarks in 3-dimensional $U(1)$ lattice gauge theory.

For a free field theory the fields $\Phi^k$ on each multigrid layer $\Lambda_k$ are not coupled to fields of the other layers. The original critical theory on the continuum space $\mathbf{R}^d$ decomposes into (an infinite number of) noncritical theories living on multigrid layers. For non-free (non-trivial) theories the field $\Phi^k$ on layer $\Lambda_k$ are coupled to other layers.

The problem of criticality is related to the problem of ultraviolet divergencies in the continuum theory. This ultraviolet divergencies comes from the infinite large numbers of couplings to finer layers (i.e. layers with small lattice spacings). To control these large number of couplings to very fine layers, one has to show that these couplings are small.

The ultraviolet problem is related to the $\infty$-volume problem of the system translated to the multigrid $\Lambda$. Start from a lattice theory of lattice spacing $a_N = L^{-N} a_0$. Transform this theory to a theory on a multigrid $\Lambda_{\geq N} = \Lambda_0 + \Lambda_1 + \cdots + \Lambda_N$. The continuum limit $N \to \infty$ is equivalent to the thermodynamic limit $\Lambda_{\geq N} \nearrow \Lambda$.

For ultraviolet asymptotically free theories there is a chance to overcome the ultraviolet problem. A model is ultraviolet asymptotically free if its short distance asymptotics is non-interacting (free). In the multigrid language very fine layers are nearly decoupled for ultraviolet asymptotically free theories.

For massless lattice theories the situation is quite similar. Infrared divergencies comes from the infinite large number of couplings to coarser layers (i.e. layers with large lattice spacings). There is a chance to cure this infrared problem for infrared asymptotically free theories. A model is infrared asymptotically free if its large distance asymptotics is free.

Renormalization theory is necessary to cure the ultraviolet and/or infrared problem and to show asymptotic freedom. In standard perturbation theory divergencies comes from the integration over all momenta for some Feynman graphs. In a phase space expansion the Feynman graphs are represented in coordinate space. The free propagator v is decomposed into a sum of propagators $\mathrm{v}^k$, $k \in \mathbf{N}$. $\mathrm{v}^k$ contains the contributions of the high frequency range $p \in [a_{k-1}^{-1}, a_k^{-1}]$. The integral over all momenta $p$ can be replaced by a sum over all $k$. Bounds on Feynman graphs show that divergencies for renormalizable theories come from subgraphs with a small number of low frequency ($k$ small) external lines [8, 9].

Using phase space expansion G. Gallavotti and F. Nicolò invented an elegant representation for effective actions by tree graphs [15 – 17] (see also [13]). These tree graph representation were used for a perturbative renormalization theory.

In perturbative renormalization theory the bare interaction is decomposed into a renormalized interaction and counterterms. The vertices in the Feynman graphs corresponds to renormalized (running) coupling constants or to counterterm insertions. If counterterms can be chosen such that the divergencies for all Feynman graphs are cancelled a theory is called perturbativly renormalizable (superrenormalizable if only a finite number of counterterms are necessary).

Convergent phase cell expansions require the understanding of nonperturbative renormalization theory. This was briefly described in the author's article [38]. In terms of convergent expansions on a multigrid, polymer activities are composed of irreducible parts (like convergent subgraphs of Feynman graphs) and renormalization parts. By a suitable choice of counterterms in the action (which are polymer-dependent) the renormalization parts obey renormalization conditions and do not lead to divergencies.

The interaction between variable $\varphi^k(y)$ and $\varphi^j(x)$, $k > j$, on different multigrid layers is exponentially decaying with $|k-j|$ if renormalization is properly performed. The number of points $y$ of $\Lambda_k$ within the hypercube $x$ increases exponentially with $|k-j|$ (like $L^{d|k-j|}$). Therefore these correlations are small if the interaction between $\varphi^k(y)$ and $\varphi^j(x)$ decays quickly enough.



Green functions (or correlation functions) are derivatives of (logarithm of) generating functionals with respect to external fields or sources. The Euclidean generating functional functional is transformed into a partition function $Z(\Lambda)$ on the multigrid. Convergent expansion methods require to approximate the partition function $Z(\Lambda)$ of the infinite system $\Lambda$ in terms of partition functions $Z(X)$ of small subsystems $X$ of $\Lambda$. In our expansions $X$ are finite nonempty subsets of $\Lambda$, called polymers. The partition functions $Z(X)$ for all finite $X$ define truncated quantities $A(P)$, called activities. The restriction of $\Lambda$ to a finite subset $X$ introduces a combined infrared and ultraviolet cutoff. The investigation of the thermodynamic limit on the multigrid includes therefore the treatment of infrared and/or ultraviolet limit.

The correct choice of the partition functions $Z(X)$ is important for the convergence properties of the expansion. Partition functions will depend on external fields $\Psi(z)$. The field can be expressed in terms of a field $\psi$ on the multigrid. The low energy behavior of a massive continuum theory is determined by the dependence of partition functions on $\psi(x)$ for $x$ in the coarsest layer and should be nearly independent of multigrid layers with small lattice spacings.

The transformation of a general class of lattice models into a polymer system were shown by C. Gruber and H. Kunz [26].

Cluster expansion formulas have to be introduced for the representation of activities. For pair interactions the activities can be expressed by tree graph formulas [6, 22]. For non-trivial quantum field theories interactions occur which couple more than two points. More general cluster expansion formulas were introduced for this case. It was shown by the author, that a simple tree formula for general multibody interactions can be found [37]. This tree graph formula will be used in this work.

In bosonic theories the main difficulty for proving convergence is the socalled "large field problem". If the absolute values of the integration variables $\varphi(x)$ are bounded, partition functions are entire functions of coupling constants, and conventional perturbation expansion would converge. Benfatto et al. [5] presented a partition of field space into "large and small field" domains to cure this problem.

This decomposition into small and large fields were used in the works of K. Gawędzki and A. Kupiainen [18 – 20] . This technique was also used by T. Bałaban [1, 2] for a treatment of four-dimensional non-Abelian lattice gauge theory by the renormalization group approach.

In the small field region one can use standard perturbation theory to state bounds on the effective actions. In the large field region stability bounds on Boltzmannians were used. The use of small and large field domains makes the definition of expansions very complicated.

The application of stability bounds on Boltzmannians for all field configurations can be easily used in hierarchical approximation. Therefore there is no need of using small and large field domains. This is done in the present article.

For the case of complete models (without using hierarchical approximation), the correct treatment of stability bounds is more complicated than in the hierarchical case. This comes from the non-locality of the kernel of $\mathcal{A}$ and the use of renormalization subtractions.

In the present approach the large field problem is cured with the help of suitable boundary conditions for systems in $X (\subseteq \Lambda_{j-1})$. Consider a field $\mathcal{A}^{j-1}(z,x)\varphi^{j-1}(x)$ for $z \in \mathbf{R}^d$ and $x \in \Lambda_{j-1}$. Then the polymer system is defined in such a way that only fields $\mathcal{A}^{j-1}(z,x)\varphi^{j-1}(x)$

are used for renormalization subtractions such that the distance from $z$ to $\Lambda_{j-1} - X$ is larger than a fixed constant $\delta$ (in unit of lattice spacing $a_j$). Then the kernel $\mathcal{A}^{j-1}(z,x)$ is small and the field $\mathcal{A}^{j-1}(z,x)\varphi^{j-1}(x)$ can be controlled by suitable bounds for $x \in \Lambda_{j-1} - X$. For $x \in X$ the fields can be bounded by using stability bounds on Boltzmannians. This procedure presented here overcomes the problem of large field problem.

Expansions for $\Phi^4$-models are presented in detail, and convergence is proven in this work. In three (and less) space dimensions, this model is superrenormalizable. The continuum limit exists and the expansions are convergent [3,4,35]. Superrenormalizable theories are much easier to control than merely renormalizable ones.

Four dimensional $\Phi^4$-theory is renormalizable in perturbation theory, but not asymptotically free in the ultraviolet [40,14]. It is believed that the continuum limit does not exist. It is shown that the four-dimensional massless lattice model is asymptotically free in the infrared [11,20]. Models which are asymptotically free in the infrared involves the same technical problems as the treatment of a massive continuum theory which is asymptotically free in the ultraviolet. Convergence of our expansions for a massless four-dimensional lattice $\Phi^4$-theory will be proven in this paper for sufficiently weak coupling. This is the first convergence proof for a computable expansion in a renormalizable model.

## ORGANIZATION OF THE PAPER

In chapter 2 the multigrid formalism is introduced. The transformation of a functional integral into a statistical mechanical system which lives on a multigrid is presented. It is shown that free (non-interacting) fields on different layers are independent.

Chapter 3 discusses convergent polymer systems for hierarchical models.

The renormalization group of the four-dimensional massless lattice $\Phi^4$-theory in hierarchical approximation is studied in section 3.1. In this section the renormalization group transformations are controlled rigorously by running coupling constants and recursive bounds on irrelevant activities.

In section 3.2 the irrelevant activities are expressed by sums over subsets of the multigrid. It is shown using renormalization conditions that terms in this expansion which contain points on different layers are supressed by powers of $L^{-1}$.

Section 3.3 discusses the hierarchical ultraviolet problem at the example of hierarchical $\Phi^4$-theory in three dimensions. The existence of the continuum limit is proven. The treatment of hierarchical models serves as a pedagogical introduction to the discussion of the full model, presented in chapter 4.

In chapter 4 the convergent expansion for the massless $\Phi^4$-theory in four dimensions is discussed.

Section 4.1 presents the definition of a multigrid polymer system. This is done recursively for each layer of the multigrid. Polymer-dependent counterterms are introduced to define renormalized partition functions and activities.



Using the general tree formula the activities are expressed in terms of sums overt tree graphs in section 4.2. Section 4.3 states recursive bounds on the irrelevant activities and running coupling constants which are defined in section 4.1. The bounds are formulated by introduction of suitable norms. These bounds will be proven in sections 4.6-4.13. In section 4.4 stability bounds (lower bounds of the relevant interaction) are stated. Section 4.5 discusses cluster expansion formulas for vacuum counterterms and running coupling constants.

Finally, in appendix A useful formulas for polymer systems are derived. Appendix B states various properties and integral representations of multigrid operator kernels. In appendix C we prove tree graph formulas. Perturbation expansions are presented in appendix D.

## 2. THE MULTIGRID

Consider the $d$-dimensional torus $T_{N,M} \equiv a_N \mathbf{Z}^d / a_M \mathbf{Z}^d$ for $M, N \in \mathbf{Z}$, $M \leq N$, $a_j \equiv L^{-j} a$, $a > 0$ and $L \in \{2, 3, \ldots\}$. For notational simplicity suppose that $L$ is odd. Define for each $x \in T_{j,M} \equiv a_j \mathbf{Z}^d / a_M \mathbf{Z}^d$ with $M \leq j \leq N$ a block

$$b(x) \equiv \{ z \in T_{N,M} | \exists z' \in T_{N,M} : z = x + z', |z'^{\mu}| \leq \frac{a_j}{2} \ \forall \mu \in \{1, \ldots, d\} \}.$$

The multigrid $\Lambda$ with base space $T_{N,M}$ (= *base*) is the set of all blocks, i.e.

$$\Lambda = \Lambda_N + \Lambda_{N-1} + \cdots + \Lambda_{M+1} + \Lambda_M$$

with

$$\Lambda_j \equiv \{ b(x) | \ x \in T_{j,M} \}.$$

$\Lambda_j$ is called the $j$th layer of the multigrid $\Lambda$. Performing the limit $\lim_{N \to \infty}$ or/and $\lim_{M \to -\infty}$ we obtain the definitions for the multigrid with base space $\mathbf{R}^d / a_M \mathbf{Z}^d$, $a_N \mathbf{Z}^d$, $\mathbf{R}^d$. The base space can be covered by the $d$-dimensional hypercubes $b(x)$ of side length $a_j$. For notational simplification we will write $b(x) = x$. For $y \in \Lambda_j$, $x \in \Lambda_{j-1}$ define

$$y \underline{\in} x \rightleftharpoons \exists m \in \mathbf{Z}^d : \ |m^{\mu}| \leq \frac{L}{2}, \ y = x + m a_j$$

and

$$\overline{x} \equiv \{ y \in \Lambda_j | y \underline{\in} x \}$$

$$[y] \equiv x \quad \text{iff } y \underline{\in} x.$$

For a field $\Phi : \ T_{N,M} \to \mathbf{C}$ ($\mathbf{C}$ = set of complex numbers) define the block spin field $C^j \Phi : \ \Lambda_j \to \mathbf{C}$ by

$$C^j \Phi(y) \equiv \frac{1}{Vol(y)} \int_{z \underline{\in} y} \Phi(z) \equiv L^{(j-N)d} \sum_{z \underline{\in} y} \Phi(z) \tag{2-1}$$

for all $y \in \Lambda_j$. $C^j$ is called the block average operator. The block average operator $C_{j-1,j}$ for fields $\Phi : \ \Lambda_j \to \mathbf{C}$ is defined by

$$(C_{j-1,j} \Phi)(x) \equiv \frac{1}{Vol(x)} \int_{y \underline{\in} x} \Phi(y) \equiv L^{-d} \sum_{y \underline{\in} x} \Phi(y) \tag{2-2}$$

for all $x \in \Lambda_{j-1}$. We have

$$C^{j-1} = C_{j-1,j} C^j. \tag{2-3}$$

Suppose that v is a free propagator on $T_{N,M}$, for example $\mathrm{v} \equiv (-\triangle_{M,N})^{-1}$, $\triangle_{M,N}$ = periodic lattice Laplacian on $T_{N,M}$. The block spin propagator $u_j$ is defined by

$$u_j \equiv C^j \mathrm{v} C^{j*}. \tag{2-4}$$

Define

$$\mathcal{A}^j \equiv \mathrm{v} C^{j*} u_j^{-1} \tag{2-5}$$



$$\mathcal{A}_{j,j-1} \equiv u_j C^*_{j-1,j} u^{-1}_{j-1}. \tag{2-6}$$

We have for $l < j$

$$\mathcal{A}^l = \mathcal{A}^j \mathcal{A}_{j,j-1} \mathcal{A}_{j-1,j-2} \ldots \mathcal{A}_{l+1,l}. \tag{2-7}$$

Consider the split

$$\Phi = \sum_{j=N}^{M-1} (\mathcal{A}^j C^j - \mathcal{A}^{j-1} C^{j-1}) \Phi + \mathcal{A}^M C^M \tag{2-8}$$

for $\Phi : \ T_{N,M} \to \mathbf{C}$. Define

$$\Phi^j = \begin{cases} (\mathcal{A}^j C^j - \mathcal{A}^{j-1} C^{j-1}) \Phi, & \text{for } N \geq j \geq M+1 \\ \mathcal{A}^M C^M \Phi, & \text{for } j = M. \end{cases} \tag{2-9}$$

$\Phi^j$ is called the fluctuation field of $\Phi$. The defining relation for the (normalized) Gaussian measure $d\mu_{\mathrm{v}}(\Phi)$ is

$$\int d\mu_{\mathrm{v}}(\Phi) \exp\{ i \int_{z \in base} J(z) \Phi(z) \} = \exp\{ \int_{z_1, z_2 \in base} J(z_1) \mathrm{v}(z_1, z_2) J(z_2) \}.$$

Expectation values for free field theories are defined by

$$< O(\Phi) >_{\mathrm{v}} \equiv \int d\mu_{\mathrm{v}}(\Phi) O(\Phi).$$

The following lemma shows that the fluctuation fields $\Phi^j$ and $\Phi^l$ are Gaussian independent for $j \neq l$.

LEMMA 1. *For $z, z' \in T_{N,M}$ and $N \leq j, l \leq M$ we have*

$$< \Phi^j(z) \Phi^l(z') >_{\mathrm{v}} = \delta_{j,l} \mathrm{v}^j(z, z') = \begin{cases} \mathrm{v}^j(z, z'), & \text{for } j = l \\ 0, & \text{for } j \neq l \end{cases} \tag{2-10}$$

*with*

$$\mathrm{v}^j(z, z') = \begin{cases} \mathcal{A}^j (u_j - u_j C^*_{j-1,j} u^{-1}_{j-1} C_{j-1,j} u_j) \mathcal{A}^{j*}(z, z'), & \text{for } N \geq j \geq M+1 \\ \mathcal{A}^M u_M \mathcal{A}^{M*}(z, z'), & \text{for } j = M \end{cases} \tag{2-11}$$

$\mathrm{v}^j$ is called the fluctuation propagator.

*Proof :* Suppose that $l < j$. Then we have

$$< \Phi^j(z) \Phi^l(z') >_{\mathrm{v}} = (\mathcal{A}^j C^j - \mathcal{A}^{j-1} C^{j-1}) \mathrm{v} (\mathcal{A}^l C^l - \mathcal{A}^{l-1} C^{l-1})^*(z, z') = \mathcal{A}^j (C^j - \mathcal{A}_{j,j-1} C^{j-1}) \mathrm{v} (C^l - \mathcal{A}_{l,l-1} C^{l-1})^* \mathcal{A}^{l*}(z, z'). \tag{2-12}$$

Since

$$(C^l - \mathcal{A}_{l,l-1} C^{l-1}) = (C_{l,l+1} \ldots C_{j-2,j-1} - \mathcal{A}_{l,l-1} C_{l-1,l} \ldots C_{j-2,j-1}) C^{j-1}$$

and

$$(C^j - \mathcal{A}_{j,j-1} C^{j-1}) \mathrm{v} C^{j-1*} = C^j \mathrm{v} C^{j-1*} - \mathcal{A}_{j,j-1} u_{j-1} = u_j C^*_{j-1,j} - u_j C^*_{j-1,j} = 0$$

we obtain by eq. (12)

$$< \Phi^j(z)\Phi^l(z') >_v = 0.$$

Furthermore

$$< \Phi^j(z)\Phi^j(z') >_v = (\mathcal{A}^j C^j - \mathcal{A}^{j-1}C^{j-1})v(\mathcal{A}^j C^j - \mathcal{A}^{j-1}C^{j-1})^*(z,z') =$$
$$\mathcal{A}^j(C^j - \mathcal{A}_{j,j-1}C^{j-1})v(C^j - \mathcal{A}_{j,j-1}C^{j-1})^*\mathcal{A}^{j*}(z,z') =$$
$$= \mathcal{A}^j(u_j - u_j C^*_{j-1,j}u^{-1}_{j-1}C_{j-1,j}u_j)\mathcal{A}^{j*}(z,z'). \qquad (2\text{-}13)$$

This completes the proof. √

For $N \geq j \geq M$ the fluctuation field $\Phi^j$ can be expressed by a multigrid field $\varphi^j : \Lambda_j \to \mathbf{C}$

$$\Phi^j = \mathcal{A}^j\varphi^j \qquad (2\text{-}14)$$

with

$$\varphi^j = D^j\Phi^j \qquad (2\text{-}15)$$

where

$$D^j \equiv C^j - \mathcal{A}_{j,j-1}C^{j-1} \qquad (2\text{-}16)$$

setting $\mathcal{A}_{M,M-1} \equiv 0$. Since $C_{j-1,j}D^j = 0$ the block average of $\varphi^j$ is zero, i.e.

$$C_{j-1,j}\varphi^j = 0. \qquad (2\text{-}17)$$

Analogously, we have for $N \geq j \geq M+1$

$$C^j\Phi^j = 0$$

and for the fluctuation propagators

$$\int_{z' \in x} \mathrm{v}^j(z,z') = 0$$

for all $x \in \Lambda_{j-1}$. Because of the Gaussian independence of fluctuation fields on different multigrid layers the Gaussian measure factorizes. Therefore the Gaussian integral can be rewritten

$$\int d\mu_{\mathrm{v}}(\Phi)F(\Phi) = \int [\prod_{j=N}^{M} d\mu_{\mathrm{v}^j}(\Phi^j)]F(\sum_{j=N}^{M}\Phi^j) = \int [\prod_{j=N}^{M} d\mu_{v^j}(\varphi^j)]F(\sum_{j=N}^{M}\mathcal{A}^j\varphi^j) \qquad (2\text{-}18)$$

where

$$v^j = \begin{cases} u_j - u_j C^*_{j-1,j}u^{-1}_{j-1}C_{j-1,j}u_j, & \text{for } N \geq j \geq M+1 \\ u_M, & \text{for } j = M \end{cases} \qquad (2\text{-}19)$$

and by lemma 1

$$\mathrm{v}^j = \mathcal{A}^j v^j \mathcal{A}^{j*}. \qquad (2\text{-}20)$$

The kernels of $\mathrm{v}^j$ and $v^j$ are exponentially decaying with decay length $a_{j-1}$ and the kernel of $\mathcal{A}^j$ is exponentially decaying with decay length $a_j$ (see appendix B), i.e. there exists constants $K_1$ and $K_2$ (uniformly in $j$) such that

$$|\mathrm{v}^j(z_1,z_2)| \leq K_1 a_j^{2-d}\exp\{-K_2 a_{j-1}^{-1}|z_1 - z_2|\} \qquad (2\text{-}21a)$$
$$|v^j(y_1,y_2)| \leq K_1 a_j^{2-d}\exp\{-K_2 a_{j-1}^{-1}|y_1 - y_2|\} \qquad (2\text{-}21b)$$
$$|\mathcal{A}^j(z,y)| \leq K_1 a_j^{-d}\exp\{-K_2 a_j^{-1}|z - y|\}. \qquad (2\text{-}21c)$$

Eq. (18) is the starting point for renormalization group procedures and the definitions of multigrid polymer systems.



# 3. CONVERGENT POLYMER SYSTEMS FOR HIERARCHICAL MODELS

In this chapter we will study the nonperturbative renormalization group analysis of a hierarchical model at the example of the Euclidean $\Phi^4$-theory in three and four dimensions. This chapter serves as an introduction to the analysis of the full model (see chapter 3). In the hierarchical model approximation the fluctuation fields are supposed to be constant on blocks of a multigrid. This leads to a great simplification of the renormalization group equations. The fluctuation propagators do not couple different blocks on a multigrid layer. Therefore the effective actions can be expressed as a sum over all blocks for each layer. Equivalently, the partition functions factorizes for each renormalization group step. In the language of polymer systems we see that only monomer activities are nonzero. The coupling between different layers remains in the hierarchical model approximation. Each block $x$ of the multigrid is coupled to the $L^4$ next finer blocks which fit in block $x$. These couplings come from the non free part of the action (= interaction). Renormalization conditions and the introduction of running (effective) coupling constants is required to show that the coupling of different layers is weak. For a weak coupling of different layers the infinite application of the renormalization group transformation is not leading to divergencies and the infrared and/or ultraviolet limit can be shown to exist.

The hierarchical model approximation simplifies the so-called "large field problem", one of the main problems in the renormalization group analysis. If the large field contributions are not properly taken into account the renormalization group will not lead to nonperturbative convergent results. Large field contributions are controlled by a lower bound on the interactions (stability bound). The special locality property for the effective interactions in hierarchical approximation is the reason for an easier treatment of the large field problem. The large field contributions are dominated, by the approach presented in this chapter, without using a large field split.

In the first section of this chapter, the renormalization group flow of the running coupling constants in the hierarchical approximation is rigorously studied. Bounds on irrelevant remainder terms, which do not depend on the number of renormalization group steps, are represented. The iteration of the renormalization group equation leads to an expansion on the multigrid. This expansion and bounds on terms of the multigrid expansion are represented in section two. In section three the ultraviolet problem at the example of $\Phi^4$–theory in three dimensions is studied.

## 3.1 Hierarchical Renormalization Group Transformations

In this section, we consider the massless $\Phi^4$-theory in $d = 4$ dimensions in hierarchical approximation (without using a large field split). The (effective) partition function $Z_j(\Psi)$ for $j \leq 0$ is the result of $|j|$ renormalization group steps. The renormalization group starts with

$$Z_0(\Psi) = \exp\{-V(\Psi)\} \qquad (1\text{-}1)$$

where $V$ is a local interaction. The renormalization group equations for a hierarchical model in $d$ dimensions on the multigrid are

$$Z_{j-1}(\Psi) = \int d\mu_{v^j}(\Phi)Z_j(\Phi + \Psi)/(\Psi = 0) \qquad (1\text{-}2)$$

with

$$\mathrm{v}^j = \mathcal{A}^j v^j \mathcal{A}^{j*} \tag{1-3}$$

where

$$v^j(y_1,y_2) = \gamma a_j^{2-d}\delta_{y_1,y_2} \tag{1-4a}$$

$$\mathcal{A}^j(z,y) = a_j^{-d}\chi_y(z) \tag{1-4b}$$

for all $j \le 0$ and $y, y_1, y_2 \in \Lambda_j$ and $z \in base$, $\gamma$ positive constant. $\chi_y$ is the characteristic function of block $y$. The kernel of $\mathrm{v}^j$ is ultralocal on layer $\Lambda_j$, i.e. $\mathrm{v}^j(z_1,z_2) = 0$ unless there exists a block $y \in \Lambda_j$ such that $z_1, z_2 \in y$. Since $\mathrm{v}(z_1,z_2)$ and $(-\triangle)^{-1}(z_1,z_2)$ for $z_1 \neq z_2$ are of order $O(|z_1 - z_2|^{2-d})$ we see that $\mathrm{v}$ is a good approximation for $(-\triangle)^{-1}$. Suppose that the external field $\Psi$ in $Z_j(\Psi)$ is constant on blocks of $\Lambda_j$. In the following we introduce a dimensionless notation to show the simplicity of the renormalization group equation in hierarchical approximation. We see that the partition function $Z_j(\Psi)$ depends only on one complex variable $\Psi$. Replacing $Z_k(\Psi)$ by $Z_k(a_k^{d/2-1}\Psi)$ for all $k \le 0$ and $\Psi$ by $a_{j-1}^{1-d/2}\Psi$ in eq. (1-2) we obtain

$$Z_{j-1}(\Psi) = \left[\int d\mu_\gamma(\Phi) Z_j(\Phi + L^{1-d/2}\Psi)\right]^{L^d} \exp\{-e_{j-1}\} \tag{1-5}$$

where the vacuum energy counterterm is given by

$$e_{j-1} = L^d \ln \int d\mu_\gamma(\Phi) Z_j(\Phi). \tag{1-6}$$

For the special case of the massless $\Phi^4$-theory in 4 dimensions we consider the multigrid $\Lambda$ with base space $a\mathbf{Z}^4$. Suppose that the renormalization group equations start with an interaction

$$V(\Psi) = \frac{m^2}{2}\Psi^2 + \frac{\lambda}{4!}\Psi^4. \tag{1-7}$$

Define for all $j \le 0$ the Taylor-coefficients of the free energy $-\ln Z_j(\Psi)$

$$m_j^2 \equiv -\frac{\partial^2}{\partial\Psi^2}\ln Z_j(\Psi)|_{\Psi=0} \tag{1-8a}$$

$$\lambda_j \equiv -\frac{\partial^4}{\partial\Psi^4}\ln Z_j(\Psi)|_{\Psi=0} \tag{1-8b}$$

$$r_j \equiv -\frac{\partial^6}{\partial\Psi^6}\ln Z_j(\Psi)|_{\Psi=0} \tag{1-8c}$$

$$q_j \equiv -\frac{\partial^8}{\partial\Psi^8}\ln Z_j(\Psi)|_{\Psi=0}. \tag{1-8d}$$

Obviously, we have for the starting coefficients

$$m_0^2 = m^2, \quad \lambda_0 = \lambda, \quad r_0 = 0, \quad q_0 = 0. \tag{1-9}$$

In the following we consider the case $d = 4$. The Taylor-coefficients defined above are called running coupling constants. They depend on the bare coupling constants $m^2, \lambda$. The bare coupling constant $\lambda$ is supposed to be small and we will see that $m^2$ (for a massless theory) can be chosen such that $\lim_{j\to-\infty} m_j^2(m^2,\lambda) = 0$ for small $\lambda$. The next proposition gives the renormalization group flow of the running coupling constants $m_j^2,\ \lambda_j,\ , r_j,\ q_j$



PROPOSITION 1. *Suppose that $\lambda$ is small. For all $j \le 0$ there exist constants $c_-^m, c_+^m, c_-^\lambda, c_+^\lambda$, $c_r, c_q$ (uniformly in $j$) and closed intervals $[\alpha_{-1},\beta_{-1}]\supset_{\neq}[\alpha_{-2},\beta_{-2}]\supset_{\neq}\dots\supset_{\neq}[\alpha_j,\beta_j]$ such that $m_j^2 = m_j^2(m^2)$ depends on $m^2$ in the following way*

$$m_j^2([\alpha_j,\beta_j]) + \frac{1}{2}\gamma\sum_{i=j}^{-\infty} L^{2(i-j)}\lambda_j \in [c_-^m\lambda_j^{3/2}, c_+^m\lambda_j^{3/2}] \tag{1-10}$$

and

$$\lambda_j \in \left(\frac{c_-^\lambda}{|j-1|}, \frac{c_+^\lambda}{|j-1|}\right), \quad |r_j + 10\gamma\sum_{i=j+1}^{0} L^{-2(i-j)}\lambda_j^2| \le c_r\lambda_j^{5/2}, \quad |q_j| \le c_q\lambda_j^3. \tag{1-11}$$

By proposition 1 follows that for the choice

$$m^2 = \lim_{j\to-\infty}\alpha_j = \lim_{j\to-\infty}\beta_j$$

the limits $j \to -\infty$ of the running coupling constants are zero. The renormalization group equations for the Taylor-coefficients are given in the following lemma

LEMMA 2. *For all $j \le 0$ we have*

$$m_{j-1}^2 = L^2[m_j^2 + \frac{1}{2}\gamma\lambda_j - \gamma(m_j^2)^2] + Q_{2,2}^j \tag{1-12a}$$

$$\lambda_{j-1} = \lambda_j - 5\gamma^2\lambda_j^2 + \frac{1}{2}\gamma r_j - 4\gamma\lambda_j m_j^2 + \frac{1}{8}\gamma^2 q_j - \frac{5}{2}\gamma^2 r_j m_j^2 + Q_{3,4}^j \tag{1-12b}$$

$$r_{j-1} = L^{-2}[r_j - 10\gamma\lambda_j^2 + \frac{1}{2}\gamma q_j - 4\gamma r_j m_j^2] + Q_{2,6}^j \tag{1-12c}$$

$$q_{j-1} = L^{-4}q_j + Q_{1,8}^j \tag{1-12d}$$

*with*

$$Q_{N,k}^j \equiv -L^4\frac{\partial^k}{\partial\Psi^k}\frac{1}{(N-1)!}\int_0^1 ds(1-s)^{N-1}\partial_s^N \ln\int d\mu_{s\gamma}(\Phi)Z_j(\Phi + L^{-1}\Psi)|_{\Psi=0} \tag{1-13}$$

*Proof :* All terms but the last one on the right hand sides of eqs. (12a-d) are evaluated by perturbation theory. We will show only the first equation (12b) explicitly. Consider the Taylor expansion in $\gamma$

$$\ln\int d\mu_\gamma(\Phi)Z_j(\Phi+\Psi) = \sum_{i=0}^{N-1} C_i^j(\Psi)\gamma^i + \frac{1}{(N-1)!}\int_0^1 ds(1-s)^{N-1}$$
$$\partial_s^N \ln\int d\mu_{s\gamma}(\Phi)Z_j(\Phi + L^{-1}\Psi). \tag{1-14}$$

We have for the first three orders

$$C_0^j(\Psi) = \ln Z_j(\Psi)$$

$$C_1^j(\Psi) = \frac{1}{2}\frac{\partial^2}{\partial\Psi^2}\ln Z_j(\Psi) + \frac{1}{2}\left(\frac{\partial}{\partial\Psi}\ln Z_j(\Psi)\right)^2$$

$$C_2^j(\Psi) = \frac{1}{8}\frac{\partial^4}{\partial\Psi^4}\ln Z_j(\Psi) + \frac{1}{2}\left(\frac{\partial^3}{\partial\Psi^3}\ln Z_j(\Psi)\right)\left(\frac{\partial}{\partial\Psi}\ln Z_j(\Psi)\right) + \frac{1}{4}\left(\frac{\partial^2}{\partial\Psi^2}\ln Z_j(\Psi)\right)^2.$$

The renormalization group equation (1-5) implies

$$\ln Z_{j-1} = L^4 \ln\left[\int d\mu_\gamma(\Phi) Z_j(\Phi + L^{-1}\Psi)\right] - \epsilon_{j-1}$$

Using (1-14) with $N = 2$, we obtain

$$\begin{aligned} m_{j-1}^2 &= -\frac{\partial^2}{\partial\Psi^2}\ln Z_{j-1}(\Psi)|_{\Psi=0} = -L^{-2}\Big[\frac{\partial^2}{\partial\Psi^2}\Big[\ln Z_j(\Psi) + \\ &\quad + \frac{\gamma}{2}\frac{\partial^2}{\partial\Psi^2}\ln Z_j(\Psi) + \frac{\gamma}{2}\left(\frac{\partial}{\partial\Psi}\ln Z_j(\Psi)\right)^2\Big] + \\ &\quad + L^4\frac{\partial^2}{\partial\Psi^2}\int_0^1 ds(1-s)\partial_s^2 \ln\int d\mu_{s\gamma}(\Phi) Z_j(\Phi + L^{-1}\Psi)\Big]|_{\Psi=0} = \\ &= L^2[m_j^2 + \frac{1}{2}\gamma\lambda_j - \gamma(m_j^2)^2] + Q_{2,2}^j. \end{aligned}$$

The last equality follows from the definition (1-8) of the running coupling constants. Eqs. (1-12b,c,d) are similarly shown . √

The next lemma shows that a bound on $Q_{N,k}^j$ suffices to control the renormalization group flow (1-10,11) of the running coupling constants.

LEMMA 3. *Suppose that there exist a constant $K$ such that*

$$|Q_{2,2}^j| \le K\lambda_j^{3/2}, \quad \max(|Q_{3,4}^j|, |Q_{2,6}^j|, |Q_{1,8}^j|) \le K\lambda_j^{5/2} \tag{1-15}$$

*for all $j \le 0$. Then the assertion of proposition 1 is valid.*

*Proof : (by induction)* For $j = 0$ the assertion of proposition 1 is trivial. Suppose that for $j < 0$ there exist constants $c_-^m, c_+^m, c_-^\lambda, c_+^\lambda, c_r, c_q$ and closed intervals $[\alpha_{-1}, \beta_{-1}] \supset_{\neq} [\alpha_{-2}, \beta_{-2}] \supset_{\neq} \dots \dots \supset_{\neq} [\alpha_j, \beta_j]$ such that $m_j^2 = m_j^2(m^2)$ depends on $m^2$ in the following way

$$m_j^2([\alpha_j,\beta_j]) + \frac{1}{2}\gamma\sum_{i=j}^{-\infty} L^{2(i-j)}\lambda_j \in [c_-^m\lambda_j^{3/2}, c_+^m\lambda_j^{3/2}]. \tag{1-10}$$

and

$$\lambda_j \in \left(\frac{c_-^\lambda}{|j-1|}, \frac{c_+^\lambda}{|j-1|}\right), \quad |r_j + 10\gamma\sum_{i=j+1}^{0} L^{-2(i-j)}\lambda_j^2| \le c_r\lambda_l^{5/2}, \quad |q_j| \le c_q\lambda_j^3 \tag{1-11}$$



We will show that the assertion of proposition 1 holds for $j-1$. For that the renormalization group flow of the running coupling constants (1-12a-d) will be used. By eq. (1-12b) we have

$$\begin{aligned}\lambda_{j-1} - \lambda_j = -5\gamma^2\sum_{i=j}^{0} L^{-2(i-j)}\lambda_j^2 + 2\gamma^2\sum_{i=-\infty}^{j} L^{2(i-j)}\lambda_j^2 + \frac{1}{2}\gamma r_j' - 4\gamma\lambda_j m_j'^2 + \\ \frac{1}{8}\gamma^2 q_j - \frac{5}{2}\gamma^2 r_j m_j^2 + Q_{3,4}^j\end{aligned} \tag{1-16}$$

with

$$m_j'^2 = m_j^2 + \frac{1}{2}\gamma\lambda_j, \quad r_j' = r_j + 10\gamma\sum_{i=j+1}^{0} L^{-2(i-j)}\lambda_j^2.$$

By induction hypothesis and eq.(1-16) follows

$$\lambda_{j-1}^{-1} = \lambda_j^{-1}(1 + \gamma_j\lambda_j + O(\lambda_j^{3/2})) = \lambda_j^{-1} + \gamma_j + O(\lambda_j^{1/2})$$

with

$$b_j = 3\gamma^2 + 5\gamma^2 L^{-4} + 5\gamma^2\sum_{i=j+1}^{0} L^{-2(i-j)} - 2\gamma^2\sum_{i=-\infty}^{j-1} L^{2(i-j)}.$$

Define

$$c_1 \equiv \inf_{j\le 0} b_j > 0, \quad c_2 \equiv \sup_{j\le 0} b_j > 0.$$

Then by induction hypothesis

$$\begin{aligned}\lambda_{j-1}^{-1} &\le c_-^{-1}|j-1| + \gamma_j + O(\lambda_j^{1/2}) \le c_-^{-1}|j-2| \\ \lambda_{j-1}^{-1} &\ge c_+^{-1}|j-1| + \gamma_j + O(\lambda_j^{1/2}) \ge c_+^{-1}|j-2|\end{aligned}$$

for

$$c_2 + O\left(\frac{1}{|j-1|}\right) \le c_-^{-1}, \quad c_1 + O\left(\frac{1}{|j-1|}\right) \ge c_+^{-1}.$$

Using eq. (1-12a) and induction hypothesis we get for $\alpha \in [\alpha_j, \beta_j]$

$$\begin{aligned} m_{j-1}^2(\alpha) + \frac{1}{2}\gamma\sum_{i=-\infty}^{j} L^{2(i-(j-1))}\lambda_{j-1} = L^2\big[m_j^2(\alpha) + \frac{1}{2}\gamma\sum_{i=-\infty}^{j} L^{2(i-j)}\lambda_j\big] + \\ + \frac{1}{2}\gamma\sum_{i=-\infty}^{j-1} L^{2(i-(j-1))}(\lambda_{j-1} - \lambda_j) - \gamma(m_j^2(\alpha))^2 + Q_{2,2}^j. \end{aligned}$$

Since

$$\lambda_{j-1} - \lambda_j = -b_j\lambda_j^2 + O(\lambda_j^{5/2}) \tag{1-17}$$

and $|Q_{2,2}^j| \le K\lambda_j^{3/2}$ we obtain

$$m_{j-1}^2([\alpha_j,\beta_j]) + \frac{1}{2}\gamma\sum_{i=j-1}^{-\infty} L^{2(i-(j-1))}\lambda_{j-1} \supsetneq [c_-^m\lambda_{j-1}^{3/2}, c_+^m\lambda_{j-1}^{3/2}].$$

Thus there exist a closed interval $[\alpha_{j-1},\beta_{j-1}]\subsetneq[\alpha_j,\beta_j]$ with

$$m_{j-1}^2([\alpha_{j-1},\beta_{j-1}])+\frac{1}{2}\gamma\sum_{i=j-1}^{-\infty}L^{2(i-(j-1))}\lambda_{j-1}=[c_-^m\lambda_{j-1}^{3/2},c_+^m\lambda_{j-1}^{3/2}].$$

From eq. (1-12c) follows

$$r_{j-1}+10\gamma\sum_{i=j}^{0}L^{-2(i-(j-1))}\lambda_{j-1}^2=$$

$$=L^{-2}[r_j+10\gamma\sum_{i=j+1}^{0}L^{-2(i-j)}\lambda_j^2]+$$

$$10\gamma L^{-2}\sum_{i=j}^{0}L^{-2(i-j)}(\lambda_{j-1}^2-\lambda_j^2)+\frac{1}{2}\gamma L^{-2}q_j-4\gamma L^{-2}r_jm_j^2+Q_{2,6}^j. \tag{1-18}$$

Thus using eqs. (1-17),(1-18) and induction hypothesis

$$|r_{j-1}+10\gamma\sum_{i=j}^{0}L^{-2(i-(j-1))}\lambda_{j-1}^2|\leq L^{-2}c_r\lambda_j^{5/2}+K'\lambda_j^{5/2}\leq c_r\lambda_{j-1}^{5/2}.$$

From eqs. (1-12d),(1-17) and induction hypothesis follows

$$|q_{j-1}|\leq L^{-4}|q_j|+K\lambda_j^{5/2}\leq(L^{-4}c_q+K)\lambda_j^{5/2}\leq c_q\lambda_{j-1}^{5/2}.$$

This completes the proof. $\surd$

It remains to estimate $Q_{N,k}^j$. For that we will need suitable bounds on $Z_j(\Psi)$. To prove the bounds (1-15) on the remainder terms $Q_{N,k}^j$ split the partition function

$$Z_j(\Psi)=Z_j^{rel}(\Psi)+R_j(\Psi) \tag{1-19}$$

where

$$Z_j^{rel}(\Psi)\equiv\exp\{-V_j^{rel}(\Psi)\}\equiv\exp\{-\frac{1}{2}m_j^2\Psi^2-\frac{\lambda_j}{4!}\Psi^4\}. \tag{1-20}$$

$Z_j^{rel}(\Psi)$ is called the relevant partition function and $R_j(\Psi)$ is called the irrelevant activity. We will show that $Z_j^{rel}$ is a good approximation for $Z_j$. By the definition of the running coupling constants $m_j^2,\lambda_j$ (see eqs. (1-8a,b)) it follows that $R_j(\Psi)$ is of order $O(\Psi^6)$. In each renormalization group step a $\Psi$-variable gives a factor $L^{-1}$ (cp. eq. (1-5)). As will be seen later, we loose a volume factor $L^4$ for each renormalization group step. Thus each renormalization group step delivers a factor $L^4L^{-6}=L^{-2}$ for $R_j$ and we see that $R_j$ behaves like an irrelevant operator.

The proof of the bound on the remainder terms is inductive and organized as follows. In lemma 7 (with the help of lemma 6) it will be shown that the bound (1-15) on $Q_{N,k}^j$ follows by a suitable bound on the irrelevant activity $R_j$. This bound on $R_j$ is proven by induction.



The start of induction is trivial since $R_N\equiv 0$. Lemma 4 gives a stability bound on $V_j^{rel}$ supposing $m_j^2=O(\lambda_j)$. Lemma 5 shows that $\exp\{\frac{\gamma'}{2}\Phi^2\}$ is Gauss-integrable for $\gamma'<\gamma$. This will be needed to prove a suitable bound on the vacuum energy counterterm $e_{j-1}$ supposing the bound (1-24) on $R_j$ (lemma 8). Lemma 9-12 are required for the proof of the induction step, i.e. the bound on $R_j$ implies the bound on $R_{j-1}$ supposing bounds on the running coupling constants (lemma 13).

$Z_j^{rel}$ is estimated by the following stability bound.

LEMMA 4. *Suppose that* $|m_j^2|\leq c_m\lambda_j$. *Then we have for all* $\Psi\in\mathbf{R}$, $\chi\in\mathbf{C}$ *and* $c\geq 0$ *and* $j\leq 0$

$$Re\ V_j^{rel}(\Psi+\chi)\geq c\lambda_j^{1/2}\Psi^2-c_m\lambda_j|\chi|^2-6\lambda_j|\chi|^4-\frac{1}{96}(48c+c_m\lambda_j^{1/2})^2. \tag{1-21}$$

*Proof :* Since

$$(\Psi+Re\ \chi)^4\geq\frac{1}{2}\Psi^4-144(Re\ \chi)^4$$

and for all $c\geq 0$

$$\frac{1}{2}\Psi^4\geq-\frac{1}{2}c^2+c\Psi^2$$

we have

$$Re\ V_j^{rel}(\Psi+\chi)\geq\frac{1}{2}m_j^2(\Psi+Re\ \chi)^2-\frac{1}{2}m_j^2(Im\ \chi)^2+\frac{\lambda_j}{4!}(\Psi+Re\ \chi)^4-$$

$$-\frac{6\lambda_j}{4!}(\Psi+Re\ \chi)^2(Im\ \chi)^2+\frac{\lambda_j}{4!}(Im\ \chi)^4\geq$$

$$\geq-c_m\lambda_j\Psi^2-c_m\lambda_j|\chi|^2+\frac{c\lambda_j}{4!2}\Psi^2-\frac{144}{4!}\lambda_j|\chi|^4-\frac{\lambda_j}{4!4}c^2.$$

Replace $c$ by $4!2\lambda_j^{-1/2}c+c_m$. Then

$$Re\ V_j^{rel}(\Psi+\chi)\geq c\lambda_j^{1/2}\Psi^2-c_m\lambda_j|\chi|^2-6\lambda_j|\chi|^4-\frac{1}{96}(48c+c_m\lambda_j^{1/2})^2.\ \surd$$

The next lemma shows that a Boltzmann factor $\exp\{\frac{\gamma'}{2}\Phi^2\}$ is Gauss-integrable for $\gamma'<\gamma$.

LEMMA 5. *For all real* $\gamma,\gamma'$, $\gamma$ *positive and* $\gamma'<\gamma$ *we have*

$$\int d\mu_\gamma(\Phi)\exp\{\frac{\gamma'}{2}\Phi^2\}=\sqrt{\frac{\gamma}{\gamma-\gamma'}}. \tag{1-22}$$

*Proof :*

$$\int d\mu_\gamma(\Phi)\exp\{\frac{\gamma'}{2}\Phi^2\}=\sqrt{\frac{\gamma}{2\pi}}\int_{-\infty}^{\infty}d\Phi\quad\exp\{-\frac{\gamma-\gamma'}{2}\Phi^2\}=$$

$$=\sqrt{\frac{\gamma}{2\pi}}\sqrt{\frac{2\pi}{\gamma-\gamma'}}=\sqrt{\frac{\gamma}{\gamma-\gamma'}}.\ \surd$$

LEMMA 6. *Suppose that $|m_j^2| \le c_m\lambda_j$. For small $\lambda_j$ there exists a constant $K$ such that*

$$\Big| \int d\mu_\gamma(\Phi) Z_j^{rel}(\Phi) - 1\Big| \le K\lambda_j \tag{1-23}$$

*for all $j \le 0$.*

*Proof :* Using lemma 5 we obtain

$$\begin{aligned}\Big| \int d\mu_\gamma(\Phi) Z_j^{rel}(\Phi) - 1\Big| &= \Big| \int_0^1 ds \int d\mu_\gamma(\Phi) V_j^{rel}(\Phi) \exp\{-sV_j^{rel}(\Phi)\}\Big| \le \\ &\le \frac{|m_j^2|}{2} \int d\mu_\gamma(\Phi)\Phi^2 \exp\{\frac{c_m\lambda_j}{2}\Phi^2\} + \frac{\lambda_j}{4!}\int d\mu_\gamma(\Phi)\Phi^4 \exp\{\frac{c_m\lambda_j}{2}\Phi^2\} \le \\ &\le \frac{c_m\lambda_j}{2}\frac{2}{c}\int d\mu_\gamma(\Phi)\exp\{\frac{c+c_m\lambda_j}{2}\Phi^2\} + \frac{\lambda_j}{4!}\frac{8}{c^2}\exp\{\frac{c+c_m\lambda_j}{2}\Phi^2\} \le \\ &\le \frac{c_m\lambda_j}{c}\Big(\frac{\gamma}{\gamma - c - c_m\lambda_j}\Big)^{1/2} + \frac{\lambda_j}{3c^2}\Big(\frac{\gamma}{\gamma - c - c_m\lambda_j}\Big)^{1/2}. \quad \surd\end{aligned}$$

The following lemma represents a necessary condition on $R_j$ for the bound on $Q^j_{N,k}$. For bounds on derivatives of analytic functions we will use Cauchy's inequality. The Cauchy inequality is

$$\Big|\frac{\partial^n}{\partial\Psi^n}F(\Psi)\Big| \le \frac{n!}{\kappa^n} \sup_{\chi\in\mathbf{C}:\ |\chi|\le\kappa} |F(\Psi+\chi)|$$

for analytic functions $F : \mathbf{C} \to \mathbf{C}$, $\kappa > 0$, $\Psi \in \mathbf{C}$ and $n \in \mathbf{N}$.

LEMMA 7. *Suppose that there exists a constant $K$ and $c_2, c_3$ such that*

$$|R_j(\Psi + \chi)| \le K\lambda_j^{1/2}\exp\{-c\lambda_j^{1/2}\Psi^2\} \tag{1-24}$$

*for all $\Psi \in \mathbf{R}$, $\chi \in \mathbf{C}$ with $|\chi| \le \lambda_j^{-1/4}c_3$. Then there exists for all $N, k \in \mathbf{N}$ a constant $K_{N,k}$, uniformly in $j$, such that*

$$|Q^j_{N,k}| \le K_{N,k}\lambda_j^{\frac{N}{2}+\frac{k}{4}}. \tag{1-25}$$

*Proof :* We have for $s \in [0,1]$, $\chi \in \mathbf{C}$, $|\chi| \le \lambda_j^{-1/4}c_3$

$$\ln \int d\mu_{s\gamma}(\Phi) Z_j(\Phi + L^{-1}\chi) = \sum_{n\ge 1} \frac{(-1)^{n+1}}{n}\Big[\int d\mu_{s\gamma}(\Phi)(Z_j(\Phi + L^{-1}\chi) - 1)\Big]^n \tag{1-26}$$

and

$$\partial_s \int d\mu_{s\gamma}(\Phi)F(\Phi+\chi) = \frac{1}{2}\gamma\frac{\partial^2}{\partial\chi^2}\int d\mu_{s\gamma}(\Phi)F(\Phi+\chi). \tag{1-27}$$

By Cauchy's inequality, eqs. (1-24),(1-26),(1-27) and lemma 6 follows the assertion. $\surd$

*Remark :* The supposition of lemma 3 holds if the bound (1-24) is valid.



The renormalization group equation for $R_j$ is, using (1-5) and (1-19),

$$\begin{aligned}R_{j-1}(\Psi) = &-Z_{j-1}^{rel}(\Psi) + \Big[\int d\mu_\gamma(\Phi) Z_j^{rel}(\Phi + L^{-1}\Psi)\Big]^{L^4}\exp\{-e_{j-1}\} + \\ &+ \Big\{\sum_{k=1}^{L^4}\binom{L^4}{k}\Big[\int d\mu_\gamma(\Phi) Z_j^{rel}(\Phi + L^{-1}\Psi)\Big]^{L^4-k}\Big[\int d\mu_\gamma(\Phi) R_j(\Phi + L^{-1}\Psi)\Big]^k\Big\}\exp\{-e_{j-1}\}.\end{aligned} \tag{1-28}$$

The sum of the first and the second term on the right hand side of eq. (1-28) nearly cancel, as will be seen in lemma 9. Define for $s \in [0,1]$ the interpolated interaction by

$$V_{j,s}^{rel}(\Psi) \equiv \frac{1}{2}m_{j,s}^2\Psi^2 + \frac{\lambda_{j,s}}{4!}\Psi^4 \tag{1-29}$$

where

$$m_{j,s}^2 \equiv sm_j^2 + (1-s)L^{-2}m_{j-1}^2 \tag{1-30a}$$

$$\lambda_{j,s} \equiv s\lambda_j + (1-s)\lambda_{j-1}. \tag{1-30b}$$

Then we have

$$Z_{j-1}^{rel}(\Psi) = \exp\{-L^4 V_{j,s}^{rel}(L^{-1}\Psi)\}|_{s=0} \tag{1-31a}$$

$$Z_j^{rel}(\Psi) = \exp\{-V_{j,s}^{rel}(\Psi)\}|_{s=1} \tag{1-31b}$$

and for the sum of the first and the second term on the right hand side of eq. (1-28) we may write

$$\begin{aligned}\delta Z_j^{rel}(\Psi) &\equiv -Z_{j-1}^{rel}(\Psi) + \Big[\int d\mu_\gamma(\Phi) Z_j^{rel}(\Phi + L^{-1}\Psi)\Big]^{L^4}\exp\{-e_{j-1}\} = \\ &= \int_0^1 ds\,\partial_s\Big[\int d\mu_{s\gamma}(\Phi)\exp\{-V_{j,s}^{rel}(\Phi + L^{-1}\Psi)\}\Big]^{L^4}.\end{aligned} \tag{1-32}$$

The following lemma presents a bound on vacuum energy counterterms $e_{j-1}$.

LEMMA 8. *Suppose that for all $\Psi \in \mathbf{R}$, $\chi \in \mathbf{C}$, $|\chi| \le \lambda_j^{-1/4}c_3$*

$$|R_j(\Psi + \chi)| \le K\lambda_j^{1/2}\exp\{-c\lambda_j^{1/2}\Psi^2\}. \tag{1-24}$$

*Then there exists a constant $K_e$ (not dependent on $j$ and $\lambda$) such that*

$$|e_{j-1}| \le K_e\lambda_j. \tag{1-33}$$

*Proof :* Since $Z_j(0) = 1$ we get by definition (1-6) and (1-13)

$$\begin{aligned}e_{j-1} &= L^4 \ln \int d\mu_\gamma(\Phi) Z_j(\Phi) = \\ &= L^4\Big[\frac{1}{2}\gamma\frac{\partial^2}{\partial\Psi^2}Z_j(\Psi)|_{\Psi=0} + \int_0^1 ds\,\partial_s \ln\int d\mu_{s\gamma}(\Phi)Z_j(\Phi)\Big] = -\frac{1}{2}\gamma m_j^2 - Q^j_{1,0}.\end{aligned}$$

Lemma 7 implies the assertion. $\surd$

LEMMA 9. *Suppose that for all* $\Psi \in \mathbf{R},\ \chi \in \mathbf{C},\ |\chi| \le \lambda_j^{-1/4} c_3$

$$|R_j(\Psi+\chi)| \le K\lambda_j^{1/2} \exp\{-c\lambda_j^{1/2}\Psi^2\}. \tag{1–24}$$

*Then there exists a constant* $K'$ *and* $c$ *such that*

$$|\delta Z_j^{rel}(\Psi+\chi)| \le K'\lambda_j^{1/2} \exp\{-c\lambda_j^{1/2}\Psi^2\}. \tag{1–34}$$

*Proof :* We have

$$\partial_s V_{j,s}^{rel}(\Psi) = \frac{1}{2}(m_j^2 - L^{-2}m_{j-1}^2)\Psi^2 + \frac{\lambda_j - \lambda_{j-1}}{4!}\Psi^4. \tag{1–35}$$

Lemma 7 and eqs.(1-34,35) imply

$$|m_j^2 - L^{-2}m_{j-1}^2| \le c_1\lambda_j \tag{1–36}$$

$$|\lambda_j - \lambda_{j-1}| \le c_2\lambda_j^2. \tag{1–37}$$

Thus for $|\chi| \le \lambda_j^{-1/4} c_3$

$$|\partial_s V_{j,s}^{rel}(\Phi + L^{-1}(\Psi+\chi))| \le \frac{1}{2}c_1\lambda_j(\Phi + L^{-1}(\Psi+\chi))^2 + \frac{c_2\lambda_j^2}{4!}(\Phi + L^{-1}(\Psi+\chi))^4 \le$$
$$\le \frac{3}{2}c_1\lambda_j(\Phi^2 + L^{-2}\Psi^2) + \frac{27c_2\lambda_j^2}{4!}(\Phi^4 + L^{-4}\Psi^4) + \frac{3}{2}c_1\lambda_j^{1/2}c_3^2 + \frac{27c_2\lambda_j^2}{4!}c_3^4. \tag{1–38}$$

Furthermore

$$\delta Z_j^{rel}(\Psi+\chi) = \int_0^1 ds L^4 \Big\{ \frac{1}{2}\gamma\Big[\frac{\partial^2}{\partial\chi^2}\int d\mu_{s\gamma}(\Phi)\exp\{-V_{j,s}^{rel}(\Phi + L^{-1}\Psi + L^{-1}\chi)\}\Big]$$
$$\Big[\int d\mu_{s\gamma}(\Phi)\exp\{-V_{j,s}^{rel}(\Phi + L^{-1}\Psi + L^{-1}\chi)\}\Big]^{L^4-1}\exp\{-se_{j-1}\} -$$
$$-\int d\mu_{s\gamma}(\Phi)\partial_s V_{j,s}^{rel}(\Phi + L^{-1}\Psi + L^{-1}\chi)\exp\{-V_{j,s}^{rel}(\Phi + L^{-1}\Psi + L^{-1}\chi)\}$$
$$\Big[\int d\mu_{s\gamma}(\Phi)\exp\{-V_{j,s}^{rel}(\Phi + L^{-1}\Psi + L^{-1}\chi)\}\Big]^{L^4-1}\exp\{-se_{j-1}\}\Big\} -$$
$$- e_{j-1}\int_0^1 ds\Big[\int d\mu_{s\gamma}(\Phi)\exp\{-V_{j,s}^{rel}(\Phi + L^{-1}\Psi + L^{-1}\chi)\}\Big]^{L^4}\exp\{-se_{j-1}\}.$$

Using a similar stability bound for $V_{j,s}^{rel}$ like the bound (1-21) on $V_j$, inequality (1-38) and lemma 5 and 8 we obtain the assertion. √

The following lemma is easily proven, using the stability bound on $Z_j^{rel}$ (see lemma 4) and lemma 5.



LEMMA 10. *Suppose that* $|m_j^2| \le c_m\lambda_j$. *For small* $\lambda_j$ *there exists a constant* $c$ *and* $K_c$ *such that*

$$\Big|\int d\mu_\gamma(\Phi) Z_j^{rel}(\Phi + L^{-1}(\Psi+\chi))\Big| \le K_c \exp\{-cL^{-4}\lambda_j^{1/2}\Psi^2\}. \tag{1–39}$$

*for all* $\Psi \in \mathbf{R},\ \chi \in \mathbf{C},\ |\chi| \le \lambda_j^{-1/4} c_3$.

The sum of all terms of the representation (1-28) for $R_{j-1}$ which contains at least two $R_j$-factors is estimated in the following lemma.

LEMMA 11. *Suppose that* $|m_j^2| \le c_m\lambda_j$ *and*

$$|R_j(\Psi+\chi)| \le K\lambda_j^{1/2}\exp\{-c\lambda_j^{1/2}\Psi^2\} \tag{1–24}$$

*for all* $\Psi \in \mathbf{R},\ \chi \in \mathbf{C},\ |\chi| \le \lambda_j^{-1/4} c_3$. *Then there exists for small* $\lambda_j$ *a constant* $K''$ *such that*

$$\Big|\sum_{k=2}^{L^4}\binom{L^4}{k}\Big[\int d\mu_\gamma(\Phi) Z_j^{rel}(\Phi + L^{-1}(\Psi+\chi))\Big]^{L^4-k}\Big[\int d\mu_\gamma(\Phi) R_j(\Phi + L^{-1}(\Psi+\chi))\Big]^k\Big| \le$$
$$\le K''\lambda_j \exp\{-c\lambda_j^{1/2}\Psi^2\}. \tag{1–40}$$

*Proof :* Using lemma 10 and the supposition (1-24) we obtain

$$\Big|\sum_{k=2}^{L^4}\binom{L^4}{k}\Big[\int d\mu_\gamma(\Phi) Z_j^{rel}(\Phi + L^{-1}(\Psi+\chi))\Big]^{L^4-k}\Big[\int d\mu_\gamma(\Phi) R_j(\Phi + L^{-1}(\Psi+\chi))\Big]^k\Big| \le$$
$$\le \sum_{k=2}^{L^4}\binom{L^4}{k} K_c^{L^4-k}\exp\{-c(1-kL^{-4})\lambda_j^{1/2}\Psi^2\} K^k\lambda_j^{k/2}$$
$$\Big[\int d\mu_\gamma(\Phi)\exp\{-c\lambda_j^{1/2}(\Phi + L^{-1}\Psi)^2\}\Big]^k \le$$
$$\le \sum_{k=2}^{L^4}\binom{L^4}{k} K_c^{L^4-k}(\lambda_j^{1/2}\widetilde{K})^k \exp\{-c(1-kL^{-4})\lambda_j^{1/2}\Psi^2 - \frac{c}{2}kL^{-2}\Psi^2\} \le$$
$$\le K_c^{L^4}\sum_{k=2}^{L^4}\binom{L^4}{k}(\lambda_j^{1/2}\widetilde{K}K_c^{-1})^k \exp\{-c\lambda_j^{1/2}\Psi^2\} \le$$
$$\le K_c^{L^4}\big[(1+\lambda_j^{1/2}\widetilde{K}K_c^{-1})^{L^4} - 1 - L^4\lambda_j^{1/2}\widetilde{K}K_c^{-1}\big]\exp\{-c\lambda_j^{1/2}\Psi^2\} \le$$
$$\le K''\lambda_j\exp\{-c\lambda_j^{1/2}\Psi^2\}.$$ √

The sum of all terms of the representation (1-28) for $R_{j-1}$ which contains at least two $R_j$-

factors or no $R_j$-factor at all is

$$\delta R_{j-1}(\Psi) \equiv \delta Z_j^{rel}(\Psi) + \Big[\int d\mu_\gamma(\Phi) Z_j^{rel}(\Phi + L^{-1}\Psi)\Big]^{L^4} (\exp\{-e_{j-1}\} - 1) +$$
$$+ \Big\{\sum_{k=2}^{L^4} \binom{L^4}{k} \Big[\int d\mu_\gamma(\Phi) Z_j^{rel}(\Phi + L^{-1}\Psi)\Big]^{L^4-k} \Big[\int d\mu_\gamma(\Phi) R_j(\Phi + L^{-1}\Psi)\Big]^k\Big\} \exp\{-e_{j-1}\} +$$
$$+ \int_0^1 ds \partial_s L^4 \Big[\int d\mu_{s\gamma}(\Phi) Z_j^{rel}(\Phi + L^{-1}\Psi)\Big]^{L^4-1} \Big[\int d\mu_\gamma(\Phi) R_j(\Phi + L^{-1}\Psi)\Big] \exp\{-e_{j-1}\} +$$
$$+ L^4 Z_j^{rel}(L^{-1}\Psi)^{L^4-1} R_j(L^{-1}\Psi)(\exp\{-e_{j-1}\} - 1). \tag{1-41}$$

Then we have

$$R_{j-1}(\Psi) = R_{j-1}^{div}(\Psi) + \delta R_{j-1}(\Psi) \tag{1-42}$$

with

$$R_{j-1}^{div}(\Psi) \equiv L^4 Z_j^{rel}(L^{-1}\Psi)^{L^4-1} R_j(L^{-1}\Psi). \tag{1-43}$$

$R_{j-1}^{div}$ is the part of $R_{j-1}$ which contains one and only one $R_j$-factor. $R_{j-1}^{div}$ is the most "dangerous" part for recursive bounds. It can only be estimated by using that $R_j(\Psi)$ is of order $O(\Psi^6)$. Application of lemmata 8,9,10 and 11 yields

**Lemma 12.** *There exists a constant $K$ and $c$ such that if $|m_j^2| \le c_m\lambda_j$, $|\lambda_j - \lambda_{j-1}| \le O(\lambda_j^2)$ and*

$$|R_j(\Psi + \chi)| \le K\lambda_j^{1/2} \exp\{-c\lambda_j^{1/2}\Psi^2\} \tag{1-24}$$

*for all $\Psi \in \mathbf{R}$, $\chi \in \mathbf{C}$, $|\chi| \le \lambda_j^{-1/4} c_3$, then we have*

$$|\delta R_{j-1}(\Psi + \chi)| \le \frac{1}{2} K\lambda_{j-1}^{1/2} \exp\{-c\lambda_{j-1}^{1/2}\Psi^2\}. \tag{1-44}$$

*for all $j \le 0$.*

It remains to estimate $R_{j-1}^{div}$. For that we will use the following renormalization conditions

$$R_j(\Psi)|_{\Psi=0} = \frac{\partial^2}{\partial\Psi^2} R_j(\Psi)|_{\Psi=0} = \frac{\partial^4}{\partial\Psi^4} R_j(\Psi)|_{\Psi=0} = 0. \tag{1-45}$$

The next lemma shows that the bound (1-24) on $R_j$ iterates under the renormalization group equation.

**Lemma 13.** *There exists a constant $K$ and $c$ such that if $|m_j^2| \le c_m\lambda_j$, $|\lambda_j - \lambda_{j-1}| \le O(\lambda_j^2)$ and*

$$|R_j(\Psi + \chi)| \le K\lambda_j^{1/2} \exp\{-c\lambda_j^{1/2}\Psi^2\} \tag{1-24}$$

*for all $\Psi \in \mathbf{R}$, $\chi \in \mathbf{C}$, $|\chi| \le \lambda_j^{-1/4} c_3$, then we have for $\lambda$ small and $L$ large enough*

$$|R_{j-1}(\Psi + \chi)| \le K\lambda_{j-1}^{1/2} \exp\{-c\lambda_{j-1}^{1/2}\Psi^2\}. \tag{1-46}$$



*for all $j \le 0$.*

*Proof:* We have to show that

$$|R_{j-1}^{div}(\Psi + \chi)| \le \frac{1}{2} K\lambda_{j-1}^{1/2} \exp\{-c\lambda_{j-1}^{1/2}\Psi^2\}. \tag{1-47}$$

Then lemma 12 concludes the assertion. By Taylor expansion and eqs. (1-45)

$$R_j(L^{-1}(\Psi + \chi)) = \exp\{-\frac{1}{3} cL^{-2}\lambda_{j-1}^{1/2}(\Psi + \chi)^2\} \frac{1}{5!} L^{-6}(\Psi + \chi)^6$$
$$\int_0^1 ds(1-s)^5 \frac{\partial^6}{\partial\widetilde{\chi}^6} \Big\{\exp\{\frac{1}{3} c\lambda_{j-1}^{1/2}(sL^{-1}(\Psi + \chi) + \widetilde{\chi})^2\} R_j(sL^{-1}(\Psi + \chi) + \widetilde{\chi})\Big\}\Big|_{\widetilde{\chi}=0}. \tag{1-48}$$

Cauchy's inequality yields, $|\chi| \le \lambda_j^{-1/4} c_3$,

$$|R_j(L^{-1}(\Psi + \chi))| \le \exp\{-\frac{1}{3} cL^{-2}\lambda_{j-1}^{1/2} Re(\Psi + \chi)^2\} L^{-6} |\Psi + \chi|^6 c_3^{-6}$$
$$(1 - L^{-4})^{-6} \lambda_j^{3/2} \exp\{c(1 - L^{-4})^2 c_3^2 + c\lambda_{j-1}^{1/2}|\chi|^2\}. \tag{1-49}$$

Lemma 4 gives for all $\widetilde{c} \ge 0$

$$|Z_j^{rel}(L^{-1}(\Psi + \chi))| \le \exp\{-\widetilde{c}L^{-2}\lambda_j^{1/2}\Psi^2 + c_m\lambda_j^{1/2} L^{-2} c_3 +$$
$$+ 6c_3^4 L^{-4} + \frac{1}{96}(48\widetilde{c} + c_m\lambda_l^{1/2})^2\}. \tag{1-50}$$

(1-43),(1-49) and (1-50) imply

$$|R_{j-1}^{div}(\Psi + \chi)| \le L^{-2}(1 - L^{-4})^{-6}\lambda_j^{3/2} c_3^{-6} |\Psi + \chi|^6$$
$$\exp\Big\{-\widetilde{c}L^{-2}(L^4 - 1)\lambda_j^{1/2}\Psi^2 - \frac{1}{6} cL^{-2}\lambda_{j-1}^{1/2}\Psi^2 + [c_m\lambda_j^{1/2} L^{-2} c_3 +$$
$$+ 6c_3^4 L^{-4} + \frac{1}{96}(48\widetilde{c} + c_m\lambda_j^{1/2})^2](L^4 - 1) + c(1 - L^{-4})^2 c_3^2 + cc_3^2\Big\}.$$

Choose $\widetilde{c} = qc$ with $q = \frac{L^2}{L^4-1}(2 - \frac{L^{-2}}{6})$. Then we have $q \le 3L^{-2}$ and

$$\widetilde{c}L^{-2}(L^4 - 1) + \frac{1}{6} cL^{-2} = 2c.$$

Thus

$$|R_{j-1}^{div}(\Psi + \chi)| \le L^{-2}(1 - L^{-4})^{-6}\lambda_j^{3/2} c_3^{-6} \exp\{Q\} |\Psi + \chi|^6 \exp\{-c\lambda_{j-1}^{1/2}|\Psi + \chi|^2\} \exp\{-c\lambda_{j-1}^{1/2}\Psi^2\}$$

with

$$Q = \big[c_m\lambda_j^{1/2} L^{-2} c_3 + 6c_3^4 L^{-4} + \frac{1}{96}(3L^{-2}c + c_m\lambda_j^{1/2})^2\big]^2 (L^4 - 1) +$$
$$+ c(1 - L^{-4})^2 c_3^2 + 2cc_3^2.$$

For $L$ large enough we see that our assertion (1-47) holds. $\surd$

We have proven by induction proposition 1 and

COROLLARY 14. *For $\lambda$ small and $L$ large enough there exist positive constants $c, c_3, K$ such that*

$$|R_j(\Psi+\chi)| \le K\lambda_j^{1/2}\exp\{-c\lambda_j^{1/2}\Psi^2\} \qquad (1\text{–}24)$$

*for all $\Psi \in \mathbf{R}$, $\chi \in \mathbf{C}$, $|\chi| \le \lambda_j^{-1/4}c_3$, and all $j \le 0$.*

The next section will show that the supposition of large $L$ is not necessary for the bounds.

### 3.2 Hierarchical Multigrid Expansions

The iteration of the renormalization group equation for $R_j$ leads to an expansion on the multigrid $\Lambda$. This expansion is a sum over all subsets of the multigrid consisting of blocks which fit into a block of $\Lambda_j$. Terms for subsets which contain large and small blocks are supressed by powers of $L^{-1}$. The dominant contributions in this expansion are coming from subsets with few large blocks.

In the following we will rewrite the renormalization group equations in terms of base space variables. Consider the multigrid $\Lambda = \Lambda_0 + \Lambda_{-1} + \Lambda_{-2} + \dots$ with base space $a\mathbf{Z}^d$. The renormalization group equations are[1]

$$Z_{j-1}(x|\Psi) = \int d\widetilde{\mu}_{v_x^j}(\Phi) Z_j(\overline{x}|\Phi+\Psi) \qquad (2\text{–}1)$$

for all $x \in \Lambda_{j-1}$ with ($\chi_y$ = characteristic function of $y$)

$$d\widetilde{\mu}_{v_x^j}(\Phi) \equiv d\mu_{v_x^j}(\Phi)\exp\{-e_{j-1}\} \qquad (2\text{–}2a)$$

$$v_x^j(z_1,z_2) \equiv \sum_{y\subseteq x} \gamma a_j^{2-d}\chi_y(z_1)\chi_y(z_2), \quad z_1, z_2 \in base \qquad (2\text{–}2b)$$

and

$$e_{j-1} \equiv \ln \int d\mu_{v_x^j}(\Phi) Z_j(\overline{x}|\Phi). \qquad (2\text{–}3)$$

The renormalization group starts with

$$Z_0(y|\Psi) = \exp\{-V_0^{rel}(y|\Psi)\} \qquad (2\text{–}4)$$

for all $y \in \Lambda_0$. Suppose that $V_j^{rel}(y|\Psi)$ is defined for all $j \le 0$ and $y \in \Lambda_j$. For the case of $\Phi^4$–theory we choose

$$V_j^{rel}(y|\Psi) \equiv \frac{1}{2}\int_{z\subseteq y} m_j^2 a_j^{-2}\Psi(z)^2 + \frac{\lambda_j}{4!}\int_{z\subseteq y}\Psi(z)^4. \qquad (2\text{–}5)$$

$\lambda_j$ and $m_j^2$ are dimensionsless. Define irrelevant activities for $y \in \Lambda_j$ by

$$R_j(y|\Psi) \equiv Z_j(y|\Psi) - Z_j^{rel}(y|\Psi) \qquad (2\text{–}6)$$

[1] $Z_j(\overline{x}|\Psi) \equiv \prod_{y\subseteq x} Z_j(y|\Psi)$



where

$$Z_j^{rel}(y|\Psi) \equiv \exp\{-V_j^{rel}(y|\Psi)\}. \qquad (2\text{–}7)$$

$R_j$ obeys the following renormalization group equations

$$\begin{aligned} R_{j-1}(x|\Psi) = {} & \sum_{y\subseteq x} Z_j^{rel}(\overline{x}-y|\Psi)R_j(y|\Psi) + \delta Z_{j-1}^{rel}(x|\Psi) + \\ & + \sum_{y\subseteq x}\int_0^1 ds\partial_s \int d\widetilde{\mu}^s_{v_x^j}(\Phi) Z_j^{rel}(\overline{x}-y|\Phi+\Psi)R_j(y|\Phi+\Psi) + \\ & + \sum_{\substack{P:\ P\subseteq \overline{x} \\ |P|\ge 2}} \int d\widetilde{\mu}_{v_x^j}(\Phi) Z_j^{rel}(\overline{x}-P|\Phi+\Psi)\prod_{y\in P} R_j(y|\Phi+\Psi) \end{aligned} \qquad (2\text{–}8)$$

where

$$\delta Z_{j-1}^{rel}(x|\Psi) \equiv -Z_{j-1}^{rel}(x|\Psi) + \int d\widetilde{\mu}_{v_x^j}(\Phi) Z_j^{rel}(\overline{x}|\Phi+\Psi) \qquad (2\text{–}9)$$

and

$$d\widetilde{\mu}^s_{v_x^j}(\Phi) \equiv d\mu_{s v_x^j}(\Phi)\exp\{-s e_{j-1}\}. \qquad (2\text{–}10)$$

Corollary 14 of the last chapter expressed in terms of base space variables gives

COROLLARY 1. *For $\lambda$ small and $L$ large there exist positive constants $c, c_3, K$ (not dependent on $j$ and $\lambda$) such that*

$$|R_j(y|\Psi+\chi)| \le K\lambda_j^{1/2}\exp\{-c\lambda_j^{1/2}a_j^{-2}\int_{z\subseteq y}\Psi^2(z)\} \qquad (2\text{–}11)$$

*for all $y \in \Lambda_j$, $\Psi: base \to \mathbf{R}$, $\chi: base \to \mathbf{C}$, $|\chi(z)| \le c_3 a_j^{-1}\lambda_j^{-1/4}$ $\forall z \in base$, and $\Psi, \chi$ are constant on blocks of $\Lambda_j$.*

$R_j$ obeys the renormalization conditions

$$R_j(y|\Psi)|_{\Psi=0} = \mathcal{D}_y^2 R_j(y|\Psi)|_{\Psi=0} = \mathcal{D}_y^4 R_j(y|\Psi)|_{\Psi=0} = 0 \qquad (2\text{–}12)$$

with

$$\mathcal{D}_y^n \equiv \frac{1}{Vol(y)}\int_{z_1\in y}\int_{z_2,\dots,z_n}\frac{\delta^n}{\delta\Psi(z_1)\dots\delta\Psi(z_n)},$$

for all $y \in \Lambda_j$. As we have seen in the last section the main contribution in the renormalization group equation for $R_{j-1}$ comes from the term which contains one and only one $R_j$-factor. We denote $R_{j-1}$ minus this term by $R_{j-1}^{conv}$. Thus for $x \in \Lambda_{j-1}$ define

$$R_{j-1}^{conv}(x|\Psi) \equiv R_{j-1}(x|\Psi) - \sum_{y\subseteq x} Z_j^{rel}(\overline{x}-y|\Psi)R_j(y|\Psi). \qquad (2\text{–}13)$$

In the sequel a renormalization group equation for $R_{j-1}^{conv}$ will be derived and $R_{j-1}$ will be expressed by $R_j^{conv}$ for $k \ge j-1$. For $0 \ge k > j$, $y \in \Lambda_k$, define

$$Z_{k,j}^{rel}(y|\Psi) \equiv Z_{j+1}^{rel}([y]_j - [y]_{j+1}|\Psi) Z_{j+2}^{rel}([y]_{j+1} - [y]_{j+2}|\Psi) \ \dots \ Z_k^{rel}([y]_{k-1} - y|\Psi) \qquad (2\text{–}14)$$

where $[y]_j$ is the block in $\Lambda_j$ which contains the smaller block $y \in \Lambda_k$, i.e.

$$[y]_j \equiv \{x \in \Lambda_j | \; \exists z \in base: \; z \underline{\in} y \text{ and } z \underline{\in} x\}.$$

$Z^{rel}_{k,j}$ obeys the following recursion relations

$$Z^{rel}_{k,j-1}(y|\Psi) = Z^{rel}_j([y]_{j-1} - [y]_j|\Psi) Z^{rel}_{k,j}(y|\Psi) \tag{2-15}$$

and

$$Z^{rel}_{k,j-1}(y|\Psi) = Z^{rel}_{k-1,j}([y]_{k-1}|\Psi) Z^{rel}_k([y]_{k-1} - y|\Psi). \tag{2-16}$$

Using the renormalization group equation (1-8) and the definition for $Z^{rel}_{k,j}$ we see that $R^{conv}_{j-1}$ obeys the following recursion relation for $x \in \Lambda_{j-1}$

$$R^{conv}_{j-1}(x|\Psi) = \delta Z^{rel}_{j-1}(x|\Psi) + \sum_{k=j}^{-1} \sum_{\substack{y \in \Lambda_k: \\ [y]_{j-1} = x}} \int_0^1 ds \partial_s \int d\tilde{\mu}^s_{v^j_*}(\Phi) Z^{rel}_{k,j-1}(y|\Phi + \Psi)$$
$$R^{conv}_k(y|\Phi + \Psi) + \sum_{n \geq 2} \frac{1}{n!} \sum_{k_1,\dots,k_n = j}^{-1} \sum_{\substack{y_1 \in \Lambda_{k_1},\dots,y_k \in \Lambda_{k_n}: \\ [y_1]_{j-1} = \dots = [y_n]_{j-1} = x \\ [y_1]_j = \dots = [y_n]_j \text{ distinct}}}$$
$$\int d\tilde{\mu}_{v^j_*}(\Phi) Z^{rel}_j(\overline{x} - \sum_{i=1}^n [y_i]_j|\Phi + \Psi) \prod_{i=1}^n [Z^{rel}_{k_i,j-1}(y_i|\Phi + \Psi) R^{conv}_{k_i}(y_i|\Phi + \Psi)]. \tag{2-17}$$

$R_{j-1}$ can be expressed in terms of $R^{conv}_k, k \geq j$, by iteration of

$$R_{j-1}(x|\Psi) = R^{conv}_{j-1}(x|\Psi) + \sum_{y \underline{\in} x} Z^{rel}_j(\overline{x} - y|\Psi) R_j(y|\Psi). \tag{2-18}$$

The result of this iteration reads

$$R_{j-1}(x|\Psi) = \sum_{k=j-1}^{-1} \sum_{\substack{y \in \Lambda_k: \\ [y]_{j-1} = x}} R_{k,j-1}(y|\Psi) \tag{2-19}$$

with

$$R_{k,j-1}(y|\Psi) = \begin{cases} R^{conv}_{j-1}(y|\Psi), & \text{for } k = j-1 \\ Z^{rel}_{k,j-1}(y|\Psi) R^{conv}_k(y|\Psi), & \text{for } -1 \geq k > j-1 \end{cases} \tag{2-20}$$

for $y \in \Lambda_k$. $R^{conv}_j$ and $R_{k,j}$ obeys the renormalization conditions

$$R^{conv}_j(y|\Psi)|_{\Psi=0} = \mathcal{D}^2_y R^{conv}_j(y|\Psi)|_{\Psi=0} = \mathcal{D}^4_y R^{conv}_j(y|\Psi)|_{\Psi=0} = 0 \tag{2-21}$$

$$R_{k,j}(y'|\Psi)|_{\Psi=0} = \mathcal{D}^2_y R_{k,j}(y'|\Psi)|_{\Psi=0} = \mathcal{D}^4_y R_{k,j}(y'|\Psi)|_{\Psi=0} = 0 \tag{2-22}$$



for all $y \in \Lambda_j, y' \in \Lambda_k$. By eq. (17) follows immediately the recursion relation for $R_{k,j-1}(x|\Psi)$, with $x \in \Lambda_k$

$$R_{k,j-1}(x|\Psi) = Z^{rel}_{k,j-1}(x|\Psi) \Big\{ \delta Z^{rel}_{j-1}(x|\Psi) +$$
$$\sum_{h=k+1}^{-1} \sum_{\substack{y \in \Lambda_h: \\ [y]_k = x}} \int_0^1 ds \partial_s \int d\tilde{\mu}^s_{v^j_*}(\Phi) R_{h,k}(y|\Phi + \Psi) +$$
$$\sum_{n \geq 2} \frac{1}{n!} \sum_{h_1,\dots,h_n = k+1}^{-1} \sum_{\substack{y_1 \in \Lambda_{h_1},\dots,y_k \in \Lambda_{h_n}: \\ [y_1]_k = \dots = [y_n]_k = x \\ [y_1]_{k+1} = \dots = [y_n]_{k+1} \text{ distinct}}} \int d\tilde{\mu}_{v^j_*}(\Phi)$$
$$Z^{rel}_j(\overline{x} - \sum_{i=1}^n [y_i]_{k+1}|\Phi + \Psi) \prod_{i=1}^n [R_{h_i,k}(y_i|\Phi + \Psi)] \Big\}. \tag{2-23}$$

The following stability bound on $Z^{rel}_{k,j-1}$ is required for the bounds on the irrelevant activities.

LEMMA 2. *For small $\lambda$ and $k \geq j$, $y \in \Lambda_k$, $\Psi$ : $base \to \mathbf{R}$, $\chi$ : $base \to \mathbf{C}$, $|\chi(z)| \leq c_3 a^{-1}_{j-1} \lambda^{-1/4}_{j-1}$ $\forall z \in base$, there exists constants $c, c_0$ (not dependent on $j, \lambda$) such that*

$$|Z^{rel}_{k,j-1}(y|\Psi + \chi)| \leq \exp\{-c a^{-2}_{j-1} \lambda^{1/2}_{j-1} \int_{z \underline{\in} [y]_{j-1} - y} \Psi^2(z) + (1 - L^{-4}) c_0\}. \tag{2-24}$$

*Proof :* We have

$$Re\ (\Psi + \chi)^2(z) = (\Psi + Re\ \chi)^2(z) - (Im\ \chi)^2(z) \tag{2-25}$$

$$Re\ (\Psi + \chi)^4(z) = (\Psi + Re\ \chi)^4(z) - 6(\Psi + Re\ \chi)^2(z)(Im\ \chi)^2(z) + (Im\ \chi)^4(z) \tag{2-26}$$

and

$$\frac{1}{2}(\Psi + Re\ \chi)^4(z) \geq -\frac{1}{2}c^2 + c(\Psi + Re\ \chi)^2(z) \tag{2-27}$$

$$-6(\Psi + Re\ \chi)^2(z)(Im\ \chi)^2(z) \geq -\frac{1}{2}(\Psi + Re\ \chi)^4(z) - 36(Im\ \chi)^4(z). \tag{2-28}$$

Eqs. (26),(27) and (28) gives

$$Re\ (\Psi + \chi)^4(z) \geq \frac{1}{2}(\Psi + Re\ \chi)^4(z) - 35(Im\ \chi)^4(z) \geq$$
$$\geq c(\Psi + Re\ \chi)^2(z) - 35(Im\ \chi)^4(z) - \frac{1}{2}c^2. \tag{2-29}$$

Thus, using $|m^2_l| \leq c_m \lambda_l$, (25) and (29)

$$Re\ \Big\{\frac{\lambda_l}{4!}(\Psi + \chi)^4(z) + \frac{m^2_l a^{-2}_l}{2}(\Psi + \chi)^2(z)\Big\} \geq (\frac{\lambda_l c}{4!} - \frac{1}{2}c_m a^{-2}_l \lambda_l)(\Psi + Re\ \chi)^2(z) -$$
$$- \frac{35\lambda_l}{4!}(Im\ \chi)^4(z) - \frac{\lambda_l c^2}{4!2} - \frac{1}{2}c_m a^{-2}_l \lambda_l (Im\ \chi)^2(z). \tag{2-30}$$

The lower bound (30) implies for $y \in \Lambda_l$ and $c \geq 4! c_m a_l^{-2}$, $|\chi(z)| \leq c_3 a_{j-1}^{-1} \lambda_{j-1}^{-1/4}$

$$Re \; \{ \frac{\lambda_l}{4!} \int_{z \underline{\in} y} (\Psi + \chi)^4(z) + \frac{m_l^2 a_l^{-2}}{2} \int_{z \underline{\in} y} (\Psi + \chi)^2(z) \} \geq \frac{c \lambda_l}{4!2} \int_{z \underline{\in} y} \Psi^2(z) -$$

$$- \frac{35}{4!} c_3^4 \frac{\lambda_l}{\lambda_{j-1}} (\frac{a_l}{a_{j-1}})^4 - \frac{\lambda_l c^2}{4!2} a_l^4 - \frac{1}{2} c_m c_3^2 \frac{\lambda_l}{\lambda_{j-1}^{1/2}} (\frac{a_l}{a_{j-1}})^2 - \frac{c \lambda_l}{4!2} \frac{c_3^2 \lambda_{j-1}^{-1/2}}{a_{j-1}^2}. \tag{2-31}$$

We have used $(\Psi + Re\; \chi)^2(z) \geq \frac{1}{2}\Psi^2(z) - (Re\; \chi)^2(z)$. Thus, replacing $c$ by $4!2 a_{j-1}^{-2} \lambda_{j-1}^{-1/2} c$ in (31) we obtain

$$\sum_{l=j}^{k} Re \; \{ \frac{\lambda_j}{4!} \int_{z \underline{\in} [y]_{l-1} - [y]_l} (\Psi + \chi)^4(z) + \frac{m_l^2 a_l^{-2}}{2} \int_{z \underline{\in} [y]_{l-1} - [y]_l} (\Psi + \chi)^2(z) \} \geq$$

$$\geq c \lambda_{j-1}^{1/2} a_{j-1}^{-2} \int_{z \underline{\in} [y]_{j-1} - y} \Psi^2(z) - (L^4 - 1) \sum_{l=j}^{k} [ \frac{35}{4!} c_3^4 \frac{c_+^\lambda}{c_-^\lambda} \frac{|j-2|}{|l-1|} L^{4(j-1-l)} -$$

$$- 4!2 c^2 \frac{c_+^\lambda}{c_-^\lambda} \frac{|j-2|}{|l-1|} L^{4(j-1-l)} - \frac{1}{2} c_m c_3^2 \lambda_l^{1/2} (\frac{c_+^\lambda}{c_-^\lambda})^{1/2} (\frac{|j-2|}{|l-1|})^{1/2} L^{2(j-1-l)} -$$

$$- c c_3^2 \frac{c_+^\lambda}{c_-^\lambda} \frac{|j-2|}{|l-1|} L^{4(j-1-l)} ] \geq c a_{j-1}^{-2} \lambda_{j-1}^{1/2} \int_{z \underline{\in} [y]_{j-1} - y} \Psi^2(z) - (1 - L^{-4}) c_0. \tag{2-32}$$

This completes the proof. √

The next lemma presents a bound on $R_{k,j-1}$.

LEMMA 3. *For small $\lambda$ there exist constants $K, c$ (not dependent on $j$ and $\lambda$) such that for all fields $\Psi$ : base $\to$ **R**, $\chi$ : base $\to$ **C**, $|\chi(z)| \leq c_3 a_{j-1}^{-1} \lambda_{j-1}^{-1/4}$ $\forall z \in$ base, and $\Psi, \chi$ are constant on blocks of $\Lambda_{j-1}$ and $-1 \geq k \geq j$, $y \in \Lambda_k$,*

$$|R_{k,j-1}(y|\Psi + \chi)| \leq K L^{6(j-1-k)} (\frac{\lambda_k}{\lambda_{j-1}})^{3/2} \lambda_k^{1/2} \exp\{ -c a_{j-1}^{-2} \lambda_{j-1}^{1/2} \int_{z \underline{\in} [y]_{j-1}} \Psi^2(z) \}. \tag{2-33}$$

*Proof :* Suppose that

$$|R_k^{conv}(y|\Psi + \chi')| \leq K_{conv} \lambda_k^{1/2} \exp\{ -c \lambda_k^{1/2} a_k^{-2} \int_{z \underline{\in} y} \Psi^2(z) \} \tag{2-34}$$

holds for all $\chi'$ : base $\to$ **C** with $|\chi'(z)| \leq c_3 a_k^{-1} \lambda_k^{-1/4}$ $\forall z \in$ base. By lemma 2 we can find constants $c, c_0$ (not dependent on $j$ and $\lambda$) such that for $y \in \Lambda_k$

$$|Z_{k,j-1}^{rel}(y|\Psi + \chi)| \leq \exp\{ -2c a_{j-1}^{-2} \lambda_{j-1}^{1/2} \int_{z \underline{\in} [y]_{j-1} - y} \Psi^2(z) + (1 - L^{-4}) c_0 \}. \tag{2-35}$$

Define

$$\widetilde{R}_k^{conv}(y|\Psi) = \exp\{ 2c a_{j-1}^{-2} \lambda_{j-1}^{1/2} \int_{z \underline{\in} y} \Psi^2(z) \} R_k^{conv}(y|\Psi). \tag{2-36}$$



Then $\widetilde{R}_k^{conv}$ fulfils the renormalization conditions. Therefore

$$R_k^{conv}(y|\Psi + \chi) = \exp\{ -2c a_{j-1}^{-2} \lambda_{j-1}^{1/2} \int_{z \underline{\in} y} (\Psi + \chi)^2(z) \}$$

$$\frac{1}{5!} \int_0^1 ds (1-s)^5 (\Psi + \chi)^6 \frac{\partial^6}{\partial \alpha^6} \widetilde{R}_k^{conv}(y|s(\Psi + \chi) + \alpha)|_{\alpha = 0}. \tag{2-37}$$

Furthermore there exists a constant $q$ such that

$$|\lambda_{l-1}^{1/4} - \lambda_l^{1/4}| = |(\lambda_{l-1} - \lambda_l)(\lambda_{l-1}^{1/2} + \lambda_l^{1/2})^{-1} (\lambda_{l-1}^{1/4} + \lambda_l^{1/4})^{-1}| \leq q \lambda_{l-1}^{5/4}.$$

Thus

$$|\lambda_l^{-1/4} - \lambda_{l-1}^{-1/4}| = |\lambda_{l-1}^{1/4} - \lambda_l^{1/4}| / |\lambda_l \lambda_{l-1}|^{1/4} \leq q \lambda_{l-1}^{3/4}. \tag{2-38}$$

Furthermore

$$a_k^{-1} \lambda_k^{-1/4} - a_{j-1}^{-1} \lambda_{j-1}^{-1/4} = \sum_{l=j}^{k} (a_l^{-1} \lambda_l^{-1/4} - a_{l-1}^{-1} \lambda_{l-1}^{-1/4}) =$$

$$= \sum_{l=j}^{k} [ a_{l-1}^{-1} (\lambda_l^{-1/4} - \lambda_{l-1}^{-1/4}) + (a_l^{-1} - a_{l-1}^{-1}) \lambda_l^{-1/4} ] \geq$$

$$\geq \sum_{l=j}^{k} [ -q \lambda_l L^{-1} a_l^{-1} \lambda_l^{-1/4} + (1 - L^{-1}) a_l^{-1} \lambda_l^{-1/4} ] \geq$$

$$\geq (1 - \frac{1 + q\lambda_k}{L}) a_k^{-1} \lambda_k^{-1/4}.$$

Thus for $|\alpha| \leq c_3 (1 - \frac{1 + q\lambda_k}{L}) a_k^{-1} \lambda_k^{-1/4}$ we obtain

$$|s\chi + \alpha| \leq c_3 (a_{j-1}^{-1} \lambda_{j-1}^{-1/4} + a_k^{-1} \lambda_k^{-1/4} - a_{j-1}^{-1} \lambda_{j-1}^{-1/4}) = c_3 a_k^{-1} \lambda_k^{-1/4}. \tag{2-39}$$

Using Cauchy's inequality and (34) we find (since $L \geq 2$)

$$|\frac{\partial^6}{\partial \alpha^6} \widetilde{R}_k^{conv}(y|s(\Psi + \chi) + \alpha)|_{\alpha = 0}| \leq 6! c_3^{-6} (1 - \frac{1 + q\lambda_k}{L})^{-6} a_k^6 \lambda_k^{3/2}$$

$$K_{conv} \lambda_k^{1/2} \exp\{ -c \lambda_k^{1/2} a_k^{-2} \int_{z \underline{\in} y} s^2 \Psi^2(z) \} \exp\{ 4c a_{j-1}^{-2} \lambda_{j-1}^{1/2} a_k^2 c_3^2 \lambda_k^{-1/2} \}. \tag{2-40}$$

By (37) and (40) follows (for $\lambda$ small such that $1 + q\lambda_k \leq (1 - 2^{-1/6}) L$

$$|Z_{k,j-1}^{rel}(y|\Psi + \chi) R_k^{conv}(y|\Psi + \chi)| \leq 2 |\Psi + \chi|^6 a_k^6 \lambda_k^{3/2} c_3^{-6} K_{conv} \lambda_k^{1/2}$$

$$\exp\{ 4c c_3^2 (\frac{a_k}{a_{j-1}})^2 \} \exp\{ -2c \lambda_{j-1}^{1/2} a_{j-1}^{-2} \int_{z \underline{\in} [y]_{j-1}} \Psi^2(z) \}. \tag{2-41}$$

We have

$$|\Psi + \chi|^6 \leq 2^5 (|\Psi|^6 + c_3^6 a_{j-1}^{-6} \lambda_{j-1}^{-3/2}) \tag{2-42}$$

and

$$|\Psi|^6 \le 3!c^{-3}\lambda_{j-1}^{-3/2}a_{j-1}^{-6}\exp\{c\lambda_{j-1}^{1/2}a_{j-1}^{-2}\int_{z\in[y]_{j-1}}\Psi^2(z)\}. \tag{2–43}$$

By (41),(42) and (43) follows

$$|Z^{rel}_{k,j-1}(y|\Psi+\chi)R^{conv}_k(y|\Psi)| \le 2K_{conv}L^{6(j-1-k)}\Big(\frac{\lambda_k}{\lambda_{j-1}}\Big)^{3/2}\lambda_k^{1/2}(3!c^{-3}c_3^{-6}+1)$$
$$\exp\{4cc_3^2\Big(\frac{a_k}{a_{j-1}}\Big)^{-2}+(1-L^{-4})c_0\}\exp\{-ca_{j-1}^{-2}\lambda_{j-1}^{1/2}\int_{z\in[y]_{j-1}}\Psi^2(z)\}.$$

This proves the bound (33) on $R_{k,j-1}$ if the bound (34) on $R^{conv}_k$ is supposed. We can prove by induction that the bound (33) holds without using the supposition (34). For $j=-1$ the bound (33) is easily shown. Suppose that

$$|R_{k,l}(y'|\Psi+\chi)| \le KL^{6(l-k)}\Big(\frac{\lambda_k}{\lambda_l}\Big)^{3/2}\lambda_k^{1/2}\exp\{-ca_l^{-2}\lambda_l^{1/2}\int_{z\in[y']_l}\Psi^2(z)\} \tag{2–44}$$

holds for all $y\in\Lambda_k$ and $k,l$ with $-1\ge k\ge l\ge j+1$ and $|\chi'(z)|\le c_3a_l^{-1}\lambda_l^{-1/4}$ $\forall z\in base$ and $\Psi,\chi'$ constant on blocks of $\Lambda_l$. By the representation (19) we find that the bound (44) implies

$$|R_l(y|\Psi+\chi')| \le K'\lambda_l^{1/2}\exp\{-c\lambda_l^{1/2}a_l^{-2}\int_{z\in y}\Psi^2(z)\}$$

This bound and the definition of $R^{conv}_k$ implies the bound (33). This finishes the proof. √

Let us remark that the supposition of large $L$ is not required.

Lemma 4. *There exists a constant $q_\lambda$ (not dependent on $j$) such that*

$$\lambda_k \le q_\lambda|k-j|\lambda_j \tag{2–45}$$

*for all $j<k\le 0$.*

*Proof :* Proposition 1 implies

$$\frac{\lambda_k}{\lambda_j} \le \frac{c_+^\lambda}{c_-^\lambda}\frac{|j-1|}{|k-1|}.$$

The assertion (45) follows from $|j-1|/|k-1|\le 2|k-j|$. √

$R^{conv}_{j-1}$ and $R_j$ are related by

$$R^{conv}_{j-1}(x|\Psi) = \delta Z^{rel}_{j-1}(x|\Psi)+\sum_{y\in x}\int_0^1 ds\partial_s\int d\tilde\mu^s_{v_*^j}(\Phi)Z^{rel}_j(\bar x-y|\Phi+\Psi)R_j(y|\Phi+\Psi)+$$
$$+\sum_{\substack{P:\,P\subseteq\bar x\\|P|\ge 2}}\int d\tilde\mu_{v_*^j}(\Phi)Z^{rel}_j(\bar x-P|\Phi+\Psi)\prod_{y\in P}R_j(y|\Phi+\Psi). \tag{2–46}$$



The bound (34) on $R_{k,j-1}$ and lemma 4 imply

$$|R_{k,j-1}(y|\Psi+\chi)| \le KK_{conv}\lambda_{j-1}^{1/2}|k-(j-1)|^2L^{-6(k-(j-1))}$$
$$\exp\{-ca_{j-1}^{-2}\lambda_{j-1}^{1/2}\int_{z\in[y]_{j-1}}\Psi^2(z)\}. \tag{2–47}$$

As a result of using renormalization conditions a convergence factor $|k-(j-1)|^2L^{-6(k-(j-1))}$ is gained. The number of all blocks $y$ of $\Lambda_k$ which are contained in the block $x\in\Lambda_{j-1}$ (i.e. $[y]_{j-1}=x$) is $L^{4(k-(j-1))}$. Thus by (17)

$$\sum_{\substack{y\in\Lambda_k:\\ [y]_{j-1}=x}}|R_{k,j-1}(y|\Psi)| \le KK_{conv}\lambda_{j-1}^{1/2}|k-(j-1)|^2L^{-2(k-(j-1))}$$
$$\exp\{-ca_{j-1}^{-2}\lambda_{j-1}^{1/2}\int_{z\in[y]_{j-1}}\Psi^2(z)\}. \tag{2–48}$$

Therefore each term of the representation (19) is supressed by a factor $|k-(j-1)|^2L^{-2(k-(j-1))}$ and the dominant contributions for $R_{j-1}$ comes from the largest blocks. We have the following bound for the error in the representation (19) if we truncate the sum up to layer $\Lambda_h$

$$|R_{j-1}(x|\Psi)-\sum_{k=j-1}^{h}\sum_{\substack{y\in\Lambda_k:\\ [y]_{j-1}=x}}R_{k,j-1}(y|\Psi)| \le O(|h-(j-1)|^2L^{-2(h-(j-1))}) \tag{2–49}$$

for all $j-1\le h\le -1$.

In the remaining part of this section we want to represent $R_{j-1}(x|\Psi)$ for $x\in\Lambda_{j-1}$ by an expansion on the multigrid and estimate the terms in the expansion. We start with the definition

$$R^0_{j-1}(x|\Psi) \equiv [Z_{j-1}(x|\Psi)-Z_j(\bar x|\Psi)]-[Z^{rel}_{j-1}(x|\Psi)-Z^{rel}_j(\bar x|\Psi)]. \tag{2–50}$$

for $x\in\Lambda_{j-1}$ and $j\le 0$ and for $y\in\Lambda_0$ define $R^0_0(y|\Psi)\equiv 0$. The renormalization group equation (8) implies

$$R^0_{j-1}(x|\Psi) = \delta Z^{rel}_{j-1}(x|\Psi)+\sum_{\substack{P:\,P\subseteq\bar x\\|P|\ge 2}}\int_0^1 ds\partial_s\int d\tilde\mu^s_{v_*^j}(\Phi)Z^{rel}_j(\bar x-P|\Phi+\Psi)$$
$$\prod_{y\in P}R_j(y|\Phi+\Psi). \tag{2–51}$$

For a subset $Q$ of $\Lambda_{\ge j-1}\equiv\Lambda_0+\Lambda_{-1}+\cdots+\Lambda_{j-1}$ consisting of disjoint blocks, i.e. $|\{y\in Q|z\in y\}|\le 1$ $\forall z\in base$, define recursively

$$\tilde Z^{rel}_0(Q\cap\Lambda_0|\Psi)\equiv 1 \tag{2–52a}$$

and

$$\tilde Z^{rel}_{j-1}(Q|\Psi) \equiv Z^{rel}_j([Q]_{j-1}-[Q]_j|\Psi)\tilde Z^{rel}_j(Q\cap\Lambda_{\ge j}|\Psi) \tag{2–52b}$$

where

$$[Q]_{j-1} \equiv \{x \in \Lambda_{j-1} |\ \exists y \in Q : [y]_{j-1} = x\}.$$

$R^0_{j-1}$ obeys the renormalization conditions

$$R^0_{j-1}(x|\Psi)|_{\Psi=0} = \mathcal{D}^2_y R^0_{j-1}(x|\Psi)|_{\Psi=0} = \mathcal{D}^4_y R^0_{j-1}(x|\Psi)|_{\Psi=0} = 0 \tag{2-53}$$

for all $x \in \Lambda_{j-1}$. The next lemma expresses $R_{j-1}$ in terms of $R^0_k, k \geq j$.

LEMMA 5. *For $x \in \Lambda_{j-1},\ j \leq 0$, we have*[1]

$$R_{j-1}(x|\Psi) = \sum'_{\substack{Q \subseteq \Lambda_{\geq j-1}: \\ [Q]_{j-1}=x}} \widetilde{Z}^{rel}_{j-1}(Q|\Psi) \prod_{y \in Q} R^0_{j_y}(y|\Psi) \tag{2-54}$$

*where the sum $\sum'$ goes over all $Q \subseteq \Lambda_{\geq j-1}$ with $[Q]_{j-1} = x$ and $|\{y \in Q|\ z \underline{\in} y\}| \leq 1$ for all $z \in base$.*

*Proof (by induction) :* We will use that

$$R_{j-1}(x|\Psi) = R^0_{j-1}(x|\Psi) + \sum_{P:\ \emptyset \neq P \subseteq \overline{x}} Z^{rel}_j(\overline{x} - P|\Psi) \prod_{y \in P} R_j(y|\Psi). \tag{2-55}$$

For $x \in \Lambda_{-1}$ we have

$$R_{-1}(x|\Psi) = R^0_{-1}(x|\Psi) = \sum'_{\substack{Q \subseteq \Lambda_{\geq -1}: \\ [Q]_{-1}=x}} \widetilde{Z}^{rel}_{-1}(Q|\Psi) \prod_{y \in Q} R^0_{j_y}(y|\Psi).$$

Suppose that

$$R_j(y|\Psi) = \sum'_{\substack{U \subseteq \Lambda_{\geq j}: \\ [U]_j=y}} \widetilde{Z}^{rel}_j(U|\Psi) \prod_{y \in U} R^0_{j_y}(y|\Psi) \tag{2-56}$$

holds for all $y \in \Lambda_j$. (56) and (55) imply

$$R_{j-1}(x|\Psi) = R^0_{j-1}(x|\Psi) + \sum_{P:\ \emptyset \neq P \subseteq \overline{x}} Z^{rel}_j(\overline{x} - P|\Psi) \prod_{y \in P} [\sum'_{\substack{U \subseteq \Lambda_{\geq j}: \\ [U]_j=x}} \widetilde{Z}^{rel}_j(U|\Psi) \prod_{y' \in U} R^0_{j_{y'}}(y'|\Psi)]. \tag{2-57}$$

[1] $j_y \equiv j$ if $y \in \Lambda_j$



Thus

$$R_{j-1}(x|\Psi) = \sum'_{\substack{Q \subseteq \Lambda_{\geq j-1}: \\ Q=\{x\}}} \widetilde{Z}^{rel}_{j-1}(Q|\Psi) R^0_{j-1}(x|\Psi) + \sum'_{\substack{Q \subseteq \Lambda_{\geq j-1}: \\ [Q]_{j-1}=x, Q \neq \{x\}}} \widetilde{Z}^{rel}_{j-1}(Q|\Psi) \prod_{y \in Q} R^0_{j_y}(y|\Psi) =$$

$$= \sum'_{\substack{Q \subseteq \Lambda_{\geq j-1}: \\ [Q]_{j-1}=x}} \widetilde{Z}^{rel}_{j-1}(Q|\Psi) \prod_{y \in Q} R^0_{j_y}(y|\Psi). \ \surd$$

We will use the notations

$$Q_x \equiv \{y \in Q|\ y \leq x\}, \quad y \leq x \rightleftharpoons \forall z \in base : \textit{if } z \underline{\in} y \textit{ then } z \underline{\in} x$$

and

$$\Psi \in \mathcal{F}_{j-1} \equiv \{\Phi :\ base \to \mathbf{C}|\ \Phi \textit{ is constant on blocks } x \in \Lambda_{j-1}\}$$

for $Q \subseteq \Lambda_{\geq j-1}$ and $x \in \Lambda_{j-1}, y \in \Lambda_k$.

In the following irrelevant activities $\widetilde{R}_{j-1}(Q|\Psi)$ will be defined recursively for all $Q \subseteq \Lambda_{\geq j-1}$ with $Q_x = Q \neq \emptyset,\ x \in \Lambda_{j-1}$. For a field $\Psi : base \to \mathbf{C}$ and a functional $F$ define

$$\mathcal{L}F(\Psi) \equiv F(\Psi)|_{\Psi=0} + \frac{1}{2} \int_{z_1,z_2} \Psi(z_1)\Psi(z_2) \frac{\delta^2}{\delta\Psi(z_1)\delta\Psi(z_2)} F(\Psi)|_{\Psi=0} + \frac{1}{4!} \int_{z_1,\ldots,z_4} \frac{\delta^4}{\delta\Psi(z_1)\ldots\delta\Psi(z_4)} F(\Psi)|_{\Psi=0} \tag{2-58}$$

and

$$\mathcal{L}_x \equiv Z^{rel}_{j-1}(x|\Psi)\mathcal{L}Z^{rel}_{j-1}(x|\Psi)^{-1} \tag{2-59}$$

for $x \in \Lambda_{j-1}$. For $y \in \Lambda_0$ define

$$\widetilde{R}_0(y|\Psi) = 0 \tag{2-60a}$$

and for $Q = \{x\}$ define

$$\widetilde{R}_{j-1}(Q|\Psi) \equiv (1 - \mathcal{L}_x)\delta Z^{rel}_{j-1}(x|\Psi). \tag{2-60b}$$

Suppose that $Q \cap \Lambda_{\geq j} \neq \emptyset$ and $x \in Q$. Then define

$$\widetilde{R}_{j-1}(Q|\Psi) \equiv (1 - \mathcal{L}_x) \int_0^1 ds \partial_s \int d\widetilde{\mu}^s_{v^j_x}(\Phi) Z^{rel}_j(\overline{x} - P|\Phi + \Psi)\widetilde{R}_j(Q'_y|\Phi + \Psi) \tag{2-60c}$$

where

$$P \equiv \{y \in \Lambda_j | Q'_y \neq \emptyset\}, \quad Q' \equiv Q - \{x\}.$$

For $Q \subseteq \Lambda_{\geq j}$ define

$$\widetilde{R}_{j-1}(Q|\Psi) \equiv Z^{rel}_j(\overline{x} - P|\Phi + \Psi)\widetilde{R}_j(Q'_y|\Phi + \Psi). \tag{2-60d}$$

The operator $(1 - \mathcal{L}_x)$ is used in the definition of $\widetilde{R}_{j-1}$ such that $\widetilde{R}_{j-1}$ obeys the renormalization conditions. The multigrid expansion for $R_{j-1}$ is given in the following lemma.

LEMMA 6. *For* $x \in \Lambda_{j-1},\ j \le 0$, *and* $\Psi \in \mathcal{F}_{j-1}$

$$R_{j-1}(x|\Psi) = \sum_{\substack{Q \subseteq \Lambda_{\ge j-1}: \\ Q_x = Q}} \widetilde{R}_{j-1}(Q|\Psi). \tag{2-61}$$

*Proof (by induction) :* Since $R_0 \equiv 0$ we have for $x \in \Lambda_{-1},\ \Psi \in \mathcal{F}_{-1}$

$$\widetilde{R}_{j-1}(Q|\Psi) = \delta Z^{rel}_{-1}(x|\Psi) = (1-\mathcal{L}_x)\delta Z^{rel}_{-1}(x|\Psi) = \sum_{\substack{Q \subseteq \Lambda_{\ge -1}: \\ Q_x = Q}} \widetilde{R}_{j-1}(Q|\Psi). \tag{2-62}$$

The second equality of (62) follows from

$$\mathcal{L}_x \delta Z^{rel}_{-1}(x|\Psi) = 0, \tag{2-63}$$

i.e. $\delta Z^{rel}_{-1}(x|\Psi)$ obeys renormalization conditions. Suppose that

$$R_j(y|\Psi) = \sum_{\substack{U \subseteq \Lambda_{\ge j}: \\ U_y = U}} \widetilde{R}_j(U|\Psi) \tag{2-64}$$

holds for all $y \in \Lambda_j$ and $\Psi \in \mathcal{F}_j$. Eq. (8) imply

$$R_{j-1}(x|\Psi) = \delta Z^{rel}_{j-1}(x|\Psi) + \sum_{P:\ \emptyset \ne P \subseteq \overline{x}} \Big[\int_0^1 ds\partial_s \int d\widetilde{\mu}^s_{v_s^j}(\Phi) Z^{rel}_j(\overline{x}-P|\Phi+\Psi) \prod_{y\in P} R_j(y|\Phi+\Psi) + Z^{rel}_j(\overline{x}-P|\Psi)\prod_{y\in P} R_j(y|\Psi)\Big]. \tag{2-65}$$

Since $R_{j-1}$ obeys renormalization conditions and $\mathcal{F}_{j-1} \subseteq \mathcal{F}_j$ we get by definition (60a-d) and (65)

$$R_{j-1}(x|\Psi) = \sum_{\substack{Q \subseteq \Lambda_{\ge j-1}: \\ Q_x = Q, x \in Q}} \widetilde{R}_{j-1}(Q|\Psi) + \sum_{\substack{Q \subseteq \Lambda_{\ge j}: \\ Q_x = Q}} \widetilde{R}_{j-1}(Q|\Psi).$$

This implies the expansion (61). $\checkmark$

For a nonempty set $Q \subseteq \Lambda_{\ge j-1}$ denote the set of all maximal elements of $Q$ by $\max(Q)$. For a set $Y \subseteq \Lambda_{\ge j-1}$ such that $y \le y'$ implies $y = y'$ for all $y, y' \in Y$ define

$$\partial(Y) \equiv \min\{y' \in \Lambda_{\ge j-1} |\ |\{y \in Y |\ y < y'\}| \ge 2\} \tag{2-66}$$

if $\exists y' \in \Lambda_{\ge j-1}$ with $|\{y \in Y|\ y < y'\}| \ge 2$ and $\partial(Y) = \emptyset$ otherwise . The following definition will be useful to formulate bounds on terms $\widetilde{R}_{j-1}(Q|\Psi)$. The *hierarchical (j-1)-tree* $\tau(Q)$ *with core* $Q$ for a finite nonempty set $Q \subseteq \Lambda_{\ge j-1}$ with $Q_x = Q$ consists of the vertex set $\mathcal{V}_{j-1}(Q)$ and lines $(yy')$ iff $y < y',\ y, y' \in \mathcal{V}(Q)$. $\mathcal{V}_{j-1}(Q)$ is recursively defined by

$$\mathcal{V}_{j-1}(Q) \equiv \{x\} \cup \begin{cases} \emptyset, & \text{for } Q = \{x\} \\ \sum_{n\ge1} \partial^n(\max(Q')) + \sum_{y\in\max(Q')} \mathcal{V}_{j_y}(Q'_y), & \text{for } Q' = Q - \{x\} \ne \emptyset \end{cases} \tag{2-67}$$



with $\partial^1 = \partial,\ \partial^{n+1} = \partial\partial^n,\ \forall n \ge 1$. The set of all lines of $\tau(Q)$ is denoted by $L(Q)$. Define

$$L_{\max}(Q) \equiv L(Q - \sum_{y\in\max(Q')} Q'_y + \max(Q')) \tag{2-68}$$

where $Q' = Q - \{x\}$. For $l = (y'y),\ y' > y$ define

$$Z^{rel}(l|\Psi) \equiv Z^{rel}_{j_y, j_{y'}}(y|\Psi) \tag{2-69}$$

and

$$|l| \equiv j_y - j_{y'}. \tag{2-70}$$

The definition (60a-d) implies for all nonempty $Q \subseteq \Lambda_{\ge j-1}$ with $Q_x = Q$

$$\widetilde{R}_{j-1}(Q|\Psi) = (1-\mathcal{L}_x)\delta Z^{rel}_{j-1}(x|\Psi) \tag{2-71a}$$

for $Q = \{x\}$

$$= (1-\mathcal{L}_x)\int_0^1 ds\partial_s \int d\widetilde{\mu}^s_{v_s^j}(\Phi) \prod_{l\in L_{\max}(Q)} Z^{rel}(l|\Phi+\Psi) \prod_{y\in\max(Q')} \widetilde{R}_{j_y}(Q'_y|\Phi+\Psi) \tag{2-71b}$$

for $Q' = Q - \{x\} \ne \emptyset$

$$= \prod_{l\in L_{\max}(Q)} Z^{rel}(l|\Psi) \prod_{y\in\max(Q')} \widetilde{R}_{j_y}(Q'_y|\Psi) \tag{2-71c}$$

for $Q \subseteq \Lambda_{\ge j}$. The proof of the following lemma is contained in the proof of lemma 3.

LEMMA 7. *Suppose that for* $y \in \Lambda_k$ *and* $\Psi, \chi'$ : *base* $\to \mathbf{C}$ *with* $|\chi'(z)| \le c_3 a_k^{-1}\lambda_k^{-1/4}$ *for all* $z \in$ *base and* $F_{y,c'}$ : $\{\Phi :$ *base* $\to \mathbf{C}\} \to \mathbf{C},\ c' \in \mathbf{R}$

$$|F_{y,c'}(\Psi+\chi')| \le K_y \exp\{-c'\lambda_k^{1/2}a_k^{-2}\int_{z\in y}\Psi^2(z)\} \tag{2-72}$$

*and*

$$\mathcal{L}F_{y,c'} = 0. \tag{2-73}$$

*Then there exist constants* $K, c$ *(not dependent on* $j$ *and* $\lambda$*) such that for all* $j \le k,\ \chi, \Psi$ : *base* $\to \mathbf{C}$ *with* $|\chi(z)| \le c_3 a_{j-1}^{-1}\lambda_{j-1}^{-1/4}$ *and* $\Psi, \chi$ *are constant on blocks of* $\Lambda_{j-1}$

$$|Z^{rel}_{k,j-1}(y|\Psi+\chi)F_{y,c}(\Psi+\chi)| \le KK_yL^{6(j-1-k)}\Big(\frac{\lambda_k}{\lambda_{j-1}}\Big)^{3/2} \exp\{-ca_{j-1}^{-2}\lambda_{j-1}^{1/2}\int_{z\in[y]_{j-1}}\Psi^2(z)\}. \tag{2-74}$$

LEMMA 8. *For all nonempty* $Y \subseteq \Lambda_{\geq j-1}$, $x \in \Lambda_{j-1}$, $j \leq 0$, *with* $Y_x = Y$, $\chi, \Psi \in \mathcal{F}_{j-1}$, $\Psi$ *real*, $|\chi(z)| \leq c_3 a_{j-1}^{-1} \lambda_{j-1}^{-1/4}$ *for all* $z \in$ *base there exist constants* $K, c$ *(uniformly in* $j$*) such that*

$$|\widetilde{R}_{j-1}(Y|\Psi + \chi)| \leq \prod_{l \in L_{\max}(Y)} [K|l|^{3/2} L^{-6|l|}] \prod_{y \in \max(Y)} B(Y_y)$$
$$\exp\{-c a_{j-1}^{-2} \lambda_{j-1}^{1/2} \int_{z \underline{\in} x} \Psi^2(z)\} \tag{2-75}$$

*if*

$$|\widetilde{R}_{j_y}(Y_y|\Psi + \chi)| \leq B(Y_y) \exp\{-c \lambda_{j_y}^{1/2} a_{j_y}^{-2} \int_{z \underline{\in} y} \Psi^2(z)\} \tag{2-76}$$

*holds for all* $y \in \max(Y)$.

The recursive bound (75) on $\widetilde{R}_{j-1}$ is proven in the same way as lemma 3. Iteration of the bound (75) shows by induction the following bound on $\widetilde{R}_{j-1}$.

PROPOSITION 9. *For small* $\lambda$ *and all nonempty* $Q \subseteq \Lambda_{\geq j-1}$, $x \in \Lambda_{j-1}$, $j \leq 0$, *with* $Q_x = Q$, $\chi, \Psi \in \mathcal{F}_{j-1}$, $\Psi$ *real*, $|\chi(z)| \leq c_3 a_{j-1}^{-1} \lambda_{j-1}^{-1/4}$ *for all* $z \in$ *base there exist constants* $K_1, K_2, c$ *(uniformly in* $j$*) such that*

$$|\widetilde{R}_{j-1}(Q|\Psi + \chi)| \leq \prod_{l \in L(Q)} [K_1 |l|^{3/2} L^{-6|l|}] \prod_{y \in Q} [K_2 \lambda_{j_y}^{1/2}] \exp\{-c a_{j-1}^{-2} \lambda_{j-1}^{1/2} \int_{z \underline{\in} x} \Psi^2(z)\}. \tag{2-77}$$

The bound (77) shows that each line $l \in L(Q)$ in the hierarchical tree $\tau(Q)$ gives a factor $K_1|l|^{3/2} L^{-6|l|}$ and each element $y$ of $Q$ gives a factor $K_2 \lambda_{j_y}^{1/2}$. The stability bound for $\widetilde{R}_{j-1}$ is the same as before. Thus we see that $\widetilde{R}_{j-1}(Q|\Psi + \chi)$ becomes small if the hierarchical tree $\tau(Q)$ contains lines with large $|l|$ or if $Q$ consists of a large number of blocks. By proposition 9 it can be easily shown that the sum in the representation (61) is uniformly bounded in $j$.

### 3.3 Hierarchical Ultraviolet Problem

In this section we consider the hierarchical ultraviolet problem at the example of 3-dimensional $\Phi^4$-theory. For $N \in \mathbf{N}$ and a cube $x_0$ of $\mathbf{R}^3$ with volume $a^3$ ($x \in \Lambda_0$) define the partition function

$$Z_0^{(N)}(x_0|\Psi) \equiv \int d\mu_{v_{x_0}^{[1,N]}}(\Phi) \exp\{-\frac{1}{2}\int_{z \underline{\in} x_0} m_N^2 a_N^{-2} (\Phi + \Psi)^2(z) - \frac{\lambda_N a_N^{-1}}{4!} \int_{z \underline{\in} x_0} (\Phi + \Psi)^4(z)\}. \tag{3-1}$$

The free propagator $v_{x_0}^{[1,N]}$ is given by

$$v_{x_0}^{[1,N]} \equiv \sum_{k=1}^{N} \sum_{\substack{x' \in \Lambda_{k-1}: \\ x' \leq x_0}} v_{x'}^k \tag{3-2}$$



$$v_{x'}^k(z_1, z_2) \equiv \sum_{y_1, y_2 \underline{\in} x'} \gamma a_k^{2-d} B_{y_1} B_{y_2} \chi_{y_1}(z_1) \chi_{y_2}(z_2) \tag{3-3}$$

for $x' \in \Lambda_{k-1}$, $z_1, z_2 \in \mathbf{R}^3$, $d = 3$, and $B_y \in \{-1, 1\}$ with $\sum_{y \underline{\in} x'} B_y = 0$. The $B$'s are chosen in such a way that

$$\int_{z_2 \underline{\in} x'} v_{x'}^k(z_1, z_2) = 0 \tag{3-4}$$

holds for all $x' \in \Lambda_{k-1}$, i.e. this is the condition that the block average of fluctuation fields is vanishing (cp. chapter 1). The bare mass squared $m_N^2$ consists in $d = 3$ dimensions of the following perturbative counterterms

$$m_N^2 a_N^{-2} = \frac{1}{2}\lambda a^{-d} \int_{z \underline{\in} x_0} v^{[1,N]}(z, z) - \frac{1}{3!}\lambda^2 a^{-d} \int_{z_1, z_2 \underline{\in} x_0} v^{[1,N]}(z_1, z_2)^3. \tag{3-5}$$

$\lambda = \lambda_N a_N^{-1}$ is the coupling constant. For $j \in \{1, \ldots, N\}$ and $x \in \Lambda_{j-1}$ define the effective partition functions

$$Z_{j-1}^{(N)}(x|\Psi) \equiv \int d\mu_{v_x^{[j,N]}}(\Phi) \exp\{-V_N(x|\Phi + \Psi)\}/(\Psi = 0) \tag{3-6}$$

with bare interaction

$$V_N(x|\Psi) \equiv \frac{1}{2}\int_{z \underline{\in} x} m_N^2 a_N^{-2} \Psi(z)^2 - \frac{\lambda_N a_N^{-1}}{4!} \int_{z \underline{\in} x} \Psi(z)^4 \tag{3-7}$$

and

$$v_{x_0}^{[j,N]} \equiv \sum_{k=j}^{N} \sum_{\substack{x' \in \Lambda_{k-1}: \\ x' \leq x_0}} v_{x'}^k. \tag{3-8}$$

The renormalization group equations are

$$Z_{j-1}^{(N)}(x|\Psi) \equiv \int d\mu_{v_x^j}(\Phi) \prod_{y \underline{\in} x} Z_j^{(N)}(y|\Phi + \Psi) \exp\{-e_{j-1}\}, \quad x \in \Lambda_{j-1} \tag{3-9}$$

with vacuum energy counterterm

$$e_{j-1} \equiv \ln \int d\mu_{v_x^j}(\Phi) \prod_{y \underline{\in} x} Z_j^{(N)}(y|\Phi). \tag{3-10}$$

The effective (or running) mass squared is defined by

$$m_{j-1}^{(N)2} \equiv -a_{j-1}^{2-d} \frac{\partial^2}{\partial \Psi^2} \ln Z_{j-1}^{(N)}(x|\Psi)|_{\Psi=0} \tag{3-11}$$

for arbitrary $x \in \Lambda_{j-1}$. Perturbation expansions gives

$$m_{j-1}^{(N)2} a_{j-1}^{-2} = m_N^2 a_N^{-2} + \frac{1}{2}\lambda a_{j-1}^{-d} \int_{z \underline{\in} x'} v^{[j,N]}(z, z) - \frac{1}{3!}\lambda^2 a_{j-1}^{-d} \int_{z_1, z_2 \underline{\in} x'} v^{[j,N]}(z_1, z_2)^3 + O(\gamma^5). \tag{3-12}$$

For the derivation of eq.(12) the condition (4) was used. Insertion of eq.(5) into eq.(12) yields

$$m_{j-1}^{(N)2}a_{j-1}^{-2} = \frac{1}{2}\lambda a^{-d}\int_{z\subseteq x} \mathrm{v}^{[1,j-1]}(z,z) - \frac{1}{3!}\lambda^2 a^{-d}\int_{z_1,z_2\subseteq x}\left[\mathrm{v}^{[j,N]}(z_1,z_2)^3 - \mathrm{v}^{[1,N]}(z_1,z_2)^3\right] + O(\gamma^5) \tag{3-13}$$

for $x \in \Lambda_0$. Define the relevant interaction for $y \in \Lambda_j$ by

$$V_j^{(N),rel}(y|\Psi) \equiv \frac{1}{2}m_j^{(N)2}a_j^{-2}\int_{z\subseteq y}\Psi(z)^2 - \frac{\lambda_j a_j^{-1}}{4!}\int_{z\subseteq y}\Psi(z)^4, \quad y\in\Lambda_j \tag{3-14}$$

and the relevant partition function

$$Z_j^{(N),rel}(y|\Psi) \equiv \exp\{-V_j^{(N),rel}(y|\Psi)\}. \tag{3-15}$$

Define irrelevant activities by

$$R_j^{(N)}(y|\Psi) \equiv -Z_j^{(N),rel}(y|\Psi) + Z_j^{(N)}(y|\Psi). \tag{3-16}$$

The irrelevant activities obey the following renormalization conditions

$$R_j^{(N)}(y|\Psi)|_{\Psi=0} = 0, \quad \frac{\partial^2}{\partial\Psi^2}R_j^{(N)}(y|\Psi)|_{\Psi=0} = 0. \tag{3-17}$$

The aim of this section is to prove the following bounds.

Proposition 1. *For all $N\in\mathbf{N}$ and $j\in\{0,1,\ldots,N\}$ and small $\lambda$ there exist constants $K,c,c_m,c_3$ (not dependent on $\lambda, j$ and $N$) such that*

$$|m_j^{(N)2}| \le c_m\lambda_j \tag{$M_j$}$$

$$|R_j^{(N)}(y|\Psi+\chi)| \le K\lambda_j^{1/2}\exp\{-c\lambda_j^{1/2}a_j^{-2}\int_{z\subseteq y}\Psi^2(z)\} \tag{$C_j$}$$

*and for all $h\in\mathbf{N}$ there exist constants $K', c'_m$ (not dependent on $\lambda, j, N$ and $h$) such that*

$$|m_j^{(N)2} - m_j^{(N+h)2}| \le c'_m\lambda_j L^{j-N} \tag{$DM_j$}$$

$$|R_j^{(N)}(y|\Psi+\chi) - R_j^{(N+h)}(y|\Psi+\chi)| \le K'\lambda_j^{1/2}L^{j-N}\exp\{-c\lambda_j^{1/2}a_j^{-2}\int_{z\subseteq y}\Psi^2(z)\} \tag{$DC_j$}$$

*holds for all $y\in\Lambda_j, \Psi:\mathbf{R}^3\to\mathbf{R}$, $\chi:\mathbf{R}^3\to\mathbf{C}$, $|\chi(z)|\le c_3 a_j^{-1/2}\lambda_j^{1/4}$ $\forall z\in\mathbf{R}^3$, and $\Psi,\chi$ are constant on blocks of $\Lambda_j$.*

By proposition 1 we see that the UV-limit $N\to\infty$ exists. Define

$$m_j^{(\infty)2} \equiv \lim_{N\to\infty} m_j^{(N)2}, \quad R_j^{(\infty)}(y|\Psi) \equiv \lim_{N\to\infty} R_j^{(N)}(y|\Psi).$$



Corollary 2. *For all $j\in\mathbf{N}$ and small $\lambda$ there exist constants $K,c,c_m,c_3$ (not dependent on $\lambda, j$) such that*

$$|m_j^{(\infty)2}| \le c_m\lambda_j$$

$$|R_j^{(\infty)}(y|\Psi+\chi)| \le K\lambda_j^{1/2}\exp\{-c\lambda_j^{1/2}a_j^{-2}\int_{z\subseteq y}\Psi^2(z)\}$$

*holds for all $y\in\Lambda_j, \Psi:\mathbf{R}^3\to\mathbf{R}$, $\chi:\mathbf{R}^3\to\mathbf{C}$, $|\chi(z)|\le c_3 a_j^{-1/2}\lambda_j^{1/4}$ $\forall z\in\mathbf{R}^3$, and $\Psi,\chi$ are constant on blocks of $\Lambda_j$.*

By eq.(13) the lowest order perturbation theoretic terms of the effective mass squared $m_{j-1}^{(N)2}$ obey the following bounds (uniform in $N$).

Lemma 3. *For all $j-1\in\{0,1,\ldots,N\}$ there exist constants $K_{1,2}, K_{3,2}$ (not dependent on $N, j$) such that*

$$|\gamma\partial_\gamma m_{j-1}^{(N)2}|_{\gamma=0}| \le K_{1,2}\lambda_{j-1}, \quad |\frac{1}{3!}\gamma^3\partial_\gamma^3 m_{j-1}^{(N)2}|_{\gamma=0}| \le K_{3,2}\lambda_{j-1}^2 \tag{3-18}$$

*with $\lambda_{j-1} = \lambda a_{j-1}$.*

*Proof :* We have by eq.(13)

$$\gamma\partial_\gamma m_{j-1}^{(N)2}|_{\gamma=0} = \frac{1}{2}\lambda a^{-d}\int_{z\subseteq x} v^{[1,j-1]}(z,z),$$

$$\frac{1}{3!}\gamma^3\partial_\gamma^3 m_{j-1}^{(N)2}|_{\gamma=0} = \frac{1}{3!}\lambda^2 a^{-d}\int_{z_1,z_2\in x}\left[v^{[j,N]}(z_1,z_2)^3 - v^{[1,N]}(z_1,z_2)^3\right].$$

There exists a constant $C$ not dependent on $N$ such that

$$|\mathrm{v}^{[1,j-1]}(z,z)| \le \sum_{k=1}^{j-1}\gamma a_k^{-1} \le C a_{j-1}^{-1}$$

and

$$|\int_{z_2\in x}\left[\mathrm{v}^{[j,N]}(z_1,z_2)^3 - \mathrm{v}^{[1,N]}(z_1,z_2)^3\right]| \le$$

$$\le 3!\sum_{\substack{k_1,k_2,k_3:\\ N\ge k_1\ge k_2\ge k_3,\ k_3\le j-1}} |\int_{z_2\subseteq x}\mathrm{v}^{k_1}(z_1,z_2)\mathrm{v}^{k_2}(z_1,z_2)\mathrm{v}^{k_3}(z_1,z_2)| \le$$

$$\le 3!\,\gamma^3\sum_{\substack{k_1,k_2,k_3:\\ N\ge k_1\ge k_2\ge k_3,\ k_3\le j-1}} a_{k_1}^d a_{k_1}^{-1}a_{k_2}^{-1}a_{k_3}^{-1} \le C.$$

This proves the assertion. √

To estimate the remainder term on the right hand side of eq.(13) define

$$W_j^{(L,E)} \equiv \frac{1}{Vol(y)}\frac{1}{L!}\partial_\gamma^L\partial_\Psi^E \ln Z_j^{(N)}(y|\Psi)|_{\substack{\Psi=0\\ \gamma=0}} \tag{3-19}$$

for $E, L \in \mathbf{N},\ y \in \Lambda_j$. Using the perturbation expansion (D-2) of appendix D we get recursive bounds for the coefficients $W_j^{(L,E)}$

$$|W_{j-1}^{(L,E)}| \le |W_j^{(L,E)}| + \sum_{m\ge 1} a_j^{(m-1)d} \sum_{\substack{L_0,\dots,L_m:\\ \sum_{a=0}^m L_a = L}} \sum_{\substack{2L_0=\sum_{b=1}^m I_b:\\ \gamma(I_1,\dots,I_m)\ conn.}} \frac{(\gamma a_j^{2-d})^{L_0}}{L_0! 2^{L_0}} \sum_{\substack{E_1,\dots,E_m:\\ \sum_{a=1}^m E_a = E}} \frac{E!}{\prod_{a=1}^m E_a!} \prod_{a=1}^m |W_j^{(L_a, E_a+|I_a|)}|. \tag{3-20}$$

We have used the following notation. Disjoint subsets $I_a$ and $I_b$ of $\underline{2L_0} = \{1,\dots,2L_0\}$ are called *compatible* ($I_a \sim I_b$) iff there exists no $i \in \{1,\dots,L_0\}$ such that $2i \in I_a$ and $2i-1 \in I_b$ or $2i-1 \in I_a$ and $2i \in I_b$. Consider a partition $\sum_{a=1}^m I_a = \underline{2L_0}$. We define a graph $\gamma(I_1,\dots,I_m)$ with vertices $I_1,\dots,I_m$ by lines $(I_a I_b)$, $a \ne b$, if $I_a$ and $I_b$ are not compatible. The sum in (20) goes over all $I_1,\dots,I_m$ such that the graph $\gamma(I_1,\dots,I_m)$ is connected. It can be easily seen that $W_j^{(L,E)} = 0$ if $2L + E \notin 4\mathbf{N}$. Define for $L, E \in \mathbf{N},\ y \in \Lambda_j$, Taylor remainder terms by

$$W_j^{(>L,E)} \equiv \frac{1}{Vol(y)} \frac{1}{L!} \int_0^1 ds (1-s)^L \gamma^{L+1} \partial_\gamma^{L+1} \partial_\Psi^E Z_{j,s\gamma}^{(N)}(y|\Psi)|_{\Psi=0}. \tag{3-21}$$

Let $L(E) \in \mathbf{N}$ for $E \in \mathbf{N}$. Using the perturbation expansion $(D-2)$ of appendix D we get recursive bounds for $W_j^{(>L,E)}$

$$|W_{j-1}^{(>L,E)}| \le |W_j^{(>L,E)}| + \sum_{m\ge1} a_j^{(m-1)d} \sum_{I:\ I\subseteq \underline{m}} \sum_{\substack{E_1,\dots,E_m:\\ \sum_{a=1}^m E_a=E}} \frac{E!}{\prod_{a=1}^m E_a!} \sum_{L_0=1}^{L} \sum_{\substack{2L_0=\sum_{b=1}^m I_b:\\ \gamma(I_1,\dots,I_m)\ conn.}} \sum_{\substack{\{L_a, a\in \underline{m}-I,\ L_a \le L(E_a+|I_a|)\}\\ L_0+\sum_{a\in \underline{m}-I} L_a + \sum_{a\in I}(L(E_a+|I_a|)+1) > L}} \frac{(\gamma a_j^{2-d})^{L_0}}{L_0! 2^{L_0}} \prod_{a\in \underline{m}-I} |W_j^{(L_a,E_a+|I_a|)}| \prod_{b\in I} |W_j^{(>L_b,E_b+|I_b|)}| + |Q_{>L,E}^{j-1}| \tag{3-22}$$

with

$$Q_{>L,E}^{j-1} \equiv \frac{1}{Vol(x)} \frac{1}{L!} \int_0^1 ds (1-s)^L \partial_s^{L+1} \partial_\Psi^E \ln \int d\mu_{s v_x^j}(\Phi) \prod_{y\subseteq x} Z_j^{(N)}(y|\Phi+\Psi)|_{\Psi=0}. \tag{3-23}$$

for $x \in \Lambda_{j-1}$.

We want to prove by induction the following bounds

$$|W_j^{(L,E)}| \le K_{L,E}\, a_j^{E(\frac{d}{2}-1)-d} (C\lambda_j)^{\frac{2L+E}{4}} \tag{$A_j$}$$

and

$$|W_j^{(>L(E),E)}| \le K_E\, a_j^{E(\frac{d}{2}-1)-d} (C\lambda_j)^{\frac{2(L(E)+1)+E}{4}} \tag{$B_j$}$$



with $L(E) \equiv \frac{8-E}{2},\ E \in \{2,4,6,8\}$. Since

$$W_j^{(1,2)} = -\gamma \partial_\gamma m_{j-1}^{(N)2}|_{\gamma=0} a_j^{-2}, \quad W_j^{(3,2)} = -\frac{1}{3!}\gamma^3 \partial_\gamma^3 m_{j-1}^{(N)2}|_{\gamma=0} a_j^{-2} \tag{3-24}$$

we see by lemma 3 that $(A_j)$ holds for $(L,E) = (1,2),(3,2)$.

The inductive proof of proposition 1 is organized in the following way. In lemma 4 it is shown that $(A_j)$ and $(B_j)$ imply $(A_{j-1})$, $(B_{j-1})$ if a suitable bound on $Q_{>L(E),E}^{j-1}$ is supposed. Lemma 5 shows that $(A_j)$ and $(B_j)$ imply $(M_j)$. Lemma 6 gives a stability bound on $Z_{k,j-1}^{rel}$ ( defined by eq.(27) ). Lemma 7 presents a bound on the vacuum energy counterterm $e_{j-1}$ and the flow of the effective mass squared $m_j^2$. if the bound $(C_j)$ is supposed. Lemma 9 together with lemma 8 implies the bound $(C_{j-1})$ if the bound $(C_k)$ holds for all $k \ge j$. Lemma 10 closes the inductive proof by proving the supposition of lemma 4 (bound on $Q_{>L(E),E}^{j-1}$).

LEMMA 4. *Suppose that for* $L(E) \equiv \frac{8-E}{2},\ E \in \{2,4,6,8\}$ *the bounds*

$$|Q_{>L(E),E}^{j-1}| \le \widetilde{K}_E\, a_j^{E(\frac{d}{2}-1)-d} (C\lambda_j)^{\frac{2(L(E)+1)+E}{4}} \tag{3-25}$$

*hold. Then there exist constants* $K_{L,E}$ *such that the bounds* $(A_j)$ *and* $(B_j)$ *imply the bounds* $(A_{j-1})$ *and* $(B_{j-1})$ *for all* $(L,E)$ *with* $L \le L(E),\ E \in \{2,4,6,8\}$.

*Proof :* For $L < L(E)$ we see that the rhs of eq.(20) contains only $W_j^{(L',E')}$-terms with $E' \in \{2,4,6,8\}$ and $L' \le L(E')$. Since

$$a_j^{E(\frac{d}{2}-1)-d} \lambda_j^{\frac{2L+E}{4}} = L^{D(L,E)} a_{j-1}^{E(\frac{d}{2}-1)-d} \lambda_{j-1}^{\frac{2L+E}{4}}$$

with degree of divergence

$$D(L,E) \equiv d - E\frac{d}{4} - \frac{L}{2}(4-d)$$

it is easily seen that eq.(20) together with the bounds $(A_j)$ imply the bounds $(A_{j-1})$ for all $(L,E)$ with $E \in \{2,4,6,8,\}, L \le L(E)$ and $D(L,E) < 0$ (for suitable constants $K_{L,E}$). For $d=3$ dimensions the case $D(L,E) \ge 0$ is equivalent to $(L,E) \in \{(1,2),(3,2)\}$. For this case $(A_{j-1})$ was already shown by lemma 3. For the cases $E \in \{2,4,6,8\}, L = L(E)$, we see that the rhs of eq.(22) contains only $W_j^{(L',E')}$-terms with $E' \in \{2,4,6,8\}$ and $L' \le L(E')$. Since for $E \ge 2$

$$D(L(E)+1,E) = d - E\frac{d}{4} - (\frac{5}{2} - \frac{E}{4})(4-d) = -\frac{E}{2} + \frac{1}{2} < 0$$

we see by eq.(22) that the bounds $(A_j, B_j)$ and (25) imply $(B_{j-1})$. $\surd$

In the following we supress the superscript $(N)$. For $y \in \Lambda_k,\ j,k \in \{0,\dots,N\}, j<k$ define

$$Z_{k,j}^{rel}(y|\Psi) \equiv Z_{j+1}^{rel}([y]_j - [y]_{j+1}|\Psi) Z_{j+2}^{rel}([y]_{j+1} - [y]_{j+2}|\Psi) \ \dots\ Z_k^{rel}([y]_{k-1} - y|\Psi) \tag{3-26}$$

and for $x \in \Lambda_{j-1}$

$$\delta R_{j-1}(x|\Psi) \equiv R_{j-1}(x|\Psi) - \sum_{y\subseteq x} Z_j^{rel}(\overline{x} - y|\Psi) R_j(y|\Psi). \tag{3-27}$$

Eqs. (26) and (27) yield the following representation for $R_{j-1}$

$$R_{j-1}(x|\Psi) = \delta R_{j-1}(x|\Psi) + \sum_{k=j}^{N-1} \sum_{\substack{y\in\Lambda_k \\ y\leq x}} Z^{rel}_{k,j-1}(y|\Psi)\delta R_k(y|\Psi). \qquad (3\text{-}28)$$

By the following representation of the effective mass squared

$$m_j^{(N)2} = a_j^2 [W_j^{(1,2)} + W_j^{(3,2)} + W_j^{(>3,2)}] \qquad (3\text{-}29)$$

we can show

LEMMA 5. *Suppose that the bounds $(A_j)$ and $(B_j)$ hold. Then there exists a constant $c_m$ such that*

$$|m_j^{(N)2}| \leq c_m \lambda_j. \qquad (M_j)$$

In the same way as in the proof for lemma 2.2 we can show

LEMMA 6. *Suppose that there exists a constant $c_m$ such that*

$$|m_i^{(N)2}| \leq c_m \lambda_i \qquad (M_i)$$

*for all $i \in \{j, \ldots, N\}$. Then for small $\lambda$, $k \geq j$, $y \in \Lambda_k$, $\Psi: \mathbf{R}^3 \to \mathbf{R}$, $\chi: \mathbf{R}^3 \to \mathbf{C}$, $|\chi(z)| \leq c_3 a_{j-1}^{-1/2} \lambda_{j-1}^{-1/4}$ $\forall z \in \mathbf{R}^3, c_3 > 0$, there exist constants $c, c_0$ (not dependent on $j, \lambda$) such that*

$$|Z^{rel}_{k,j-1}(y|\Psi+\chi)| \leq \exp\{-c a_{j-1}^{-2} \lambda_{j-1}^{1/2} \int_{z\in[y]_{j-1}-y} \Psi^2(z) + (1-L^{-3})c_0\}. \qquad (3\text{-}30)$$

For $y \in \Lambda_j$, $\Psi: \mathbf{R}^3 \to \mathbf{R}$, $\chi: \mathbf{R}^3 \to \mathbf{C}$, $|\chi(z)| \leq c_3 a_j^{-1/2} \lambda_j^{-1/4}$ $\forall z \in \mathbf{R}^3, c_3 > 0$ and $\Psi, \chi$ constant on blocks of $\Lambda_j$ consider the bound

$$|R_j(y|\Psi+\chi)| \leq K\lambda_j^{1/2} \exp\{-c\lambda_j^{1/2} a_j^{-2} \int_{z\in y} \Psi^2(z)\} \qquad (C_j)$$

For some positive constants $K$ and $c$ (not dependent on $j$ and $\lambda$).

LEMMA 7. *Suppose that the bound $(C_j)$ holds. Then there exists for small $\lambda$ constants $c_e$ and $c'_m$ such that*

$$|e_{j-1}| \leq c_e \lambda_{j-1}. \qquad (3\text{-}31)$$

and

$$|m_{j-1}^2 a_{j-1}^{-2} - m_j^2 a_j^{-2}| \leq c'_m \lambda_{j-1} a_{j-1}^{-2}. \qquad (3\text{-}32)$$

*Proof :* The vacuum energy counterterm reads

$$e_{j-1} = \ln \int d\mu_{v_*^j}(\Phi) \prod_{y\in z} Z_j(y|\Phi) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} [\int d\mu_{v_*^j}(\Phi) \prod_{y\in z} Z_j(y|\Phi) - 1]^n. \qquad (3\text{-}33)$$



The definition (16) of the irrelevant activity gives

$$\int d\mu_{v_*^j}(\Phi) \prod_{y\in z} Z_j(y|\Phi) - 1 = \int d\mu_{v_*^j}(\Phi)[\prod_{y\in z} Z_j^{rel}(y|\Phi) - 1] +$$
$$+ \sum_{P:\, \emptyset\neq P\subseteq \bar{z}} \int d\mu_{v_*^j}(\Phi) Z_j^{rel}(\bar{z} - P|\Phi) \prod_{y\in P} R_j(y|\Phi). \qquad (3\text{-}34)$$

The bound $(C_j)$, lemma 3 and $R_j(y|0) = 0$ and eq.(34) imply (for a suitable $(\lambda, j)$- independent constant $C$)

$$|\int d\mu_{v_*^j}(\Phi) \prod_{y\in z} Z_j(y|\Phi) - 1| \leq C\lambda_{j-1}. \qquad (3\text{-}35)$$

Thus by eq.(33) there exists a constant $c_e$ (not dependent on $\lambda$ and $j$) such that

$$|e_{j-1}| \leq \sum_{n\geq 1} \frac{(C\lambda_{j-1})^n}{n} \leq c_e \lambda_{j-1}. \qquad (3\text{-}36)$$

By definition (11) of the effective mass squared we have

$$m_{j-1}^2 a_{j-1}^{-2} - m_j^2 a_j^{-2} = -a_{j-1}^{-d} \frac{\partial^2}{\partial \Psi^2} [\ln Z_{j-1}(x|\Psi) - \ln \prod_{y\in z} Z_j(y|\Psi)]|_{\Psi=0} =$$
$$= -a_{j-1}^{-d} \frac{\partial^2}{\partial \Psi^2} \int_0^1 ds \partial_s \ln \int d\mu_{sv_*^j}(\Phi) \prod_{y\in z} Z_j(y|\Phi+\Psi)|_{\Psi=0}.$$

Cauchy's inequality, bound $(C_j)$ and the definition of $R_j$ imply the bound (32). ✓

By the definition (27) of $\delta R_{j-1}$ follows

$$\delta R_{j-1}(x|\Psi) = -Z_{j-1}^{rel}(x|\Psi) + \int d\mu_{v_*^j}(\Phi) Z_j^{rel}(\bar{x}|\Phi+\Psi) \exp\{-e_{j-1}\} +$$
$$+ \sum_{P:\emptyset\neq P\subseteq \bar{x}} \int_0^1 ds \partial_s \int d\mu_{sv_*^j}(\Phi) Z_j^{rel}(\bar{x} - P|\Phi+\Psi) \prod_{y\in P} R_j(y|\Phi+\Psi) \exp\{-se_{j-1}\} +$$
$$+ \sum_{\substack{P:\, P\subseteq \bar{x} \\ |P|\geq 2}} Z_j^{rel}(\bar{x} - P|\Psi) \prod_{y\in P} R_j(y|\Psi) \qquad (3\text{-}37)$$

LEMMA 8. *Suppose that the bound $(C_j)$ holds. For small $\lambda$, $x \in \Lambda_{j-1}$, $\Psi: \mathbf{R}^3 \to \mathbf{R}$, $\chi: \mathbf{R}^3 \to \mathbf{C}$, $|\chi(z)| \leq c_3 a_j^{-1/2} \lambda_j^{1/4}$ $\forall z \in \mathbf{R}^3$, and $\Psi, \chi$ are constant on blocks of $\Lambda_{j-1}$ there exists a constant $K'$ such that*

$$|\delta R_{j-1}(x|\Psi+\chi)| \leq K' \lambda_{j-1}^{1/2} \exp\{-c\lambda_{j-1}^{1/2} a_{j-1}^{-2} \int_{z\in x} \Psi^2(z)\}. \qquad (3\text{-}38)$$

$K'$ is independent of $\lambda$ and $j$ and depends on $K$ in such a way that $K'/K$ becomes small for large $K$ and small $\lambda$.

*Proof :* For the first two terms of the rhs of eq.(37) we have

$$-Z^{rel}_{j-1}(x|\Psi)+\int d\mu_{v^j_z}(\Phi)Z^{rel}_j(\overline{x}|\Phi+\Psi)\exp\{-e_{j-1}\}=$$
$$=\int_0^1 ds\partial_s\Big[\int d\mu_{sv^j_z}(\Phi)\prod_{y\in z}Z^{rel}_{j,s}(y|\Phi+\Psi)\exp\{-se_{j-1}\}\Big] \tag{3-39}$$

with

$$Z^{rel}_{j,s}(y|\Psi)\equiv\exp\{-\frac{1}{2}m^2_j(s)a_j^{-2}\int_{z\in y}\Psi^2(z)-\frac{\lambda}{4!}\int_{z\in y}\Psi^4(z)\} \tag{3-39a}$$

$$m^2_j(s)\equiv sm^2_j+(1-s)m^2_{j-1}L^{-2}. \tag{3-39b}$$

Lemma 7 and eq.(39) imply (using Cauchy's inequality)

$$|-Z^{rel}_{j-1}(x|\Psi)+\int d\mu_{v^j_z}(\Phi)Z^{rel}_j(\overline{x}|\Phi+\Psi)\exp\{-e_{j-1}\}|\le$$
$$\le K_{rel}\lambda^{1/2}_{j-1}\exp\{-c\lambda^{1/2}_{j-1}a^{-2}_{j-1}\int_{z\in x}\Psi^2(z)\} \tag{3-40}$$

for a $(\lambda,j)$-independent constant $K_{rel}$. The third term on the right hand side of eq.(37) is supressed by an additional factor $\lambda^{1/2}_{j-1}$ (since the $s$-derivative works on $\int d\mu_{sv^j_z}$ or brings down a factor $e_{j-1}$). The fourth term on the right hand side of eq.(37) contains at least two $R_j$-factors. Thus this term is also supressed by $\lambda^{1/2}_{j-1}$. The final bound is

$$|\delta R_{j-1}(x|\Psi+\chi)|\le(K_{rel}+K''\lambda^{1/2}_{j-1})\lambda^{1/2}_{j-1}\exp\{-c\lambda^{1/2}_{j-1}a^{-2}_{j-1}\int_{z\in x}\Psi^2(z)\}.$$

This shows the bound (38) for $\delta R_{j-1}$ with $K'=K_{rel}+K''\lambda^{1/2}_{j-1}$ and $K'/K$ becomes small for large $K$ (since $K_{rel}$ does not depend on $K$). $\surd$

LEMMA 9. *Suppose that $(C_k)$ holds for $k\in\{j,\dots,N\}$. Then the bound $(C_{j-1})$ holds for small $\lambda$ and $K$ large enough.*

*Proof :* By eq.(17) and (27) $\delta R_{j-1}$ obeys the renormalization conditions

$$\delta R_{j-1}(x|\Psi)|_{\Psi=0}=0,\quad \frac{\partial^2}{\partial\Psi^2}\delta R_j(x|\Psi)|_{\Psi=0}=0. \tag{3-41}$$

For $k\in\{j,\dots,N\}$, $y\in\Lambda_k$ we have

$$\delta R_k(y|\Psi)=Z^{rel}_k(y|\Psi)\frac{1}{3!}\int_0^1 ds(1-s)^3\Psi^4\frac{\partial^4}{\partial\chi^4}Z^{rel}_k(y|\chi)^{-1}\delta R_k(y|\chi)|_{\chi=s\Psi}.$$



Cauchy's inequality yields

$$|\delta R_k(y|\Psi+\chi)|\le|Z^{rel}_k(y|\Psi+\chi)||\Psi+\chi|^4\frac{1}{3!}\int_0^1 ds(1-s)^3\frac{4!}{\kappa^4}$$
$$\sup_{\widetilde{\chi}\in\mathbf{C},\,|\widetilde{\chi}|\le\kappa}|Z^{rel}_k(y|s\Psi+\chi+\widetilde{\chi})^{-1}\delta R_k(y|s\Psi+\chi+\widetilde{\chi}). \tag{3-42}$$

Furthermore we have

$$|\chi+\widetilde{\chi}|\le c_3a^{-1/2}_{j-1}\lambda^{-1/4}_{j-1}+\kappa\le c_3a_k^{-1/2}\lambda_k^{-1/4}$$

for

$$\kappa\le c_3(a_k^{-1/2}\lambda_k^{-1/4}-a^{-1/2}_{j-1}\lambda^{-1/4}_{j-1}).$$

Thus for $\kappa=c_3(1-L^{-3/4})a_k^{-1/2}\lambda_k^{-1/4}$ we can use lemma 6 for estimating $\delta R_k$ on the rhs of eq.(42). This gives

$$|\delta R_k(y|\Psi+\chi)|\le CK'a_k^2\lambda_k^{3/2}|\Psi+\chi|^4\int_0^1 ds(1-s)^3\exp\{-(1-s^4)^{1/2}c\lambda_k^{1/2}a_k^{-2}\int_{z\in y}\Psi^2(z)\}$$
$$\exp\{-c\lambda_k^{1/2}a_k^{-2}\int_{z\in y}\Psi^2(z)\}\le C'K'(\frac{a_k}{a_{j-1}})^2(\frac{\lambda_k}{\lambda_{j-1}})^{3/2}\lambda^{1/2}_{j-1}\exp\{-c\lambda_k^{1/2}a_k^{-2}\int_{z\in y}\Psi^2(z)\}. \tag{3-43}$$

In the last bound of (43) we have used the integrability of $(1-s)^3(1-s^4)^{-2}$ over the interval $[0,1]$. The bounds (30) of lemma 6 and eq.(28) imply

$$|R_{j-1}(x|\Psi+\chi)|\le\big[K'+\sum_{k=j}^{N-1}C''K'L^{-\frac{1}{2}(k-(j-1))}\big]\lambda^{1/2}_{j-1}\exp\{-c\lambda^{1/2}_{j-1}a^{-2}_{j-1}\int_{z\in x}\Psi^2(z)\} \tag{3-44}$$

with $C''=C'\exp\{(1-L^{-3})c_0\}$. Thus for $K$ large enough the bound (44) implies $(C_{j-1})$. $\surd$

LEMMA 10. *Suppose that the bound $(C_j)$ holds. Then there exists a constant $\widetilde{K}_E$ (not dependent on $j$ and $\lambda$) such that*

$$|Q^{j-1}_{>L(E),E}|\le\widetilde{K}_E a_j^{E(\frac{d}{2}-1)-d}(C\lambda_j)^{\frac{2(L(E)+1)+E}{4}} \tag{3-25}$$

*for all $E=\{2,4,6,8\}$, $L(E)=\frac{8-E}{2}$.*

*Proof :* Consider

$$Q^{j-1}_{>L(E),E}\equiv a^{-d}_{j-1}\frac{1}{L(E)!}\int_0^1 ds(1-s)^{L(E)}\partial_s^{L(E)+1}\partial^E_\Psi\ln\int d\mu_{sv^j_z}(\Phi)\prod_{y\in z}Z^{(N)}_j(y|\Phi+\Psi)|_{\Psi=0}.$$

Using the change of covariance lemma

$$\partial_s\int d\mu_{sv^j_z}(\Phi)F(\Phi+\Psi)=\frac{1}{2}\int_{z_1,z_2}\frac{\delta}{\delta\Psi(z_1)}v^j_z(z_1,z_2)\frac{\delta}{\delta\Psi(z_2)}\int d\mu_{v^j_z}(\Phi)F(\Phi+\Psi)$$

we see that each $s$-derivative is equivalent to two $\Psi$-derivatives. Thus the total number of $\Psi$-derivatives in the definition of $Q^{j-1}_{>L(E),E}$ is $E+2(L(E)+1)$. $Z_j(y|\Psi+\chi)$ is analytic for $\chi$ in $\{z \in \mathbb{C} | \; |z| \leq c_3 a_j^{1-d/2}\lambda_j^{-1/4}\}$. Thus by the bound $(C_j)$ and Cauchy's inequality we obtain a factor $a_j^{-1+d/2}\lambda_j^{1/4}$ for each $\Psi$-derivative. Thus for the total number of $\Psi$-derivatives we get a factor $a_j^{1/2}\lambda_j^{1/4})^{E+2(L(E)+1)}$. Each free propagator $v^j_z$ gives a factor $a_j^{2-d}$. There are $L(E)+1$ free propagators giving a factor $a_j^{(2-d)(L(E)+1)}$. Thus we obtain as total factor $a_j^{E(d/2-1)-d}\lambda_j^{\frac{2(L(E)+1)+E}{4}}$ for $Q^{j-1}_{>L(E),E}$. This proves (25).///$surd$ This proves (25) for $d=3$. ✓

*Proof of Proposition 1 (by induction) :*

Firstly, we will prove $(M_j)$ and $(C_j)$ by induction. Since $R_N \equiv 0$ the bound $(C_N)$ holds. Lemma 3, the definition of $W_N^{(L,E)}$ and $W_N^{(>L(E),E)}=0$ show that $(A_N)$ and $(B_N)$ are valid. Furthermore, lemma 5 implies the bound $(M_N)$.

*Induction hypothesis :* $(A_k)$, $(B_k)$, $(C_k)$, $(M_k)$ hold for all $k \in \{j, j+1, \ldots, N\}$.

By lemma 10 we see that the bound (25) for $Q^{j-1}_{L(E),E}$ holds. Thus by lemma 4 and induction hypothesis we see that $(A_{j-1})$ and $(B_{j-1})$ is valid. Therefore lemma 5 implies $(M_{j-1})$ and lemma 9 implies $(C_{j-1})$. This completes the proof for the bound of $(M_j)$ and $(C_j)$.

We will prove now $(DM_j)$ and $(DC_j)$ by induction. By the first part of proposition 1 we have

$$|m_N^{(N)2} - m_N^{(N+h)2}| \leq 2c_m\lambda_N \leq c'_m \lambda_N L^{N-N}$$

for $c' \geq 2c_m$ and

$$|R_N^{(N)}(y|\Psi+\chi) - R_j^{(N+h)}(y|\Psi+\chi)| = |R_j^{(N+h)}(y|\Psi+\chi)| \leq$$
$$\leq K'\lambda_N^{1/2} L^{N-N} \exp\{-c\lambda_N^{1/2} a_N^{-2} \int_{z \in y} \Psi^2(z)\}.$$

This proves $(DM_N)$ and $(DC_N)$. Define

$$\widetilde{W}_j^{(L,E)} \equiv \frac{1}{Vol(y)} \frac{1}{L!} \partial^L_\gamma \partial^E_\Psi [\ln Z_j^{(N)}(y|\Psi) - \ln Z_j^{(N+h)}(y|\Psi+\chi)]|_{\substack{\Psi=0\\ \gamma=0}}$$

and analogously $\widetilde{W}_j^{(>L,E)}$ and $\widetilde{Q}^{j-1}_{>L,E}$ and

$$\widetilde{m}_j^2 \equiv m_j^{(N)2} - m_j^{(N+h)2}.$$

Consider the bounds

$$|\widetilde{W}_j^{(L,E)}| \leq K_{L,E} a_j^{E(\frac{d}{2}-1)-d} (C\lambda_j)^{\frac{2L+E}{4}} L^{j-N} \qquad (DA_j)$$

and

$$|\widetilde{W}_j^{(>L(E),E)}| \leq K_E a_j^{E(\frac{d}{2}-1)-d} (C\lambda_j)^{\frac{2(L(E)+1)+E}{4}} L^{j-N} \qquad (DB_j)$$



for $E \in \{2,4,6,8\}$, $L \leq L(E) \equiv \frac{8-E}{2}$.

*Induction hypothesis :* $(DA_k)$, $(DB_k)$, $(DC_k)$, $(DM_k)$ hold for all $k \in \{j, j+1, \ldots, N\}$.

Consider

$$\widetilde{Q}^{j-1}_{>L,E} \equiv \frac{1}{Vol(x)} \frac{1}{L!} \int_0^1 ds (1-s)^L \partial_s^{L+1} \partial^E_\Psi [\ln \int d\mu_{sv^j_x}(\Phi) \prod_{y \in x} Z_j^{(N)}(y|\Phi+\Psi) -$$
$$- \ln \int d\mu_{sv^j_x}(\Phi) \prod_{y \in x} Z_j^{(N+h)}(y|\Psi+\Psi)]|_{\Psi=0}. \qquad (3\text{–}45)$$

Perform the $s$- and $\Psi$- derivatives and use the change of covariance lemma for the rhs of eq.(45). By induction hypothesis we obtain

$$|\widetilde{Q}^{j-1}_{>L(E),E}| \leq \widetilde{K}_E a_j^{E(\frac{d}{2}-1)-d} (C\lambda_j)^{\frac{2(L(E)+1)+E}{4}} L^{j-N} \qquad (3\text{–}46)$$

for a suitable constant $\widetilde{K}_E$ (not dependent on $\lambda, j, N$ and $h$). Using bounds on $\widetilde{W}_j^{(L,E)}$ and $\widetilde{W}_j^{(>L(E),E)}$ analogous to (20) and (22) then the bounds (46),$(DA_j)$,$(DB_j)$ imply the bounds $(DA_{j-1})$,$(DB_{j-1})$. The relation

$$\widetilde{m}_j^2 = a_j^2 [\widetilde{W}_j^{(1,2)} + \widetilde{W}_j^{(3,2)} + \widetilde{W}_j^{(>3,2)}]$$

and $(DA_{j-1})$,$(DB_{j-1})$ imply $(DM_{j-1})$. Define for $y \in \Lambda_j$

$$\widetilde{R}_j(y|\Psi) = R_N^{(N)}(y|\Psi) - R_j^{(N+h)}(y|\Psi)$$

and for $x \in \Lambda_{j-1}$

$$\delta\widetilde{R}_{j-1}(x|\Psi) \equiv \widetilde{R}_{j-1}(x|\Psi) - \sum_{y \in x} Z_j^{(N),rel}(\bar{x}-y|\Psi)\widetilde{R}_j(y|\Psi).$$

Thus

$$\widetilde{R}_{j-1}(x|\Psi) = \delta\widetilde{R}_{j-1}(x|\Psi) + \sum_{k=j}^{N-1} \sum_{\substack{y \in \Lambda_k:\\ y \leq x}} Z_{k,j-1}^{(N),rel}(y|\Psi)\delta\widetilde{R}_k(y|\Psi) +$$
$$+ \sum_{\substack{y \in \Lambda_N:\\ y \leq x}} Z_{N,j-1}^{(N),rel}(y|\Psi)\delta\widetilde{R}_N(y|\Psi).$$

Since $\delta\widetilde{R}_k$ and $\widetilde{R}_k$ obey the renormalization conditions we see in the same way as in the proof of lemma 9 that $(DC_j)$ implies $(DC_{j-1})$. This completes the proof. ✓

# 4. MULTIGRID POLYMER SYSTEMS FOR EUCLIDEAN QUANTUM FIELD THEORY

In the second chapter it was shown that for weakly coupled euclidean quantum field theories in hierarchical approximation the partition function can be controlled by recursive bounds for running coupling constants and irrelevant remainder terms (irrelevant activities). It was shown that the coupling between different layers is small by using renormalization conditions for the irrelevant remainder terms. In the hierarchical model approximation the free propagator is local and therefore there is no coupling between different blocks of each layer. For each renormalization group step it was sufficient to consider the partition function for one block only. The running coupling constants were also completely defined by the effective partition function on a single block of the corresponding multigrid layer.

In this chapter we want to remove the restriction of hierarchical approximation and consider the full model. The problems of ultraviolet and/or infrared properties will be shown to be equivalent to the question of convergence of cluster expansions for polymer systems on the multigrid. The condition of convergence is related to the condition that the polymer activities are small for extensive polymers (i.e. polymers which contain a large number of blocks and/or contains blocks which are far apart). In other words the partition function ( or effective partition functions) should be well approximated by small subsystems of the multigrid.

Renormalization conditions are required to show that different layers of the multigrid are weakly coupled. To obtain renormalization conditions for *all* polymer activities, requires the introduction of polymer-dependent counterterms. This leads to a repolymerization of the polymer system with not renormalized activities and to the introduction of renormalized activities (i.e. activities which obey renormalization conditions). The differences between renormalized and not renormalized activities are small. Nevertheless, the repolymerization procedure has to be performed for each renormalization group step to avoid a blow up of such small differences.

## 4.1. Layerwise Polymer System

In this section a polymer system for each layer of the multigrid will be defined recursively. Euclidean $\Phi_4^4$-theory on a lattice will be used as an explicit example.

Consider a euclidean quantum field theory of one real field $\Phi$ on *base* (= lattice,$\mathbf{R}^d$,torus) with action

$$S(\Phi) = S_0(\Phi) + V(\Phi)$$

$$S_0(\Phi) = \frac{1}{2}\int_{z\in base} [\nabla_\mu \Phi(z)]^2 = \frac{1}{2}(\Phi, -\Delta\Phi)$$

$$V(\Phi) = \int_{z\in base} \mathcal{V}(\Phi(z)).$$



$\Delta$ is the corresponding Laplacian on *base*. $S_0$ is the action of a massless free field theory, with propagator

$$\mathrm{v} = (-\Delta)^{-1}.$$

$\mathcal{V}$ is a real valued function bounded below. For an external source $J$ the partition function (or generating function for Green functions) is

$$Z[J] = \int \prod_{z\in base} d\Phi(z) \exp\{-S(\Phi) + (J,\Phi)\}/(J=0)$$

$$(J,\Phi) = \int_{z\in base} J(z)\Phi(z).$$

For renormalization group calculations the following partition function for an external field $\Psi$ will be more useful

$$Z(\Psi) = \int \prod_{z\in base} d\Phi(z) \exp\{-S(\Phi+\Psi)\}/(\Psi=0).$$

The two partition functions are related by

$$Z(\Psi) = \exp\{-\frac{1}{2}(J,\mathrm{v}J)\} Z[J]|_{J=\mathrm{v}^{-1}\Psi}.$$

$Z(\Psi)$ is the generating function of the free-propagator-amputated Green functions. The partition function $Z(\Psi)$ can be rewritten by

$$Z(\Psi) = \int d\mu_{\mathrm{v}}(\Phi) \exp\{-V(\Phi+\Psi)\}/(\Psi=0) \qquad (1\text{–}1)$$

where $d\mu_{\mathrm{v}}(\Phi)$ is the Gaussian measure for a free field theory with propagator v, i.e.

$$d\mu_{\mathrm{v}}(\Phi) = \frac{1}{\mathcal{N}} \prod_{z\in base} d\Phi(z) \exp\{-S_0(\Phi)\}.$$

$\mathcal{N}$ is a normalization factor such that $\int d\mu_{\mathrm{v}}(\Phi) = 1$.

For renormalization group calculations an expansion over scales of the free propagator

$$\mathrm{v} = \sum_{j\in\mathbf{Z}} \mathrm{v}^j$$

is used. The integral kernel of $\mathrm{v}^j$ should obey the following bounds (for $d$ dimensions)

$$|\mathrm{v}^j(z_1,z_2)| \le K_1 a_j^{2-d} \exp\{-K_2 a_{j-1}^{-1}|z_1 - z_2|\}$$

$$|z_1 - z_2| \equiv \sum_{\mu=1}^{d} |z_1^\mu - z_2^\mu|$$

for $z_1, z_2 \in base, a_j = L^{-j}a$. $L$ is a scaling factor larger than one and $a$ is a fundamental length. $K_1$ and $K_2$ are $L$–independent positive constants. The decay length $a_{j-1}$ of $v^j$ increases if $j$ decreases. If *base* is a lattice then $a$ is the corresponding lattice spacing and an expansion over scales of the free propagator with $v^j = 0$ for $j > 0$ will be used. A theory on the $d$-dimensional Euclidean space $\mathbf{R}^d$ (=*base*) can be regularized by replacing the free propagator v in the definition (1-1) for the partition function by

$$v^{(j_I,j_U)} \equiv \sum_{i=j_I}^{j_U} v^i$$

with $j_I \leq j_U$ . For $j_U < \infty$ we have a theory with ultraviolet cutoff and for $j_I => -\infty$ we have a theory with infrared cutoff. The partition function can be recursively computed by[1]

$$Z_{j_U}(\Psi) = \exp\{-V(\Psi)\}$$

$$Z_{j-1}(\Psi) = \int d\mu_{v^j}(\Phi) Z_j(\Phi + \Psi)/(\Psi = 0)$$

for $j_I \leq j \leq j_U$ . Then the partition function for a theory with UV- and IR-cutoff reads

$$Z^{(j_I,j_U)}(\Psi) \equiv \int d\mu_{v^{(j_I,j_U)}}(\Phi) \exp\{-V(\Phi + \Psi)\}/(\Psi = 0) = Z_{j_I-1}(\Psi).$$

To be more specific consider the $\lambda\Phi^4$–theory on the $d$-dimensional lattice $(a\mathbf{Z})^d$ (=*base*). We will use the lattice notations

$$\int_{z\in(a\mathbf{Z})^d} = a^d \sum_{z\in(a\mathbf{Z})^d}, \qquad \nabla_\mu f(z) = \frac{1}{a}[f(z+e_\mu) - f(z)], \qquad -\Delta = \nabla_\mu \nabla_{-\mu}$$

where $e_\mu$ is the (lattice) vector of length $a$ in $\mu$- direction. The action is

$$S(\Phi) = \frac{1}{2}\int_{z\in base} [\nabla_\mu \Phi(z)]^2 + V(\Phi)$$

$$V(\Phi) = \frac{1}{2} m_0^2 \int_{z\in base} \Phi(z)^2 + \frac{\lambda_0}{4!}\int_{z\in base} \Phi(z)^4.$$

For the expansion of the free propagator of the free propagator $v = (-\Delta)^{-1}$ we take the decomposition

$$v = \sum_{j\leq 0} v^j, \qquad v^j = \mathcal{A}^j v^j \mathcal{A}^{j*}$$

of the propagator v on the multigrid $\Lambda = \Lambda_0 + \Lambda_{-1} + \Lambda_{-2} + \ldots$ (see chapter 2). The renormalization group equations are

$$Z_0(\Psi) = \exp\{-V(\Psi)\}, \tag{1-2a}$$

[1] Suppose that $V(\Psi)|_{\Psi=0} = 0$



$$Z_{j-1}(\Psi) = \int d\mu_{v^j}(\Phi) Z_j(\Phi + \Psi)/(\Psi = 0). \tag{1-2b}$$

The $\lambda\Phi^4$–theory in 4 dimensions has one relevant operator $\int \Phi^2$ and two marginal operators $\int(\nabla\Phi)^2$ , $\int \Phi^4$ which are invariant under $\Phi \to -\Phi$. The effective interaction is defined by

$$V_j(\Psi) \equiv -\ln Z_j(\Psi).$$

Running coupling constants for the relevant and marginal operators are defined by[1]

$$m_j^2 \equiv \mathcal{D}_y^2 V^j(\Psi)|_{\Psi=0} \tag{1-3a}$$

$$\beta_j^{\mu\nu} \equiv \mathcal{D}_y^{\mu\nu} V^j(\Psi)|_{\Psi=0} \tag{1-3b}$$

$$\lambda_j \equiv \mathcal{D}_y^4 V^j(\Psi)|_{\Psi=0} \tag{1-3c}$$

$y \in \Lambda_j$ , $\mu, \nu \in \{1, \ldots, 4\}$ , where

$$\mathcal{D}_y^2 \equiv \frac{1}{Vol(y)} \int_{z_1 \in y} \int_{z_2 \in base} \frac{\delta^2}{\delta\Psi(z_1)\delta\Psi(z_2)} \tag{1-4a}$$

$$\mathcal{D}_y^{\mu\nu} \equiv -\frac{1}{2Vol(y)} \int_{z_1 \in y} \int_{z_2 \in base} (z_1^\mu - z_2^\mu)(z_1^\nu - z_2^\nu) \frac{\delta^2}{\delta\Psi(z_1)\delta\Psi(z_2)} \tag{1-4b}$$

$$\mathcal{D}_y^4 \equiv \frac{1}{Vol(y)} \int_{z_1 \in y} \int_{z_2,z_3,z_4 \in base} \frac{\delta^4}{\delta\Psi(z_1)\ldots\delta\Psi(z_4)}. \tag{1-4c}$$

Our aim of this section is to find the following representation for the partition functions

$$Z_j(\Psi) = \sum_{\mathbf{P}\in\mathcal{P}(\Lambda_j)} Z_j^{rel}(\Lambda_j - supp\ \mathbf{P}|\Psi) \prod_{P\in\mathbf{P}} R_j^{ren}(P|\Psi), \tag{1-5}$$

where $\mathcal{P}(\Lambda_j)$ consists of sets which consist of disjoint finite nonempty subsets of $\Lambda_j$, i.e.

$$\mathcal{P}(\Lambda_j) \equiv \{\emptyset\} \cup \{\mathbf{P} | \exists n \in \mathbf{N} - \{0\} : \mathbf{P} = \{P_1, \ldots, P_n\}, P_a \subseteq \Lambda_j \text{ finite and nonempty and } P_a \cap P_b = \emptyset \text{ for } a \neq b\}$$

and $supp\ \mathbf{P} = \sum_{i=1}^n P_i$ for $\mathbf{P} = \{P_1, \ldots, P_n\}$ . $Z_j^{rel}$ is called *relevant partition function* and $R_j^{ren}$ is called *renormalized irrelevant activity*. $R_j^{ren}$ is not uniquely defined by eq. (1-5). For the definition of $R_j^{ren}$ we will introduce functions $R_j$ which obey the same relation. The representation (1-5) should obey the following conditions

1. $Z_j^{rel}$ factorizes on $\Lambda_j$ , i.e. for all $Y \subseteq \Lambda_j$

$$Z_j^{rel}(Y|\Psi) = \prod_{y\in\Lambda_j} Z_j^{rel}(y|\Psi).$$

[1] The definition for $m_j^2$, $\beta_j^{\mu\nu}$ and $\lambda_j$ is independent of $y \in \Lambda_j$ because of translation invariance of the effective theory on layer $\Lambda_j$. By euclidean lattice invariance we have $\beta_j^{\mu\nu} = \beta_j \delta^{\mu\nu}$

2. For all $y \in \Lambda_j$ we have

$$Z_j^{rel}(y|\Psi) = \exp\{-V_j^{rel}(y|\Psi)\},$$

where $V_j^{rel}$ consists of no irrelevant operators and $V_j^{rel}(y|0) = 0$,

3. $Z_j^{rel}(P|\Psi)$ and $R_j^{ren}(P|\Psi)$ depends only on $\Psi(z)$ for $z \underline{\in} P$.
4. $R_j^{ren}$ obeys the renormalization conditions

$$\begin{aligned} R_j^{ren}(P|\Psi)|_{\Psi=0} &= 0, & \mathcal{D}_z^2 R_j^{ren}(P|\Psi)|_{\Psi=0} &= 0 \\ \mathcal{D}_y^{\mu\nu} R_j^{ren}(P|\Psi)|_{\Psi=0} &= 0, & \mathcal{D}_y^4 R_j^{ren}(P|\Psi)|_{\Psi=0} &= 0 \end{aligned} \tag{1-6}$$

for all $z \in base$ , $y \in \Lambda_j$ , $P \subseteq \Lambda_j$ ( $P$ nonempty and finite ), $\mu,\nu \in \{1,\ldots,4\}$ , where

$$\mathcal{D}_z^2 \equiv \int_{z \in base} \frac{\delta^2}{\delta\Psi(z)\delta\Psi(z_2)}.$$

The definition of running coupling constants (1-3a,b,c) and conditions 1.-4. imply for $z \underline{\in} y \in \Lambda_j$

$$\mathcal{D}_z^2 V_j^{rel}(y|\Psi)|_{\Psi=0} = m_j^2 \tag{1-7a}$$

$$\mathcal{D}_y^{\mu\nu} V_j^{rel}(y|\Psi)|_{\Psi=0} = \beta_j^{\mu\nu} \tag{1-7b}$$

$$\mathcal{D}_y^4 V_j^{rel}(y|\Psi)|_{\Psi=0} = \lambda_j. \tag{1-7c}$$

Define for $j < 0$ and $y \in \Lambda_j$

$$V_j^{rel}(y|\Psi) \equiv \frac{1}{2}\int_{z \underline{\in} y} m_j^2 \Psi(z)^2 + \frac{1}{2}\sum_{\mu,\nu=1}^{4} \beta_j^{\mu\nu} \int_{z \underline{\in} y} (\nabla_\mu^{per}\Psi(z))(\nabla_\nu^{per}\Psi(z)) + \frac{\lambda_j}{4!}\int_{z \underline{\in} y} \Psi(z)^4, \tag{1-8}$$

where

$$\nabla_\mu^{per}\Psi(z) \equiv \begin{cases} \frac{1}{a}[\Psi(z + ae_\mu) - \Psi(z)], & \text{for } z + ae_\mu \underline{\in} y(z) \\ \frac{1}{(L-1)a}[\Psi(z + (ae_\mu - Lae_\mu) - \Psi(z)], & \text{for } z + ae_\mu \underline{\notin} y(z) \end{cases}$$

$y(z)$ denotes the block of $\Lambda_{-1}$ which contains $z$, i.e. $y(z) \in \Lambda_{-1}$ such that $z \underline{\in} y(z)$. Then $V_j^{rel}$ satisfies the relations (1-7a,b,c). We have defined the wave function term in such a way that $V_j^{rel}(y|\Psi)$ depends only on $\Psi(z)$ for $z \underline{\in} y$. If we would have used $\nabla_\mu$ instead of $\nabla_\mu^{per}$ this condition would not hold. The relevant partition functions $Z_j^{rel}$ are defined by

$$Z_j^{rel}(Y|\Psi) \equiv \exp\{-V_j^{rel}(Y|\Psi)\} \tag{1-9}$$

for all $Y \subseteq \Lambda_j$.

We call a finite nonempty subset $P$ of $\Lambda_j$ a *polymer* of $\Lambda_j$. Two polymers $P_1$ and $P_2$ are called compatible if $P_1 \cap P_2 = \emptyset$, i.e. $P_1$ and $P_2$ are disjoint.

In the following the *renormalized irrelevant activity* $R_j^{ren}(P|\Psi)$ will be recursively defined for all polymers $P$ of $\Lambda_j$.



The definition of the polymer system starts on the finest multigrid layer $\Lambda_0$. Then only the bare coupling constants $m_0^2$, $\beta_0^{\mu\nu}$, and $\lambda_0$ are given. The effective partition function $Z_0$ is equal to $Z_0^{rel}$. The representation (1-5) for $Z_0$ is easily fulfilled for the choice $R_0^{ren}(P|\Psi) = 0$ for all polymers $P$ of $\Lambda_0$. The definition of $R_j^{ren}$ presented here is recursive. $R_{j-1}^{ren}$ depends on $R_j^{ren}$ and the running coupling constants $m_{j-1}^2$, $\beta_{j-1}^{\mu\nu}$, $\lambda_{j-1}$. For intermediary steps auxiliary activities $R_j$ are introduced. $R_j$ is normalized, but does not obey renormalization conditions.

For a polymer $X$ of $\Lambda_{j-1}$ the free propagator $\mathrm{v}_X^j$ (restricted to $X \subseteq \Lambda_{j-1}$) is defined by

$$\mathrm{v}_X^j \equiv \mathcal{A}_X v_X^j \mathcal{A}_X^{j*}, \tag{1-10}$$

where $\mathcal{A}_X^j$ and $v_X^j$ are defined by integral kernels

$$v_X^j(y_1,y_2) \equiv \begin{cases} v^j(y_1,y_2), & \text{for } y_1,y_2 \underline{\in} X \\ 0, & \text{otherwise} \end{cases} \tag{1-11}$$

$$\mathcal{A}_X^j(z,y) \equiv \begin{cases} \mathcal{A}^j(z,y), & \text{for } z,y \underline{\in} X \\ 0, & \text{otherwise} \end{cases} \tag{1-12}$$

for all $y, y_1, y_2 \in \Lambda_j, z \in base$. The substitution $v_X^j$ for $v^j$ replaces the infinite dimensional Gaussian integral $\int d\mu_{vj}(\Phi)$ (the integration variables are $\varphi(y)$ for all $y \in \Lambda_j$) by a finite dimensional integral (integration variables are $\varphi(y)$ for all $y \underline{\in} X$). This can be seen by the following integral representation of a Gaussian expectation value

$$\begin{aligned} &\int d\mu_{vj}(\Phi)F(\Phi) = \int d\mu_{vj}(\varphi)F(\mathcal{A}^j\varphi) = \\ &= \int [\prod_{y\in\Lambda_j} \frac{d\varphi(y)dq(y)}{2\pi}] F(\mathcal{A}^j\varphi)\exp\{-\frac{1}{2}\int_{y_1,y_2\in\Lambda_j} q(y_1)v^j(y_1,y_2)q(y_2) + i\int_{y\in\Lambda_j} q(y)\varphi(y)\}. \end{aligned} \tag{1-13}$$

The second substitution $\mathcal{A}_X^j$ for $\mathcal{A}^j$ is necessary to maintain the locality property for $R_j^{ren}$ (i.e. $R_j^{ren}(P|\Psi)$ depends only on $\Psi(z)$ for $z \underline{\in} P$).

$R_j^{ren}$ and $R_j$ will be uniquely defined by partition functions $Z_j^{ren}(Y|\Psi)$ and $Z_j(Y|\Psi)$ for all finite subsets $Y$ of $\Lambda_j$. These relations are

$$Z_j(Y|\Psi) = \sum_{\mathbf{P}\in\mathcal{P}(Y)} Z_j^{rel}(Y - supp\ \mathbf{P}|\Psi) \prod_{P\in\mathbf{P}} R_j(P|\Psi). \tag{1-14}$$

$$Z_j^{ren}(Y|\Psi) = \sum_{\mathbf{P}\in\mathcal{P}(Y)} Z_j^{rel}(Y - supp\ \mathbf{P}|\Psi) \prod_{P\in\mathbf{P}} R_j^{ren}(P|\Psi) \tag{1-15}$$

for all finite subsets $Y$ of $\Lambda_j$, where $\mathcal{P}(Y)$ consists of sets which consist of disjoint nonempty subsets of $Y$, i.e.

$$\mathcal{P}(Y) \equiv \{\emptyset\} \cup \{\mathbf{P}|\exists n \in \mathbf{N} - \{0\} : \mathbf{P} = \{P_1,\ldots,P_n\}, P_a \subseteq Y \text{ finite and nonempty and } P_a \cap P_b = \emptyset \text{ for } a \neq b\}$$

and $supp\ \mathbf{P} = \sum_{i=1}^n P_i$ for $\mathbf{P} = \{P_1,\ldots,P_n\}$ .

If $Z_j(Y|\Psi)$ is defined for all finite $Y \subset \Lambda_j$ then the activities $R_j$ are uniquely defined. This can be seen as follows. For $y \in \Lambda_j$ we have

$$R_j^{ren}(y|\Psi) = Z_j^{ren}(y|\Psi) - Z_j^{rel}(y|\Psi). \tag{1-16}$$

Suppose that $|Y| = N$ and $R_j^{ren}(P|\Psi)$ is defined for all polymers $P$ with $|P| < N$. Then $R_j^{ren}(Y|\Psi)$ is defined by the following relation

$$R_j^{ren}(Y|\Psi) = Z_j^{ren}(Y|\Psi) - \sum_{\substack{\mathbf{P} \in \mathcal{P}(Y):\\ \forall P \in \mathbf{P}:\ |P|<|Y|}} Z_j^{rel}(Y - supp\ \mathbf{P}|\Psi) \prod_{P \in \mathbf{P}} R_j^{ren}(P|\Psi). \tag{1-17}$$

The right hand side of the above relation contains only $R_j(P|\Psi)$ with $|P| < N$.

Define for the first renormalization group step

$$Z_{-1}(X|\Psi) \equiv \int d\mu_{v_X^0}(\Phi) Z_0^{rel}(\overline{X}|\Phi + \Psi) \exp\{-e_{-1}(X)\} \tag{1-18}$$

where

$$e_{j-1}(X) \equiv \ln \int d\mu_{v_X^0}(\Phi) Z_0^{rel}(X|\Phi). \tag{1-19}$$

for all finite $X \subseteq \Lambda_{-1}$. We have chosen $e_{-1}(X)$ such that $Z_{-1}(X|\Psi)|_{\Psi=0} = 1$. The new running coupling constants are obtained by

$$m_{-1}^2 = -\mathcal{D}_z^2 \ln Z_{-1}(\Lambda_{-1}|\Psi)|_{\Psi=0} \tag{1-20a}$$

$$\beta_{-1}^{\mu\nu} = -\mathcal{D}_y^{\mu\nu} \ln Z_{-1}(\Lambda_{-1}|\Psi)|_{\Psi=0} \tag{1-20b}$$

$$\lambda_{-1} = -\mathcal{D}_y^4 \ln Z_{-1}(\Lambda_{-1}|\Psi)|_{\Psi=0} \tag{1-20c}$$

for all $z \in base$ and $y \in \Lambda_{-1}$. This defines the relevant interaction $V_{-1}^{rel}$ and the relevant partition function $Z_{-1}^{rel}$. The not renormalized irrelevant activities $R_{-1}$ are now uniquely determined by eq.(1-14) for $j = 0$ as explained above.

The renormalization conditions does not hold for the activities $R_{-1}$. For renormalization cancellations it is necessary to have renormalized activities in each renormalization group step. In the following it is explained how to define renormalized irrelevant activities in a general step.

Suppose the running coupling constants $m_j^2, \beta_j^{\mu\nu}, \lambda_j$ and the not normalized activities $R_j(P|\Psi)$ are determined for all polymers $P$ of $\Lambda_j$. Furthermore, suppose that $R_j(P|\Psi)|_{\Psi=0} = 0$. We want to define renormalized activities $R_j^{ren}(P|\Psi)$ for all polymers $P$ of $\Lambda_j$.

We have seen that the renormalized irrelevant activities $R_j^{ren}(P|\Psi)$ are uniquely defined if the renormalized partition function $Z_j^{ren}(Y|\Psi)$ are defined for all finite subsets $Y$ of $\Lambda_j$ (cp. eqs.(1-16,17)). Using eq.(1-15),$R_j^{ren}(P|\Psi)$ obeys the renormalization conditions (1-6) for all polymers $P$ of $\Lambda_j$ if $Z_j^{ren}$ is normalized and obeys for all finite subsets $Y$ of $\Lambda_j$ the exact renormalization conditions

$$m_j^2 = -\mathcal{D}_z^2 \ln Z_j^{ren}(Y|\Psi)|_{\Psi=0} \tag{1-21a}$$

$$\beta_j^{\mu\nu} = -\mathcal{D}_y^{\mu\nu} \ln Z_j^{ren}(Y|\Psi)|_{\Psi=0} \tag{1-21b}$$



$$\lambda_j = -\mathcal{D}_y^4 \ln Z_j^{ren}(Y|\Psi)|_{\Psi=0} \tag{1-21c}$$

for $z \underline{\in} Y$ , $y \in Y$ , $\mu,\nu \in \{1,\dots,4\}$. This is obtained by introduction of polymer-dependent counterterms. Define for $Y' \subseteq Y$

$$\delta V_j(Y',Y|\Psi) \equiv \sum_{y \in Y'} \delta V_j(y,Y|\Psi) \tag{1-22a}$$

with

$$\delta V_j(y,Y|\Psi) \equiv \frac{1}{2}\int_{z \underline{\in} y} \delta m_j^2(z|Y)\Psi(z)^2 + \frac{1}{2}\sum_{\mu,\nu=1}^{4} \delta\beta_j^{\mu\nu}(y|Y) \int_{z \underline{\in} y} (\nabla_\mu^{per}\Psi(z))(\nabla_\nu^{per}\Psi(z)) + \frac{1}{4!}\delta\lambda_j(y|Y)\int_{z \underline{\in} y} \Psi(z)^4. \tag{1-22b}$$

and

$$Z_j^{cou}(Y',Y|\Psi) \equiv \exp\{-\delta V_j(Y',Y|\Psi)\}. \tag{1-23}$$

The polymer-dependent counterterms are

$$\delta m_j^2(z|Y) \equiv \sum_{P:\emptyset \neq P \subseteq Y} \widetilde{\delta m_j^2}(z|P) \tag{1-24a}$$

$$\delta\beta_j^{\mu\nu}(y|Y) \equiv \sum_{P:\emptyset \neq P \subseteq Y} \widetilde{\delta\beta_j^{\mu\nu}}(y|P) \tag{1-24b}$$

$$\delta\lambda_j(y|Y) \equiv \sum_{P:\emptyset \neq P \subseteq Y} \widetilde{\delta\lambda_j}(y|P) \tag{1-24c}$$

with

$$\widetilde{\delta m_j^2}(z|P) \equiv \mathcal{D}_z^2 R_j(P|\Psi)|_{\Psi=0} \tag{1-25a}$$

$$\widetilde{\delta\beta_j^{\mu\nu}}(y|P) \equiv \mathcal{D}_y^{\mu\nu} R_j(P|\Psi)|_{\Psi=0} \tag{1-25b}$$

$$\widetilde{\delta\lambda_j}(y|P) \equiv \mathcal{D}_y^4 R_j(P|\Psi)|_{\Psi=0} - \frac{3}{Vol(y)}\int_{z \underline{\in} y}\int_{z' \underline{\in} P}\Big[\sum_{\substack{P_1,P_2 \subseteq P\\ P_1 \cap P_2 \neq \emptyset,\, P_1 \cup P_2 = P}} \mathcal{D}_z^2 R_j(P_1|\Psi)|_{\Psi=0}\, \mathcal{D}_{z'}^2 R_j(P_2|\Psi)|_{\Psi=0}\Big] \tag{1-25c}$$

for $z \underline{\in} \Lambda_j$ , $y \in \Lambda_j$ and $\mu,\nu \in \{1,\dots,4\}$ Finally, define the *renormalized partition function* $Z_j^{ren}$ by

$$Z_j^{ren}(Y|\Psi) \equiv \sum_{\mathbf{P} \in \mathcal{P}(Y)} Z_j^{rel}(Y - supp\ \mathbf{P}|\Psi) Z_j^{cou}(Y - supp\ \mathbf{P}, Y|\Psi) \prod_{P \in \mathbf{P}} R_j(P|\Psi). \tag{1-26}$$

Then we see that $Z_j^{ren}(y|\Psi)$ obeys the exact renormalization conditions (1-6) for all finite $Y \subseteq \Lambda_j$. This completes the definition of $R_j^{ren}$ for given $R_j$ and $m_j^2$, $\beta_j^{\mu\nu}$, $\lambda_j$. This procedure is called *renormalization group step*.

Up to now we have only defined $R_{-1}$, $m^2_{-1}$, $\beta^{\mu\nu}_{-1}$, $\lambda_{-1}$ and by the repolymerization step the renormalized activities $R^{ren}_{-1}$. In the following we want to define $R_{j-1}$ by $R^{ren}_j$, $m^2_j$, $\beta^{\mu\nu}_j$, $\lambda_j$.

Before we are going to define $Z_{j-1}(X|\Psi)$ for all finite $X \subseteq \Lambda_{j-1}$ let us explain how to use the renormalization conditions for renormalization cancellations. This will serve as a motivation for later definitions.

Consider the second order term of a Taylor expansion in $\Psi$ of the renormalized activity $R^{ren}_j(P|\Psi)$

$$R^{(2)}_j(P|\Psi) \equiv \int_{z_1,z_2 \underline{\in} P} \Psi(z_1)\Psi(z_2)\frac{\delta^2}{\delta\Psi(z_1)\delta\Psi(z_2)} R^{ren}_j(P|\Psi)|_{\Psi=0}. \tag{1-27}$$

Using the renormalization condition

$$\mathcal{D}^2_z R^{ren}_j(P|\Psi)|_{\Psi=0} = 0$$

we see that $R^{(2)}_j(P|\Psi)$ can be rewritten

$$R^{(2)}_j(P|\Psi) = -\frac{1}{2}\int_{z_1,z_2 \underline{\in} P} [\Psi(z_1) - \Psi(z_2)]^2 \frac{\delta^2}{\delta\Psi(z_1)\delta\Psi(z_2)} R^{ren}_j(P|\Psi)|_{\Psi=0}. \tag{1-28}$$

Suppose that $\Psi$ is smooth on length scale $a_{j-1}$. Then we get by eq.(1-28) a suppression factor $(a_j/a_{j-1}))^2 = L^{-2}$. For further renormalization group steps $\Psi$ has to be replaced by $\mathcal{A}^{j-1}\varphi$. $\mathcal{A}^{j-1}\varphi$ is smooth on length scale $a_{j-1}$ and the supression factor $L^{-2}$ is gained. But the multigrid field $\varphi$ is not bounded. The base space fields $\Psi$ are well localized in the Taylor expansion (1-27) for $R^{ren}_j(P|\Psi)$ and can be easily bounded using stability bounds. After application of the renormalization conditions (see eq.(1-28)) we have to bound fields $\varphi$ which live on the multigrid layer $\Lambda_{j-1}$. The field $\varphi(x)$ can be estimated for $x \in X$, where $X$ is some polymer of $\Lambda_{j-1}$ which contains the polymer $P$. Fields $\varphi(x)$ for $x \in \Lambda_{j-1}$, where the distance from $x$ to $P$ is larger than a fixed length $\delta$ (in units of $a_j$), can be controlled by using the fact that $\mathcal{A}^j(z,x)$ is small for $z \underline{\in} P$. Therefore if in the definition of $Z_{j-1}(X|\Psi)$ only activities $R^{ren}_j(P|\Psi)$ occur for polymers $P$ such that all blocks $x \in \Lambda_{j-1} - X$ have a distance to $P$ larger than $\delta$ (in units of $a_j$) then the multigrid fields $\varphi$ can be controlled after the use of renormalization cancellations.

After these remarks on large field problem and renormalization we will now define $Z^c_j(X|\Psi)$ for all finite $X \subseteq \Lambda_{j-1}$. For all finite subsets $X$ of $\Lambda_{j-1}$ and a positive real number $\delta$ define[1]

$$Z^c_j(X|\Psi) = \sum_{\mathbf{P}\in\mathcal{P}_\delta(X)} Z^{rel}_j(\overline{X} - supp\ \mathbf{P}|\Psi) \prod_{P\in\mathbf{P}} R^{ren}_j(P|\Psi) \tag{1-29}$$

where

$$\mathcal{P}_\delta(X) \equiv \{\mathbf{P} \in \mathcal{P}(\overline{X}) | U_\delta(supp\ \mathbf{P}) \subseteq \overline{X}\} \tag{1-30}$$

and for $Y \subseteq \Lambda_j$ the $\delta$- neighborhood of $Y$ is defined by

$$U_\delta(Y) \equiv \{y \in \Lambda_j | dist(y, Y) < \delta a_j\}. \tag{1-31}$$

[1] $\overline{X} \equiv \{y \in \Lambda_j | \exists x \in X : y \underline{\in} x\}$



Furthermore define for all finite $X \subseteq \Lambda_{j-1}$

$$Z_{j-1}(X|\Psi) \equiv \int d\mu_{v^j_X}(\Phi) Z^c_j(X|\Phi + \Psi)\exp\{-e_{j-1}(X)\} \tag{1-32}$$

where

$$e_{j-1}(X) \equiv \ln \int d\mu_{v^j_X}(\Phi) Z^c_j(X|\Phi). \tag{1-33}$$

Then $R_{j-1}$ is defined by the following relation

$$Z_{j-1}(X|\Psi) = \sum_{\mathbf{Q}\in\mathcal{P}(X)} Z^{rel}_{j-1}(X - supp\ \mathbf{Q}|\Psi) \prod_{Q\in\mathbf{Q}} R_{j-1}(Q|\Psi). \tag{1-34}$$

for all polymers $X$ of $\Lambda_{j-1}$ (cp. eq. (1-14)). $Z^{rel}_{j-1}$ on the right hand side of eq.(1-34) is defined by

$$m^2_j = -\mathcal{D}^2_z \ln Z_{j-1}(\Lambda_{j-1}|\Psi)|_{\Psi=0} \tag{1-35a}$$

$$\beta^{\mu\nu}_j = -\mathcal{D}^{\mu\nu}_x \ln Z_{j-1}(\Lambda_{j-1}|\Psi)|_{\Psi=0} \tag{1-35b}$$

$$\lambda_j = -\mathcal{D}^4_x \ln Z_{j-1}(X|\Psi)|_{\Psi=0} \tag{1-35c}$$

for $z \underline{\in} X$ , $x \in X$ , $\mu, \nu \in \{1, \ldots, 4\}$.

Let us repeat the main steps for the definition of the polymer system. We have started with the known relevant partition function $Z^{rel}_0$. Then we have defined $Z_{-1}(X|\Psi)$ for all finite $X \subseteq \Lambda_{-1}$. Then we have determined the running coupling constants $m^2_{-1}$, $\beta^{\mu\nu}_{-1}$, $\lambda_{-1}$. By relation (1-14) $R_{-1}(P|\Psi)$ was defined for all polymers $P$ of $\Lambda_{-1}$. In a repolymerization step we have defined $R^{ren}_{-1}$. We have supposed that $R^{ren}_j$ and $m^2_j$, $\beta^{\mu,\nu}_j$, $\lambda_j$ are given. Then we have defined $Z^c_j(X|\Psi)$ for all finite $X \subseteq \Lambda_{j-1}$ by suitable boundary conditions. $Z_{j-1}(X|\Psi)$ was defined as a Gaussian expectation value of $Z^c_j(X|\Psi)$. Then we have defined the running coupling constants $m^2_{j-1}$, $\beta^{\mu\nu}_{j-1}$, $\lambda_{j-1}$ and by this the activities $R_{j-1}$. By a repolymerization group step we could define $R^{ren}_{j-1}$ since $R_{j-1}$ and $m^2_{j-1}$, $\beta^{\mu\nu}_{j-1}$, $\lambda_{j-1}$ are defined. This finishes our recursive definition of the polymer system on the multigrid.

For further applications the following definitions are introduced. Define activities $A_j$, $A^c_j$, $A^{ren}_j$ for all polymers $X$ of $\Lambda_{j-1}$ and $Y$ of $\Lambda_j$ by

$$Z_j(Y|\Psi) = \sum_{Y=\sum P} \prod_P A_j(P|\Psi) \tag{1-36a}$$

$$Z^c_j(X|\Psi) = \sum_{X=\sum Q} \prod_Q A^c_j(Q|\Psi) \tag{1-36b}$$

$$Z^{ren}_j(Y|\Psi) = \sum_{Y=\sum P} \prod_P A^{ren}_j(P|\Psi). \tag{1-36c}$$

Eqs. (1-14),(1-15) and

$$Z^{rel}_j(Y|\Psi) = \prod_{y\in\Lambda_j} Z^{rel}_j(y|\Psi)$$

imply for all polymers $P$ of $\Lambda_j$

$$A_j(P|\Psi) = \delta_{1,|P|} Z_j^{rel}(P|\Psi) + R_j(P|\Psi) \qquad (1\text{–}37a)$$

$$A_j^{ren}(P|\Psi) = \delta_{1,|P|} Z_j^{rel}(P|\Psi) + R_j^{ren}(P|\Psi). \qquad (1\text{–}37b)$$

Furthermore, define $R_j^c$ such that

$$A_j^c(Q|\Psi) = \delta_{1,|Q|} Z_j^{rel}(Q|\Psi) + R_j^c(Q|\Psi) \qquad (1\text{–}37c)$$

for all polymers $Q$ of $\Lambda_{j-1}$. For the smallest scale $j = 0$ we set $R_0 \equiv R_0^{ren} \equiv R_0^c \equiv 0$.

*Summary* :

(i): For all finite subsets $Y$ of $\Lambda_j$ partition functions $Z_j$ are defined by

$$Z_j(Y|\Psi) = \sum_{\mathbf{P}\in\mathcal{P}(Y)} Z_j^{rel}(Y - supp\ \mathbf{P}|\Psi) \prod_{P\in\mathbf{P}} R_j(P|\Psi).$$

(ii) Repolymerization step : For all finite subsets $Y$ of $\Lambda_j$ partition functions $Z_j^{ren}$ are defined by

$$Z_j^{ren}(Y|\Psi) \equiv \sum_{\mathbf{P}\in\mathcal{P}(Y)} Z_j^{rel}(Y - supp\ \mathbf{P}|\Psi) Z_j^{cou}(Y - supp\ \mathbf{P}, Y|\Psi) \prod_{P\in\mathbf{P}} R_j(P|\Psi).$$

$Z_j^{cou}$ is chosen such that the renormalization conditions

$$m_j^2 = -\mathcal{D}_z^2 \ln Z_j^{ren}(Y|\Psi)|_{\Psi=0}$$

$$\beta_j^{\mu\nu} = -\mathcal{D}_y^{\mu\nu} \ln Z_j^{ren}(Y|\Psi)|_{\Psi=0}$$

$$\lambda_j = -\mathcal{D}_y^4 \ln Z_j^{ren}(Y|\Psi)|_{\Psi=0}$$

are fulfilled for all polymers $Y$ of $\Lambda_j$ and all $z \underline{\in} Y$ , $y \in Y$ and $\mu, \nu \in \{1, \ldots, 4\}$. $R_j^{ren}$ is defined by the relation

$$Z_j^{ren}(Y|\Psi) = \sum_{\mathbf{P}\in\mathcal{P}(Y)} Z_j^{rel}(Y - supp\ \mathbf{P}|\Psi) \prod_{P\in\mathbf{P}} R_j^{ren}(P|\Psi)$$

for all finite subsets $Y$ of $\Lambda_j$ .

(iii) Coarsening step : For all finite subsets $X$ of $\Lambda_{j-1}$ define partition functions $Z_j^c$ by

$$Z_j^c(X|\Psi) \equiv \sum_{\mathbf{P}\in\mathcal{P}_s(X)} Z_j^{rel}(\overline{X - supp\ \mathbf{P}}|\Psi) \prod_{P\in\mathbf{P}} R_j^{ren}(P|\Psi).$$

(iv) Integration step : For all finite subsets $X$ of $\Lambda_{j-1}$ define partition functions $Z_{j-1}$ by

$$Z_{j-1}(X|\Psi) \equiv \int d\mu_{v_X^j}(\Phi) Z_j^c(X|\Phi + \Psi) \exp\{-e_{j-1}(X)\}.$$



$e_{j-1}$ is chosen such that the normalization conditions

$$Z_{j-1}(X|\Psi)|_{\Psi=0} = 1$$

are fulfilled for all finite subsets $X$ of $\Lambda_{j-1}$.

To obtain bounds for the various activities more explicit relations for activities are required. This will be done in the next section.

## 4.2. Cluster Expansion Formulas for Activities

Explicit relations for activities are represented in this section using cluster expansion techniques. A simple general tree formula is introduced [37] which generalizes the tree formula of Glimm, Jaffe, Spencer, Brydges and Federbush [4, 22].

Suppose that for each finite subset $X$ of a denumerable set $\Lambda = \{x_1, x_2, \ldots\}$ a partition function $Z(X)$ is defined. For each pair $(x_a x_b)$ of distinct elements of $\Lambda$ associate a real variable $t_{x_a x_b}$ and define $t$-dependent partition functions $Z_t(X)$ such that

(i) $Z_t(X)$ depends only on $t_{x_a x_b}$ if $x_a, x_b \in X$.

(ii) $Z_t(X) = 1$ if $t_{x_a x_b} = 1$ for all $x_a, x_b \in X$.

(iii) $Z_t(X_1 + X_2) = Z_t(X_1) Z_t(X_2)$ if $t_{x_a x_b} = 0$ for all pairs $(x_a x_b)$ with $x_a \in X_1$ and $x_b \in X_2$.

(iv) $Z_t(X)$ is differentiable.

$Z_t(X)$ is called an *interpolated partition function.* For $n \in \mathbf{N}$ , $n \leq 2$ , introduce mappings $\eta$ from $\{2, \ldots, n\}$ into $\{1, \ldots, n-1\}$, providing $\eta(i) < i$ , real variables $s_1, \ldots, s_{n-1}$, and monomial

$$f(\eta|s) \equiv \prod_{i=2}^{n} [s_{i-2} s_{i-3} \cdots s_{\eta(i)}]. \qquad (2\text{–}1)$$

For all finite nonempty subsets $X$ of $\Lambda$ with $|X| = n$ and $x \in X$ introduce bijective mappings $\tilde{x}$ from $\{1, \ldots, n\}$ into $X$ , $|X| = n$ , such that $\tilde{x}(1) = x$ . For a given mapping $\tilde{x}$ and real variables $s_1, \ldots, s_{n-1}$ we define

$$t[\tilde{x}, s]_{\tilde{x}(i)\tilde{x}(j)} = s_i s_{i+1} \cdots s_{j-1} \qquad (2\text{–}2)$$

for all $i, j \in \{1, \ldots, n\}$ , $i < j$. For mappings $\eta$ and $\tilde{x}$ define a tree graph $\tau(\tilde{x}, \eta)$ by

$$\tau(\tilde{x}, \eta) \equiv \{(\tilde{x}(i)\tilde{x}(\eta(i)))| i \in \{2, \ldots, n\}\}. \qquad (2\text{–}3)$$

For all finite nonempty subsets $X$ of $\Lambda$ define activities $A(P)$ by the relations

$$Z(X) = \sum_{X=\sum P} \prod_{P} A(P). \qquad (2\text{–}4)$$

Then we have

PROPOSITION 2.1. *For $n \geq 2$ and $X \subseteq \Lambda$ , $|X| = n$ , we have*

$$A(X) = \sum_{\tau \in T(X)} \sum_{\substack{\tilde{z},\eta: \\ \tau(\tilde{z},\eta)=\tau}} \int_0^1 ds_1 \dots ds_{n-1} f(\eta|s) \{ \prod_{(ab)\in\tau} \frac{\partial}{\partial t_{ab}} \} Z_t(X)|_{t=t[\tilde{z},s]}. \tag{2-5}$$

*$T(X)$ denotes the set of all trees with vertex set $X$.*

For a proof see [7] or app. C. Associate for each nonempty subset $P$ of $\Lambda$ a tree $\tau(P) \in T(P)$ and define

$$Z_t(X) \equiv \sum_{X=\sum P} \prod_P [ \prod_{(xy)\in\tau(P)} t_{xy} ].$$

With this definition we find, using proposition 2.1,

LEMMA 2.2. *For all $\tau \in T(X)$ , $|X| = n \leq 2$ , we have*

$$\sum_{\substack{\tilde{z},\eta: \\ \tau(\tilde{z},\eta)=\tau}} \int_0^1 ds_1 \dots ds_{n-1} f(\eta|s) = 1. \tag{2-6}$$

For a further application of proposition 2.1 suppose that the partition function $Z$ is represented by

$$Z(X) = \sum_{n\geq1} \frac{1}{n!} \sum_{\substack{Q_1,\dots,Q_n \subseteq X: \\ \sum Q_i = X}} \widehat{Z}(Q_1,\dots,Q_n), \tag{2-7}$$

where $\widehat{Z}(Q_1,\dots,Q_n)$ is defined for all mutually disjoint finite nonempty subsets $Q_a$ of $\Lambda$. Define an interpolated partition function $\widehat{Z}_t(Q_1,\dots,Q_n)$ for real variables $t_{ab}$ , $a,b \in \{1,\dots,n\}, a \neq b$ , such that

(i') $\widehat{Z}_t(Q_i, i \in I)$ depends only on $t_{ab}$ if $a,b \in I$ for $I \subseteq \underline{n} \equiv \{1,\dots,n\}$.

(ii') $\widehat{Z}_t(Q_i, i \in I) = 1$ if $t_{ab} = 1$ for all $a,b \in I$ and $I \subseteq \underline{n}$ .

(iii') $\widehat{Z}_t(Q_i, i \in I_1 + I_2) = \widehat{Z}_t(Q_i, i \in I_1)\widehat{Z}_t(Q_i, i \in I_2)$ if $t_{ab} = 0$ for all pairs $(ab)$ with $a \in I_1$ , $b \in I_2$ .

(iv') $\widehat{Z}_t$ is differentiable.

By proposition 2.1 we obtain follows

$$\widehat{Z}(Q_1,\dots,Q_n) = \sum_{\underline{n}=\sum I} \prod_I \widehat{A}(Q_i, i \in I), \tag{2-8}$$

where the sum runs over all partitions of $\underline{n}$ into nonempty subsets and

$$\widehat{A}(Q_1,\dots,Q_n) = \sum_{\tau\in T_n} \sum_{\substack{\pi,\eta: \\ \tau=\tau(\pi,\eta)}} \int_0^1 ds\ f(\eta|s) \{ \prod_{(ab)\in\tau} \frac{\partial}{\partial t_{ab}} \} \widehat{Z}_t(Q_1,\dots,Q_n)|_{t=t[\pi,s]}. \tag{2-9}$$

$\pi$ runs over all permutations of $\{1,\dots,n\}$ with $\pi(1) = 1$ and $T_n$ denotes the set of all trees with vertex set $\{1,\dots,n\}$ .



LEMMA 2.3.

$$A(X) = \sum_{n\geq1} \frac{1}{n!} \sum_{\substack{Q_1,\dots,Q_n \subseteq X: \\ \sum Q_i = X}} \widehat{A}(Q_1,\dots,Q_n). \tag{2-10}$$

In the following the activities for the layerwise polymer system defined in the last section are represented by tree formulas.

1. Repolymerization step :

For polymers $P, Y$ of $\Lambda_j$ with $P \subseteq Y$ , define

$$A_j(P,Y|\Psi) \equiv \delta_{1,|P|} Z_j^{rel}(P|\Psi) Z_j^{cou}(P,Y|\Psi) + R_j(P|\Psi) \tag{2-11}$$

and for all polymers $P_1,\dots,P_n$ with $\sum P_i = Y$ define

$$\widehat{Z}_j^{ren}(P_1,\dots,P_n|\Psi) \equiv \prod_{i=1}^n A_j(P_i,Y|\Psi). \tag{2-12}$$

Then eq. (1-15) reads

$$Z_j^{ren}(Y|\Psi) = \sum_{n\geq1} \frac{1}{n!} \sum_{\substack{P_1,\dots,P_n \subseteq Y: \\ \sum P_i = Y}} \widehat{Z}_j(P_1,\dots,P_n|\Psi). \tag{2-13}$$

For $P, P_1,\dots,P_n \subseteq Y$ with $\sum P_i = Y$ , $z \underline{\in} Y$ , $y \in Y$ , $\mu,\nu \in \{1,\dots,4\}$ define

$$\widetilde{\delta m^2_{j,t}}(z|P) \equiv \{ \prod_{\substack{(ab): \\ z\underline{\in}P_a,\ P\cap P_b\neq\emptyset}} t_{ab} \} \widetilde{\delta m^2_j}(z|P) \tag{2-14a}$$

$$\widetilde{\delta\beta^{\mu\nu}_{j,t}}(y|P) \equiv \{ \prod_{\substack{(ab): \\ y\in P_a,\ P\cap P_b\neq\emptyset}} t_{ab} \} \widetilde{\delta\beta^{\mu\nu}_j}(y|P) \tag{2-14b}$$

$$\widetilde{\delta\lambda_{j,t}}(y|P) \equiv \{ \prod_{\substack{(ab): \\ y\in P_a,\ P\cap P_b\neq\emptyset}} t_{ab} \} \widetilde{\delta\lambda_j}(y|P) \tag{2-14c}$$

and

$$\delta m^2_{j,t}(z|Y) \equiv \sum_{P:\emptyset\neq P\subseteq Y} \widetilde{\delta m^2_{j,t}}(z|P) \tag{2-15a}$$

$$\delta\beta^{\mu\nu}_{j,t}(y|Y) \equiv \sum_{P:\emptyset\neq P\subseteq Y} \widetilde{\delta\beta^{\mu\nu}_{j,t}}(y|P) \tag{2-15b}$$

$$\delta\lambda_{j,t}(y|Y) \equiv \sum_{P:\emptyset\neq P\subseteq Y} \widetilde{\delta\lambda_{j,t}}(y|P) \tag{2-15c}$$

and interpolated polymer-dependent counterterms

$$\delta V_{j,t}(P,Y|\Psi) \equiv \sum_{y\in P} \Big\{ \frac{1}{2} \int_{z\underline{\in}y} \delta m^2_{j,t}(z|Y)\Psi(z)^2 + \frac{1}{2} \sum_{\mu,\nu=1}^4 \delta\beta^{\mu\nu}_{j,t}(y|Y) \int_{z\underline{\in}y} (\nabla^{per}_\mu \Psi(z))(\nabla^{per}_\nu \Psi(z)) + \frac{\delta\lambda_{j,t}(y|Y)}{4!} \int_{z\underline{\in}y} \Psi(z)^4 \Big\} \tag{2-16}$$

$$Z^{cou}_{j,t}(P,Y|\Psi) \equiv \exp\{-\delta V_{j,t}(P,Y|\Psi)\} \tag{2-17}$$

$$A_{j,t}(P,Y|\Psi) \equiv \delta_{1,|P|} Z^{rel}_j(P|\Psi) Z^{cou}_{j,t}(P,Y|\Psi) + R_j(P|\Psi) \tag{2-18}$$

and

$$\widehat{Z}^{ren}_{j,t}(P_1,\dots,P_n|\Psi) \equiv \prod_{i=1}^{n} A_{j,t}(P_i,Y|\Psi). \tag{2-19}$$

$\widehat{Z}^{ren}_{j,t}$ obeys the conditions i′ − iv′. Thus we have for the activities $A^{ren}_j$ of $Z^{ren}_j$ (cp. eq. (1-36))

$$A^{ren}_j(Y|\Psi) = A_j(Y,Y|\Psi) + \sum_{n\geq 2}\frac{1}{n!} \sum_{\substack{P_1,\dots,P_n\subseteq Y:\\ \sum P_i = Y}} \sum_{\tau\in T_n} \sum_{\substack{\pi,\eta:\\ \tau=\tau(\pi,\eta)}} \int_0^1 ds\ f(\eta|s)\{\prod_{(ab)\in\tau}\frac{\partial}{\partial t_{ab}}\} \prod_{i=1}^{n} A_{j,t}(P_i,Y|\Psi)|_{t=t[\pi,s]}, \tag{2-20}$$

or equivalently

$$R^{ren}_j(Y|\Psi) = \delta_{1,|Y|} Z^{rel}_j(Y|\Psi)[Z^{cou}_j(Y,Y|\Psi) - 1] + R_j(Y|\Psi) + \sum_{n\geq 2}\frac{1}{n!} \sum_{\substack{P_1,\dots,P_n\subseteq Y:\\ \sum P_i = Y}} \sum_{\tau\in T_n} \sum_{\substack{\pi,\eta:\\ \tau=\tau(\pi,\eta)}} \int_0^1 ds\ f(\eta|s)\{\prod_{(ab)\in\tau}\frac{\partial}{\partial t_{ab}}\} \prod_{i=1}^{n} A_{j,t}(P_i,Y|\Psi)|_{t=t[\pi,s]}. \tag{2-21}$$

2. Coarsening step :

For a polymer $P$ of $\Lambda_j$ a polymer $[P]$ of $\Lambda_{j-1}$ is defined by

$$[P] \equiv \{x \in \Lambda_{j-1} | \exists y \in P :\ y \underline{\in} x\}.$$

For a polymer $X$ of $\Lambda_{j-1}$ and a polymer $P$ of $\Lambda_j$ with $P \subseteq \overline{X}$ define

$$A^{ren}_j(P,X|\Psi) \equiv \begin{cases} \delta_{1,|P|} Z^{rel}_j(P|\Psi), & \text{for } [U_\delta(P)] \not\subseteq X \\ A^{ren}_j(P|\Psi), & \text{otherwise.} \end{cases} \tag{2-22}$$

Then we have (cp. eq. (1-29))

$$Z^c_j(X|\Psi) \sum_{\overline{X}=\sum P} \prod_P A^{ren}_j(P,X|\Psi). \tag{2-23}$$

For sets $P_1,\dots,P_n$ the Venn-diagram $\gamma(P_1,\dots,P_n)$ consists of vertices $P_1,\dots,P_n$ and lines $(P_aP_b)$ iff $P_a \cap P_b \neq \emptyset$. Furthermore define for a polymer $X$ of $\Lambda_{j-1}$

$$\mathcal{P}^c_\delta(X) \equiv \{\mathbf{P} \in \mathcal{P}_\delta(X) | \mathbf{P} = \{P_1,\dots,P_n\},\ \gamma([U_\delta(P_1)],\dots,[U_\delta(P_n)]) \text{ is connected and } [U_\delta(supp\ \mathbf{P})] = X\}. \tag{2-24}$$



Eqs. (2-23) and (1-36b) imply

$$A^c_j(X|\Psi) = \sum_{\mathbf{P}\in\mathcal{P}^c_\delta(X)} \prod_{P\in\mathbf{P}} A^{ren}_j(P,X|\Psi), \tag{2-25}$$

or equivalently

$$R^c_j(X|\Psi) \equiv -\delta_{1,|X|} Z^{rel}_j(\overline{X}|\Psi) + A^c_j(X|\Psi) = \sum_{\mathbf{P}\in\mathcal{P}^c_\delta(X)} Z^{rel}_j(\overline{X} - supp\ \mathbf{P}|\Psi) \prod_{P\in\mathbf{P}} R^{ren}_j(P|\Psi). \tag{2-26}$$

3. Integration step :

For polymers $Q_1,\dots,Q_n,X$ of $\Lambda_{j-1}$ with $\sum Q_i = X$ define

$$\widehat{Z}_{j-1}(Q_1,\dots,Q_n|\Psi) \equiv \int d\mu_{v^j_X}(\Phi) \prod_{i=1}^{n} A^c_j(Q_i|\Phi+\Psi) \exp\{-e_{j-1}(X)\} \tag{2-27}$$

and

$$v^j_X[t](y_1,y_2) \equiv t_{ab} v^j_X(y_1,y_2) \tag{2-28a}$$

$$\mathcal{A}^j_X[t](z,y) \equiv t_{ab}\mathcal{A}^j_X(z,y) \tag{2-28b}$$

$$\mathrm{v}^j_X[t] \equiv \mathcal{A}^j_X[t] v^j_X[t] \mathcal{A}^{j*}_X[t] \tag{2-28c}$$

if $y_1 \underline{\in} Q_a$ , $y_2 \underline{\in} Q_b$ and $z \underline{\in} Q_a$ , $y \underline{\in} Q_b$. For a polymer $Q \subseteq X$ define

$$\widetilde{e}_{j-1,t}(Q) \equiv \{\prod_{\substack{(ab):\\ Q\cap Q_a,\ Q\cap Q_b\neq\emptyset}} t_{ab}\}\ \widetilde{e}_{j-1}(Q) \tag{2-29a}$$

$$e_{j-1,t}(X) \equiv \sum_{Q:\emptyset\neq Q\subseteq X} \widetilde{e}_{j-1,t}(Q). \tag{2-29b}$$

The interpolated partition function $\widehat{Z}_{j-1,t}$ is defined by

$$\widehat{Z}_{j-1,t}(Q_1,\dots,Q_n|\Psi) \equiv \int d\mu_{\mathrm{v}^j_X[t]}(\Phi) \prod_{i=1}^{n} A^c_j(Q_i|\Phi+\Psi) \exp\{-e_{j-1,t}(X)\} \tag{2-30}$$

and fulfils conditions (i′) - (iv′) . Thus we have for the activity $A_{j-1}$ of $Z_{j-1}$

$$A_{j-1}(X|\Psi) \equiv \int d\mu_{\mathrm{v}^j_X}(\Phi) A^c_j(X|\Phi+\Psi)\exp\{-e_{j-1}(X)\} + \sum_{n\geq 2}\frac{1}{n!} \sum_{\substack{Q_1,\dots,Q_n\subseteq X:\\ \sum Q_i = X}} \sum_{\tau\in T_n} \sum_{\substack{\pi,\eta:\\ \tau=\tau(\pi,\eta)}} \int_0^1 ds\ f(\eta|s)\{\prod_{(ab)\in\tau}\frac{\partial}{\partial t_{ab}}\} \int d\mu_{\mathrm{v}^j_X[t]}(\Phi) \prod_{i=1}^{n} A^c_j(Q_i|\Phi+\Psi)\exp\{-e_{j-1,t}(X)\}|_{t=t[\pi,s]}, \tag{2-31}$$

or equivalently

$$R_{j-1}(X|\Psi) = \delta_{1,|X|} \int d\mu_{v^j_X}(\Phi)\left[Z_j^{rel}(\overline{X}|\Phi+\Psi)\exp\{-e_{j-1}(X)\} - Z_{j-1}^{rel}(X|\Psi)\right] +$$
$$+ \int d\mu_{v^j_X}(\Phi) R_j^c(X|\Phi+\Psi)\exp\{-e_{j-1}(X)\} +$$
$$+ \sum_{n\geq 2}\frac{1}{n!} \sum_{\substack{Q_1,\ldots,Q_n\subseteq X:\\ \sum Q_i = X}} \sum_{\tau\in T_n} \sum_{\substack{\pi,\eta:\\ \tau=\tau(\pi,\eta)}} \int_0^1 ds\ f(\eta|s)\{\prod_{(ab)\in\tau}\frac{\partial}{\partial t_{ab}}\} \tag{2-32}$$
$$\int d\mu_{v^j_{X}[t]}(\Phi)\prod_{i=1}^n A_j^c(Q_i|\Phi+\Psi)\exp\{-e_{j-1,t}(X)\}|_{t=t[\pi,s]}.$$

*Summary* :

(i) Repolymerization step :

$$R_j^{ren}(Y|\Psi) = \delta_{1,|Y|} Z_j^{rel}(Y|\Psi)[Z_j^{cou}(Y,Y|\Psi)-1] + R_j(Y|\Psi) + \sum_{n\geq 2}\frac{1}{n!}$$
$$\sum_{\substack{P_1,\ldots,P_n\subseteq Y:\\ \sum P_i = Y}} \sum_{\tau\in T_n} \sum_{\substack{\pi,\eta:\\ \tau=\tau(\pi,\eta)}} \int_0^1 ds\ f(\eta|s)\{\prod_{(ab)\in\tau}\frac{\partial}{\partial t_{ab}}\}\prod_{i=1}^n A_{j,t}(P_i,Y|\Psi)|_{t=t[\pi,s]}, \tag{2-21}$$

with

$$A_{j,t}(P,Y|\Psi) = \delta_{1,|P|} Z_j^{rel}(P|\Psi) Z_{j,t}^{cou}(P,Y|\Psi) + R_j(P|\Psi). \tag{2-18}$$

(ii) Coarsening step :

$$R_j^c(X|\Psi) = \sum_{\mathbf{P}\in\mathcal{P}^c_\delta(X)} Z_j^{rel}(\overline{X} - supp\ \mathbf{P}|\Psi)\prod_{P\in\mathbf{P}} R_j^{ren}(P|\Psi). \tag{2-26}$$

(iii) Integration step :

$$R_{j-1}(X|\Psi) = \delta_{1,|X|} \int d\mu_{v^j_X}(\Phi)\left[Z_j^{rel}(\overline{X}|\Phi+\Psi)\exp\{-e_{j-1}(X)\} - Z_{j-1}^{rel}(X|\Psi)\right] +$$
$$+ \int d\mu_{v^j_X}(\Phi) R_j^c(X|\Phi+\Psi)\exp\{-e_{j-1}(X)\} +$$
$$+ \sum_{n\geq 2}\frac{1}{n!} \sum_{\substack{Q_1,\ldots,Q_n\subseteq X:\\ \sum Q_i = X}} \sum_{\tau\in T_n} \sum_{\substack{\pi,\eta:\\ \tau=\tau(\pi,\eta)}} \int_0^1 ds\ f(\eta|s)\{\prod_{(ab)\in\tau}\frac{\partial}{\partial t_{ab}}\} \tag{2-32}$$
$$\int d\mu_{v^j_{X}[t]}(\Phi)\prod_{i=1}^n A_j^c(Q_i|\Phi+\Psi)\exp\{-e_{j-1,t}(X)\}|_{t=t[\pi,s]}.$$

with

$$A_j^c(Q|\Psi) = \delta_{1,|Q|} Z_j^{rel}(\overline{Q}|\Psi) + R_j^c(Q|\Psi). \tag{2-33}$$



### 4.3. Recursive Bounds on the Layerwise Polymer System

In this section bounds on the irrelevant activities and running coupling constants are represented.

To state recursive bounds, let us introduce various definitions and notations. For a polymer $P$ of $\Lambda_j$, $d$-dimensional *base*, and a constant $\epsilon$, $0<\epsilon<1$ define[1]

$$\mathcal{K}_j(P) \equiv \Big\{\chi:\ base\to\mathbf{C}\ |\ \forall\ z,z_1,z_2 \underline{\in} y,\ y\in P:\ |\chi(z)|\leq a_j^{1-d/2}\lambda_j^{-1/4},$$
$$|\chi(z_1)-\chi(z_2)|\leq a_j^{-d/2}|z_1-z_2|\lambda_j^{-1/4},$$
$$|\chi_2(z_1,z_2)|\leq a_j^{-d/2-1+\epsilon}|z_1-z_2|^{2-\epsilon}\lambda_j^{-1/4}, \tag{3-1a}$$
$$|\nabla_\mu\chi(z_1)-\nabla_\mu\chi(z_2)|\leq a_j^{-d/2-1}|z_1-z_2|\ \lambda_j^{-1/4}$$
$$\forall\mu\in\{1,\ldots,d\}\Big\}$$

$$\mathcal{F}_j(P)\equiv\{\Psi: base\to\mathbf{C}\ |\ \exists\ \varphi:\Lambda_j\to\mathbf{R}\ ,\ \chi\in c_3\mathcal{K}_j(P):\ \Psi=\mathcal{A}^j\varphi+\chi\}. \tag{3-1b}$$

For a polymer $Y$ of $\Lambda_j$ define the minimal number of $\delta$-neighborhoods which are a covering of $Y$

$$\rho(Y)\equiv\min\{|P|\ |\ P\subseteq Y,\cup_{y\in P}U_\delta(y)\supseteq Y\} \tag{3-2}$$

and

$$n_Y\equiv\frac{1}{2}[\frac{\rho(Y)-1}{4}]+\frac{1}{2}, \tag{3-3}$$

where

$$[r]\equiv\max\{m\in\mathbf{Z}|\ m\leq r\}$$

for $r\in\mathbf{R}$. For a polymer $P$ of $\Lambda_j$ define a tree distance by

$$L_{tree}(P)\equiv\min_{\tau\in T(P)}\{\sum_{(yy')\in\tau} dist(y,y')\}, \tag{3-4}$$

where $T(P)$ is the set of all tree graphs with vertex set $P$ and

$$dist(y,y')\equiv\sum_{\mu=1}^d dist(y^\mu,y'^\mu)\equiv\sum_{\mu=1}^d \min_{z^\mu\underline{\in}y^\mu,z'^\mu\underline{\in}y'^\mu}|z^\mu-z'^\mu|. \tag{3-5}$$

For positive constants $c,C_1,C_2,C_3$ and $i\in\mathbf{Z}_-\equiv\{0,-1,-2,\ldots\}$ define

$$C_{i,C_1,C_2}(P)\equiv(C_1\lambda_i)^{n_P}C_2^{\rho(P)} \tag{3-6}$$

$$T_{i,c}(P)\equiv\exp\{-ca_i^{-1}L_{tree}(P)\} \tag{3-7}$$

[1] $\chi_2(z_1,z_2)\equiv\chi(z_1)-\chi(z_2)-\sum_{\mu=1}^d(z_1^\mu-z_2^\mu)\nabla_\mu\chi(z)|_{z=z_2}$, $|z_1-z_2|\equiv\sum_{\mu=1}^d|z_1^\mu-z_2^\mu|$

and for $\varphi : \Lambda_i \to \mathbf{R}$ , $K_2 > 0$,[2]

$$S^{in}_{i,c}(P|\varphi) \equiv \exp\left\{-ca_i^{-2}\lambda_i^{1/2} \int_{z \underline{\in} P} (\mathcal{A}^i\varphi)^2(z)\right\} \tag{3-8}$$

$$S^{out}_{i,c,K_2}(P|\varphi) \equiv \exp\left\{ca_i^{-2}\lambda_i^{1/2} \sum_{y:\ y\in\Lambda_j - P}{}' \int_{z\underline{\in} y} \exp\{-\frac{K_2}{2}a_i^{-1}dist(z,P)\}(\mathcal{A}^i\varphi)^2(z)\right\} \tag{3-9}$$

where the sum $\sum'$ runs over all $y \in \Lambda_j - P$ with

$$a_i^{-2}\lambda_i^{1/2} \int_{z\underline{\in} y} \exp\{-\frac{K_2}{2}a_i^{-1}dist(z,P)\}(\mathcal{A}^i\varphi)^2(z) \geq 1.$$

For positive constants $c_1, c_2$ define

$$S_{i,c_1,c_2,K_2}(P|\varphi) \equiv S^{in}_{i,c_1}(P|\varphi) S^{out}_{i,c_2,K_2}(P|\varphi). \tag{3-10}$$

Suppose we have defined $A(P|\Psi)$ for all polymers $P$ of $\Lambda_j$ and all fields $\Psi : base \to \mathbf{C}$. Define for positive constants $K_2, C_1, C_2, \gamma, c_1, c_2, c_3, c$ the norm

$$\|A\|_i^{\tilde{c}} \equiv \sup_{y\in\Lambda_i}\left\{\sum_{P: y\in P} \sup_{\substack{\varphi:\Lambda_i\to\mathbf{R}\\ \chi\in c_3\mathcal{K}_i(P)}} \{|A(P|\Psi)|_{\psi=\mathcal{A}^i\varphi+\chi}/B_i^{\tilde{c}}(P|\varphi)\}\right\} \tag{3-11}$$

with $\tilde{c} = (C_1, C_2, \gamma, c_1, c_2, c_3, c, K_2)$ and

$$B_i^{\tilde{c}}(P|\varphi) \equiv C_{i,C_1,C_2}(P)\ T_{i,c}(P)\ S_{i,c_1,c_2,K_2}(P|\varphi)\exp\{-\gamma|P|\}. \tag{3-12}$$

From the above definitions follows

LEMMA 3.1. *For all polymers $X$ of $\Lambda_{j-1}$ and $\varphi :\ \Lambda_{j-1} \to \mathbf{R}$, $\varphi' = \mathcal{A}_{j,j-1}\varphi$ we have*

(a) $C_{j,C_1,C_2}(X) = \left(\frac{\lambda_j}{\lambda_{j-1}}\right)^{n_X} C_{j-1,C_1,C_2}(X)$.

(b) $S^{in}_{j,c_1}(X|\varphi')$ *is monotonly decreasing in $c_1$ and*

$$S^{in}_{j,c_1}(X|\varphi') = S^{in}_{j-1,L^2c_1(\frac{\lambda_j}{\lambda_{j-1}})^{1/2}}(X|\varphi).$$

(c) $S^{out}_{j,c_2,K_2}(X|\varphi')$ *is monotonly increasing in $c_2$ and monotonly decreasing in $K_2$ and*

$$S^{out}_{j,c_2,K_2}(X|\varphi') = S^{out}_{j-1,L^2c_2(\frac{\lambda_j}{\lambda_{j-1}})^{1/2},LK_2}(X|\varphi).$$

(d) $T_{j,c}(X) = T_{j-1,Lc}(X)$.

(e) $-L_{tree}(\overline{X}) \leq -L_{tree}(X)$.

[2] $dist(z,P) \equiv \inf\{|z-y| \mid y \in P\}$



(f) $\mathcal{K}_{j-1}(X) \subseteq (\frac{\lambda_j}{\lambda_{j-1}})^{1/4}L^{-1}\mathcal{K}_j(\overline{X})$.

(g) *For*

$$\tilde{c}' = (C_1, C_2, \gamma, L^2c_1(\frac{\lambda_j}{\lambda_{j-1}})^{1/2}, L^2c_2(\frac{\lambda_j}{\lambda_{j-1}})^{1/2}, Lc_3(\frac{\lambda_j}{\lambda_{j-1}})^{-1/4}, L^{-1}c, LK_2)$$

*we have*

$$B^{\tilde{c}'}_{j-1}(X|\varphi) = B^{\tilde{c}}_j(X|\varphi')(\frac{\lambda_j}{\lambda_{j-1}})^{-n_X}.$$

Suppose that the following constant is positive and finite

$$q_\lambda \equiv \sup_j\{\frac{1}{\lambda_j}\ln(\frac{\lambda_j}{\lambda_{j-1}})\}. \tag{3-13}$$

LEMMA 3.2. *For all positive constants $\tilde{c} = (C_1, C_2, \gamma = 0, c_1, c_2, c_3, c, K_2)$ and*

$$\tilde{c}' = (C_1, C_2, \gamma - q_\lambda\lambda_j, L^2c_1(\frac{\lambda_j}{\lambda_{j-1}})^{1/2}, L^2c_2(\frac{\lambda_j}{\lambda_{j-1}})^{1/2}, Lc_3(\frac{\lambda_j}{\lambda_{j-1}})^{-1/4}, Lc, LK_2)$$

*and polymer $X$ of $\Lambda_{j-1}$, $\varphi : \Lambda_{j-1} \to \mathbf{R}$, $\varphi' = \mathcal{A}_{j,j-1}\varphi$ we have.*

$$B^{\tilde{c}'}_{j-1}(X|\varphi) \geq B^{\tilde{c}}_j(X|\varphi').$$

*Proof* : Lemma 3.1(g) and

$$(\frac{\lambda_j}{\lambda_{j-1}})^{-n_X} \geq \exp\{-q_\lambda\lambda_j n_X\}$$

and

$$n_X = \frac{1}{2}[\frac{\rho(X)-1}{4}] + \frac{1}{2} \leq \frac{1}{8}(\rho(X)-1) + 1 \leq \frac{1}{8}(|X|-1) + 1 \leq |X|$$

prove our lemma. $\surd$

For the polymer system defined in section 1 with $\lambda_0 = \lambda$, $\beta_0 = 0$ we have the following bound on the irrelevant activities :

THEOREM 3.3. *For small $\lambda$ and large $L$ and a suitable bare mass $m_0$ there exist positive constants $\tilde{c} = (C_1, C_2, \gamma = 0, c_1, c_2, c_3, c, K_2)$ and $\epsilon$ with $C_1$ large $C_2$ and $\epsilon$ small such that*

$$\|R^{ren}_j\|_j^{\tilde{c}} \leq \epsilon. \tag{3-14}$$

*for all $j \leq 0$.*

THEOREM 3.4. *Suppose that $\lambda$ is small and $L$ large and $\epsilon$ is a small positive constant. There exist positive constants $c_-, c_+, c_\beta, \tilde{c}$ and intervals $[\alpha_0,\beta_0]\supsetneq[\alpha_1,\beta_1]\supsetneq\dots\supsetneq[\alpha_j,\beta_j]\supsetneq\dots$ such that*

$$m_j^2([\alpha_j,\beta_j]) + s_j\lambda_j = a_j^{-2}\lambda_j^{3/2}[-\tilde{c},\tilde{c}] \tag{3-15a}$$

$$|\beta_j^{\mu\nu}| \le c_\beta \lambda_j^{1-\epsilon} \tag{3-15b}$$

$$\lambda_j \in \left(\frac{c_-}{|j-1|}, \frac{c_+}{|j-1|}\right) \tag{3-15c}$$

*for all* $j \le 0$ and $\mu,\nu \in \{1,\dots,4\}$ *with*

$$s_j \equiv \sum_{k=-\infty}^{j} \delta s_k \tag{3-16a}$$

$$\delta s_k \equiv \frac{1}{Vol(x)} \int_{z \in x} \mathrm{v}^k(z,z), \quad x \in \Lambda_{k-1}. \tag{3-16b}$$

*Furthermore*

$$m_{j-1}^2 - m_j^2 = \delta s_j \lambda_j + Q_m \tag{3-17a}$$

$$\beta_{j-1}^{\mu\nu} - \beta_j^{\mu\nu} = Q_\beta \tag{3-17b}$$

$$\lambda_{j-1} - \lambda_j = -\gamma_j \lambda_j^2 + Q_\lambda \tag{3-17c}$$

*with*

$$\gamma_j = \frac{3}{2}\frac{1}{Vol(x)} \int_{z_1 \in x} \int_{z_2} \Big[\sum_{k=j}^{0} \mathrm{v}^k(z_1,z_2)\Big]^2, \quad x \in \Lambda_{j-1} \tag{3-17d}$$

*and there exists a constant* $K$ *such that*

$$|Q_m| \le K a_j^{-2} \lambda_j^{3/2}, \quad |Q_\beta| \le K\lambda_j^2, \quad |Q_\lambda| \le K \lambda_j^{5/2}. \tag{3-18}$$

For a polymer $X$ of $\Lambda_{j-1}$ define

$$R_{j-1}^{div}(X|\Psi) \equiv \sum_{\substack{P:\ [U_\delta(P)]=X \\ \rho(P)=1}} Z_j^{rel}(\overline{X} - P|\Psi) R_j^{ren}(P|\Psi) \tag{3-19}$$

and

$$\delta R_j^c \equiv R_j^c - R_{j-1}^{div} \tag{3-20a}$$

$$\delta R_{j-1} \equiv R_{j-1} - R_{j-1}^{div} \tag{3-20b}$$

$$\delta R_{j-1}^{ren} \equiv R_{j-1}^{ren} - R_{j-1}^{div}. \tag{3-20c}$$

Theorem 3.3 can be proven by the following lemmata.

LEMMA 3.5. *For small* $\lambda$ *and large* $L$ *there exists* $\tilde{c} = (C_1, C_2, \gamma = 0, c_1, c_2, c_3, c, K_2)$ *for all* $j \le 0$ *such that if*

$$\|R_j^{ren}\|_j^{\tilde{c}} \le \epsilon,$$

*then we have*

$$\|R_{j-1}^{div}\|_{j-1}^{\tilde{c}} \le \frac{1}{4}\epsilon.$$



LEMMA 3.6. *For small* $\lambda$ *and large* $L$ *there exists* $\tilde{c} = (C_1, C_2, \gamma = 0, c_1, c_2, c_3, c, K_2)$ *for all* $j \le 0$ *such that if*

$$\|R_j^{ren}\|_j^{\tilde{c}} \le \epsilon$$

*there exist constants* $\tilde{c}_i = (C_1^{(i)}, C_2^{(i)}, \gamma^{(i)}, c_1^{(i)}, c_2^{(i)}, c_3^{(i)}, c^{(i)}, K_2), i = 1, 2,$ *such that*

$$\|\delta R_j^c\|_j^{\tilde{c}_1} \le \frac{1}{4}\|R_j^{ren}\|_j^{\tilde{c}} \tag{3-21}$$

$$\|\delta R_{j-1}\|_{j-1}^{\tilde{c}_2} \le \frac{1}{2}\epsilon \tag{3-22}$$

$$\|\delta R_{j-1}^{ren}\|_{j-1}^{\tilde{c}} \le \frac{1}{4}\epsilon + \|\delta R_{j-1}\|_{j-1}^{\tilde{c}_2}. \tag{3-23}$$

*Proof of theorem 3.3 (by induction) :* For $j = 0$ the assertion is trivial. Suppose that (3-14) holds for $j$. Then, using lemma 3.5 and 3.6 and eqs. (3-20) , we obtain

$$\|R_{j-1}^{ren}\|_{j-1}^{\tilde{c}} \le \|R_{j-1}^{div}\|_{j-1}^{\tilde{c}} + \|\delta R_{j-1}^{ren}\|_{j-1}^{\tilde{c}} \le \frac{1}{4}\epsilon + \frac{1}{4}\epsilon + \|\delta R_{j-1}\|_{j-1}^{\tilde{c}_2} \le$$
$$\le \epsilon$$

for $\lambda$ small and $L$ large enough. $\surd$

Lemma 3.5 and 3.6 and theorem 3.4 will be proven in the following sections. The inductive proof is organized as follows. In section 4.4 a lower bound on the relevant interaction $V_j^{rel}$ (stability bound) is presented. Section 4.6 states a bound on $\delta R_j^c$ and proves (3-21) of lemma 3.6 (lemma 6.10). Section 4.7 presents a proof of lemma 3.5. In sections 4.8,9,11 we proof bounds on $\delta R_{j-1}$ for the integration step and (3-22) of lemma 3.6 is proven (proposition 11.5). Section 4.10 proves theorem 3.4. In section 4.9 and 4.12 vacuum energy counterterms and polymer-dependent counterterms are estimated. Section 4.13 presents bounds on $\delta R_{j-1}^{ren}$ for the repolymerization step and proves (3-23) of lemma 3.6 (lemma 13.3). This will complete the proof.

### 4.4. Stability Bound

In this section a lower bound on the relevant interaction $V_j^{rel}$ (stability bound) is represented.

To establish bounds on the relevant interaction $V_j^{rel}$ we need decay properties for the kernels $\mathcal{A}^j$.

LEMMA 4.1. *For all* $\epsilon$ *with* $0 < \epsilon < 1$ *there exist positive constants* $K_1$ *and* $K_2$ *such that for all* $z, z_1, z_2 \in base,\ y \in \Lambda_j$ *and* $\mu \in \{1,\dots,d\}$

$$|\mathcal{A}^j(z,y)| \le K_1 a_j^{-d} \exp\{-K_2 a_j^{-1}|z-y|\},$$

$$|\mathcal{A}^j(z_1,y) - \mathcal{A}^j(z_2,y)| \le K_1 a_j^{-d-1}|z_1 - z_2|[\exp\{-K_2 a_j^{-1}|z_1-y|\} + \exp\{-K_2 a_j^{-1}|z_2-y|\}],$$

$$|\mathcal{A}_2^j(z_1,z_2;y)| \le K_1 a_j^{-d-2+\epsilon}|z_1-z_2|^{2-\epsilon}[\exp\{-K_2 a_j^{-1}|z_1-y|\}+\exp\{-K_2 a_j^{-1}|z_2-y|\}],$$

where

$$\mathcal{A}_2^j(z_1,z_2;y) \equiv \mathcal{A}^j(z_1,y)-\mathcal{A}^j(z_2,y)-\sum_{\mu=1}^{d}(z_1^\mu - z_2^\mu)\nabla_\mu \mathcal{A}^j(z,y)|_{z=z_2}.$$

This lemma is proven in appendix B.

For all complex fields $\Psi$ : $base \to \mathbf{C}$ the following lemma gives a bound on the real part of the quartic interaction.

LEMMA 4.2. *For $y \in \Lambda_j$ and $c \ge 0$ we have*

$$Re\,[\frac{\lambda_j}{4!}\int_{z\Subset y}\Psi(z)^4] \ge ca_j^{-2}\lambda_j^{1/2}\int_{z\Subset y}(Re\Psi)^2(z) - \frac{17\lambda_j}{4!}\int_{z\Subset y}(Im\Psi)^4(z) - 12c^2. \tag{4-1}$$

*Proof :* We have

$$Re\Psi(z)^4 = (Re\Psi(z))^4 - 6(Re\Psi(z))^2(Im\Psi(z))^2 + (Im\Psi(z))^4 \ge \frac{1}{2}(Re\Psi(z))^4 - 17(Im\Psi(z))^4 \tag{4-2}$$

and

$$\frac{\lambda_j}{2*4!}\int_{z\Subset y}(\Psi(z))^4 \ge ca_j^{-2}\lambda_j^{1/2}\int_{z\Subset y}(Re\Psi)^2(z) - 12c^2. \tag{4-3}$$

Eqs. (4-2), (4-3) imply the lower bound (4-1). $\surd$

LEMMA 4.3. *There exist constants $K$ and $K_2$ such that for $y \in \Lambda_j$, $\Psi = \mathcal{A}^j\varphi + \chi$ with $\varphi : \Lambda_j \to \mathbf{R}$ and $\chi \in \mathcal{K}_j(y)$ and $\mu,\nu \in \{1,\ldots,d\}$*

$$|\int_{z\Subset y}(\nabla_\mu^{per}\Psi(z))(\nabla_\nu^{per}\Psi(z))| \le Ka_j^{-2}\int_{y'\in\Lambda_j}\exp\{-K_2 a_j^{-1}dist(y,y')\}\varphi(y')^2 + 2\lambda_j^{-1/2}. \tag{4-4}$$

*Proof :* We have

$$|\nabla_\mu^{per}\Psi(z)\nabla_\nu^{per}\Psi(z)| \le \frac{1}{2}[|\nabla_\mu^{per}\Psi(z)|^2 + |\nabla_\nu^{per}\Psi(z)|^2] \tag{4-5}$$

and

$$|\nabla_\mu^{per}\Psi(z)|^2 \le 2[(\nabla_\mu^{per}(\mathcal{A}^j\varphi)(z))^2 + |\nabla_\nu^{per}\chi(z)|^2] \tag{4-6}$$

By lemma 4.1 we obtain for $z\Subset y$, $y' \in \Lambda_j$

$$|\nabla_\mu^{per}\mathcal{A}^j(z,y')| \le K_1 a_j^{-d-1}\exp\{-K_2 a_j^{-1}dist(y,y')\}. \tag{4-7}$$

Thus

$$\int_{z\Subset y}(\nabla_\mu^{per}\mathcal{A}^j\varphi(z))^2 \le \int_{z\Subset y}a_j^{2d}\sum_{y_1,y_2\in\Lambda_j}(\nabla_\mu^{per}\mathcal{A}^j(z,y_1))(\nabla_\mu^{per}\mathcal{A}^j(z,y_2))\varphi(y_1)\varphi(y_2) \le$$

$$\le K_1^2 a_j^{d-2}\sum_{y_1,y_2\in\Lambda_j}\exp\{-K_2 a_j^{-1}dist(y,y_1)\}\exp\{-K_2 a_j^{-1}dist(y,y_2)\}\varphi^2(y_1) \le$$

$$\frac{1}{2}Ka_j^{-2}\int_{y'\in\Lambda_j}\exp\{-K_2 a_j^{-1}dist(y,y')\}\varphi^2(y') \tag{4-8}$$



for a suitable constant $K$. For $\chi \in \mathcal{K}_j(y)$ we have

$$|\nabla_\mu^{per}\chi(z)| \le a_j^{-d/2}\lambda_j^{-1/4}. \tag{4-9}$$

Thus

$$\int_{z\Subset y}|\nabla_\mu^{per}\chi(z)|^2 \le \lambda_j^{-1/2}. \tag{4-10}$$

(4-5),(4-6),(4-8) and (4-10) imply the assertion . $\surd$

Using lemma 4.2 and 4.3 we see that for the relevant interaction

$$V_j^{rel}(y|\Psi) \equiv \frac{1}{2}\int_{z\Subset y}m_j^2\Psi(z)^2 + \frac{1}{2}\sum_{\mu,\nu=1}^{4}\beta_j^{\mu\nu}\int_{z\Subset y}(\nabla_\mu^{per}\Psi(z))(\nabla_\nu^{per}\Psi(z)) + \frac{\lambda_j}{4!}\int_{z\Subset y}\Psi(z)^4$$

the following bound is valid.

LEMMA 4.4. *Suppose that*

$$|m_j^2| \le c_m a_j^{-2}\lambda_j, \qquad |\beta_j| \le c_\beta\lambda_j.$$

*There exist constants $K$ and $K_2$ such that for $\lambda_j$ small and $\Psi = \mathcal{A}^j\varphi + \chi$ with $\varphi : \Lambda_j \to \mathbf{R}$ and $\chi \in \mathcal{K}_j(y)$ and $c > 0$*

$$Re\ V_j^{rel}(y|\Psi) \ge ca_j^{-2}\lambda_j^{1/2}\int_{z\Subset y}(Re\Psi)^2(z) - \frac{17\lambda_j}{4!}\int_{z\Subset y}(Im\Psi)^4(z) -$$

$$- Kc_\beta\lambda_j a_j^{-2}\int_{y'\in\Lambda_j}\exp\{-K_2 a_j^{-1}dist(y,y')\}\varphi(y')^2 - 15c^2. \tag{4-11}$$

Using

$$a_j^d\varphi^2(y) \le \int_{z\Subset y}(\mathcal{A}^j\varphi)^2(z)$$

and lemma 4.4 we obtain

PROPOSITION 4.5. *Suppose that*

$$|m_j^2| \le c_m a_j^{-2}\lambda_j, \qquad |\beta_j| \le c_\beta\lambda_j.$$

*There exist constants $K_\beta$ and $K_0$ such that for $\lambda_j$ small and for all polymers $P$ of $\Lambda_j$ and fields $\Psi = \mathcal{A}^j\varphi + \chi$ with $\varphi : \Lambda_j \to \mathbf{R}$, $\chi \in \mathcal{K}_j(P)$, and $c > 0$*

$$Re\ V_j^{rel}(P|\Psi) \ge ca_j^{-2}\lambda_j^{1/2}\int_{z\Subset P}(Re\Psi)^2(z) - \frac{17\lambda_j}{4!}\int_{z\Subset P}(Im\Psi)^4(z) -$$

$$- K_\beta c_\beta\lambda_j a_j^{-2}\sum_{y\in\Lambda_j - P}{}'\int_{z\Subset y}\exp\{-\frac{K_2}{2}a_j^{-1}dist(z,P)\}(\mathcal{A}^j\varphi)^2(z) - 15c^2|P| \tag{4-12}$$

*where the sum* $\sum'$ *runs over all* $y \in \Lambda_j - P$ *with*

$$\lambda_j^{1/2} a_j^{-2} \int_{z \in y} \exp\{-\frac{K_2}{2} a_j^{-1} dist(z,P)\}(A^j\varphi)^2(z) \geq 1.$$

### 4.5. Cluster Expansion Formulas for Vacuum Energy Counterterms and Running Coupling Constants

In the following a cluster expansion formula for the vacuum energy counterterms $\tilde{e}_{j-1}$ is represented. A cluster $C = (P_1^{k_1}, \dots, P_n^{k_n})$ is a collection of polymers $P_i$ with multiplicity $k_i \in \mathbf{N}^* = \mathbf{N} - \{0\}$ such that the graph $\gamma(C)$ is connected. The graph $\gamma(C)$ consists of $\sum_{i=1}^n k_i$ vertices, where $k_i$ vertices are represented by the polymer $P_i$. The vertices $P_a$ and $P_b$ are connected by a line in $\gamma(C)$ if $P_a \cap P_b \neq \emptyset$. For a cluster $C$ we will use the notations

$$C! \equiv k_1! \dots k_n!, \quad a(C) \equiv \sum_{G:\ G \subseteq \gamma(C)} (-1)^{|G|} \tag{5-1}$$

for $C = (P_1^{k_1}, \dots, P_n^{k_n})$ and $G \subseteq \gamma(C)$ means that $G$ is a subgraph of $\gamma(C)$ such that all vertices of $\gamma(C)$ are connected by at least one line of $G$. $|G|$ denotes the number of lines of $G$. For a polymer $X$ the set of all clusters $C$ in $X$ is denoted by $\mathcal{C}(X)$. Suppose that $M(P)$ is a complex number for each polymer. Then we have

LEMMA 5.1.

$$\ln \Big[ \sum_{\mathbf{P} \in \mathcal{P}(X)} \prod_{P \in \mathbf{P}} M(P) \Big] = \sum_{C \in \mathcal{C}(X)} \frac{a(C)}{C!} \prod_{P \in C} M(P). \tag{5-2}$$

This lemma is proven in app. A.

Define not normalized partition functions $Z_{j-1}^n$ and activities $A_{j-1}^n$ by

$$Z_{j-1}^n(X|\Psi) \equiv \int d\mu_{v_X^j}(\Phi) Z_j^c(X|\Phi+\Psi) = \sum_{X = \sum Q} \prod_Q A_{j-1}^n(Q|\Psi) \tag{5-3}$$

for all polymers $X$ of $\Lambda_{j-1}$. Since (cp. (1-33))

$$e_{j-1}(X) = \ln\ Z_{j-1}^n(X|\Psi)|_{\Psi=0}$$

we see by lemma 5.1

LEMMA 5.2. *For all polymers* $X$ *of* $\Lambda_{j-1}$ *we have*

$$\tilde{e}_{j-1}(X) = \sum_{\substack{C \in \mathcal{C}(X): \\ supp\ C = X}} \frac{a(C)}{C!} \prod_{Q \in C} M_{j-1}^n(Q|\Psi)|_{\Psi=0} \tag{5-4}$$



*where*

$$M_{j-1}^n(Q|\Psi) \equiv -\delta_{1,|Q|} + A_{j-1}^n(Q|\Psi). \tag{5-5}$$

The following lemma presents a relation for normalized activities $A_{j-1}$ and not normalized activities $A_{j-1}^n$. For polymers $Q_1, \dots, Q_n$ of $\Lambda_{j-1}$ define

$$E_{j-1}(Q_1, \dots, Q_n) \equiv \prod_{i=1}^n \exp\{-e_{j-1}(Q_i)\} \Big[ \delta_{1,n} + \sum_{m \geq 1} \sum_{Q'_1, \dots, Q'_m \subseteq \sum Q_i}' \prod_{a=1}^m [\exp\{-\tilde{e}_{j-1}(Q'_i)\} - 1] \Big]$$

where the sum $\sum'$ runs over all $Q'_1, \dots, Q'_m \subseteq \sum Q_i$ such that $Q'_a \not\subseteq Q_i$ for all $a \in \{1, \dots, m\}$, $i \in \{1, \dots, n\}$, $Q'_a \neq Q'_b$, for all $a, b$ with $a \neq b$ and $\gamma(Q_1, \dots, Q_n, Q'_1, \dots, Q'_m)$ connected .

LEMMA 5.3. *For all polymers* $X$ *of* $\Lambda_{j-1}$ *we have*

$$A_{j-1}(X|\Psi) = \sum_{n \geq 1} \frac{1}{n!} \sum_{\substack{Q_1, \dots, Q_n \\ \sum Q_i = X}} \prod_{i=1}^n A_{j-1}^n(Q_i|\Psi) E_{j-1}(Q_1, \dots, Q_n).$$

Using the cluster expansion formula (proposition 2.1) we obtain the following representation for not renormalized activities

LEMMA 5.4. *For all polymers* $X$ *of* $\Lambda_{j-1}$ *we have*

$$A_{j-1}^n(X|\Psi) = \sum_{k=0}^N A_{j-1}^{n,k}(X|\Psi) + A_{j-1}^{n,\geq N+1}(X|\Psi)$$

*where for* $k \in \mathbf{N}^*$

$$A_{j-1}^{n,k}(X|\Psi) = \delta_{1,k} \Big[ R_j^c(X|\Psi) + \int_0^1 ds\ \partial_s \int d\mu_{sv_X^j}(\Phi) R_j^c(X|\Phi+\Psi) \Big] +$$
$$+ \sum_{n \geq 2} \frac{1}{n!} \sum_{\substack{Q_1, \dots, Q_n \\ \sum Q_i = X}} \sum_{\substack{I:\ \emptyset \neq I \subseteq \underline{n} \\ |I| = k}} \sum_{\substack{\pi, \eta: \\ \tau = \tau(\pi, \eta)}} \int_0^1 ds\ f(\eta|s) \{ \prod_{(ab) \in \tau} \frac{\partial}{\partial t_{ab}} \}$$
$$\int d\mu_{v_X^j[t]}(\Phi) \prod_{i \in \underline{n} - I} [\delta_{1,|Q_i|} Z_j^{rel}(Q_i|\Phi+\Psi)] \prod_{i \in I} R_j^c(Q_i|\Phi+\Psi)|_{t = t[\pi, s]}$$

*and* $A_{j-1}^{n,0}(X|\Psi)$ *is the activity for the partition functions*

$$Z_{j-1}^{n,0}(X|\Psi) \equiv \int d\mu_{v_X^j}(\Phi) Z_j^{rel}(X|\Phi+\Psi) = \sum_{X = \sum Q} \prod_Q A_{j-1}^{n,0}(Q|\Psi).$$

The running coupling constants are represented by not normalized activities and vacuum energy counterterms in the following lemma.

LEMMA 5.5. *For all $x \in \Lambda_{j-1}$ we have*

$$m_{j-1}^2 = -\sum_{n\geq 1} \frac{1}{(n-1)!} \sum_{\substack{Q_1,\dots,Q_n \subseteq \Lambda_{j-1}:\\ x\in Q_1,\ Q_a\cap Q_b=\emptyset,\ a\neq b}} \mathcal{D}_x^2 A_{j-1}^n(Q_1|\Psi)|_{\Psi=0} \prod_{i=2}^n A_{j-1}^n(Q_i|\Psi)|_{\Psi=0}$$
$$E_{j-1}(Q_1,\dots,Q_n)$$

$$\beta_{j-1}^{\mu\nu} = -\sum_{n\geq 1} \frac{1}{(n-1)!} \sum_{\substack{Q_1,\dots,Q_n \subseteq \Lambda_{j-1}:\\ x\in Q_1,\ Q_a\cap Q_b=\emptyset,\ a\neq b}} \mathcal{D}_x^{\mu\nu} A_{j-1}^n(Q_1|\Psi)|_{\Psi=0} \prod_{i=2}^n A_{j-1}^n(Q_i|\Psi)|_{\Psi=0}\,.$$
$$E_{j-1}(Q_1,\dots,Q_n)$$

$$\lambda_{j-1} = -\sum_{n\geq 1} \frac{1}{(n-1)!} \sum_{\substack{Q_1,\dots,Q_n \subseteq \Lambda_{j-1}:\\ x\in Q_1,\ Q_a\cap Q_b=\emptyset,\ a\neq b}} \Big[\mathcal{D}_x^4 A_{j-1}^n(Q_1|\Psi)|_{\Psi=0} \prod_{i=2}^n A_{j-1}^n(Q_i|\Psi)|_{\Psi=0}$$
$$+3(n-1)\int_{z\in base} \mathcal{D}_x^2 A_{j-1}^n(Q_1|\Psi)|_{\Psi=0}\, \mathcal{D}_z^2 A_{j-1}^n(Q_2|\Psi)|_{\Psi=0} \prod_{i=3}^n A_{j-1}^n(Q_i|\Psi)|_{\Psi=0}\Big]$$
$$E_{j-1}(Q_1,\dots,Q_n) - \frac{3}{Vol(x)}\int_{x\in base} \sum_{\substack{X_1,X_2:\\ X_1\cap X_2\neq\emptyset}} \sum_{n_1\geq 1}\sum_{n_2\geq 1} \frac{1}{(n_1-1)!}\frac{1}{(n_2-1)!}$$
$$\sum_{\substack{Q_1^{(1)},\dots,Q_n^{(1)}\subseteq X_1:\\ x\in Q_1^{(1)},\ \sum Q_a^{(1)}=X_1}} \sum_{\substack{Q_1^{(2)},\dots,Q_n^{(2)}\subseteq X_2:\\ z\in Q_1^{(2)},\ \sum Q_a^{(2)}=X_2}} \mathcal{D}_x^2 A_{j-1}^n(Q_1^{(1)}|\Psi)|_{\Psi=0}$$
$$\prod_{i=2}^{n_1} A_{j-1}^n(Q_i^{(1)}|\Psi)|_{\Psi=0}\quad \mathcal{D}_z^2 A_{j-1}^n(Q_1^{(2)}|\Psi)|_{\Psi=0} \prod_{i=2}^{n_2} A_{j-1}^n(Q_i^{(2)}|\Psi)|_{\Psi=0}$$
$$E_{j-1}(Q_1^{(1)},\dots,Q_n^{(1)})\ E_{j-1}(Q_1^{(2)},\dots,Q_n^{(2)}).$$

*Proof :* By the definition of the running coupling constants (see eqs. (1-3a,b,c)) we have for arbitrary $x \in \Lambda_{j-1}$

$$m_{j-1}^2 = -\mathcal{D}_x^2 \ln\ Z_{j-1}(\Psi)|_{\Psi=0} = -\mathcal{D}_x^2 \sum_{X:\emptyset\neq X\subseteq\Lambda_{j-1}} A_{j-1}(X|\Psi)|_{\Psi=0}$$

$$\beta_{j-1}^{\mu\nu} = -\mathcal{D}_x^{\mu\nu} \ln\ Z_{j-1}(\Psi)|_{\Psi=0} = -\mathcal{D}_x^{\mu\nu} \sum_{X:\emptyset\neq X\subseteq\Lambda_{j-1}} A_{j-1}(X|\Psi)|_{\Psi=0}$$

$$\lambda_{j-1} = -\mathcal{D}_x^4 \ln\ Z_{j-1}(\Psi)|_{\Psi=0} = -\mathcal{D}_x^4 \sum_{X:\emptyset\neq X\subseteq\Lambda_{j-1}} A_{j-1}(X|\Psi)|_{\Psi=0}$$
$$-\frac{3}{Vol(x)}\int_{z\in x}\int_{z'\in base} \sum_{\substack{Q_1,Q_2:\\ Q_1\cap Q_2\neq\emptyset}} \mathcal{D}_z^2 A_{j-1}(Q_1|\Psi)|_{\Psi=0}\, \mathcal{D}_{z'}^2 A_{j-1}(Q_2|\Psi)|_{\Psi=0}.$$

Using lemma 5.4 we see that the assertion is valid. ✓



## 4.6. Coarsening Step : Bounds on $\delta R_j^c$

In this section it will be shown how to gain small factors from the terms $C_2^{\rho(P)}$ (for $C_2$ small) and $T_{j,c}(P)$ for all polymers $P$ of $\Lambda_j$ in a coarsening step (lemma 6.1-4). This will lead to the proof of inequality (3-21a) of lemma 3.6. Furthermore, this shows that renormalization subtractions are only necessary for small polymers $P$ ($\rho(P) = 1$).

We suppose that $\delta$ is of order $\ln L$. We call $y_1, y_2 \in \Lambda_j$ neighbors if there exists $\mu \in \{1,\dots,d\}$ such that $|y_1^\mu - y_2^\mu| \leq a_j$. $P_1, P_2 \subseteq \Lambda_j$ are called neighbors if there exists $y_1 \in P_1$ and $y_2 \in P_2$ such that $y_1$ and $y_2$ are neighbors. $Y \subseteq \Lambda_j$ is called connected if there exists no disjoint partition $Y = P_1 + P_2$ such that $P_1$ and $P_2$ are nonempty and not neighbors.

LEMMA 6.1. *For all $P \subseteq \Lambda_j$ with $[P]$ connected we have for $L$ large enough*

$$\rho(P) \geq 2\rho([U_\delta(P)]) - 1. \tag{6-1}$$

*Proof :* For $L$ large enough we have for all $y_1, y_2 \in \Lambda_j$ with $[y_1]$ and $[y_2]$ neighbors

$$U_\delta([y_1]) \supseteq [U_\delta(y_1)] \cup [U_\delta(y_2)]. \tag{6-2}$$

Suppose that $P = \{y_1,\dots,y_M\}$ and $M \geq N = \rho(P)$ and

$$\cup_{i=1}^N U_\delta(y_i) \supseteq P.$$

$\{[y_1],\dots,[y_N]\}$ is connected for large $L$. There exists a tree $\tau \in T_N$ such that $[y_a],[y_b]$ are neighbors for all $(ab) \in \tau$. Then there exists a subset $|I| \leq \frac{N+1}{2}$ such that for all $i \in \{1,\dots,N\}$ there exists $i' \in I$ such that $(ii') \in \tau$. Thus

$$\cup_{i\in I} U_\delta([y_i]) \supseteq \cup_{i=1}^N [U_\delta(y_i)] \supseteq [U_\delta(P)].\ ✓$$

LEMMA 6.2. *For $P \subseteq \Lambda_j$ and a disjoint partition $P = P_1 + \dots P_m$ such that $[P_a]$ is connected for all $a \in \{1,\dots,m\}$ and $[P_a],[P_b]$ are not neighbors for all $a,b \in \{1,\dots,m\}$ and $a \neq b$ we have*

$$\exp\{-ca_j^{-1}L_{tree}(P)\} \leq \exp\{-cdL(m-1)\}. \tag{6-3}$$

*Proof :* Consider the tree graph $\tau_P \in T(P)$ with

$$L_{tree}(P) = \sum_{(yy')\in\tau_P} dist(y,y')$$

There exist $m-1$ lines $(yy')$ of $\tau_P$ such that $[y],[y']$ are not neighbors, i.e.

$$dist(y^\mu, y'^\mu) \geq a_{j-1} \quad \forall\mu \in \{1,\dots,d\}.$$

Thus

$$dist(y,y') \geq da_{j-1}$$

for $m-1$ lines $(yy')$ of $\tau_P$. This shows the assertion . ✓

LEMMA 6.3. *For a constant $C_2$ with $C_2 \le e^{-2cLd}$ and a polymer $P$ of $\Lambda_j$ with $\rho(P) \ge 2$ we have*

$$C_2^{\rho(P)} \exp\{-ca_j^{-1} L_{tree}(P)\} \le \exp\{-\frac{cLd}{2u_j(\delta)}|P|\}\ C_2^{\rho(|U_\delta(P)|)} \tag{6-4}$$

*where* $u_j(\delta) \equiv |U_\delta(y)|,\ y \in \Lambda_j$.

*Proof :* Consider the two cases $[P]$ is connected (a) and $[P]$ is not connected (b).
(a): Suppose that $[P]$ is connected. By lemma 6.1 we have

$$\rho(P) - \rho([U_\delta(P)]) \ge \frac{1}{2}(\rho(P)-1) \ge \frac{1}{4}\rho(P)$$

for $\rho(P) \ge 2$. Thus

$$C_2^{\rho(P)} \exp\{-ca_j^{-1} L_{tree}(P)\} \le C_2^{\rho(P)-\rho([U_\delta(P)])} C_2^{\rho([U_\delta(P)])} \le C_2^{\frac{1}{4}\rho(P)} C_2^{\rho([U_\delta(P)])}. \tag{6-4'}$$

Since

$$\rho(P) \ge |P|/u_j(\delta)$$

we obtain by (6-4') the relation (6-4) for $C_2^{1/4} \le e^{-\frac{cLd}{2}}$.
(b): Suppose that $P = P_1 + \ldots P_m,\ m \ge 2$, and $[P_a]$ connected and $[P_a], [P_b]$ are not neighbors for all $a, b \in \{1, \ldots, m\}, a \ne b$. Using (for $L$ large enough)

$$\rho(P) = \sum_{i=1}^{m} \rho(P_i)$$

and lemma 6.2, we obtain

$$C_2^{\rho(P)} \exp\{-ca_j^{-1} L_{tree}(P)\} \le \exp\{-\frac{1}{2}cdLm\} \prod_{a=1}^{m} C_2^{\rho(P_a)}. \tag{6-5}$$

Furthermore

$$\prod_{a=1}^{m} C_2^{\rho(P_a)} \le \prod_{a:\ \rho(P_a)\ge 2} e^{-\frac{cLd}{2u_j(\delta)}|P_a|} C_2^{\rho([U_\delta(P)])} \tag{6-6}$$

and

$$\exp\{-\frac{1}{2}cdLm\} \le \prod_{a:\ \rho(P_a)=1} e^{-\frac{cLd}{2u_j(\delta)}|P_a|}. \tag{6-7}$$

By (6-5),(6-6) and (6-7) follows the assertion. $\surd$

LEMMA 6.4. *For $\mathbf{P} \in \mathcal{P}_\delta^c(X)$ with $\sum_{P\in\mathbf{P}} \rho(P) \ge 2$ and $C_2 \le e^{-2cdL}$ we have*

$$\prod_{P\in\mathbf{P}} [C_{j,C_1,C_2}(P) T_{j,c}(P)] \le \left[\prod_{P\in\mathbf{P}} \exp\{-\frac{cLd}{8u_j(\delta)}|P|\}\right] C_{j,C_1,C_2}(X) T_{j-1,\frac{3}{2}c}(X). \tag{6-8}$$

*Proof :* For $\mathbf{P} \in \mathcal{P}_\delta^c(X)$ we have for $L \ge 2$

$$\prod_{P\in\mathbf{P}} T_{j,c}(P) \le \left[\prod_{P\in\mathbf{P}} T_{j,(1-\frac{3}{2}L^{-1})c}(P)\right] T_{j-1,\frac{3}{2}c}(X) \le T_{j,\frac{1}{4}c}(\sum_{P\in\mathbf{P}} P) T_{j-1,\frac{3}{2}c}(X).$$



Using

$$\sum_{P\in\mathbf{P}} n_P \ge n_X$$

and lemma 6.3, we obtain the assertion. $\surd$

For $P \subseteq \Lambda_j, i \le j, \varphi : \Lambda_i \to \mathbf{R}$, define

$$S_{i,c}^{\prime in}(P|\varphi) \equiv \exp\left\{-ca_i^{-2}\lambda_i^{1/2} \sum_{y\in P}{}' \int_{z\underline{\in} y} (\mathcal{A}^i\varphi)^2(z)\right\} \tag{6-9}$$

where the sum $\sum'$ runs over all $y \in P$ with

$$a_i^{-2}\lambda_i^{1/2} \int_{z\underline{\in} y} (\mathcal{A}^i\varphi)^2(z) \ge 1.$$

LEMMA 6.5. *For all polymers $X$ of $\Lambda_{j-1}$ and $\mathbf{P} \in \mathcal{P}_\delta^c(X)$, $X \subseteq \Lambda_{j-1}$, positive constants $c_2, K_2$, and a function $\varphi : \Lambda_j \to \mathbf{R}$ we have*

$$\prod_{P\in\mathbf{P}} S_{j,c_2,K_2}^{out}(P|\varphi) \le S_{j,c_2',K_2/3}^{out}(\overline{X}|\varphi)[S_{j,q_1c_2}^{\prime in}(\overline{X}|\varphi)]^{-1} \tag{6-10}$$

with

$$q_n \equiv \sup_{z\in base} \{\sum_{y\in\Lambda_j} \exp\{-\frac{K_2}{2n}a_j^{-1} dist(z,y)\}\},\quad c_2' \equiv q_3 \exp\{-\frac{K_2}{6}\delta\}c_2.$$

*Proof :* We have

$$\ln[\prod_{P\in\mathbf{P}} S_{j,c_2,K_2}^{out}(P|\varphi)] \equiv \sum_{P\in\mathbf{P}} c_2 a_j^{-2}\lambda_j^{1/2} \sum_{y\in\Lambda_j - P}{}' \int_{z\underline{\in} y} \exp\{-\frac{K_2}{2}a_j^{-1} dist(z,P)\}(\mathcal{A}^j\varphi)^2(z)$$
$$\le \sum_{P\in\mathbf{P}} c_2 a_j^{-2}\lambda_j^{1/2} \sum_{y\in\Lambda_j}{}' \int_{z\underline{\in} y} \exp\{-\frac{K_2}{2}a_j^{-1} dist(z,P)\}(\mathcal{A}^j\varphi)^2(z). \tag{6-11}$$

For $z\underline{\in}\Lambda_j - \overline{X}$ we have

$$\sum_{P\in\mathbf{P}} \exp\{-\frac{K_2}{2}a_j^{-1} dist(z,P)\} \le \sum_{y\in\Lambda_j} \exp\{-\frac{K_2}{6}a_j^{-1} dist(z,y)\} \exp\{-\frac{K_2\delta}{6}\}$$
$$\exp\{-\frac{K_2}{6}a_j^{-1} dist(z,\overline{X})\} \tag{6-12}$$

and for $z\underline{\in}\overline{X}$

$$\sum_{P\in\mathbf{P}} \exp\{-\frac{K_2}{2}a_j^{-1} dist(z,P)\} \le \sum_{y\in\Lambda_j} \exp\{-\frac{K_2}{2}a_j^{-1} dist(z,y)\} \tag{6-13}$$

The assertion follows from (6-11),(6-12) and (6-13). $\surd$

Define for $P \subseteq \Lambda_j$, $\varphi : \Lambda_j \to \mathbf{R}$, and a constant $c_4$

$$S^4_{j,c_4}(P|\varphi) \equiv \exp\{-c_4\lambda_j \int_{z \subseteq y} (\mathcal{A}^j\varphi)^4(z)\}. \tag{6–14}$$

and for $\chi$ : $base \to \mathbf{C}$

$$S^{im}_{i,c}(P|\chi) \equiv \exp\{c\lambda_i \int_{z \subseteq P} |Im\chi|^4(z)\}. \tag{6–15}$$

LEMMA 6.6. *There exist constants $K_\beta$ and $K_0$ such that for all polymers $P$ of $\Lambda_j$, $\Psi = \mathcal{A}^j\varphi + \chi$, $\varphi :\ \Lambda_j \to \mathbf{R}$, $\chi :\ base \to \mathbf{C}$ and $c_4 = (3/4)^3 \frac{1}{4!4}$ and for all positive $c'$*

$$|Z^{rel}_j(P|\Psi)| \leq S^{in}_{j,c'}(P|\varphi) S^4_{j,c_4}(P|\varphi) S^{out}_{j,K_\beta c_\beta \lambda_j^{1/2},K_2}(P|\varphi) S^{im}_{j,\frac{17}{4!}}(P|\chi) \exp\{K_0 c'^2 |P|\}. \tag{6–16}$$

*Proof* : Use the stability bound (4-12). $\surd$

LEMMA 6.7. *For all polymers $X$ of $\Lambda_{j-1}$, $\mathbf{P} \in \mathcal{P}^c_\delta(X)$, $\Psi = \mathcal{A}^j\varphi + \chi$, $\varphi :\ \Lambda_j \to \mathbf{R}$, $\chi :\ base \to \mathbf{C}$, $\sum_{P\in\mathbf{P}} \rho(P) \geq 2$, $C_2 \leq e^{-2cdL}$ and all positive $c'$ there exist constants $K_\beta$, $K_0$, $\tilde{c} = (C_1, C_2, \gamma = 0, c_1, c_2, c_3, c, K_2)$ such that for all $j \leq 0$*

$$|Z^{rel}_j(\overline{X} - supp\ \mathbf{P}|\Psi)| \prod_{P\in\mathbf{P}} \tilde{B^c_j}(P|\varphi) \leq \left[\prod_{P\in\mathbf{P}} \exp\{-\frac{cLd}{8u_j(\delta)}|P|\}\right] S^{im}_{j,\frac{17}{4!}}(\overline{X} - supp\ \mathbf{P}|\chi)$$
$$\exp\{K_0 c'^2 L^{-4} |\overline{X} - supp\ \mathbf{P}|\} \hat{\hat{B}}{}^c_j(\overline{X}|\varphi) \tag{6–17}$$

with

$$\hat{\hat{c}} = (C_1, C_2, \gamma = 0, \hat{c}_1, \hat{c}_2, \hat{c}_3, \hat{c}, \hat{K}_2),$$
$$\hat{c}_1 = \min(L^{-2}c', c_1 - q_1c_2 - K_\beta c_\beta \lambda_j^{1/2}),\ \hat{c}_2 = c'_2 + K_\beta c_\beta \lambda_j^{1/2},$$
$$\hat{c}_3 = c_3,\ \hat{K}_2 = K_2/3,\ \hat{c} = \frac{3}{2}L^{-1}c,\ c'_2 = q_3 \exp\{-\frac{K_2\delta}{6}\}c_2,$$
$$q_n = \sup_{z\in base}\{\sum_{y\in\Lambda_j} \exp\{-\frac{K_2}{2n}a_j^{-1} dist(z,y)\}\},\quad q_1c_2 \leq c_4 = (\frac{3}{4})^3 \frac{1}{4!4}.$$

*Proof* : Lemma 6.4 and 6.5 imply

$$\prod_{P\in\mathbf{P}} \tilde{B^c_j}(P|\varphi) \leq \left[\prod_{P\in\mathbf{P}} \exp\{-\frac{cLd}{8u_j(\delta)}|P|\}\right] C_{j,c_1,c_2}(X)\ T_{j-1,\frac{3}{2}c}(X) S^{in}_{j,c_1-q_1c_2}(supp\ \mathbf{P}|\varphi)$$
$$S^{out}_{j,c'_2,K_2/3}(\overline{X}|\varphi)[S'^{in}_{j,q_1c_2}(\overline{X} - supp\ \mathbf{P}|\varphi)]^{-1} \tag{6–18}$$

with

$$q_n \equiv \sup_{z\in base}\{\sum_{y\in\Lambda_j} \exp\{-\frac{K_2}{2n}a_j^{-1} dist(z,y)\}\},\quad c'_2 \equiv q_3 \exp\{-\frac{K_2}{6}\delta\}c_2.$$



Using for $y \in \Lambda_j$

$$a_j^{-2}\lambda_j^{1/2} \int_{z\subseteq y} (\mathcal{A}^j\varphi)^2(z) \leq [a_j^{-2}\lambda_j^{1/2} \int_{z\subseteq y} (\mathcal{A}^j\varphi)^2(z)]^2 \leq$$
$$\leq \int_{z\subseteq y} (a_j^{-2}\lambda_j^{1/2})^2 \int_{z\subseteq y} (\mathcal{A}^j\varphi)^4(z) = \lambda_j \int_{z\subseteq y} (\mathcal{A}^j\varphi)^4(z) \tag{6–19}$$

if

$$a_j^{-2}\lambda_j^{1/2} \int_{z\subseteq y} (\mathcal{A}^j\varphi)^2(z) \geq 1,$$

we obtain

$$[S'^{in}_{j,q_1c_2}(\overline{X} - supp\ \mathbf{P}|\varphi)]^{-1} S^4_{j,c_4}(\overline{X} - supp\ \mathbf{P}|\varphi) \leq 1$$

for $q_1c_2 \leq c_4$. From lemma 6.6, (6-18) and (6-19) follows

$$|Z^{rel}_j(\overline{X} - supp\ \mathbf{P}|\Psi)| \prod_{P\in\mathbf{P}} \tilde{B^c_j}(P|\varphi) \leq \left[\prod_{P\in\mathbf{P}} \exp\{-\frac{cLd}{8u_j(\delta)}|P|\}\right] C_{j,c_1,c_2}(X)\ T_{j-1,c}(X)$$
$$S^{in}_{j,c_1-q_1c_2-K_\beta c_\beta\lambda_j^{1/2}}(supp\ \mathbf{P}|\varphi) S^{in}_{j,c'L^{-2}}(\overline{X} - supp\ \mathbf{P}|\varphi) S^{out}_{j,c'_2+K_\beta c_\beta \lambda_j^{1/2},K_2/3}(\overline{X}|\varphi)$$
$$S^{im}_{j,\frac{17}{4!}}(\overline{X} - supp\ \mathbf{P}|\chi) \exp\{K_0 c'^2 L^{-4}|\overline{X} - supp\ \mathbf{P}|\} \leq$$
$$\leq \left[\prod_{P\in\mathbf{P}} \exp\{-\frac{cLd}{8u_j(\delta)}|P|\}\right] S^{im}_{j,\frac{17}{4!}}(\overline{X} - supp\ \mathbf{P}|\chi) \exp\{K_0 c'^2 L^{-4}|\overline{X} - supp\ \mathbf{P}|\} \hat{\hat{B}}{}^c_j(\overline{X}|\varphi)$$

for

$$\hat{\hat{c}} = (C_1, C_2, \gamma = 0, \hat{c}_1, \hat{c}_2, \hat{c}, \hat{K}_2),$$
$$\hat{c}_1 = \min(L^{-2}c', c_1 - q_1c_2 - K_\beta c_\beta \lambda_j^{1/2}),\ \hat{c}_2 = c'_2 + K_\beta c_\beta \lambda_j^{1/2}$$
$$K'_2 = K_2/3,\ \hat{c} = \frac{3}{2}L^{-1}c,\ c'_2 = q_3 \exp\{-\frac{K_2\delta}{6}\}c_2.\quad \surd$$

For $\mathbf{P} \in \mathcal{P}^c_\delta(X)$, $X \subseteq \Lambda_{j-1}$ we have

$$|X| \leq 2^d |supp\ \mathbf{P}|.$$

Thus we obtain from lemma 6.7

LEMMA 6.8.

$$|Z^{rel}_j(\overline{X} - supp\ \mathbf{P}|\Psi)| \prod_{P\in\mathbf{P}} \tilde{B^c_j}(P|\varphi) \leq$$
$$\exp\{-\gamma|X|\}\ S^{im}_{j,\frac{17}{4!}}(\overline{X} - supp\ \mathbf{P}|\chi) \prod_{P\in\mathbf{P}} \exp\{-\frac{cLd}{24u_j(\delta)}|P|\}\ \hat{\hat{B}}{}^c_j(\overline{X}|\varphi) \tag{6–20}$$

for $\gamma = \frac{cLd}{24u_j(\delta)}2^{-d}$, and $K_0\ 2^d c'^2 \leq \frac{cLd}{24u_j(\delta)}$.

Define

$$K_v(\delta) \equiv L^{-d} \sum_{y:\ x\in[U_\delta(y)]} 1 \tag{6–21}$$

for $x \in \Lambda_{j-1}$.

LEMMA 6.9. *Suppose that $B(P) \in \mathbf{R}_+ \equiv \{r \in \mathbf{R} | r \geq 0\}$ for all polymers $P$ of $\Lambda_j$ and*

$$[\gamma'^{-1} 2^6 L^4 K_v(\delta)]^2 \; b < 1.$$

*Then we have for large $L$, positive $\gamma'$ and $x \in \Lambda_{j-1}$*

$$\sum_{n\geq 2} \frac{1}{(n-1)!} \sum_{\substack{P_1,\ldots,P_n:\; x\in[U_\delta(P_1)] \\ \mathbf{P}=\{P_1,\ldots,P_n\}\in\mathcal{P}^c_\delta}} \prod_{i=1}^{n} [B(P_i) e^{-\gamma'|P_i|}] \leq$$

$$\leq -\ln\Big\{1 - (\gamma'^{-1} 2^6 L^4 K_v(\delta))^2 \; b\Big\} \; 4K_v(\delta) L^4 \; b \tag{6-22}$$

for

$$b \equiv \sup_{y\in\Lambda_j} \{ \sum_{P:\; y\in P} B(P)\} < \infty.$$

*Proof :* For $\{P_1,\ldots,P_n\} = \mathbf{P} \in \mathcal{P}^c_\delta$ define a tree $\tau(\mathbf{P}) \in T_n$ such that $[U_\delta(P_a)] \cap [U_\delta(P_b)] \neq \emptyset$ for all $(ab) \in \tau$. Denote the left hand side of (6-22) by $I$, i.e.

$$I = \sum_{n\geq 2} \frac{1}{(n-1)!} \sum_{\tau\in T_n} \sum_{\substack{\mathbf{P}=\{P_1,\ldots,P_n\}\in\mathcal{P}^c_\delta: \\ \tau=\tau(\mathbf{P}),\; x\in[U_\delta(P_1)]}} \prod_{i=1}^{n} [B(P_i) e^{-\gamma'|P_i|}].$$

For $d_1,\ldots,d_n$ with $\sum_{i=1}^n d_i = 2(n-1)$ define

$$T(d_1,\ldots,d_n) \equiv \{\tau \in T_n | \; \forall i \in \{1,\ldots,n\} : \; d_i \text{ lines of } \tau \text{ emerge from vertex } i\}. \tag{6-23}$$

Then we have

$$I = \sum_{n\geq 2} \frac{1}{(n-1)!} \sum_{\substack{d_1,\ldots,d_n\in\mathbf{N}-\{0\}: \\ \sum d_i = 2(n-1)}} \sum_{\tau\in T(d_1,\ldots,d_n)} \sum_{\substack{\mathbf{P}=\{P_1,\ldots,P_n\}\in\mathcal{P}^c_\delta: \\ \tau=\tau(\mathbf{P}),\; x\in[U_\delta(P_1)]}} \prod_{i=1}^{n} [B(P_i) e^{-\gamma'|P_i|}].$$

Since for a polymer $P$ of $\Lambda_j$ and $L$ large enough

$$\sum_{P':\; [U_\delta(P')]\cap[U_\delta(P)]\neq\emptyset} B(P') \leq \sum_{y\in P} \; \sum_{y':\; [U_\delta(y')]\cap[U_\delta(y)]\neq\emptyset} \; \sum_{P':\; y'\in P'} B(P') \leq |P| 2^d K_v(\delta) L^4 \; b$$

and for $m \in \mathbf{N}$

$$|P|^m \exp\{-\gamma'|P|\} \leq \gamma'^{-m} m!$$

we obtain

$$I \leq \sum_{n\geq 2} \frac{1}{(n-1)!} \sum_{\substack{d_1,\ldots,d_n\in\mathbf{N}-\{0\}: \\ \sum d_i = 2(n-1)}} \sum_{\tau\in T(d_1,\ldots,d_n)} K_v(\delta) L^4 (\gamma'^{-1} 2^d K_v(\delta) L^4)^{d_1}$$

$$\prod_{i=2}^{n} [\gamma'^{-1} 2^d K_v(\delta) L^4]^{d_i - 1} \; d_1 \prod_{i=1}^{n} (d_i - 1)! \; b^n.$$



According to

$$\sum_{\substack{d_1,\ldots,d_n: \\ \sum d_i = 2(n-1)}} 1 \leq 4^n \tag{6-24}$$

and Cayley's theorem

$$|T(d_1,\ldots,d_n)| = \frac{(n-2)!}{\prod_{i=1}^n (d_i - 1)!} \tag{6-25}$$

$I$ is bounded by

$$I \leq \sum_{n\geq 2} \frac{1}{n-1} 4K_v(\delta) L^4 b [(\gamma'^{-1} 2^6 L^4 K_v(\delta))^2 \; b]^{n-1} =$$

$$-\ln\Big\{1 - [\gamma'^{-1} 2^6 L^4 K_v(\delta)]^2 \; b\Big\} \; 4K_v(\delta) L^4 \; b. \; \checkmark$$

LEMMA 6.10. *For small $\lambda$ and large $L$ there exist $(L,\lambda)$–independent positive constants*

$$\widetilde{c} = (C_1, C_2, 0, c_1, c_2, c_3, c, K_2)$$

and

$$\widetilde{c}_1 = (C_1, C_2, \gamma' = \frac{cLd2^{-d}}{24u_j(\delta)}, c_1^{(1)}, c_2^{(1)}, c_3^{(1)}, c^{(1)}, K_2/3)$$

*with*

$$c_1^{(1)} = \min(L^{-2}c', c_1 - q_1 c_2 - K_\beta c_\beta \lambda_j^{1/2}), \; c_2^{(1)} = c_2' + K_\beta c_\beta \lambda_j^{1/2},$$

$$c_3^{(1)} \leq \min(c_3, [(1-2^{-d})\frac{4!}{17}\gamma']^{1/4} L^{-1}), \; c^{(1)} = \frac{3}{2} L^{-1} c, \; c_2' = q_3 \exp\{-\frac{K_2\delta}{6}\} c_2,$$

$$c' = [K_0^{-1} \frac{cLd2^{-d}}{24u_j(\delta)}]^{1/2}, \; q_n = \sup_{z\in base} \{ \sum_{y\in\Lambda_j} \exp\{-\frac{K_2}{2n} a_j^{-1} dist(z,y)\}\}$$

*such that if*

$$\|R_j^{ren}\|_j^{\widetilde{c}} \leq \epsilon$$

*then we have*

$$\|\delta R_j^c\|_j^{\widetilde{c}_1} \leq \frac{1}{4} \|R_j^{ren}\|_j^{\widetilde{c}}$$

*for all $j \leq 0$.*

*Proof :* For a polymer $X$ of $\Lambda_{j-1}$ we have

$$\delta R_j^c(X|\Psi) = \sum_{\substack{\mathbf{P}\in\mathcal{P}^c_\delta(X): \\ \sum \rho(P)\geq 2}} Z_j^{rel}(\overline{X} - supp\; \mathbf{P}|\Psi) \prod_{P\in\mathbf{P}} R_j^{ren}(P|\Psi).$$

Using lemma 6.8 we obtain for $x \in \Lambda_{j-1}$, $\gamma' = \frac{cLd2^{-d}}{24u_j(\delta)}$ and $\chi \in c_3^{(1)}\mathcal{K}_j(X)$, $c_3^{(1)} \leq (1 - 2^{-d})[\frac{4!}{17}\gamma']^{1/4}L^{-1}$

$$\sum_{X:\, x\in X} |\delta R_j^c(X|\Psi)|/\widetilde{B}_j^{c_1}(X|\varphi) \leq \sum_{X:\, x\in X} \sum_{\substack{P:\, |U_\delta(P)|=X:\\ \rho(P)\geq 2}} \exp\{-\gamma'|P|\}|R_j^{ren}(P|\Psi)|/\widetilde{B}_j^c(P|\varphi) +$$

$$+ \sum_{n\geq 2} \frac{1}{(n-1)!} \sum_{\substack{P_1,\dots,P_n:\, x\in(U_\delta(P_1))\\ P=(P_1,\dots,P_n)\in \mathcal{P}_\delta^c}} \prod_{i=1}^{n}\Big[\exp\{-\gamma'|P_i|\}|R_j^{ren}(P_i|\Psi)|/\widetilde{B}_j^c(P_i|\varphi)\Big]. \tag{6-26}$$

For the first term on the right hand side of (6-26) we have

$$\sum_{X:\, x\in X} \sum_{\substack{P:\, |U_\delta(P)|=X:\\ \rho(P)\geq 2}} \exp\{-\gamma'|P|\}|R_j^{ren}(P|\Psi)|/\widetilde{B}_j^c(P|\varphi) \leq K_v(\delta)L^4 \exp\{-2\gamma'\}\|R_j^{ren}\|_j^{\widetilde{c}}. \tag{6-27}$$

For the second term on the right hand side of (6-26) we obtain for small $\lambda$ , using lemma 6.9,

$$\sum_{\substack{P_1,\dots,P_n:\, x\in(U_\delta(P_1))\\ P=(P_1,\dots,P_n)\in \mathcal{P}_\delta^c}} \prod_{i=1}^{n}\Big[\exp\{-\gamma'|P_i|\}|R_j^{ren}(P_i|\Psi)|/\widetilde{B}_j^c(P_i|\varphi)\Big] \leq$$

$$\leq -\ln[1-(\gamma'^{-1}2^6L^4K_v(\delta))^2\|R_j^{ren}\|_j^{\widetilde{c}}]\; 4K_v(\delta)L^4\|R_j^{ren}\|_j^{\widetilde{c}}. \tag{6-28}$$

Thus for small $\lambda$ and large $L$ we obtain by (6-27) and (6-28) the assertion . √

Corollary 6.11. *For small $\lambda$ and large $L$ there exist $(L,\lambda)$–independent positive constants*

$$\widetilde{c} = (C_1, C_2, \gamma = 0, c_1, c_2, c_3, c, K_2)$$

*with*

$$\widetilde{c}' = (C_1, C_2, \gamma', 8L^{-2}c_1, 2c_2', 9L^{-1}c_3, \frac{3}{2}L^{-1}c, K_2/3)$$

$$\gamma' = \frac{cLd2^{-d}}{24u_j(\delta)}, \quad c_2' = q_3\exp\{-\frac{K_2\delta}{6}\}c_2,$$

$$q_n = \sup_{z\in base}\{\sum_{y\in\Lambda_j}\exp\{-\frac{K_2}{2n}a_j^{-1}dist(z,y)\}\}$$

$$q_3\exp\{-\frac{K_2\delta}{6}\} \leq \frac{1}{18}L^{-2}$$

*such that if*

$$\|R_j^{ren}\|_j^{\widetilde{c}} \leq \epsilon$$

*then we have*

$$\|\delta R_j^c\|_j^{\widetilde{c}'} \leq \frac{1}{4}\|R_j^{ren}\|_j^{\widetilde{c}}$$



*for all $j \leq 0$.*

*Proof :* Using lemma 6.10 we obtain for $x \in \Lambda_{j-1}$, $\varphi : \Lambda_j \to \mathbf{R}$, $\Psi = \mathcal{A}^j\varphi + \chi$, $\chi \in c_3^{(1)}\mathcal{K}_j(\overline{X})$ with

$$c_1^{(1)} = \min(L^{-2}c', c_1 - q_1c_2 - K_\beta c_\beta\lambda_j^{1/2}),\quad c_2^{(1)} = c_2' + K_\beta c_\beta \lambda_j^{1/2},$$

$$c_3^{(1)} \leq \min(c_3, [(1-2^{-d})\frac{4!}{17}\gamma']^{1/4}L^{-1}),\quad c^{(1)} = \frac{3}{2}L^{-1}c,\quad c_2' = q_3\exp\{-\frac{K_2\delta}{6}\}c_2,$$

$$c' = [K_0^{-1}\frac{cLd2^{-d}}{24u_j(\delta)}]^{1/2}$$

the following bound

$$\|\delta R_j^c\|_j^{\widetilde{c}_1} \leq \frac{1}{4}\|R_j^{ren}\|_j^{\widetilde{c}}.$$

For small $\lambda$ and large $L$ we have

$$8L^{-2}c_1 \leq c_1^{(1)},\quad 2c_2' \geq c_2^{(1)},\quad 9L^{-1}c_3 \leq c_3^{(1)}.$$

This proves the assertion. √

## 4.7. Renormalization Subtractions : Bounds on $R_{j-1}^{div}$

In this section it will be shown how to use renormalization conditions to bound $R_{j-1}^{div}$. By renormalization subtraction a power of $L^{-1}$ is gained. This will lead to a proof of lemma 3.5 (lemma 7.7). Properties of the smooth fields of $\mathcal{K}_{j-1}(X)$ and $\mathcal{F}_{j-1}(X)$ will be proven (lemma 7.1-4).

For a field $\Psi$ and $z, z_0 \in base$ define

$$\Psi_{z_0}(z) \equiv \Psi(z) - \Psi(z_0),\quad \Psi_{2,z_0}(z) \equiv \Psi_2(z,z_0) \equiv \Psi(z) - \Psi(z_0) - $$
$$- \sum_{\mu=1}^{d}(z^\mu - z_0^\mu)\nabla_\mu\Psi(z')|_{z'=z_0}.$$

Using renormalization conditions for $R_j^{ren}$ we obtain

Lemma 7.1. *For all polymers $P$ of $\Lambda_j$ and fields $\Psi$ we have*

$$\int_{z_1,z_2}\Psi(z_1)\Psi(z_2)\frac{\delta^2}{\delta\chi(z_1)\delta\chi(z_2)}R_j^{ren}(P|\chi)|_{\chi=0} = \int_{z_1,z_2}\Big[2\sum_{\mu=1}^{d}(z_1^\mu - z_0^\mu)$$
$$\nabla_\mu\Psi(z)|_{z=z_0}\;\Psi_{2,z_0}(z_2) + \Psi_{2,z_0}(z_1)\Psi_{2,z_0}(z_2)\Big]\frac{\delta^2}{\delta\chi(z_1)\delta\chi(z_2)}R_j^{ren}(P|\chi)|_{\chi=0} \tag{7-1}$$

and

$$\int_{z_1,\dots,z_4} \Psi(z_1)\dots\Psi(z_4)\frac{\delta^4}{\delta\chi(z_1)\dots\delta\chi(z_4)}R_j^{ren}(P|\chi)|_{\chi=0} =$$

$$\sum_{a=1}^{4}\int_{z_1,\dots,z_4} [\prod_{i:\ i<a} \Psi(z_i)]\Psi_{z_0}(z_a)[\prod_{i':\ i'>a} \Psi(z_{i'})]\frac{\delta^4}{\delta\chi(z_1)\dots\delta\chi(z_4)}R_j^{ren}(P|\chi)|_{\chi=0}. \tag{7-2}$$

For the use of renormalization cancellations we need smoothness properties for the external fields.

LEMMA 7.2. *For $z_0 \in P$, $P$ polymer of $\Lambda_j$, $X$ polymer of $\Lambda_{j-1}$ and $\chi \in \mathcal{K}_{j-1}(X)$, $P \subseteq \overline{X}$, we have*

$$L^{d/2}\frac{a_j}{dist(P)}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}\chi_{z_0} \in \mathcal{K}_j(P) \tag{7-3a}$$

$$\frac{1}{2}L^{d/2+1-\epsilon}(\frac{a_j}{dist(P)})^{2-\epsilon}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}\chi_{2,z_0} \in \mathcal{K}_j(P) \tag{7-3b}$$

$$L^{d/2}\frac{a_j}{dist(P)}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}\sum_{\mu=1}^{d}((.)^\mu - z_0^\mu)\nabla_\mu\chi_{z_0} \in \mathcal{K}_j(P) \tag{7-3c}$$

*Proof* : For $z, z_1, z_2 \underline{\in} y$, $y \in P$ we have

$$|L^{d/2}\frac{a_j}{dist(P)}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}\chi_{z_0}(z)| \le$$
$$\le L^{d/2}\frac{a_j}{dist(P)}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}a_{j-1}^{-d/2}|z - z_0|\lambda_{j-1}^{-1/4} \le a_j^{1-d/2}\lambda_j^{-1/4},$$

$$L^{d/2}\frac{a_j}{dist(P)}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}|\chi_{z_0}(z_1) - \chi_{z_0}(z_2)| \le$$
$$\le L^{d/2}\frac{a_j}{dist(P)}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}a_{j-1}^{-d/2}|z_1 - z_2|\lambda_{j-1}^{-1/4} \le a_j^{-d/2}\lambda_j^{-1/4}|z_1 - z_2|$$

and

$$L^{d/2}\frac{a_j}{dist(P)}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}|\nabla_\mu\chi_{z_0}(z_1) - \nabla_\mu\chi_{z_0}(z_2)| \le$$
$$\le L^{d/2}\frac{a_j}{dist(P)}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}|\nabla_\mu\chi(z_1) - \nabla_\mu\chi(z_2)| \le$$
$$\le L^{d/2}\frac{a_j}{dist(P)}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}a_{j-1}^{-d/2-1}|z_1 - z_2|\lambda_{j-1}^{-1/4} \le a_j^{-d/2-1}\lambda_j^{-1/4}|z_1 - z_2|$$

and

$$L^{d/2}\frac{a_j}{dist(P)}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}|(\chi_{z_0})_2(z_1, z_2)| \le L^{d/2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}|\chi_2(z_1, z_2)| \le$$
$$\le L^{d/2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}a_{j-1}^{-d/2-1+\epsilon}|z_1 - z_2|^{2-\epsilon}\lambda_{j-1}^{-1/4} \le L^{-1+\epsilon}a_j^{-d/2-1+\epsilon}\lambda_j^{-1/4}|z_1 - z_2|^{2-\epsilon}\lambda_j^{-1/4}.$$



This shows (7-3a). Furthermore

$$|L^{d/2+1-\epsilon}(\frac{a_j}{dist(P)})^{2-\epsilon}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}\chi_{2,z_0}(z)| \le$$
$$\le L^{d/2+1-\epsilon}a_j^{2-\epsilon}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}a_{j-1}^{-d/2-1+\epsilon}\lambda_{j-1}^{-1/4} \le a_j^{1-d/2}\lambda_j^{-1/4}$$

and

$$|L^{d/2+1-\epsilon}(\frac{a_j}{dist(P)})^{2-\epsilon}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}[\chi_{2,z_0}(z_2) - \chi_{2,z_0}(z_1)]| =$$
$$= |L^{d/2+1-\epsilon}(\frac{a_j}{dist(P)})^{2-\epsilon}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}[\chi_2(z_2, z_1) + \sum_{\mu=1}^{d}(z_2^\mu - z_1^\mu)(\nabla_\mu\chi(z_0) - \nabla_\mu\chi(z_1))]| \le$$
$$\le L^{d/2+1-\epsilon}(\frac{a_j}{dist(P)})^{2-\epsilon}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}[a_{j-1}^{-d/2-1+\epsilon}|z_1 - z_2|^{2-\epsilon}\lambda_{j-1}^{-1/4} +$$
$$+ |z_1 - z_2|a_{j-1}^{-d/2-1}|z_0 - z_1|\lambda_{j-1}^{-1/4}] \le a_j^{-d/2-1+\epsilon}\lambda_j^{-1/4}|z_1 - z_2|a_j^{1-\epsilon} + a_j^{-d/2-1}\lambda_j^{-1/4}|z_1 - z_2|a_j \le$$
$$\le 2a_j^{-d/2}\lambda_j^{-1/4}|z_1 - z_2|$$

and

$$|L^{d/2+1-\epsilon}(\frac{a_j}{dist(P)})^{2-\epsilon}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}[\nabla_\mu\chi_{2,z_0}(z_1) - \nabla_\mu\chi_{2,z_0}(z_2)]| =$$
$$= |L^{d/2+1-\epsilon}(\frac{a_j}{dist(P)})^{2-\epsilon}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}|\nabla_\mu\chi(z_1) - \nabla_\mu\chi(z_2)| \le$$
$$\le L^{d/2+1-\epsilon}(\frac{a_j}{dist(P)})^{2-\epsilon}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}[a_{j-1}^{-d/2-1+\epsilon}|z_1 - z_2|\lambda_{j-1}^{-1/4} \le a_j^{-d/2-1}\lambda_j^{-1/4}, |z_1 - z_2|$$

$$|L^{d/2+1-\epsilon}(\frac{a_j}{dist(P)})^{2-\epsilon}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}\chi_{2,z_0}(z_1, z_2)| = L^{d/2+1-\epsilon}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}|\chi(z_1, z_2)| \le$$
$$\le |L^{d/2+1-\epsilon}\lambda_j^{-1/4}a_{j-1}^{-d/2-1+\epsilon}|z_1 - z_2|^{2-\epsilon} = a_j^{-d/2-1+\epsilon}\lambda_j^{-1/4}|z_1 - z_2|^{2-\epsilon}.$$

This shows (7-3b). Furthermore

$$|L^{d/2}\frac{a_j}{dist(P)}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}\sum_{\mu=1}^{d}(z^\mu - z_0^\mu)\nabla_\mu\chi(z_0)| \le$$
$$\le L^{d/2}\frac{a_j}{dist(P)}(\frac{\lambda_{j-1}}{\lambda_j})^{1/4}a_{j-1}^{-d/2}\lambda_{j-1}^{-1/4} \le a_j^{1-d/2}\lambda_j^{-1/4}.$$

This shows (7-3c). √

Define for $\varphi : \Lambda_{j-1} \to \mathbf{R}$, $X \subseteq \Lambda_{j-1}$ and positive constants $K', K_2, c$

$$K_{j-1,c}(X|\varphi) \equiv K'c^{-1/2}(\frac{\lambda_j}{\lambda_{j-1}})^{1/4}\exp\Big\{c\lambda_{j-1}^{1/2}a_{j-1}^{-2}$$
$$\sum_{x\in\Lambda_{j-1}}{}' \int_{z\underline{\in}x} \exp\{-K_2 a_{j-1}^{-1} dist(z, X)\}(\mathcal{A}^{j-1}\varphi)(z)^2\Big\} \tag{7-4}$$

where the sum $\sum'$ goes over all $x \in \Lambda_{j-1}$ such that

$$\lambda_{j-1}^{1/2} a_{j-1}^{-2} \int_{z \underline{\in} x} \exp\{-K_2 a_{j-1}^{-1} dist(z,X)\}(\mathcal{A}^{j-1}\varphi)(z)^2 \geq 1.$$

**Lemma 7.3.** *There exists a constant $K'$ ( $L$-independent ) such that*

$$K_{j-1,c}(X|\varphi)^{-1} L^{d/2-1} \mathcal{A}^{j-1}\varphi \in \frac{1}{4}\mathcal{K}_j(P) \tag{7-5a}$$

$$K_{j-1,c}(X|\varphi)^{-1} L^{d/2} \frac{a_j}{dist(P)} \sum_{\mu=1}^{d} ((.)^\mu - z_0^\mu)(\nabla_\mu \mathcal{A}^{j-1}\varphi)(z_0) \in \frac{1}{4}\mathcal{K}_j(P) \tag{7-5b}$$

$$K_{j-1,c}(X|\varphi)^{-1} L^{d/2} \frac{a_j}{dist(P)} (\mathcal{A}^{j-1}\varphi)_{z_0} \in \frac{1}{4}\mathcal{K}_j(P) \tag{7-5c}$$

$$K_{j-1,c}(X|\varphi)^{-1} L^{d/2-\epsilon} \left(\frac{a_j}{dist(P)}\right)^{2-\epsilon} (\mathcal{A}^{j-1}\varphi)_{2,z_0} \in \frac{1}{4}\mathcal{K}_j(P) \tag{7-5d}$$

*for all $\varphi : \Lambda_{j-1} \to \mathbf{R}$, polymers $P$ of $\Lambda_j$ and polymers $X$ of $\Lambda_{j-1}$ with $P \subseteq X$, $X \subseteq \Lambda_{j-1}$ and $z_0 \underline{\in} P$.*

*Proof :* For $z \underline{\in} P$ and $C > 0$ we have

$$|\mathcal{A}^{j-1}\varphi(z)| \leq \sum_{x \in \Lambda_{j-1}} K_1 \exp\{-K_2 a_{j-1}^{-1} dist(z,x)\}|\varphi(x)| \leq$$

$$\leq K_1 K C^{-1/2} \sup_{x \in \Lambda_{j-1}} \left\{ (C \exp\{-K_2 a_{j-1}^{-1} dist(x,P)\}\varphi(x)^2)^{1/2} \right\} \leq$$

$$\leq K_1 K C^{-1/2} \exp\left\{ \frac{1}{2} \sum_{x \in \Lambda_{j-1}} C \exp\{-K_2 a_{j-1}^{-1} dist(x,P)\}\varphi(x)^2 \right\}.$$

Using

$$\int_{z \underline{\in} x} (\mathcal{A}^{j-1}\varphi)^2(z) \geq Vol(x)\varphi(x)^2 = a_{j-1}^d \varphi(x)^2 \tag{7-6}$$

we obtain for a suitable $\tilde{z} \underline{\in} x \in \Lambda_{j-1}$

$$\lambda_{j-1}^{1/2} a_{j-1}^{-2} \int_{z \underline{\in} x} \exp\{-K_2 a_{j-1}^{-1} dist(z,X)\}(\mathcal{A}^{j-1}\varphi)(z)^2 \geq$$

$$\geq \lambda_{j-1}^{1/2} a_{j-1}^{-2} \exp\{-K_2 a_{j-1}^{-1} dist(\tilde{z},X)\} a_{j-1}^d \varphi(x)^2.$$

Thus for $x \in \Lambda_{j-1}$ with

$$\lambda_{j-1}^{1/2} a_{j-1}^{-2} \int_{z \underline{\in} x} \exp\{-K_2 a_{j-1}^{-1} dist(z,X)\}(\mathcal{A}^{j-1}\varphi)(z)^2 \leq 1$$

we have

$$|\varphi(x)| \leq \lambda_{j-1}^{-1/4} a_{j-1}^{1-d/2} \exp\{\frac{K_2}{2} a_{j-1}^{-1} dist(\tilde{z},X)\}$$



and for $z \underline{\in} P$

$$\sum_{x \in \Lambda_{j-1}}'' K_1 \exp\{-K_2 a_{j-1}^{-1} dist(z,x)\}|\varphi(x)| \leq$$

$$\leq \sum_{x \in \Lambda_{j-1}} K_1 \lambda_{j-1}^{-1/4} a_{j-1}^{1-d/2} \exp\{-K_2 a_{j-1}^{-1}(dist(z,x) - \frac{1}{2} dist(\tilde{z},X))\} \leq$$

$$\leq \tilde{K} a_{j-1}^{1-d/2} \lambda_{j-1}^{-1/4}.$$

The sum $\sum''$ goes over all $x \in \Lambda_{j-1}$ with

$$\lambda_{j-1}^{1/2} a_{j-1}^{-2} \int_{z \underline{\in} x} \exp\{-K_2 a_{j-1}^{-1} dist(z,x)\}(\mathcal{A}^{j-1}\varphi)(z)^2 \leq 1.$$

Thus

$$|\mathcal{A}^{j-1}\varphi(z)| \leq \tilde{K} a_{j-1}^{1-d/2} \lambda_{j-1}^{-1/4} +$$

$$+ K_1 K C^{-1/2} \exp\left\{ \frac{1}{2} \sum_{x \in \Lambda_{j-1}}' C \exp\{-K_2 a_{j-1}^{-1} dist(x,P)\}\varphi(x)^2 \right\}.$$

where the sum $\sum'$ goes over all $x \in \Lambda_{j-1}$ with

$$\lambda_{j-1}^{1/2} a_{j-1}^{-2} \int_{z \underline{\in} x} \exp\{-K_2 a_{j-1}^{-1} dist(z,x)\}(\mathcal{A}^{j-1}\varphi)(z)^2 \geq 1.$$

Using (7-6) we obtain

$$\sum_{x \in \Lambda_{j-1}}' \exp\{-K_2 a_{j-1}^{-1} dist(x,P)\}\varphi(x)^2 \leq$$

$$\leq \sum_{x \in \Lambda_{j-1}}' a_{j-1}^{-d} \int_{z \underline{\in} x} \exp\{-K_2 a_{j-1}^{-1} dist(x,P)\}(\mathcal{A}^{j-1}\varphi)(z)^2 \leq$$

$$\leq \exp\{K_2\} \sum_{x \in \Lambda_{j-1}}' a_{j-1}^{-d} \int_{z \underline{\in} x} \exp\{-K_2 a_{j-1}^{-1} dist(z,X)\}(\mathcal{A}^{j-1}\varphi)(z)^2.$$

For $C = 2\exp\{-K_2\} a_{j-1}^{d-2} \lambda_{j-1}^{1/2} c$ we obtain

$$|\mathcal{A}^{j-1}\varphi(z)| \leq K a_{j-1}^{1-d/2} \lambda_{j-1}^{-1/4} \Big[ 1 + c^{-1/2} \exp\{c a_{j-1}^{-2} \lambda_{j-1}^{1/2}$$

$$\sum_{x \in \Lambda_{j-1}}' \int_{z \underline{\in} x} \exp\{-K_2 a_{j-1}^{-1} dist(z,X)\}(\mathcal{A}^{j-1}\varphi)(z)^2\} \Big]$$

According to the decay properties for the $\mathcal{A}^{j-1}$–kernel we see that (7-5a) follows for $K'$ large enough ($L$–independent ). The relations (7-5a,b,c,d) are similarly shown. ✓

Lemma 7.2 and 7.3 imply

COROLLARY 7.4. *For polymers $X$ of $\Lambda_{j-1}$, $P \subseteq \overline{X}$, $z_0 \underline{\in} P$ and $\Psi = \mathcal{A}^{j-1}\varphi + \chi$, $\varphi : \Lambda_{j-1} \to \mathbf{R}$, $\chi \in \mathcal{K}_{j-1}(X)$ we have*

$$K_{j-1,c}(X|\varphi)^{-1} L^{d/2-1} \Psi \in \frac{1}{2}\mathcal{K}_j(P) \tag{7-7a}$$

$$K_{j-1,c}(X|\varphi)^{-1} L^{d/2} \frac{a_j}{dist(P)} \sum_{\mu=1}^{d} ((.)^\mu - z_0^\mu)(\nabla_\mu \Psi)(z_0) \in \frac{1}{2}\mathcal{K}_j(P) \tag{7-7b}$$

$$K_{j-1,c}(X|\varphi)^{-1} L^{d/2} \frac{a_j}{dist(P)} (\Psi)_{z_0} \in \frac{1}{2}\mathcal{K}_j(P) \tag{7-7c}$$

$$K_{j-1,c}(X|\varphi)^{-1} L^{d/2-\epsilon} \left(\frac{a_j}{dist(P)}\right)^{2-\epsilon} (\Psi)_{2,z_0} \in \frac{1}{2}\mathcal{K}_j(P). \tag{7-7d}$$

Define

$$Z_j^{qu}(P|\varphi) \equiv \exp\{V_j^{qu}(P|\varphi)\} \tag{7-8a}$$

$$V_j^{qu}(P|\varphi) \equiv 3c_1 a_{j-1}^{-2} \lambda_{j-1}^{1/2} \int_{z \underline{\in} P} (\mathcal{A}^{j-1}\varphi)^2(z) \tag{7-8b}$$

and

$$\widetilde{R}_j^{ren}(P|\varphi,\Psi) \equiv Z_j^{qu}(P|\varphi) R_j^{ren}(P|\Psi). \tag{7-9}$$

LEMMA 7.5. *For $\Psi = \mathcal{A}^{j-1}\varphi + \chi$, $\varphi : \Lambda_{j-1} \to \mathbf{R}$, $\chi \in c_3\mathcal{K}_{j-1}(X)$, $X$ polymer of $\Lambda_{j-1}$ and a polymer $P \subseteq \overline{X}$ there exists a constant $\Theta \in [0,1]$ such that*

$$\begin{aligned}
|\widetilde{R}_j^{ren}(P|\varphi,\Psi)| \le{} & 5\max(c_3^{-2}, c_3^{-4}) K_{j-1,c}(X|\varphi)^4 \exp\{\tfrac{1}{3}V_j^{qu}(P|\varphi)\} L^{-d} L^{-1+\epsilon} \\
& (a_j^{-1} dist(P))^{4-2\epsilon} \sup_{\chi \in c_3\mathcal{K}_j(P)} |R_j^{ren}(P|\chi)| + c_3^{-6} K_{j-1,c}(X|\varphi)^6 L^{-d} L^{6-2d} \\
& \sup_{|\alpha|\le 1} |Z_j^{qu}(P|\Theta\varphi + \alpha c_3 K_{j-1,c}(X|\varphi)^{-1} L^{d/2-1}\varphi) \\
& \qquad R_j^{ren}(P|\Theta\Psi + \alpha K_{j-1,c}(X|\varphi)^{-1} L^{d/2-1}\Psi)|.
\end{aligned}$$

*Proof :* Define for $\alpha \in \mathbf{C}$

$$\widetilde{R}_{j,\alpha}^{ren}(P|\varphi,\Psi) \equiv \widetilde{R}_j^{ren}(P|\alpha\varphi,\alpha\Psi).$$

Taylor expansion gives

$$\widetilde{R}_j^{ren}(P|\varphi,\Psi) = \frac{1}{2}\frac{\partial^2}{\partial\alpha^2}\widetilde{R}_{j,\alpha}^{ren}(P|\varphi,\Psi)|_{\alpha=0} + \frac{1}{4!}\frac{\partial^4}{\partial\alpha^4}\widetilde{R}_{j,\alpha}^{ren}(P|\varphi,\Psi)|_{\alpha=0} + \frac{1}{6!}\frac{\partial^6}{\partial\alpha^6}\widetilde{R}_{j,\alpha}^{ren}(P|\varphi,\Psi)|_{\alpha=\Theta}$$



for a suitable $\Theta \in [0,1]$. According to lemma 7.1 and notation (7-9)

$$\widetilde{R}_j^{ren}(P|\varphi,\Psi) = \frac{1}{2}\int_{z_1,z_2} [2\sum_{\mu=1}^{d}(z_1^\mu - z_0^\mu)\nabla_\mu\Psi(z)|_{z=z_0}\Psi_{2,z_0}(z_2) + \Psi_{2,z_0}(z_1)\Psi_{2,z_0}(z_2)]$$

$$\frac{\delta^2}{\delta\chi(z_1)\delta\chi(z_2)} R_j^{ren}(P|\chi)|_{\chi=0}(1 + V_j^{qu}(P|\varphi)) + \frac{1}{4!}\sum_{a=1}^{4}\int_{z_1,\dots,z_4} [\prod_{i:\ i<a}\Psi(z_i)]\Psi_{z_0}(z_a)[\prod_{i':\ i'<a}\Psi(z_{i'})]$$

$$\frac{\delta^4}{\delta\Psi(z_1)\dots\delta\Psi(z_4)} R_j^{ren}(P|\chi)|_{\chi=0} + \frac{1}{6!}\frac{\partial^6}{\partial\alpha^6} Z_j^{qu}(P|(\Theta+\alpha)\varphi) R_j^{ren}(P|(\Theta+\alpha)\Psi)|_{\alpha=0} =$$

$$\begin{aligned}
= \frac{1}{2}(1 + V_j^{qu}(P|\varphi))\Big[ & 2\frac{\partial^2}{\partial\alpha_1\partial\alpha_2} R_j^{ren}(P|\alpha_1 \sum_{\mu=1}^{d}((.)^\mu - z_0^\mu)\nabla_\mu\Psi(z_0) + \alpha_2\Psi_{2,z_0}(.))|_{\alpha_1=\alpha_2=0} + \\
& + \frac{\partial^2}{\partial\alpha^2} R_j^{ren}(P|\alpha\Psi_{2,z_0}(.))|_{\alpha=0}\Big] + \frac{1}{3!}\frac{\partial^4}{\partial\alpha_1\partial\alpha_2^3} R_j^{ren}(P|\alpha_1\Psi_{z_0}(.) + \alpha_2\Psi(.))|_{\alpha_1=\alpha_2=0} + \\
& + \frac{1}{6!}\frac{\partial^6}{\partial\alpha^6} Z_j^{qu}(P|(\Theta+\alpha)\varphi) R_j^{ren}(P|(\Theta+\alpha)\Psi)|_{\alpha=0}.
\end{aligned}$$

Corollary 7.4 and Cauchy's inequality yields

$$\begin{aligned}
|R_j^{ren}(P|\varphi,\Psi)| \le{} & c_3^{-2}\exp\{\tfrac{1}{3}V_j^{qu}(P|\varphi)\}\Big[2K_{j-1,c}(X|\varphi)^2 L^{-(1+d-\epsilon)}(a_j^{-1}\ dist(P))^{3-\epsilon} + \\
& + 2K_{j-1,c}(X|\varphi)^2 L^{-(2+d-2\epsilon)}(a_j^{-1}\ dist(P))^{4-2\epsilon}\Big] \sup_{\chi \in c_3\mathcal{K}_j(P)} |R_j^{ren}(P|\chi)| + \\
& + c_3^{-4} K_{j-1,c}(X|\varphi)^4 L^{3-2d} a_j^{-1}\ dist(P) \sup_{\chi \in c_3\mathcal{K}_j(P)} |R_j^{ren}(P|\chi)| + \\
& + c_3^{-6} K_{j-1,c}(X|\varphi)^6 L^{6-3d} \sup_{|\alpha|\le 1} |Z_j^{qu}(P|\Theta\varphi + \alpha c_3 K_{j-1,c}(X|\varphi)^{-1} L^{d/2-1}\varphi) \\
& \qquad R_j^{ren}(P|\Theta\varphi + \alpha c_3 K_{j-1,c}(X|\varphi)^{-1} L^{d/2-1}\varphi)|.
\end{aligned}$$

This proves our lemma . ✓

Define

$$\widetilde{K}_{j-1,c}(X|\varphi) = \exp\Big\{c\lambda_{j-1}^{1/2} a_{j-1}^{-2} \sum_{x \in \Lambda_{j-1}}{}' \int_{z \underline{\in} x} \exp\{-K_2 a_{j-1}^{-1}\ dist(z,X)\}(\mathcal{A}^{j-1}\varphi)^2(z)\Big\}. \tag{7-10}$$

LEMMA 7.6. *For $\varphi : \Lambda_{j-1} \to \mathbf{R}$, $\Theta \in [0,1]$, $\alpha \in \mathbf{C}$ with $|\alpha| \le 1$, and*

$$c' = 3c_1 L^{-2}\left(\frac{\lambda_{j-1}}{\lambda_j}\right)^{1/2},\ q \ge 12,\ K_\beta c_\beta \lambda_j^{1/2} \le 6cL^{-2}\left(\frac{\lambda_{j-1}}{\lambda_j}\right)^{1/2},$$

$$c_1 = qc,\ c_2 \le c_4 = \left(\frac{3}{4}\right)^3 \frac{1}{4!4},\ c_2' = \exp\{-\frac{K_2}{6}\delta\}c_2$$

we have for all polymers $X$ of $\Lambda_{j-1}$ and $P \subseteq \overline{X}$, $\varphi' = \mathcal{A}_{j,j-1}\varphi$

$$S^{in}_{j,c'}(\overline{X} - P|\varphi') S^{4}_{j,c_4}(\overline{X} - P|\varphi') S^{out}_{j,K_\beta c_\beta \lambda_j^{1/2},K_2}(\overline{X} - P|\varphi') Z^{qu}_j(P|\varphi)^{-1} \widetilde{K}_{j-1,c}(X|\varphi)^6$$

$$|Z^{qu}_j(P|\Theta\varphi + \alpha c_3 K_{j-1,c}(X|\varphi)^{-1} L^{d/2-1}\varphi)| S^{in}_{j,c_1}(P|\Theta\varphi') S^{out}_{j,c_2,K_2}(P|\Theta\varphi') \leq$$

$$\leq \exp\{\frac{3}{8} c_1 c_3^2 (\frac{\lambda_{j-1}}{\lambda_j})^{1/2} L^{-2}|P|\} S^{in}_{j,2c_1L^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2}}(\overline{X}|\varphi')$$

$$S^{out}_{j,c_2'+K_\beta c_\beta \lambda_j^{1/2},2K_2/3}(\overline{X}|\varphi') S^{out}_{j-1,\frac{8c_1}{q},2K_2}(X|\varphi).$$

*Proof :* a) For $\alpha = \alpha_1 + i\alpha_2 \in \mathbf{C}$, $|\alpha| \leq 1$, we have

$$|Z^{qu}_j(P|\Theta\varphi + \alpha c_3 K_{j-1,c}(X|\varphi)^{-1} L^{d/2-1}\varphi)| \leq$$

$$\leq Z^{qu}_j(P|2^{1/2}\Theta\varphi) Z^{qu}_j(P|2^{1/2}\alpha_1 c_3 K_{j-1,c}(X|\varphi)^{-1} L^{d/2-1}\varphi).$$

Since $\alpha_1 K_{j-1,c}(X|\varphi)^{-1} L^{d/2-1}\varphi \in \frac{c_3}{4}\mathcal{K}_j(P)$ we obtain

$$|Z^{qu}_j(P|\Theta\varphi + \alpha c_3 K_{j-1,c}(X|\varphi)^{-1} L^{d/2-1}\varphi)| \leq$$

$$\leq Z^{qu}_j(P|2^{1/2}\Theta\varphi) \exp\{\frac{3}{8} c_1 c_3^2 (\frac{\lambda_{j-1}}{\lambda_j})^{1/2} L^{-2}|P|\}.$$

b) Since $6L^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2} \leq 1$ we have

$$Z^{qu}_j(P|2^{1/2}\Theta\varphi) S^{in}_{j,c_1}(P|\Theta\varphi') =$$

$$= \exp\left\{6c_1 a_{j-1}^{-2}\lambda_{j-1}^{1/2} \int_{z\in P} (\Theta \mathcal{A}^{j-1}\varphi)^2(z) - \dot{c}_2 a_j^{-2}\lambda_j^{1/2} \int_{z\in P} (\Theta \mathcal{A}^{j-1}\varphi)^2(z)\right\} \leq 1.$$

c) For $c_2' = \exp\{-\frac{K_2}{6}\delta\} c_2$ we have

$$S^{out}_{j,c_2,K_2}(P|\Theta\varphi') \leq S^{out}_{j,c_2',2K_2/3}(\overline{X}|\Theta\varphi')[S'^{in}_{j,c_2}(\overline{X} - P|\Theta\varphi')]^{-1}.$$

d) For $c_2 \leq c_4$ we have

$$[S'^{in}_{j,c_2}(\overline{X} - P|\Theta\varphi')]^{-1} S^4_{j,c_4}(\overline{X} - P|\varphi') \leq 1.$$

e) For $c' = 2c_1L^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2}$ we have

$$S^{in}_{j,c'}(\overline{X} - P|\varphi') Z^{qu}_j(P|\varphi')^{-1} = S^{in}_{j,c'}(\overline{X}|\varphi').$$

f) We have

$$\widetilde{K}_{j-1,c}(X|\varphi)^6 \leq [S'^{in}_{j-1,6c}(X|\varphi)]^{-1} S^{out}_{j-1,6c,2K_2}(X|\varphi)$$

and

$$[S'^{in}_{j-1,6c}(X|\varphi)]^{-1} \leq [S'^{in}_{j,6cL^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2}}(\overline{X}|\varphi')]^{-1}$$

and for $6cL^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2} + K_\beta c_\beta \lambda_j^{1/2} \leq c_1 L^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2}$

$$S^{in}_{j,c'}(\overline{X}|\varphi')[S'^{in}_{j,6cL^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2}}(\overline{X}|\varphi')]^{-1}[S'^{in}_{j,K_\beta c_\beta \lambda_j^{1/2}}(\overline{X}|\varphi')]^{-1} \leq$$

$$\leq S'^{in}_{j,2c_1L^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2}}(\overline{X}|\varphi')]^{-1}$$

g) We have

$$S^{out}_{j,K_\beta c_\beta \lambda_j^{1/2},K_2}(\overline{X} - P|\varphi') \leq S^{out}_{j,K_\beta c_\beta \lambda_j^{1/2},K_2}(\overline{X}|\varphi')[S^{in}_{j,K_\beta c_\beta \lambda_j^{1/2}}(\overline{X}|\varphi')]^{-1}$$

a)-g) verify the assertion. $\surd$



Lemma 7.7. *For $\varphi : \Lambda_{j-1} \to \mathbf{R}$, $\Theta \in [0,1]$, $\alpha \in \mathbf{C}$ with $|\alpha| \leq 1$, and*

$$c' = 2c_1L^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2},\ q \geq 12,\ c_1 = qc,\ K_\beta c_\beta \lambda_j^{1/2} \leq 2cL^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2},$$

*and all polymers $X$ of $\Lambda_{j-1}$ and $P \subseteq \overline{X}$, $\varphi' = \mathcal{A}_{j,j-1}\varphi$*

$$Z^{qu}_j(P|\varphi)^{-2/3} \widetilde{K}_{j-1,c}(X|\varphi)^4 S^{in}_{j,c'}(\overline{X} - P|\varphi') S^{out}_{j,K_\beta c_\beta \lambda_j^{1/2}}(\overline{X} - P|\varphi') \leq$$

$$\leq S^{in}_{j,\frac{3}{2}c_1L^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2}}(\overline{X}|\varphi') S^{out}_{j,K_\beta c_\beta \lambda_j^{1/2},K_2}(\overline{X}|\varphi') S^{out}_{j-1,\frac{4c_1}{q},2K_2}(X|\varphi).$$

*Proof :* a) Since $c' = 2c_1L^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2}$ we have

$$Z^{qu}_j(P|\varphi)^{-2/3} S^{in}_{j,c'}(\overline{X} - P|\varphi') = S^{in}_{j,c'}(\overline{X}|\varphi').$$

b) We have

$$\widetilde{K}_{j-1,c}(X|\varphi)^4 \leq [S'^{in}_{j-1,4c}(X|\varphi)]^{-1} S^{out}_{j-1,4c,2K_2}(X|\varphi).$$

c) We have

$$S^{out}_{j,K_\beta c_\beta \lambda_j^{1/2},K_2}(\overline{X} - P|\varphi') \leq S^{out}_{j,K_\beta c_\beta \lambda_j^{1/2},K_2}(\overline{X}|\varphi')[S^{in}_{j,K_\beta c_\beta \lambda_j^{1/2}}(\overline{X}|\varphi')]^{-1}$$

d) We have

$$[S'^{in}_{j-1,4c}(X|\varphi)]^{-1} \leq [S'^{in}_{j,4cL^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2}}(\overline{X}|\varphi')]^{-1}$$

e) For $4cL^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2} + K_\beta c_\beta \lambda_j^{1/2} \leq \frac{1}{4}c' = \frac{c_1}{2}L^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2}$ we have

$$S^{in}_{j,c'}(\overline{X}|\varphi')[S'^{in}_{j,4cL^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2}}(\overline{X}|\varphi')]^{-1}[S'^{in}_{j,K_\beta c_\beta \lambda_j^{1/2}}(\overline{X}|\varphi')]^{-1} \leq$$

$$\leq S'^{in}_{j,\frac{3}{2}c_1L^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2}}(\overline{X}|\varphi')$$

a)-e) imply the assertion. $\surd$

Lemma 7.8. *There exist positive constants $K_0$ and $K_\beta$ such that for all polymers $X$ of $\Lambda_{j-1}$ and $P \subseteq \overline{X}$ with $[U_\delta(P)] = X$ , $\rho(P) = 1$ and $\Psi = \mathcal{A}^{j-1}\varphi + \chi$, $\chi \in c_3\mathcal{K}_{j-1}(X)$ and positive constants $c', c_4 = (3/4)^3\frac{1}{4!4}$ we have*

$$|Z^{rel}_j(\overline{X} - P|\Psi)| \leq S^{in}_{j,c'}(\overline{X} - P|\varphi') S^4_{j,c_4}(\overline{X} - P|\varphi') S^{out}_{j,K_\beta c_\beta \lambda_j^{1/2},K_2}(\overline{X} - P|\varphi')$$

$$\exp\{\frac{17\lambda_j c_3^4 2^d}{4!\lambda_{j-1}} + K_0 c'^2 (2L)^d\}.$$

*Proof :* Using lemma 6.6, we get

$$|Z^{rel}_j(\overline{X} - P|\Psi)| \leq S^{in}_{j,c'}(\overline{X} - P|\varphi') S^4_{j,c_4}(\overline{X} - P|\varphi') S^{out}_{j,K_\beta c_\beta \lambda_j^{1/2},K_2}(\overline{X} - P|\varphi')$$

$$S^{im}_{j,\frac{17}{4!}}(\overline{X} - P|\chi) \exp\{K_0 c'^2 |\overline{X} - P|\}.$$

Since

$$|\overline{X} - P| \leq |\overline{X}| = L^d|X| \leq (2L)^d$$

and $\chi \in c_3\mathcal{K}_{j-1}(X)$ we have

$$S^{im}_{j,\frac{17}{4!}}(\overline{X} - P|\chi) \exp\{K_0 c'^2|\overline{X} - P|\} \leq \exp\{\frac{17\lambda_j c_3^4 2^d}{4!\lambda_{j-1}} + K_0 c'^2 (2L)^d\}.$$

This proves the assertion. $\surd$

LEMMA 7.9. *For small $\lambda$ and large $L$ there exist constants*

$$\widetilde{c} = (C_1, C_2, \gamma = 0, c_1, c_2, c_3, c, K_2), \quad c_2 \le c_4 = (\frac{3}{4})^3 \frac{1}{4!4}$$

$$\widetilde{c}' = (C_1, C_2, \gamma = 0, \frac{3}{2}c_1, c_2, c_3, Lc, 2K_2)$$

*such that if*

$$\|R_j^{ren}\|_j^{\widetilde{c}} \le \epsilon$$

*then we have*

$$\|R_{j-1}^{div}\|_{j-1}^{\widetilde{c}'} \le \frac{1}{4}\epsilon$$

*for all* $j \le 0$.

*Proof :* Using lemma 7.5 we obtain

$$|Z_j^{rel}(\overline{X} - P|\Psi) R_j^{ren}(P|\Psi)| = |Z_j^{rel}(\overline{X} - P|\Psi) Z_j^{qu}(P|\varphi)^{-1} \widetilde{R}_j^{ren}(P|\varphi, \Psi)| \le I_1 + I_2$$

with

$$I_1 = 5\max(c_3^{-2}, c_3^{-4})(K')^4 c^{-2} \frac{\lambda_j}{\lambda_{j-1}} L^{-d} \; L^{-1+\epsilon} \; (a_j^{-1} dist(P))^{4-2\epsilon} |Z_j^{rel}(\overline{X} - P|\Psi)| \; Z_j^{qu}(P|\varphi)^{-2/3} \widetilde{K}_{j-1,c}(X|\varphi)^4 \sup_{\chi \in c_3 \mathcal{K}_j(P)} |R_j^{ren}(P|\chi)|$$

and

$$I_2 = c_3^{-6}(K')^6 c^{-3} L^{-d} \; L^{6-2d} |Z_j^{rel}(\overline{X} - P|\Psi)| Z_j^{qu}(P|\varphi)^{-1} \widetilde{K}_{j-1,c}(X|\varphi)^6 \sup_{|\alpha| \le 1} |Z_j^{qu}(P|\Theta\varphi + \alpha c_3 K_{j-1,c}(X|\varphi)^{-1} \; L^{d/2-1}\varphi) \; R_j^{ren}(P|\Theta\Psi + \alpha K_{j-1,c}(X|\varphi)^{-1} \; L^{d/2-1}\Psi)|.$$

Using lemma 7.7 and 7.8 we obtain

$$I_1 \le 5\max(c_3^{-2}, c_3^{-4})(K')^4 c^{-2} \frac{\lambda_j}{\lambda_{j-1}} L^{-d} \; L^{-1+\epsilon} \; (K_v'\delta)^{4-2\epsilon} \sup_{\chi \in c_3 \mathcal{K}_j(P)} |R_j^{ren}(P|\chi)| \; \exp\{\frac{17\lambda_j c_3^4 2^d}{4!\lambda_{j-1}} + K_0 c'^2 (2L)^d\} S^{in}_{j,\frac{3}{2}c_1 L^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2}}(\overline{X}|\varphi') \; S^{out}_{j,K_\beta c_\beta \lambda_j^{1/2},K_2}(\overline{X}|\varphi') S^{out}_{j-1,\frac{4c_1}{q},2K_2}(X|\varphi).$$

with

$$K_v' \equiv a_j^{-1} dist(U_\delta(y))/\delta$$

for $c' = 2c_1 L^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2}$, $q \ge 12$, $c_1 = qc$, and $6K_\beta c_\beta \lambda_j^{1/2} \le c_1 L^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2}$. For $5c_1/q \le c_2$ and $\lambda_j$ small we have

$$S^{in}_{j,\frac{3}{2}c_1 L^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2}}(\overline{X}|\varphi') S^{out}_{j,K_\beta c_\beta \lambda_j^{1/2},K_2}(\overline{X}|\varphi') S^{out}_{j-1,\frac{4c_1}{q},2K_2}(X|\varphi) \le$$
$$\le S^{in}_{j-1,\frac{3}{2}c_1}(X|\varphi) S^{out}_{j-1,c_2,2K_2}(X|\varphi).$$



Thus

$$I_1 \le 5\max(c_3^{-2}, c_3^{-4})(K')^4 q^2 c_1^{-2} \frac{\lambda_j}{\lambda_{j-1}} L^{-d} \; L^{-1+\epsilon} \; (K_v'\delta)^{4-2\epsilon} \sup_{\chi \in c_3 \mathcal{K}_j(P)} |R_j^{ren}(P|\chi)| \; \exp\{\frac{17\lambda_j c_3^4 2^d}{4!\lambda_{j-1}} + K_0 c'^2 (2L)^d\} S^{in}_{j-1,\frac{3}{2}c_1}(X|\varphi) S^{out}_{j-1,c_2,2K_2}(X|\varphi).$$

Using lemma 7.6 and 7.8 we obtain

$$I_2 \le c_3^{-6}(K')^6 c^{-3} L^{-d} \; L^{6-2d} \exp\{\frac{17\lambda_j c_3^4 2^d}{4!\lambda_{j-1}} + K_0 c'^2 (2L)^d\} \; \exp\{\frac{3}{8} c_1 c_3^2 (\frac{\lambda_{j-1}}{\lambda_j})^{1/2} L^{-2} |P|\} S^{in}_{j,2c_1 L^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2}}(\overline{X}|\varphi') \; S^{out}_{j,c_2'+K_\beta c_\beta \lambda_j^{1/2},2K_2/3}(\overline{X}|\varphi') S^{out}_{j-1,\frac{6c_1}{q},2K_2}(X|\varphi).$$
$$\sup_{|\alpha| \le 1} \left\{ |R_j^{ren}(P|\Theta\Psi + \alpha K_{j-1,c}(X|\varphi)^{-1} \; L^{d/2-1}\Psi)| / [S^{in}_{j,c_1}(P|\Theta\varphi') S^{out}_{j,c_2}(P|\Theta\varphi')] \right\}.$$

for

$$c' = 3c_1 L^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2}, \; q \ge 12, \; K_\beta c_\beta \lambda_j^{1/2} \le 6cL^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2},$$

$$c_1 = qc, \; c_2 \le c_4 = (\frac{3}{4})^3 \frac{1}{4!4}, \; c_2' = \exp\{-\frac{K_2}{6}\delta\} c_2.$$

For $6c_1/q \le c_2/2$, $\lambda_j$ small and $L$ large we have

$$S^{in}_{j,2c_1 L^{-2}(\frac{\lambda_{j-1}}{\lambda_j})^{1/2}}(\overline{X}|\varphi') S^{out}_{j,c_2'+K_\beta c_\beta \lambda_j^{1/2},2K_2/3}(\overline{X}|\varphi') S^{out}_{j-1,\frac{6c_1}{q},2K_2}(X|\varphi) \le$$
$$\le S^{in}_{j-1,\frac{3}{2}c_1}(X|\varphi) S^{out}_{j-1,c_2,2K_2}(X|\varphi).$$

Thus

$$I_2 \le c_3^{-6}(K')^6 c^{-3} L^{-d} \; L^{6-2d} \exp\{\frac{17\lambda_j c_3^4 2^d}{4!\lambda_{j-1}} + K_0 c'^2 (2L)^d\} \; \exp\{\frac{3}{8} c_1 c_3^2 (\frac{\lambda_{j-1}}{\lambda_j})^{1/2} L^{-2} u_j(\delta)\} S^{in}_{j-1,\frac{3}{2}c_1}(X|\varphi) S^{out}_{j-1,c_2,2K_2}(X|\varphi).$$
$$\sup_{\chi' \in c_3 \mathcal{K}_j(P)} \left\{ |R_j^{ren}(P|\Theta \mathcal{A}^j \varphi' + \chi')| / [S^{in}_{j,c_1}(P|\Theta\varphi') S^{out}_{j,c_2}(P|\Theta\varphi')] \right\}.$$

By (7-11) and (7-12) follows

$$\sup_{\substack{\varphi:\ \Lambda_{j-1}\to\mathbf{R}\\ x\in c_3\mathcal{K}_{j-1}(X)}} \left\{ |R^{div}_{j-1}(X|\Psi)|_{\Psi=\mathcal{A}^{j-1}\varphi+\chi} / B^{\widetilde{c}'}_{j-1}(X|\varphi) \right\} \le$$

$$\le \Big[ 5\max(c_3^{-2}, c_3^{-4})(K')^4 q^2 c_1^{-2} \frac{\lambda_j}{\lambda_{j-1}} L^{-d} L^{-1+\epsilon} (K'_v\delta)^{4-2\epsilon}$$

$$\exp\{\frac{17\lambda_j c_3^4 2^d}{4!\lambda_{j-1}} + K_0 c_1^2 \frac{\lambda_{j-1}}{\lambda_j} 2^d 4\} +$$

$$+ c_3^{-6}(K')^6 q^3 c_1^{-3} L^{-d} L^{6-2d} \exp\{\frac{17\lambda_j c_3^4 2^d}{4!\lambda_{j-1}} + K_0 c_1^2 \frac{\lambda_{j-1}}{\lambda_j} 2^d 4\}$$

$$\exp\{\frac{3}{8} c_1 c_3^2 (\frac{\lambda_{j-1}}{\lambda_j})^{1/2} L^{-2} u_j(\delta)\} \Big]$$

$$\sum_{\substack{P:\ [U_\delta(P)]=X\\ \rho(P)=1}} \sup_{\substack{\varphi':\ \Lambda_j\to\mathbf{R}\\ x'\in c_3\mathcal{K}_{j-1}(X)}} \left\{ |R_j(P|\varphi')|_{\Psi=\mathcal{A}^{j-1}\varphi+\chi} / B^{\widetilde{c}}_j(P|\varphi') \right\}.$$

Since

$$\sum_{X:\ x\in X} \sum_{\substack{P:\ [U_\delta(P)]=X\\ \rho(P)=1}} (\ldots) \le \sum_{y:\ x\in[U_\delta(y)]} \sum_{P:\ y\in P} (\ldots)$$

and (cp. (6-21))

$$\sum_{y:\ x\in[U_\delta(y)]} 1 = K_v(\delta)\, L^d$$

we obtain

$$\|R^{div}_{j-1}\|^{\widetilde{c}'}_{j-1} \le A\, \|R^{ren}_j\|^{\widetilde{c}}_j$$

with

$$A = \Big[ 5\max(c_3^{-2}, c_3^{-4})(K')^4 q^2 c_1^{-2} \frac{\lambda_j}{\lambda_{j-1}} L^{-1+\epsilon} (K'_v\delta)^{4-2\epsilon}$$

$$\exp\{\frac{17\lambda_j c_3^4 2^d}{4!\lambda_{j-1}} + K_0 c_1^2 \frac{\lambda_{j-1}}{\lambda_j} 2^d 4\} +$$

$$+ c_3^{-6}(K')^6 q^3 c_1^{-3} L^{-2} \exp\{\frac{17\lambda_j c_3^4 2^d}{4!\lambda_{j-1}} + K_0 c_1^2 \frac{\lambda_{j-1}}{\lambda_j} 2^d 4\}$$

$$\exp\{\frac{3}{8} c_1 c_3^2 (\frac{\lambda_{j-1}}{\lambda_j})^{1/2} L^{-2} u_j(\delta)\} \Big] K_v(\delta).$$

Since $c_1, c_3, c_4$ are $L$-independent and $\delta \propto \ln L$ we have for large $L$

$$A \le \frac{1}{4}.$$

Thus

$$\|R^{div}_{j-1}\|^{\widetilde{c}'}_{j-1} \le \frac{1}{4}\epsilon. \quad \checkmark$$

*Remark* : Lemma 7.9 implies lemma 3.5.



LEMMA 7.10. *For small $\lambda$ and large $L$ there exist positive constants $K_0$ and*

$$\widetilde{c} = (C_1, C_2, \gamma, c_1, c_2, c_3, c, K_2)$$

*such that*

$$\|R^{div}_{j-1}\|^{\widetilde{c}'}_j \le K_v L^d \exp\{\frac{17}{4!} c_3^4 (2L)^4 + K_0 c_1^2 (2L)^4 + \gamma\} \|R^{ren}_j\|^{\widetilde{c}}_j$$

*for all positive $\gamma$, and all $j \le 0$ and*

$$\widetilde{c}' = (C_1, C_2, \gamma, \frac{1}{2} c_1, c'_2, c_3, c, K_2/3)$$

$$c'_2 = q_3 \exp\{-\frac{K_2\delta}{6}\} c_2,$$

$$q_n = \sup_{z\in base} \{ \sum_{y\in\Lambda_j} \exp\{-\frac{K_2}{2n} a_j^{-1} dist(z,y)\} \}.$$

*Proof* : For $x \in \Lambda_{j-1}$, $\varphi : \Lambda_j \to \mathbf{R}$, $\Psi = \mathcal{A}^j\varphi + \chi$, $\chi \in c_3\mathcal{K}_j(\overline{X})$ we have

$$\sum_{X:\ x\in X} \frac{|R^{div}_{j-1}(X|\Psi)|}{B^{\widetilde{c}'}_j(X|\varphi)} \le \sum_{X:\ x\in X} \sum_{\substack{P:\ [U_\delta(P)]=X\\ \rho(P)=1}} \frac{|Z^{rel}_j(\overline{X} - P|\Psi) R^{ren}_j(P|\Psi)|}{B^{\widetilde{c}'}_j(X|\varphi)}.$$

Lemma 6.5 implies for $[U_\delta(P)] = X$

$$S_{j,c_1,c_2,K_2}(\overline{X} - P|\varphi)\, S_{j,c_1,c_2,K_2}(P|\varphi) \le S_{j,c_1-q_1c_2,c'_2,K_2/3}(\overline{X}|\varphi). \tag{7-13}$$

Lemma 6.6 implies

$$|Z^{rel}_j(\overline{X} - P|\Psi)| \le S_{j,c_1,K_\beta c_\beta \lambda_j^{1/2},K_2}(\overline{X} - P|\varphi) \exp\{[\frac{17}{4!} c_3^4 + K_0 c_1^2] |\overline{X} - P|\}.$$

Since $|[U_\delta(P)]| \le 2^d$ if $\rho(P) = 1$ for $L$ large we obtain

$$\frac{|Z^{rel}_j(\overline{X} - P|\Psi)|}{B^{\widetilde{c}'}_j(X|\varphi)} \le \exp\{(2L)^d [\frac{17}{4!} c_3^4 + K_0 c_1^2]\} \frac{S_{j,c_1,c_2,K_2}(\overline{X} - P|\varphi)}{B^{\widetilde{c}'}_j(X|\varphi)}.$$

Thus eq. (7-13) implies

$$\frac{|Z^{rel}_j(\overline{X} - P|\Psi)|}{B^{\widetilde{c}'}_j(X|\varphi)} \le \exp\{(2L)^d [\frac{17}{4!} c_3^4 + K_0 c_1^2]\} \frac{S_{j,c_1-q_1c_2,c'_2,K_2/3}(\overline{X}|\varphi)}{B^{\widetilde{c}'}_j(X|\varphi)} \cdot \frac{1}{S_{j,c_1,c_2,K_2}(P|\varphi)} \le$$

$$\le \exp\{(2L)^d [\frac{17}{4!} c_3^4 + K_0 c_1^2]\} \exp\{\gamma 2^d\} \frac{1}{B^{\widetilde{c}}_j(P|\varphi)}.$$

Thus

$$\sum_{X:\ x\in X} \frac{|R^{div}_{j-1}(X|\Psi)|}{B^{\widetilde{c}'}_j(X|\varphi)} \le \exp\{(2L)^d [\frac{17}{4!} c_3^4 + K_0 c_1^2] + \gamma\} \sum_{y:\ x\in[U_\delta(y)]} \sum_{P:\ y\in P} \frac{|R^{ren}_j(P|\Psi)|}{B^{\widetilde{c}}_j(P|\varphi)} \le$$

$$\le K_v L^d \exp\{\frac{17}{4!} c_3^4 (2L)^4 + K_0 c_1^2 (2L)^4 + \gamma\} \|R^{ren}_j\|^{\widetilde{c}}_j. \quad \checkmark$$

## 4.8. Integration Step (I) : Bounds on $M^n_{j-1}$

In the following sections bounds for the integration step are presented. In this section not normalized activities $M^n_{j-1}$ are necessary to find bounds on vacuum energy counterterms $e_{j-1}$. This will be performed in the next section 4.9. Section 4.11 will close the bounds for the integration step by proving a bound on normalized irrelevant activities $R_{j-1}$.

Define for a polymer $X$ of $\Lambda_{j-1}$ the Gaussian expectation value of the relevant partition function

$$\widehat{Z}^n_{j-1}(X|\Psi) \equiv \int d\mu_{v^j_X}(\Phi) Z^{rel}_j(X|\Phi+\Psi) = \sum_{X=\sum Q} \prod_Q \widehat{A}^n_{j-1}(Q|\Psi), \tag{8-1a}$$

$$\widehat{M}^n_{j-1}(X|\Psi) \equiv -\delta_{1,|X|} + \widehat{A}^n_{j-1}(X|\Psi), \tag{8-1b}$$

$$\delta M^n_{j-1}(X|\Psi) \equiv A^n_{j-1}(X|\Psi) - \widehat{A}^n_{j-1}(X|\Psi). \tag{8-1c}$$

Using the tree graph formula we have for a polymer $X = \{x_1, \ldots, x_N\}$ of $\Lambda_{j-1}$

$$\widehat{M}^n_{j-1}(X|\Psi) = \delta_{1,|X|}[\int d\mu_{v^j_X}(\Phi) Z^{rel}_j(X|\Phi+\Psi) - 1] + \\ + \sum_{\tau\in T_N} \sum_{\substack{\pi,\eta: \\ \tau=\tau(\pi,\eta)}} \int_0^1 ds\ f(\eta|s) \sum_{m\geq 1} \sum_{\substack{(\sigma_1,\ldots,\sigma_m): \\ \sum_{i=1}^m \sigma_i=\tau}} \\ \int d\mu_{v^j_X[t]}(\Phi) \prod_{i=1}^m [(\prod_{(ab)\in\sigma_i} \frac{\partial}{\partial t_{ab}}) \frac{1}{2}\triangle_{v^j_X[t]}] \\ \prod_{i=1}^N Z^{rel}_j(\{x_i\}|\Phi+\Psi)|_{t=t[\pi,s]} \tag{8-2}$$

and for the part of the activity which contains at least one $R^c_j$–term

$$\delta M^n_{j-1}(X|\Psi) = R^c_j(X|\Psi) + \int_0^1 d\gamma \frac{1}{2}\triangle_{v^j_X} \int d\mu_{\gamma v^j_X}(\Phi) R^c_j(X|\Phi+\Psi) + \\ + \sum_{N\geq 2} \frac{1}{N!} \sum_{\substack{Q_1,\ldots,Q_N\subseteq X: \\ \sum Q_i = X}} \sum_{\substack{I:\ \emptyset\neq I\subseteq \underline{N}: \\ |Q_a|=1\ \forall a\in \underline{N}-I}} U(Q_1,\ldots,Q_N; I|\Psi) \tag{8-3}$$

with

$$U(Q_1,\ldots,Q_N;I|\Psi) \equiv \sum_{\tau\in T_N} \sum_{\substack{\pi,\eta: \\ \tau=\tau(\pi,\eta)}} \int_0^1 ds\ f(\eta|s) \sum_{m\geq 1} \sum_{\substack{(\sigma_1,\ldots,\sigma_m): \\ \sum_{i=1}^m \sigma_i=\tau}} \\ \int d\mu_{v^j_X[t]}(\Phi) \prod_{i=1}^m [(\prod_{(ab)\in\sigma_i} \frac{\partial}{\partial t_{ab}}) \frac{1}{2}\triangle_{v^j_X[t]}] \\ \prod_{i\in \underline{N}-I} Z^{rel}_j(Q_i|\Phi+\Psi) \prod_{i'\in I} R^c_j(Q_{i'}|\Phi+\Psi)|_{t=t[\pi,s]} \tag{8-4}$$



where

$$\triangle_{v^j_X[t]} \equiv \left(\frac{\delta}{\delta\Phi}, v^j_X[t]\frac{\delta}{\delta\Phi}\right) \equiv \int_{z_1,z_2} \frac{\delta}{\delta\Phi(z_1)} v^j_X[t](z_1,z_2) \frac{\delta}{\delta\Phi(z_2)}. \tag{8-5}$$

We have used the change of covariance lemma and eq. (1-37). $(\sigma_1,\ldots,\sigma_m)$ denotes a (not ordered) $m$–tuple of subgraphs of $\tau$.

The following definition will be useful to estimate tree graph formulas for activities.

DEFINITION 8.1. *For $m,n\geq 1$ and $Q_1,\ldots,Q_n, P_1,\ldots,P_m, X$ polymers of $\Lambda_{j-1}$ with $\sum_{i=1}^n Q_i = X$ and $P_a \subseteq X$ $\forall a\in \underline{m}$ we call $[P_1,\ldots P_m]$ a connectivity graph with respect to $Q_1,\ldots,Q_n$ if*

(a) $P_1\cap Q_1 \neq \emptyset$ and $P_1 \cap X - Q_1 \neq \emptyset$

(b) *For all $k\in\{2,\ldots,m\}$ we have $P_k \cap \sum_{a\in \underline{m}:\ Q_a\cap \bigcup_{b=1}^{k-1} P_b\neq\emptyset} Q_a \neq \emptyset$*

(c) *There exists an injective function $f:\ \{1,\ldots,m\}\to\{1,\ldots,n\}$ such that*

$$\forall a\in\{1,\ldots m\}:\ P_a\cap Q_{f(a)}\neq\emptyset.$$

(d) $\forall a\in\{1,\ldots,n\}\ \exists b\in\{1,\ldots,m\}:\ Q_a\cap P_b\neq\emptyset.$

*For the special case $|Q_i| = 1$ for all $i\in\{1,\ldots,n\}$ and $Q_1 = \{x\}$ we call $[P_1,\ldots,P_m]$ a connectivity graph in $X$ with root $x$. The set of all connectivity graphs with respect to $Q_1,\ldots,Q_n$ is denoted by $\mathcal{CG}(Q_1,\ldots,Q_n)$ and the set of all connectivity graphs in $X$ with root $x$ is denoted by $\mathcal{CG}_x(X)$. The set of all connectivity graphs with root $x$ is defined by*

$$\mathcal{CG}_x \equiv \bigcup_X \mathcal{CG}_x(X).$$

The following lemma will be used to bound external fields.

LEMMA 8.2. *For $P_1,\ldots,P_N, Y$ polymers of $\Lambda_j$ with $\sum_{i=1}^N P_i = Y$ and $\varphi:\ \Lambda_j\to\mathbf{R}$ we have*

$$\prod_{i=1}^N S^{out}_{j,c_2,K_2}(P_i|\varphi) \leq S^{out}_{j,c_2q'_2,K_2/2}(Y|\varphi)[S^{in}_{j,q'_2c_2}(Y|\varphi)]^{-1}$$

*and*

$$\prod_{i=1}^N S_{j,c_1,c_2,K_2}(P_i|\varphi) \leq S_{j,c_1-q'_2c_2,c_2q'_2,K_2/2}(Y|\varphi)$$

*with*

$$q'_2 \equiv \sup_{z\in base} \{\sum_{y\in\Lambda_j} \exp\{-\frac{K_2}{4}a_j^{-1} dist(z,y)\}\}.$$

*Proof :* We have

$$\sum_{i=1}^{N} c_2 a_j^{-2}\lambda_j^{1/2} \sum_{y\in\Lambda_j - P_i} \int_{z\underline{\in} z} \exp\{-\frac{K_2}{2}a_j^{-1} dist(z,P_i)\}(\mathcal{A}^j\varphi)^2(z) \le$$

$$\le \sum_{y\in\Lambda_j} c_2 a_j^{-2}\lambda_j^{1/2} \int_{z\underline{\in}\Lambda_j} \exp\{-\frac{K_2}{4}a_j^{-1} dist(z,y) - \frac{K_2}{4}a_j^{-1} dist(z,Y)\}(\mathcal{A}^j\varphi)^2(z) \le$$

$$\le c_2 q_2' a_j^{-2}\lambda_j^{1/2} \int_{z\underline{\in}\Lambda_j} \exp\{-\frac{K_2}{4}a_j^{-1} dist(z,Y)\}(\mathcal{A}^j\varphi)^2(z) =$$

$$= c_2 q_2' a_j^{-2}\lambda_j^{1/2} \int_{z\underline{\in}\Lambda_j - Y} \exp\{-\frac{K_2}{4}a_j^{-1} dist(z,Y)\}(\mathcal{A}^j\varphi)^2(z) +$$

$$+ c_2 q_2' a_j^{-2}\lambda_j^{1/2} \int_{z\underline{\in} Y} (\mathcal{A}^j\varphi)^2(z).$$

This proves the assertion. √ With the help of the definition 8.1 of connectivity graphs the following bound on $U(Q_1,\ldots,Q_N;I|\Psi)$ is obtained.

LEMMA 8.3. *For positive constants $\tilde{c} = (C_1, C_2, \gamma, c_1, c_2, c_3, c, K_2)$, $c \le \frac{K_2}{4}L^{-1}$, $N \ge 2$ and polymers $Q_1,\ldots,Q_N$ of $\Lambda_{j-1}$ with $\sum_{i=1}^N Q_i = X$, $I \subseteq \underline{N}$ with $|Q_a| = 1$ $\forall a \in \underline{N} - I$ and $\Psi = \mathcal{A}^{j-1}\varphi + \chi$ with $\varphi : \Lambda_{j-1} \to \mathbf{R}$, $\chi \in \frac{1}{2}c_3\mathcal{K}_j(\overline{X})$ there exists a $(\lambda, C_1)$-independent constant $\tilde{K}$ such that*

$$|U(Q_1,\ldots,Q_N;I|\Psi)| \le \sum_{m\ge 1} \frac{(C_1\lambda_j)^{\frac{m}{2}\delta_{2m,X,1}}}{m!} \sum_{\substack{P_1,\ldots,P_m \subseteq \Lambda_{j-1}:\ |P_i|\le 4 \\ \{P_1,\ldots,P_m\}\in \mathcal{CG}(Q_1,\ldots,Q_N)}} (\tilde{K}C_1^{-1/2})^m$$

$$\Big[\prod_{i=1}^m \exp\{-\frac{K_2}{4}a_{j-1}^{-1}L_{tree}(P_i)\}\Big]\prod_{i\in I}\Big[\sup_{\substack{\chi\in c_3\mathcal{K}_j(Q_i) \\ \tilde{\varphi}:\ \Lambda_j\to\mathbf{R}}} \Big|\frac{R_j^c(Q_i|\mathcal{A}^j\tilde{\varphi}+\chi)}{B_j^{\tilde{c}}(Q_i|\varphi)}\Big|\Big] B_j^{\tilde{c}'}(X|\varphi')$$

*with $\varphi' = \mathcal{A}_{j,j-1}\varphi$ and*

$$\tilde{c}' = (C_1, C_2, \gamma, \frac{1}{2}c_1 - 2q_2', 2c_2q_2', c_3, c, K_2/2)$$

$$q_2' \equiv \sup_{z\in base}\Big\{\sum_{y\in\Lambda_{j-1}} \exp\{-\frac{K_2}{4}a_j^{-1}dist(z,y)\}\Big\}.$$

*Proof :* Consider $(\sigma_1,\ldots,\sigma_m)$ with $\sum_{i=1}^m \sigma_i = \tau$. For each $i \in \{1,\ldots,m\}$ define $a_i \in V(\sigma_i)$ (= vertex set of graph $\sigma_i$) such that $a_i \ne a_{i'}$ for $i \ne i'$. For $y\underline{\in}Q_a$ define $c(y) \equiv a$. We have ($\chi_y$ = characteristic function )

$$(\prod_{(ab)\in\sigma_i}\frac{\partial}{\partial t_{ab}})\Delta_{v_X^j[t]}F(\Phi) = \sum_{\substack{y_1,\ldots,y_4\in\overline{X}: \\ \{c(y_1),\ldots,c(y_4)\}\supseteq V(\sigma_i)}} (\prod_{(ab)\in\sigma_i}\frac{\partial}{\partial t_{ab}})\, t_{c(y_1)c(y_3)}t_{c(y_1)c(y_2)}t_{c(y_2)c(y_4)}$$

$$\exp\{-\frac{3K_2}{4}a_j^{-1}|y_1-y_3|\}a_j^{d-2}v^j(y_1,y_2)\exp\{-\frac{3K_2}{4}a_j^{-1}|y_2-y_4|\}$$

$$\frac{\partial^2}{\partial\alpha^i_{y_1y_3}\partial\beta^i_{y_2y_4}}F(\Phi(.) + a_j^{1+\frac{d}{2}}\exp\{\frac{3K_2}{4}a_j^{-1}|y_1-y_3|\}\mathcal{A}^j(.,y_1)\alpha^i_{y_1y_3}\chi_{y_3}(.) +$$

$$+ a_j^{1+\frac{d}{2}}\exp\{\frac{3K_2}{4}a_j^{-1}|y_2-y_4|\}\mathcal{A}^j(.,y_2)\beta^i_{y_2y_4}\chi_{y_4}(.))|_{\substack{\alpha\equiv 0\\ \beta\equiv 0}}.$$



For $y_1^{(i)},\ldots,y_4^{(i)} \in \overline{X}$ with $\{c(y_1^{(i)}),\ldots,c(y_4^{(i)})\} \supseteq V(\sigma_i)$, $i \in \{1,\ldots,m\}$ define $\tilde{y}_i \in Q_a$ $\{y_1^{(i)},\ldots,y_4^{(i)}\}$ and

$$\chi(\alpha,\beta;\{y_1^{(1)},\ldots,y_4^{(1)}\},\ldots,\{y_1^{(m)},\ldots,y_4^{(m)}\})(z) \equiv a_j^{1+\frac{d}{2}}\sum_{i=1}^m\Big[\alpha^i_{y_1^{(i)}y_3^{(i)}}$$

$$\exp\{\frac{3K_2}{4}a_j^{-1}|y_1^{(i)}-y_3^{(i)}| - \frac{K_2}{8}a_{j-1}^{-1}|\tilde{y}_i - y_3^{(i)}|\}\mathcal{A}^j(z,y_1^{(i)})\chi_{y_3^{(i)}}(z) +$$

$$+ (1\to 2,\ 3\to 4,\ \alpha\to\beta)\Big]$$

Then we obtain

$$U(Q_1,\ldots,Q_N;I|\Psi) = \sum_{\tau\in\mathcal{T}_N}\sum_{\substack{\pi,\eta: \\ \tau=\tau(\pi,\eta)}}\int_0^1 ds\ f(\eta|s)\sum_{m\ge 1}\frac{1}{2^m}\sum_{\substack{(\sigma_1,\ldots,\sigma_m): \\ \sum_{i=1}^m\sigma_i=\tau}}$$

$$\sum_{\substack{y_1^{(1)},\ldots,y_4^{(1)}\in\overline{X}: \\ \{c(y_1^{(1)}),\ldots,c(y_4^{(1)})\}\supseteq V(\sigma_1)}} \cdots \sum_{\substack{y_1^{(m)},\ldots,y_4^{(m)}\in\overline{X}: \\ \{c(y_1^{(m)}),\ldots,c(y_4^{(m)})\}\supseteq V(\sigma_m)}} \int d\mu_{v_X^j[t]}(\tilde{\varphi})$$

$$\prod_{i=1}^m\Big[(\prod_{(ab)\in\sigma_i}\frac{\partial}{\partial t_{ab}})t_{c(y_1^{(i)})c(y_3^{(i)})}t_{c(y_1^{(i)})c(y_2^{(i)})}t_{c(y_2^{(i)})c(y_4^{(i)})}$$

$$\exp\{-\frac{3K_2}{4}a_j^{-1}|y_1^{(i)}-y_3^{(i)}| + \frac{K_2}{8}a_{j-1}^{-1}|\tilde{y}_i - y_3^{(i)}|\}\ a_j^{d-2}v^j(y_1^{(i)},y_2^{(i)})$$

$$\exp\{-\frac{3K_2}{4}a_j^{-1}|y_2^{(i)}-y_4^{(i)}| + \frac{K_2}{8}a_{j-1}^{-1}|\tilde{y}_i - y_4^{(i)}|\}\ \frac{\partial^2}{\partial\alpha^i_{y_1^{(i)}y_3^{(i)}}\partial\beta^i_{y_2^{(i)}y_4^{(i)}}}\Big]$$

$$\prod_{i\in\underline{N}-I} Z_j^{rel}(Q_i|\mathcal{A}^j[t]\tilde{\varphi} + \Psi + \chi(\alpha,\beta;\{y_1^{(1)},\ldots,y_4^{(1)}\},\ldots,\{y_1^{(m)},\ldots,y_4^{(m)}\}))$$

$$\prod_{i'\in I} R_j^c(Q_i'|\mathcal{A}^j[t]\tilde{\varphi} + \Psi + \chi(\alpha,\beta;\{y_1^{(1)},\ldots,y_4^{(1)}\},\ldots,\{y_1^{(m)},\ldots,y_4^{(m)}\})))|_{\substack{\alpha\equiv 0,\ \beta\equiv 0 \\ t=t(\pi,s)}}$$

For $c_3$ large enough and $|\alpha|$, $|\beta| \le \lambda_j^{-1/4}$ we have

$$\chi(\alpha,\beta;\{y_1^{(1)},\ldots,y_4^{(1)}\},\ldots,\{y_1^{(m)},\ldots,y_4^{(m)}\}) \in \frac{c_3}{2}\mathcal{K}_j(\overline{X}).$$

For $z\underline{\in}Q_a$ we have

$$\mathcal{A}_X^j[t]\tilde{\varphi}(z) = \int_{y\in X} t_{a,c(y)}\mathcal{A}^j(z,y)\tilde{\varphi}(y) = (\mathcal{A}^j\tilde{\varphi}_{X,a}[t])(z)$$

with

$$\tilde{\varphi}_{X,a}[t](y) = t_{a,c(y)}\tilde{\varphi}(y).$$

Thus for $a \in \{1,\ldots,N\}$

$$R_j^c(Q_a|\mathcal{A}^j[t]\tilde{\varphi} + \Psi + \chi(\alpha,\beta;\{y_1^{(1)},\ldots,y_4^{(1)}\},\ldots,\{y_1^{(m)},\ldots,y_4^{(m)}\})) = R_j^c(Q_a|\mathcal{A}^j\varphi' + \chi'$$

with

$$\varphi' = \mathcal{A}_{j,j-1}\varphi + \tilde{\varphi}_{X,a}[t] : \Lambda_j \to \mathbf{R}$$

$$\chi' = \chi + \chi(\alpha, \beta; \{y_1^{(1)}, \ldots, y_4^{(1)}\}, \ldots, \{y_1^{(m)}, \ldots, y_4^{(m)}\}) \in c_3\mathcal{K}_j(Q_a).$$

Using

$$S_{j,c}^{in}(P|\varphi + \tilde{\varphi}) \leq S_{j,\frac{1}{2}c}^{in}(P|\varphi)[S_{j,c}^{in}(P|\tilde{\varphi})]^{-1}$$

$$S_{j,c}^{out}(P|\varphi + \tilde{\varphi}) \leq S_{j,2c}^{out}(P|\varphi) S_{j,2c}^{out}(P|\tilde{\varphi})$$

and (cp. lemma 2.2)

$$\sum_{\substack{\pi,\eta: \\ \tau=\tau(\pi,\eta)}} \int_0^1 ds\ f(\eta|s) = 1$$

and Cauchy's inequality we obtain

$$|U(Q_1, \ldots, Q_N; I|\Psi)| \leq \sum_{m\geq 1} \frac{1}{m!} \sum_{\substack{P_1,\ldots,P_m \subset \Lambda_{j-1}:\ |P_i|\leq 4 \\ [P_1,\ldots,P_m]\in \mathcal{CG}(Q_1,\ldots,Q_N)}} (\tilde{K}\lambda_j^{1/2})^m$$

$$\left[\prod_{i=1}^m \exp\{-\frac{K_2}{2} a_{j-1}^{-1} L_{tree}(P_i)\}\right] \prod_{i\in \underline{N}-I} \left[\sup_{\substack{\chi\in c_3\mathcal{K}_j(Q_i) \\ \varphi:\ \Lambda_j\to\mathbf{R}}} \left|\frac{Z_j^{rel}(Q_i|\mathcal{A}^j\varphi + \chi)}{S_{j,c_1,c_2,K_2}(Q_i|\varphi)}\right|\right]$$

$$\prod_{i\in I}\left[\sup_{\substack{\chi\in c_3\mathcal{K}_j(Q_i) \\ \varphi:\ \Lambda_j\to\mathbf{R}}} \left|\frac{R_j^c(Q_i|\mathcal{A}^j\varphi + \chi)}{S_{j,c_1,c_2,K_2}(Q_i|\varphi')}\right|\right] \prod_{i=1}^N S_{j,\frac{1}{2}c_1,2c_2,K_2}(Q_i|\varphi') \tag{8–6}$$

with $\varphi' = \mathcal{A}_{j,j-1}\varphi$. According to lemma 8.2 we have

$$\prod_{i=1}^N S_{j,\frac{1}{2}c_1,2c_2,K_2}(Q_i|\varphi') \leq S_{j,\frac{1}{2}c_1-2q_2'c_2,2c_2q_1',K_2}(X|\varphi'). \tag{8–7}$$

Define

$$X_1 \equiv \sum_{a\in I} Q_a.$$

Use that $[P_1, \ldots, P_m] \in \mathcal{CG}(Q_1, \ldots, Q_N)$ and $|P_a| \leq 4$ for all $a \in \{1, \ldots, m\}$ implies $m \geq [\frac{N-1}{4}] + 1$. Thus

$$\frac{m}{2} + n_{X_1} \geq n_X$$

and for $C_1\lambda_j < 1$

$$(C_1\lambda_j)^{m/2} \prod_{i\in I} (C_1\lambda_j)^{n_{Q_i}} \leq (C_1\lambda_j)^{n_X}.$$

For $C_2 > 1$ and $c \leq \frac{K_2}{4}L^{-1}$ we obtain

$$(C_1\lambda_j)^{m/2} \prod_{i=1}^m \exp\{-\frac{K_2}{4} a_{j-1}^{-1} L_{tree}(P_i)\}[\prod_{i\in I} C_{j,C_1,C_2}(Q_i) T_{j,c}(Q_i)] \leq C_{j,C_1,C_2}(X) T_{j,c}(X).$$



For the special case $n_X = \frac{1}{2}$ we have

$$\prod_{i\in I} (C_1\lambda_j)^{n_{Q_i}} \leq (C_1\lambda_j)^{n_X}.$$

Therefore

$$(C_1\lambda_j)^{m/2} \prod_{i\in I} (C_1\lambda_j)^{n_{Q_i}} \leq (C_1\lambda_j)^{\frac{m}{2}\delta_{2n_X,1}} C_{j,C_1,C_2}(X) T_{j,c}(X). \tag{8–8}$$

(8-6),(8-7) and (8-8) imply the assertion. √

Consider the following decomposition for the part of the activity $\delta M_{j-1}^n$ which contains at least one $R_j^c$-term

$$\delta M_{j-1}^n = R_j^c + \delta_1 M_{j-1}^n + \delta_2 M_{j-1}^n$$

where

$$\delta_1 M_{j-1}^n(X|\Psi) \equiv \int_0^1 d\gamma \frac{1}{2}\triangle_{v_X^j} \int d\mu_{\gamma v_X^j}(\Phi) R_j^c(X|\Phi + \Psi) \tag{8–9}$$

and

$$\delta_2 M_{j-1}^n(X|\Psi) \equiv \sum_{N\geq 2} \frac{1}{N!} \sum_{\substack{Q_1,\ldots,Q_N\subseteq X: \\ \sum Q_i = X}} \sum_{\substack{I:\ \emptyset\neq I\subseteq \underline{N}: \\ |Q_a|=1\ \forall a\in \underline{N}-I}} U(Q_1, \ldots, Q_N; I|\Psi). \tag{8–10}$$

In the next lemma $\delta_1 M_{j-1}^n$ and $\delta_2 M_{j-1}^n$ are estimated. $\delta_1 M_{j-1}^n$ is trivially bounded using Cauchy's inequality.

LEMMA 8.4. *For positive constants $\tilde{c} = (C_1, C_2, \gamma, c_1, c_2, c_3, c, K_2)$, $c \leq \frac{K_2}{4}L^{-1}$ and large $\gamma$ there exists a constant $K$ (not dependent on $\lambda_j$) such that*

$$\|\delta_1 M_{j-1}^n\|_j^{\tilde{c}'} \leq K\lambda_j^{1/2} \|R_j^c\|_j^{\tilde{c}}$$

*with*

$$\tilde{c}' = (C_1, C_2, \frac{1}{2}\gamma, \frac{1}{2}c_1, 2c_2, \frac{1}{2}c_3, c, K_2).$$

Using lemma 8.3 we want to estimate $\delta_2 M_{j-1}^n$. Consider the following decomposition

$$\delta_2 M_{j-1}^n(X|\Psi) = \delta_2' M_{j-1}^n(X|\Psi) + \delta_2'' M_{j-1}^n(X|\Psi) \tag{8–11}$$

where

$$\delta_2' M_{j-1}^n(X|\Psi) = \begin{cases} \delta_2 M_{j-1}^n(X|\Psi), & \text{for } n_X = \frac{1}{2} \\ 0, & \text{for } n_X > \frac{1}{2}. \end{cases} \tag{8–11a}$$

LEMMA 8.5. *For positive constants $\tilde{c} = (C_1, C_2, \gamma, c_1, c_2, c_3, c, K_2)$, $c \leq \frac{K_2}{4}L^{-1}$ and large $\gamma$ there exists a constant $K$ (not dependent on $\lambda_j$) such that*

$$\|\delta_2' M_{j-1}^n\|_j^{\tilde{c}'} \leq K\lambda_j^{1/2} \|R_j^c\|_j^{\tilde{c}} \tag{8–12}$$

*and*

$$\|\delta_2'' M_{j-1}^n\|_j^{\tilde{c}'} \leq K C_1^{-1} \|R_j^c\|_j^{\tilde{c}} \tag{8–13}$$

with

$$\tilde{c}' = (C_1, C_2, \frac{1}{2}\gamma, \frac{1}{2}c_1 - q_2' c_2, 2c_2 q_2', \frac{1}{2}c_3, c, K_2/2)$$

$$q_2' \equiv \sup_{z \in base} \{ \sum_{y \in \Lambda_j} \exp\{-\frac{K_2}{4} a_j^{-1} dist(z,y)\} \}.$$

*Proof :* $[P_1, \dots, P_m] \in \mathcal{CG}(Q_1, \dots, Q_N)$ and $|P_a| \leq 4$ for all $a \in \{1, \dots, m\}$ implies $m \geq [\frac{N-1}{4}] + 1$. For $\tilde{c}'' = (C_1, C_2, \gamma, \frac{1}{2}c_1 - q_2' c_2, 2c_2 q_2', \frac{1}{2}c_3, c, K_2/2)$ we have

$$\sum_{X:\, x \in X} \frac{|\delta_2' M_{j-1}^n(X|\Psi)|}{B_j^{\tilde{c}'}(X|\varphi')} \leq \sum_{N \geq 2} \frac{1}{(N-1)!} \sum_{\substack{Q_1, \dots, Q_N \subseteq \Lambda_{j-1}: \\ x \in Q_1}} \sum_{\substack{I:\, \emptyset \neq I \subseteq \underline{N}: \\ |Q_a| = 1\ \forall a \in \underline{N} - I}} \exp\{-\frac{1}{2}\gamma|X|\} \frac{|U(Q_1, \dots, Q_N; I|\Psi)|}{B_j^{\tilde{c}''}(X|\varphi')}.$$

Using lemma 8.3 and the definition of connectivity graphs (definition 8.1) we obtain

$$\sum_{X:\, x \in X} \frac{|\delta_2' M_{j-1}^n(X|\Psi)|}{B_j^{\tilde{c}'}(X|\varphi')} \leq \sum_{N \geq 2} \frac{1}{(N-1)!} \sum_{\substack{Q_1, \dots, Q_N \subseteq \Lambda_{j-1}: \\ x \in Q_1}} \sum_{\substack{I:\, \emptyset \neq I \subseteq \underline{N}: \\ |Q_a| = 1\ \forall a \in \underline{N} - I}} \exp\{-\frac{1}{2}\gamma|supp\, \mathcal{G}|\}$$
$$\sum_{m \geq 1} \sum_{\substack{P_1, \dots, P_m \subseteq \Lambda_{j-1}:\ |P_i| \leq 4 \\ [P_1, \dots, P_m] \in \mathcal{CG}(Q_1, \dots, Q_N)}} (\tilde{K}\lambda_j^{-1/2})^m$$
$$\Big[\prod_{i=1}^m \exp\{-\frac{K_2}{4} a_{j-1}^{-1} L_{tree}(P_i)\}\Big] \prod_{i \in I} \Big[\sup_{\substack{x \in c_3 \mathcal{K}_j(Q_i) \\ \varphi:\, \Lambda_j \to \mathbb{R}}} |\frac{R_j^c(Q_i|\mathcal{A}^j\varphi + \chi)}{B_j^{\tilde{c}}(Q_i|\varphi)}|\Big]$$

According to the definition of $\mathcal{CG}_x$ we obtain

$$\sum_{X:\, x \in X} \frac{|\delta_2' M_{j-1}^n(X|\Psi)|}{B_j^{\tilde{c}''}(X|\varphi')} \leq \sum_{N \geq 2} \sum_{M:\ M - N \geq [\frac{N-1}{4}] + 1} \frac{1}{(M-1)!} \sum_{\substack{J \subseteq \underline{M}: \\ |J| = N}} \sum_{k=1}^N \sum_{\substack{I \subseteq J: \\ |I| = k}}$$
$$\sum_{\substack{\mathcal{G} \in \mathcal{CG}_x: \\ \mathcal{G} = [U_1, \dots, U_M]}} \exp\{-\frac{1}{2}\gamma|supp\, \mathcal{G}|\}(\tilde{K}\lambda_j^{1/2})^{M-N} \Big[\prod_{a \in \underline{M} - J} \exp\{-\frac{K_2}{4} a_{j-1}^{-1} L_{tree}(U_a)\}\Big]$$
$$\Big[\prod_{b \in I} \sup_{\substack{x \in c_3 \mathcal{K}_j(Q_i) \\ \varphi:\, \Lambda_j \to \mathbb{R}}} |\frac{R_j^c(U_b|\mathcal{A}^j\varphi + \chi)}{B_j^{\tilde{c}}(U_b|\varphi)}|\Big] \Big[\prod_{c \in J - I} \delta_{1,|U_c|}\Big]. \tag{8-14}$$

For $\mathcal{G} = [U_1, \dots, U_M] \in \mathcal{CG}_x$ define a tree $\tau(\mathcal{G}) \in T_M$ such that $U_a \cap U_b \neq \emptyset$ if $(ab) \in \tau$. Then we have

$$\sum_{\substack{\mathcal{G} \in \mathcal{CG}_x: \\ \mathcal{G} = [U_1, \dots, U_M]}} (\dots) = \sum_{\substack{d_1, \dots, d_M \in \mathbb{N} - \{0\}: \\ \sum d_i = 2(M-1)}} \sum_{\tau \in T(d_1, \dots, d_M)} \sum_{\substack{\mathcal{G} \in \mathcal{CG}_x:\ \tau(\mathcal{G}) = \tau \\ \mathcal{G} = [U_1, \dots, U_M]}} (\dots). \tag{8-15}$$



Performing the summation over all connectivity graphs for fixed trees we obtain for $I \subseteq J \subseteq \underline{M}$, $|J| = N$, $|I| = k$, $\tau \in T(d_1, \dots, d_M)$

$$\sum_{\substack{\mathcal{G} \in \mathcal{CG}_x:\ \tau(\mathcal{G}) = \tau \\ \mathcal{G} = [U_1, \dots, U_M]}} \exp\{-\frac{1}{2}\gamma|supp\, \mathcal{G}|\} \Big[\prod_{a \in \underline{M} - J} \exp\{-\frac{K_2}{4} a_{j-1}^{-1} L_{tree}(U_a)\}\Big]$$
$$\Big[\prod_{b \in I} \sup_{\substack{x \in c_3 \mathcal{K}_j(Q_i) \\ \varphi:\, \Lambda_j \to \mathbb{R}}} |\frac{R_j^c(U_b|\mathcal{A}^j\varphi + \chi)}{B_j^{\tilde{c}}(U_b|\varphi)}|\Big] \Big[\prod_{c \in J - I} \delta_{1,|U_c|}\Big] \leq$$
$$(2\gamma^{-1})^{d_1} \prod_{i=2}^M [2\gamma^{-1}]^{d_i - 1} d_1 \prod_{i=1}^M (d_i - 1)!\, b^{M-N} (\|R_j^c\|_j^{\tilde{c}})^k$$

with

$$b = \sum_{U:\, x \in U} \exp\{-\frac{K_2}{4} a_{j-1}^{-1} L_{tree}(U)\}. \tag{8-16}$$

Using

$$\sum_{\substack{d_1, \dots, d_M \in \mathbb{N} - \{0\}: \\ \sum d_i = 2(M-1)}} 1 \leq 4^M$$

and Cayley's Theorem

$$\sum_{\tau \in T(d_1, \dots, d_M)} 1 \leq \frac{(M-2)!}{\prod_{i=1}^M (d_i - 1)!}$$

we obtain according to (8-15) and (8-16)

$$\sum_{\substack{\mathcal{G} \in \mathcal{CG}_x: \\ \mathcal{G} = [U_1, \dots, U_M]}} \exp\{-\frac{1}{2}\gamma|supp\, \mathcal{G}|\} \Big[\prod_{a \in \underline{M} - J} \exp\{-\frac{K_2}{4} a_{j-1}^{-1} L_{tree}(U_a)\}\Big]$$
$$\Big[\prod_{b \in I} \sup_{\substack{x \in c_3 \mathcal{K}_j(Q_i) \\ \varphi:\, \Lambda_j \to \mathbb{R}}} |\frac{R_j^c(U_b|\mathcal{A}^j\varphi + \chi)}{B_j^{\tilde{c}}(U_b|\varphi)}|\Big] \Big[\prod_{c \in J - I} \delta_{1,|U_c|}\Big] \leq$$
$$\leq (M-1)!(2\gamma^{-1})^{M-1} 4^M b^{M-N} (\|R_j^c\|_j^{\tilde{c}})^k. \tag{8-17}$$

(8-14) and (8-17) imply

$$\sum_{X:\, x \in X} \frac{|\delta_2' M_{j-1}^n(X|\Psi)|}{B_j^{\tilde{c}''}(X|\varphi')} \leq \sum_{N \geq 2} \sum_{M:\ M - N \geq [\frac{N-1}{4}] + 1} \sum_{k=1}^N \frac{\gamma}{2}(\tilde{K}\lambda_j^{1/2} b)^{M-N}$$
$$(24\gamma^{-1})^M (\|R_j^c\|_j^{\tilde{c}})^k \leq \sum_{N \geq 2} \sum_{M'=0}^\infty \frac{\gamma}{2}(\tilde{K}\lambda_j^{1/2} b\, 24\gamma^{-1})^{M'}$$
$$(\tilde{K}\lambda_j^{1/2} b)^{[\frac{N-1}{4}]+1} (24\gamma^{-1})^{N + [\frac{N-1}{4}] + 1} \frac{\|R_j^c\|_j^{\tilde{c}}}{1 - \|R_j^c\|_j^{\tilde{c}}}.$$

We have used

$$\sum_{\substack{J \subseteq \underline{M}: \\ |J| = N}} \sum_{\substack{I \subseteq J: \\ |I| = k}} 1 \leq 3^M.$$

For $\lambda_j$ small we see that (8-12) holds. (8-13) can be similarly shown. $\surd$

The last two lemmata of this section present improved bounds on activities for polymers which contain more than one element.

Define for a polymer $X$ of $\Lambda_{j-1}$

$$\widetilde{M}^n_{j-1}(X|\Psi) \equiv (1-\delta_{1,|X|})M^n_{j-1}(X|\Psi) \tag{8-18}$$

and

$$\widetilde{\widehat{M}}^n_{j-1}(X|\Psi) \equiv (1-\delta_{1,|X|})\widehat{M}^n_{j-1}(X|\Psi). \tag{8-19}$$

Recall that $\widehat{M}^n_{j-1}$ depends only on $Z^{rel}_j$.

$\widetilde{\widehat{M}}^n_{j-1}$ is estimated in the following lemma.

LEMMA 8.6. *For small $\lambda_j$ and large $L$ there exist $\tilde{c} = (C_1, C_2, \gamma, c_1, c_2, c_3, c, K_2)$ and and $K$ (not dependent on $\lambda_j$ and $C_1$) such that*

$$\|\widetilde{\widehat{M}}^n_{j-1}\|^{\tilde{c}}_j \le K C_1^{-1/2}$$

*for all $j \le 0$.*

For a polymer $X$ of $\Lambda_{j-1}$ with $|X| \ge 2$ we have for small $\lambda_j$

$$\widetilde{M}^n_{j-1}(X|\Psi) = \widetilde{\widehat{M}}^n_{j-1}(X|\Psi) = R^c_j(X|\Psi) + \delta_1 M^n_{j-1}(X|\Psi) + \delta'_2 M^n_{j-1}(X|\Psi) + \delta''_2 M^n_{j-1}(X|\Psi). \tag{8-20}$$

Define

$$\delta\widetilde{M}^n_{j-1}(X|\Psi) \equiv \widetilde{M}^n_{j-1}(X|\Psi) - R^{div}_{j-1}(X|\Psi). \tag{8-21}$$

Using lemma 8.4,8.5 and 8.6 we obtain

LEMMA 8.7. *For small $\lambda_j$ and large $L$ there exists $\tilde{c} = (C_1, C_2, \gamma, c_1, c_2, c_3, c, K_2)$ and $K$ (not dependent on $\lambda_j$ and $C_1$) such that*

$$\|\widetilde{M}^n_{j-1}\|^{\tilde{c}'}_{j-1} \le \|R^c_j\|^{\tilde{c}'}_{j-1} + K C_1^{-1}\|R^c_j\|^{\tilde{c}}_j + K C_1^{-1/2}$$

*and*

$$\|\delta\widetilde{M}^n_{j-1}\|^{\tilde{c}'}_{j-1} \le \|\delta R^c_j\|^{\tilde{c}'}_{j-1} + K C_1^{-1}\|R^c_j\|^{\tilde{c}}_j + K C_1^{-1/2}$$

*for all $j \le 0$ with*

$$\tilde{c}' = (C_1, C_2, \frac{1}{3}\gamma, \frac{L^2}{3}c_1, 3L^2 c_2 q'_2, \frac{L}{3}c_3, Lc, \frac{L}{2}K_2)$$

$$q'_2 \equiv \sup_{z\in base}\Big\{\sum_{y\in\Lambda_j}\exp\{-\frac{K_2}{4}a_j^{-1}dist(z,y)\}\Big\}.$$



## 4.9. Bounds on Vacuum Energy Counterterms

The bounds presented are related to the method introduced by C. Cammarota [7] (see also [39] and for a different approach [29]).

The not normalized activities $M^n_{j-1}$ for zero external field $\Psi$ determines the vacuum energy counterterms $e_{j-1}$. In this section the vacuum energy counterterms $e_{j-1}$ are bounded by suitable bounds on $M^n_{j-1}$.

Throughout this section we omit the argument $\Psi$ in the various notations for activities if the external field $\Psi$ is zero.

We have for the vacuum energy counterterms (cp. lemma 5.2)

$$\tilde{e}_{j-1}(X) = M^n_{j-1}(X|\Psi)|_{\Psi=0} + \sum_{\substack{C\in c(X):\,|C|\ge 2\\ supp\ C=X}} \frac{a(C)}{C!}\prod_{Q\in C} M^n_{j-1}(Q|\Psi)|_{\Psi=0}. \tag{9-1}$$

For a polymer $Q$ of $\Lambda_{j-1}$ and $\tilde{c} = (C_1, C_2, \gamma, c)$ define

$$B^{\tilde{c}}_i(Q) \equiv C_{i,C_1,C_2}(Q)\ T_{i,c}(Q)\ \exp\{-\gamma|Q|\}. \tag{9-2}$$

Suppose we have defined $A(Q|\Psi)$ for all polymers $Q$ in $\Lambda_{j-1}$ Define the norm

$$|||A|||^{\tilde{c}}_i \equiv \sup_{x\in\Lambda_{j-1}}\Big\{\sum_{Q:x\in Q}\frac{|A(Q)|}{B^{\tilde{c}}_i(Q)}\Big\}. \tag{9-3}$$

Define for the part of the not normalized activity $M^n_{j-1}$ which contains no $R^c_j$-factors

$$\widehat{M}^n_{j-1}(X) \equiv \widehat{M}^n_{j-1}(X|\Psi)|_{\Psi=0} \tag{9-4a}$$

and

$$\widehat{M}^{n'}_{j-1}(X) \equiv \begin{cases}\widehat{M}^n_{j-1}(X) & \text{for } n_X = \frac{1}{2}\\ 0 & \text{for } n_X > \frac{1}{2}\end{cases} \tag{9-4b}$$

$$\widehat{M}^{n''}_{j-1}(X) \equiv \widehat{M}^n_{j-1}(X) - \widehat{M}^{n'}_{j-1}(X). \tag{9-4c}$$

The norms $|||.|||$ for $\widehat{M}^{n'}_{j-1}$ and $\widehat{M}^{n''}_{j-1}$ are easily bounded using the same techniques as developped in section 4.8.

LEMMA 9.1. *For small $\lambda_j$ there exists $\tilde{c} = (C_1, C_2, \gamma, c)$ and $K$ such that*

$$|||\widehat{M}^{n'}_{j-1}|||^{\tilde{c}}_j \le K\lambda_j^{1/2}C_1^{-1/2}, \quad |||\widehat{M}^{n''}_{j-1}|||^{\tilde{c}}_j \le K C_1^{-1}$$

*for all $j \le 0$.*

By definition (8-1),(8-9),(8-11) we have for a polymer $X$ of $\Lambda_{j-1}$

$$\delta M^n_{j-1}(X) = \delta_1 M^n_{j-1}(X) + \delta'_2 M^n_{j-1}(X) + \delta''_2 M^n_{j-1}(X) \tag{9-5}$$

and

$$R_j^c(X) = 0.$$

For $n_X = \frac{1}{2}$ we have

$$M_{j-1}^{n'}(X) \equiv M_{j-1}^{n}(X) = \widehat{M}_{j-1}^{n'}(X) + \delta_1 M_{j-1}^{n}(X) + \delta_2' M_{j-1}^{n}(X) \tag{9-6}$$

and for $n_X > \frac{1}{2}$ we have

$$M_{j-1}^{n''}(X) \equiv M_{j-1}^{n}(X) = \widehat{M}_{j-1}^{n''}(X) + \delta_1 M_{j-1}^{n}(X) + \delta_2'' M_{j-1}^{n}(X). \tag{9-7}$$

Using lemma 8.4,8.5,8.6 we obtain

**Lemma 9.2.** *For small $\lambda_j$ there exists $\tilde{c} = (C_1, C_2, \gamma, c)$ and $K$ such that*

$$|||M_{j-1}^{n'}|||_j^{\tilde{c}'} \le K\lambda_j^{1/2}[C_1^{-1/2} + 2|||R_j^c|||_j^{\tilde{c}}]$$

$$|||M_{j-1}^{n''}|||_j^{\tilde{c}'} \le K[C_1^{-1} + (\lambda_j^{1/2} + C_1^{-1})|||R_j^c|||_j^{\tilde{c}}]$$

*for all $j \le 0$ and $\tilde{c}' = (C_1, C_2, \frac{1}{2}\gamma, c)$.*

For a polymer $X$ of $\Lambda_{j-1}$ define

$$\tilde{e}'_{j-1}(X) \equiv \begin{cases} \tilde{e}_{j-1}(X) & \text{for } n_X = \frac{1}{2} \\ 0 & \text{for } n_X > \frac{1}{2} \end{cases} \tag{9-8}$$

$$\tilde{e}''_{j-1}(X) \equiv \tilde{e}_{j-1}(X) - \tilde{e}'_{j-1}(X). \tag{9-9}$$

**Lemma 9.3.** *For small $\lambda_j$ there exists $\tilde{c} = (C_1, C_2, \gamma, c)$ such that*

$$|||\tilde{e}'_{j-1}|||_j^{\tilde{c}'} \le 2|||M_{j-1}^{n'}|||_j^{\tilde{c}} \tag{9-10}$$

$$\max(|||\tilde{e}_{j-1}|||_j^{\tilde{c}'}, |||\tilde{e}''_{j-1}|||_j^{\tilde{c}'}) \le 2|||M_{j-1}^{n}|||_j^{\tilde{c}} \tag{9-11}$$

*for all $j \le 0$ and $\tilde{c}' = (C_1, C_2, \frac{1}{2}\gamma, c)$.*

*Proof :* We have

$$\tilde{e}'_{j-1}(X) = M_{j-1}^{n'}(X) + \sum_{\substack{C \in \mathcal{C}(X):\, |C| \ge 2 \\ \text{supp } C = X}} \frac{a(C)}{C!} \prod_{Q \in C} M_{j-1}^{n'}(Q).$$

Thus, using $|a(C)| \le |\tau(C)|$, (for a proof see [39], pp.35-37)

$$\sum_{X:\, x \in X} \frac{|\tilde{e}'_{j-1}(X)|}{B_j^{\tilde{c}'}(X)} \le \sum_{X:\, x \in X} \Bigg\{ \frac{|M_{j-1}^{n'}(X)|}{B_j^{\tilde{c}}(X)} +$$

$$+ \sum_{N \ge 2} \sum_{\tau \in T_N} \sum_{\substack{C \in \mathcal{C}(X):\, |C| = N \\ \text{supp } C = X,\ \gamma(C) \supseteq \tau}} \frac{1}{C!} \frac{a(C)}{\tau(C)} \prod_{Q \in C} \frac{|M_{j-1}^{n'}(Q)|}{B_j^{\tilde{c}'}(Q)} \Bigg\} \le$$

$$\le |||M_{j-1}^{n'}|||_j^{\tilde{c}'} + \sum_{N \ge 2} \sum_{\tau \in T_N} \frac{1}{(N-1)!} \sum_{\substack{Q_1, \dots, Q_N:\, x \in Q_1 \\ \tau(Q_1, \dots, Q_N) = \tau}} \left[ \prod_{i=1}^{N} \exp\{-\frac{\gamma}{2}|Q_i|\} \frac{|M_{j-1}^{n'}(Q_i)|}{B_j^{\tilde{c}}(Q_i)} \right]$$



where $\tau(Q_1, \dots, Q_N)$ is a tree graph with vertex set $\{1, \dots, N\}$ and lines $(ab) \in \tau$ if $Q_a \cap Q_b \ne \emptyset$. By Cayley's Theorem we have

$$|||\tilde{e}'_{j-1}|||_j^{\tilde{c}'} \le |||M_{j-1}^{n'}|||_j^{\tilde{c}'} + \sum_{N \ge 2} \frac{1}{(N-1)!} \sum_{\substack{d_1, \dots, d_N \in \mathbb{N} - \{0\}: \\ \sum d_i = 2(N-1)}}$$

$$\sum_{\tau \in T(d_1, \dots, d_N)} d_1 \prod_{i=2}^{N} (d_i - 1)! (12\gamma^{-1})^{N-1} (|||M_{j-1}^{n'}|||_j^{\tilde{c}})^N \le$$

$$\le |||M_{j-1}^{n'}|||_j^{\tilde{c}} + \sum_{N \ge 2} \frac{\gamma}{2} (8\gamma^{-1} |||M_{j-1}^{n'}|||_j^{\tilde{c}})^N.$$

Thus for $8\gamma^{-1}|||M_{j-1}^{n'}|||_j^{\tilde{c}} < 1$

$$|||\tilde{e}'_{j-1}|||_j^{\tilde{c}'} \le |||M_{j-1}^{n'}|||_j^{\tilde{c}} + \frac{32}{\gamma}(|||M_{j-1}^{n'}|||_j^{\tilde{c}})^2 \frac{1}{1 - 8\gamma^{-1}|||M_{j-1}^{n'}|||_j^{\tilde{c}}}.$$

This proves (9-10).(9-11) can be similarly shown. ✓

**Lemma 9.4.** *For small $\lambda$ there exists $K$, $\gamma$ and $\tilde{c} = (C_1, C_2, 0, c_1, c_2, c_3, c, K_2)$ and $\epsilon$, with $C_1$ large , $C_2$ and $\epsilon$ small, such that if*

$$\|R_j^{ren}\|_j^{\tilde{c}} \le \epsilon$$

*then we have*

$$|||\tilde{e}'_{j-1}|||_j^{\tilde{c}'} \le K\lambda_j^{1/2}$$

$$\max(|||\tilde{e}_{j-1}|||_j^{\tilde{c}'}, |||\tilde{e}''_{j-1}|||_j^{\tilde{c}'}) \le KC_1^{-1}$$

*for all $j \le 0$ and $\tilde{c}' = (C_1, C_2, \gamma, Lc)$.*

*Proof :* Use lemma 6.10 , lemma 9.2 and 9.3 and

$$|||R_j^c|||_{j-1}^{\tilde{c}} = |||\delta R_j^c|||_{j-1}^{\tilde{c}}. \quad ✓$$

### 4.10. Bounds on Running Coupling Constants : Induction Step

In this section we will prove bounds on $m_{j-1}^2, \beta_{j-1}^{\mu\nu}, \lambda_{j-1}$ supposing bounds on $m_j^2, \beta_j^{\mu\nu}, \lambda_j$ and $R_j^{ren}$. Consider the Taylor expansion in the free propagator $v^j$ up to order $N$. Use that

$$\ln \int d\mu_{v^j}(\Phi) Z_j(\Phi + \Psi) = \sum_{i=0}^{N-1} C_i^j(\Psi) + Q_N^j(\Psi) \tag{10-1}$$

with

$$C_0^j(\Psi) = \ln Z_j(\Psi)$$

$$C_1^j(\Psi) = \frac{1}{2}\int_{z_1,z_2} \mathrm{v}^j(z_1,z_2)\frac{\delta^2}{\delta\Psi(z_1)\delta\Psi(z_2)}\ln Z_j(\Psi) + \frac{1}{2}\int_{z_1,z_2} \mathrm{v}^j(z_1,z_2)\Big[\frac{\delta}{\delta\Psi(z_1)}\ln Z_j(\Psi)\Big]\Big[\frac{\delta}{\delta\Psi(z_2)}\ln Z_j(\Psi)\Big]$$

$$C_2^j(\Psi) = \frac{1}{8}\int_{z_1,z_2,z_3,z_4} \mathrm{v}^j(z_1,z_2)\mathrm{v}^j(z_3,z_4)\frac{\delta^4}{\delta\Psi(z_1)\delta\Psi(z_2)\delta\Psi(z_3)\delta\Psi(z_4)}\ln Z_j(\Psi) + \frac{1}{2}\int_{z_1,z_2,z_3,z_4} \mathrm{v}^j(z_1,z_2)\mathrm{v}^j(z_3,z_4)\Big[\frac{\delta^3}{\delta\Psi(z_1)\delta\Psi(z_2)\delta\Psi(z_3)}\ln Z_j(\Psi)\Big]\Big[\frac{\delta}{\delta\Psi(z_4)}\ln Z_j(\Psi)\Big] + \frac{1}{4}\int_{z_1,z_2,z_3,z_4} \mathrm{v}^j(z_1,z_2)\mathrm{v}^j(z_3,z_4)\Big[\frac{\delta^2}{\delta\Psi(z_1)\delta\Psi(z_3)}\ln Z_j(\Psi)\Big]\Big[\frac{\delta^2}{\delta\Psi(z_2)\delta\Psi(z_4)}\ln Z_j(\Psi)\Big]$$

and remainder term

$$Q_N^j(\Psi) = \frac{1}{(N-1)!}\int_0^1 ds \quad (1-s)^{N-1}\partial_s^N \ln\int d\mu_{s\mathrm{v}^j}(\Phi)Z_j(\Phi+\Psi).$$

By (10-1), $x \in \Lambda_{j-1}$, the renormalization group equations for $m_{j-1}^2, \beta_{j-1}^{\mu\nu}, \lambda_{j-1}$ read

$$m_{j-1}^2 = -\mathcal{D}_x^2 \ln\int d\mu_{\mathrm{v}^j}(\Phi)Z_j(\Phi+\Psi)|_{\Psi=0} = m_j^2 - \frac{1}{2}\int_{z_1\in x}\int_{z_2}\int_{z_1',z_2'} \mathrm{v}^j(z_1',z_2') \frac{\delta^4}{\delta\Psi(z_1)\delta\Psi(z_2)\delta\Psi(z_1')\delta\Psi(z_2')}\ln Z_j(\Psi)|_{\Psi=0} - \int_{z_1\in x}\int_{z_2}\int_{z_1',z_2'} \mathrm{v}^j(z_1',z_2') \Big[\frac{\delta^2}{\delta\Psi(z_1')\delta\Psi(z_1)}\ln Z_j(\Psi)\Big]\Big[\frac{\delta^2}{\delta\Psi(z_2')\delta\Psi(z_2)}\ln Z_j(\Psi)\Big]|_{\Psi=0} - \mathcal{D}_x^2 Q_2^j(\Psi)|_{\Psi=0} \tag{10–2}$$

and

$$\beta_{j-1}^{\mu\nu} = \beta_j^{\mu\nu} - \frac{1}{2}\mathcal{D}_x^{\mu\nu}\int_{z_1,z_2} \mathrm{v}^j(z_1,z_2)\frac{\delta^2}{\delta\Psi(z_1)\delta\Psi(z_2)}\ln Z_j(\Psi) - \frac{1}{2}\mathcal{D}_x^{\mu\nu}\int_{z_1,z_2} \mathrm{v}^j(z_1,z_2) \Big[\frac{\delta}{\delta\Psi(z_1)}\ln Z_j(\Psi)\Big]\Big[\frac{\delta}{\delta\Psi(z_2)}\ln Z_j(\Psi)\Big]|_{\Psi=0} - \mathcal{D}_x^{\mu\nu}Q_2^j(\Psi)|_{\Psi=0} \tag{10–3}$$



and

$$\lambda_{j-1} = \lambda_j - \frac{1}{2}r_j^j - \frac{1}{Vol(x)}\int_{z_1\in x}\int_{z_2,z_3,z_4}\int_{z_1',z_2'} \mathrm{v}^j(z_1',z_2')\Big[\frac{\delta^2}{\delta\Psi(z_1)\delta\Psi(z_1')}\ln Z_j(\Psi) \frac{\delta^4}{\delta\Psi(z_2)\delta\Psi(z_2')\delta\Psi(z_3)\delta\Psi(z_4)}\ln Z_j(\Psi)|_{\Psi=0} - 3(z_1\leftrightarrow z_2)\Big] - \frac{1}{8Vol(x)} \int_{z_1\in x}\int_{z_2,z_3,z_4}\int_{z_1',\dots,z_4'} \mathrm{v}^j(z_1',z_2')\mathrm{v}^j(z_3',z_4')\frac{\delta^4}{\delta\Psi(z_1)\dots\delta\Psi(z_4)}\frac{\delta^4}{\delta\Psi(z_1')\dots\delta\Psi(z_4')}\ln Z_j(\Psi)|_{\Psi=0} - \frac{3}{2Vol(x)}\int_{z_1\in x}\int_{z_2,z_3,z_4}\int_{z_1',\dots,z_4'} \mathrm{v}^j(z_1',z_2')\mathrm{v}^j(z_3',z_4')\frac{\delta^4}{\delta\Psi(z_1)\delta\Psi(z_2)\delta\Psi(z_1')\delta\Psi(z_3')}\ln Z_j(\Psi)|_{\Psi=0} \frac{\delta^4}{\delta\Psi(z_3)\delta\Psi(z_4)\delta\Psi(z_2')\delta\Psi(z_4')}\ln Z_j(\Psi)|_{\Psi=0} - \frac{1}{2Vol(x)}\int_{z_1\in x}\int_{z_2,z_3,z_4}\int_{z_1',\dots,z_4'} \mathrm{v}^j(z_1',z_2') \mathrm{v}^j(z_3',z_4')\frac{\delta^4}{\delta\Psi(z_1)\dots\delta\Psi(z_4)}\frac{\delta^2}{\delta\Psi(z_1')\delta\Psi(z_3')}\ln Z_j(\Psi)|_{\Psi=0}\frac{\delta^2}{\delta\Psi(z_1')\delta\Psi(z_2')}\ln Z_j(\Psi)|_{\Psi=0} - \, + \cdots - \mathcal{D}_x^4 Q_3^j(\Psi)|_{\Psi=0} \tag{10–4}$$

with

$$r_j^l \equiv \frac{1}{Vol(x)}\int_{z_1\in x}\int_{z_2,z_3,z_4}\int_{z_1',z_2'} \mathrm{v}^l(z_1',z_2')\frac{\delta^4}{\delta\Psi(z_1)\dots\delta\Psi(z_4)}\frac{\delta^2}{\delta\Psi(z_1')\delta\Psi(z_2')}\ln Z_j(\Psi)|_{\Psi=0} \tag{10–5}$$

for $x \in \Lambda_{l-1}, l \le j \le 0$. Define

$$g_{j,\mu\nu}^l \equiv \mathcal{D}_x^{\mu\nu}\int_{z_1,z_2} \mathrm{v}^l(z_1,z_2)\frac{\delta^2}{\delta\Psi(z_1)\delta\Psi(z_2)}\ln Z_j(\Psi)|_{\Psi=0}. \tag{10–6}$$

Lemma 10.1. *Suppose that $\lambda$ is small and $L$ large and $\epsilon,\epsilon'$ are small positive constants such that there exist $\tilde{c} = (C_1, C_2, \gamma = 0, c_1, c_2, c_3, c, K_2)$, positive constants $c_-, c_+, c_\beta, \tilde{c}$ and intervals $[\alpha_0,\beta_0]\supset_{\neq}[\alpha_1,\beta_1]\supset_{\neq}\dots\supset_{\neq}[\alpha_j,\beta_j]\supset_{\neq}\dots$ with*

$$\|R_j^{ren}\|_j^{\tilde{c}} \le \epsilon$$

and

$$m_j^2([\alpha_j,\beta_j]) + s_j\lambda_j = a_j^{-2}\lambda_j^{3/2}[-\tilde{c},\tilde{c}] \tag{$A_j$}$$

$$|\beta_j^{\mu\nu}| \le c_\beta\lambda_j^{1-\epsilon} \tag{$B_j$}$$

$$\lambda_j \in \Big(\frac{c_-}{|j-1|}, \frac{c_+}{|j-1|}\Big) \tag{$C_j$}$$

$$|g_{j,\mu\nu}^l| \le c_g L^{2(l-j)}\lambda_j^2 \tag{$D_j$}$$

$$|r_j^l - 6\sum_{k=j+1}^{0}\frac{1}{Vol(x)}\int_{z_1\in x}\int_{z_2} \mathrm{v}^l(z_1,z_2)\mathrm{v}^k(z_1,z_2)\lambda_j^2| \le c_r L^{2(l-j)}\lambda_j^{5/2} \tag{$E_j$}$$

*for all $j \le 0$ and $\mu,\nu \in \{1,\dots,4\}$ with*

$$s_j \equiv \sum_{k=-\infty}^{j} \delta s_k \tag{3-16a}$$

$$\delta s_k \equiv \frac{1}{Vol(x)} \int_{z \in x} \mathrm{v}^k(z,z), \quad x \in \Lambda_{k-1}. \tag{3-16b}$$

*Then we have $(A_{j-1})$, $(B_{j-1})$, $(C_{j-1})$, $(D_{j-1})$, $(E_{j-1})$ and*

$$m_{j-1}^2 - m_j^2 = \delta s_j \lambda_j + Q_m \tag{3-17a}$$

$$\beta_{j-1}^{\mu\nu} - \beta_j^{\mu\nu} = Q_\beta \tag{3-17b}$$

$$\lambda_{j-1} - \lambda_j = -\gamma_j \lambda_j^2 + Q_\lambda \tag{3-17c}$$

*with*

$$\gamma_j = \frac{3}{2} \frac{1}{Vol(x)} \int_{z_1 \in x} \int_{z_2} \Big[\sum_{k=j}^{0} \mathrm{v}^k(z_1,z_2)\Big]^2, \quad x \in \Lambda_{j-1} \tag{3-17d}$$

*and there exists a constant $K$ such that*

$$|Q_m| \le K a_j^{-2} \lambda_j^{3/2}, \quad |Q_\beta| \le K\lambda_j^2, \quad |Q_\lambda| \le K\lambda_j^{5/2}. \tag{3-18}$$

*Proof (by induction) :* In the following use the decomposition

$$\ln Z_{j-1}(\Psi) = \ln \widehat{Z}_{j-1}^{rel}(\Psi) + U_{j-1}(\Psi) \tag{10-7}$$

where

$$\widehat{Z}_{j-1}^{rel}(\Psi) = \int d\mu_{vi}(\Phi) Z_j^{rel}(\Phi + \Psi). \tag{10-8}$$

We have the following representation of $\ln Z_{j-1}$ (for a proof see appendix A)

$$\ln Z_{j-1}(\Psi) = \sum_{X:\ X \subseteq \Lambda_{j-1}} \ \sum_{C \in \mathcal{C}(X):\ supp\ C = X} \frac{a(C)}{C!} \prod_{Q \in C} M_{j-1}^n(Q|\Psi)|_{\Psi=0}. \tag{10-9}$$

Using the decomposition

$$M_{j-1}^n(X|\Psi) \equiv \widehat{M}_{j-1}^n(X|\Psi) + \delta M_{j-1}^n(X|\Psi)$$

we obtain

$$U_{j-1}(\Psi) = \sum_{X:\ X \subseteq \Lambda_{j-1}} \ \sum_{C \in \mathcal{C}(X):\ supp\ C = X} \frac{a(C)}{C!} \Big[\prod_{Q \in C} M_{j-1}^n(Q|\Psi) - \prod_{Q \in C} \widehat{M}_{j-1}^n(Q|\Psi)\Big]|_{\Psi=0}. \tag{10-10}$$

Using this equation we can bound derivatives of $U_{j-1}(\Psi)$ by bounds on $M_{j-1}^n$ and $\widehat{M}_{j-1}^n$ (cp. section 4.8). The technique for obtaining bounds on derivatives of $U_{j-1}(\Psi)$ is the same as in sections 4.8 an 4.9 and is not repeated here.



We have for $x \in \Lambda_{j-1}$

$$r_{j-1}^l = r_j^l + \frac{6}{Vol(x)} \int_{z_1 \in x} \int_{z_2} \mathrm{v}^l(z_1,z_2) \mathrm{v}^k(z_1,z_2) \lambda_j^2 + Q_{r,x}^l \tag{10-11}$$

with

$$|Q_{r,x}^l| \le K L^{2(l-j)} \lambda_j^{5/2}. \tag{10-12}$$

By (10-4) and $(E_j)$ we obtain (3-17c).$(E_j)$ and (10-11) imply for $x' \in \Lambda_{j-1}$

$$r_{j-1}^l - 6 \sum_{k=j}^{0} \frac{1}{Vol(x')} \int_{z_1 \in x'} \int_{z_2} \mathrm{v}^l(z_1,z_2) \mathrm{v}^k(z_1,z_2) \lambda_{j-1}^2 =$$

$$= \sum_{x \in x'} L^{-4} \Big\{ r_j^l - 6 \sum_{k=j+1}^{0} \frac{1}{Vol(x)} \int_{z_1 \in x} \int_{z_2} \mathrm{v}^l(z_1,z_2) \mathrm{v}^k(z_1,z_2) \lambda_j^2 -$$

$$- 6 \sum_{k=j}^{0} \frac{1}{Vol(x)} \int_{z_1 \in x} \int_{z_2} \mathrm{v}^l(z_1,z_2) \mathrm{v}^k(z_1,z_2) (\lambda_{j-1}^2 - \lambda_j^2) + Q_{r,x}^l \Big\}.$$

Thus, using (3-17c), we obtain

$$|r_{j-1}^l - 6 \sum_{k=j}^{0} \frac{1}{Vol(x')} \int_{z_1 \in x'} \int_{z_2} \mathrm{v}^l(z_1,z_2) \mathrm{v}^k(z_1,z_2) \lambda_{j-1}^2| \le$$

$$\le c_r L^{2(l-j)} \lambda_j^{5/2} + K' L^{2(l-j)} \lambda_j^3 + K L^{2(l-j)} \lambda_j^{5/2} \le$$

$$\le c_r L^{2(l-(j-1))} \lambda_{j-1}^{5/2}$$

for $c_r$ large and $\lambda_j$ small enough. This proves $(E_{j-1})$. For $l \le j-1, \mu,\nu \in \{1,\dots,d\}$ we have

$$g_{j-1,\mu\nu}^l = g_{j,\mu\nu}^l + Q_g$$

with

$$|Q_g| \le K L^{2(l-j)} \lambda_j^2.$$

Thus

$$|g_{j-1,\mu\nu}^l| \le c_g L^{2(l-j)} \lambda_j^2 + K L^{2(l-j)} \lambda_j^2$$

and, using (3-17c), we see

$$|g_{j-1,\mu\nu}^l| \le c_g L^{2(l-(j-1))} \lambda_{j-1}^2 \tag{$D_{j-1}$}$$

for $c_g$ large enough.(3-17c) implies

$$\lambda_{j-1}^{-1} = \lambda_j^{-1}(1 + \gamma_j \lambda_j + O(\lambda_j^{3/2})) = \lambda_j^{-1} + \gamma_j + O(\lambda_j^{1/2}).$$

Thus

$$\lambda_{j-1}^{-1} \le c_-^{-1}|j-1| + \gamma_j + O(\lambda_j^{1/2}) \le c_-^{-1}|j-2|$$

$$\lambda_{j-1}^{-1} \ge c_+^{-1}|j-1| + \gamma_j + O(\lambda_j^{1/2}) \ge c_+^{-1}|j-2|$$

for $\gamma_j + O(\lambda_j^{1/2}) \in (c_+^{-1}, c_-^{-1})$. This proves

$$\lambda_{j-1} \in \left(\frac{c_-}{|j-2|}, \frac{c_+}{|j-2|}\right). \qquad (C_{j-1})$$

$(D_j)$ and (10-3) imply (3-17b). From (3-17b,c) and $(B_j, C_j)$ follows

$$\left|\frac{\beta_{j-1}}{\lambda_{j-1}}\right| \le \left(\left|\frac{\beta_j}{\lambda_j}\right| + K\lambda_j\right)(1 + O(\lambda_j)) \le (c_\beta \lambda_j^{-\epsilon} + K\lambda_j)(1 + O(\lambda_j)) \le c_\beta \lambda_{j-1}^{-\epsilon} \qquad (10\text{-}13)$$

for $c_\beta$ large and $\lambda_j$ small enough. This proves $(B_{j-1})$. Using (10-2) we obtain (3-17a) and

$$m_{j-1}^2 + s_{j-1}\lambda_{j-1} = m_j^2 + s_j\lambda_j + s_{j-1}(\lambda_{j-1} - \lambda_j) + Q_m.$$

Thus, using (3-17a), we obtain for large $\widetilde{c}$

$$\begin{aligned} m_{j-1}^2([\alpha_j, \beta_j]) + s_{j-1}\lambda_{j-1} &\supseteq \\ \supseteq [-a_j^{-2}\lambda_j^{3/2}\widetilde{c} + a_j^{-2}\lambda_j^{3/2}K &+ K' a_j^{-2}\lambda_j^2, a_j^{-2}\lambda_j^{3/2}\widetilde{c} - a_j^{-2}\lambda_j^{3/2}K - K' a_j^{-2}\lambda_j^2] \supseteq \\ &\supseteq L^2 a_{j-1}^{-2}\lambda_j^{3/2}[-\widetilde{c} + K + K'\lambda_j^{1/2}, \widetilde{c} - K - K'\lambda_j^{1/2}] \underset{\neq}{\supset} a_{j-1}^{-2}\lambda_{j-1}^{3/2}[-\widetilde{c}, \widetilde{c}]. \end{aligned}$$

Thus there exists a closed interval $[\alpha_{j-1}, \beta_{j-1}]$ with

$$m_{j-1}^2([\alpha_{j-1}, \beta_{j-1}]) + s_{j-1}\lambda_{j-1} = a_{j-1}^{-2}\lambda_{j-1}^{3/2}[-\widetilde{c}, \widetilde{c}]. \qquad (A_{j-1})$$

This completes the proof. ✓



### 4.11. Integration Step (II) : Bounds on $R_{j-1}$

This section closes the bounds for the integration step. In proposition 11.5 the bound (3-22) of lemma 3.6 will be proven.

For monomers $X$ of $\Lambda_{j-1}$ eqs. (1-25a) and (1-21) imply

$$\begin{aligned} R_{j-1}(X|\Psi) = A_{j-1}(X|\Psi) - Z_{j-1}^{rel}(X|\Psi) &= \\ = \int d\mu_{v_X^j}(\Phi)[Z_j^{rel}(\overline{X}|\Phi + \Psi) &- Z_{j-1}^{rel}(X|\Psi)] + \\ + \int d\mu_{v_X^j}(\Phi) Z_j^{rel}(\overline{X}|\Phi + \Psi)&[\exp\{-e_{j-1}(X)\} - 1] + \\ &+ \int d\mu_{v_X^j}(\Phi) R_j^c(X|\Phi + \Psi)\exp\{-e_{j-1}(X)\} \end{aligned} \qquad (11\text{-}1)$$

and for polymers $X$ of $\Lambda_{j-1}$ with $|X| > 1$, eqs. (1-21),(1-25a),(5-3),(5-5) and the general tree formula (2-5)

$$R_{j-1}(X|\Psi) = A_{j-1}(X|\Psi) = M_{j-1}^n(X|\Psi)\exp\{-e_{j-1}(X)\} + \delta' R_{j-1}(X|\Psi) \qquad (11\text{-}2)$$

with

$$\begin{aligned} \delta' R_{j-1}(X|\Psi) = \sum_{N\ge 2}\frac{1}{N!} \sum_{\substack{Q_1,\dots,Q_N \subseteq X:\\ \sum Q_i = X}} \sum_{\tau \in T_N} \sum_{\substack{\pi,\eta:\\ \tau = \tau(\pi,\eta)}} \int_0^1 ds\ f(\eta|s) \\ \left\{\prod_{(ab)\in\tau}\frac{\partial}{\partial t_{ab}}\right\} \prod_{i=1}^N A_{j-1}^n(Q_i|\Psi)\exp\{-e_{j-1,t}(X)\}|_{t=t(\pi,s)}. \end{aligned} \qquad (11\text{-}3)$$

We have

$$A_{j-1}^n(X|\Psi) = \begin{cases} \int d\mu_{v_X^j}(\Phi) Z_j^{rel}(\overline{X}|\Phi + \Psi) + \int d\mu_{v_X^j}(\Phi) R_j^c(X|\Phi + \Psi), & \text{for } |X| = 1 \\ M_{j-1}^n(X|\Psi), & \text{for } |X| > 1. \end{cases} \qquad (11\text{-}4)$$

Define for a polymer $X$ of $\Lambda_{j-1}$

$$\widetilde{M}_{j-1}^n(X|\Psi) \equiv (1 - \delta_{1,|X|}) M_{j-1}^n(X|\Psi). \qquad (11\text{-}5)$$

Lemma 11.1. *For positive constants $K_0$ and $\widetilde{c} = (C_1, C_2, \gamma, c_1, c_2, c_3, c, K_2)$ and*

$$D \equiv \exp\{-\frac{\gamma}{4}\}\left[\exp\{\frac{17}{4!}c_3^4 + K_0 c_1^2\} + (C_1\lambda_j)^{1/2} C_2 [\|R_j^c\|_{j-1}^{\widetilde{c}} + \|\delta_1 M_{j-1}^n\|_{j-1}^{\widetilde{c}}]\right] \qquad (11\text{-}6)$$

*such that*

$$D \le \frac{\gamma}{96}, \quad \|\widetilde{M}_{j-1}^n\|_{j-1}^{\widetilde{c}} \le \frac{1}{2}D, \quad |||\widetilde{e}_{j-1}|||_{j-1}^{\widetilde{c}} \le \frac{\gamma}{96}$$

we have for small $\lambda_j$

$$\|\delta' R_{j-1}\|_{j-1}^{\tilde{c}'} \le 24 \||\tilde{e}_{j-1}\||_{j-1}^{\tilde{c}}$$

with

$$\tilde{c}' = (C_1, C_2, \frac{1}{2}\gamma, \frac{1}{2}c_1, q_2' c_2, c_3, c, K_2/2)$$

$$q_2' \equiv \sup_{z \in base} \{ \sum_{x \in \Lambda_{j-1}} \exp\{-\frac{K_2}{4} a_{j-1}^{-1} dist(z,x)\} \}.$$

*Proof :* For polymers $P, Q_1, \ldots, Q_N, X$ of $\Lambda_{j-1}$ with $\sum_{i=1}^N Q_i = X$ define

$$C(P) \equiv \{a \in \{1, \ldots, N\} | \; Q_a \cap P \neq \emptyset\}.$$

We have

$$\delta' R_{j-1}(X|\Psi) = \sum_{N \ge 2} \frac{1}{N!} \sum_{\substack{Q_1,\ldots,Q_N \subseteq X: \\ \sum Q_i = X}} \sum_{m \ge 1} \sum_{P_1,\ldots,P_m \subseteq X} \sum_{\tau \in T_N} \sum_{\substack{\pi,\eta: \\ \tau = \tau(\pi,\eta)}} \int_0^1 ds \; f(\eta|s)$$
$$\sum_{\substack{(\sigma_1,\ldots,\sigma_m): \\ \sum_{i=1}^m \sigma_i = \tau}} \prod_{i=1}^m [(\prod_{(ab) \in \sigma_i} \frac{\partial}{\partial t_{ab}})(-\tilde{e}_{j-1,t}(P_i))]$$
$$\prod_{i \in N} A_{j-1}^n(Q_i|\Psi) \; \exp\{-e_{j-1,t}(X)\}|_{t=t(\pi,s)}. \tag{11-7}$$

Using for $\varphi : \Lambda_{j-1} \to \mathbf{R}$ (see lemma 8.2)

$$\prod_{i=1}^N S_{j-1,c_1,c_2,K_2}(Q_i|\varphi) \le S_{j-1,c_1-q_2'c_2,q_2'c_2,K_2/2}(X|\varphi) \tag{11-8}$$

and $|e_{j-1,t}(X)| \le K\lambda_j^{1/2}|X|$ with $K$ not dependent on $\lambda_j$ we obtain for $\Psi = \mathcal{A}^{j-1}\varphi + \chi$, $\chi \in c_3\mathcal{K}_{j-1}(X)$, and $K\lambda_j^{1/2} \le \gamma/4$, $\frac{1}{2}c_1 \ge q_2'c_2$

$$\frac{|\delta' R_{j-1}(X|\Psi)|}{B_{j-1}^{\tilde{c}'}(X|\varphi)} \le \sum_{N \ge 2} \frac{1}{N!} \sum_{I \subseteq \underline{N}} \sum_{m \ge 1}$$
$$\sum_{\substack{Q_1,\ldots,Q_N \subseteq X: \\ \sum Q_i = X}} \sum_{\tau \in T_N} \sum_{\substack{\pi,\eta: \\ \tau = \tau(\pi,\eta)}} \int_0^1 ds \; f(\eta|s)$$
$$\exp\{-\frac{\gamma}{4}|X|\} \sum_{\substack{(\sigma_1,\ldots,\sigma_m): \\ \sum_{i=1}^m \sigma_i = \tau}} \sum_{\substack{P_1,\ldots,P_m \subseteq X \\ V(\sigma_i) \supseteq C(P_i)}} [\prod_{i=1}^m \frac{|\tilde{e}_{j-1}(P_i)|}{B_{j-1}^{\tilde{c}}(P_i)}]$$
$$[\prod_{i \in \underline{N}-I} \delta_{1,|Q_i|} \frac{|\int d\mu_{v_{Q_i}^j}(\Phi)[Z_j^{rel}(Q_i|\Phi+\Psi) + R_j^c(Q_i|\Phi+\Psi)]|}{S_{j-1,c_1,c_2,K_2}(Q_i|\varphi)}]$$
$$[\prod_{i \in I}(1-\delta_{1,|Q_i|}) \frac{|M_{j-1}^n(Q_i|\Psi)|}{B_{j-1}^{\tilde{c}}(Q_i|\varphi)}]. \tag{11-9}$$



Using the stability bound (see lemma 6.6) for $\varphi' = \mathcal{A}_{j,j-1}\varphi$, $x \in \Lambda_{j-1}$,

$$|Z_j^{rel}(\overline{x}|\Psi)| \le S_{j,c'}^{in}(\overline{x}|\varphi) S_{j,K_\beta c_\beta \lambda_j^{1/2},K_2}^{out}(\overline{x}|\varphi) S_{j,\frac{17}{4!}}^{im}(\overline{x}|\chi) \exp\{K_0 c'^2 |\overline{x}|\} \le$$
$$\le S_{j-1,L^2(\frac{\lambda_j}{\lambda_{j-1}})^{1/2}c'}^{in}(x|\varphi) S_{j-1,L^2\frac{\lambda_j}{\lambda_{j-1}})^{1/2}K_\beta c_\beta \lambda_j^{1/2},K_2}^{out}(x|\varphi)$$
$$\exp\{\frac{17}{4!}c_3^4\} \exp\{K_0 c'^2 |\overline{x}|\} \tag{11-10}$$

we obtain for $c' = L^{-2}(\frac{\lambda_j}{\lambda_{j-1}})^{1/2} K_\beta c_1$, $c_2 \ge L^2(\frac{\lambda_j}{\lambda_{j-1}})^{1/2} K_\beta c_\beta \lambda_j^{1/2}$

$$|Z_j^{rel}(\overline{x}|\Psi)| \le S_{j-1,c_1,c_2,K_2}(x|\varphi) \exp\{\frac{17}{4!}c_3^4 + K_0 c_1^2\} \tag{11-11a}$$

and for $c' = 2L^{-2}(\frac{\lambda_j}{\lambda_{j-1}})^{1/2} K_\beta c_1$, $c_2 \ge 2L^2(\frac{\lambda_j}{\lambda_{j-1}})^{1/2} K_\beta c_\beta \lambda_j^{1/2}$

$$|Z_j^{rel}(\overline{x}|\Psi)| \le S_{j-1,2c_1,\frac{1}{2}c_2,K_2}(x|\varphi) \exp\{\frac{17}{4!}c_3^4 + 4K_0 c_1^2\}. \tag{11-11b}$$

Thus by (11-11b)

$$|\frac{\partial}{\partial s} \int d\mu_{sv_x^j}(\Phi) Z_j^{rel}(\overline{x}|\Phi+\Psi)| = |\frac{1}{2}\triangle_{v_x^j} \int d\mu_{sv_x^j}(\Phi) Z_j^{rel}(\overline{x}|\Phi+\Psi)| \le$$
$$\le K\lambda_j^{1/2} S_{j-1,c_1,c_2,K_2}(x|\varphi). \tag{11-12}$$

Furthermore by eq. (11-11) and (11-12)

$$|\int d\mu_{v_x^j}(\Phi) Z_j^{rel}(\overline{x}|\Phi+\Psi)| = |Z_j^{rel}(\overline{x}|\Psi) + \int_0^1 ds \frac{\partial}{\partial s} \int d\mu_{sv_x^j}(\Phi) Z_j^{rel}(\overline{x}|\Phi+\Psi)| \le$$
$$\le \left[\exp\{\frac{17}{4!}c_3^4 + K_0 c_1^2\} + K\lambda_j^{1/2}\right] S_{j-1,c_1,c_2,K_2}(x|\varphi) \tag{11-13}$$

and

$$|\int d\mu_{v_x^j}(\Phi) R_j^c(x|\Phi+\Psi)| = |R_j^c(x|\Psi) + \int_0^1 ds \frac{1}{2}\triangle_{v_x^j} \int d\mu_{sv_x^j}(\Phi) R_j^c(x|\Phi+\Psi)| \le$$
$$\le |R_j^c(x|\Psi)| + |\delta_1 M_{j-1}^n(x|\Psi)|. \tag{11-14}$$

Thus for $\lambda_j$ small

$$\frac{|\int d\mu_{v_x^j}(\Phi)[Z_j^{rel}(\overline{x}|\Phi+\Psi) + R_j^c(x|\Phi+\Psi)]|}{S_{j-1,c_1,c_2,K_2}(x|\varphi)} \le$$
$$\le 2\exp\{\frac{17}{4!}c_3^4 + K_0 c_1^2\} + (C_1\lambda_j)^{1/2} C_2 [\|R_j^c\|_{j-1}^{\tilde{c}} + \|\delta_1 M_{j-1}^n\|_{j-1}^{\tilde{c}}]. \tag{11-15}$$

Using the definition of connectivity graphs (definition 8.1), lemma 2.2 and (11-9) we obtain for $x \in \Lambda_{j-1}$

$$\sum_{x\in X} \frac{|\delta' R_{j-1}(X|\Psi)|}{B^{\tilde{c}'}_{j-1}(X|\varphi)} \leq \sum_{M\geq 2} \frac{1}{(M-1)!} \sum_{\mathcal{G}=[U_1,\ldots,U_M]}^{\mathcal{G}\in\mathcal{CG}_x:} \sum_{N=2}^{M-1}\sum_{k=0}^{N} \sum_{\substack{J\subseteq \underline{M}:\\ |J|=N}} \sum_{\substack{I\subseteq J:\\ |I|=k}} \exp\{-\frac{\gamma}{4}|supp\ \mathcal{G}|\}$$
$$D^{N-k}\Big[\prod_{a\in \underline{M}-J} \frac{|\widetilde{e}_{j-1}(U_a)|}{B^{\tilde{c}}_{j-1}(U_a)}\Big]\Big[\prod_{b\in I}(1-\delta_{1,|U_b|})\frac{|M^n_{j-1}(U_b|\Psi)|}{B^{\tilde{c}}_{j-1}(U_b|\varphi)}\Big] \prod_{c\in J-I} \delta_{1,|U_c|} \tag{11-16}$$

with

$$D \equiv \exp\{-\frac{\gamma}{4}\gamma\}\Big[\exp\{\frac{17}{4!}c_3^4 + K_0c_1^2\} + (C_1\lambda_j)^{1/2}C_2[\|R^c_j\|^{\tilde{c}}_{j-1} + \|\delta_1 M^n_{j-1}\|^{\tilde{c}}_{j-1}]\Big]. \tag{11-17}$$

For $\mathcal{G} = [U_1,\ldots,U_M] \in \mathcal{CG}_x$ define a tree $\tau(\mathcal{G}) \in T_M$ such that $U_a \cap U_b \neq \emptyset$ if $(ab) \in \tau$. Then we have

$$\sum_{\mathcal{G}=[U_1,\ldots,U_M]}^{\mathcal{G}\in\mathcal{CG}_x:} (\ldots) = \sum_{\substack{d_1,\ldots,d_M\in\mathbb{N}-\{0\}:\\ \sum d_i=2(M-1)}} \sum_{\tau\in T(d_1,\ldots,d_M)} \sum_{\mathcal{G}=[U_1,\ldots,U_M]}^{\mathcal{G}\in\mathcal{CG}_x:\ \tau(\mathcal{G})=\tau} (\ldots). \tag{11-18}$$

Performing the summation over all connectivity graphs for fixed trees we obtain for $I \subseteq J \subsetneq \underline{M}$, $|J| = N$, $|I| = k$, $\tau \in T(d_1,\ldots,d_M)$

$$\sum_{\mathcal{G}=[U_1,\ldots,U_M]}^{\mathcal{G}\in\mathcal{CG}_x:\ \tau(\mathcal{G})=\tau} \exp\{-\frac{1}{4}\gamma|supp\ \mathcal{G}|\}\Big[\prod_{a\in\underline{M}-J}\frac{|\widetilde{e}_{j-1}(U_a)|}{B^{\tilde{c}}_{j-1}(U_a)}\Big]$$
$$\Big[\prod_{b\in I}(1-\delta_{1,|U_b|})\frac{|M^n_{j-1}(U_b|\Psi)|}{B^{\tilde{c}}_{j-1}(U_b|\varphi)}\Big]\Big[\prod_{c\in J-I}\delta_{1,|U_c|}\Big] \leq$$
$$(4\gamma^{-1})^{d_1}\prod_{i=2}^{M}[4\gamma^{-1}]^{d_i-1}d_1\prod_{i=1}^{M}(d_i-1)!(|||\widetilde{e}_{j-1}|||^{\tilde{c}}_{j-1})^{M-N}(\|\widetilde{M}^n_{j-1}\|^{\tilde{c}}_{j-1})^k. \tag{11-19}$$

Using

$$\sum_{\substack{d_1,\ldots,d_M\in\mathbb{N}-\{0\}:\\ \sum d_i=2(M-1)}} 1 \leq 4^M \tag{11-20}$$

and Cayley's theorem

$$\sum_{\tau\in T(d_1,\ldots,d_M)} 1 \leq \frac{(M-2)!}{\prod_{i=1}^M(d_i-1)!} \tag{11-21}$$

we obtain according to (11-18) and (11-19)

$$\sum_{\mathcal{G}=[U_1,\ldots,U_M]}^{\mathcal{G}\in\mathcal{CG}_x:} \exp\{-\frac{1}{4}\gamma|supp\ \mathcal{G}|\}\Big[\prod_{a\in\underline{M}-J}\frac{|\widetilde{e}_{j-1}(U_a)|}{B^{\tilde{c}}_{j-1}(U_a)}\Big]$$
$$\Big[\prod_{b\in I}(1-\delta_{1,|U_b|})\frac{|M^n_{j-1}(U_b|\Psi)|}{B^{\tilde{c}}_{j-1}(U_b|\varphi)}\Big]\Big[\prod_{c\in J-I}\delta_{1,|U_c|}\Big] \leq$$
$$(M-1)!(4\gamma^{-1})^{M-1}4^M(|||\widetilde{e}_{j-1}|||^{\tilde{c}}_{j-1})^{M-N}(\|\widetilde{M}^n_{j-1}\|^{\tilde{c}}_{j-1})^k. \tag{11-22}$$



(11-16) and (11-22) imply

$$\|\delta' R_{j-1}\|^{\tilde{c}'}_{j-1} \leq \sum_{M\geq 2}\sum_{N=2}^{M-1}\sum_{k=0}^{N} 3^M D^{N-k}(4\gamma^{-1})^{M-1}4^M$$
$$(|||\widetilde{e}_{j-1}|||^{\tilde{c}}_{j-1})^{M-N}(\|\widetilde{M}^n_{j-1}\|^{\tilde{c}}_{j-1})^k. \tag{11-23}$$

We have used

$$\sum_{\substack{J\subseteq\underline{M}:\\ |J|=N}}\sum_{\substack{I\subseteq J:\\ |I|=k}} 1 \leq 3^M.$$

Thus

$$\|\delta' R_{j-1}\|^{\tilde{c}'}_{j-1} \leq \sum_{M\geq 2}\sum_{N=2}^{M-1}\sum_{k=0}^{N}\frac{\gamma}{4}(48\gamma^{-1})^M D^N(|||\widetilde{e}_{j-1}|||^{\tilde{c}}_{j-1})^{M-N}(\|\widetilde{M}^n_{j-1}\|^{\tilde{c}}_{j-1}/D)^k. \tag{11-24}$$

Since

$$\|\widetilde{M}^n_{j-1}\|^{\tilde{c}}_{j-1} \leq \frac{1}{2}D \tag{11-25}$$

we have

$$\|\delta' R_{j-1}\|^{\tilde{c}'}_{j-1} \leq \sum_{M\geq 2}\sum_{N=2}^{M-1}\frac{\gamma}{2}(48\gamma^{-1})^M D^N(|||\widetilde{e}_{j-1}|||^{\tilde{c}}_{j-1})^{M-N}$$
$$\leq \sum_{M'\geq 1}\sum_{N\geq 2}\frac{\gamma}{2}(48\gamma^{-1}D)^N(48\gamma^{-1}|||\widetilde{e}_{j-1}|||^{\tilde{c}}_{j-1})^{M'} \leq$$
$$\leq 2\gamma(48\gamma^{-1}D)^2 48\gamma^{-1}|||\widetilde{e}_{j-1}|||^{\tilde{c}}_{j-1} \tag{11-26}$$

for

$$D \leq \frac{\gamma}{96}, \quad |||\widetilde{e}_{j-1}|||^{\tilde{c}}_{j-1} \leq \frac{\gamma}{96}.$$

This proves the assertion. √

For polymers $X$ of $\Lambda_{j-1}$ define

$$\delta\widetilde{R}_{j-1}(X|\Psi) \equiv (1-\delta_{1,|X|})[R_{j-1}(X|\Psi) - R^{div}_{j-1}(X|\Psi)] =$$
$$= \delta\widetilde{M}^n_{j-1}(X|\Psi) + \widetilde{M}^n_{j-1}(X|\Psi)[\exp\{-e_{j-1}(X)\} - 1] + \delta' R_{j-1}(X|\Psi). \tag{11-27}$$

LEMMA 11.2. *For positive constants $K_0$, $\tilde{c} = (C_1, C_2, \gamma, c_1, c_2, c_3, c, K_2)$ and*

$$D \equiv \exp\{-\frac{\gamma}{4}\}\Big[\exp\{\frac{17}{4!}c_3^4 + K_0c_1^2\} + (C_1\lambda_j)^{1/2}C_2[\|R^c_j\|^{\tilde{c}}_{j-1} + \|\delta_1 M^n_{j-1}\|^{\tilde{c}}_{j-1}]\Big]$$

*such that*

$$D \leq \frac{\gamma}{96}, \quad \|\widetilde{M}^n_{j-1}\|^{\tilde{c}}_{j-1} \leq \frac{1}{2}D, \quad |||\widetilde{e}_{j-1}|||^{\tilde{c}}_{j-1} \leq \frac{\gamma}{96}$$

we have for small $\lambda_j$

$$\|\delta\widetilde{R}_{j-1}\|_{j-1}^{\widetilde{c}'} \le \|\delta\widetilde{M}^n_{j-1}\|_{j-1}^{\widetilde{c}'} + K\lambda_j^{1/2}\|\widetilde{M}^n_{j-1}\|_{j-1}^{\widetilde{c}} + 24|||\widetilde{e}_{j-1}|||_{j-1}^{\widetilde{c}}.$$

with

$$\widetilde{c}' = (C_1, C_2, \frac{1}{2}\gamma, \frac{1}{2}c_1, q_2'c_2, c_3, c, K_2/2)$$

$$q_2' \equiv \sup_{z\in base}\Big\{\sum_{x\in\Lambda_{j-1}} \exp\{-\frac{K_2}{4}a_{j-1}^{-1}dist(z,x)\}\Big\}.$$

LEMMA 11.3. *For $\lambda_j$ small and $L$ large there exists*

$$\widetilde{c} = (C_1, C_2, \gamma, c_1, c_2, c_3, c, K_2)$$

*(not dependent on $\lambda_j$ ) and $\epsilon$ such that if*

$$\|R_j^{ren}\|_j^{\widetilde{c}} \le \epsilon$$

*then*

$$\|\delta\widetilde{R}_{j-1}\|_{j-1}^{\widetilde{c}_2} \le \frac{\epsilon}{3}$$

*for all $j\le 0$ and*

$$\widetilde{c}_2 = (C_1, C_2, \frac{1}{6}\gamma', \frac{4}{3}c_1, 6L^2q_3(q_2')^2\exp\{\frac{K_2\delta}{6}\}c_2, 3c_3, \frac{3}{2}c, \frac{L}{12}K_2)$$

$$\gamma' = \frac{cLd2^{-d}}{24u_j(\delta)}, q_2' \equiv \sup_{z\in base}\Big\{\sum_{x\in\Lambda_{j-1}}\exp\{-\frac{K_2}{4}a_{j-1}^{-1}dist(z,x)\}\Big\},$$

$$q_3 = \sup_{z\in base}\Big\{\sum_{y\in\Lambda_j}\exp\{-\frac{K_2}{6}a_j^{-1}dist(z,y)\}\Big\}.$$

*Proof :* By lemma 8.7 follows

$$\|\delta\widetilde{M}^n_{j-1}\|_{j-1}^{\widetilde{c}_2} \le \|\delta R_j^c\|_{j-1}^{\widetilde{c}_2} + KC_1^{-1}\|R_j^c\|_j^{\widetilde{c}_2''} + KC_1^{-1/2} \qquad (11\text{–}28)$$

with

$$\widetilde{c}_2'' = (C_1, C_2, \frac{1}{2}\gamma', 4L^{-2}c_1, 2q_3q_2'\exp\{-\frac{K_2\delta}{6}\}c_2, 9L^{-1}c_3, \frac{3}{2}L^{-1}c, K_2/6)$$

and

$$\|\widetilde{M}^n_{j-1}\|_{j-1}^{\widetilde{c}_2'} \le KC_1^{-1}\|R_j^c\|_j^{\widetilde{c}_1} + KC_1^{-1/2} \qquad (11\text{–}29)$$

with

$$\widetilde{c}_2' = (C_1, C_2, \frac{1}{3}\gamma', \frac{8}{3}c_1, 6L^2q_3, q_1'\exp\{-\frac{K_2\delta}{6}\}c_2, 3c_3, \frac{3}{2}c, \frac{L}{6}K_2)$$

and

$$\widetilde{c}_1 = (C_1, C_2, \gamma', 8L^{-2}c_1, 2q_3\exp\{-\frac{K_2\delta}{6}\}c_2, 9L^{-1}c_3, L^{-1}c, K_2/3).$$



According to lemma 11.2 we have

$$\|\delta\widetilde{R}_{j-1}\|_{j-1}^{\widetilde{c}_2} \le \|\delta\widetilde{M}^n_{j-1}\|_{j-1}^{\widetilde{c}_2} + K\lambda_j^{1/2}\|\widetilde{M}^n_{j-1}\|_{j-1}^{\widetilde{c}_2'} + 24|||\widetilde{e}_{j-1}|||_{j-1}^{\widetilde{c}_2'}. \qquad (11\text{–}30)$$

(11-28),(11-29) and (11-30) imply

$$\|\delta\widetilde{R}_{j-1}\|_{j-1}^{\widetilde{c}_2} \le \|\delta R_j^c\|_{j-1}^{\widetilde{c}_2} + KC_1^{-1}\|R_j^c\|_j^{\widetilde{c}_2''} + KC_1^{-1/2} + K^2C_1^{-1}\lambda_j^{1/2}\|R_j^c\|_j^{\widetilde{c}_1} + K^2C_1^{-1/2}\lambda_j^{1/2} + 24|||\widetilde{e}_{j-1}|||_{j-1}^{\widetilde{c}_2'}. \qquad (11\text{–}31)$$

By lemma 7.10 we have

$$\|R_j^c\|_j^{\widetilde{c}_1} \le \|\delta R_j^c\|_j^{\widetilde{c}_1} + \|R_j^{div}\|_j^{\widetilde{c}_1} \le \|\delta R_j^c\|_j^{\widetilde{c}_1} + K\|R_j^{ren}\|_j^{\widetilde{c}^{(iv)}} \qquad (11\text{–}32)$$

with

$$\widetilde{c}^{(iv)} = (C_1, C_2, 0, 16L^{-2}c_1, 2c_2, 9L^{-1}c_3, L^{-1}c, K_2/3).$$

Since for $L$ large

$$\|R_j^{ren}\|_j^{\widetilde{c}^{(iv)}} \le \|R_j^{ren}\|_j^{\widetilde{c}} \qquad (11\text{–}33)$$

we obtain by (11-32)

$$\|R_j^c\|_j^{\widetilde{c}_1} \le \|\delta R_j^c\|_j^{\widetilde{c}_1} + K\|R_j^{ren}\|_j^{\widetilde{c}}. \qquad (11\text{–}34)$$

Furthermore

$$\|R_j^c\|_j^{\widetilde{c}_2''} \le \|\delta R_j^c\|_j^{\widetilde{c}_2''} + \|R_j^{div}\|_j^{\widetilde{c}_2''} \le \|\delta R_j^c\|_j^{\widetilde{c}_1} + K\|R_j^{ren}\|_j^{\widetilde{c}} \qquad (11\text{–}35)$$

and

$$\|\delta R_j^c\|_{j-1}^{\widetilde{c}_2} \le \|\delta R_j^c\|_j^{\widetilde{c}_1}. \qquad (11\text{–}36)$$

(11-31),(11-32),(11-34) and (11-36) imply

$$\|\delta\widetilde{R}_{j-1}\|_{j-1}^{\widetilde{c}_2} \le \|\delta R_j^c\|_j^{\widetilde{c}_1} + KC_1^{-1}[\|\delta R_j^c\|_j^{\widetilde{c}_1} + K\|R_j^{ren}\|_j^{\widetilde{c}}] + KC_1^{-1/2} + K^2C_1^{-1}\lambda_j^{1/2}[\|\delta R_j^c\|_j^{\widetilde{c}_1} + K\|R_j^{ren}\|_j^{\widetilde{c}}] + K^2C_1^{-1/2}\lambda_j^{1/2} + 24|||\widetilde{e}_{j-1}|||_{j-1}^{\widetilde{c}_2'}. \qquad (11\text{–}37)$$

By lemma 9.4 we have

$$|||\widetilde{e}_{j-1}|||_{j-1}^{\widetilde{c}_2'} \le KC_1^{-1}. \qquad (11\text{–}38)$$

Thus, using (11-37),(11-38),cor. 6.11 and

$$\|R_j^{ren}\|_j^{\widetilde{c}} \le \epsilon, \qquad (11\text{–}39)$$

we obtain for large $C_1$

$$\|\delta\widetilde{R}_{j-1}\|_{j-1}^{\widetilde{c}_2} \le \frac{\epsilon}{4} + KC_1^{-1}[\frac{\epsilon}{4}K\epsilon] + KC_1^{-1/2} + K^2C_1^{-1}\lambda_j^{1/2}[\frac{\epsilon}{4}K\epsilon] + K^2C_1^{-1}\lambda_j^{1/2} + 24KC_1^{-1} \le \frac{\epsilon}{3}.$$

This completes the proof. $\surd$

For monomers $X$ of $\Lambda_{j-1}$ we have

$$\delta R^{mon}_{j-1}(X|\Psi) \equiv \delta_{1,|X|}[R_{j-1}(X|\Psi) - R^{div}_{j-1}(X|\Psi)] = \sum_{i=1}^{5} \delta R^{(i)}_{j-1}(X|\Psi) \tag{11-40}$$

with

$$\delta R^{(1)}_{j-1}(X|\Psi) \equiv \int d\mu_{v^j_x}(\Phi)[Z^{rel}_j(\overline{X}|\Phi+\Psi) - Z^{rel}_{j-1}(X|\Psi)] \tag{11-40a}$$

$$\delta R^{(2)}_{j-1}(X|\Psi) \equiv \int d\mu_{v^j_x}(\Phi) Z^{rel}_j(\overline{X}|\Phi+\Psi)[\exp\{-e_{j-1}(X)\} - 1] \tag{11-40b}$$

$$\delta R^{(3)}_{j-1}(X|\Psi) \equiv \int d\mu_{v^j_x}(\Phi)\delta R^c_j(X|\Phi+\Psi)\exp\{-e_{j-1}(X)\} \tag{11-40c}$$

$$\delta R^{(4)}_{j-1}(X|\Psi) \equiv \int d\mu_{v^j_x}(\Phi)[R^{div}_j(X|\Phi+\Psi) - R^{div}_{j-1}(X|\Psi)] \tag{11-40d}$$

$$\delta R^{(5)}_{j-1}(X|\Psi) \equiv \int d\mu_{v^j_x}(\Phi) R^{div}_{j-1}(X|\Phi+\Psi)[\exp\{-e_{j-1}(X)\} - 1]. \tag{11-40e}$$

For $s \in [0,1]$ define $s$–dependent coupling constants

$$m^2_{j,s} \equiv s m^2_j + (1-s) m^2_{j-1} \tag{11-41a}$$

$$\beta^{\mu\nu}_{j,s} \equiv s\beta^{\mu\nu}_j + (1-s)\beta^{\mu\nu}_{j-1} \tag{11-41b}$$

$$\lambda_{j,s} \equiv s\lambda_j + (1-s)\lambda_{j-1}. \tag{11-41c}$$

Define $Z^{rel}_{j,s}$ for $x \in \Lambda_{j-1}$ by

$$Z^{rel}_{j,s}(\overline{x}|\Psi) \equiv \exp\{-V^{rel}_{j,s}(\overline{x}|\Psi)\} \tag{11-42}$$

$$V^{rel}_{j,s}(\overline{x}|\Psi) \equiv \frac{1}{2}\int_{z\in x} m^2_{j,s}\Psi(z)^2 + \frac{1}{2}\sum_{\mu,\nu=1}^{4}\beta^{\mu\nu}_{j,s}\int_{z\in x}(\nabla^{per}_\mu\Psi(z))(\nabla^{per}_\nu\Psi(z)) + \frac{\lambda_{j,s}}{4!}\int_{z\in x}\Psi(z)^4. \tag{11-43}$$

Then we have

$$\delta R^{(1)}_{j-1}(X|\Psi) = \int_0^1 ds\ \partial_s \int d\mu_{sv^j_x}(\Phi) Z^{rel}_{j,s}(\overline{X}|\Phi+\Psi). \tag{11-44}$$

Application of lemmata 10.3,9.4,7.9,7.10 and corollary 6.11 yields

LEMMA 11.4. *For $\lambda_j$ small and $L$ large there exists*

$$\widetilde{c} = (C_1, C_2, \gamma, c_1, c_2, c_3, c, K_2)$$

*(not dependent on $\lambda_j$) and small $\epsilon$ such that if*

$$\|R^{ren}_j\|^{\widetilde{c}}_j \le \epsilon$$



*then we have*

$$\|\delta R^{mon}_{j-1}\|^{\widetilde{c}_2}_{j-1} \le \frac{\epsilon}{6}$$

*for all $j \le 0$ and*

$$\widetilde{c}_2 = (C_1, C_2, \frac{1}{6}\gamma', \frac{4}{3}c_1, 6L^2 q_3(q_2')^2\exp\{-\frac{K_2\delta}{6}\}c_2, 3c_3, \frac{3}{2}c, \frac{L}{12}K_2)$$

$$\gamma' = \frac{cLd2^{-d}}{24u_j(\delta)}, q_2' \equiv \sup_{z\in base}\Big\{\sum_{x\in\Lambda_{j-1}}\exp\{-\frac{K_2}{4}a^{-1}_{j-1}dist(z,x)\}\Big\},$$

$$q_3 = \sup_{z\in base}\Big\{\sum_{y\in\Lambda_j}\exp\{-\frac{K_2}{6}a^{-1}_j dist(z,y)\}\Big\}.$$

Since

$$\delta R_{j-1}(X|\Psi) = \delta R^{mon}_{j-1}(X|\Psi) + \delta\widetilde{R}_{j-1}(X|\Psi)$$

we see that lemma 11.3 and 11.4 imply

PROPOSITION 11.5. *For $\lambda_j$ small and $L$ large there exists*

$$\widetilde{c} = (C_1, C_2, \gamma, c_1, c_2, c_3, c, K_2)$$

*(not dependent on $\lambda_j$ ) and small $\epsilon$ such that if*

$$\|R^{ren}_j\|^{\widetilde{c}}_j \le \epsilon$$

*then*

$$\|\delta R_{j-1}\|^{\widetilde{c}_2}_{j-1} \le \frac{\epsilon}{2}$$

*for all $j \le 0$ and*

$$\widetilde{c}_2 = (C_1, C_2, \frac{1}{6}\gamma', \frac{4}{3}c_1, 6L^2 q_3(q_2')^2\exp\{-\frac{K_2\delta}{6}\}c_2, 3c_3, \frac{3}{2}c, \frac{L}{12}K_2)$$

$$\gamma' = \frac{cLd2^{-d}}{24u_j(\delta)},\ q_2' \equiv \sup_{z\in base}\Big\{\sum_{x\in\Lambda_{j-1}}\exp\{-\frac{K_2}{4}a^{-1}_{j-1}dist(z,x)\}\Big\},$$

$$q_3 = \sup_{z\in base}\Big\{\sum_{y\in\Lambda_j}\exp\{-\frac{K_2}{6}a^{-1}_j dist(z,y)\}\Big\}.$$

*Remark* : Proposition 11.5 implies inequality (3-22) of Lemma 3.6.

## 4.12. Bounds on Polymer-dependent Counterterms

For $z \in base$, $x \in \Lambda_{j-1}$, and a polymer $X$ of $\Lambda_{j-1}$ the polymer-dependent counterterms read (since $R^{div}_{j-1}$ obeys renormalization conditions)

$$\widetilde{\delta m^2_{j-1}}(z|X) \equiv \mathcal{D}^2_z \delta R_{j-1}(X|\Psi)|_{\Psi=0} \tag{12–1a}$$

$$\widetilde{\delta\beta^{\mu\nu}_{j-1}}(x|X) \equiv \mathcal{D}^{\mu\nu}_x \delta R_{j-1}(X|\Psi)|_{\Psi=0} \tag{12–1b}$$

$$\widetilde{\delta\lambda_{j-1}}(x|X) \equiv \mathcal{D}^4_x \delta R_{j-1}(X|\Psi)|_{\Psi=0} - \\ - \frac{3}{Vol(x)} \int_{z \underline{\in} x} \int_{z' \underline{\in} X} \Bigg[ \sum_{\substack{Q_1,Q_2 \subseteq X \\ Q_1 \cap Q_2 \neq \emptyset, Q_1 \cup Q_2 = X}} \mathcal{D}^2_z \delta R_{j-1}(Q_1|\Psi)|_{\Psi=0} \mathcal{D}^2_{z'} \delta R_j(Q_2|\Psi)|_{\Psi=0} \Bigg]. \tag{12–1c}$$

Define

$$\delta\tilde{V}^2_{j-1}(x,X|\Psi) \equiv \frac{1}{2} \int_{z \underline{\in} x} \widetilde{\delta m^2_{j-1}}(z|X)\Psi(z)^2. \tag{12–2}$$

LEMMA 12.1. *There exist constants $K$ and $K'$ such that*

$$\sum_{X:\, x \in X} |\delta\tilde{V}^2_{j-1}(x,X|\Psi)| \le K\lambda_{j-1} a^{-2}_{j-1} \int_{z \in base} \exp\{-K_2 a^{-1}_{j-1} dist(z,x)\}(\mathcal{A}^j\varphi)^2(z) + K'\lambda^{1/2}_{j-1} \tag{12–3}$$

*for all polymers $X$ of $\Lambda_{j-1}$, $x \in \Lambda_{j-1}$ , $\Psi = \mathcal{A}^{j-1}\varphi + \chi$, $\chi \in c_3\mathcal{K}_{j-1}(x)$, $\varphi:\ \Lambda_{j-1} \to$* **R** *and all $j \le 0$.*

*Proof :* We have

$$\delta\tilde{V}^2_{j-1}(x,X|\Psi) = A_1(X) + A_2(X) + A_3(X) \tag{12–4}$$

with

$$A_1(X) \equiv \frac{1}{2} \int_{z \underline{\in} x} \int_{x_1,x_2 \in \Lambda_{j-1}} \widetilde{\delta m^2_{j-1}}(z|X)\mathcal{A}^{j-1}(z,x_1)\mathcal{A}^{j-1}(z,x_2)\varphi(x_1)\varphi(x_2) \tag{12–5a}$$

$$A_2(X) \equiv \int_{z \underline{\in} x} \int_{x' \Lambda_{j-1}} \widetilde{\delta m^2_{j-1}}(z|X)\mathcal{A}^{j-1}(z,x')\varphi(x')\chi(z) \tag{12–5b}$$

$$A_3(X) \equiv \frac{1}{2} \int_{z \underline{\in} x} \widetilde{\delta m^2_{j-1}}(z|X)\chi^2(z). \tag{12–5c}$$

Since

$$\widetilde{\delta m^2_{j-1}}(z|X) = \mathcal{D}^2_z \delta R_{j-1}(X|\Psi)|_{\Psi=0} = \int_{z' \in base} \frac{\delta^2}{\delta\Psi(z)\delta\Psi(z')} \delta R_{j-1}(X|\Psi)|_{\Psi=0} \tag{12–6}$$



we obtain

$$A_1(X) = \frac{1}{2} \int_{x_1,x_2 \in \Lambda_{j-1}} \frac{\partial^2}{\partial\alpha_1\partial\alpha_2} \delta R_{j-1}(X|\alpha_1\chi_x(.)\mathcal{A}^{j-1}(.,x_1)\mathcal{A}^{j-1}(.,x_2) + \alpha_2)|_{\alpha_1=\alpha_2=0}$$

$$\varphi(x_1)\varphi(x_2) = \frac{1}{2} K_1^2 a^{-2-d}_{j-1} \lambda^{1/2}_{j-1} \int_{x_1,x_2 \in \Lambda_{j-1}} \delta R_{j-1}(X|\alpha_1\chi_x(.)a^{1+\frac{3d}{2}}_{j-1} K_1^{-2} \lambda^{-1/2}_{j-1}$$

$$\exp\{K_2 a^{-1}_{j-1}[dist(x,x_1) + dist(x,x_2)]\}\mathcal{A}^{j-1}(.,x_1)\mathcal{A}^{j-1}(.,x_2) + \alpha_2 a^{1-d/2}_{j-1}\lambda^{-1/4}_{j-1})|_{\alpha_1=\alpha_2=0}$$

$$\prod_{a=1}^{2} [\exp\{-K_2 a^{-1}_{j-1} dist(x,x_a)\}\varphi(x_a)].$$

Using Cauchy's inequality we see that there exists a constant $K$ such that

$$|A_1(X)| \le K\lambda^{1/2}_{j-1} a^{-2}_{j-1} \sup_{\chi \in c_3\mathcal{K}_{j-1}(x)} |\delta R_{j-1}(X|\chi)| \int_{x' \in \Lambda_{j-1}} \exp\{-K_2 a^{-1}_{j-1} dist(x,x')\}\varphi(x')^2. \tag{12–7}$$

Furthermore

$$A_2(X) = \int_{x' \in \Lambda_{j-1}} \frac{\partial^2}{\partial\alpha_1\partial\alpha_2} \delta R_{j-1}(X|\alpha_1\chi_x(.)\mathcal{A}^{j-1}(.,x')\chi(.) + \alpha_2)|_{\alpha_1=\alpha_2=0}\varphi(x') =$$

$$= K_1 a^{-1-d/2}_{j-1} \lambda^{1/4}_{j-1} \int_{x' \in \Lambda_{j-1}} \frac{\partial^2}{\partial\alpha_1\partial\alpha_2} \delta R_{j-1}(X|\alpha_1\chi_x(.)a^d_{j-1} K_1^{-1} \mathcal{A}^{j-1}(.,x')\chi(.) +$$

$$+ \alpha_2 a^{1-d/2}_{j-1} \lambda^{-1/4}_{j-1})|_{\alpha_1=\alpha_2=0} \exp\{-K_2 a^{-1}_{j-1} dist(x,x')\}\varphi(x').$$

Using Cauchy's inequality we obtain

$$|A_2(X)| \le K a^{-1-d/2}_{j-1} \lambda^{1/4}_{j-1} \sup_{\chi \in c_3\mathcal{K}_{j-1}(x)} |\delta R_{j-1}(X|\chi)| \int_{x' \in \Lambda_{j-1}} \exp\{-K_2 a^{-1}_{j-1} dist(x,x')\}|\varphi(x')|. \tag{12–8}$$

Furthermore

$$|A_3(X)| = \frac{1}{2} |\frac{\partial^2}{\partial\alpha^2} \delta R_{j-1}(X|\alpha\chi_x\chi)| \le K a^{-1-d/2}_{j-1} \lambda^{1/4}_{j-1} \sup_{\chi \in c_3\mathcal{K}_{j-1}(x)} |\delta R_{j-1}(X|\chi)|. \tag{12–9}$$

(12-4,7,8,9) and Proposition 11.5 imply the assertion. √

LEMMA 12.2. *For $x \in \Lambda_{j-1}$ and $\chi \in \mathcal{K}_{j-1}(x)$ we have*

$$\chi^2 \in 4a^{1-d/2}_{j-1} \lambda^{-1/4}_{j-1} \mathcal{K}_{j-1}(x). \tag{12–10}$$

*Proof :* Suppose that $z, z_1, z_2 \in x$.

a) $|\chi^2(z)| \le [a^{1-d/2}_{j-1} \lambda^{-1/4}_{j-1}]|\chi^2(z)|$.

b) $|\chi^2(z_1) - \chi^2(z_2)| \le [2a^{1-d/2}_{j-1} \lambda^{-1/4}_{j-1}]|\chi(z_1) - \chi(z_2)|$.

c) For $z + ae_\mu \underline{\in} x$ we have

$$|\nabla_\mu \chi(z)| = a^{-1}|\chi(z + ae_\mu) - \chi(z)| \le a^{1-d/2}_{j-1} \lambda^{-1/4}_{j-1}$$

and for $z + a e_\mu \not\underline{\in} x$ we have

$$|\nabla_\mu \chi(z)| = |\nabla_\mu \chi(z) - \nabla_\mu \chi(z - a e_\mu) + \nabla_\mu \chi(z - a e_\mu)| \leq$$
$$a a_{j-1}^{-d/2-1} \lambda_{j-1}^{-1/4} + a_{j-1}^{-d/2} \lambda_{j-1}^{-1/4} \leq (1 + L^{j-1}) a_{j-1}^{-d/2} \lambda_{j-1}^{-1/4}.$$

Thus

$$|\nabla_\mu \chi(z)| \leq (1 + L^{j-1}) a_{j-1}^{-d/2} \lambda_{j-1}^{-1/4}. \tag{12-11}$$

Without loss of generality suppose that $z_1 \neq z_2$. We have

$$(\chi^2)_2(z_1, z_2) = [\chi(z_1) + \chi(z_2)]\chi_2(z_1, z_2) - \sum_{\mu=1}^{d} (z_1^\mu - z_2^\mu) \nabla_\mu \chi(z)|_{z=z_2}$$
$$[a \nabla_\mu \chi(z)|_{z=z_2} + \chi(z_2) - \chi(z_1)]. \tag{12-12}$$

Since for $z_1 \neq z_2$

$$a \leq |z_1 - z_2|^{1-\epsilon} a_{j-1}^{\epsilon} \tag{12-13}$$

we obtain,using (12-11) and (12-13),

$$|(\chi^2)_2(z_1, z_2)| \leq 2 a_{j-1}^{1-d/2} \lambda_{j-1}^{-1/4} |(\chi_2(z_1, z_2)| + |z_1 - z_2| (1 + L^{j-1}) a_{j-1}^{-d/2} \lambda_{j-1}^{-1/4}$$
$$|z_1 - z_2|^{1-\epsilon} a_{j-1}^{\epsilon} L^{j-1} a_{j-1}^{-d/2} \lambda_{j-1}^{-1/4} + |z_1 - z_2| (1 + L^{j-1}) a_{j-1}^{-d/2} \lambda_{j-1}^{-1/4}$$
$$|z_1 - z_2|^{1-\epsilon} a_{j-1}^{\epsilon} a_{j-1}^{-d/2} \lambda_{j-1}^{-1/4} \leq (3 + 2L^{j-1}) a_{j-1}^{1-d/2} \lambda_{j-1}^{-1/4}$$
$$a_{j-1}^{-d/2-1+\epsilon} \lambda_{j-1}^{-1/4} |z_1 - z_2|^{2-\epsilon} \leq [4 a_{j-1}^{1-d/2} \lambda_{j-1}^{-1/4}] a_{j-1}^{-d/2-1+\epsilon} \lambda_{j-1}^{-1/4} |z_1 - z_2|^{2-\epsilon}.$$

This completes the proof. $\surd$

Define for $\varphi : \Lambda_{j-1} \to \mathbf{R}$ and positive constants $q, c$ and a polymer $X$ of $\Lambda_{j-1}$

$$K_{j-1,c,q}(X|\varphi) \equiv q\, c^{-1/2} \exp\{c \lambda_{j-1}^{1/2} a_{j-1}^{-2} \sum_{x \in \Lambda_{j-1}} \int_{z \underline{\in} x}$$
$$\exp\{-2K_2 a_{j-1}^{-1} dist(z, X)\} (\mathcal{A}^{j-1}\varphi)^2(z)\}. \tag{12-14}$$

LEMMA 12.3. *There exists a $(\lambda, L)$–independent constant $q$ such that for $\chi \in \mathcal{K}_{j-1}(X)$ and $\varphi : \Lambda_{j-1} \to \mathbf{R}$*

$$K_{j-1,c,q}(X|\varphi)^{-1} (\mathcal{A}^{j-1}\varphi + \chi) \in \mathcal{K}_{j-1}(X). \tag{12-15}$$

*Proof :* For $z \underline{\in} X$ and all positive $C$

$$|(\mathcal{A}^{j-1}\varphi)(z)| \leq \sum_{x \in \Lambda_{j-1}} K_1 \exp\{-K_2 a_{j-1}^{-1} dist(z, x)\} |\varphi(x)|$$
$$C^{-1/2} \exp\{\frac{1}{2} \sum_{x \in \Lambda_{j-1}} C \exp\{-2K_2 a_{j-1}^{-1} dist(z, x)\} \varphi(x)^2\}. \tag{12-16}$$



Using

$$\int_{z \underline{\in} x} (\mathcal{A}^{j-1}\varphi)^2(z) \geq a_{j-1}^{d} \varphi(x)^2 \tag{12-17}$$

we obtain for $z \underline{\in} X$

$$\exp\{-2K_2 a_{j-1}^{-1} dist(z, x)\} \varphi(x)^2 \leq a_{j-1}^{-d} \int_{z' \underline{\in} x} \exp\{-2K_2 a_{j-1}^{-1} dist(z, x)\} (\mathcal{A}^{j-1}\varphi)^2(z') \leq$$
$$\leq a_{j-1}^{-d} \int_{z' \underline{\in} x} \exp\{-2K_2 a_{j-1}^{-1} dist(X, x)\} (\mathcal{A}^{j-1}\varphi)^2(z') \leq$$
$$\leq \exp\{2K_2\} a_{j-1}^{-d} \int_{z \underline{\in} x} \exp\{-2K_2 a_{j-1}^{-1} dist(z, X)\} (\mathcal{A}^{j-1}\varphi)^2(z). \tag{12-18}$$

(12-16) and (12-18) imply for $C = 2c \lambda_{j-1}^{1/2} a_{j-1}^{1-d/2}$

$$|(\mathcal{A}^{j-1}\varphi)(z)| \leq \frac{1}{2^{1/2}} \exp\{K_2\} c^{-1/2} \lambda_{j-1}^{-1/4} a_{j-1}^{1-d/2}$$
$$\exp\{c \lambda_{j-1}^{1/2} a_{j-1}^{-2} \sum_{x \in \Lambda_{j-1}} \int_{z \underline{\in} x} \exp\{-2K_2 a_{j-1}^{-1} dist(z, X)\} (\mathcal{A}^{j-1}\varphi)^2(z)\}$$
$$K_{j-1,c,q}(X|\varphi) \lambda_{j-1}^{-1/4} a_{j-1}^{1-d/2}$$

with $q = \frac{1}{2^{1/2}} \exp\{K_2\}$. $\surd$

Lemma 12.2 and 12.3 imply for $\Psi = \mathcal{A}^{j-1}\varphi + \chi$, $\chi \in c_3 \mathcal{K}_{j-1}(x)$, $x \in \Lambda_{j-1}$,

$$c_3 a_{j-1}^{d/2-1} \lambda_{j-1}^{1/4} K_{j-1,c,q}(x|\varphi)^{-2} \Psi^2 \in c_3 \mathcal{K}_{j-1}(x) \tag{12-19}$$

for a suitable constant $q$.

LEMMA 12.4. *There exists a $(\lambda, L)$–independent constant $q$ such that for $x \in \Lambda_{j-1}$ and all polymers $X$ of $\Lambda_{j-1}$, $\Psi = \mathcal{A}^{j-1}\varphi + \chi$, $\chi \in \mathcal{K}_{j-1}(x)$, $\varphi : \Lambda_{j-1} \to \mathbf{R}$ and all $j \leq 0$*

$$|\delta \tilde{V}_{j-1}^2(x, X|\Psi)| \leq K_{j-1,c,q}(X|\varphi)^2 \sup_{\chi \in c_3 \mathcal{K}_{j-1}(x)} |\delta R_{j-1}(X|\chi)|. \tag{12-20}$$

*Proof :* Define $q$ such that

$$\varphi' = 4 c_3^{-1} a_{j-1}^{d/2-1} \lambda_{j-1}^{1/4} K_{j-1,c,q}(x|\varphi)^{-2} \Psi^2 \in c_3 \mathcal{K}_{j-1}(x)$$

Then

$$|\int_{z \underline{\in} x} \widetilde{\delta m_{j-1}^2}(z|X) \varphi'(z)| = |\int_{z \underline{\in} x} \int_{z' \in base} \frac{\delta^2}{\delta \Psi(z) \delta \Psi(z')} \delta R_{j-1}(X|\Psi)|_{\Psi=0} \varphi'(z)| =$$
$$= |\frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} \delta R_{j-1}(X|\alpha_1 \chi_x(.) \varphi'(.) + \alpha_2)|_{\alpha_1=\alpha_2=0}| \leq$$
$$\leq 4 c_3^{-1} a_{j-1}^{d/2-1} \lambda_{j-1}^{1/4} \sup_{\chi \in c_3 \mathcal{K}_{j-1}(x)} |\delta R_{j-1}(X|\chi)|.$$

Since

$$|\delta \tilde{V}_{j-1}^2(x, X|\Psi)| \leq \frac{1}{4} c_3 a_{j-1}^{1-d/2} \lambda_{j-1}^{-1/4} K_{j-1,c,q}(x|\varphi)^2 \int_{z \underline{\in} x} \widetilde{\delta m_{j-1}^2}(z|X) \varphi'(z)$$

we obtain the assertion. $\surd$

LEMMA 12.5. *For $x \in \Lambda_{j-1}$ and all polymers $X$ of $\Lambda_{j-1}$ we have*

$$|\delta\widetilde{\beta^{\mu\nu}_{j-1}}(x|X)| \le 16c_3^{-2}\lambda_{j-1}^{1/2} \sup_{\chi\in c_3\mathcal{K}_{j-1}(x)} |\delta R_{j-1}(X|\chi)|, \tag{12-21a}$$

$$|\delta\widetilde{\lambda_{j-1}}(x|X)| \le 8c_3^{-4}\lambda_{j-1}a_{j-1}^{d-4} \sup_{\chi\in c_3\mathcal{K}_{j-1}(x)} |\delta R_{j-1}(X|\chi)|. \tag{12-21b}$$

*Proof :* For $z_0 \underline{\in} x$ we have

$$\delta\widetilde{\beta^{\mu\nu}_{j-1}}(x|X) = \frac{1}{Vol(x)}\int_{z_1\underline{\in} x}\int_{z_2\in base}(z_1^\mu - z_2^\mu)(z_1^\nu - z_2^\nu)\frac{\delta^2}{\delta\Psi(z_1)\delta\Psi(z_2)}\delta R_{j-1}(X|\Psi)|_{\Psi=0} =$$
$$= \frac{1}{Vol(x)}\frac{\partial^2}{\partial\alpha_1\partial\alpha_2}\Big[\delta R_{j-1}(X|((.)^\mu - z_0^\mu)((.)^\nu - z_0^\nu)\chi_x(.)\alpha_1 + \alpha_2)+$$
$$+ \delta R_{j-1}(X|\chi_x(.)\alpha_1 + (z_0^\mu - (.)^\mu)(z_0^\nu - (.)^\nu)\alpha_2)+$$
$$+ \delta R_{j-1}(X|((.)^\mu - z_0^\mu)\chi_x(.)\alpha_1 + ((.)^\nu - z_0^\nu)\alpha_2)+$$
$$+ \delta R_{j-1}(X|((.)^\nu - z_0^\nu)\chi_x(.)\alpha_1 + ((.)^\mu - z_0^\mu)\alpha_2)\Big]|_{\alpha_1=\alpha_2=0}.$$

Using Cauchy's inequality we obtain (12-21a). Furthermore

$$\delta\widetilde{\lambda_{j-1}}(x|X) = \frac{1}{Vol(x)}\int_{z_1\underline{\in} x}\int_{z_2,z_3,z_4\in base}\frac{\delta^4}{\delta\Psi(z_1)\dots\delta\Psi(z_4)}\delta R_{j-1}(X|\Psi)|_{\Psi=0} =$$
$$= \frac{1}{Vol(x)}\frac{\partial^4}{\partial\alpha_1\partial\alpha_2^3}\delta R_{j-1}(X|\chi_x(.)\alpha_1 + \alpha_2)|_{\alpha_1=\alpha_2=0}.$$

Cauchy's inequality yields (12-23b). $\surd$

LEMMA 12.6. *For $x \in \Lambda_{j-1}, \Psi = \mathcal{A}^{j-1}\varphi + \chi,\ \varphi :\ \Lambda_{j-1} \to \mathbf{R},\ \chi \in \mathcal{K}_{j-1}(x)$ we have*

$$|\int_{z\underline{\in} x}(\nabla^{per}_\mu\Psi(z))(\nabla^{per}_\nu\Psi(z))| \le c_3^2 K_{j-1,c,q}(X|\varphi)^2\lambda_{j-1}^{-1/2} \tag{12-22a}$$

$$|\int_{z\underline{\in} x}\Psi(z)^4| \le c_3^4 K_{j-1,c,q}(X|\varphi)^4 a_{j-1}^{4-d}\lambda_{j-1}^{-1}. \tag{12-22b}$$

*Proof :* Lemma 12.3 implies

$$|\Psi(z)| \le c_3 K_{j-1,c,q}(X|\varphi)a_{j-1}^{1-d/2}\lambda_{j-1}^{-1/4}$$

$$|\nabla^{per}_\mu\Psi(z)| \le c_3 K_{j-1,c,q}(X|\varphi)a_{j-1}^{d/2}\lambda_{j-1}^{-1/4}. \quad \surd$$

Define

$$\delta\widetilde{V}^{\mu\nu}_{j-1}(x,X|\Psi) \equiv \frac{1}{2}\delta\beta^{\mu\nu}_j(x|X)\int_{z\underline{\in} x}(\nabla^{per}_\mu\Psi(z))(\nabla^{per}_\nu\Psi(z)) \tag{12-23}$$

and

$$\delta\widetilde{V}^4_{j-1}(x,X|\Psi) \equiv \frac{1}{4!}\delta\lambda_j(x|X)\int_{z\underline{\in} x}\Psi(z)^4. \tag{12-24}$$

Lemma 12.4,12.5,12.6 imply



LEMMA 12.7. *There exists a $(\lambda, L)$–independent constant $q$ such that for all $x \in \Lambda_{j-1}$, $\Psi = \mathcal{A}^{j-1}\varphi + \chi,\ \ \varphi :\ \Lambda_{j-1} \to \mathbf{R},\ \chi \in \mathcal{K}_{j-1}(x)$ and all polymers $X$ of $\Lambda_{j-1}$ and positive $c$*

$$|\delta\widetilde{V}^2_{j-1}(x,X|\Psi)| \le K_{j-1,c,q}(X|\varphi)^2 \sup_{\chi\in c_3\mathcal{K}_{j-1}(x)} |\delta R_{j-1}(X|\chi)|$$

$$|\delta\widetilde{V}^{\mu\nu}_{j-1}(x,X|\Psi)| \le K_{j-1,c,q}(X|\varphi)^2 \sup_{\chi\in c_3\mathcal{K}_{j-1}(x)} |\delta R_{j-1}(X|\chi)|$$

$$|\delta\widetilde{V}^4_{j-1}(x,X|\Psi)| \le K_{j-1,c,q}(X|\varphi)^4 \sup_{\chi\in c_3\mathcal{K}_{j-1}(x)} |\delta R_{j-1}(X|\chi)|.$$

### 4.13. Repolymerization Step : Bounds on $\delta R^{ren}_{j-1}$

In this section the bounds for the repolymerization step are proven. Lemma 13.3 implies the bound (3-23) of lemma 3.6.

By eqs. (2-18),(2-21),(3-20b,c) we have for all polymers $X$ of $\Lambda_{j-1}$

$$\delta R^{ren}_{j-1}(X|\Psi) = \delta_1 R^{ren}_{j-1}(X|\Psi) + \delta_2 R^{ren}_{j-1}(X|\Psi) \tag{13-1}$$

with

$$\delta_1 R^{ren}_{j-1}(X|\Psi) \equiv \delta_{1,|X|}Z^{rel}_{j-1}(X|\Psi)[Z^{cou}_{j-1}(X,X|\Psi) - 1] + \delta R_{j-1}(X|\Psi) \tag{13-2}$$

$$\delta_2 R^{ren}_{j-1}(X|\Psi) = \sum_{N\ge 2}\frac{1}{N!}\sum_{\substack{Q_1,\dots,Q_N\subseteq X:\\ \sum Q_i = X}}\ \sum_{\substack{I:\ \emptyset\ne I\subseteq \underline{N}:\\ |Q_a|=1\ \forall a\in I}}\ \sum_{\tau\in T_N}\ \sum_{\substack{\pi,\eta:\\ \tau=\tau(\pi,\eta)}}\int_0^1 ds\ f(\eta|s)$$
$$\{\prod_{i=1}^m\prod_{(ab)\in\tau}\frac{\partial}{\partial t_{ab}}\}[\prod_{a\in I}Z^{rel}_{j-1}(Q_a|\Psi)Z^{cou}_{j-1,t}(Q_a,X|\Psi)]\prod_{b\in\underline{N}-I}R_{j-1}(Q_a|\Psi)|_{t=t[\pi,s]}. \tag{13-3}$$

LEMMA 13.1. *For $\lambda_j$ small and $L$ large there exist positive constants*

$$\widetilde{c} = (C_1, C_2, \gamma = 0, c_1, c_2, c_3, c, K_2)$$

*and small $\epsilon$ such that if*

$$\|R^{ren}_j\|^{\widetilde{c}}_j \le \epsilon$$

*then*

$$\|\delta_2 R^{ren}_{j-1}\|^{\widetilde{c}}_{j-1} \le \frac{\epsilon}{8}$$

*for all $j \le 0$.*

*Proof :* By eq. (13-3) we obtain

$$\delta_2 R^{ren}_{j-1}(X|\Psi) = \sum_{N\ge 2}\frac{1}{N!}\sum_{\substack{Q_1,\dots,Q_N\subseteq X:\\ \sum Q_i = X}}\ \sum_{\substack{I:\ \emptyset\ne I\subseteq \underline{N}:\\ |Q_a|=1\ \forall a\in I}}\ \sum_{\tau\in T_N}\sum_{m\ge 1}\ \sum_{\substack{(\sigma_1,\dots,\sigma_m):\\ \sum_{i=1}^m\sigma_i=\tau}}$$
$$\sum_{X_1,\dots,X_m\subseteq X}\ \sum_{\substack{\pi,\eta:\\ \tau=\tau(\pi,\eta)}}\int_0^1 ds\ f(\eta|s)\prod_{i=1}^m[(\prod_{(ab)\in\sigma_i}\frac{\partial}{\partial t_{ab}})(-\sum_{a\in I}\delta\widetilde{V}_{j-1,t}(Q_a,X_i|\Psi)]$$
$$[\prod_{a\in I}Z^{rel}_{j-1}(Q_a|\Psi)Z^{cou}_{j-1,t}(Q_a,X|\Psi)]\prod_{b\in\underline{N}-I}R_{j-1}(Q_a|\Psi)|_{t=t[\pi,s]}.$$

Using Lemma 2.2 and (3-20c)

$$|\delta_2 R^{ren}_{j-1}(X|\Psi)| \le \sum_{N\ge 2}\frac{1}{N!}\sum_{\substack{Q_1,\dots,Q_N\subseteq X:\\ \sum Q_i = X}}\ \sum_{\substack{I:\ \emptyset\ne I\subseteq \underline{N}:\\ |Q_a|=1\ \forall a\in I}}\ \sum_{\substack{J:\ \emptyset\ne J\subseteq \underline{N}-I:\\ |Q_a|\le 2^d\ \forall a\in \underline{N}-I-J}}$$

$$\sum_{\tau\in T_N}\sum_{m\ge 1}\sum_{\substack{(\sigma_1,\dots,\sigma_m):\\ \sum_{i=1}^m \sigma_i=\tau}}\ \sum_{\substack{X_1,\dots,X_m\subseteq X\\ V(\sigma_a)\subseteq C(X_a)}}\ \prod_{i=1}^m\Big[\sum_{x\in X_i}|\delta\widetilde V_{j-1}(x,X_i|\Psi)|\Big]$$

$$\Big[\prod_{a\in I}|Z^{rel}_{j-1}(Q_a|\Psi)|\exp\{|\delta\widetilde V_{j-1}(Q_a,X_i|\Psi)|\}\Big]\prod_{b\in J}|\delta R_{j-1}(Q_b|\Psi)|$$

$$\prod_{c\in \underline{N}-I-J}|\delta R^{ren}_{j-1}(Q_c|\Psi)| \tag{13-4}$$

with $C(P)\equiv\{a\in\{1,\dots,N\}|\ Q_a\cap P\ne\emptyset\}$ and $V(\sigma)\equiv$ vertex set of $\sigma$. We have used that for each polymer $P$ of $\Lambda_j$ with $\rho(P)=1$ the inequality

$$|[U_\delta(P)]|\le 2^d$$

holds. Using lemmata 12.1,12.4 and the stability bound (proposition 4.5), we see that there exist constants $K, K_2, c_1, c_2$ such that for all $x\in\Lambda_{j-1}$

$$|Z_{j-1}(x|\Psi)\exp\{|\delta\widetilde V_{j-1}(x,X|\Psi)|\}\le K\, S_{j-1,\frac{5}{4}c_1+\frac{1}{2}c_2,\frac{1}{2}q_2'^{-1}c_2,2K_2}(x|\varphi) \tag{13-5}$$

with

$$q_2' \equiv \sup_{z\in base}\Big\{\sum_{x\in\Lambda_{j-1}}\exp\{-\frac{K_2}{4}a_{j-1}^{-1}dist(z,x)\}\Big\}.$$

For large $L$ we have for the term of the sum of (13-4)

$$\Big[\prod_{a\in I}|Z^{rel}_{j-1}(Q_a|\Psi)|\exp\{|\delta\widetilde V_{j-1}(Q_a,X_i|\Psi)|\}\Big]\prod_{b\in J}|\delta R_{j-1}(Q_b|\Psi)|$$

$$\prod_{c\in\underline{N}-I-J}|\delta R^{ren}_{j-1}(Q_c|\Psi)|\le K^{|I|}\Big[\prod_{a\in\underline{N}}S_{j-1,\frac{5}{4}c_1+\frac{1}{2}c_2,\frac{1}{2}q_2'^{-1}c_2,2K_2}(Q_a|\varphi)\Big]$$

$$\Big[\prod_{b\in J}\frac{|\delta R_{j-1}(Q_b|\Psi)|}{S_{j-1,\frac{4}{3}c_1,6L^2q_3(q_2')^2\exp\{-\frac{K_2\delta}{6}\}c_2,\frac{L}{12}K_2}(Q_b|\varphi)}\Big]$$

$$\Big[\prod_{c\in\underline{N}-I-J}\frac{|R^{div}_{j-1}(Q_c|\Psi)|}{S_{j-1,\frac{3}{2}c_1,\frac{1}{2}q_2'^{-1}c_2,2K_2}(Q_c|\varphi)}\Big] \tag{13-6}$$

with

$$q_3 = \sup_{z\in base}\Big\{\sum_{y\in\Lambda_j}\exp\{-\frac{K_2}{6}a_j^{-1}dist(z,y)\}\Big\}.$$

By Lemma 8.2 follows

$$\prod_{a\in\underline{N}}S_{j-1,\frac{5}{4}c_1+\frac{1}{2}c_2,\frac{1}{2}q_2'^{-1}c_2,2K_2}(Q_a|\varphi)]\le S_{j-1,\frac{5}{4}c_1,\frac{1}{2}c_2,K_2}(X|\varphi)\le$$

$$\le S^{in}_{j-1,\frac{1}{4}c_1}(X|\varphi)[S^{out}_{j-1,\frac{1}{2}c_2,K_2}(X|\varphi)]^{-1}S_{j-1,c_1,c_2,K_2}(X|\varphi). \tag{13-7}$$



(13-4,5,6,7) and Lemma 12.5 imply

$$|\delta_2 R^{ren}_{j-1}(X|\Psi)| \le \sum_{N\ge 2}\frac{1}{N!}\sum_{\substack{Q_1,\dots,Q_N\subseteq X:\\ \sum Q_i = X}}\ \sum_{\substack{I:\ \emptyset\ne I\subseteq \underline{N}:\\ |Q_a|=1\ \forall a\in I}}\ \sum_{\substack{J:\ \emptyset\ne J\subseteq \underline{N}-I:\\ |Q_a|\le 2^d\ \forall a\in \underline{N}-I-J}}$$

$$\sum_{\tau\in T_N}\sum_{m\ge 1}\sum_{\substack{(\sigma_1,\dots,\sigma_m):\\ \sum_{i=1}^m \sigma_i=\tau}}\ \sum_{\substack{X_1,\dots,X_m\subseteq X\\ V(\sigma_a)\subseteq C(X_a)}}\ \sum_{x_1\in X_1}\cdots\sum_{x_m\in X_m}\prod_{i=1}^m[K_{j-1,c_i,q}(X_i|\varphi)^4$$

$$\sup_{\chi\in c_3K_{j-1}(X_i)}|\delta R_{j-1}(X_i|\chi)|]K^{|I|}S^{in}_{j-1,\frac{1}{4}c_1}(X|\varphi)[S^{out}_{j-1,\frac{1}{2}c_2,K_2}(X|\varphi)]^{-1}$$

$$S_{j-1,c_1,c_2,K_2}(X|\varphi)\Big[\prod_{b\in J}\frac{|\delta R_{j-1}(Q_b|\Psi)|}{S_{j-1,\frac{4}{3}c_1,6L^2q_3(q_2')^2\exp\{\frac{K_2\delta}{6}\}c_2,\frac{L}{12}K_2}(Q_b|\varphi)}\Big]$$

$$\Big[\prod_{c\in\underline{N}-I-J}\frac{|R^{div}_{j-1}(Q_c|\Psi)|}{S_{j-1,\frac{3}{2}c_1,\frac{1}{2}q_2'^{-1}c_2,2K_2}(Q_c|\varphi)}\Big] \tag{13-8}$$

for all positive $c_i>0,\ i\in\{1,\dots,m\}$, and a suitable constant $q>1$. Since $V(\sigma_a)\subseteq C(X_a)$ we can choose $x_a'\in X_a$ for all $a\in\{1,\dots,m\}$ such that

$$x_a'\ne x_b'\quad\text{if}\quad a\ne b. \tag{13-9}$$

We can choose, using (13-9),

$$c_i\equiv\exp\{-\tilde c a_{j-1}^{-1}dist(x_i,x_i')\}b$$

such that

$$\prod_{i=1}^m K_{j-1,c_i,q}(X_i|\varphi)^4 S^{in}_{j-1,\frac{1}{4}c_1}(X|\varphi)[S^{out}_{j-1,\frac{1}{2}c_2,K_2}(X|\varphi)]^{-1}\le$$

$$\le q^{4m}b^{-2m}\prod_{i-1}^m\exp\{4\tilde c a_{j-1}^{-1}dist(x_i,x_i')\}. \tag{13-10}$$

According to (13-8) and (13-10)

$$\frac{|\delta_2 R^{ren}_{j-1}(X|\Psi)|}{\widetilde B^c_{j-1}(X|\varphi)}\le\sum_{N\ge 2}\frac{1}{N!}\sum_{\substack{Q_1,\dots,Q_N\subseteq X:\\ \sum Q_i = X}}\ \sum_{\substack{I:\ \emptyset\ne I\subseteq \underline{N}:\\ |Q_a|=1\ \forall a\in I}}\ \sum_{\substack{J:\ \emptyset\ne J\subseteq \underline{N}-I:\\ |Q_a|\le 2^d\ \forall a\in \underline{N}-I-J}}$$

$$\sum_{\tau\in T_N}\sum_{m\ge 1}\sum_{\substack{(\sigma_1,\dots,\sigma_m):\\ \sum_{i=1}^m \sigma_i=\tau}}\ \sum_{X_1,\dots,X_m\subseteq X}\ \sum_{x_1\in X_1}\cdots\sum_{x_m\in X_m}K^{|I|}q^{4m}c^{-2m}\Big[\prod_{i-1}^m\exp-\frac{\gamma'}{6}|X_i|\Big\}$$

$$\sup_{\chi\in c_3K_{j-1}(X_i)}\Big\{\frac{|\delta R_{j-1}(X_i|\chi)|}{\widetilde B^{c_2}_{j-1}(X_i|\varphi)}\Big\}\Big[\prod_{b\in J}\frac{|\delta R_{j-1}(Q_b|\Psi)|}{\widetilde B^{c_2}_{j-1}(Q_b|\varphi)}\exp\{-\frac{\gamma'}{6}|Q_b|\}T_{j-1,\frac{1}{2}c}(Q_b)\Big]$$

$$\Big[\prod_{c\in\underline{N}-I-J}\frac{|\delta R^{div}_{j-1}(Q_c|\Psi)|}{\widetilde B^{c_2}_{j-1}(Q_c|\varphi)}\Big] \tag{13-11}$$

with

$$\tilde{c}_2 = (C_1, C_2, \frac{1}{6}\gamma', \frac{4}{3}c_1, 6L^2 q_3(q_2')^2 \exp\{-\frac{K_2\delta}{6}\}c_2, 3c_3, \frac{3}{2}c, \frac{L}{12}K_2)$$

$$\gamma' = \frac{cLd2^{-d}}{24u_j(\delta)},$$

$$\tilde{c}' = (C_1, C_2, \gamma = 0, \frac{3}{2}c_1, \frac{1}{2}q_2^{l-1}c_2, c_3, Lc, 2K_2).$$

Using the definition of connectivity graphs (definition 8.1) and (13-11), we obtain for large $L$

$$\frac{|\delta_2 R^{ren}_{j-1}(X|\Psi)|}{B^{\tilde{c}}_{j-1}(X|\varphi)} \le \sum_{M\ge 2} \frac{1}{(M-1)!}\exp\{-\frac{\gamma'}{6}\} \sum^{\mathcal{G}\in\mathcal{CG}_x:}_{\mathcal{G}=[U_1,\ldots,U_M]} \sum_{N=0}^{M-1}\sum_{k=0}^{N} \sum_{\substack{J\subseteq \underline{M}:\\ |J|=N}} \sum_{\substack{I\subseteq J:\\ |I|=k}}$$
$$[\prod_{a\in\underline{M}-J} \exp\{-\frac{\gamma'}{10}|U_a|\} \sup_{\chi\in c_3 K_{j-1}(U_a)} \{\frac{|\delta R_{j-1}(U_a|\chi)|}{B^{\tilde{c}_2}_{j-1}(U_a|\varphi)}\}$$
$$[\prod_{b\in I} \frac{|\delta R_{j-1}(U_b|\Psi)|}{B^{\tilde{c}_2}_{j-1}(U_b|\varphi)} \exp\{-\frac{\gamma'}{10}|U_b|\}][\prod_{c\in J-I} \chi(|U_c|\le 2^d)$$
$$\frac{|\delta R^{div}_{j-1}(U_c|\Psi)|}{B^{\tilde{c}_2}_{j-1}(U_c|\varphi)}] \tag{13-12}$$

For $\mathcal{G} = [U_1, \ldots, U_M] \in \mathcal{CG}_x$ define a tree $\tau(\mathcal{G}) \in T_M$ such that $U_a \cap U_b \ne \emptyset$ if $(ab) \in \tau$. Then we have

$$\sum^{\mathcal{G}\in\mathcal{CG}_x:}_{\mathcal{G}=[U_1,\ldots,U_M]} (\ldots) = \sum_{\substack{d_1,\ldots,d_M\in\mathbb{N}-\{0\}:\\ \sum d_i = 2(M-1)}} \sum_{\tau\in T(d_1,\ldots,d_M)} \sum^{\mathcal{G}\in\mathcal{CG}_x:\ \tau(\mathcal{G})=\tau}_{\mathcal{G}=[U_1,\ldots,U_M]} (\ldots). \tag{13-13}$$

Performing the summation over all connectivity graphs for fixed trees we obtain for $I \subseteq J \subseteq \underline{M}$, $|J| = N$, $|I| = k$, $\tau \in T(d_1, \ldots, d_M)$

$$\sum^{\mathcal{G}\in\mathcal{CG}_x:\ \tau(\mathcal{G})=\tau}_{\mathcal{G}=[U_1,\ldots,U_M]} [\prod_{a\in\underline{M}-J} \exp\{-\frac{\gamma'}{10}|U_a|\} \sup_{\chi\in c_3 K_{j-1}(U_a)} \{\frac{|\delta R_{j-1}(U_a|\chi)|}{B^{\tilde{c}_2}_{j-1}(U_a|\varphi)}\}$$
$$[\prod_{b\in I} \frac{|\delta R_{j-1}(U_b|\Psi)|}{B^{\tilde{c}_2}_{j-1}(U_b|\varphi)} \exp\{-\frac{\gamma'}{10}|U_b|\}][\prod_{c\in J-I} \chi(|U_c|\le 2^d)$$
$$\frac{|\delta R^{div}_{j-1}(U_c|\Psi)|}{B^{\tilde{c}_2}_{j-1}(U_c|\varphi)}] \le$$
$$\le d_1 \prod_{i-1}^{M} (d_i - 1)!(\|\delta R_{j-1}\|^{\tilde{c}_2}_{j-1})^{M-N+k} (\|R^{div}_{j-1}\|^{\tilde{c}'}_{j-1})^{N-k}. \tag{13-14}$$

Using

$$\sum_{\substack{d_1,\ldots,d_M\in\mathbb{N}-\{0\}:\\ \sum d_i = 2(M-1)}} 1 \le 4^M \tag{13-15}$$



and Cayley's Theorem

$$\sum_{\tau\in T(d_1,\ldots,d_M)} 1 \le \frac{(M-2)!}{\prod_{i=1}^M (d_i - 1)!} \tag{13-16}$$

we obtain according to (11-13) and (11-14)

$$\sum^{\mathcal{G}\in\mathcal{CG}_x:}_{\mathcal{G}=[U_1,\ldots,U_M]} \sum_{N=0}^{M-1}\sum_{k=0}^{N} \sum_{\substack{J\subseteq \underline{M}:\\ |J|=N}} \sum_{\substack{I\subseteq J:\\ |I|=k}} [\prod_{a\in\underline{M}-J} \exp\{-\frac{\gamma'}{10}|U_a|\} \sup_{\chi\in c_3 K_{j-1}(U_a)} \{\frac{|\delta R_{j-1}(U_a|\chi)|}{B^{\tilde{c}_2}_{j-1}(U_a|\varphi)}]$$
$$[\prod_{b\in I} \frac{|\delta R_{j-1}(U_b|\Psi)|}{B^{\tilde{c}_2}_{j-1}(U_b|\varphi)} \exp -\frac{\gamma'}{10}|U_b|\}][\prod_{c\in J-I} \chi(|U_c|\le 2^d) \frac{|\delta R^{div}_{j-1}(U_c|\Psi)|}{B^{\tilde{c}_2}_{j-1}(U_c|\varphi)}] \le$$
$$\le 4^M (M-1)!(\|\delta R_{j-1}\|^{\tilde{c}_2}_{j-1})^{M-N+k} (\|R^{div}_{j-1}\|^{\tilde{c}'}_{j-1})^{N-k}. \tag{13-17}$$

(13-12) and (13-17) imply

$$\|\delta_2 R^{ren}_{j-1}\|^{\tilde{c}}_{j-1} \le \sum_{M\ge 1}\sum_{N=0}^{M-1}\sum_{k=0}^{N} 12^M \exp\{\frac{-\gamma'}{6}\}(\|\delta R_{j-1}\|^{\tilde{c}_2}_{j-1})^{M-N+k}(\|R^{div}_{j-1}\|^{\tilde{c}'}_{j-1})^{N-k}.$$

We have used

$$\sum_{\substack{J\subseteq \underline{M}:\\ |J|=N}} \sum_{\substack{I\subseteq J:\\ |I|=k}} 1 \le 3^M.$$

Thus

$$\|\delta_2' R^{ren}_{j-1}\|^{\tilde{c}'}_{j-1} \le \sum_{M\ge 1} 12^M \exp\{\frac{-\gamma'}{6}\}[(\|\delta R_{j-1}\|^{\tilde{c}_2}_{j-1})^M + (\|\delta R_{j-1}\|^{\tilde{c}_2}_{j-1} +$$
$$+ (\|R^{div}_{j-1}\|^{\tilde{c}'}_{j-1})^M].$$

By Proposition 11.5 and Lemma 7.9

$$\|\delta_2' R^{ren}_{j-1}\|^{\tilde{c}'}_{j-1} \le \sum_{M\ge 1} \exp\{\frac{-\gamma'}{6}\}[(6\epsilon)^M + (9\epsilon)^M] \le \exp\{\frac{-\gamma'}{6}\}[\frac{6\epsilon}{1-6\epsilon} + \frac{9\epsilon}{1-9\epsilon}] \le \frac{\epsilon}{8}.$$

for $L$ large and $\epsilon$ small enough. ✓

**Lemma 13.2.** *For $\lambda_j$ small and $L$ large there exists*

$$\tilde{c} = (C_1, C_2, \gamma = 0, c_1, c_2, c_3, c, K_2)$$

*and small $\epsilon$ such that if*

$$\|R^{ren}_j\|^{\tilde{c}}_j \le \epsilon$$

*then*

$$\|\delta_1 R^{ren}_{j-1}\|^{\tilde{c}}_{j-1} \le \frac{\epsilon}{8} + \|\delta R_{j-1}\|^{\tilde{c}}_{j-1}$$

*for all $j \le 0$.*

Lemma 13.1 and 13.2 imply

LEMMA 13.3. *For $\lambda_j$ small and $L$ large there exists*

$$\tilde{c} = (C_1, C_2, \gamma = 0, c_1, c_2, c_3, c, K_2)$$

*and small $\epsilon$ such that if*

$$\|R_j^{ren}\|_j^{\tilde{c}} \leq \epsilon$$

*then*

$$\|\delta R_{j-1}^{ren}\|_{j-1}^{\tilde{c}} \leq \frac{\epsilon}{4} + \|\delta R_{j-1}\|_{j-1}^{\tilde{c}}$$

*for all $j \leq 0$.*

*Remark :* Comparing the constants $\tilde{c}_2$ of section 4.11 with $\tilde{c}$ we see that $\|\delta R_{j-1}^{ren}\|_{j-1}^{\tilde{c}} \leq \|\delta R_{j-1}^{ren}\|_{j-1}^{\tilde{c}_2}$. Lemma 13.3 implies inequality (3-23) of Lemma 3.6.

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# APPENDICES

## A. Polymer Systems

Useful relations for general polymer systems are presented in this section. Lemma 5.1 of section 4.5 is proven in this appendix. Let $\Lambda$ be a denumerable set and consider the commutative algebra $\mathcal{A}_\Lambda \equiv \mathcal{A}$ generated by elements $\Theta_x$ for all $x \in \Lambda$ and a unit element 1 which obey the following relation

$$\Theta_x \Theta_x = 0 \tag{A–1}$$

for all $x \in \Lambda$. An element $\widehat{Z}$ may be represented by

$$\widehat{Z} = \sum_{P:\ P \subseteq \Lambda} Z(P)\Theta_P \tag{A–2}$$

where

$$\Theta_P \equiv \prod_{x \in P} \Theta_x, \quad \Theta_\emptyset \equiv 1 \tag{A–3}$$

and $Z(P)$ are complex numbers. For $X \subseteq \Lambda$ the subalgebra $\mathcal{A}_X$ of $\mathcal{A}$ is the algebra generated by 1 and $\Theta_x$ for all $x \in X$. For $\widehat{Z} \in \mathcal{A}$ define $\widehat{Z}(X) \in \mathcal{A}_X$ by

$$\widehat{Z}(X) \equiv \sum_{P:\ P \subseteq X} Z(P)\Theta_P \tag{A–4}$$

and define the functions exp and ln : $\mathcal{A} \to \mathcal{A}$ by

$$\exp \widehat{Z} \equiv \sum_{n \geq 0} \frac{1}{n!} \widehat{Z}^n, \qquad \ln(1 + \widehat{Z}) \equiv \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} \widehat{Z}^n. \tag{A–5}$$

Integration is defined by

$$\int d\Theta_x \Theta_y = \delta_{xy} \equiv \begin{cases} 1, & \text{for } x = y \\ 0, & \text{for } x \neq y \end{cases}, \quad \int d\Theta_x = 0. \tag{A–6}$$

We will use the notation $\int d\Theta_X \equiv \prod_{x \in X} \int d\Theta_x$ for all $X \subseteq \Lambda$. $\widehat{Z} \in \mathcal{A}$ is called a generalized partition function if $\widehat{Z}(\emptyset) = 0$. $\widehat{A} = \ln \widehat{Z}$ is called the generalized activity. Then we have $\widehat{A}(\emptyset) = 0$ . $Z(X) \equiv \int d\Theta_X \widehat{Z}$ and $A(X) \equiv \int d\Theta_X \widehat{A}$ are called the partition function resp. activity of $X \subseteq \Lambda$.

LEMMA A.1. *Let $X$ be a finite nonempty subset of $\Lambda$ and $\widehat{Z}$ a generalized partition function. For the generalized activity $\widehat{A} = \ln \widehat{Z}$ we have the following relations*[1]

$$Z(X) = \sum_{X = \sum P} \prod_P A(P) \tag{A–7}$$

[1] The sums run over all partitions of $X$ into disjoint nonempty sets

$$A(X) = \sum_{n\geq 1}(n-1)!(-1)^{n-1} \sum_{X=\sum_{i=1}^n P_i} \prod_{i=1}^n Z(P_i) \tag{A-8}$$

and

$$Z(X) = \sum_{P:\ x\in P\subseteq X} A(P)Z(X-P) \tag{A-9}$$

for all $x \in X$.

*Proof :* Using $\widehat{Z} = \exp\{\widehat{A}\}$, we get

$$Z(X) = \int d\Theta_X \exp\{\widehat{A}\} = \int d\Theta_X [\sum_{n\geq 0} \frac{1}{n!}[\sum_P A(P)\Theta_P]^n] =$$

$$= \sum_{n\geq 0} \frac{1}{n!} \sum_{\substack{P_1,\dots,P_n:\\ X=\sum_{i=1}^n P_i}} \prod_{i=1}^n A(P_i).$$

This shows relation (7). Furthermore,

$$A(X) = \int d\Theta_X \ln\{\widehat{Z}\} = \int d\Theta_X [\sum_{n\geq 1} \frac{(-1)^{n+1}}{n}[\sum_P Z(P)\Theta_P]^n] =$$

$$= \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \sum_{\substack{P_1,\dots,P_n:\\ X=\sum_{i=1}^n P_i}} \prod_{i=1}^n Z(P_i).$$

This shows relation (8). For $x \in X$ we have

$$Z(X) = \int d\Theta_X \widehat{Z} = \int d\Theta_X \exp\{\sum_{P:\ P\subseteq X} A(P)\Theta_P\} =$$

$$= \int d\Theta_X [1 + \sum_{P:\ x\in P\subseteq X} A(P)\Theta_P] \exp\{\sum_{P:\ P\subseteq X-x} A(P)\Theta_P\} =$$

$$= \sum_{P:\ x\in P\subseteq X} A(P)Z(X-P).$$

This shows relation (8) and completes the proof. √

For $\widehat{O} \in \mathcal{A}$ define expectation values for all finite nonempty subsets $X$ of $\Lambda$

$$< \widehat{O} >_X \equiv \int d\Theta_X [\sum_{Q:\ Q\subseteq X} \Theta_Q]\widehat{O} \tag{A-10}$$

and

$$<< \widehat{O} >>_X \equiv < \exp\{\widehat{M}(X)\}\widehat{O} >_X / < \exp\{\widehat{M}(X)\} >_X \tag{A-11}$$

where

$$\widehat{M} \equiv \sum_P [-\delta_{1,|P|} + A(P)]\Theta_P. \tag{A-12}$$



LEMMA A.2. *For all finite nonempty subsets $X$ of $\Lambda$ and $P \subseteq X$ we have*

$$< \Theta_P >_X = 1, \quad << \Theta_P >>_X = Z(X-P)/Z(X) \tag{A-13}$$

and

$$Z(X) = < \exp\{\widehat{M}(X)\} >_X = 1 + \sum_{Q:\ \emptyset\neq Q\subseteq X} \sum_{Q=\sum P} \prod_P M(P). \tag{A-14}$$

*with* $M(P) = \int d\Theta_P \widehat{M}$.

*Proof :* We have

$$< \Theta_P >_X = \int d\Theta_X [\sum_{R:\ P\subseteq R\subseteq X} \Theta_R] = 1$$

and

$$Z(X) = \int d\Theta_X \widehat{Z} = \int d\Theta_X \exp\{\sum_{x\in X} \Theta_x\} \exp\{\sum_{P:\ \emptyset\neq P\subseteq X} M(P)\Theta_P\} =$$

$$= \int d\Theta_X \sum_{Q:\ Q\subseteq X} \Theta_Q \exp\{\widehat{M}(X)\} = < \exp\{\widehat{M}\} >_X .$$

Furthermore

$$<< \Theta_P >>_X = \int d\Theta_X [\sum_{Q:\ Q\subseteq X-P} \Theta_{Q+P}] \exp\{\widehat{M}(X)\}/Z(X) =$$

$$= \sum_{Q:\ Q\subseteq X-P} \int d\Theta_{X-P}\Theta_Q \exp\{\widehat{M}(X-P)\}/Z(X) = Z(X-P)/Z(X). \ \checkmark$$

$\rho_X(P) \equiv Z(X-P)/Z(X)$ is called the reduced correlation function. For $\widehat{O}_1,\dots,\widehat{O}_N \in \mathcal{A}$, $N \geq 2$, and finite nonempty subsets $X$ of $\Lambda$ define recursively truncated expectation values by

$$< \widehat{O}_1; \dots; \widehat{O}_N >_X \equiv < \widehat{O}_1 \dots \widehat{O}_N >_X - \sum_{n\geq 2} \sum_{\underline{N}=\sum_{a=1}^n I_a} \prod_{a=1}^n < \prod_{i\in I_a} [\widehat{O}_i;\ ] >_X . \tag{A-15}$$

LEMMA A.3. *For all $P_1,\dots,P_N \subseteq X$, $N \geq 2$, with $X$ finite nonempty subset of $\Lambda$ we have*

$$< \prod_{i=1}^N \Theta_{P_i} >_X = \begin{cases} 1, & \text{for } P_1,\dots,P_N \text{ disjoint} \\ 0, & \text{otherwise} \end{cases} \tag{A-16}$$

and[1]

$$< \prod_{i=1}^N [\Theta_{P_i};\ ] >_X = \begin{cases} \sum'_{G\subseteq\gamma(P_1,\dots,P_N)} (-1)^{|G|}, & \text{for } \gamma(P_1,\dots,P_N) \text{ connected} \\ 0, & \text{otherwise.} \end{cases} \tag{A-17}$$

[1] For notation see section 4.5

*Proof :* Eq. (16) follows immediately from lemma A.2. For $P, Q \subseteq \Lambda$ define

$$\delta(P,Q) \equiv \begin{cases} 1, & \text{for } P \cap Q = \emptyset \\ 0, & \text{for } P \cap Q \neq \emptyset \end{cases}$$

Then we have

$$< \prod_{i=1}^{N} \Theta_{P_i} >_X = \prod_{(ij)} \delta(P_i, P_j) = \sum_{n \geq 1} \sum_{\underline{N} = \sum_{a=1}^{n} I_a} \prod_{a=1}^{n} [ \sum_{G \subseteq \gamma(P_i, i \in I_a)} \prod_{(ij) \in I_a} [\delta(P_i, P_j) - 1]].$$

By definition (15) follows eq. (17). √

LEMMA A.4. *For all finite nonempty subsets $X$ of $\Lambda$ we have*

$$\ln Z(X) = \sum_{N \geq 1} \frac{1}{N!} < \prod_{a=1}^{N} [\widehat{M};\ ] >_X = \sum_{N \geq 1} \sum_{P_1, \ldots, P_N \subseteq X} < \prod_{a=1}^{N} [\Theta_{P_a};\ ] >_X \prod_{a=1}^{N} M(P_a). \quad \text{(A-18)}$$

*Proof :*

$$Z(X) = < \exp\{\widehat{M}\} >_X = \sum_{N \geq 0} \frac{1}{N!} < \widehat{M}^N >_X =$$

$$= \sum_{N \geq 0} \sum_{\substack{k_1, \ldots, k_N \in \mathbb{N}: \\ \sum_{m=1}^{N} m k_m = N}} \prod_{m=1}^{N} [\frac{1}{k_m!} (\frac{1}{m!} < \prod_{i=1}^{m} [\widehat{M};\ ] >_X)^{k_m}] =$$

$$= \prod_{m=1}^{\infty} [\sum_{k \geq 0} \frac{1}{k!} (\frac{1}{m!} < \prod_{i=1}^{m} [\widehat{M};\ ] >_X)^{k}] =$$

$$= \exp\{ \sum_{m \geq 1} \frac{1}{m!} < \prod_{i=1}^{m} [\widehat{M};\ ] >_X \}. \ \surd$$

Lemma A.4 shows that we have the following representation for the combinatoric coefficient $a(C), C = (P_1, \ldots, P_N)$

$$a(C) = < \prod_{i=1}^{N} [\Theta_{P_i};\ ] >_X . \quad \text{(A-19)}$$

Lemma A.3 and A.4 and eq.(19) prove lemma 5.1 of section 4.5.

## B. Multigrid Operators

In this appendix we consider kernels of multigrid operators and prove their exponential decay.

For $M \leq j \leq 0,\ a_j = L^{-j} a$, define the tori

$$T_j \equiv a_j \mathbf{Z}^d / a_M \mathbf{Z}^d$$



$$B_j \equiv \frac{2\pi}{a_M} \mathbf{Z}^d / \frac{2\pi}{a_j} \mathbf{Z}^d .$$

We have for $y \in T_j$ and $p \in B_j$[1]

$$\frac{1}{(2\pi)^d} \int_{p \in B_j} \exp\{ipy\} = a_j^{-d} \delta_{z,0} \equiv \delta(z) \quad \text{(B-1a)}$$

$$\int_{y \in T_j} \exp\{ipy\} = a_M^d \delta_{p,0} \equiv (2\pi)^d \delta(p). \quad \text{(B-1b)}$$

For notational simplicity suppose that $L$ is odd. Define

$$\int_{z \underline{\in} y} F(z) \equiv a^d \sum_{\substack{m \in \mathbf{Z}^d \\ |m^\mu| < L^{-j}/2}} F(y + ma)$$

for $y \in T_j$ and periodic functions $F$, $F(z + a_M m) = F(z)$, $\forall m \in \mathbf{Z}^d$. For $y \in T_k$ and $x \in T_j$, $k > j$, we write $y \underline{\in} x$ if there exists $m \in \mathbf{Z}^d$ with $|m^\mu| < L^{k-j}/2$ such that $y = x + m a_k$. Define for $z \in T_0,\ y \in T_j,\ x \in T_{j-1},\ y_1, y_2 \in T_j$

$$C^j(y,z) \equiv \frac{1}{Vol(y)} \chi_y(z) = \begin{cases} \frac{1}{Vol(y)}, & \text{for } z \underline{\in} y \\ 0, & \text{otherwise} \end{cases} \quad \text{(B-2)}$$

$$C_{j-1,j}(x,y) \equiv \frac{1}{Vol(x)} \chi_x(y) = \begin{cases} \frac{1}{Vol(x)}, & \text{for } y \underline{\in} x \\ 0, & \text{otherwise} \end{cases} \quad \text{(B-3)}$$

and

$$u_j(y_1, y_2) \equiv a_j^{-2d} \int_{z_1 \underline{\in} y_1} \int_{z_2 \underline{\in} y_2} \mathrm{v}(z_1, z_2) \equiv C^j u_j C^{j*}(y_1, y_2) \quad \text{(B-4)}$$

$$\mathcal{A}^j(z,y) \equiv \frac{1}{Vol(y)} \int_{y' \in T_j} \int_{\substack{z' \in T_0: \\ z' \underline{\in} y'}} \mathrm{v}(z, z') u_j^{-1}(y', y) \equiv \mathrm{v} C^{j*} u_j^{-1}(z,y) \quad \text{(B-5)}$$

$$\mathcal{A}_{j-1,j}(y,x) \equiv \frac{1}{Vol(x)} \int_{x' \in T_{j-1}} \int_{\substack{y' \in T_j: \\ y' \underline{\in} x'}} u_j(y,y') u_{j-1}^{-1}(x',x) \equiv u_j C_{j-1,j}^{*} u_{j-1}^{-1}(y,x) \quad \text{(B-6)}$$

$$v^j(y_1, y_2) \equiv (u_j - \mathcal{A}_{j,j-1} u_{j-1} \mathcal{A}_{j,j-1}^{*})(y_1, y_2). \quad \text{(B-7)}$$

Define Fourier transforms by

$$\widetilde{\mathrm{v}}(p) \equiv \int_{z \in T_0} \mathrm{v}(0,z) \exp\{ipz\} \equiv \frac{a^2}{\sum_{\mu=1}^{d} \sin^2 \frac{p^\mu a}{2}} \quad \text{(B-8)}$$

$$\widetilde{u}_j(p) \equiv u_j(0,y) \exp\{ipy\} \quad \text{(B-9)}$$

$$\widetilde{\mathcal{A}}_z^j(p) \equiv \int_{y \in T_j} \mathcal{A}^j(z,y) \exp\{ip(y-z)\} \quad \text{(B-10)}$$

[1] We use the notations $\int_{p \in B_j} = (\frac{2\pi}{a_M})^d \sum_{p \in B_j}$, $\int_{z \in T_j} = a_j^d \sum_{z \in T_j}$

$$\widetilde{A}^y_{j,j-1}(p) \equiv \int_{x\in T_{j-1}} A^j(y,x)\exp\{ip(x-y)\} \tag{B-11}$$

$$\widetilde{v}^j(p;m_1,m_2) \equiv \int_{x_2\in T_{j-1}} v^j(x_1+m_1a_j, x_2+m_2a_j)\exp\{ip(x_2-x_1)+ip(m_2-m_1)a_j\} \tag{B-12}$$

for $m_1, m_2 \in \mathbf{Z}^d, |m_1^\mu|, |m_2^\mu| < \frac{L}{2}$. Furthermore define

$$F_j(p) \equiv \prod_{\mu=1}^{d}\left[\frac{\sin\frac{p^\mu a_j}{2}}{\sin\frac{p^\mu a}{2}}\right], \quad F_{j-1,j}(p) = \frac{F_{j-1}(p)}{F_j(p)}. \tag{B-13}$$

LEMMA B.1. For $y \in \Lambda_j$ we have

$$\int_{z\underline{\in} y} \exp\{ipz\} = a^d F_j(p)\exp\{ipy\}. \tag{B-14}$$

*Proof :*

$$\int_{z\underline{\in} y} \exp\{ipz\} = a^d \sum_{\substack{m\in \mathbf{Z}^d\\ |m^\mu|<L^{-j}/2}} \exp\{ip(y+ma)\} =$$

$$= a^d \prod_{\mu=1}^{d}\Big[\sum_{m^\mu:\ |m^\mu|<L^{-j}/2} (\exp\{ip^\mu a\})^{m^\mu}\Big]\exp\{ipy\} = a^d F_j(p)\exp\{ipy\}. \quad \surd$$

LEMMA B.2. For $z \in$ *base*, $y \in \Lambda_j$, $m_1, m_2 \in \mathbf{Z}^d$ with $|m_1^\mu|, |m_2^\mu| < L/2$ we have

$$\widetilde{u}^j(p) = L^{2jd} \sum_{\substack{m\in \mathbf{Z}^d:\\ |m^\mu|<L^{-j}/2}} F_j^2(p+\frac{2\pi}{a_j}m)\widetilde{v}(p+\frac{2\pi}{a_j}m) \tag{B-15}$$

$$\widetilde{A}^j_z(p) = L^{jd} \sum_{\substack{m\in \mathbf{Z}^d:\\ |m^\mu|<L^{-j}/2}} F_j(p+\frac{2\pi}{a_j}m)\frac{\widetilde{v}(p+\frac{2\pi}{a_j}m)}{\widetilde{u}_j(p)}\exp\{i\frac{2\pi}{a_j}mz\} \tag{B-16}$$

$$\widetilde{A}^y_{j,j-1}(p) = L^{-d} \sum_{\substack{m\in \mathbf{Z}^d:\\ |m^\mu|<L/2}} F_{j-1,j}(p+\frac{2\pi}{a_{j-1}}m)\frac{\widetilde{u}_j(p+\frac{2\pi}{a_{j-1}}m)}{\widetilde{u}_{j-1}(p)}\exp\{i\frac{2\pi}{a_{j-1}}my\} \tag{B-17}$$

$$\widetilde{v}^j(p;m_1,m_2) = \frac{L^{-2d}}{2} \sum_{\substack{n_1,n_2\in \mathbf{Z}^d:\\ |n_1^\mu|,|n_2^\mu|<L/2}} \frac{\widetilde{u}_j(p+\frac{2\pi}{a_{j-1}}n_1)\widetilde{u}_j(p+\frac{2\pi}{a_{j-1}}n_2)}{\widetilde{u}_{j-1}(p)}$$
$$\left[F_{j-1,j}(p+\frac{2\pi}{a_{j-1}}n_1) - F_{j-1,j}(p+\frac{2\pi}{a_{j-1}}n_2)\right]^2 \exp\{i\frac{2\pi}{L}(n_1m_1-n_2m_2)\}. \tag{B-18}$$



*Proof :* We have

$$\widetilde{u}_j(p) = \int_{y\in T_j} u_j(0,y)\exp\{ipy\} =$$
$$= \int_{y\in T_j} a_j^{-2d}\int_{z_1\underline{\in} 0}\int_{z_2\underline{\in} y} v(z_1,z_2)\exp\{ipy\} =$$
$$= \int_{y\in T_j} a_j^{-2d}\int_{z_1\underline{\in} 0}\int_{z_2\underline{\in} y}\int_{p'\in B_0} \widetilde{v}(p')\exp\{ip'(z_1-z_2)+ipy\} =$$
$$= L^{2jd}\int_{y\in T_j}\int_{p'\in B_0}\widetilde{v}(p')F_j^2(p)\exp\{i(p-p')y\} =$$
$$= L^{2jd}\sum_{\substack{m\in \mathbf{Z}^d:\\ |m^\mu|<L^{-j}/2}} F_j^2(p+\frac{2\pi}{a_j}m)\widetilde{v}(p+\frac{2\pi}{a_j}m)$$

and

$$\widetilde{A}^j_z(p) = \int_{y\in T_j} A^j(z,y)\exp\{ip(y-z)\} =$$
$$= \int_{y\in T_j} a_j^{-d}\int_{y'\in T_j}\int_{z'\underline{\in} y'} v(z,z')u_j^{-1}(y',y)\exp\{ip(y-z)\} =$$
$$= a_j^{-d}\int_{y\in T_j}\int_{y'\in T_j}\int_{z'\underline{\in} y'}\int_{p'\in B_0}\int_{p''\in B_j}\frac{\widetilde{v}(p')}{\widetilde{u}_j(p'')}\exp\{ip''(y'-y)+ip'(z-z')+ip(y-z)\} =$$
$$= a_j^{-d}\int_{y'\in T_j}\int_{z'\underline{\in} y'}\int_{p'\in B_0}\frac{\widetilde{v}(p')}{\widetilde{u}_j(p)}\exp\{ipy'+ip'(z-z')-ipz\} =$$
$$= L^{jd}\int_{y'\in T_j}\int_{p'\in B_0}\frac{\widetilde{v}(p')}{\widetilde{u}_j(p)}F_j(p')\exp\{ipy'+ip'z-ipz-ip'y'\} =$$
$$= L^{jd}\sum_{\substack{m\in \mathbf{Z}^d:\\ |m^\mu|<L^{-j}/2}} F_j(p+\frac{2\pi}{a_j}m)\frac{\widetilde{v}(p+\frac{2\pi}{a_j}m)}{\widetilde{u}_j(p)}\exp\{i\frac{2\pi}{a_j}mz\}$$

and

$$\widetilde{A}^y_{j,j-1}(p) = \int_{x\in T_{j-1}} A^j(y,x)\exp\{ip(x-y)\} =$$
$$= \int_{x\in T_{j-1}} a_{j-1}^{-d}\int_{x'\in T_{j-1}}\int_{y'\underline{\in} x'} u_j(y,y')u_{j-1}^{-1}(x',x)\exp\{ip(x-y)\} =$$
$$= L^{-d}\int_{x'\in T_{j-1}}\int_{p'\in B_j}\frac{\widetilde{u}_j(p')}{\widetilde{u}_{j-1}(p)}\exp\{ipx'+ip'y-ipy-ip'x'\} =$$
$$= L^{-d}\sum_{\substack{m\in \mathbf{Z}^d:\\ |m^\mu|<L/2}} F_{j-1,j}(p+\frac{2\pi}{a_{j-1}}m)\frac{\widetilde{u}_j(p+\frac{2\pi}{a_{j-1}}m)}{\widetilde{u}_{j-1}(p)}\exp\{i\frac{2\pi}{a_{j-1}}my\}.$$

The last equality is proven by

$$\int_{x'\in T_{j-1}} u_j C^*_{j-1,j} u_{j-1}^{-1} C_{j-1,j} u_j (x+m_1a_j, x'+m_2a_j)\exp\{ip(x'-x)-ip(m_2-m_1)a_j\} =$$

$$L^{-2d}\sum_{\substack{n_1,n_2\in \mathbf{Z}^d:\\ |n_1^\mu||n_2^\mu|<L/2}} \frac{\widetilde{u}_j(p+\frac{2\pi}{a_{j-1}}n_1)\widetilde{u}_j(p+\frac{2\pi}{a_{j-1}}n_2)}{\widetilde{u}_{j-1}(p)} F_{j-1,j}(p+\frac{2\pi}{a_{j-1}}n_1)F_{j-1,j}(p+\frac{2\pi}{a_{j-1}}n_2)$$

$$\exp\{i\frac{2\pi}{L}(n_1m_1-n_2m_2)\}$$

and

$$\widetilde{u}^{j-1}(p) = L^{-2d}\sum_{\substack{n'\in \mathbf{Z}^d:\\ |n'^\mu|<L/2}} F^2_{j-1,j}(p+\frac{2\pi}{a_{j-1}}n')\widetilde{u}_j(p+\frac{2\pi}{a_{j-1}}n').$$

Thus

$$\widetilde{v}^j(p;m_1,m_2) = \int_{x'\in T_{j-1}} (u_j - u_jC^*_{j-1,j}u_{j-1}^{-1}C_{j-1,j}u_j)(x+m_1a_j,x'+m_2a_j)$$

$$\exp\{ip(x'-x)+ip(m_2-m_1)a_j\} =$$

$$= \frac{L^{-2d}}{2}\sum_{\substack{n_1,n_2\in \mathbf{Z}^d:\\ |n_1^\mu|,|n_2^\mu|<L/2}} \frac{\widetilde{u}_j(p+\frac{2\pi}{a_{j-1}}n_1)\widetilde{u}_j(p+\frac{2\pi}{a_{j-1}}n_2)}{\widetilde{u}_{j-1}(p)}$$

$$[F_{j-1,j}(p+\frac{2\pi}{a_{j-1}}n_1)-F_{j-1,j}(p+\frac{2\pi}{a_{j-1}}n_2)]^2\exp\{i\frac{2\pi}{L}(n_1m_1-n_2m_2)\}. \surd$$

LEMMA B.3. *There exist constants $q$, $0<q<1$ and $c$ such that for all $j\leq 0$, $m\in \mathbf{Z}^d$ with $|m^\mu|<L^{-j}/2$ and $|Re p^\mu|<\frac{\pi}{a_j}$, $|Im p^\mu|<q\frac{\pi}{a_j}$ for all $\mu\in\{1,\dots,d\}$*

$$|F_j(p+\frac{2\pi}{a_j}m)|\leq cL^{-jd}\prod_{\mu=1}^d\frac{1}{1+|m^\mu|} \tag{B-19}$$

$$|\frac{\widetilde{v}(p+\frac{2\pi}{a_j}m)}{\widetilde{u}_j(p)}|\leq\frac{c}{1+|m|^2} \tag{B-20}$$

$$|\frac{\widetilde{u}_j(p+\frac{2\pi}{a_j}m)}{\widetilde{u}_{j-1}(p)}|\leq\frac{c}{1+|m|^2} \tag{B-21}$$

*and for $m\neq 0$*

$$|\widetilde{u}_j(p+\frac{2\pi}{a_j}m)|\leq\frac{ca_j^2}{1+|m|^2} \tag{B-22}$$

*and for $j<0$*

$$F_{j,j+1}(p+\frac{2\pi}{a_j}m)|\leq cL^d\prod_{\mu=1}^d\frac{1}{1+|m^\mu|}. \tag{B-23}$$



*Proof :* We have

$$|\frac{\sin\frac{p^\mu a_j}{2}}{\sin\frac{p^\mu a}{2}}|^2 = \frac{\sin^2\frac{Re\ p^\mu a_j}{2}+\sinh^2\frac{Im\ p^\mu a_j}{2}}{\sin^2\frac{Re\ p^\mu a}{2}+\sinh^2\frac{Im\ p^\mu a}{2}}\leq$$

$$\leq\frac{\sin^2\frac{Re\ p^\mu a_j}{2}}{\sin^2\frac{Re\ p^\mu a}{2}}+\frac{\sinh^2\frac{Im\ p^\mu a_j}{2}}{\sinh^2\frac{Im\ p^\mu a}{2}}\leq(K\frac{a_j}{a})^2=(KL^{-j})^2$$

for $p^\mu\in C_{j,q}\equiv\{z\in\mathbf{C}|\ |Re\ z|\leq\frac{\pi}{a_j},\ |Im\ z|\leq q\frac{\pi}{a_j}\}$. Thus

$$|F_j(p)|\leq K^dL^{-jd}. \tag{B-24}$$

Use that for $|x|\leq\frac{\pi}{2}$

$$|\sin x|\geq(1-\frac{\pi}{4})|x|$$

and

$$\frac{L^j\pi|m^\mu|}{2}\leq|L^j(\frac{Re\ p^\mu a_j}{2}+\pi m^\mu)|\leq\frac{\pi}{2}.$$

Thus

$$\sin(\frac{Re\ p^\mu a}{2}+\pi m^\mu)\geq\frac{\pi}{4}(1-\frac{\pi}{4})L^j(1+|m^\mu|) \tag{B-25}$$

for $m^\mu\neq 0$. Thus, using (24) and (25),

$$|F_j(p+\frac{2\pi}{a_j}m)|\leq cL^{-jd}\prod_{\mu=1}^d\frac{1}{1+|m^\mu|}.$$

For $p$ real and $|p^\mu|\leq\frac{\pi}{a_j}+\frac{\widetilde{q}\pi}{a_j}$ ($\widetilde{q}$ small) we have

$$|\widetilde{v}(p+\frac{2\pi}{a_j}m)|\leq K_1a_j^2(1+|m|^2)^{-1}$$

and

$$|\widetilde{u}_j(p)|\geq K_2a_j^2.$$

Thus

$$|\frac{\widetilde{v}(p+\frac{2\pi}{a_j}m)}{\widetilde{u}_j(p)}|\leq\frac{K}{1+|m|^2}.$$

Since

$$|\frac{\widetilde{v}(p+\frac{2\pi}{a_j}m)}{\widetilde{u}_j(p)}|\leq\frac{K'}{1+|m|^2}$$

for $p\in Q\equiv\{z\in\mathbf{C}|\ \exists x\in\mathbf{R},z'\in\mathbf{C}:\ z=x+z',|x|\leq\frac{\pi}{a_j},|z'|\leq\frac{\widetilde{q}\pi}{a_j}\}$ and $\widetilde{v}(p+\frac{2\pi}{a_j}m)/\widetilde{u}_j(p)$ is analytic in $Q$ we obtain (20). (21),(22) and (23) are similarly shown. $\surd$

*Proof of Lemma 4.1 :* We have

$$\mathcal{A}^j(z,y) = \int_{p\in B_j} \tilde{\mathcal{A}}^j_z(p)\exp\{ip(z-y)\} = \sum_{m\in\mathbf{Z}^d}\int_{p\in\mathbf{R}^d:\ |p^\mu|\le\frac{\pi}{a_j}} \tilde{\mathcal{A}}^j_z(p)\exp\{ip(z-y+ma_M)\} \equiv$$
$$\equiv \sum_{m\in\mathbf{Z}^d} \mathcal{A}^j_{cont}(z, y+ma_M). \tag{B-26}$$

By lemma 3 we have

$$|\tilde{\mathcal{A}}^j_z(p)| \le C \tag{B-27}$$

for $p^\mu \in C_{j,q} \equiv \{z\in\mathbf{C}|\ |Re\ z| \le \frac{\pi}{a_j},\ |Im\ z| \le q\frac{\pi}{a_j}\}$ and $\tilde{\mathcal{A}}^j_z(p)$ is analytic in $C_{j,q}$. Lifting the contour in $p$–integration shows

$$|\mathcal{A}^j_{cont}(z,y)| \le C' a_j^{-d}\exp\{-\frac{q\pi}{2}a_j^{-1}|z-y|\} \tag{B-28}$$

and by (26)

$$|\mathcal{A}^j(z,y)| \le K_1 a_j^{-d}\exp\{-K_2 a_j^{-1}|z-y|\}. \tag{B-29}$$

We have for $p\in C_{j,q}$

$$|\tilde{\mathcal{A}}^j_{z_1}(p) - \tilde{\mathcal{A}}^j_{z_2}(p)| = |L^{jd} \sum_{\substack{m\in\mathbf{Z}^d:\\ |m^\mu|<L^{-j}/2}} F_j(p+\frac{2\pi}{a_j}m)\frac{\tilde{v}(p+\frac{2\pi}{a_j}m)}{\tilde{u}_j(p)}$$

$$(\exp\{i\frac{2\pi}{a_j}mz_1\} - \exp\{i\frac{2\pi}{a_j}mz_2\})| \le \sum_{\substack{m\in\mathbf{Z}^d:\\ |m^\mu|<L^{-j}/2}} C^2\left[\prod_{\mu=1}^d \frac{1}{1+|m^\mu|}\right]\frac{1}{1+|m|^2}\frac{2\pi}{a_j}|m||z_1-z_2| \le$$
$$\le K' a_j^{-1}|z_1-z_2|. \tag{B-30}$$

Furthermore

$$\mathcal{A}^j(z_1,y) - \mathcal{A}^j(z_2,y) \le \int_{p\in B_j}(\tilde{\mathcal{A}}^j_{z_1}(p)\exp\{ipz_1\} - \tilde{\mathcal{A}}^j_{z_2}(p)\exp\{ipz_2\})\exp\{-ipy\} =$$
$$\sum_{m\in\mathbf{Z}^d}\int_{p\in\mathbf{R}^d:\ |p^\mu|\le\frac{\pi}{a_j}} [(\tilde{\mathcal{A}}^j_{z_1}(p) - \tilde{\mathcal{A}}^j_{z_2}(p))\exp\{ipz_1\}+$$
$$+\tilde{\mathcal{A}}^j_{z_2}(p))(\exp\{ipz_1\} - \exp\{ipz_2\})]\exp\{-ip(y+ma_M)\}. \tag{B-31}$$

By (30,31) and lifting the contour in $p$–integration follows

$$|\mathcal{A}^j(z_1,y) - \mathcal{A}^j(z_2,y)| \le K_1 a_j^{-d-1}|z_1-z_2|[\exp\{-K_2a_j^{-1}|z_1-y|\} + \exp\{-K_2a_j^{-1}|z_2-y|\}]. \tag{B-32}$$

Define

$$\tilde{\mathcal{A}}^j_2(z_1,z_2;p) \equiv \tilde{\mathcal{A}}^j_{z_1}(p) - \tilde{\mathcal{A}}^j_{z_2}(p) - \sum_{\mu=1}^d (z_1^\mu - z_2^\mu)\nabla_\mu\tilde{\mathcal{A}}^j_z(p)|_{z=z_2}. \tag{B-33}$$



Then we get

$$|\tilde{\mathcal{A}}^j_2(z_1,z_2;p)| \le \sum_{\substack{m\in\mathbf{Z}^d:\\ |m^\mu|<L^{-j}/2}} C^2\left[\prod_{\mu=1}^d\frac{1}{1+|m^\mu|}\right]\frac{1}{1+|m|^2}(\frac{2\pi}{a_j})^2|m|^2|z_1-z_2|^2 \le$$
$$K'|j|^d a_j^{-2}|z_1-z_2|^2. \tag{B-34}$$

This implies

$$|\mathcal{A}^j_2(z_1,z_2;y)| \le K_1 a_j^{-d-2+\epsilon}|z_1-z_2|^{2-\epsilon}[\exp\{-K_2a_j^{-1}|z_1-y|\} + \exp\{-K_2a_j^{-1}|z_2-y|\}].\ \surd \tag{B-35}$$

LEMMA B.4. *There exist positive constants $K_1$ and $K_2$ such that for all $j\le 0$ and $\mu,\nu\in\{1,\dots,d\}$ and $z\in base, y\in\Lambda_j$*

$$|\nabla_\mu\mathcal{A}^j(z,y)| \le K_1 a_j^{-d-1}\exp\{-K_2a_j^{-1}|z-y|\} \tag{B-36}$$

$$|\nabla_\mu\nabla_\nu\mathcal{A}^j(z,y)| \le K_1|j|^d a_j^{-d-2}\exp\{-K_2a_j^{-2}|z-y|\}. \tag{B-37}$$

LEMMA B.5. *There exist positive constants $K_1$ and $K_2$ such that for all $j\le 0$ and $z_1,z_2\in base, y_1,y_2\in\Lambda_j$*

$$|v^j(y_1,y_2)| \le K_1 a_j^{2-d}\exp\{-K_2a_{j-1}^{-1}|y_1-y_2|\} \tag{B-38}$$

$$|\mathrm{v}^j(z_1,z_2)| \le K_1 a_j^{2-d}\exp\{-K_2a_{j-1}^{-1}|z_1-z_2|\} \tag{B-39}$$

*Proof :* We have for $y_1 = m_1a_j + x_1,\ y_2 = m_2a_j + x_2,\ |m_1^\mu|, |m_1^\mu| < L/2$

$$v^j(y_1,y_2) = \int_{p\in B_{j-1}} \tilde{v}^j(p;m_1,m_2)\exp\{ip(y_1-y_2)\} = \sum_{m\in\mathbf{Z}^d} v^j_{cont}(y_1, y_2+ma_M) \tag{B-40}$$

where

$$v^j_{cont}(y_1,y_2) = \sum_{m\in\mathbf{Z}^d}\int_{p\in\mathbf{R}^d:\ |p^\mu|\le\frac{\pi}{a_{j-1}}} \tilde{v}^j(p;m_1,m_2)\exp\{ip(y_1-y_2)\}. \tag{B-41}$$

By lemma 3 follows

$$|\tilde{v}^j(p;m_1,m_2)| \le K' \sum_{\substack{n_1\in\mathbf{Z}^d:\\ |n_1^\mu|<L/2}} \sum_{\substack{n_2\in\mathbf{Z}^d:\\ n_2\neq 0, |n_2^\mu|<L/2}} \frac{a_j^2}{(1+|n_1|^2)(1+|n_2|^2)} \le K a_j^2. \tag{B-42}$$

Lifting the contour in $p$–integration by $iK_2a_{j-1}$ yields the bound (38). (38) and (29) implies (39). $\surd$

LEMMA B.6. *For $y \in \Lambda_j$, $z \in base$, $x \in \Lambda_{j-1}$ we have*

$$\int_{y' \in \Lambda_j} U_j(y,y') = \widetilde{v}(0), \qquad \int_{y \in \Lambda_j} \mathcal{A}^j(z,y) = 1$$

$$\int_{x \in \Lambda_{j-1}} \mathcal{A}_{j,j-1}(y,x) = 1. \qquad \int_{y' \in \Lambda_j} v^j(y,y') = 0.$$

*Proof* : The assertion is implied by

$$\widetilde{u}^j(0) = \widetilde{v}(0), \ \widetilde{\mathcal{A}}^j_z(0) = 1, \ \widetilde{\mathcal{A}}^y_{j,j-1}(0) = 1, \ \widetilde{v}^j(0; m_1, m_2) = 0$$

for all $z \in base$, $y \in \Lambda_j$, $m_1, m_2 \in \mathbf{Z}^d$, $|m_1^\mu|, |m_2^\mu| < L/2$. $\surd$

## C. Tree Graph Formulas

In this appendix we give a proof of proposition 2.1 (section 4.2). Let $\Lambda$ be a denumerable set (e.g. lattice $\mathbf{Z}^d$). For finite subsets $X = \{x_1, \ldots, x_m\}$ of $\Lambda$ consider partition functions

$$Z(X) = \exp\Big\{ \sum_{(xy) \in X^*} u_{xy} + \sum_{x \in X} u_x \Big\} \tag{C-1}$$

where $X^* \equiv \{(x_i x_j) | i,j \in \{1, \ldots, m\}, i < j\}$ and $u_{xy} = u_{yx}$. Polymer activities $A$ are defined by the relations

$$Z(X) = \sum_{X = \sum P} \prod_P A(P) \tag{C-2}$$

for all finite nonempty subsets of $\Lambda$. For the next lemma suppose $u_x = 0 \ \forall x \in \Lambda$.

LEMMA C.1. *For $X = \{x_1, \ldots, x_m\}$, and all $N \geq 1$ we have*

$$Z(X) = \sum_{n=1}^{N-1} Q_n(X) + R_N(X) \tag{C-3}$$

*with*

$$Q_n(X) \equiv \sum_{\substack{\widetilde{x}: \underline{n} \to X: \\ \widetilde{x}(1) = x, \ \widetilde{x} \text{ injective}}} \ \sum_{\substack{\eta: \{2,\ldots,n\} \to \{1,\ldots,n-1\}: \\ \eta(i) < i \ \forall i}} \int_0^1 ds_1 \ldots ds_{n-1} \prod_{i=2}^n [s_{i-2} \ldots s_{\eta(i)} u_{\widetilde{x}(i)\widetilde{x}(\eta(i))}]$$
$$\exp\Big\{ \sum_{\substack{i,j \in \underline{n}: \\ i<j}} u_{\widetilde{x}(i)\widetilde{x}(j)} s_i s_{i+1} \ldots s_{j-1} + \sum_{(xy) \in (X - \widetilde{x}(\underline{n}))^*} u_{xy} \Big\} \tag{C-4}$$



and

$$R_N(X) \equiv \sum_{\substack{\widetilde{x}: \underline{N} \to X: \\ \widetilde{x}(1) = x, \ \widetilde{x} \text{ injective}}} \ \sum_{\substack{\eta: \{2,\ldots,N\} \to \{1,\ldots,N-1\}: \\ \eta(i) < i \ \forall i}} \int_0^1 ds_1 \ldots ds_{N-1} \prod_{i=2}^N [s_{i-2} \ldots s_{\eta(i)} u_{\widetilde{x}(i)\widetilde{x}(\eta(i))}]$$
$$\exp\Big\{ \sum_{\substack{i,j \in \underline{N}: \\ i<j}} u_{\widetilde{x}(i)\widetilde{x}(j)} s_i s_{i+1} \ldots s_{j-1} + \sum_{\substack{i \in \underline{N}: \\ x' \in X - \widetilde{x}(\underline{N})}} u_{\widetilde{x}(i)x'} s_i s_{i+1} \ldots s_{N-1} + \sum_{(xy) \in (X - \widetilde{x}(\underline{N}))^*} u_{xy} \Big\}. \tag{C-5}$$

*Proof (by induction)* : For $N = 1$ we have $Z(X) = R_1(X)$, i.e. eq. (3) holds for $N = 1$. Suppose that eq. (3) is valid. We have

$$\exp\Big\{ \sum_{\substack{i,j \in \underline{N}: \\ i<j}} u_{\widetilde{x}(i)\widetilde{x}(j)} s_i s_{i+1} \ldots s_{j-1} + \sum_{\substack{i \in \underline{N}: \\ x' \in X - \widetilde{x}(\underline{N})}} u_{\widetilde{x}(i)x'} s_i s_{i+1} \ldots s_{N-1} + \sum_{(xy) \in (X - \widetilde{x}(\underline{N}))^*} u_{xy} \Big\} =$$
$$= \exp\Big\{ \sum_{\substack{i,j \in \underline{N}: \\ i<j}} u_{\widetilde{x}(i)\widetilde{x}(j)} s_i s_{i+1} \ldots s_{j-1} + \sum_{(xy) \in (X - \widetilde{x}(\underline{n}))^*} u_{xy} \Big\} + \int_0^1 ds_N \partial_{s_N}$$
$$\exp\Big\{ \sum_{\substack{i,j \in \underline{N}: \\ i<j}} u_{\widetilde{x}(i)\widetilde{x}(j)} s_i s_{i+1} \ldots s_{j-1} + \sum_{\substack{i \in \underline{N}: \\ x' \in X - \widetilde{x}(\underline{N})}} u_{\widetilde{x}(i)x'} s_i s_{i+1} \ldots s_N + \sum_{(xy) \in (X - \widetilde{x}(\underline{N}))^*} u_{xy} \Big\} =$$
$$= \exp\Big\{ \sum_{\substack{i,j \in \underline{N}: \\ i<j}} u_{\widetilde{x}(i)\widetilde{x}(j)} s_i s_{i+1} \ldots s_{j-1} + \sum_{(xy) \in (X - \widetilde{x}(\underline{N}))^*} u_{xy} \Big\} +$$
$$+ \sum_{\widetilde{x}(N+1) \in X - \widetilde{x}(\underline{N})} \ \sum_{\eta(N+1) \in \underline{N}} s_i \ldots s_{N-1} \, u_{\widetilde{x}(i)\widetilde{x}(N+1)} \int_0^1 ds_N \partial_{s_N}$$
$$\exp\Big\{ \sum_{\substack{i,j \in \underline{N+1}: \\ i<j}} u_{\widetilde{x}(i)\widetilde{x}(j)} s_i s_{i+1} \ldots s_{j-1} + \sum_{\substack{i \in \underline{N+1}: \\ x' \in X - \widetilde{x}(\underline{N+1})}} u_{\widetilde{x}(i)x'} s_i s_{i+1} \ldots s_N +$$
$$+ \sum_{(xy) \in (X - \widetilde{x}(\underline{N+1}))^*} u_{xy} \Big\}. \tag{C-6}$$

Insertion of eq. (6) into (5) gives

$$R_N(X) = Q_N(X) + R_{N+1}(X)$$

and by induction hypothesis

$$Z(X) = \sum_{n=1}^{N-1} Q_n(X) + R_N(X) = \sum_{n=1}^{N} Q_n(X) + R_{N+1}(X). \ \surd$$

Since $R_N(X) = 0$ for $N > |X|$ we obtain by lemma C.1

$$Z(X) = \sum_{P: \ x \in P \subseteq X} A(P) Z(X - P) \tag{C-7}$$

where $A(P)$ is represented by the following tree graph formula

LEMMA C.2.

$$A(P) = \sum_{\substack{\tilde{x}:\underline{n} \to P: \\ \tilde{x}(1)=x,\ \tilde{x}\ \text{injective}}} \sum_{\substack{\eta:\{2,\dots,n\}\to\{1,\dots,n-1\}: \\ \eta(i)<i\ \forall i}} \int_0^1 ds_1 \dots ds_{n-1} f(\eta|s) \prod_{i=2}^{n} [u_{\tilde{x}(i)\tilde{x}(\eta(i))}]$$

$$\exp\{ \sum_{\substack{i,j \in \underline{n} \\ i<j}} u_{\tilde{x}(i)\tilde{x}(j)} f[\tilde{x}, s]_{\tilde{x}(i)\tilde{x}(j)} + \sum_{x \in P} u_x \} \qquad \text{(C-8)}$$

*for all $P$ with $x \in P$, $|P| = n$.*

*Proof of Proposition 2.1 :* Define a translation operator $T[t]$ by

$$T_X[t] Z_{t'}(X) = Z_{t+t'}(X).$$

For transparency let us write

$$T_X[t] = \exp\{ \sum_{(xy)\in X^*} t_{xy} \frac{\partial}{\partial t_{xy}} \}.$$

Lemma C.2 implies

$$T_X[t] = \sum_{X=\sum P} \prod_P \hat{T}_P[t]$$

with

$$\hat{T}_P[t] = \sum_{\tau \in T(P)} \sum_{\substack{\tilde{x},\eta: \\ \tau(\tilde{x},\eta)=\tau}} \int_0^1 ds_1 \dots ds_{n-1} f(\eta|s) \{ \prod_{(xy)\in\tau} \frac{\partial}{\partial t_{xy}} \} T_P[t[\tilde{x}, s]].$$

Thus, using property (iii) for $Z_t(X)$,

$$Z(X) = T_X[t] Z_0(X) = \sum_{X=\sum P} \prod_P [\hat{T}_P[t] Z_0(P)].$$

Therefore

$$A(P) = \hat{T}_P[t] Z_0(P) = \sum_{\tau \in T(P)} \sum_{\substack{\tilde{x},\eta: \\ \tau(\tilde{x},\eta)=\tau}} \int_0^1 ds_1 \dots ds_{n-1} f(\eta|s) \{ \prod_{(xy)\in\tau} \frac{\partial}{\partial t_{xy}} \} Z_t(P)|_{t=t[\tilde{x},s]}. \ \surd$$



## D. Perturbation Expansions

Consider the partition function

$$Z(\Psi) = \int d\mu_v(\Phi) \exp\{W(\Phi + \Psi)\}$$

with free propagator v, interaction $W$ and external field $\Psi : \mathbf{R}^d \to \mathbf{C}$. In this appendix formal expansions in the number of free propagators and in the number of interactions for the partition function $Z$ and the free energy $\ln Z$ are presented.

### D.1. Free Propagator Expansion

PROPOSITION D.1. *The formal expansion in the number of the free propagator for the partition function reads*

$$Z(\Psi) = \exp\{W(\Psi)\} \Big\{ 1 + \sum_{L=1}^{\infty} \sum_{m\geq 1} \sum_{\underline{2L}=\sum_{a=1}^{m} I_a} \frac{1}{L!} \int_{z_1,\dots z_{2L}} \prod_{i=1}^{L} [\frac{1}{2} v(z_{2i-1}, z_{2i})]$$

$$\prod_{a=1}^{m} [( \prod_{i\in I_a} \frac{\delta}{\delta\Psi(z_i)} ) W(\Psi)] \Big\}. \qquad \text{(D-1)}$$

*Proof :* We have

$$Z(\Psi) = \exp\{\frac{1}{2}\Delta_v\} \exp\{W(\Psi)\}$$

with

$$\Delta_v \equiv (\frac{\partial}{\partial\Psi}, v \frac{\partial}{\partial\Psi}) \equiv \int_{z_1,z_2} \frac{\delta}{\delta\Psi(z_1)} v(z_1, z_2) \frac{\delta}{\delta\Psi(z_2)}.$$

Thus

$$Z(\Psi) = \Big\{ \sum_{L=0}^{\infty} \frac{1}{L!} (\frac{1}{2}\Delta_v)^L \Big\} \exp\{W(\Psi)\} =$$

$$= \Big\{ 1 + \sum_{L=1}^{\infty} \frac{1}{L!} \int_{z_1,\dots z_{2L}} \prod_{i=1}^{L} [\frac{1}{2} v(z_{2i-1}, z_{2i})]$$

$$\prod_{i=1}^{2L} \frac{\delta}{\delta\Psi(z_i)} \Big\} \exp\{W(\Psi)\}.$$

Since

$$\prod_{i=1}^{2L} \frac{\delta}{\delta\Psi(z_i)} \exp\{W(\Psi)\} = \exp\{W(\Psi)\} \sum_{m\geq 1} \sum_{\underline{2L}=\sum_{a=1}^{m} I_a} \prod_{a=1}^{m} [( \prod_{i\in I_a} \frac{\delta}{\delta\Psi(z_i)} ) W(\Psi)]$$

we obtain our assertion. $\surd$

Disjoint subsets $I_a$ and $I_b$ of $\underline{2L} = \{1, \dots, 2L\}$ are called *compatible* ($I_a \sim I_b$) iff there exists no $i \in \{1, \dots, L\}$ such that $2i \in I_a$ and $2i-1 \in I_b$ or $2i-1 \in I_a$ and $2i \in I_b$. Consider a partition $\sum_{a=1}^{m} I_a = \underline{2L}$. We define a graph $\gamma(I_1, \dots, I_m)$ with vertices $I_1, \dots, I_m$ by lines $(I_a I_b)$, $a \neq b$, if $I_a$ and $I_b$ are not compatible.

PROPOSITION D.2. *The formal expansion in the number of free propagators for the free energy reads*

$$\ln Z(\Psi) = W(\Psi) + \sum_{L=1}^{\infty}\sum_{m\geq 1} \sum_{\substack{2L=\sum_{a=1}^{m} l_a \\ \tau(l_1,\ldots,l_m)\ \text{connected}}} \frac{1}{L!}\int_{z_1,\ldots,z_{2L}} \prod_{i=1}^{L}[\frac{1}{2}v(z_{2i-1},z_{2i})] \prod_{a=1}^{m}[(\prod_{i\in I_a}\frac{\delta}{\delta\Psi(z_i)})W(\Psi)]. \qquad \text{(D-2)}$$

There is a simple graphical representation for the series expansion (D-2) of the free energy. For each term in the sum on the right hand side of eq.(D-2) draw $m$ bubbles and $L$ lines representing the $m$ interaction terms $W$ resp. the free propagators v. Connect two bubbles $W$ by a line v if they are connected by a propagator kernel in the corresponding term.

### D.2. Interaction Expansion

For expectation values $< \cdots >$ define truncated expectation values by

$$< W_1 >^T \equiv < W_1 > \qquad \text{(D-3a)}$$

$$< W_1; \ldots; W_N >^T \equiv < W_1 \ldots W_N > - \sum_{n\geq 2}\sum_{\underline{N}=\sum_{a=1}^{n} I_a} \prod_{a=1}^{n} < \prod_{i\in I_a}[W_i; \,] >^T \qquad \text{(D-3b)}$$

for all $N \geq 2$. Define the Gaussian expectation value

$$< \cdots >_v \equiv \int d\mu_v(\Phi)(\ldots). \qquad \text{(D-4)}$$

PROPOSITION D.3. *The formal expansion in the number of interactions $W$ for the free energy is*

$$\ln Z(\Psi) = \sum_{N\geq 1}\frac{1}{N!} < \prod_{a=1}^{N}[W; \,] >_v . \qquad \text{(D-5)}$$

*Proof :*

$$Z(\Psi) = < \exp\{W\} >_v = \sum_{M\geq 0}\frac{1}{M!} < W^M >_v =$$

$$= \sum_{M\geq 0}\sum_{\substack{k_1,\ldots,k_M\in\mathbf{N}: \\ \sum_{m=1}^{M} mk_m = M}} \prod_{m=1}^{M}[\frac{1}{k_m!}(\frac{1}{m!} < \prod_{i=1}^{m}[W; \,] >_v)^{k_m}] =$$

$$= \prod_{m=1}^{\infty}[\sum_{k\geq 0}\frac{1}{k!}(\frac{1}{m!} < \prod_{i=1}^{m}[W; \,] >_v)^k] =$$

$$= \prod_{m=1}^{\infty}\{\exp\{\sum\frac{1}{m!} < \prod_{i=1}^{m}[W; \,] >_v\}.$$

Taking the logarithm gives the assertion. √



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