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Abstract

A Selberg trace formula is derived for the Laplace-Beltrami operator on bordered Riemann surfaces with Dirichlet or Neumann boundary conditions using a construction via the compact double of the surface, for which the standard trace formula is valid. We discuss applications of the trace formula to spectral functions of the Laplace-Beltrami operator and calculate its functional determinant.

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1 Introduction

In recent years the Selberg trace formula [1-3] has become notably popular among physicists. There are two fields in physics, where Riemann surfaces occur and the trace formula has been successfully applied: quantum chaology [4,5] and string theory [6]. In the first field it was discovered [4] that Gutzwiller's periodic-orbit theory for the semiclassical quantization of a classically chaotic system becomes exact for a particle sliding freely on a Riemann surface of genus $g \geq 2$ (Hadamard-Gutzwiller model). The corresponding periodic-orbit formula is just Selberg's trace formula. This then has been intensively applied there [5].

The second striking application of the Selberg trace formula has been string theory. In Polyakov's path integral approach [6], where the string partition function is given as an integral over all world sheets of the string, there occurs the functional determinant of the Laplace-Beltrami operator on the world sheet as a result of the integration over the embedding functions into space-time. It is now possible to evaluate this determinant using the Selberg trace formula and express it through Selberg's zeta function [6,7]. Also, the ghost determinant appearing in string theory may be expressed analogously. In fact, the determinants of higher-rank Laplace-like operators allow such a treatment, too [8]. Even in the case of the superstring a "super-analogue" of Selberg's trace formula has been derived and applied [9]. Thus the use of Selberg techniques has become a whole business in physics.

In this paper we derive a trace formula for bordered Riemann surfaces, that is we study the trace of certain integral operators associated with the Laplace-Beltrami operator obeying Dirichlet or Neumann boundary conditions. Our interest in that subject mainly stems from string theory, but we hope that one will be able to use the formula also in quantum chaology, e.g. to study the quantum Sinai billiard.

In string theory, one way to define off-shell string scattering amplitudes is to use a functional integral over bordered world sheets [10], the bordering curves being the incoming and outgoing string states of the scattering process. Then naturally the functional determinant of the Laplace-Beltrami operator occurs and the question arises, how to deal with it. Blau et al. [11] used for the first time a version of Selberg's trace formula for bordered surfaces, but restricted their attention to surfaces of genus zero.

Our purpose in this paper is to derive a trace formula for bordered Riemann surfaces of arbitrary genus. We proceed by constructing a compact Riemann surface through doubling the original one and by using the well-known trace formula for that cocompact case. To deal with the boundary conditions on the bordered surface properly, we devide the eigenfunctions of the Laplace-Beltrami operator on the doubled surface into symmetry classes according to their reflection property introduced by the doubling procedure. We combine methods that have previously been used in [11] and by Venkov [12], who studied the trace formula for (anti-) symmetric eigenfunctions of the Laplace-Beltrami operator on the fundamental domain of the modular group $PSL(2,\mathbb{Z})$ in the Poincaré upper half-plane.

Our paper is organized as follows: First we explain the construction of the doubled surfaces and derive the trace formula. Then we study the trace of the heat kernel and the MP-zeta function of the Dirichlet-(Neumann-)Laplace-Beltrami operator and give some of their properties. In addition to the usual Selberg zeta function (on the doubled surface) we introduce functions that effectively take care of the correct boundary conditions. We are then in a position to express the functional determinant of the Dirichlet-(Neumann-)Laplace-Beltrami operator by these functions or by the respective heat kernels. These are the formulae that are used in [13] to study the on-shell limit of off-shell string scattering amplitudes.

2 Derivation of the Trace Formula

Before we start to derive the trace formula we briefly recall how bordered Riemann surfaces are most conveniently dealt with [14]. The idea is to lift the discussion to an appropriately chosen compact surface, since the theory of compact Riemann surfaces is well developed and comparatively easy to handle.

Let $\bar{\Sigma}$ be a compact Riemann surface of genus g and $d_1, \dots, d_n$ conformal, non-overlapping discs on $\bar{\Sigma}$. Then $\Sigma := \bar{\Sigma}\backslash\{d_1, \dots, d_n\}$ is a bordered Riemann surface of signature (g, n). $c_i := \partial d_i$ are the n components of $\partial\Sigma$. Now one takes a copy $I\Sigma$ of Σ, a mirror image, and glues both surfaces together along $\partial\Sigma$ and $\partial(I\Sigma)$. Technically this is done in terms of local coordinates in the following way: Let z be a local coordinate in the neighbourhood of $P \in \Sigma$. Then $-\bar{z}$ is taken as the local coordinate in the neighbourhood of the mirror image P' of P on $I\Sigma$. The corresponding points on $\partial\Sigma$ and $\partial(I\Sigma)$ respectively are identified and thus have purely imaginary coordinates. The reflection I: $P \to P'$ in $\partial\Sigma$ then is an anticonformal involution ($I^2 = 1$) on the doubled surface $\hat{\Sigma} := \Sigma \cup I\Sigma$. Furthermore $\Sigma = \hat{\Sigma}/I$ and $\hat{\Sigma}$ is a compact Riemann surface of genus $\hat{g} = 2g + n - 1$. The uniformization theorem for compact Riemann surfaces now states that $\hat{\Sigma}$, for $\hat{g} \geq 2$, may be represented as $\hat{\Sigma} \simeq \hat{\Gamma}\backslash\mathcal{H}$, where $\hat{\Gamma}$ is the Fuchsian group of $\hat{\Sigma}$ and $\mathcal{H}$ is the Poincaré upper half-plane, $\mathcal{H} = \{z = x+iy | y > 0\}$, endowed with the hyperbolic metric $ds^2 = y^{-2}(dx^2 + dy^2)$. $\hat{\Gamma}$ then is a discrete subgroup of $PSL(2, \mathbb{R})$ that solely consits of hyperbolic elements. In several circumstances it is advantegeous to represent $\hat{\Sigma}$ by a fundamental domain $\hat{\mathcal{F}} \subset \mathcal{H}$ for $\hat{\Gamma}$.

To construct a convenient fundamental domain and representation of the involution I on it it is advantegeous to view $\hat{\Sigma}$ as a symmetric Riemann surface with reflection symmetry I. For such surfaces the Fuchsian groups are well investegated by Sibner [15]. He shows that $\hat{\mathcal{F}}$ may be chosen as the interior of a fundamental polygon in $\mathcal{H}$ with $4\hat{g} + 2n - 2$ edges, which is symmetric with respect to the imaginary axes. The involution I is being represented by $z \to -\bar{z}$, that is a reflection in the symmetry axis of $\hat{\mathcal{F}}$. One of the bordering curves, say c_n, is mapped onto the imaginary axis and the others are among the edges of the fundamental polygon. The advantage of this construction is that one can work directly on $\hat{\mathcal{F}}$, with I, viewed as a mapping of complex numbers, being formally identical on $\hat{\Sigma}$ and $\hat{\mathcal{F}}$.

On $\mathcal{H}$ the Laplace-Beltrami operator takes the form $\Delta = y^2(\partial_x^2 + \partial_y^2)$, hence it commutes with I. Therefore the eigenfunctions of $-\Delta$ can be simultaneously chosen as eigenfunctions of I. The odd functions (with respect to I) on $\hat{\Sigma}$ are exactly the (antisymmetric) continuations of the functions on Σ that satisfy Dirichlet boundary conditions on $\partial\Sigma$, and the even functions are in the same way related to the functions on Σ that satisfy Neumann boundary conditions. In this paper we will explicitly deal with theDirichlet case and mention only from time to time how the Neumann case looks like. In the trace formula the difference is just a few signs. Thus from now on we concentrate on odd functions on $\hat{\Sigma}$ or $\hat{\mathcal{F}}$ respectively. They may be constructed from functions defined on the whole of $\mathcal{H}$ via Poincaré series. Let $f_0 \in C(\mathcal{H})$ be continuous, then

$$f(z) := \sum_{\gamma\in\hat{\Gamma}}[f_0(\gamma z) - f_0(\gamma I(z))] = \sum_{\gamma\in\hat{\Gamma}}[f_0(\gamma z) - f_0(\gamma(-\bar{z}))] \qquad (1)$$

is such an odd, $\hat{\Gamma}$-automorphic function, i.e. $f(\gamma z) = f(z)$ for all $\gamma \in \hat{\Gamma}$, and $f(Iz) = -f(z)$.

In the Selberg trace formula one considers traces of integral operators, whose spectra are related to the spectrum of $-\Delta$. Let $\Phi \in C_c^\infty(\mathbb{R})$ be a smooth function with compact support. Then for $z, z' \in \mathcal{H}$

$$k(z, z') := \Phi\left(\frac{|z - z'|^2}{yy'}\right) \qquad (2)$$

is called a point-pair invariant. For $M \in PSL(2, \mathbb{R})$, which acts on $z \in \mathcal{H}$ as a fractional linear transformation, it follows that $k(Mz, Mz') = k(z, z')$ and also $k(Iz, Iz') = k(z, z')$. Let $\psi \in C^\infty(\mathcal{H})$ be an eigenfunction of $-\Delta$, $-\Delta\psi = \lambda\psi$, then it is simultaneously an eigenfunction of the integral operator L (see e.g. [2]),

$$(L\psi)(z) := \int_{\mathcal{H}} d\mu(z')k(z, z')\psi(z') = \Lambda(\lambda)\psi(z) , \qquad (3)$$

where $d\mu(z) := \frac{dxdy}{y^2}$ is the Poincaré measure on $\mathcal{H}$, and Λ depends only on Φ. Now we form the integral kernel

$$\hat{K}(z, z') := \frac{1}{2}\sum_{\gamma\in\hat{\Gamma}}[k(z, \gamma z') - k(z, \gamma(-\bar{z}'))] , \qquad (4)$$

which defines an integral operator on $L_2(\hat{\mathcal{F}})$. If $f \in L_2(\hat{\mathcal{F}})$ is an odd eigenfunction of $-\Delta$ with eigenvalue λ, then we compute

$$\begin{aligned}(\hat{L}f)(z) &:= \int_{\hat{\mathcal{F}}} d\mu(z')\hat{K}(z, z')f(z') \\ &= \frac{1}{2}(Lf)(z) - \frac{1}{2}(Lf)(-\bar{z}) \\ &= \Lambda(\lambda)f(z) ,\end{aligned} \qquad (5)$$

hence f is also an eigenfunction of $\hat{L}$, with eigenvalue $\Lambda(\lambda)$. $-\Delta$ on $\hat{\Sigma}$ has a discrete spectrum, $0 < \lambda_1 \leq \lambda_2 \leq \dots$, and $\hat{K}$ can be expanded in (odd) eigenfunctions φ_n of $-\Delta$,

$$\hat{K}(z, z') = \sum_{n=1}^{\infty}\Lambda(\lambda_n)\varphi_n(z)\varphi_n(z') . \qquad (6)$$

As usual, one defines $\lambda = p^2 + \frac{1}{4}$, $\Lambda(\lambda) = h(p)$, and gets as the trace of $\hat{L}$

$$\begin{aligned}Tr\,\hat{L} = \sum_{n=1}^{\infty}\Lambda(\lambda_n) &= \sum_{n=1}^{\infty}h(p_n) \\ &= \int_{\hat{\mathcal{F}}} d\mu(z)\hat{K}(z, z) \\ &= \frac{1}{2}\sum_{\gamma\in\hat{\Gamma}}\int_{\hat{\mathcal{F}}} d\mu(z)k(z, \gamma z) - \frac{1}{2}\sum_{\rho\in\hat{\Gamma}I}\int_{\hat{\mathcal{F}}} d\mu(z)k(z, \rho z) .\end{aligned} \qquad (7)$$

The first sum on the r.h.s. of (7) is $\frac{1}{2}$ times the r.h.s. of the usual Selberg trace formula for the compact surface $\hat{\Sigma}$ and the result of its evaluation is well known to be [1,2]

$$\sum_{\gamma\in\hat{\Gamma}}\int_{\hat{\mathcal{F}}} d\mu(z)k(z, \gamma z) = (\hat{g} - 1)\int_{-\infty}^{+\infty} dp\, h(p)p\tanh(\pi p) + \sum_{\{\gamma\}_p}\sum_{k=1}^{\infty}\frac{l(\gamma)}{2\sinh(\frac{k}{2}l(\gamma))}g(kl(\gamma)) . \quad (8)$$

In order that all the sums and integrals converge absolutely, $h(p)$ has to be an even function, holomorphic in the strip $|Im\,p| \leq \frac{1}{2} + \varepsilon$, $\varepsilon > 0$, and has to decrease faster than $|p|^{-2}$ at infinity. $g(x) := \int_{-\infty}^{+\infty} \frac{dp}{2\pi} h(p) e^{ipx}$ is the Fourier-transform of $h(p)$, and the sum over $\{\gamma\}_p$ runs over all primitive conjugacy classes in $\hat{\Gamma}$. $l(\gamma)$ is the length of the closed geodesic on $\hat{\Sigma}$ that corresponds to the hyperbolic conjugacy class $\{\gamma\}_p$.

We now evaluate the second sum on the r.h.s. of (7). To this end we first study the set $\hat{\Gamma}I$ of reflections. We divide the primitive reflections in $\hat{\Gamma}I$ into two classes as discussed in [11]. Here a $\rho \in \hat{\Gamma}I$ is called primitive , if it cannot be written as a power of another reflection. Since $\rho \in \hat{\Gamma}I$ implies that $\rho^2 \in \hat{\Gamma}$, the reflections were classified in [11] according to the properties of their squares. Three cases are to be distinguished:

- $\rho = \rho_e$, $\rho_e^2 = 1$. These are the pure refletions in a geodesic.
- $\rho = \rho_i$, $\rho_i^2 \in \{C_i\}_{\hat{\Gamma}}$, $i = 1, \ldots, n$. The $\{C_i\}_{\hat{\Gamma}}$ are the conjugacy classes of the C_i in $\hat{\Gamma}$, which correspond to the closed geodesics c_i on $\hat{\Sigma}$.
- $\rho = \rho_p$, ρ_p^2 being a primitive element in $\hat{\Gamma}$ and $\rho_p^2 \notin \{C_i\}_{\hat{\Gamma}}$.

There are, however, no pure refletions in $\hat{\Gamma}$ that do not lie in the second class. To see this write $\rho_i = \gamma_i I$, $\gamma_i \in \hat{\Gamma}$. As $\rho_i c_i = c_i$ it follows that $Ic_i = \gamma_i^{-1} c_i$. Hence γ_i is the hyperbolic transformation that identifies the edge of $\hat{\mathcal{F}}$ in one half of $\mathcal{H}$ with its corresponding edge, which, by the symmetric construction of $\hat{\mathcal{F}}$, lies in the other half of $\mathcal{H}$. Now, the γ_i are the only transformations in $\hat{\Gamma}$ that identify edges in different halves of $\mathcal{H}$. Thus the ρ_i are the only primitive pure reflections.

We now split the sum over the $\rho \in \hat{\Gamma}I$ according to the above classification and get all elements out of the primitive ones by summing over all powers of the primitive reflections. There one must only take odd powers, since an even power of a reflection is a hyperbolic transformation. We define $I_{\hat{\mathcal{F}}}(\rho) := \int_{\hat{\mathcal{F}}} d\mu(z) k(z, \rho z)$ as a shorthand and get

$$\sum_{\rho \in \hat{\Gamma}} I_{\hat{\mathcal{F}}}(\rho) = \sum_{i=1}^{n} \sum_{\rho_i} \sum_{k=0}^{\infty} I(\rho_i^{2k+1}) + \sum_{\rho_p} \sum_{k=0}^{\infty} I_{\hat{\mathcal{F}}}(\rho_p^{2k+1}) \,. \tag{9}$$

With these expressions one can repeat the manipulations carried out in [11]:

$$\begin{aligned} \sum_{\rho} I_{\hat{\mathcal{F}}}(\rho^{2k+1}) &= \sum_{\{\rho\}} \sum_{\sigma \in \{\rho\}} I_{\hat{\mathcal{F}}}(\sigma^{2k+1}) \\ &= \sum_{\{\rho\}} \sum_{\gamma \in Z(\rho^2) \backslash \hat{\Gamma}} I_{\hat{\mathcal{F}}}(\gamma^{-1} \rho^{2k+1} \gamma) \\ &= \int_{Z(\rho^2) \backslash \mathcal{H}} d\mu(z) k(z, \rho^{2k+1} z) \,. \end{aligned} \tag{10}$$

Here $Z(\rho^2) := \{\gamma \in \hat{\Gamma} | \gamma \rho^2 \gamma^{-1} = \rho^2\}$ and $Z(\rho) := \{\gamma \in \hat{\Gamma} | \gamma \rho \gamma^{-1} = \rho\}$ are the centralizers of ρ^2 and ρ respectively. One easily finds that they are equal, $Z(\rho) = Z(\rho^2)$.

We now conjugate ρ^2 in such a way that it acts as a dilatation by a factor $N(\rho^2) = e^{l(\rho^2)}$, where $l(\rho^2)$ is the length of the closed geodesic on $\hat{\Sigma}$ corresponding to $\{\rho^2\}$. Then ρ itself acts as $\rho z = e^{l(\rho^2)/2}(-\bar{z})$. A fundamental domain for the centralizer $Z(\rho^2)$ is given by $\{z \in \mathcal{H}; 1 \leq$

$y < N(\rho^2)\}$. Thus

$$\begin{aligned} \sum_{\rho \in \hat{\Gamma}I} \int_{\hat{\mathcal{F}}} d\mu(z) k(z, \rho z) &= 2 \sum_{\{\rho_i\}} \sum_{k=0}^{\infty} \int_1^{N(C_i)} \frac{dy}{y^2} \int_0^{\infty} dx\ \Phi\left(\frac{|z + N^k(C_i)\bar{z}|^2}{N^k(C_i) y^2}\right) \\ &\quad + 2 \sum_{\{\rho_p\}} \sum_{k=0}^{\infty} \int_1^{N(\rho_p^2)} \frac{dy}{y^2} \int_0^{\infty} dx\ \Phi\left(\frac{|z + N^{k+1/2}(\rho_p^2)\bar{z}|^2}{N^{k+1/2}(\rho_p^2) y^2}\right) \,. \end{aligned} \tag{11}$$

A manipulation well-known from the proof of the Selberg trace formula in the cocompact case leads to (see e.g. [2])

$$\int_1^{N_0} \frac{dy}{y^2} \int_0^{\infty} dx\ \Phi\left(\frac{|z + N\bar{z}|^2}{Ny^2}\right) = \frac{1}{2} \frac{\ln N_0}{N^{1/2} + N^{-1/2}} g(\ln N) \,. \tag{12}$$

We use (12) in (11) and insert $N(\rho^2) = e^{l(\rho^2)}$. In the first sum of (11) we seperate the $k = 0$ term and introduce $L := \sum_{i=1}^{n} l(c_i)$ as the total length of $\partial\Sigma$. The lengths $l(c_i)$ are twofold degenerate, since C_i and C_i^{-1} both have to be included into the sum. We collect all this and formulate the trace formula as a theorem.

Theorem: Let $h(p)$ be an even function, analytic in the strip $|Im\,p| \leq \frac{1}{2} + \varepsilon$, $\varepsilon > 0$, and decreasing faster than $|p|^{-2}$ at infinity, with Fourier-transform $g(x) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} h(p)\, e^{ipx}$. Let $\lambda_n = p_n^2 + \frac{1}{4}$ be the eigenvalues of the Dirichlet-Laplace-Beltrami operator on the bordered Riemann surface Σ of signature (g, n), then ($\hat{g} = 2g + n - 1$, $\hat{g} \geq 2$)

$$\begin{aligned} \sum_{n=1}^{\infty} h(p_n) &= \frac{\hat{g} - 1}{2} \int_{-\infty}^{+\infty} dp h(p) p \tanh(\pi p) + \sum_{\{\gamma\}_p} \sum_{k=1}^{\infty} \frac{l(\gamma)}{4 \sinh(kl(\gamma)/2)} g(kl(\gamma)) \\ &\quad - \sum_{\{\rho_p\}} \sum_{k=0}^{\infty} \frac{l(\rho_p^2)}{4 \cosh[(k + 1/2) l(\rho_p^2)/2]} g((k + 1/2) l(\rho_p^2)) \\ &\quad - \frac{L}{4} g(0) - \sum_{i=1}^{n} \sum_{k=1}^{\infty} \frac{l(c_i)}{2 \cosh[kl(c_i)/2]} g(kl(c_i)) \,. \end{aligned} \tag{13}$$

REMARKS:

1. The function $h(p)$ has to satisfy the same conditions as in the cocompact case of (8), because the same manipulations have been done in the derivations of both formulae.
2. The case of Neumann boundary conditions can be treated in complete analogy to the Dirichlet case. One then has to study functions which are even under the reflection I on $\hat{\Sigma}$. The relevant integral kernel for which one has to evaluate the trace is given by

$$\hat{K}^N(z, z') := \frac{1}{2} \sum_{\gamma \in \hat{\Gamma}} [k(z, \gamma z') + k(z, \gamma(-\bar{z}'))] \,. \tag{14}$$

The only difference to (13) is that the minus signs in front of the three last terms change into plus signs.

3 Application to Heat Kernels, Zeta Functions and Determinants

The first spectral function that we want to study with the trace formula is the trace of the heat kernel for $\frac{\partial}{\partial t} - \Delta$. For $t > 0$ this is defined by (D=Dirichlet)

$$\theta_D(t) := Tr\, e^{t\Delta} = \sum_{n=1}^{\infty} e^{-\lambda_n t} \,. \qquad (15)$$

Inserting the function $h(p) = e^{-(p^2+\frac{1}{4})t}$ into the trace formula, we obtain $\theta_D(t) = \sum_{k=1}^{5} \theta_D^{(k)}(t)$, where the terms on the r.h.s. are labelled according to their appearence in (13):

$$\begin{aligned}
\theta_D^{(1)}(t) &= \frac{\hat{g}-1}{4\sqrt{\pi}} \frac{e^{-t/4}}{t^{3/2}} \int_0^\infty du \frac{u e^{u^2/4t}}{\sinh(u/2)} \,. \qquad (16)\\
\theta_D^{(2)}(t) &= \frac{1}{8\sqrt{\pi}} \frac{e^{-t/4}}{\sqrt{t}} \sum_{\{\gamma\}_p} \sum_{k=1}^{\infty} \frac{l(\gamma)}{\sinh(kl(\gamma)/2)} e^{-k^2 l^2(\gamma)/4t}\\
\theta_D^{(3)}(t) &= -\frac{1}{8\sqrt{\pi}} \frac{e^{-t/4}}{\sqrt{t}} \sum_{\{\rho_p\}} \sum_{k=0}^{\infty} \frac{l(\rho_p^2)}{\cosh((k+1/2)l(\rho_p^2)/2)} e^{-k^2 l^2(\rho_p^2)/4t}\\
\theta_D^{(4)}(t) &= -\frac{L}{8\sqrt{\pi}} \frac{e^{-t/4}}{\sqrt{t}}\\
\theta_D^{(5)}(t) &= -\frac{1}{4\sqrt{\pi}} \frac{e^{-t/4}}{\sqrt{t}} \sum_{i=1}^{n} \sum_{k=1}^{\infty} \frac{l(c_i)}{\cosh(kl(c_i)/2)} e^{-k^2 l^2(c_i)/4t} \,.
\end{aligned}$$

The first thing to study is the small-t asymptotics. One notices that the contributions $\theta_D^{(2)}$, $\theta_D^{(3)}$ and $\theta_D^{(5)}$ decrease exponentially for $t \to 0+$. Thus only $\theta_D^{(1)}$ and $\theta_D^{(4)}$ are relevant for the small-t asymptotics, giving

$$\theta_D(t) = \frac{\hat{g}-1}{2t} - \frac{L}{8\sqrt{\pi}} \frac{1}{\sqrt{t}} - \frac{1}{6}(\hat{g}-1) + O(\sqrt{t}) \,, \quad t \to 0+ \,. \qquad (17)$$

In the case of Neumann boundary conditions the term proportional to $t^{-\frac{1}{2}}$ changes its sign. The result (17) is exactly what one expects from the general expansion given by McKean and Singer [16].

The functional determinant of the Laplace-Beltrami operator will be defined by the method of zeta function regularization. Therefore we need to investigate the zeta function of Minakshisundaram-Pleijel (MP-zeta function), which is for $Re\, s > 1$ defined as

$$\zeta_D(s) := \sum_{n=1}^{\infty} \lambda_n^{-s} \,. \qquad (18)$$

In terms of this function the determinant of $-\Delta$ is defined to be $det(-\Delta)_D := e^{-\zeta_D'(0)}$. This definition requires an analytical continuation of ζ_D to $s = 0$, which is possible, because ζ_D is a meromorphic function with only a simple pole at $s = 1$.

We take $h(p) = (p^2 - (\sigma - \frac{1}{2}))^{-s}$, $Re\, s$, $Re\, \sigma > 1$, to use it in the trace formula. Then $\zeta_D(s) = \lim_{\sigma \to 1+} \sum_{n=1}^{\infty} h(p_n)$. Again we label the terms on the r.h.s. of the trace formula by

$\zeta_D^{(1)}, \ldots, \zeta_D^{(5)}$. They are given by

$$\begin{aligned}
\zeta_D^{(1)}(s) &= \frac{\hat{g}-1}{s-1} \frac{\pi}{2} \int_0^\infty dp \frac{(p^2+1/4)^{1-s}}{\cosh^2(\pi p)} \qquad (19)\\
\zeta_D^{(2)}(s) &= \frac{1}{2\Gamma(s)} \lim_{\sigma \to 1+} I(s;\sigma)\\
\zeta_D^{(3)}(s) &= -\frac{1}{2\Gamma(s)} \lim_{\sigma \to 1+} J(s;\sigma)\\
\zeta_D^{(4)}(s) &= -\frac{L}{\sqrt{\pi}} 2^{2s-4} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\\
\zeta_D^{(5)}(s) &= -\frac{1}{2\Gamma(s)} \lim_{\sigma \to 1+} H(s;\sigma) \,,
\end{aligned}$$

where we have introduced the functions $I(s;\sigma)$ and $J(s;\sigma)$, which are both entire functions of s for $Re\, \sigma > 1$, whereas $H(s;\sigma)$ is entire in s for $Re\, \sigma > 0$. These functions are defined by

$$\begin{aligned}
I(s;\sigma) &:= \frac{(2\sigma-1)^{\frac{1}{2}-s}}{\sqrt{4\pi}} \sum_{\{\gamma\}_p} \sum_{k=1}^{\infty} \frac{l}{\sinh(\frac{k}{2}l)} (kl)^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(kl(\sigma - \frac{1}{2})) \qquad (20)\\
J(s;\sigma) &:= \frac{(2\sigma-1)^{\frac{1}{2}-s}}{\sqrt{4\pi}} \sum_{\{\rho_p\}} \sum_{k=0}^{\infty} \frac{l}{\cosh((k+\frac{1}{2})\frac{l}{2})} ((k+\frac{1}{2})l)^{s-\frac{1}{2}} K_{s-\frac{1}{2}}((k+\frac{1}{2})l(\sigma-\frac{1}{2}))\\
H(s;\sigma) &:= \frac{(2\sigma-1)^{\frac{1}{2}-s}}{\sqrt{\pi}} \sum_{i=1}^{n} \sum_{k=1}^{\infty} \frac{l_i}{\cosh(\frac{k}{2}l_i)} (kl_i)^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(kl_i(\sigma-\frac{1}{2})) \,.
\end{aligned}$$

$K_\nu(z)$ is a modified Bessel function, and we drop the argument of $l(\gamma)$ and $l(\rho_p^2)$ whenever it may not lead to confusion; $l_i := l(c_i)$. (19) serves as an analytical continuation to $s = 0$, since the pole term has been explicitly extracted in $\zeta_D^{(1)}$.

In principle one could now study $\zeta_D'(0)$, but to do this properly we first investegate $I(0;\sigma)$, $J(0;\sigma)$ and $H(0;\sigma)$ for $Re\, \sigma > 1$ or $Re\, \sigma > 0$ respectively. After rearranging the k-summations in (20) we arrive at

$$\begin{aligned}
I(0;\sigma) &= -\ln \prod_{\{\gamma\}_p} \prod_{n=0}^{\infty} \left(1 - e^{-(\sigma+n)l}\right) = -\ln Z(\sigma) \qquad (21)\\
J(0;\sigma) &= -\ln \prod_{\{\rho_p\}} \prod_{n=0}^{\infty} \left(\frac{(1-e^{-l(n+\frac{1}{2}\sigma)})(1+e^{-l(n+\frac{1}{2}(\sigma+1))})}{(1-e^{-l(n+\frac{1}{2}(\sigma+1))})(1+e^{-l(n+\frac{1}{2}\sigma)})} \right) := -\ln Y(\sigma)\\
H(0;\sigma) &= -\ln \prod_{i=1}^{n} \prod_{m=0}^{\infty} \left(\frac{1-e^{-l_i(\sigma+2m)}}{1-e^{-l_i(\sigma+(2m+1))}} \right)^2 =: -\ln X(\sigma) \,.
\end{aligned}$$

$Z(\sigma)$ is the usual Selberg zeta function (on the doubled surface $\hat{\Sigma}$) and $Y(\sigma)$, $X(\sigma)$ are two new functions similar to $Z(\sigma)$, that come from the additional terms in the trace formula, which take care of the boundary conditions.

In a next step we would like to discuss the analytic properties of the newly introduced functions $X(s)$ and $Y(s)$. Therefore we study the regularized resolvent of $-\Delta$, which can be achieved by the choice $h(p) = [p^2 + (s-\frac{1}{2})^2]^{-1} - [p^2 + (\sigma - \frac{1}{2})^2]^{-1}$, for $Re\, s,\ Re\, \sigma > 1$, in the trace formula. The l.h.s. then gives

$$\lim_{\sigma\to 1+} \sum_{n=1}^{\infty} h(p_n) = \sum_{n=1}^{\infty} \left\{ \frac{1}{\lambda_n + s(s-1)} - \frac{1}{\lambda_n} \right\} . \tag{22}$$

Before performing the limit $\sigma \to 1+$ one obtains from the trace formula, by expressing the r.h.s. in terms of $Z(s)$, $Y(s)$ and $X(s)$,

$$\begin{aligned}\sum_{n=1}^{\infty} \left[\frac{1}{\lambda_n + s(s-1)} - \frac{1}{\lambda_n + \sigma(\sigma-1)} \right] &= -(\hat{g}-1)[\psi(s) - \psi(\sigma)] \\ &+ \frac{1}{2}\frac{1}{2s-1}\left\{ \frac{Z'(s)}{Z(s)} - \frac{Y'(s)}{Y(s)} - \frac{X'(s)}{X(s)} \right\} - \frac{L}{4}\frac{1}{2s-1} \\ &- \frac{1}{2}\frac{1}{2\sigma-1}\left\{ \frac{Z'(\sigma)}{Z(\sigma)} - \frac{Y'(\sigma)}{Y(\sigma)} - \frac{X'(\sigma)}{X(\sigma)} \right\} + \frac{L}{4}\frac{1}{2\sigma-1} ,\end{aligned} \tag{23}$$

where $\psi(z) := \Gamma'(z)/\Gamma(z)$ denotes the digamma function. It is known how $Z(\sigma)$ behaves in the limit $\sigma \to 1+$ [7],

$$\lim_{\sigma\to 1+} \left[\frac{1}{2\sigma-1}\frac{Z'(\sigma)}{Z(\sigma)} - \frac{1}{\sigma(\sigma-1)} \right] = \frac{1}{2}\frac{Z''(1)}{Z'(1)} - 1 =: B , \tag{24}$$

and thus the behaviour of $X(\sigma)$ and $Y(\sigma)$ may be deduced from the fact that the limit $\sigma \to 1+$ yields a finite result on the l.h.s. of (23). From (21) it is clear that

$$\frac{X'(1)}{X(1)} = -\frac{d}{d\sigma}H(0;\sigma)|_{\sigma=1} = \sum_{i=1}^{n}\sum_{k=1}^{\infty} 2l_i \frac{e^{-kl_i}}{1+e^{-kl_i}} \tag{25}$$

is finite and positive, as (21) converges for $Re\,\sigma > 0$ and all the summands are positive. Hence also the limit

$$\lim_{\sigma\to 1+} \left\{ \frac{1}{2\sigma-1}\left[\frac{Z'(\sigma)}{Z(\sigma)} - \frac{Y'(\sigma)}{Y(\sigma)} \right] \right\} =: B + A \tag{26}$$

exists, which together with (24) defines the constant A. From (26) one may draw the behaviour of the logarithmic derivative of $Y(\sigma)$ at $\sigma = 1$,

$$\frac{Y'(\sigma)}{Y(\sigma)} = \frac{1}{\sigma-1} - A + 1 + O(\sigma-1) , \quad \sigma \to 1 . \tag{27}$$

Therefore $Y(\sigma)$ itself has a simple zero at $\sigma = 1$ and $Y'(1)$ is finite. We now define $\gamma_D := (\hat{g}-1)\gamma + \frac{B+A}{2} - \frac{L}{4} - \frac{1}{2}\frac{X'(1)}{X(1)}$, where γ is Euler's constant. It may be shown that $\gamma_D = FP\zeta_D(1) := \lim_{s\to 1}(\zeta_D(s) - \frac{1}{2}\frac{\hat{g}-1}{s-1})$ is the finite part of $\zeta_D(s)$ at $s = 1$. With all these definitions the limit $\sigma \to 1+$ of (23) yields

$$\begin{aligned}\sum_{n=1}^{\infty} \left[\frac{1}{\lambda_n + s(s-1)} - \frac{1}{\lambda_n} \right] &= -(\hat{g}-1)\psi(s) - \frac{L}{4}\frac{1}{2s-1} - \gamma_D \\ &+ \frac{1}{2}\frac{1}{2s-1}\left\{ \frac{Z'(s)}{Z(s)} - \frac{Y'(s)}{Y(s)} - \frac{X'(s)}{X(s)} \right\} .\end{aligned} \tag{28}$$

This regularized trace of the resolvent is a meromorphic function of s and thus defines a meromorphic continuation of the function $X(s)Y(s)$ to all $s \in \mathbb{C}$. Knowing the analytic properties of the Selberg zeta function [2] one can obtain the poles and zeros of $X(s)Y(s)$ from (28). Denote the eigenvalues of $-\Delta$ on $\hat{\Sigma}$ by $\tau_n = \frac{1}{4} + r_n^2$ with multiplicities δ_n, then the non-trivial zeros of $Z(s)$ are $s_n = \frac{1}{2} \pm ir_n$. Furthermore denote the Dirichlet eigenvalues as before by $\lambda_n = \frac{1}{4} + p_n^2$ with mulitiplicities d_n and the Neumann eigenvalues as $\lambda_n^{(N)} = \frac{1}{4} + p_n^{(N)2}$ with multiplicities $d_n^{(N)}$. Then $X(s)Y(s)$ has poles of order d_n at $s_n = \frac{1}{2} \pm ip_n$ and zeros at $s_n = 0, 1, \frac{1}{2} \pm ip_n^{(N)}$ of order $1, 1, d_n^{(N)}$ respectively.

Another way of obtaining the analytic properties of $X(s)Y(s)$ is to express this product through determinant functions. Therefore define

$$\mathcal{D}_i(z) := det'(-\Delta + z)_i , \tag{29}$$

where the index $i = D, N, \Delta$ indicates, whether we take the case of Dirichlet or Neumann boundary conditions on $\partial\Sigma$ or $-\Delta$ on the whole of $\hat{\Sigma}$, and the prime denotes the omission of possible zero modes. The obvious relation $\mathcal{D}_\Delta(z) = \mathcal{D}_D(z)\mathcal{D}_N(z)$ is fulfilled for all z. This can be obtained using the regularization

$$\mathcal{D}_D(z) = \mathcal{D}_D(0)e^{\gamma_D z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{\lambda_n}\right) e^{-\frac{z}{\lambda_n}} \right] , \tag{30}$$

which is valid for all z. The other determinant functions are defined analogously. Using the representation [7]

$$Z(s) = s(s-1)\mathcal{D}_\Delta(s(s-1))e^{2C(\hat{g}-1)}\left[(2\pi)^{1-s}e^{s(s-1)}G(s)G(s+1)\right]^{2(\hat{g}-1)} , \tag{31}$$

with $C := \frac{1}{4} - \frac{1}{2}\ln 2\pi - 2\zeta'(-1)$, $\zeta(s)$ denoting Riemann's zeta function, and $G(z)$ being Barnes' double gamma function [17], one derives by integrating (28) in s,

$$X(s)Y(s) = X(1)Y'(1)s(s-1)e^{-\frac{L}{2}(s-1)}\frac{\mathcal{D}_D(0)}{\mathcal{D}_N(0)}\frac{\mathcal{D}_N(s(s-1))}{\mathcal{D}_D(s(s-1))} . \tag{32}$$

This can be pushed even a bit further by eliminating the unknown constants $X(1)$ and $Y'(1)$. To do this we study the limit $s \to \infty$ in (32).

Since $X(s)$ and $Y(s)$ are defined by Euler products (21), they both converge to unity for $s \to \infty$, $\lim_{s\to\infty} X(s)Y(s) = 1$. On the r.h.s. of (32) there occur the determinant functions, for which the same limit has to be investigated. Sarnak showed [18], that if the trace of a heat kernel has the small-t asymptotics $\theta(t) = \frac{a}{t} + \frac{b}{\sqrt{t}} + c + O(\sqrt{t})$, $t \to 0+$, then the corresponding determinant function has the asymptotics

$$\ln det(-\Delta + z) = az - (az - c)\ln z + 2b\sqrt{\pi}\sqrt{z} + f(z) , \quad z \to \infty , \tag{33}$$

where $\lim_{z\to\infty} f(z) = 0$. In our case this means,

$$\begin{aligned}\ln\frac{det(-\Delta + s(s-1))_N}{det(-\Delta + s(s-1))_D} &= \ln\frac{s(s-1)\mathcal{D}_N(s(s-1))}{\mathcal{D}_D(s(s-1))} \\ &= \frac{L}{2}\sqrt{s(s-1)} + f(s) .\end{aligned} \tag{34}$$

Thus

$$1 = \lim_{s\to\infty} X(s)Y(s) = X(1)Y'(1)\frac{\mathcal{D}_D(0)}{\mathcal{D}_N(0)}e^{\frac{L}{4}} \ . \tag{35}$$

We insert (35) into (32) and obtain the desired representation for $X(s)Y(s)$ in terms of determinant functions,

$$X(s)Y(s) = s(s-1)\frac{\mathcal{D}_N(s(s-1))}{\mathcal{D}_D(s(s-1))}e^{-\frac{L}{4}(2s-1)} \ , \tag{36}$$

which clearly exhibits the analytic properties that we already discussed above.

To evaluate the functional determinant of the Dirichlet-Laplace-Beltrami operator we require an analytic continuation of $X(s)$, $Y(s)$ and $Z(s)$ to $s = 1$. For Selberg's zeta function there is one available (see eq. (40) below)[2] and we want to recall this formula here. In exactly the same manner a similar representation for $X(s)$ and $Y(s)$ may be obtained as well. For $Z(s)$ one starts with McKean's integral representation [19]

$$\frac{Z'(s)}{Z(s)} = 2(2s-1)\int_0^\infty dt\ e^{-s(s-1)t}\theta_D^{(2)}(t) \ , \quad Re\, s > 1 \ . \tag{37}$$

The additional factor of two on the r.h.s. stems from the fact that our $\theta_D^{(2)}(t)$ is $\frac{1}{2}$ times McKean's $\vartheta(t)$.

We integrate this logarithmic derivative from $s > 1$ to $\sigma > s$ and after that perform $\sigma \to \infty$. (Notice that $\lim_{s\to\infty} Z(s) = 1$.) This gives for $Re\, s > 1$

$$Z(s) = \exp\left\{-2\int_0^\infty \frac{dt}{t}\theta_D^{(2)}(t)e^{-s(s-1)t}\right\} \ . \tag{38}$$

Splitting the region of integration and using ([20], p. 342)

$$\int_1^\infty \frac{dt}{t}e^{-xt} = E_1(x) = -\gamma - \ln x - \sum_{n=1}^\infty \frac{(-x)^n}{n\, n!} \tag{39}$$

in (38) yields

$$Z(s) = s(s-1)\exp\left\{\gamma + \sum_{n=1}^\infty \frac{[s(1-s)]^n}{n\, n!} - 2\int_0^1 \frac{dt}{t}\theta_D^{(2)}(t)e^{-s(s-1)t} - 2\int_1^\infty \frac{dt}{t}e^{-s(s-1)t}\left[\theta_D^{(2)}(t) - \frac{1}{2}\right]\right\} \ . \tag{40}$$

The large-t asymptotics that may be derived from the trace formula expressions, $\theta_D^{(2)}(t) = \frac{1}{2} + O(e^{-\alpha t})$, $t \to \infty$, $\alpha := min(r_1, \frac{1}{4})$, shows that the integrals in the exponential of (40) converge for $Re\, s > \frac{1}{2} + \sqrt{\frac{1}{4} + (Im\, s)^2 - \alpha}$. Since $\alpha > 0$ the representation (40) is valid in a neighbourhood of $s = 1$.

In complete analogy to this reasoning one can also find a representation for $Y(s)$ that converges in the vicinity of $s = 1$ and reads

$$Y(s) = s(s-1)\exp\left\{\gamma + \sum_{n=1}^\infty \frac{[s(1-s)]^n}{n\, n!} + 2\int_0^1 \frac{dt}{t}\theta_D^{(3)}(t)e^{-s(s-1)t} + 2\int_1^\infty \frac{dt}{t}e^{-s(s-1)t}\left[\theta_D^{(3)}(t) + \frac{1}{2}\right]\right\} \ . \tag{41}$$

[2] Formula (40) is an unpublished result of Aurich and Steiner.

(41) has the same region of convergence as (40). The representation (21) for $X(s)$ is valid for $s = 1$ anyway, thus it can most easily be rewritten in terms of $\theta_D^{(5)}$. For $Re\, s > \frac{1}{2}$ one gets

$$X(s) = \exp\left\{2\int_0^\infty \frac{dt}{t}e^{-s(s-1)t}\theta_D^{(5)}(t)\right\} \ . \tag{42}$$

(40), (41) and (42) together can now be used to express the determinant of $-\Delta$ in terms of $X(1)$, $Y'(1)$ and $Z'(1)$ or through θ_D respectively.

To evaluate $det'(-\Delta)_D$ we have to differentiate (19) at $s = 0$. Using the analytic properties of $I(s;\sigma)$, $J(s;\sigma)$ and $H(s;\sigma)$ derived above we get

$$\zeta_D'(0) = (\hat{g}-1)C + \frac{L}{8} + \ln\sqrt{X(1)} + \frac{1}{2}\lim_{\sigma\to 1+}[I(0;\sigma) - J(0;\sigma)] \ , \tag{43}$$

where we used the well-known result [6,7] $\zeta_D^{(1)\prime}(0) = (\hat{g}-1)C$. As $Y(\sigma)$ and $Z(\sigma)$ both have simple zeros at $\sigma = 1$ the result for the determinant is

$$det'(-\Delta)_D = \sqrt{\frac{Z'(1)}{X(1)Y'(1)}}\exp\left[-(\hat{g}-1)C - \frac{L}{8}\right] \ . \tag{44}$$

Using (40), (41) and (42) the prefactor of the exponential can be expressed through the trace of the heat kernel,

$$det'(-\Delta)_D = \exp\left\{-\int_0^\infty \frac{dt}{t}\left[\theta_D^{(2)}(t) + \theta_D^{(3)}(t) + \theta_D^{(5)}(t)\right]\right\}\exp\left[-(\hat{g}-1)C - \frac{L}{8}\right] \ . \tag{45}$$

In [13] we use (45) to express a part of the integrand in the formula for the scattering amplitudes in terms of the lengths l_i of the bordering curves c_i, that are the external string states of the scattering process.

The case of Neumann boundary conditions can easily be derived from (35),

$$\begin{aligned} det'(-\Delta)_N &= det'(-\Delta)_D X(1)Y'(1)e^{\frac{L}{4}} \\ &= \sqrt{X(1)Y'(1)Z'(1)}\exp\left[-(\hat{g}-1)C + \frac{L}{8}\right] \ . \end{aligned} \tag{46}$$

4 Summary

In this paper we derived a Selberg trace formula for bordered Riemann surfaces. This formula allowed us to express functions of the eigenvalues of the Laplace-Beltrami operator, endowed with either Dirichlet or Neumann conditions on the surface's boundary, through the lengths of the closed geodesics on the compact double of the surface. On the other hand this could be viewed as a trace formula for reflection symmetric Riemann surfaces concerning the spectral problem of the Laplace-Beltrami operator on either even or odd functions under the symmetry operation.

We discussed spectral functions of $-\Delta$ for both Dirichlet and Neumann boundary conditions and evaluated the respective functional determinants. In addition we gave an expression that involved the relation between determinant functions of either cases.

Since in our final formula (45) the dependence of the determinant on the lengths of the bordering curves is made explicit, this result may be used in string theory to study the dependence of scattering amplitudes on the lengths of the external string states. Although this result was our main motivation to study the trace formula, we hope to have presented another example of an exact (non-semiclassical) periodic-orbit formula for quantum chaology.

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