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The On-shell Limit of Bosonic Off-shell String Scattering Amplitudes

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Abstract

We investigate the on-shell limit of off-shell amplitudes for the scattering of n bosonic strings. The amplitudes are defined by Polyakov path integrals over bordered world sheets. The on-shell limit is obtained by letting the lengths of the bordering curves shrink to zero. In the leading approximation to this limit it is shown that the amplitudes aquire poles in any order of string perturbation theory at the masses of free string eigenstates. We thus generalize previous results to the case of arbitrary genus, i.e. to an arbitrary number of loops.

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1 Introduction

Usually, in first quantized string theory scattering amplitudes for on-shell particle states are defined as correlation functions of vertex operators in the corresponding two-dimensional conformal field theory. This approach has the disadvantage that no direct off-shell extension is possible, since a naive continuation to off-shell momenta produces a conformal anomaly [1]. Therefore, a different definition of off-shell amplitudes is required. If one does not want to rely on the yet not too well developed string field theory, the Polyakov path integral offers a convenient way of defining the required amplitudes. In this approach one integrates over all world sheets that interpolate between the given string configurations in the embedding space-time. As a first step, Cohen et al. [2] intensively investigated the propagator for the closed bosonic string. The precise mathematical definition of the path integral for the general case, especially a thorough discussion of the boundary conditions, has been given by Jaskólski [3].

The on-shell limit of these amplitudes is defined as follows: One stretches the cylinder-like regions that are attached to the "body" of the world sheet for the scattering process and which correspond to the incoming and outgoing non-interacting string configurations to infinity. One obtains a conformally equivalent picture, if one projects the cylinder-like regions down on the "body" of the world sheet. One then ends up with holes cut in that "body". The stretching to infinity corresponds to a shrinking of those holes to punctures. The scattering amplitudes defined above are functionals of the external string states and the on-shell limit consists of the limit, in which the lengths of the bordering curves of the world sheet shrink to zero.

Thus, first of all, a path integral over bordered world sheets is required. This has been beautifully studied by Alvarez [4] and Jaskólski [3]. Then the remaining finite-dimensional integral over the moduli space of bordered Riemann surfaces has to be investigated in the limit of shrinking bordering curves. For the tree approximation this has been done by Blau et al. [5-7]. Their result was that the amplitudes aquire poles at the masses of the string eigenstates in the lowest approximation to that limit.

In this paper the generalization to arbitrary order in string perturbation theory will be derived and it will be shown that the one-loop result pertains to any number of loops. After that we discuss what happens when other curves on the world sheet shrink that is when intermediate states are put on mass-shell. The result will be that poles occur in the appropriate kinematical variables as is expected from the tree approximation given in [5-7]. In both cases we give up the restriction to spheres with holes as world sheets and study surfaces of arbitrary genus with holes. Our analysis will heavily rely on Selberg trace formula techniques. The derivation of the Selberg trace formula for bordered Riemann surfaces of arbitrary genus is presented elsewhere [8].

Our paper is organized as follows: First we review the definition of the scattering amplitudes. Then we present the treatment of bordered Riemann surfaces via their compact double with emphasis on the period matrix and the moduli for these surfaces. After that we study the behaviour of the scattering amplitudes in the on-shell limit. The discussion of the shrinking of external curves will be followed by an investigation of internal ones. In the last section we summarize our results.

2 The String Scattering Amplitudes

We want to study the scattering of n closed, oriented bosonic strings. Thus n closed, oriented curves $c_1, \ldots, c_n$ in space-time may be given. For ease of computation we work in flat Euclidean space-time and with Euclidean signature metrics on the world sheets.

The scattering amplitude is defined as a path integral over all surfaces that interpolate between $c_1, \ldots, c_n$, weighted with the Boltzmann factor associated with Polyakov's action

$$S_P[X,g] = \frac{1}{2}\int d^2\sigma \sqrt{g} g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\mu \ . \tag{1}$$

The world sheet Σ is parametrized by the embedding functions $X^\mu(\sigma)$ into space-time and carries the Riemannian metric $g_{\alpha\beta}$. The scattering amplitude is then

$$A(c_1, \ldots, c_n) = \sum_{p=0}^{\infty} A_p(c_1 \ldots, c_n) \ , \quad A_p(c_1, \ldots, c_n) = \int \mathcal{D}\pi \int \frac{\mathcal{D}X\mathcal{D}g}{Vol(H)} e^{-S_P[X,g]} \tag{2}$$

Here the sum runs over the genera of the surfaces and $\mathcal{D}\pi$ denotes an integration over all parametrizations of the bordering curves c_i. $H = Weyl(\Sigma) \times Diff(\Sigma)$ is the symmetry group of the action, whose volume has to be devided out in order not to overcount the degrees of freedom. To perform the path integration properly one needs additional counter terms to the action [4]. In the following we shall work in the critical dimension $d = 26$ and shall use a gauge-fixed version, so that we do not explicitly state the counter terms.

First one carries out the X-integration, where the appropriate boundary conditions have to be taken into account. This was carefully done in [2-4]. One splits $X^\mu = X^\mu_{cl} + Y^\mu$, where X^μ_{cl} is a solution of the classical equation of motion $\Delta_g X^\mu_{cl} = 0$; $\Delta_g = g^{-\frac{1}{2}}\partial_\alpha g^{\alpha\beta} g^{\frac{1}{2}} \partial_\beta$ is the Laplace-Beltrami operator on Σ. Y^μ satisfies Dirichlet boundary conditions on $\partial\Sigma = \cup_{i=1}^n c_i$. Thus

$$\int \mathcal{D}X \, e^{-S_P[X,g]} = const. \, [det'(-\Delta)]^{-13} \, e^{-S_P[X_{cl},g]} \ , \tag{3}$$

where Δ is the Dirichlet-Laplace-Beltrami operator on Σ. In the following, $S_P[X_{cl}, g]$ will also be denoted as the "classical" action S_{cl}.

The by now well-known gauge-fixing procedure in the integration over the metrics [9] leads then to an integral over the moduli space $\mathcal{M}_{p,n}$ of bordered Riemann surfaces of genus p with n bordering curves,

$$A_p(c_1, \ldots, c_n) = \int \mathcal{D}\pi \int_{\mathcal{M}_{p,n}} d\mu_{WP} \, N \, [det'(P_1^\dagger P_1)]^{1/2} [det'(-\Delta)]^{-13} e^{-S_{cl}} \ , \tag{4}$$

where $d\mu_{WP}$ is the Weil-Petersson measure on $\mathcal{M}_{p,n}$, N denotes a normalization factor, and $P_1^\dagger P_1$ is the well-known ghost operator. Notice that we are going to discuss "point-like" string states so that the integration over boundary parametrizations presents no problems since it will disappear in further calculations. In general, the formal symbol $\mathcal{D}\pi$ is not well-defined. For a thorough study of this problem and a way out in the cases not under investigation here, see [3].

In the following, $det'(-\Delta)$ will be expressed in terms of the Selberg zeta function, using a Selberg trace formula for bordered surfaces [8]. The "classical" action is expressible via the period matrix for the double of the surface Σ, and $d\mu_{WP}$ is being constructed using the moduli space of the doubled surfaces. When all this is done, we will be able to study the on-shell limit of the expression (4). As stated above, this limit consists of letting the lengths $l_i = l(c_i)$ of the bordering curves c_i, $i = 1, \ldots, n$ shrink to zero. After that, n punctures x_i will be left on the surface and one will be able to Fourier-transform the then x_i-dependent amplitudes. This introduces n momenta p_i and one is left with the momentum-dependent amplitudes. Also the integration over the parametrizations of the bordering curves will be superfluous in this limit.

The only restriction on our result is that the genus p of the world sheet and the number n of bordering curves must satisfy the inequality $\hat{p} := 2p - n - 1 \geq 2$. So the only cases that are excluded are $p = 0$, $n = 1$ and $p = 0$, $n = 2$, respectively. The latter case corresponds to the propagator, which was investigated by different means in [2].

3 Bordered Riemann Surfaces

To deal with the path integral over bordered surfaces Σ we construct the compact double $\hat{\Sigma}$ of Σ. This is a well-known method in mathematics to treat such surfaces. We would like to introduce the necessary concepts and fix our notation in this chapter.

Let $\tilde{\Sigma}$ be a compact Riemann surface of genus p with canonical homology basis $\{a_1, b_1, \ldots, a_p, b_p\}$. Then $\Sigma := \tilde{\Sigma} \backslash \{d_1, \ldots, d_n\}$, d_i being conformal discs, is a bordered Riemann surface with bordering curves $c_i := \partial d_i$. (p, n) is called the signature of Σ. Then a homology basis for Σ is given by $\{a_1, b_1, \ldots, a_p, b_p, c_1, \ldots, c_{n-1}\}$.

To construct the double $\hat{\Sigma}$ of Σ one takes a mirror image $I\Sigma$ of Σ. If $z = z(P)$ is a local coordinate in the vincinity of $P \in \Sigma$, then $-\bar{z}$ is a coordinate in the vincinity of the mirror image P' of P on $I\Sigma$. One now identifies corresponding points on the boundary $\partial\Sigma$ and $\partial(I\Sigma)$ resp.. Thus the map $I: P \to P'$ acts as an involution ($I^2 = 1$) on $\hat{\Sigma}$; it is a reflection in $\partial\Sigma$ and $\hat{\Sigma} = \Sigma \cup I\Sigma$, $\Sigma = \hat{\Sigma}/I$. Then the double $\hat{\Sigma}$ turns out to be a compact Riemann surface of genus $\hat{p} = 2p + n - 1$ [10].

From now on we will only deal with those surfaces Σ, whose double $\hat{\Sigma}$ has genus $\hat{p} \geq 2$. Therefore $\hat{\Sigma}$ may carry a hyperbolic metric according to the uniformisation theorem for Riemann surfaces. The restriction of this metric to Σ is called *intrinsic metric*. There is a theorem [10], which states that the bordering curves c_i of Σ are geodesics in the intrinsic metric.

An alternative point of view may be to look at $\hat{\Sigma}$ as a symmetric surface with an anti-conformal symmetry I. For such surfaces the Fuchsian groups are well investigated [11]. For example, in [11] Sibner essentially shows that one can construct a fundamental domain $\hat{\mathcal{F}}$ in the Poincaré upper half-plane $\mathcal{H}$ for the Fuchsian group $\hat{\Gamma}$ that respects the symmetry I in a direct manner. It is shown, that one can construct a hyperbolic polygon with $4\hat{p} + 2n - 2$ edges with convex, simply connected interior that may serve as a fundamental domain. The involution I on $\hat{\Sigma}$ is represented as a reflection in the imaginary axis, $I: z \to -\bar{z}$. One of the bordering curves, say c_n, is mapped on the imaginary axis, the others are among the edges of the fundamental polygon.

An important quantity of $\hat{\Sigma}$ that is needed in the subsequent calculations is the period matrix $\hat{\Omega}$. This will be constructed as in [12]. Let $\{a_1, b_1, \ldots, a_p, b_p, c_1, \ldots, c_{n-1}\}$ be a canonical homology basis for Σ. Then build up a corresponding basis $\{\hat{a}_1, \hat{b}_1, \ldots, \hat{a}_{\hat{p}}, \hat{b}_{\hat{p}}\}$ for $\hat{\Sigma}$ by adding the mirror images of the elements of the basis for Σ.

- $\hat{a}_i = a_i, \quad \hat{b}_i = b_i$ for $i = 1, \ldots, p$

- $\hat{a}_{p+i} = c_i$ for $i = 1, \ldots, n-1$
- $\hat{a}_{p+n-1+i} = I(a_i), \quad \hat{b}_{p+n-1+i} = -I(b_i)$ for $i = 1, \ldots, p$.

$\hat{b}_{p+i}, \quad i = 1, \ldots, n-1$ have to be chosen appropriately to give a canonical homology basis for $\hat{\Sigma}$. In a completely analogous manner [12] a cohomology basis $\{\hat{\omega}_1, \ldots, \hat{\omega}_{\hat{p}}\}$ has to be formed out of the Abelian differentials of the first kind on $\hat{\Sigma}$. Then the period matrix

$$\hat{\Omega}_{ij} = \int_{\hat{b}_i} \hat{\omega}_j \tag{5}$$

for $\hat{\Sigma}$ has the form

$$\hat{\Omega} = \begin{pmatrix} \Omega & b & c \\ b^t & i\tau & -\bar{b}^t \\ -\bar{c} & -\bar{b} & -\bar{\Omega} \end{pmatrix} , \tag{6}$$

where Ω is the period matrix for Σ. c is a $p \times p$ matrix. b is a $p \times (n-1)$ matrix and τ is a real $(n-1) \times (n-1)$ matrix, that will play an important role in studying the "classical" action. Here we only mention that τ, as $\hat{\Omega}$, is symmetric and has a positive definite imaginary part.

Now we want to introduce moduli for the considered bordered Riemann surfaces. To this end we first study the moduli space $\mathcal{M}_{\hat{p}}$ of the doubled (compact) Riemann surfaces of genus $\hat{p} = 2p + n - 1$. In this space we introduce the so called Fenchel–Nielsen (FN) coordinates [13]. To do this one cuts a genus $\hat{p}$ surface along $3\hat{p} - 3$ closed geodesics into $2\hat{p} - 2$ "pairs of pants", which are surfaces of signature $(0, 3)$. Twisting the "pairs of pants" along such a cut by τ and gluing them together, yields again a new Riemann surface. The $3\hat{p} - 3$ lengths l_i of the closed geodesics $\{\alpha_1, \ldots, \alpha_{3\hat{p}-3}\}$ in the partition of $\hat{\Sigma}$ into the "pairs of pants" together with their possible twists τ_i form a set of $6\hat{p} - 6$ real coordinates for $\mathcal{M}_{\hat{p}}$. Wolpert [13] showed that the Weil-Petersson measure on $\mathcal{M}_{\hat{p}}$ in terms of the FN coordinates reads

$$d\mu_{WP} = \prod_{i=1}^{3\hat{p}-3} d\tau_i \wedge dl_i \; . \tag{7}$$

In the following we give the construction of moduli for Σ out of the FN parameters for $\hat{\Sigma}$. We choose the partition of $\hat{\Sigma}$ into "pairs of pants" in a way that respects the symmetry I.

First we cut $\hat{\Sigma}$ along $c_1, \ldots, c_n$ into Σ and $I\Sigma$. Now, to construct FN coordinates for the surface Σ of signature (p, n) one needs a partition into $2p - 2 + n$ "pairs of pants" by cutting Σ along $3p - 3 + n$ appropriate closed geodesics [13]. We take any such partition of Σ and the mirror image of it for $I\Sigma$ to get $2(6p - 6 + 2n)$ FN moduli. Altogether we cut $\hat{\Sigma}$ along $2(3p - 3 + n) + n = 3\hat{p} - 3$ closed geodesics into $2(2p - 2 + n) = 2\hat{p} - 2$ "pairs of pants", as it is required. This partition is invariant under the symmetry I, since the c_i form the fixed point set of I and, by construction, with α_i also $I\alpha_i$ lies in it.

Now we form complex moduli $t^1, \ldots, t^{3\hat{p}-3}$; $t^k = t_1^k + it_2^k$. For the closed geodesics $\alpha_i = c_i$, $i = 1, \ldots, n$ we take t^i as the complex parameters that are connected with the FN parameters of α_i via $l_i = \frac{2\pi^2}{\ln|t^{i-1}|}$ and $\tau_i = \pi\frac{\arg t^i}{\ln|t^{i-1}|}$. This is the usual way to parametrize a "degenerating collar" [14]. To define t^k, $k = n+1, \ldots, 3\hat{p} - 3$, we use the symmetrical definition of the corresponding α_k and $I\alpha_k$. Call the differentials built from the FN coordinates corresponding to the curves $I\alpha_k$ Idl_k and $Id\tau_k$ respectively, and define $dt_1^k := dl_k + Idl_k$, $dt_2^k := d\tau_k + Id\tau_k$, $dt_1^{k+3p-3+n} := dl_k - Idl_k$ and $dt_2^{k+3p-3+n} := d\tau_k - Id\tau_k$; $k = n+1, \ldots, 3p - 3 + 2n$. This gives exactly $6\hat{p} - 6$ real moduli for $\hat{\Sigma}$.

For Σ one needs $6p - 6 + 3n$ real moduli [15]. These can be taken from those defined above for $\hat{\Sigma}$. In [4] it is shown that, because of the proper boundary conditions in the path integral over bordered surfaces, the quadratic differentials on the surfaces have to fulfil Neumann boundary conditions on $\partial\Sigma$. Since the holomorphic quadratic differentials form the cotangent space to $\mathcal{M}_{\hat{p}}$ in the point corresponding to $\hat{\Sigma}$, resp. to $\mathcal{M}_{p,n}$ in Σ, they are isomorphic to the differentials dt, where t is a moduli parameter. Thus the $dt_1^k = dl_k + Idl_k$ and $dt_2^k = d\tau_k + Id\tau_k$, $k = n+1, \ldots, 3p - 3 + 2n$, among the moduli for $\hat{\Sigma}$ have the correct boundary behaviour. We take as the missing parameters the $t_1^k = l_k$, $k = 1, \ldots, n$, which are natural to describe Σ, as they are the lengths of Σ's bordering curves. The corresponding $t_2^k = \tau_k$ are not needed, so one may neglect them by setting $t^k = t_1^k$, $k = 1, \ldots, n$, real. Thus we get a set $t^k = t_1^k + it_2^k$ of $6p - 6 + 3n$ real moduli for Σ, whose corresponding quadratic differentials have the proper boundary behaviour.

Having chosen these parameters for $\mathcal{M}_{p,n}$, we go back to $\mathcal{M}_{\hat{p}}$. For this space of compact Riemann surfaces Masur [16] investigated the Hermitian form

$$H = \sum_{i,j=1}^{3\hat{p}-3} h_{ij} dt^i \otimes \overline{dt^j}, \quad h_{ij} := H\left(\frac{\partial}{\partial t_1^i}, \frac{\partial}{\partial t_1^j}\right) . \tag{8}$$

The real Riemannian metric constructed from this form is

$$(g_{ij}) = (Re\, h_{ij}) = \left(< \frac{\partial}{\partial t_1^i}, \frac{\partial}{\partial t_1^j} > \right) . \tag{9}$$

Now we take the t^k's from above to define the Weil-Petersson metric on $\mathcal{M}_{\hat{p}}$. Call the Weil-Petersson metric tensor (G_{ij}), then

- $i, j = 1, \ldots, 3p - 3 + 2n$:
$$G_{ij} = g_{ij}$$
- $i, j = n+1, \ldots, 3p - 3 + 2n$:
$$G_{i+3p-3+n, j+3p-3+n} = < \frac{\partial}{\partial t_2^i}, \frac{\partial}{\partial t_2^j} > = g_{3\hat{p}-3+i, 3\hat{p}-3+j} = g_{ij}$$
- $i = 1, \ldots, 3p - 3 + 2n$, $j = n+1, \ldots, 3p - 3 + 2n$:
$$G_{i,j+3p-3+n} = < \frac{\partial}{\partial t_1^i}, \frac{\partial}{\partial t_2^j} > = g_{i,3\hat{p}-3+j} = Im\, h_{ij}.$$
- $i = n+1, \ldots, 3p - 3 + 2n$, $j = 1, \ldots, 3p - 3 + 2n$:
$$G_{i+3p-3+n,j} = -Im\, h_{ij}$$

Therefore the metric tensor aquires the form

$$(G_{ij}) = \begin{pmatrix} g_{ij} & g_{ij} & Im\, h_{i,j-3p+3-n} \\ g_{ij} & g_{ij} & Im\, h_{i,j-3p+3-n} \\ -Im\, h_{i-3p+3-n,j} & -Im\, h_{i-3p+3-n,j} & g_{i-3p+3-n,j-3p+3-n} \end{pmatrix} . \tag{10}$$

This representation will be used for the Weil-Petersson metric on $\mathcal{M}_{p,n}$ in the subsequent calculations.

4 The On-shell Limit of the Scattering Amplitudes

The first ingredient entering the path integral of the scattering amplitudes, whose on-shell limit we are going to study, is the "classical" action

$$S_{cl} = \frac{1}{2}\int d^2\sigma\sqrt{g}g^{\alpha\beta}\partial_\alpha X^\mu_{cl}\partial_\beta X^\mu_{cl} \ . \tag{11}$$

This will be treated in a way analogous to [6]. As one might be confused by what has been presented in that paper concerning the treatment of S_{cl}, we repeat the derivation here in our notation and at the same time introduce the necessary generalizations to the case of surfaces of arbitrary genus.

We work in the conformal gauge $z = \sigma^1 + i\sigma^2$, $\bar z = \sigma^1 - i\sigma^2$, $ds^2 = e^\varphi dz d\bar z$, which reads in terms of the metric tensor $g_{zz} = 0 = g_{\bar z\bar z}$, $g_{z\bar z} = \frac{1}{2}e^\varphi$, $\sqrt{g} = \frac{1}{2}e^\varphi$. Hence

$$S_{cl} = i\int d^2z\,\partial_z X^\mu_{cl}\partial_{\bar z}X^\mu_{cl} \ . \tag{12}$$

We want to rewrite this action such that the on-shell limit can be obtained, and therefore recall that this limit consists of shrinking the n holes in the world sheets to punctures, which shall be located at the points x^μ_i. Now introduce n functions $\phi_i(z,\bar z)$ on Σ with $\partial_z\partial_{\bar z}\phi_i = 0$ and $\phi_i|_{c_j} = \delta_{ij}$. Then define

$$X^\mu_{pt} := \sum_{i=1}^n x^\mu_i\phi_i \quad \Rightarrow \quad X^\mu_{pt}|_{c_j} = x^\mu_j = const.\,. \tag{13}$$

Now split $X^\mu_{cl} = X^\mu_{pt} + V^\mu$, with some appropriate functions $V^\mu(z,\bar z)$. In the limit under discussion we have $l_k \to 0$, hence also $V^\mu \to 0$. Then define $\phi(z,\bar z) := \sum_{i=1}^\infty \phi_i(z,\bar z)$. This function satisfies $\partial_z\partial_{\bar z}\phi = 0$, thus it is an eigenfunction of $\Delta = 4\partial_z\partial_{\bar z}$ with eigenvalue zero, hence is a constant. As $\phi|_{\partial\Sigma} = 1$, it follows that $\phi \equiv 1$ on the whole of Σ. Therefore

$$\phi_n = 1 - \sum_{i=1}^{n-1}\phi_i \quad \Rightarrow \quad \partial_{z(\bar z)}\phi_n = -\sum_{i=1}^{n-1}\partial_{z(\bar z)}\phi_i \ . \tag{14}$$

If one plugs all this into the action S_{cl} one is left with

$$S_{cl} = i\sum_{k,l=1}^{n-1}(x^\mu_k - x^\mu_n)(x^\mu_l - x^\mu_n)\int d^2z\,\partial_z\phi_k\partial_{\bar z}\phi_l - i\sum_{k=1}^{n-1}A^\mu_k + O(V^2) \ , \tag{15}$$

where $A^\mu_k := \int d^2z[\partial_z\phi_k\partial_{\bar z}V^\mu + \partial_{\bar z}\phi_k\partial_z V^\mu]$, and in the on-shell limit $A^\mu_k \to 0$. Here $Re\,\partial_z\phi_k\partial_{\bar z}\phi_l$ is symmetric in k,l and $Im\,\partial_z\phi_k\partial_{\bar z}\phi_l$ is antisymmetric. Since this is to be contracted with the symmetric expression $(x_k - x_n)(x_l - x_n)$ in (15), only the real part survives.

The functions $\phi_i(z,\bar z)$ are real valued and harmonic and hence may be decomposed as

$$\phi_i(z,\bar z) = \rho_i(z) + \bar\rho_i(\bar z) = 2Re\ \rho_i(z) \ . \tag{16}$$

These functions are defined on Σ and may be antisymmetrically continued to $\hat\Sigma$. In [6] it is shown that this implies $\rho_i(0) = 0$ and that $Im\,\rho_i(z)$ is smooth, when crossing $\partial\Sigma$.

The differentials

$$\partial\phi_i := \partial_z\phi_i dz = \partial_z\rho_i dz \tag{17}$$

now form $n-1$ Abelian differentials of the first kind on $\hat\Sigma$. ($\partial\phi_n$ is not independent of those because of (14)).

As a homology basis we choose that one described in section 3, which especially contains the $c_i = \hat a_{p+i}$, $i = 1,\ldots,n-1$. The corresponding $\hat b_{p+i}$ are then chosen to be oriented such that they direct from that side of Σ, on which $Re\,\rho_i|_{\hat a_{p+i}} = \frac{1}{2}$, to the side, on which $Re\,\rho_i|_{\hat a_{p+i}} = -\frac{1}{2}$. Hence

$$\int_{\hat b_{p+i}}\partial\phi_j = \int_{\hat b_{p+i}} dz\ \partial_z\rho_j = \delta_{ij} \ . \tag{18}$$

Then we define the matrix (σ_{ij}) by

$$\sigma_{ij} := \int_{\hat a_{p+i}}\partial\phi_j = \int_{\hat a_{p+i}} dz\ \partial_z\rho_j \ . \tag{19}$$

With

$$d(\phi_l\partial_z\phi_k dz) = -\partial_z\rho_k\partial_{\bar z}\bar\rho_l d^2z \tag{20}$$

one gets

$$Re\ i\int d^2z\,\partial_z\phi_k\partial_{\bar z}\phi_l = \frac{1}{2}Im\ \sigma_{lk} \ . \tag{21}$$

(21) compared with (15) yields

$$S_{cl} = \frac{1}{2}\sum_{k,l=1}^{n-1}(x^\mu_k - x^\mu_n)(x^\mu_l - x^\mu_n)Im\ \sigma_{kl} + i\sum_{k=1}^{n-1}(x^\mu_k - x^\mu_n)A^\mu_k + O(V^2) \ . \tag{22}$$

We now Fourier-transform the scattering amplitudes in the on-shell limit, since then they only depend on the punctures x^μ_i. Therefore we study

$$\begin{aligned}
F(p_1,\ldots,p_n) \ &:= \ \int d^{26}x_1\ldots d^{26}x_n \exp[-S_{cl}]\exp\left[i\sum_{k=1}^n x_k p_k\right] \\
&= \ (2\pi)^{13(n+1)}\delta^{26}\left(\sum_{k=1}^n p_k\right)[det\ Im\ \sigma^{-1}]^{13} \\
&\quad \cdot\exp\left\{-\frac{1}{2}\sum_{k,l=1}^{n-1}(p_k - A_k)[Im\ \sigma^{-1}]_{kl}(p_l - A_l) + O(V^2)\right\} \\
&= \ (2\pi)^{13(n+1)}\delta^{26}\left(\sum_{k=1}^n p_k\right)[det\ Im\ \sigma^{-1}]^{13}\exp\left\{-\frac{1}{2}\sum_{k,l=1}^{n-1}p_k[Im\ \sigma^{-1}]_{kl}p_l\right\}e^{-S_{reg}} \ ,
\end{aligned} \tag{23}$$

where we collect all the expressions that do not occur explicitly in the regular S_{reg}. The amplitudes then aquire the form

$$A(p_1,\ldots,p_n) = \sum_{p=0}^\infty\int d\mu_{WP}\ const.\ [det'(P_1^\dagger P_1)]^{\frac{1}{2}}[det'(-\Delta)]^{-13}F(p_1,\ldots,p_n) \ . \tag{24}$$

We now relate $Im\,\sigma$ to the period matrix $\hat\Omega$ of $\hat\Sigma$. Let $\{\hat\omega_i | i = 1,\ldots,\hat p\}$ be the cohomology basis and $\{\hat a_i, \hat b_i | i = 1,\ldots,\hat p\}$ be the homology basis introduced in section 3. Then it is possible to expand the Abelian differentials $\partial\phi_i$ in terms of this cohomology basis,

$$\partial\phi_k(z) = \sum_{l=1}^{\hat p} A_{kl}\hat\omega_l(z) \ . \tag{25}$$

In addition to (19) we have for $j = 1, \ldots, n-1$ and $l = 1, \ldots, p, p+n, \ldots, \hat{p}$,

$$\int_{\hat{a}_l} \partial\phi_i(\cdot) = 0 . \qquad (26)$$

Now we insert (25) in (19) and (26) to obtain

$$\begin{aligned} \sigma_{ij} &= \sum_{k=1}^{\hat{p}} A_{jk} \int_{\hat{a}_{p+i}} \hat{\omega}_k(\cdot) = A_{j,p+i} \\ 0 &= \sum_{k=1}^{\hat{p}} A_{jk} \int_{\hat{a}_l} \hat{\omega}_k(\cdot) = A_{jl} . \end{aligned} \qquad (27)$$

Thus the coefficients A_{jk} are for $j = 1, \ldots, n-1$.

$$\begin{aligned} A_{j1} = \ldots = A_{jp} = A_{j,p+n} = \ldots = A_{j\hat{p}} = 0 \\ A_{j,p+1} = \sigma_{j1}, \ldots, A_{j,p+n-1} = \sigma_{j,n-1} \end{aligned} \qquad (28)$$

and we can give the expansion (25) explicitly as

$$\partial\phi_j(z) = \sum_{k=1}^{n-1} \sigma_{kj} \hat{\omega}_{p+k}(z) . \qquad (29)$$

Since the differentials $\partial\phi_i$ are normalized to the $\hat{b}$-periods (18), we conclude

$$\begin{aligned} \delta_{ij} &= \int_{\hat{b}_{p+i}} \partial\phi_j(\cdot) = \sum_{k=1}^{n-1} \sigma_{kj} \int_{\hat{b}_{p+i}} \hat{\omega}_{p+k}(\cdot) \\ &= \sum_{k=1}^{n-1} \sigma_{kj} \hat{\Omega}_{p+i,p+k} = i \sum_{k=1}^{n-1} \sigma_{kj} \tau_{ik} , \end{aligned} \qquad (30)$$

where $i\tau$ is the central block of the period matrix $\hat{\Omega}$, introduced in section 3. Thus $(Im\,\sigma^{-1})_{kl} = \tau_{kl}$, because τ is real. Comparing with the previous result for $p = 0$ in [6], we see that our extension to $p > 0$ gives the same answer with only a part of $\hat{\Omega}$ entering into the formulae, not the whole matrix.

To study $Im\,\sigma^{-1}$ in the limit of n degenerating closed curves on $\hat{\Sigma}$, we look at

$$\tau_{kl} = \frac{1}{i} \int_{\hat{b}_{p+k}} \hat{\omega}_{p+l}(\cdot) = -\frac{1}{2\pi} \int_{\hat{b}_{p+k}} \int_{\hat{b}_{p+l}} \hat{\omega}(\cdot,\cdot) . \qquad (31)$$

Here $\hat{\omega}(w,z) = dw\,dz \frac{d^2}{dw\,dz} \ln E(w,z)$ and $E(w,z)$ is the prime form of $\hat{\Sigma}$. (This was introduced by Fay, see [12].) The normalized Abelian differential $\hat{\omega}(w,z)$ of the second kind has been intensively studied in the limit of degenerating closed curves on $\hat{\Sigma}$. In the mathematical literature this limit is called the "pinching limit", on which there exists a fairly large amount of literature. We mention [12] and [17], where the period matrix is studied in the pinching limit. A more accessible presentation of that matter may also be found in [6]. But one should be aware of the confusion about $Im\,\sigma^{-1}$ and τ in [6]. Nevertheless, the final answer given there turns out to be correct, and we thus only quote the result here,

$$(Im\,\sigma^{-1})_{ij} = \pi \left[\frac{\delta_{ij}}{l_i} + \frac{1}{l_n} + const. + O(e^{-\frac{1}{l_k}}) \right] . \qquad (32)$$

Conservation of momentum, indicated by the δ-function in (23), then leads to

$$-\frac{1}{2} \sum_{k,l=1}^{n-1} p_k (Im\,\sigma^{-1})_{kl} p_l = -\frac{\pi}{2} \sum_{k=1}^{n} \frac{p_k^2}{l_k} + const.\, p_n^2 + O(e^{-\frac{1}{l_k}}) . \qquad (33)$$

The determinant of the matrix $Im\,\sigma^{-1}$ can be evaluated, and the leading term in the limit $l_k \to 0$ is

$$det(Im\,\sigma^{-1}) = \frac{\pi^{n-1}}{l_1 \ldots l_n} + \text{less divergent terms} . \qquad (34)$$

The next point is to discuss the appropriate form of the determinants in (4). We do this by applying Selberg trace formula techniques, which are well-known in the treatment of the string partition function (see e.g. [9,19]), but are not used in other cases.

In [8] we derived a trace formula for bordered Riemann surfaces well adopted to what is needed for the current problem, namely for the treatment of the Dirichlet-Laplace-Beltrami operator on Σ. In ref. [8] we obtained

$$det'(-\Delta) = \exp\left\{ -\int_0^\infty \frac{dt}{t} \left[\theta_D^{(2)}(t) + \theta_D^{(3)}(t) + \theta_D^{(5)}(t) \right] \right\} \exp\left[-(\hat{p}-1)C - \frac{L}{8} \right] , \qquad (35)$$

where $\theta_D(t) = \sum_{n=1}^{5} \theta_D^{(n)}(t)$ is the trace of the Dirichlet heat kernel for $-\Delta$, and the superscripts refer to the various terms in the trace formula of [8]. $\theta_D^{(2)}$ and $\theta_D^{(5)}$ depend on the lengths l_i of the degenerating curves, whereas $\theta_D^{(3)}$ does not. Furthermore $C = \frac{1}{4} - \frac{1}{2}\ln 2\pi - 2\zeta'(-1)$, with $\zeta(s)$ denoting Riemann's zeta function, and $L = \sum_{i=1}^{n} l_i$ is the total length of $\partial\Sigma$. We now separate the l_i-independent terms and define

$$\theta_{reg}(t) := \theta_D^{(3)}(t) + \frac{1}{8\sqrt{\pi}} \frac{e^{-\frac{t}{4}}}{\sqrt{t}} \sum_{\{\gamma\}_p \neq \{c_i\}} \sum_{k=1}^{\infty} \frac{l(\gamma)}{\sinh(kl(\gamma)/2)} e^{-k^2 l^2(\gamma)/4t} . \qquad (36)$$

With this definition the integrand in the exponential in (35) may be rewritten, using the explicit expressions for the $\theta_D^{(n)}$ given in [8], as

$$\begin{aligned} &\theta_D^{(2)}(t) + \theta_D^{(3)}(t) + \theta_D^{(5)}(t) \\ &= \theta_{reg}(t) + \frac{1}{4\sqrt{\pi}} \frac{e^{-\frac{t}{4}}}{\sqrt{t}} \sum_{i=1}^{n} \sum_{k=1}^{\infty} l_i e^{-k^2 l_i^2/4t} \left[\frac{1}{\sinh(kl_i/2)} - \frac{1}{\cosh(kl_i/2)} \right] . \end{aligned} \qquad (37)$$

We insert this into (35) and perform the t-integration in the l_i-dependent part. After rearranging the k-summation we arrive at

$$det'(-\Delta) = \exp\left\{ -(\hat{p}-1)C - \int_0^\infty \frac{dt}{t} \theta_{reg}(t) \right\} e^{-\frac{L}{8}} \prod_{i=1}^{n} \prod_{k=1}^{\infty} \left(1 - e^{-2kl_i}\right)^2 . \qquad (38)$$

The treatment of the ghost determinant will be completely analogous. In [7] it was shown that the determinant of the operator $P_1^\dagger P_1$, obeying the proper boundary conditions on $\partial\Sigma$, is equal to the square root of $det'\hat{P}_1^\dagger \hat{P}_1$, where $\hat{P}_1$ acts on vector fields defined on the compact double $\hat{\Sigma}$. This determinant can be expressed in terms of Selberg's zeta function $Z(s)$ on $\hat{\Sigma}$ through [18]

$$det'(P_1^\dagger P_1) = Z(2) e^{(\hat{p}-1)C_1} , \qquad (39)$$

with $C_1 = \frac{7}{3}\ln 2 + 3\ln 2\pi - \frac{9}{2} + 4\zeta'(-1)$. We use McKean's integral representation [19] for the logarithmic derivative of $Z(s)$ to obtain after one integration

$$Z(2) = \exp\left\{-2\int_0^\infty \frac{dt}{t} e^{-2t}\theta_D^{(2)}(t)\right\} . \tag{40}$$

In $\theta_D^{(2)}(t)$, which is McKean's $\frac{1}{2}\vartheta(t)$, we again separate the l_i-independent part as in (36) and get

$$\theta_D^{(2)}(t) = \tilde\theta_{reg}(t) + \frac{1}{4\sqrt{\pi}}\frac{e^{-\frac{t}{4}}}{\sqrt{t}}\sum_{i=1}^{n}\sum_{k=1}^{\infty}\frac{l_i}{\sinh(kl_i/2)}e^{-k^2l_i^2/4t} . \tag{41}$$

Inserting (41) in (40), performing the t-integration over the l_i-dependent part and rearranging the k-summation leads to

$$det'(P_1^\dagger P_1) = \exp\left\{(\hat p - 1)C_1 - 2\int_0^\infty \frac{dt}{t}e^{-2t}\tilde\theta_{reg}(t)\right\}\prod_{i=1}^{n}\prod_{k=2}^{\infty}(1-e^{-kl_i})^2 . \tag{42}$$

Now we use in (38) and (42) Jacobi's famous identity for the product over k occurring in both formulae to get the expression for the determinants that contains the explicit dependence on the degenerating parameters l_i,

$$\begin{aligned}[det'(P_1^\dagger P_1)]^{\frac{1}{2}}[det'(-\Delta)]^{-13} = & \left(\frac{\sqrt{2\pi}}{\pi^{13}}\right)^n \exp\left\{(\hat p - 1)(13C + \frac{1}{2}C_1)\right\} \\ & \cdot\exp\left\{\int_0^\infty \frac{dt}{t}[13\theta_{reg}(t) - e^{-2t}\tilde\theta_{reg}(t)]\right\} \\ & \cdot e^{-\frac{13L}{24}}\prod_{i=1}^{n} l_i^{\frac{23}{2}} e^{\frac{2\pi^2}{l_i}}[1+O(l_i)]\prod_{k=1}^{\infty}\left(1-e^{-\frac{2\pi^2}{l_i}}\right)^{-26}\left(1-e^{-\frac{4\pi^2 k}{l_i}}\right) .\end{aligned} \tag{43}$$

For our final purpose we expand the k-product into a series,

$$\prod_{k=1}^{\infty}\left(1-e^{-\frac{2\pi^2 k}{l_i}}\right)^{-26}\left(1-e^{-\frac{4\pi^2 k}{l_i}}\right) = \sum_{j=0}^{\infty} a_j e^{-\frac{2\pi^2}{l_i}j} , \tag{44}$$

which defines the expansion coefficients a_j.

There remains to express the Weil-Petersson measure in terms of the l_i. Masur [16] studied the limit $l_k \to 0$, which is equivalent to $t^k \to 0$, $k = 1,\ldots,n$, in (10). We recall that $t^k = t_1^k$ is real in our case (see section 3). Masur gives

- $G_{ii} = const.\ \frac{1}{|t^i|^2(\ln|t^i|^{-1})^3}$, $i = 1,\ldots,n$
- $G_{ij} = O\left(\frac{1}{|t^i||t^j|(\ln|t^i|^{-1})^3(\ln|t^j|^{-1})^3}\right)$, $i \neq j$, $i,j = 1,\ldots,n$
- G_{ij} is regular for $i,j > n$
- $G_{ij} = O\left(\frac{1}{|t^i|(\ln|t^i|^{-1})^3}\right)$, $i = 1,\ldots,n$, $j > n$.

Instead of the t^k's we want to use the l_k's, so we perform the change of coordinates $t^k \to l_k$, $k = 1,\ldots,n$, which yields

- $i = 1,\ldots,n$:

$$\tilde G_{ii} = const.\ \ln|t^i|^{-1} =: \frac{A_i}{l_i}$$

- $i \neq j$, $i,j = 1,\ldots,n$:

$$\tilde G_{ij} = const.\ \frac{1}{\ln|t^i|^{-1}\ln|t^j|^{-1}} = const.\ l_i l_j .$$

- $i,j > n$:

$$\tilde G_{ij} = \tilde G_{ij}^0 \text{ is regular}$$

- $i = 1,\ldots,n$, $j > n$:

$$\tilde G_{ij} = const.\ \frac{1}{\ln|t^i|^{-1}} = const.\ l_i .$$

where $(\tilde G_{ij})$ is the metric tensor in the new coordinates. Hence the Weil-Petersson metric tensor aquires in the pinching limit the form

$$(\tilde G_{ij}) = \begin{pmatrix} \frac{A_i}{l_i}\delta_{ij} + O(l_i l_j) & O(l_i) \\ & \\ O(l_j) & \tilde G_{ij}^0 \end{pmatrix} , \tag{45}$$

while its determinant $\tilde G = det(\tilde G_{ij})$ becomes in this limit $\tilde G \simeq \tilde G^0 \prod_{i=1}^n \frac{A_i}{l_i}$. Therefore

$$d\mu_{WP} = \sqrt{\tilde G}\prod_{i=1}^{n} dl_i \prod_{k=n+1}^{3p-3+2n} dt_1^k dt_2^k \simeq d\mu_{WP}^{reg}\prod_{i=1}^{n}\frac{dl_i}{\sqrt{l_i}} . \tag{46}$$

Together with (33) and (43) we now have everything at hand that is required to perform the integration over the leading terms in the on-shell limit of (4). Collecting the results obtained in (33), (43) and (46) we can perform the l_i-integrations. If the regular terms which are irrelevant in the on-shell limit are included in $\hat S$, we get

$$\begin{aligned} & \int d\mu_{WP}[det'(P_1^\dagger P_1)]^{\frac{1}{2}}[det'(-\Delta)]^{-13}F(p_1,\ldots,p_n) \simeq \\ & \delta^{26}\left(\sum_{k=1}^{n} p_k\right)\int d\mu_{WP}^{reg} e^{-\hat S}\prod_{k=1}^{n}\int_0^\infty \frac{dl_k}{l_k^2}\exp\left\{-\frac{\pi}{2}\frac{p_k^2}{l_k} + \frac{2\pi^2}{l_k}\right\}\sum_{j=0}^{\infty} a_j e^{-\frac{2\pi^2 j}{l_k}} \\ & = \delta^{26}\left(\sum_{k=1}^{n} p_k\right)\int d\mu_{WP}^{reg} e^{-\hat S}\prod_{k=1}^{n}\sum_{j=0}^{\infty} a_j\int_0^\infty dy_k \exp\left\{-\left[\frac{\pi}{2}p_k^2 + 2\pi^2(j-1)\right]y_k\right\} \\ & = \delta^{26}\left(\sum_{k=1}^{n} p_k\right)\prod_{k=1}^{n}\sum_{j=0}^{\infty}\frac{a_j}{p_k^2 + 4\pi(j-1)}\int d\mu_{WP}^{reg} e^{-\hat S} .\end{aligned} \tag{47}$$

This result shows that the scattering amplitudes aquire poles at $p_k^2 = 4\pi(1-j)$, $j = 0,1,2,\ldots$, in all orders p of string perturbation theory. This corresponds to the propagation of infinitely many particle states along the "external legs" of the world sheet. In the propagator case [2] it was shown that the residues at the poles are identical to the degeneracies of the open string particle states. The a_i's here differ from those degeneracies. They contain further information about the glueing of the "external legs" to the "body" of the world sheet, since only one end of these cylinder-like parts is shrunk to zero length.

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