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Fermions in Two (1+1)-Dimensional Anomalous Gauge Theories: The Chiral Schwinger Model and the Chiral Quantum Gravity

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Abstract

The fermion in the gauge invariant formulation of the chiral Schwinger model and its relation to the fermion in the anomalous formulation is studied. A gauge invariant fermion operator is constructed that does not give rise to an asymptotic fermion field. It fits in the scheme prepared by generalized Schwinger models. Singularities in the short-distance limit of the chiral Schwinger model in the anomalous formulation lead to the conclusion that it is not a promising starting point for investigations towards realistic (3+1)-dimensional gauge theories with chiral fermion content. A new anomalous (1+1)-dimensional model is studied, the chiral quantum gravity. It is proven to be consistent if only a limited number of chiral fermions couple. The fermion propagator behaves analogously to the one in the massless Thirring model. A general rule is derived for the change of the fermion operator, which is induced by the breakdown of a gauge symmetry.

CONTENTS

INTRODUCTION

Sometimes symmetries of classical Lagrangians are broken at the quantum level. In 1969 Adler, Bardeen, Bell and Jackiw [1] showed that such an anomaly emerges in the fermionic triangle diagram for the π^0-decay. Here the gauge fields couple to the conserved vector current, while it is the global axial current which is not conserved. Things become dangerous if the gauge fields couple to nonconserved chiral currents [2]. Consistency, unitarity and renormalizability of the model may be lost if the fermion content of the model is not chosen so as to cancel the total anomaly. This is the case in the Standard Model, and the necessary cancellation also causes the general believe in the existence of the top-quark and a τ-neutrino different from the e- and μ-neutrinos.

In 1971 Wess and Zumino [3] proved that special consistency conditions have to be fulfilled by the anomalies and constructed an effective action by solving them. This action has many applications, one of them is again the π^0-decay.

Since then many new facets and applications of anomalies have been found [4], but a rough classification can already be done on the basis of the topics mentioned above: There are anomalies in global currents, which are often necessary to describe physical effects, and there are anomalies in (dynamical) gauge currents, which seem to lead to inconsistent, i.e. useless, models.

This common knowledge was shaken by a development starting about 1984. D'Hoker and Farhi [6] demonstrated that a decoupling of some chiral fermions in a vector model gives rise to a model with chiral fermion content which is nevertheless still consistent, since an anomaly-cancelling Wess-Zumino term is generated by the procedure. Faddeev and Shatashvili [7] also studied anomaly cancellation by a Wess-Zumino term, and in 1985 Jackiw and Rajaraman [8] presented the first anomalous, consistent model: the chiral Schwinger model, i.e. chiral QED in (1+1) dimensions. This discovery induced to exciting new ideas about realistic (3+1)-dimensional models [9]. An anomaly could be the reason for the mass generation of gauge bosons, thereby rendering the ad-hoc Higgs boson superfluous. During the last years no essential progress has been made towards realistic nonabelian chiral (3+1)-dimensional models. There exist however investigations on an abelian theory (chiral QED) [10], which usually avoid the renormalizability question. This can be justified by referring to the anomalous model as an effective theory of a more fundamental one, which is reliable only in the low-energy domain.

In the first part of the present work I study the fermionic sector of the chiral Schwinger

model and of a chiral quantum gravity in (1+1) dimensions. For the chiral Schwinger model there is some confusion about the important question whether it has single particle states in the interacting sector. This arises from the gauge invariant formulation [12], which was developed by reconsidering the anomalous formulation of ref.[8]. I resolve this problem by demonstrating that the physical content of the gauge invariant formulation is different from the one in the anomalous formulation.

In the second part of my work I present a review over the chiral quantum gravity, a new anomalous, consistent model, which has been developed by I. Tsutsui and myself [56,57,58]. The chiral quantum gravity shares many features with the chiral Schwinger model if a special gauge is chosen. This is shown for the solution derived by bosonization and for the fermion propagator.

THE CHIRAL SCHWINGER MODEL

In the chiral Schwinger model (CSM) a fermion couples chirally to an U(1)-gauge field in (1+1) dimensions. Since chiral gauge symmetry is lost by quantization, one should expect that the CSM gives rise to an inconsistent, nonunitary quantum theory. However, in 1985 Jackiw and Rajaraman [8] proved that in spite of the obvious anomaly the model is consistent and, for special values of a regularization parameter, also unitary. The reason for this unexpected behaviour is that the anomaly awakes new dynamical degrees of freedom, which are unphysical in an analogous gauge invariant model: The longitudinal component of the gauge field. This formulation is called "anomalous" and abbreviated by AF in the present work.

Halliday et al. [11] considered a gauge invariant formulation in the spirit of Faddeev and Shatashvili [7], where a Wess-Zumino term is added to cancel the variation of the original action under gauge transformations.

The next important step towards an understanding of the CSM was done when the Wess-Zumino term was shown to arise naturally by adopting the Faddeev-Popov trick for quantizing a theory with an anomaly [12]. Instead of the usual integration over classes of gauge equivalent field configurations, one really has to integrate over all configurations. Harada and Tsutsui [12] proposed a gauge invariant standard action, which defines the gauge invariant formulation (GIF) of the CSM.

These steps were accompanied by thorough investigations of the quantization in the Hamiltonian formulation [13], which is usually believed to be more reliable than the Lagrangian formalism. In the AF second class constraints where found, which signals the breakdown of the gauge invariance, while in the GIF only first class constraints exist. Moreover, Falck and Kramer [14] showed that the AF arises within the GIF if a special gauge is fixed, the "unitary gauge". This confirmed the impression from the Lagrangian formalism that the AF should just be the GIF with the Wess-Zumino scalar set to zero [11,12].

In the original paper on the CSM, Jackiw and Rajaraman detected a pole for vanishing momentum in the photon propagator, which they related to unconfined fermions. This was an interesting suggestion, since (1+1)-dimensional QED, the Schwinger model (SM) [15], has served as a standard example for confinement, respectively screening [16]. Girotti, Rothe and Rothe [17] calculated the fermion propagator and confirmed the existence of an asymptotic fermion field and consequently single particle fermionic states. The corresponding fermion operator was shown not to carry chiral charge, so it may be at best the product of a screening procedure.

Later investigations relying on the GIF in the Landau gauge however came to different conclusions [18,19]: in fact, the GIF of the CSM closely parallels the SM. A charged fermion operator exists which is confined by a screening mechanism, and the usual gauge invariant Green's function does not give rise to fermionic single particle states. With respect to the well-established idea that the AF is just the unitary gauge of the GIF, the previous result on the fermion seemed to be gauge dependent. The situation turned out to be more involved when different explicitly gauge invariant fermionic Green's functions where presented for the GIF, which correspond to different fermionic spectra [19,20].

In the present work I want to give an answer to the question whether there exists an interacting fermion operator which gives rise to single particle states in the GIF of the CSM, as has been shown for the AF [17].

There are basically two motivations to throw light on the fermionic sector of the CSM, although thanks to bosonization in (1+1) dimensions one can solve the model without caring for fermions. First of all, the CSM may be seen as a toy model in the ambitious program to investigate the possibility of anomalous models in (3+1) dimensions, where no bosonization is at hand. Another attitude is to look at the CSM as a new interesting model in its own right, which can be compared to other well-known (1+1)-dimensional models like the SM, the Schroer model [21] or the Thirring model [22].

In some sense these two attitudes towards the CSM cannot both be respected at the same time. A thorough analysis on the fate of the fermionic states necessitates a detailed investigation of the Hilbert space structure, as they exist e.g. for the SM [38] and the Schroer model [21]. However, such rigorous results cannot be extrapolated to the situation in higher dimensions. It is demanding to present a solution to the problem in a framework which teaches us something about the possibility to construct consistent anomalous models in (3+1) dimensions. We will see that the understanding of the fermionic sector in the CSM is decisive for the question whether to rely on the AF or the GIF.

The plan for secs.1 to 7 is the following:

Section 1 contains a short survey of the solution of the CSM in the AF, including the derivation of the fermion propagator. Then these results are compared to the corresponding ones for the conventional SM.

Afterwards, in sec.2, the GIF is presented and the solution for the Landau gauge is written down. The unitary gauge is found to lead back to the AF. I write down three gauge invariant fermionic Green's functions which imply contradictory answers to the question concerning the existence of an asymptotic fermion field.

In sec.3 I consider some generalizations of the SM and the CSM. This will help us to discriminate between the unique features of the CSM and the common ones.

Section 4 contains the operator solutions of the CSM in the AF and the GIF, Landau gauge. Special care is devoted to the construction of the fermion operators, to keep track of the renormalization constants.

In sec.5 an operator transformation is used to check the relation between different fermion operators. The appearence of renormalization problems indicate that a change of the physical state space may be implicit in formal transformations. A special gauge invariant fermion operator is singled out to represent the physical content of the GIF. It does not give rise to a propagator with a pole for vanishing momentum.

Section 6 finishes the investigation of the long-distance region with an review of the different Schwinger models.

In sec.7 the short-distance region is studied. A perturbation theoretic derivation of two fermion propagators is done, with special emphazise on the physical picture behind the calculation. The advantage of the GIF compared to the AF for future considerations on (3+1)-dimensional anomalous models is stressed.

Some features of boson and fermion propagators used in the text are collected in Appendix A. The dimensional regularization scheme and details about the calculations in sec.7 are presented in Appendix B.

1. The Chiral Schwinger Model in the Anomalous Formulation

The essential features of the CSM in its anomalous formulation (AF) are presented. First the consistency and unitarity is proved, then the photon propagator and the fermion propagator are derived. At last the properties of the (conventional) SM are reviewed.

Derivation of the Effective Action

The Lagrangian of the CSM reads [1]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\partial\!\!\!/ + e\sqrt{\pi}A\!\!\!/(1-\gamma^5)\right)\psi. \tag{1.1}$$

This is invariant under the chiral gauge transformation

$$\begin{aligned} A_\mu \to A'_\mu &= A_\mu + \tfrac{1}{e}\partial_\mu\Lambda, \\ \psi \to \psi' &= \exp\{i\sqrt{\pi}\Lambda(1-\gamma^5)\}\psi. \end{aligned} \tag{1.2}$$

However, one should expect that this symmetry is lost due to the chiral anomaly as soon as the model is quantized. Possible problems connected to an anomaly in the (local) gauge current are inconsistency, nonunitarity and nonrenormalizibility. Since there is no need for conventional renormalization in our (1+1)-dimensional model, the last problem is absent. How about the other two?

The partition function is

$$Z = \frac{1}{N}\int dA d\psi d\bar{\psi} e^{iS[\bar{\psi},\psi,A]}, \tag{1.3}$$

where $S = \int d^2x\mathcal{L}$ is the classical action and N an infinite normalization constant which is dropped henceforth. I want to calculate the effective action $W[A]$,

$$e^{iW[A]} = \int d\psi d\bar{\psi} e^{iS[\bar{\psi},\psi,A]}, \tag{1.4}$$

by integrating out the fermion. In (1+1) dimensions, one can express the gauge potential A_μ as follows:

$$A_\mu = \frac{1}{e}(\partial_\mu\sigma + \epsilon_{\mu\nu}\partial^\nu\rho), \tag{1.5}$$

with the scalar fields σ and ρ. Since we have

$$\gamma^\mu A_\mu(1\pm\gamma^5) = \frac{1}{e}\gamma^\mu\partial_\mu(\sigma\mp\rho)(1\pm\gamma^5), \tag{1.6}$$

the Lagrangian (1.1) can be turned into a free one by the transformation

$$\begin{aligned} \psi \to \chi &= \exp[i\sqrt{\pi}(\sigma-\rho)(1-\gamma^5)]\psi, \\ \bar{\psi} \to \bar{\chi} &= \bar{\psi}\exp[-i\sqrt{\pi}(\sigma-\rho)(1-\gamma^5)]. \end{aligned} \tag{1.7}$$

[1] my conventions are: $\gamma^0 = \sigma^1$, $\gamma^1 = i\sigma^2$, $\gamma^5 = -\gamma^0\gamma^1$, $\epsilon^{01} = -\epsilon_{01} = 1$; $\psi = (\psi_R, \psi_L)^T$

It is well-known, however, that the fermionic measure in (1.3) is not gauge invariant under the chiral transformation (1.2) [5]. This gives rise to a Jacobian:

$$J_L[A] = \det\frac{\partial\psi}{\partial\chi}\det\frac{\partial\bar\psi}{\partial\bar\chi} = \left[\det e^{-i\sqrt{\pi}(\sigma+\rho)\gamma^5}\right]^2. \tag{1.8}$$

For an infinitesimal $\sigma+\rho$ one gets

$$\log J_L[A] = 1 - 2i\sqrt{\pi}\,\mathrm{Tr}[\gamma^5(\sigma+\rho)]. \tag{1.9}$$

Here "Tr" is defined according to Fujikawa:

$$\mathrm{Tr}[\gamma^5(\sigma+\rho)] = \int d^2x_E(\sigma+\rho)\lim_{M\to\infty}\sum_n \varphi_n^+(x)\gamma^5 e^{-\frac{\not{D}_H^2}{M^2}}\varphi_n(x), \tag{1.10}$$

where $\not{D}_H$ is the hermitian Dirac operator, and φ_n a complete set of eigenfunctions; M is a cutoff-parameter and the integration measure d^2x_E indicates that the derivation has to be done in the Euclidian. Many authors have calculated (1.10) for the CSM; I follow Falck [25], to get a consistent result when the gauge invariant CSM is employed in sec.2. The problem mainly consists in finding the correct way to introduce the regularization ambiguity which arises in the CSM for its lack of chiral gauge invariance. Here I write down only the decisive steps:

$$\begin{aligned} C(x) &= \lim_{M\to\infty}\sum_n \varphi_n^+\gamma^5 e^{-\frac{\not{D}_H^2}{M^2}}\varphi_n \\ &= \lim_{M\to\infty}\mathrm{tr}\int\frac{d^2k_E}{(2\pi)^2}e^{-ikx}\gamma^5 e^{-\frac{1}{M^2}(\not{D}_H^2-\frac{i\epsilon}{2}\gamma_E^\mu\gamma_E^\nu\tilde F_{\mu\nu})}e^{ikx} \\ &= -\frac{i\epsilon}{4\pi}\epsilon^{\mu\nu}\tilde F_{\mu\nu} = -\frac{i\epsilon}{2\pi}\epsilon^{\mu\nu}\partial_\mu B_\nu, \end{aligned} \tag{1.11}$$

where $\tilde F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, and

$$B_\mu = \frac{1}{2}\left[(r+s)g_{\mu\nu} + (r-s)\epsilon_{\mu\nu}\right]A^\mu. \tag{1.12}$$

With $A_\mu^\pm = (g_{\mu\nu}\pm\epsilon_{\mu\nu})A^\nu$ I get

$$\log J_L[A] = 1 - \epsilon\int d^2x(\sigma+\rho)\epsilon^{\mu\nu}\partial_\mu(rA_\nu^+ + sA_\nu^-). \tag{1.13}$$

If I started with a right-handed fermion coupling to the gauge field, I had $(r-s)\to(s-r)$ in (1.12) and the result for the sum of J_L and J_R would be expression (1.13) with r,s both substituted by $r+s$. Therefore one has to demand $r+s=1$, and we are left with only one free regularization parameter.

It is decisive to realize that both components of the gauge potential A_μ have to be used to define the ambiguous B_μ-field. The anomaly forbids the naive reasoning that one may forget about the right-handed sector altogether, because of the Lagrangian (1.1). If this is not respected, an inconsistency arises when the Wess-Zumino term is considered [25].

The infinitesimal form of $J_L[A]$, (1.13), may be used to derive a differential equation, and its integration gives the result

$$\log J_L[A] = -\frac{1}{2}\int d^2x\left[(r-s)(\sigma+\rho)\Box(\sigma+\rho) - 2e(\sigma+\rho)\partial^\mu(rA_\mu^- - sA_\mu^+)\right]. \tag{1.14}$$

By use of

$$\sigma+\rho = e\frac{\partial_\mu}{\Box}A^{+\mu} \tag{1.15}$$

this may be brought into the well-know nonlocal form; with the kinetic term of the gauge field the result is [8]

$$W[A] = \int d^2x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e^2sA^2 - \frac{1}{2}e^2A_\mu^+\frac{\partial_\mu\partial_\nu}{\Box}A_\nu^+\right]. \tag{1.16}$$

To get formal coincidence with the standard form used generally, one has to put $s\equiv\frac{a}{2}$. Usually the nonlocal term in (1.16) is eliminated by introducing a scalar field ϕ:

$$W[A,\phi] = \int d^2x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}ae^2A^2 - \frac{1}{2}\phi\Box\phi - e\phi\partial_\mu A^{+\mu}\right]. \tag{1.17}$$

The gauge current to which A_μ couples reads

$$J^\mu = e(g^{\mu\nu} - \epsilon^{\mu\nu})\partial_\nu\phi + e^2aA^\mu. \tag{1.18}$$

The equation of motion are for A_μ,

$$\partial_\mu F^{\mu\nu} = -J^\nu, \tag{1.19}$$

and for ϕ,

$$\Box\phi + e\partial_\mu A^{+\mu} = 0. \tag{1.20}$$

It is not obvious that the model is consistent for $a\neq 1$, i.e. $\partial_\mu\partial_\nu F^{\mu\nu} = -\partial_\mu J^\mu = 0$ by (1.18). However, this relation is fulfilled, as we will see below.

The Particle Content and the Photon Propagator

The next step is to find out what particle content the model has. Using (1.5), i.e. $e\partial_\mu F^{\mu\nu} = \epsilon^{\nu\alpha}\partial_\alpha\Box\rho$, one can rewrite the equations of motion for A_μ

$$\epsilon^{\nu\alpha}\partial_\alpha\Box\rho + \epsilon^2a(\partial^\nu\sigma + \epsilon^{\nu\alpha}\partial_\alpha\rho) + e(\partial^\nu\phi - \epsilon^{\nu\alpha}\partial_\alpha\phi) = 0, \tag{1.21}$$

and for ϕ

$$\Box\phi + \Box\sigma + \Box\rho = 0. \tag{1.22}$$

Applying $\epsilon_{\nu\beta}\partial^\beta$ to (1.21) and inserting $\Box\phi$ from (1.22) gives

$$\Box\left[\Box\rho + (a+1)\epsilon^2\rho\right] + \epsilon^2\Box\sigma = 0. \tag{1.23}$$

Take $\rho = (a-1)\sigma + h$, with $\Box h = 0$. Then one derives $\Box(\Box + m^2)\sigma = 0$; that is, σ can be split into a massive and a massless scalar: $\sigma = \sigma_m + H$. For σ_m we have

$$\begin{aligned} (\Box + m^2)\sigma_m &= 0, \\ m^2 &= \frac{e^2 a^2}{a-1} . \end{aligned} \tag{1.24}$$

By use of (1.21) one deduces $H = -\frac{1}{a}h$ and $\phi = -a\sigma_m + h$, so that A_μ can be expressed by two scalar particles, a massive (σ_m) and a massless one (h):

$$eA_\mu = \partial_\mu\left(\sigma_m - \frac{1}{a}h\right) + \epsilon_{\mu\nu}\partial^\nu\left[(a-1)\sigma_m + \frac{1}{a}h\right]. \tag{1.25}$$

By inserting this A_μ in (1.19) and taking the derivative one realizes that the model is consistent. For $a > 1$ it is also unitary. I will always suppose that this relation is fulfilled.

The effective action $W[A]$, (1.16), is quadratic in A_μ; it can be expressed as

$$\begin{aligned} W[A] &= \int d^2x \frac{1}{2} A_\mu M^{\mu\nu} A_\nu, \\ M^{\mu\nu} &= \left(\Box + (a+1)e^2\right) g^{\mu\nu} - \left(\Box + 2e^2\right)\frac{\partial^\mu\partial^\nu}{\Box} + e^2 \frac{\epsilon^{\mu\alpha}\partial_\alpha\partial^\nu + \epsilon^{\nu\alpha}\partial_\alpha\partial^\mu}{\Box} . \end{aligned} \tag{1.26}$$

The photon propagator is then easily derived by inverting $M^{\mu\nu}$:

$$P^{\mu\nu} = i[M^{-1}]^{\mu\nu} = \frac{i}{-\Box - m^2}\left(-g^{\mu\nu} + \frac{1}{a-1}\left[\left(-\frac{\Box}{e^2} - 2\right)\frac{\partial^\mu\partial^\nu}{\Box} + \frac{\epsilon^{\mu\alpha}\partial_\alpha\partial^\nu + \epsilon^{\nu\alpha}\partial_\alpha\partial^\mu}{\Box}\right]\right), \tag{1.27}$$

or, in momentum space,

$$\tilde{P}^{\mu\nu}(k) = \frac{i}{k^2 - m^2}\left(-g^{\mu\nu} - \frac{1}{a-1}\left[\left(\frac{k^2}{e^2} - 2\right)\frac{k^\mu k^\nu}{k^2} + \frac{\epsilon^{\mu\alpha}k_\alpha k^\nu + \epsilon^{\nu\alpha}k_\alpha k^\mu}{k^2}\right]\right). \tag{1.28}$$

One finds that $\tilde{P}^{\mu\nu}(k)$ has two poles with positive residuum: one for $k^2 = m^2$ and another for $k^0 - k^1 = 0$. The first one can be interpreted as indicating the existence of a massive scalar; the second one was suggested to be related to pairs of unconfined fermions [8].

THE FERMION PROPAGATOR

For later purposes I present the derivation of the fermion propagator with a general ansatz. It is given by

$$G(x-y)_{\alpha\beta} = \langle T\left(\bar{\psi}_\alpha(x)\psi_\beta(y)\right)\rangle = \int dA d\psi d\bar{\psi}\, \bar{\psi}_\alpha(x)\psi_\beta(y) e^{iS(\psi,\bar{\psi},A)}. \tag{1.29}$$

Instead of the transformation (1.7) we have

$$\begin{aligned} \psi &\to \chi = \exp[i\sqrt{\pi}(\sigma - \gamma^5\rho)]\psi, \\ \bar{\psi} &\to \bar{\chi} = \bar{\psi}\exp[-i\sqrt{\pi}(\sigma - \gamma^5\rho)]. \end{aligned} \tag{1.30}$$

Substituting this into (1.29) gives

$$G(x-y)_{\alpha\beta} = \int dA d\chi d\bar{\chi} [e^{-i\sqrt{\pi}(\sigma - \gamma^5\rho)}\chi]_\alpha(x) [\bar{\chi} e^{i\sqrt{\pi}(\sigma + \gamma^5\rho)}]_\beta(y) e^{i(W[A] - \bar{\chi} i\partial\!\!\!/ \chi)}. \tag{1.31}$$

The integration over the free fermion gives a constant factor, which can be absorped in the normalization constant N in (1.3). With $P_{R,L} = \frac{1}{2}(1 \pm \gamma^5)$ one has

$$\begin{aligned} e^{-i\sqrt{\pi}(\sigma - \gamma^5\rho)} &= e^{-i\sqrt{\pi}P_R(\sigma - \rho)} + e^{-i\sqrt{\pi}P_L(\sigma + \rho)} - 1 , \\ e^{i\sqrt{\pi}(\sigma + \gamma^5\rho)} &= e^{i\sqrt{\pi}P_R(\sigma - \rho)} + e^{i\sqrt{\pi}P_L(\sigma + \rho)} - 1 . \end{aligned} \tag{1.32}$$

Defining the right- (left-) handed propagator by

$$G_{R,L}(x-y) = P_{R,L}G(x-y)P_{L,R}, \tag{1.33}$$

I end up with

$$G_{R,L}(x-y) = G^{free}_{R,L}(x-y)\int dA\, e^{-i\sqrt{\pi}(\sigma \mp \rho)(x)}\, e^{i\sqrt{\pi}(\sigma \mp \rho)(y)} e^{iW[A]} , \tag{1.34}$$

so that the matrix character of $G(x-y)_{\alpha\beta}$ in (1.29) does not do any harm. Therefore I will now confine myself to a left-handed fermion propagator.

$\sigma + \rho$ is expressed by A^-_μ, (1.15), and $\tilde{A}(k) = \int d^2k e^{ikx} A(x)$; $\tilde{A}(k) = \tilde{A}(-k)$. Eq.(1.31) is reformulated into

$$\begin{aligned} G_L(x-y) = G^{free}_L(x-y) \int dA \exp\Bigg\{ \frac{i}{2}\int \frac{d^2k}{(2\pi)^2} \Bigg[&\tilde{A}_\mu(k)\left((-i\tilde{P})^{-1}(k)\right)^{\mu\nu}\tilde{A}_\nu(k) \\ &+ \sqrt{\pi}\underbrace{\left(-e^{ikx}\frac{i\epsilon}{k^2}(k^\mu - \epsilon^{\mu\alpha}k_\alpha) + e^{iky}\frac{i\epsilon}{k^2}(k^\mu - \epsilon^{\mu\alpha}k_\alpha)\right)}_{J^\mu(x,y;k)}\tilde{A}_\mu(k) \\ &+ \tilde{A}_\mu(k)\underbrace{\left(e^{-ikx}\frac{i\epsilon}{k^2}(k^\mu - \epsilon^{\mu\alpha}k_\alpha) - e^{-iky}\frac{i\epsilon}{k^2}(k^\mu - \epsilon^{\mu\alpha}k_\alpha)\right)}_{J^{*\mu}(x,y;k)}\sqrt{\pi}\Bigg]\Bigg\}. \end{aligned} \tag{1.35}$$

The A-integration is gaussian, and so the result can be written down directly:

$$G_L(x-y) = G^{free}_L(x-y)\exp\left\{\frac{\pi}{2}\int\frac{d^2k}{(2\pi)^2} J_\mu(x,y;k)\tilde{P}^{\mu\nu}(k)J^*_\nu(x,y;k)\right\}. \tag{1.36}$$

When the photon propagator (1.28) is substituted, one has

$$\begin{aligned} G_L(x-y) &= G^{free}_L(x-y)\exp\left\{\frac{4\pi i}{a-1}\int\frac{d^2k}{(2\pi)^2}\left(e^{-ik(x-y)} - 1\right)\frac{1}{k^2 - m^2 + i0}\right\} \\ &= G^{free}_L(x-y)\exp\left\{-\frac{4\pi i}{a-1}\tilde{\Delta}_F(x-y;m^2)\right\}. \end{aligned} \tag{1.37}$$

Here I used $\tilde{\Delta}(x;m^2) \equiv \Delta(x;m^2) - \Delta(0;m^2)$. Usually a renormalized fermion propagator is considered instead of (1.37). It is defined with the help of the wavefunction renormalization constant

$$\mathcal{Z} = \exp\left\{-\frac{4\pi i}{a-1}\int\frac{d^2k}{(2\pi)^2}\frac{1}{k^2 - m^2 - i0}\right\} \simeq \left(\frac{\Lambda^2}{m^2}\right)^{-\frac{1}{a-1}} . \tag{1.38}$$

where Λ is a ultraviolet (UV) cutoff mass. Then the renormalized propagator is $G_L^{ren} = \mathcal{Z}^{-1}G_L$. In the short-distance limit, $-(x-y)^2 \to 0$, it aquires an anomalous dimension $\frac{1}{a-1}$:

$$G_L^{ren}(x-y) = G_L^{free}(x-y)\left[-m^2(x-y)^2\right]^{-\frac{1}{a-1}}. \tag{1.39}$$

This feature is known to appear also in nonanomalous (1+1)-dimensional models like the Schroer model [21] and the Thirring model [22].

More important than the behaviour in the short-distance limit is the long-distance behaviour, $-(x-y)^2 \to \infty$. In fact, G_L^{ren} becomes a free propagator. This is usually interpreted as indicating the existence of an asymptotic fermion field, i.e. of single particle fermionic states. Moreover, the fermion does not carry chiral charge [17]; this can be easily seen in the operator formalism introduced in sec.4, so that I do not show this point in the present section.

The form of (1.37) suggests that the same fermion propagator can be derived by using a massive scalar instead of the gauge potential A_μ. The operator expression for A_μ is such that $e(1\pm\gamma^5)A_\mu = (1\pm\gamma^5)\partial_\mu\Lambda$, with a massive scalar Λ satisfying the commutation relation $[\Lambda(x),\Lambda(0)] = \frac{i}{a-1}\Delta(x;m^2)$ (see sec.4). So one may write

$$\begin{aligned} G_L(x-y) = G_L^{free}(x-y) \quad \int d\Lambda \exp\Big\{2i\int\frac{d^2k}{(2\pi)^2}\Big[\tilde\Lambda(k)(-i\tilde P_\Lambda)^{-1}(k)\tilde\Lambda(k) \\ -i\sqrt{\pi}\left(e^{ikx}-e^{iky}\right)\tilde\Lambda(k) + \tilde\Lambda(k)i\left(e^{-ikx}-e^{-iky}\right)\sqrt{\pi}\Big]\Big\} \end{aligned} \tag{1.40}$$

instead of (1.35). With the propagator

$$\tilde P_\Lambda(k) = -\frac{i}{a-1}\frac{1}{k^2-m^2}, \tag{1.41}$$

the result for G_L is again (1.37).

The Schwinger Model

To get an impression of the novel features introduced by the CSM, one has to remember the corresponding knowledge of the SM [15]. It is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\psi\left(i\partial\!\!\!/ + 2e\sqrt{\pi}A\!\!\!/\right)\psi. \tag{1.42}$$

The effective action can be directly derived from the result for the CSM, if the effective action for a right-handed fermion is added to (1.16). By use of

$$A_\mu^\pm\frac{\partial^\mu\partial^\nu}{\Box}A_\nu^\pm = -A^2 + 2A_\mu\frac{\partial^\mu\partial^\nu}{\Box}A_\nu \mp A_\mu\frac{\epsilon^{\mu\alpha}\partial_\alpha\partial^\nu + \epsilon^{\nu\alpha}\partial_\alpha\partial^\mu}{\Box}A_\nu, \tag{1.43}$$

one gets

$$W[A] = \int d^2x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + 2e^2A^2 - 2e^2A_\mu\frac{\partial_\mu\partial_\nu}{\Box}A_\nu\right]. \tag{1.44}$$

No ambiguity arises in this action, since one has to demand that the gauge invariance of (1.42) is still present in (1.44). The interesting point in (1.44) is that the originally massless gauge field A_μ has aquired a mass, $m^2 = 4e^2$. This is due to an axial anomaly, which, however, must not be mixed up with the anomaly considered above for the CSM. In fact, if we choose the representation (1.5) for A_μ and use the gauge invariance of the model to eliminate the longitudinal component, it seems that the transformation

$$\psi \to \psi' = e^{i\sqrt{\pi}\rho\gamma^5}\psi \tag{1.45}$$

makes (1.42) a free Langrangian. The mass term is now due to the noninvariance of the fermionic measure in the partition function. The difference to the CSM is that (1.45) does not belong to a current which couples to the gauge field, as has been the case for (1.2). Therefore the present anomaly does not jeopardize the consistency of the SM.

Let us look at the photon propagator. Since gauge invariance is preserved in (1.44), one has to fix a gauge. I take $\mathcal{L}_{GF} = B\partial_\mu A^\mu - \frac{\alpha}{2}B^2$, the Lorentz gauge. The photon propagator is

$$P^{\mu\nu} = \frac{i}{-\Box-m^2}\left[-g^{\mu\nu} + \frac{\partial^\mu\partial^\nu}{\Box} + \alpha(\Box+m^2)\frac{\partial^\mu\partial^\nu}{\Box^2}\right], \tag{1.46}$$

and the fermion propagator

$$\begin{aligned} G(x-y) = G^{free}(x-y)\exp\Big\{\pi i\int\frac{d^2k}{(2\pi)^2}\left(e^{-ik(x-y)}-1\right) \\ \times\left(\frac{1}{k^2-m^2+i0} - \frac{1}{k^2+i0} + \frac{\alpha}{(k^2+i0)^2}\right)\Big\}. \end{aligned} \tag{1.47}$$

For the Landau gauge, $\alpha = 0$, one has

$$G(x-y) = G^{free}(x-y)\exp\left\{-\pi i\left[\bar\Delta_F(x-y;m^2) - \bar D_F(x-y)\right]\right\}, \tag{1.48}$$

where $\bar D$ is defined analogously to $\bar\Delta$. G is both UV- and IR-convergent. The "-1" in the exponent of (1.47) plays the role of an IR-renormalization (see Appendix A). The fermion propagator does not become a free propagator in the long-distance limit. On the other hand, this is reached in the short-distance limit, i.e. there is no anomalous dimension.

The long-distance behaviour is usually interpreted to indicate screening of the fermion's charge. In fact, if the Wilson-loop

$$\mathcal{W} = \langle\exp\{ie\oint A_\mu dz^\mu\}\rangle \tag{1.49}$$

is calculated [19], one finds the following potential between static charges separated by a distance d:

$$V \propto 1 - e^{-md}, \tag{1.50}$$

which becomes constant for large d, i.e. the charge is screened. There are many investigations which confirm this screening behaviour (see e.g. [16]).

2. The Chiral Schwinger Model in the Gauge Invariant Formulation

The GIF of the CSM is presented and the fermion propagators in the Landau gauge and the unitary gauge are derived. A puzzling difference in their conventional interpretation is detected. Gauge invariant fermion propagators are proposed which also do not give a unique answer to the question whether there are fermionic single particle states in the GIF, as has been shown to be the case in the AF.

The Effective Action with Wess-Zumino Term

Soon after Jackiw and Rajaraman proposed the CSM as an anomalous gauge theory which can be quantized consistently, it was shown that there exists an alternative formulation for it, which employs a gauge invariant effective action. Instead of (1.3), Harada and Tsutsui [12] derived the equivalent partition function

$$Z^{HT} = \int d\psi d\bar{\psi}\mathcal{D}A dg\, e^{iS[\bar{\psi},\psi,A]+i\alpha_1[A,g^{-1}]}. \tag{2.1}$$

This has been gained by modifying the Faddeev-Popov procedure in accord with the lacking gauge invariance of Z, (1.3). g is an element of the chiral gauge group, $\alpha_1[A,g^{-1}]$ the Wess-Zumino term and $\mathcal{D}A \equiv dA\Delta_f[A]\delta(f[A])$.

Starting from (2.1) one can easily derive the new effective action: note that the Wess-Zumino term is defined by

$$\alpha_1[A,g^{-1}] = W[A^{g^{-1}}] - W[A]. \tag{2.2}$$

The Wess-Zumino scalar θ is connected with g by $g = \exp(-i\theta)$. α_1 can be derived by substituting $\sigma+\rho$ by $-\theta$ in (1.13), the Jacobian for an infinitesimal transformation parameter. I end up with

$$\log \alpha_1[A,\theta] = \frac{1}{2}\int d^2x[(r-s)\theta\Box\theta + 2e\theta\partial^\mu(rA_\mu^+ - sA_\mu^-)]. \tag{2.3}$$

The Lagrangian defining the GIF of the CSM is (with $s = \frac{a}{2}$, $r = 1-\frac{a}{2}$)

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\partial\!\!\!/ - e\sqrt{\pi}A\!\!\!/(1-\gamma^5)\right)\psi \\ & -\frac{1}{2}(a-1)\theta\Box\theta - e\frac{1}{2}\theta\partial^\mu\left((a-2)A_\mu^+ + aA_\mu^-\right),\end{aligned} \tag{2.4}$$

and instead of (1.17) one has

$$\begin{aligned}W[A,\phi,\theta] = \int d^2x\Big[&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}e^2aA^2 - \frac{1}{2}\phi\Box\phi - e\phi\partial_\mu A^{+\mu} \\ &-\frac{1}{2}(a-1)\theta\Box\theta - e\frac{1}{2}\theta\partial^\mu((a-2)A_\mu^+ + aA_\mu^-)\Big].\end{aligned} \tag{2.5}$$

The effective action is invariant under the local gauge transformation

$$A_\mu \to A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\Lambda;\quad \phi \to \phi' = \phi - \Lambda;\quad \theta \to \theta' = \theta - \Lambda. \tag{2.6}$$

The Landau Gauge and the Unitary Gauge

Before photon and fermion propagators can be derived from the gauge invariant effective action (2.1), one has to fix a gauge. A common choice is the Lorentz gauge, which is fixed by adding $\mathcal{L}_{GF} = B\partial_\mu A^\mu - \frac{\alpha}{2}B^2$ to the Lagrangian in (2.5). I will later confine myself to $\alpha = 0$, i.e. the Landau gauge. The gauge current reads

$$J^\mu = e(g^{\mu\nu} - \epsilon^{\mu\nu})\partial_\nu\phi + e^2aA^\mu + e\left[(a-1)g^{\mu\nu} + \epsilon^{\mu\nu}\right]\partial_\nu\theta \tag{2.7}$$

The equation of motion for B is

$$\partial_\mu A^\mu + \alpha B = 0, \tag{2.8}$$

for A_μ

$$\partial_\mu F^{\mu\nu} - \partial^\nu B + J^\nu = 0, \tag{2.9}$$

for ϕ

$$\Box\phi + e\partial_\mu A^\mu = 0, \tag{2.10}$$

and for θ

$$(a-1)\Box\theta + e[(a-1)g^{\mu\nu} - \epsilon^{\mu\nu}]\partial_\mu A_\nu = 0. \tag{2.11}$$

To get the effective action $W[A]$ from (2.5), the scalar ϕ and the Wess-Zumino scalar θ have to be integrated out,

$$\int dAd\phi d\theta e^{iW[A,\phi,\theta]} = \int dAe^{iW[A]}, \tag{2.12}$$

For ϕ this is trivial; θ is replaced by the free scalar $\tilde{\theta}$,

$$\tilde{\theta} = \sqrt{a-1}\,\theta + \frac{e}{2\sqrt{a-1}}\frac{\partial^\mu}{\Box}((a-1)A_\mu^+ + aA_\mu^-). \tag{2.13}$$

I get

$$W[A] = \int d^2x\left[\frac{1}{2}A_\nu\left(\Box + \frac{e^2a^2}{a-1}\right)\left(g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\Box}\right)A_\nu + \frac{1}{2\alpha}A_\mu\frac{\partial^\mu\partial^\nu}{\Box}A_\nu\right]. \tag{2.14}$$

By inversion one derives ($\alpha = 0$):

$$P^{\mu\nu} = \frac{i}{-\Box - m^2}\left[-g^{\mu\nu} + \frac{\partial^\mu\partial^\nu}{\Box}\right], \tag{2.15}$$

with $m^2 = \frac{a^2e^2}{a-1}$.

Now the fermion propagator can be calculated analogously to the procedure in sec.1; one has

$$
\begin{aligned}
G_L(x-y) &= G_L^{free}(x-y)\exp\left\{4\pi i\frac{a-1}{a^2}\int\frac{d^2k}{(2\pi)^2}\left(e^{-ik(x-y)}-1\right)\right.\\
&\qquad\left.\times\left(\frac{1}{k^2-m^2+i0}-\frac{1}{k^2+i0}\right)\right\} \qquad (2.16)\\
&= G_L^{free}(x-y)\exp\left\{-4\pi i\tfrac{a-1}{a^2}\left[\bar{\Delta}_F(x-y;m^2)-\bar{D}_F(x-y)\right]\right\},
\end{aligned}
$$

G_L is exactly the Landau gauge fermion propagator of the SM, (1.48), if $a=2$. It becomes a free propagator in the short-distance limit, and exhibits screening behaviour in the long-distance limit.

Another possible gauge fixing condition is given by $\mathcal{L}_{GF}=B\theta$. This unitary gauge corresponds to $\theta=0$, i.e. $g=1$, and it has been suggested that the AF is reached by fixing this gauge in the GIF (see e.g. refs.[12,14]). In fact, the equation of motion for θ, (2.11), is fulfilled in the unitary gauge by the solution for A_μ in the AF (1.25), and therefore no degrees of freedom are lost if $\theta=0$ is directly applied to the effective action. It is clear that the photon and fermion propagator in the unitary gauge are given just by (1.27) and (1.37), respectively. In contradistinction to the Landau gauge fermion propagator the unitary gauge one becomes free in the long-distance limit, indicating the existence of fermionic single particle states in the interacting sector. So we have the confusing situation that different gauges give different answers to the question whether the chiral fermion coupling to the gauge field gives rise to an asymptotic field. One may suspect that the trouble comes from interpreting gauge noninvariant objects; this is, however, not the case.

Gauge Invariant Fermion Propagator

In the GIF of the CSM the physical information should be deduced from gauge invariant quantities. There are three obvious gauge invariant fermion propagators:

$$G_L^\theta(x-y)=\int\mathcal{D}Ad\psi d\bar{\psi}d\theta\,\bar{\psi}_L(x)e^{-2ie(\theta(x)-\theta(y))}e^{iS[\psi,\bar{\psi},A]+i\alpha_1[A,\theta]}\psi_L(y), \qquad (2.17)$$

$$G_L^A(x-y)=\int\mathcal{D}Ad\psi d\bar{\psi}\,\bar{\psi}_L(x)e^{2ie\int_x^y A_\mu(z)dz^\mu}e^{iS[\psi,\bar{\psi},A]+i\alpha_1[A,\theta]}\psi_L(y), \qquad (2.18)$$

$$G_L^\phi(x-y)=\int\mathcal{D}Ad\psi d\bar{\psi}d\phi\,\bar{\psi}_L(x)e^{-2ie(\phi(x)-\phi(y))}e^{iS[\psi,\bar{\psi},A]+i\alpha_1[A,\theta]}\psi_L(y). \qquad (2.19)$$

By the transformation properties of θ, A_μ and ϕ it is clear that all three are gauge invariant. Following the standard procedure to derive an explicit expression for the propagator, one realizes that (2.17) is just the fermion propagator of the AF, which has already been identified as the unitary gauge propagator. A transformation of ψ and $\bar{\psi}$ which cancels the exponential function $\exp[-2ie(\theta(x)-\theta(y))]$ produces a Wess-Zumino term by the variation of the fermionic measure; this in turn cancels the Wess-Zumino term in (2.17) and the θ-integration can be

dropped. One ends up with the expression (1.37). This is generally valid: If all operators of the GIF are made gauge invariant with the help of a Wess-Zumino field, one has exactly the same observables as in the AF.

This specific choice to construct gauge invariant objects is however not unique; the second fermion propagator G_L^A is the usual choice for the SM [15]. Expression (2.18) necessitates a more tedious calculation, which has been done in ref.[27]. The result is hard to interprete in the long-distance region, but there is some evidence that the propagator does not become a free one. This will be confirmed later; I also postpone the interpretation of G_L^ϕ till operator solutions are available in sec.5. What is very obvious there would render necessary an involved calculation in the present framework. Then we will also see that all gauge invariant fermion operators are neutral, i.e. they do not carry chiral charge. Therefore the charge is screened in the CSM (AF and GIF) as in the SM, but the question remains whether the fermion also disappears from the spectrum.

The Problem

The main problem to be attacked in the first part of my work is whether there is an asymptotic fermion field in the left-handed sector of the GIF of the CSM. Since this question has an affirmative answer for the AF, I deal at the same time with the relation between these two formulations.

3. Generalizations of the Schwinger Model

Some generalizations of the Schwinger model are considered. This includes a model with chiral fermionic content generalizing the chiral Schwinger model, a multifermion vector Schwinger model with artificially broken gauge symmetry, the asymmetric Schwinger model with three chirally coupled fermions implying anomaly cancellation, and a Schwinger model with vector boson mass in the classical Lagrangian.

A Generalized Chiral Schwinger Model

A very general ansatz for a model consisting of fermions chirally coupled to an abelian gauge field is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{i=1}^{n_L} \bar{\psi}_{L_i}\left(i\partial\!\!\!/ + \sqrt{\pi}g_{L_i}A\!\!\!/(1-\gamma^5)\right)\psi_{L_i} + \sum_{j=1}^{n_R}\bar{\psi}_{R_j}\left(i\partial\!\!\!/ + \sqrt{\pi}g_{R_j}A\!\!\!/(1+\gamma^5)\right)\psi_{R_j}, \tag{3.1}$$

i.e. n_L left-handed fermions couple with the coupling constants g_{L_i} and n_R right-handed fermions with coupling constants g_{R_j}. The derivation of the effective action $W[A]$ can be done as in sec.1. The fermions are transformed by

$$\begin{aligned}\rho_{L_i,R_j} &= \exp\left\{i\sqrt{\pi}\frac{g_{L_i,R_j}}{e}(\tau \pm \rho)(1 \mp \gamma^5)\right\}\psi_{L_i,R_j},\\ \bar{\rho}_{L_i,R_j} &= \bar{\psi}_{L_i,R_j}\exp\left\{-i\sqrt{\pi}\frac{g_{L_i,R_j}}{e}(\sigma \pm \rho)(1 \pm \gamma^5)\right\}.\end{aligned} \tag{3.2}$$

Instead of (1.16), one has

$$\begin{aligned}W[A] = \int d^2x \Bigg[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} &+ \sum_i g_{L_i}^2\left(s_{L_i}A^2 - \frac{1}{2}A_\mu^+\frac{\partial^\mu\partial^\nu}{\Box}A_\nu^+\right)\\ &+ \sum_j g_{R_j}^2\left(r_{R_j}A^2 - \frac{1}{2}A_\mu^-\frac{\partial^\mu\partial^\nu}{\Box}A_\nu^-\right)\Bigg].\end{aligned} \tag{3.3}$$

If we suppose that the classical gauge symmetry of (3.1) is not broken at the quantum level, we have $s_{L_i} = r_{R_j} = \frac{1}{2}$ and

$$\sum_i g_{L_i}^2 = \sum_j g_{R_j}^2. \tag{3.4}$$

The experience with the CSM indicates that it may be reasonable to disregard the relation (3.4) and get nevertheless a consistent model. If $n_R \neq n_L$ I call it the generalized chiral Schwinger model (GCSM). A GIF of the GCSM is obtained by adding the Wess-Zumino term

$$\begin{aligned}\alpha_1[A,\theta] = \int d^2x \Bigg[&\sum_i\left(\frac{1}{2}(r_{L_i} - s_{L_i})\theta\Box\theta - g_{L_i}\theta\partial^\mu(r_{L_i}A_\mu^- - s_{L_i}A_\mu^+)\right)\\ &- \sum_j\left(\frac{1}{2}(r_{R_j} - s_{R_j})\theta\Box\theta - g_{R_j}\theta\partial^\mu(s_{R_j}A_\mu^- - r_{R_j}A_\mu^+)\right)\Bigg].\end{aligned} \tag{3.5}$$

Here I take the $\theta = 0$ gauge, i.e. the AF of the GCSM. The photon propagator derived from $W[A]$ is

$$\begin{aligned}P^{\mu\nu} = \frac{i}{-\Box - \tilde{m}^2}\Bigg[-g^{\mu\nu} &+ \frac{\left(\Box + 2(\sum_i g_{L_i}^2 + \sum_j g_{R_j}^2)\right)}{2e^2\left(\sum_i s_{L_i} + \sum_j r_{R_j} - \frac{1}{2}(n_L + n_R)\right)}\frac{\partial^\mu\partial^\nu}{\Box}\\ &- \frac{\left(\sum_i g_{L_i}^2 - \sum_j g_{R_j}^2\right)}{2e^2\left(\sum_i s_{L_i} + \sum_j r_{R_j} - \frac{1}{2}(n_L + n_R)\right)}\frac{\epsilon^{\mu\alpha}\partial_\alpha\partial^\nu + \epsilon^{\nu\alpha}\partial_\alpha\partial^\mu}{\Box}\Bigg],\end{aligned} \tag{3.6}$$

with

$$\tilde{m}^2 = \frac{\left(\sum_i g_{L_i}^2 - \sum_j g_{R_j}^2\right)^2}{2e^2\left(\sum_i(s_{L_i} - \frac{1}{2}) + \sum_j(r_{R_j} - \frac{1}{2})\right)} + 2e^2\left(\sum_i s_{L_i} + \sum_j r_{R_j} + \frac{1}{2}(n_L + n_R)\right). \tag{3.7}$$

To keep in contact with the CSM, I put $g_{L_i} = g_{R_j} = e$, and $s_{L_i} = r_{R_j} = \frac{a}{2}$. Then the mass can be expressed as

$$m^2 = e^2\frac{(n_L - n_R)^2 + (a^2 - 1)(n_L + n_R)^2}{(a-1)(n_L + n_R)}. \tag{3.8}$$

For $n_L = 1$, $n_R = 0$ we get $m^2 = \frac{e^2a^2}{a-1}$, the mass of the CSM. The SM mass can be derived by putting $n_R = n_L = 1$ and $a = 1$, as it should be.

When the pole structure of the Fourier transformed photon propagator is studied, one finds poles with positive residua for $k^2 = m^2$, $k^0 - k^1 = 0$ and $k^0 + k^1 = 0$. It is not possible to interprete the last two poles as has been done for the CSM, i.e. as indicating the existence of pairs of unconfined fermions. There fermionic single particle states have been found not only in the uncoupled right-handed, but also in the left-handed sector. The right- and left-handed fermion propagators of the model defined by (3.1) and (3.8) look as follows:

$$\begin{aligned}G_{L_i,R_j}(x-y) = G_{L,R}^{free}(x-y)\exp\Bigg\{\frac{4\pi i}{(a-1)(n_L + n_R)}\int\frac{d^2k}{(2\pi)^2}&\left(e^{-ik(x-y)} - 1\right)\\ &\times\frac{k^2 - 4e^2 n_{R,L}}{(k^2 + i0)(k^2 - m^2 + i0)}\Bigg\}.\end{aligned} \tag{3.9}$$

While one always has an anomalous dimension in the short-distance limit, as in the CSM, one never gets a free propagator in the left- (right-) handed sector if there are right- (left-) handed fermions coupling. This stresses the unique position of the CSM in its AF.

Vector Schwinger Models with Artificially Broken Gauge Symmetry

Now I confine myself to a vector Schwinger model (VSM) with $n_R = n_L = n$ and $g_{L_i} = g_{R_j} = e$. This model has already been studied (see e.g. [28,29]), and up to the presence of $n-1$ massless bosons in the spectrum the physical content of it was found to parallel the $n = 1$-case, i.e.

the SM. This means especially that in the Landau gauge the fermion propagator is of the form (1.48).

Here I break the gauge symmetry artificially by adopting the prescription $s_{L_i} = r_{R_j} = \frac{a}{2}$. I will henceforth use the term VSM with broken gauge symmetry (BGS). The fermion propagator now reads

$$G(x-y) = G^{free}(x-y)\exp\left\{ \frac{4\pi i}{2(a-1)n}\int \frac{d^2k}{(2\pi)^2}\left(e^{-ik(x-y)}-1\right) \times \left(\frac{1}{k^2+i0} + \frac{2e^2(a-1)n}{(k^2+i0)(k^2-m^2+i0)}\right)\right\}. \tag{3.10}$$

This form resembles the propagator for the Landau gauge VSM, up to an extra term that comes from a physical massless scalar and which produces an anomalous dimension. There is no value of a for which the propagator becomes a free one in the long-distant limit. Since the VSM (Landau gauge) part of (3.10) contains a contribution from a massless ghost scalar, it is important to realize that the new physical scalar never cancels this contribution, which would deliberate the fermion. A special VSM is of course the SM; the effective action for the SM with BGS is

$$W[A] = \int d^2x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e^2(a+1)A^2 - 2e^2 A_\mu \frac{\partial^\mu\partial^\nu}{\Box}A_\nu\right]. \tag{3.11}$$

The mass of the photon reads $m^2 = 2e^2(a+1)$. In the SM we had $m^2 = 4e^2$, i.e. $a = 1$. The photon propagator reads

$$P^{\mu\nu} = \frac{i}{-\Box - m^2}\left[-g^{\mu\nu} - \frac{1}{2e^2(a-1)}(\Box + 4e^2)\frac{\partial^\mu\partial^\nu}{\Box}\right], \tag{3.12}$$

and the fermion propagator is (3.10) with $n = 1$.

This has to be compared to the spectrum represented by the effective action (3.11). After bosonizing the nonlocal term in $W[A]$ one gets the equations of motion (for A_μ)

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu + 4e^2\partial^\nu\phi = 0 \tag{3.13}$$

and (for ϕ)

$$\Box\phi + \partial_\mu A^\mu = 0. \tag{3.14}$$

For A_μ the expression (1.5) is inserted; by applying ∂_ν to (3.13) I get $\Box\phi = \Box\sigma = 0$, i.e. $\partial_\mu A^\mu = 0$. Then I apply $\epsilon_{\nu\beta}\partial^\beta$ to eq.(3.13), and introduce $\rho = \zeta + \phi$ to solve

$$\Box\left(\Box\rho - m^2\rho\right) = 0. \tag{3.15}$$

This leads to $(\Box + m^2)\zeta = 0$ and, from (3.13),

$$\partial_\mu\sigma = -\frac{4e^2}{m^2}\partial_\mu\phi - \epsilon_{\mu\nu}\partial^\nu\zeta. \tag{3.16}$$

Substituting ρ and σ in $A_\mu = \frac{1}{e}(\partial_\mu\sigma + \epsilon_{\mu\nu}\partial^\nu\rho)$, one ends up with

$$eA_\mu = -\frac{4e^2}{m^2}\partial_\mu\phi + \epsilon_{\mu\nu}\partial^\nu\zeta. \tag{3.17}$$

That means the spectrum is equal to the one of the CSM: one massive (ζ) and one massless (ϕ) scalar. The longitudinal component of A_μ is expressed by ϕ. In contrast to the SM, this massless scalar is not spurious, i.e. it cannot be gauged away. If one realizes that $\langle T(A_\mu(x)A_\nu(y))\rangle$ is given by (3.12) it becomes obvious that the k^{-2}-term in the fermion propagator (3.10) is generated by the factor $\exp\{[\partial_\mu\Box^{-1}A^\mu, \partial_\nu\Box^{-1}A^\nu]\}$, i.e. the longitudinal component of A_μ. This factor together with the Landau gauge solution of the SM makes up expression (3.10) with $n = 1$.

A Massive Vector Model

There is another possible generalization of the SM. A mass is given to the vector field in the SM [32]:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\psi\left(i\partial\!\!\!/ + 2e\sqrt{\pi}B\!\!\!/\right)\psi + \frac{1}{2}M^2B^2, \tag{3.18}$$

with $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, i.e. B is now a massive vector potential. The gauge symmetry is explicitly broken already at the classical level. The derivation of the effective action is straightforward.

$$\begin{aligned} W[B] &= \int d^2x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left(2e^2 + \tfrac{1}{2}M^2\right)B^2 - 2e^2 B_\mu\frac{\partial^\mu\partial^\nu}{\Box}B_\nu\right] \\ &= \int d^2x \frac{1}{2}B_\mu\left[g^{\mu\nu}\left(\Box + 4e^2 + M^2\right) - 4e^2\frac{\partial^\mu\partial^\nu}{\Box}\right]B_\nu. \end{aligned} \tag{3.19}$$

The photon propagator is

$$P^{\mu\nu} = \frac{i}{-\Box - m^2}\left[-g^{\mu\nu} + \frac{\Box + 4e^2}{M^2}\frac{\partial^\mu\partial^\nu}{\Box}\right], \tag{3.20}$$

with $m^2 = 4e^2 + M^2$, and the fermion propagator reads

$$G(x-y) = G^{free}(x-y)\exp\left\{4\pi i\frac{e^2}{M^2}\int\frac{d^2k}{(2\pi)^2}\left(e^{-ik(x-y)}-1\right) \times \left(\frac{1}{k^2+i0} + \frac{M^2}{(k^2+i0)(k^2-m^2+i0)}\right)\right\}. \tag{3.21}$$

This coincides with eq.(4.35) of ref.[32], if the conventions are fitted (cp. also ref.[33], eq.(29)). There are no fermionic single particle states, although the photon propagator (3.20) contains a pole at $k^2 = 0$ in momentum space, which is related to a physical massless scalar field. One has again a contribution from a physical massless scalar in the fermion propagator, which however never exactly cancels the unphysical contribution ($M^2/m^2 < 1$). In this sense the massive vector model closely resembles the VSM with BGS. The decisive difference is that for vanishing coupling, $\epsilon \to 0$, the propagator (3.21) becomes a free one.

The Asymmetric Schwinger Model

In ref.[19] we showed that the SM can be identified with the CSM in the GIF, if one of its chiral fermions is integrated out, the corresponding scalar is related to the Wess-Zumino scalar and a freely introduced regularization parameter is given a special value. This idea can be extended by starting with the asymmetric Schwinger model (ASM) [28]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\partial\!\!\!/ + g_{L_1}\sqrt{\pi}A\!\!\!/(1-\gamma^5)\right)\psi + \bar{\psi}\left(i\partial\!\!\!/ + g_{L_2}\sqrt{\pi}A\!\!\!/(1-\gamma^5)\right)\psi + \bar{\psi}\left(i\partial\!\!\!/ + g_R\sqrt{\pi}A\!\!\!/(1+\gamma^5)\right)\psi, \quad (3.22)$$

and [30,31]

$$g_{L_1} = \epsilon, \quad g_{L_2} = \frac{2-a}{2\sqrt{a-1}}\epsilon, \quad g_R = \frac{a}{2\sqrt{a-1}}\epsilon, \quad (3.23)$$

so the mass is just $m^2 = 4g_R^2 = \frac{\epsilon^2 a^2}{a-1}$. In this model the chiral anomaly is cancelled by a properly chosen fermion content; relation (3.4) is fulfilled. In the GIF of the CSM the same effect has been reached by the bosonic content, i.e. by the introduction of the Wess-Zumino scalar.

Integrating out the two fermions coupled by g_{L_2} and g_R, one can exactly derive the Wess-Zumino term (2.3). This means that the ASM is in some sense equivalent to the GIF of the CSM, given by the Lagrangian (2.5). The correspondence is however not perfect, since ψ_{L_1} has to be treated on different footing than ψ_{L_2} and ψ_R if the (standard) effective action (2.5) shall be derived.

To analyse (3.22) I integrate out the fermions and transform the three scalar fields according to

$$\varphi \to \phi' = \phi - \frac{g}{\Box}\partial^\mu A^+_\mu. \quad (3.24)$$

The result is the equivalent bosonic Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\phi'_1\Box\phi'_1 - \frac{1}{2}\phi'_2\Box\phi'_2 - \frac{1}{2}\varphi'_3\Box\varphi'_3 - \sqrt{\pi}g_R\phi'_3\epsilon_{\mu\nu}\partial^\nu A^\mu. \quad (3.25)$$

It is equal to the result for the SM [29], apart from the appearence of two kinetic term for ϕ'_1 and φ'_2 in (3.25).

As for the fermion propagators the situation parallels the SM (and other nonanomalous Schwinger models). Although there are massless scalars in the physical spectrum of the solution, none of them couples to change the SM propagator (1.48).

Intermediate Conclusions and Comments

In the present section I introduced some generalizations of the Schwinger model to be compared to the CSM. Let us summarize the most important features.

For the GCSM in the AF it has been shown that in general it does not give rise to an asymptotic fermion field (see propagator (3.9)). An exception is represented by models where the fermions couple only with one chirality, as in the CSM.

Then I considered the VSM with BGS. The VSM (without BGS) has already been studied years ago and it is known that if n fermions couple, $n-1$ massless scalars and a massive one with mass $m^2_{\rm VSM} = nm^2_{\rm SM}$ constitute the spectrum of the solution derived by bosonization. With the artificially BGS I showed that already for $n = 1$ a massless scalar appears in addition to a massive one, whose mass depends on a regularization parameter. Moreover, thanks to the broken symmetry, the fermion propagator can be written down and it reflects the coupling of the new massless scalar to the fermion. There is an anomalous dimension in the short-distance limit, as in the AF of the CSM, but the fermion propagator does not become a free one in the long-distance region.

By using operator solutions Boyanovsky has proposed an attractive picture [23]: The ghost contribution of the SM is cancelled in the CSM by the new massless scalar and thereby the fermion becomes deconfined. In the light of the GCSM with the fermion propagator (3.9) one has to ask why this does not happen here as well. My answer is that it never happens. This cancellation appears in the operator solution of a model (like the CSM, AF), but the comparision with the SM is misleading. The change of the fermion operator in the VSM when a BGS is introduced artificially suggests that the AF fermion operator of the CSM does not originate from a cancellation of massless contributions. For the VSM with BGS we have seen that it is the longitudinal component of the gauge field which destroys the gauge invariance when we demand $a \neq 1$. It becomes physical and couples to the Landau gauge fermion of the VSM. In the AF of the CSM the left-handed part of the gauge field destroys the gauge symmetry of the free fermion model. It becomes physical and massive (maybe by a screening mechanism) and couples to the *free* fermion operator. From this point of view, the fermionic states have been there before the anomaly comes in.

While in the GIF of the CSM the gauge symmetry is restored by the Wess-Zumino scalar, the ASM is made gauge invariant by fitting the fermion content. By bosonization of two of the three fermions a close resemblence between the two models has been detected [11,30,31]. Note that the physical content of the ASM can be interpreted in more than one way. Crossing out the terms with coupling constants g_{L_2} and g_R in the Lagrangian (3.22) gives a well-defined unitary theory: The CSM in the AF. One can also eliminate the unphysical degrees of freedom of a covariant solution, which is the standard procedure ([28], see also sec.5). It is obvious that e.g. the VSM can also be treated like the ASM. Nobody would say that the physical content of a two fermion VSM is obtained by eliminating three chiral couplings, leaving the CSM in the AF. The "unitary gauge" of the GIF considered in sec.2 refers to such a procedure, and if one employs the relation between the ASM and the GIF of the CSM, one should not fix this gauge.

4. The Chiral Schwinger Model in the Operator Formalism

The operator solutions for the GIF in the Landau gauge and the unitary gauge are written down. The corresponding fermion operators are constructed with special emphazise on the necessary renormalization.

Expressing A_μ, ϕ and θ by Basic Fields

To go on I use operator solutions of the equations of motion in the Landau gauge and the unitary gauge [18,34]. In my notation they read as follows:

i) Landau Gauge :

The equations of motion to be solved are given by (2.8)-(2.11).

$$\begin{aligned} eA_\mu &= \frac{a-1}{ea^2}\epsilon_{\mu\nu}\partial^\nu F + \frac{1}{a}\epsilon_{\mu\nu}\partial^\nu H + \partial_\mu X + \frac{1}{ea}\partial_\mu B, \\ \phi &= -\frac{a-1}{ea^2}F + \frac{a-1}{a}H - X, \\ \theta &= \frac{1}{ea^2}F - \frac{1}{a}H - X. \end{aligned} \tag{4.1}$$

F is a massive scalar, while H, X and B are massless. The nonvanishing commutators of these fields read

$$\begin{aligned} [F(x), F(y)] &= im^2\Delta(x-y;m^2), \\ [H(x), H(y)] &= iD(x-y), \\ [X(x), B(y)] &= -ieD(x-y), \\ [X(x), X(y)] &= i/a\, D(x-y), \\ [B(x), B(y)] &= 0. \end{aligned} \tag{4.2}$$

From here it is clear that not all fields are physical. A subsidiary constraint has to be imposed to define the physical state space; if we demand that the Maxwell equation $\partial_\mu F^{\mu\nu} = -J^\nu$, with J^ν from (2.8), is fulfilled on the physical state space, it is

$$B^{(+)}|\text{phys}\rangle = 0. \tag{4.3}$$

Therefore only F and H belong to the physical Hilbert space, where the zero-norm states allowed by (4.3) are removed. Since the chiral charge is $Q_L = \int dx^1 \partial^0 B$, any neutral operator is physical.

ii) Unitary Gauge (AF):

The equations of motion are (1.19) and (1.20). They have already been solved in sec.1. Here

I use a different notation ($\sigma_m = \frac{1}{ea^2} f$):

$$\begin{aligned} eA_\mu &= \frac{1}{ea^2}\left[\partial_\mu f + (a-1)\epsilon_{\mu\nu}\partial^\nu f\right] - \frac{1}{a}(\partial_\mu h - \epsilon_{\mu\nu}\partial^\nu h), \\ \phi &= -\frac{1}{ea} f + h. \end{aligned} \tag{4.4}$$

f is a massive scalars and h is massless. The commutators read

$$\begin{aligned} [\, f(x), f(y)\,] &= im^2\Delta(x-y; m^2), \\ [\, h(x), h(y)\,] &= iD(x-y). \end{aligned} \tag{4.5}$$

Derivation of the Fermion Operators

Since I am mainly interested in fermions, I have to derive their expressions in the operator formalism. A useful procedure to do this has been proposed in ref.[18], and I will follow it here with special emphazise on the renormalization aspects [35].

Naturally, the fermion operator has to fulfill the Dirac equation:

$$\begin{aligned} &i\partial\!\!\!/\psi(x) + e\sqrt{\pi} : (A\!\!\!/(1-\gamma^5)\psi)(x) : = 0, \\ &: (A\!\!\!/\psi)(x) : \equiv \frac{1}{2}\gamma^\mu \lim_{\epsilon\to\infty}\left[A_\mu(x+\epsilon)\psi(x) + \psi(x)A_\mu(x-\epsilon)\right]. \end{aligned} \tag{4.6}$$

A first proposal for the form of the fermion operator is

$$\psi = : \exp\left\{ i\sqrt{\pi}(1-\gamma^5)\Lambda \right\} : \psi^{free}, \tag{4.7}$$

where Λ is defined by

$$e(1-\gamma^5)A_\mu = (g_{\mu\nu} + \epsilon_{\mu\nu})\partial^\nu\Lambda. \tag{4.8}$$

For ψ^{free} I make the standard ansatz [26]

$$\psi^{free} = : \exp\left\{ i\sqrt{\pi}(\varsigma + \gamma^5\tilde{\varsigma}) \right\} : u. \tag{4.9}$$

where ς is a massless scalar field and $\tilde{\varsigma}$ its dual field ($\partial_\mu\varsigma = \epsilon_{\mu\nu}\partial^\nu\tilde{\varsigma}$). The constant $u = [\mu/(2\pi)]^{\frac{1}{2}}$ contains the IR-cutoff μ. [2]

Eq.(4.7) is however not unique, since one can add terms of the form $(\zeta - \bar{\zeta})$ to Λ and the new ψ still fulfills (4.6). To get an unambiguous form, one has to demand that

$$J^\mu = e\sqrt{\pi} \; : \bar{\psi}\gamma^\mu(1-\gamma^5)\psi : = e^2 a A^\mu + e(g^{\mu\nu} - \epsilon^{\mu\nu})\partial_\nu\phi. \tag{4.10}$$

[2] It is reasonable to take ψ^{free} with $\varsigma = H$ as a physical basic field instead of H itself [36], because when H is also regularized by μ the dependence on μ drops out; cp.(4.15).

Here the left-hand side is defined by

$$J^\mu = e\sqrt{\pi}\lim_{\epsilon\to 0}\left[\bar{\psi}(x+\epsilon)\gamma^\mu(1-\gamma^5)\psi(x) - \langle 0|\bar{\psi}(x+\epsilon)\gamma^\mu(1-\gamma^5)\psi(x)|0\rangle\right]. \tag{4.11}$$

To deal with the regularization ambiguity, I introduce phase factors following ref.[18]:

$$\begin{aligned} P_V &\simeq 1 + ie\sqrt{\pi}\epsilon^\mu[(1+a)A_\mu + \epsilon_{\mu\nu}A^\nu], \\ P_A &\simeq 1 + ie\sqrt{\pi}\epsilon^\mu[A_\mu + (1-a)\epsilon_{\mu\nu}A^\nu]. \end{aligned} \tag{4.12}$$

This is inserted into (4.11), [3]

$$\begin{aligned} e\sqrt{\pi}\bar{\psi}(x+\epsilon)\gamma^\mu(P_V - P_A\gamma^5)\psi(x) &= \frac{1}{2}[\gamma^\mu(P_V - P_A\gamma^5)]_{\alpha\beta}[\bar{\psi}^\alpha_R(x+\epsilon), \psi^\beta_R(x)] \\ &= \frac{1}{2}\gamma^\mu\Big\{[P_V - P_A\gamma^5]_{21}[\psi^*_R(x+\epsilon)\psi_R(x) - \psi_R(x)\psi^*_R(x+\epsilon)] \\ &\quad + [P_V - P_A\gamma^5]_{12}[\psi^*_L(x+\epsilon)\psi_L(x) - \psi_L(x)\psi^*_L(x+\epsilon)]\Big\}, \end{aligned} \tag{4.13}$$

with $\bar{\psi} = \psi^+\gamma^0$. Taking ψ from (4.7) and (4.9) one has

$$\begin{aligned} \psi^*_R(x+\epsilon)\psi_R(x) &= \exp\left\{\pi[(\varsigma+\tilde{\varsigma})^{(+)}(x+\epsilon), (\varsigma+\tilde{\varsigma})^{(-)}(x)]\right\} \\ &\qquad : \left(1 + i\sqrt{\pi}\epsilon^\mu\partial_\mu(\varsigma+\tilde{\varsigma})(x)\right) : u^*u, \\ \psi_R(x)\psi^*_R(x+\epsilon) &= \exp\left\{\pi[(\varsigma+\tilde{\varsigma})^{(+)}(x), (\varsigma+\tilde{\varsigma})^{(-)}(x+\epsilon)]\right\} \\ &\qquad : \left(1 + i\sqrt{\pi}\epsilon^\mu\partial_\mu(\varsigma+\tilde{\varsigma})(x)\right) : u^*u, \\ \psi^*_L(x+\epsilon)\psi_L(x) &= \exp\left\{\pi[2\Lambda^{(+)}(x+\epsilon) + (\varsigma+\tilde{\varsigma})^{(+)}(x+\epsilon), 2\Lambda^{(-)}(x) + (\varsigma+\tilde{\varsigma})^{(-)}(x)]\right\} \\ &\qquad : \left(1 + i\sqrt{\pi}\epsilon^\mu\partial_\mu[2\Lambda(x) + (\varsigma+\tilde{\varsigma})(x)]\right) : u^*u, \\ \psi_L(x)\psi^*_L(x+\epsilon) &= \exp\left\{\pi[2\Lambda(x)^{(+)} + (\varsigma+\tilde{\varsigma})^{(+)}(x), 2\Lambda^{(-)}(x+\epsilon) + (\varsigma+\tilde{\varsigma})^{(-)}(x+\epsilon)]\right\} \\ &\qquad : \left(1 + i\sqrt{\pi}\epsilon^\mu\partial_\mu[2\Lambda(x) + (\varsigma+\tilde{\varsigma})(x)]\right) : u^*u. \end{aligned} \tag{4.14}$$

The exponential factor of the right-handed fermions reads

$$\exp\left\{2\pi\left(D^{(+)} + \tilde{D}^{(+)}\right)(\epsilon)\right\} = -\frac{1}{\mu(\epsilon^0 + \epsilon^1 - i0)}. \tag{4.15}$$

For the left-handed fermion one has in $\psi^*_L(x+\epsilon)\psi_L(x)$ the exponent

$$\exp\{2\pi\left(D^{(+)} - \tilde{D}^{(+)}\right)(\epsilon) + 2\mathcal{R}(\epsilon)\} = \frac{\exp\{2\mathcal{R}(\epsilon)\}}{\mu(\epsilon^0 - \epsilon^1 + i0)} \tag{4.16}$$

with

$$\begin{aligned} \mathcal{R}(\epsilon) = 2\pi\Big(&[\Lambda^{(+)}(x+\epsilon), \Lambda^{(-)}(x)] - \frac{1}{2}[(\varsigma+\tilde{\varsigma})^{(+)}(x-\epsilon), \Lambda^{(-)}(x)] \\ &\frac{1}{2}[\Lambda^{(+)}(x), (\varsigma+\tilde{\varsigma})^{(-)}(x+\epsilon)]\Big) \end{aligned} \tag{4.17}$$

[3] From now on I will take for granted that the $\epsilon \to 0$-limit is defined properly whenever ϵ shows up.

and for $\psi_L^*(x)\psi_L(x+\epsilon)$ an analogous expression. The expression $\mathcal{R}(\epsilon)$, (4.17), is related to renormalization; it will be discussed below. By employing a symmetric point splitting procedure, i.e. $\epsilon^\mu\epsilon^\nu\epsilon^{-2} = g^{\mu\nu}$, one finally gets for Λ in (4.7)

$$\Lambda = -\phi - \varsigma. \tag{4.18}$$

The correct ansatz for the fermion operator is

$$\psi =: \exp\left\{i\sqrt{\pi}\left[(1-\gamma^5)\Lambda + \varsigma + \gamma^5\tilde{\varsigma}\right] - (1-\gamma^5)\mathcal{R}(\epsilon)\right\} : u. \tag{4.19}$$

Let us specify the gauge now:

i) Landau Gauge:

Here I have

$$\Lambda = \frac{a-1}{ea^2}F + \frac{1}{a}H + X + \frac{1}{ea}B,$$
$$\varsigma = -H - \frac{1}{ea}B. \tag{4.20}$$

Eq.(4.17) gives

$$\mathcal{R}(\epsilon) = \pi\frac{a-1}{a^2}\left(\Delta^{(+)}(\epsilon;m^2) - D^{(+)}(\epsilon)\right). \tag{4.21}$$

The massless scalar's propagator is (IR-) regularized with μ. Therefore (4.21) produces a term of the form $\log(\frac{m^2}{\mu^2})$. It is important to realize that this term is necessary to get the free fermion propagator in the short-distance limit for the propagator corresponding to (4.19). The left-handed *non*renormalized fermion operator without $\mathcal{R}(\epsilon)$ reads

$$\psi_L^n(x) = :\exp\left\{i\sqrt{\pi}\left[-2\phi(x) + H(x) + \tilde{H}(x) + \frac{1}{ea}\left(B(x) + \tilde{B}(x)\right)\right]\right\} : \left(\frac{\mu}{2\pi}\right)^{\frac{1}{2}}. \tag{4.22}$$

ii) Unitary Gauge (AF):

One gets instead of (4.20)

$$\Lambda = \frac{1}{ea}f, \quad \varsigma = -h, \tag{4.23}$$

and for the left-handed fermion operator

$$\Psi_L(x) = \exp\left\{i\sqrt{\pi}\,\frac{2}{ea}f(x)\right\} : \exp\left\{-i\sqrt{\pi}\left[h(x) - \tilde{h}(x)\right]\right\} : v. \tag{4.24}$$

This time, the $\mathcal{R}(\epsilon)$-term in (4.17) leads to

$$(1-\gamma^5)\mathcal{R}(\epsilon) \to \frac{2\pi}{a-1}\Delta^{(+)}(\epsilon;m^2). \tag{4.25}$$

I absorbed it in expression (4.24) by not normal-ordering the $\exp\{i\sqrt{\pi}\frac{2}{ea}f\}$-term:

$$\exp\left\{i\sqrt{\pi}\frac{2}{ea}f(x)\right\} =: \exp\left\{i\sqrt{\pi}\frac{2}{ea}f(x)\right\} : \exp\left\{-\frac{2\pi}{a-1}\Delta^{(+)}(\epsilon;m^2)\right\}. \tag{4.26}$$

Although (4.24) is the correct operator which fulfills the current equation (4.10), it is obviously UV-divergent, i.e. one needs an UV-cutoff to make sense of it. Therefore the $\mathcal{R}(\epsilon)$-term has to be removed by a wavefunction renormalization. The result will be called henceforth Ψ_L^{ren}. This renormalization corresponds to the elimination of the factor $\exp\{\frac{4\pi i}{a-1}\Delta_F(0,m^2)\}$ in expression (1.37). Even more interesting than the necessity to eliminate an UV-cutoff is the fact that there is *no* need for an IR-cutoff, because that would obscure the long-distance behaviour.

5. Operator Transformations

An operator transformation from the Landau gauge fermion operator to a physical fermion operator is considered. The renormalized unitary fermion operator is shown not to be directly derivable. A different unitary operator is proposed as the adequate physical fermion operator of the GIF. Its propagator resembles the one of the VSM with BGS I considered in sec.3.

Formal Operator Transformations

In sec.2 three gauge invariant fermion propagators were written down. The first one, (2.17), was build up by the fermion operator

$$\chi_L^\theta = \exp\left\{2i\sqrt{\pi}\theta\right\}\psi_L. \tag{5.1}$$

χ_L^θ is explicitly gauge invariant. ψ_L is given by (4.19), (4.20) and (4.21). If the B-field is absorbed in ψ_L^{free}, we get

$$\chi_L^\theta(x) =: \exp\left\{i\sqrt{\pi}\frac{2}{\epsilon a}F(x) - \frac{2\pi}{a-1}\Delta^{(+)}(\epsilon;m^2)\right\} : \psi_L^{free}(x). \tag{5.2}$$

which corresponds to the nonrenormalized unitary fermion operator Ψ_L, (4.24).

Let us consider $\hat\chi_L^\theta =: \exp\{2i\sqrt{\pi}\theta\} : \psi_L$, i.e. I try to absorb the UV-renormalization constant by normal-ordering θ. I get

$$\hat\chi_L^\theta(x) =: \exp\left\{i\sqrt{\pi}\frac{2}{\epsilon a}F(x) + \mathcal{R}(\epsilon)\right\} : \psi_L^{free}(x). \tag{5.3}$$

with

$$\mathcal{R}(\epsilon) = -2\pi\frac{a-1}{a^2}\left(\Delta^{(+)}(\epsilon;m^2) - D^{(+)}(\epsilon)\right). \tag{5.4}$$

While χ_L^θ is not UV-renormalized, $\hat\chi_L^\theta$ is not IR-renormalized. Both give rise to a propagator with $p^2 = 0$-pole in momentum space, but with indefinite residuum. I guess that one can say that χ_L^θ corresponds to an asymptotic fermion field, since the UV-renormalization should not change the long-distance behaviour; however, $\hat\chi_L^\theta$ cannot be interpreted in this way. Moreover, $\hat\chi_L^\theta$ makes explicit that the transformation changes the IR-structure of the model. Although the renormalized Landau gauge fermion operator and the nonrenormalized unitary fermion operator are directly related by an operator transformation, the latter does not represent the physical content of the former.

In sec.2 we considered another gauge invariant propagator (2.19), which can be build up by

$$\chi_L^c = \exp\left\{2i\sqrt{\pi}\sigma\right\}\psi_L. \tag{5.5}$$

In this case we end up with

$$G_L^c(x-y) = G_L^{free}(x-y)\exp\left\{-4\pi\bar{D}_F(x-y)\right\}. \tag{5.6}$$

which does not become a free left-handed propagator in the long-distance limit.

Projection on the Physical State Space

Up to now I did not consider the physical state space defined by (4.3). Therefore the justification of the operators considered above is not obvious, although (or rather: because) all are physical. The physical content of the GIF seems to be ambiguous. To clarify this point, one has to define the transformation with reference to the physical state space. This procedure has been applied for different models [28,37], and for the CSM in the GIF it works as follows. The physical state space has been defined in sec.4 by $B^{(+)}|\text{phys}\rangle = 0$ for the Landau gauge. Since B generates states with zero norm, one can decompose it:

$$B = (\epsilon aX + B) - \epsilon aX, \tag{5.7}$$

with

$$\begin{aligned}
[\epsilon aX(x) + B(x), \epsilon aX(y) + B(y)] &= -i\epsilon^2 aD(x-y),\\
[\epsilon aX(x), \epsilon aX(y)] &= i\epsilon^2 aD(x-y),\\
[\epsilon aX(x) + B(x), \epsilon aX(y)] &= 0.
\end{aligned} \tag{5.8}$$

The negative frequency part of the operator $\partial_\mu\Box^{-1}A^\mu = X + \frac{1}{\epsilon a}B$ builds up an indefinite metric state space and therefore this operator has to be subtracted from the Landau gauge operators to get the correct physical operators [28,37].

Thanks to this reference to the physical state space, I get

$$\chi_L^A(x) = \exp\left\{-2i\sqrt{\pi}\left(X + \frac{1}{\epsilon a}B\right)(x)\right\}\psi_L(x), \tag{5.9}$$

and the propagator

$$\tilde G_L^A(x-y) = G_L^{free}(x-y)\exp\left\{-4\pi i\frac{a-1}{a^2}\left(\bar\Delta_F(x-y;m^2) - \bar D_F(x-y)\right) - \frac{4\pi i}{a}\bar D_F(x-y)\right\}. \tag{5.10}$$

This has the same form as the propagator for the VSM with BGS studied in sec.3, eq.(3.10). Moreover, it also parallels the physical gauge solution for the (nonanomalous) VSM and ASM [28].

The gauge invariant propagator G_L^A, (2.18), corresponds for $x_0 = y_0$ to the operator

$$\chi_L^C =: \exp\left\{i\sqrt{\pi}\left(2\frac{a-1}{\epsilon a^2}\left(F - \tilde F\right)(x) + \left(\frac{2}{a} - 1\right)\left(H - \tilde H\right)(x)\right)\right\} : \sigma_L(x), \tag{5.11}$$

where σ_L contains the B-field. This is the Coulomb gauge solution, $A_1 = 0$. For $a = 2$, χ_L^C exactly coincides with the result for the SM [15]. In the Landau gauge, the analogy has been derived only for the propagator (ψ_L^η, (4.22), still contains the H-field, which is lacking in the

SM), so it is even stronger in the present case. The Coulomb gauge propagator is known to grow exponentially for $|x_1 - y_1| \to \infty$, so that one can conclude that neither χ_L^A nor χ_L^C give rise to fermionic single particle states.

Just as it is not enough to take a gauge invariant fermion operator (resp. propagator), we see that unitarity is also not a unique criterion for the problem concerning fermionic states ($\theta = 0$-gauge and Coulomb gauge give different solutions). The only reasonable criterion is to refer to the physical state space.

Some Comments

In the light of the correct projection on the physical state space of the GIF, one may reconsider the transformation (5.1). What is the physical state space if χ_L^θ given by (5.1) is the physical fermion operator? A subsidiary constraint corresponding to this transformation seems to be

$$\left[-B + e(a-1)\left(\theta + \tilde{\theta}\right)\right]^{(+)} |\text{phys}\rangle = 0. \tag{5.12}$$

However, the massive component in θ causes severe problems. Let us choose $m^2 = 0$, $a \neq 0$. I split the constraint into the negative norm operator $-B + e(a-1)\theta$ and the positive norm operator $(a-1)\tilde{\theta}$. The fermion operator is reached by

$$\tilde{\chi}_L^\theta(x) = \exp\left\{-2i\sqrt{\pi}\left(-\theta + \frac{1}{e(a-1)}B\right)(x)\right\}\psi_L(x). \tag{5.13}$$

The propagator of $\tilde{\chi}_L^\theta$ is

$$\tilde{G}_L^\theta(x-y) = G_L^{free}(x-y)\exp\left\{-\frac{4\pi i}{a-1}\tilde{D}_F(x-y)\right\}. \tag{5.14}$$

$\tilde{\chi}_L^\theta$ can be identified with the unitary gauge fermion operator Ψ_L for $m^2 = 0$, (4.24), if

$$f = F - \frac{1}{a-1}B, \quad h = H + \frac{1}{ea}B. \tag{5.15}$$

f, h and $\tilde{\chi}_L^\theta$ are physical, as they commute with the operator defining the physical state space. The form of (5.14) and the physicality of $\tilde{\chi}_L^\theta$ gives a clear indication that we are really looking at the $m^2 \to 0$ limit of the transformation (5.1). Since in general $m^2 \neq 0$, (5.1) cannot be seen as a sensible transformation leading to the physical fermion operator, although the result is surely gauge invariant relative to the subsidiary constraint (4.3).

Here I want to comment on a very general point: On one hand, $\tilde{\chi}_L^\theta$ (and all other results of the transformations considered above) contain the physical massless scalar H, and there should be a physical free fermion operator because of the relation (4.9). On the other hand the propagator (5.14) does not become a free one in the long-distance limit. $\tilde{\chi}_L^{\theta ren}$ is roughly written as

$$\tilde{\chi}_L^{\theta ren} = \psi_L^{free} :e^{i\varphi}:, \tag{5.16}$$

with $\langle T(\varphi(x)\varphi(y))\rangle = -\frac{4\pi i}{a-1}D_F(x-y)$. Following ref.[21], this may be interpreted as referring to a direct product state space of ψ_L^{free} and φ; correspondingly, there is an asymptotic fermion field. However, if the two-point Wightman function of $\tilde{\chi}_L^{\theta ren}$ is calculated it can be shown not to give rise to a pole for vanishing momentum ([21]; cp. propagator (5.14)). This means that the state space defined by the Wightman functions of $\tilde{\chi}_L^{\theta ren}$ and φ does *not* contain these states. It has to be stressed that the existence of fermionic single particle states in the AF does not depend on a special definition of the state space, because φ is massive in that case. Therefore the "spurious" fermion states in the GIF have to be discriminated from those in the AF.

The existence of the free fermion operator referring to the (redundant) physical state space is not very interesting; without studying explicit operator solutions in sec.3, it should be obvious that *all* Schwinger models (anomalous and nonanomalous) except the SM give rise to these free fermion operators.

6. Concluding Remarks on the Fermion in the Long-Distance Region

The results of the preceding sections are reviewed.

In the preceding sections the PROBLEM formulated at the end of sec.2 has been attacked with different methods. It is now possible to give an answer: In the GIF of the CSM there is no physical fermion operator which gives rise to an asymptotic fermion field. One may formally define a gauge invariant candidate $\exp\{2i\sqrt{\pi}\theta\}\psi_L$, which leads back to the AF, but by starting from the Landau gauge solution of the CSM I showed that this operator should not be considered as a part of the GIF. Its definition is not motivated by the physical state space, and the necessity to renormalize the result of the formal transformation (5.1) refers to a change of this state space. By disregarding these criteria other gauge invariant operators would become available and the physical content of the fermionic sector of the CSM in the GIF would by ambiguous.

A correct physical fermion operator of the GIF is χ_L^A, (5.9). It has been derived from the Landau gauge solution by subtracting the operator generating indefinite metric states; this procedure has e.g. been applied to the VSM and the ASM [28] as well as to a (3+1)-dimensional massive vector model [37].

The appearance of a physical massless scalar in the solution of the CSM is a quite uninteresting feature, which has nothing to do with an anomaly. Many generalized Schwinger models contain physical massless scalars, e.g. the VSM with or without BGS and the ASM, and only the SM is an exception.

The fermion operators of the GCSM in the AF do in generally not give rise to an asymptotic fermion field. It is only for the CSM in the AF that the chiral fermion propagator of the interacting sector becomes a free one in the long-distance limit. Its form can be explained as originating from a free fermion operator, where a massive scalar is coupled by the anomaly. This point of view is motivated by considering the VSM with BGS. As a general rule I suggest that the part of the gauge field which breaks the gauge symmetry becomes physical and couples to the fermion.

We have seen that in the AF of the CSM and GCSM, and the VSM with BGS, the fermion operator gets an anomalous dimension; this property is shared by the physical operator of the CSM in the GIF (χ_L^A), which has been proposed in sec.5. For the CSM it corresponds to the ϵ^{-2}-pole in the photon propagator (1.27) (see also (3.12) for the VSM with BGS). From here one can expect that the short-distance behaviour of the CSM in the AF is problematic.

as will be explained in sec.7. This is particularly puzzling since it jeopardizes the possibility to apply perturbation theory, which is important for any progress towards the construction of (3+1)-dimensional anomalous models with chiral fermions. Moreover, one may question the validity of the CSM in the AF as a well-defined quantum field theory [45].

7. Fermions in the Short-Distance Limit

The short-distance limit of the CSM is investigated. The effective action and fermion propagators are derived perturbatively. Noncanonical features of the AF are shown by studying the Dirac equation.

Perturbation Theoretic Derivation of the Effective Action

The effective action (1.16) has been derived perturbatively in the dimensional regularization scheme by calculating the Feynman diagram 7.1a [39,40].

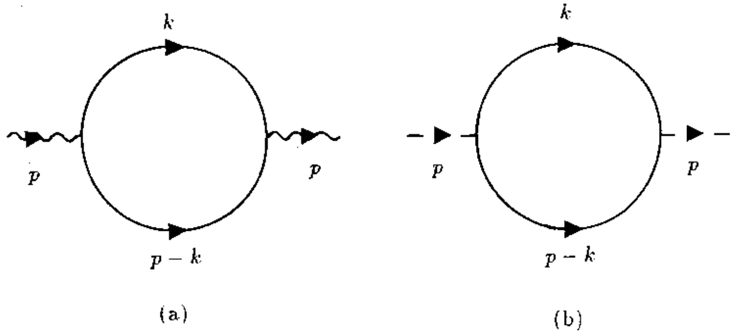

FIGURE 7.1: One-loop diagrams leading to the effective action.

With regard to later application to the chiral quantum gravity I will briefly report this derivation and then show how it has to be modified if the scalar fields σ and ρ are used instead of A_μ (see relation (1.5)). The interaction part of the Lagrangian reads

$$\begin{aligned}\mathcal{L}_{int} &= e\sqrt{\pi}\bar\psi \not{A}(1-\gamma^5)\psi \\ &= \sqrt{\pi}\bar\psi[\not\partial(\sigma+\rho)](1-\gamma^5)\psi \\ &= -\sqrt{\pi}(\sigma+\rho)[\bar\psi(\not\partial\psi)+(\not\partial\bar\psi)\psi](1-\gamma^5).\end{aligned} \tag{7.1}$$

In the usual calculation of Fig.7.1a, the vertices read $ie\sqrt{\pi}\gamma^\mu(1-\gamma^5)$. In ref.[39] the vertex has been generalized to d dimensions by

$$\gamma_\mu \to \bar\gamma_\mu = s\gamma_\mu + r\theta_{\mu\alpha}\gamma^\alpha\bar\gamma^5, \tag{7.2}$$

where $r+s = 1+\mathcal{O}(\epsilon)$, $\theta_{\mu\alpha} = \epsilon_{\mu\alpha}+\mathcal{O}(\epsilon)$, and $\epsilon \equiv d-2$. r and s coincide with the parameters introduced in sec.1. For the fermion loop in Fig.7.1a a minus sign has to be added, and a $\frac{1}{2!}$-factor comes from treating the two boson lines symmetrically. The effective action is given by

$$W[A] = \frac{1}{2}\int dpA^\mu(p)\Pi_{\mu\nu}(p)A^\nu(-p), \tag{7.3}$$

with the polarization tensor

$$\Pi_{\mu\nu}(p) = -\int \frac{d^dk}{(2\pi)^d}\frac{(p-k)^\sigma k^\rho}{k^2(p-k)^2}\underbrace{\mathrm{Tr}\left[\bar\gamma_\mu(1-\bar\gamma^5)\gamma_\sigma\bar\gamma_\nu(1-\bar\gamma^5)\gamma_\rho\right]}_{\Gamma_{\mu\sigma\nu\rho}}. \tag{7.4}$$

The result for the momentum integration can be derived with the formulas presented in Appendix B.

$$\begin{aligned}\int \frac{d^2k}{(2\pi)^2}\frac{(p-k)^\sigma k^\rho}{k^2(p-k)^2} &= -\frac{i}{(4\pi)^{\frac{d}{2}}}p^{-2}p^\sigma p^\rho(1+\mathcal{O}(\epsilon)) - \frac{i}{(4\pi)^{\frac{d}{2}}}g^{\sigma\rho}\left(\epsilon^{-1}+\mathcal{O}(\epsilon^0)\right) \\ &\equiv I_0^{\sigma\rho}+I_{-1}^{\sigma\rho}.\end{aligned} \tag{7.5}$$

For $I_0^{\sigma\rho}$ one may put $d=2$ in $\Gamma_{\mu\sigma\nu\rho}$,

$$\Gamma_{\mu\sigma\nu\rho}I_0^{\sigma\rho} = -\frac{i}{\pi}\left[2p^{-2}p_\mu p_\nu - g_{\mu\nu} + \epsilon_{\mu\alpha}p^\alpha p_\nu + \epsilon_{\nu\alpha}p^\alpha p_\mu\right]. \tag{7.6}$$

The ϵ-pole of $I_{-1}^{\sigma\rho}$ makes a more detailed analysis necessary, since the contraction over the indices σ and ρ really has to be done in d dimensions. Fortunately, this turns out to be easy when the Breitenlohner-Maison scheme (with $(\bar\gamma^5)^2=1$) is employed [41]. One ends up with

$$\Gamma_{\mu\sigma\nu\rho}I_{-1}^{\sigma\rho} = \frac{i}{\pi}(s^2-r^2)g_{\mu\nu}. \tag{7.7}$$

The effective action then coincides with (1.16), if $s-r=a$. Diagrams with more than two photon lines can be excluded by exactly the same proof as in the case of the SM [42]. While diagram 7.1a is logarithmically divergent (integral (7.5)), all higher order diagrams are convergent, since k^{-2}-factors are added. This finiteness is essential for the proof that they have a vanishing contribution [42].

Is it possible to do the same calculation with the third line in (7.1)? The vertices in Fig.7.1b read $-\sqrt{\pi}\not{p}(1-\gamma^5)$. If one would just generalize the γ-matrices as has been done above the result would be

$$W'[A] = \frac{1}{2}e^2(a-1)\int d^2xA_\mu^+\frac{\partial^\mu\partial^\nu}{\Box}A_\nu^+. \tag{7.8}$$

This is incorrect; moreover, not even the SM effective action can be derived by this way. As the mass term we are looking for is of the form $A^2 = -e^2(\sigma\Box\sigma - \rho\Box\rho)$, it is clear from (7.7) how to save the situation for the SM: instead of $\not{A}(1\pm\gamma^5) = \gamma^\mu(\partial_\mu\sigma \mp \epsilon_{\mu\nu}\partial^\nu\rho)(1\pm\bar\gamma^5)$ I use $\not{A}(1\pm\gamma^5) = \gamma^\mu(\partial_\mu\sigma \mp \theta_{\mu\nu}\partial^\mu\rho\gamma^\nu\bar\gamma^5)(1\pm\bar\gamma^5)$. For the CSM, the interacting part of the Lagrangian, (7.1), reads in momentum space

$$\sqrt{\pi}p^\mu\mathcal{V}_\mu^L(\sigma,\rho) = \sqrt{\pi}p^\mu(\sigma\bar\gamma_\mu + \rho\theta_{\mu\alpha}\bar\gamma_\alpha\bar\gamma^5)(1-\bar\gamma^5). \tag{7.9}$$

The effective action cannot be written in the form (7.3), with A_μ expressed by σ and ρ. The calculation of the trace with the new vertices (7.10) is more involved as before, but can be done along the same lines; the result is

$$\begin{aligned} W[\sigma,\rho] &= \int dp \int \frac{d^2k}{(2\pi)^2} \frac{p^\mu(p-k)^\sigma p^\nu k^\rho}{k^2(p-k)^2} \mathrm{Tr}[\mathcal{V}^L_\mu(\sigma,\rho)\gamma_\sigma \mathcal{V}^L_\nu(\sigma,\rho)\gamma_\rho] \\ &= \frac{i}{2\pi}\int dp\, p^2 \left[(s+r)^2(\sigma+\rho)^2 - (s^2-r^2)(\sigma^2-\rho^2)\right]. \end{aligned} \tag{7.10}$$

Therefore one obtains again (1.16), up to the kinetic term of A_μ. This second way to derive the complete effective action by perturbation theory will be the only one in sec.8, when I consider the chiral quantum gravity, where no interaction term like the first line of (7.1) is available.

Perturbation Theoretic Derivation of the Fermion Propagator in Landau Gauge

The exact expression of the fermion propagator has been derived in sec.2 in the configuration space. For perturbation theory, however, the momentum space is more suited. Unfortunately, it seems not to be possible to Fourier transform an expression of the kind (2.16) in closed form. Therefore I expand the exponential function to second order and do the transformation by convolutions.

$$\begin{aligned} G_L(x-y) &\simeq G_L^{free}(x-y)\left[1 + F(x-y) + \frac{1}{2!}F^2(x-y) + \ldots\right], \\ F(x-y) &\equiv 4\pi i \frac{a-1}{a^2}\int \frac{d^2k}{(2\pi)^2}\left(e^{-ik(x-y)} - 1\right)\left(\frac{1}{k^2-m^2} - \frac{1}{k^2}\right). \end{aligned} \tag{7.11}$$

The details of the Fourier transformation are given in Appendix B. In first order of $\frac{a-1}{a^2}$ I get

$$\begin{aligned} G_L^{free}(x-y)F(x-y) &\to \tilde{G}_L^{free}(p) * \tilde{F}(p) \\ &= i\frac{\not p}{p^2}\frac{a-1}{a^2}P_L\left[-\frac{m^2}{p^2} - \frac{1}{2}\frac{m^4}{p^4} + \mathcal{O}\left(\frac{m^6}{p^6}\right)\right]. \end{aligned} \tag{7.12}$$

The second order in $\frac{a-1}{a^2}$ reads

$$\begin{aligned} G_L^{free}(x-y)F^2(x-y) &\to \tilde{G}_L^{free}(p) * \tilde{F}(p) * \tilde{F}(p) \\ &= i\frac{\not p}{p^2}\left(\frac{a-1}{a^2}\right)^2 P_L\left[\frac{m^4}{p^4} + 2\frac{m^4}{p^4}\log\left(\frac{m^2}{-p^2}\right) - \mathcal{O}\left(\frac{m^6}{p^6}\right)\right]. \end{aligned} \tag{7.13}$$

By putting things together, I end up with

$$\begin{aligned} \tilde{G}_L(p) \simeq\ & \tilde{G}_L^{free}(p)\left[1 - \frac{a-1}{a^2}\left(\frac{m^2}{p^2} + \frac{1}{2}\frac{m^4}{p^4}\right)\right. \\ & \left. + \left(\frac{a-1}{a^2}\right)^2\frac{m^4}{p^4}\left(1 + 2\log\left(\frac{m^2}{-p^2}\right)\right) + \mathcal{O}\left(\frac{m^6}{p^6}\right)\right]. \end{aligned} \tag{7.14}$$

This resembles the perturbation theoretic result for the SM in the light-cone gauge [43]. However, in contrast to the calculation there one has to use the exact photon propagator from the start in the Landau gauge (for both SM and CSM). This observation is important with respect to the AF of the CSM, where no free photon propagator is available. The Feynman diagrams that contribute are shown in Fig.7.2:

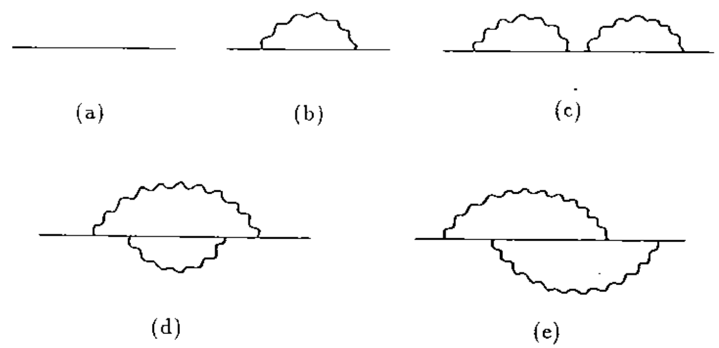

FIGURE 7.2: Diagrams contributing to the fermion propagator up to order $\mathcal{O}((\frac{a-1}{a^2})^2)$.

It is obvious that the exact photon propagator has to be used for the calculation. If the free one ($m^2 = 0$ in (2.15)) is taken, the $\frac{m^2}{p^2}$-term in (7.14) cannot be derived, because diagram 7.2b would give a zero contribution. One can suppose that the correct $m^2 \to 0$ limit (i.e. the free propagator) is not reached in the trivial sense because the coupling becomes zero, but because the momentum integrals give zero result. A cancellation takes place, as can already be inferred from the form of (2.16). The physical picture motivated by the Lagrangian (7.1) is that in the Landau gauge case a massive scalar and a massless (ghost-) scalar interact by a momentum coupling with a free fermion. They are not decoupled for vanishing mass, but their contributions cancel.

For the calculation of diagram 7.2d and especially 7.2e the algebraic programming system *Reduce* [62] has been necessary. With the photon propagator (2.15) and the vertex $e\sqrt{\pi}\gamma^\mu(1-\gamma^5)$ I get

$$\begin{aligned} D_a &= i\frac{\not p}{p^2}P_L, \\ D_b &= i\frac{\not p}{p^2}P_L\left(\frac{a-1}{a^2}\frac{m^2}{p^2}\right)\left[-1 - \frac{1}{2}\frac{m^2}{p^2} + \mathcal{O}\left(\frac{m^4}{p^4}\right)\right], \end{aligned}$$

$$D_c = i\frac{\not p}{p^2}P_L\left(\frac{a-1}{a^2}\frac{m^2}{p^2}\right)^2\left[1+\mathcal{O}\left(\frac{m^2}{p^2}\right)\right],$$
$$D_d = D_c, \tag{7.15}$$
$$D_e = i\frac{\not p}{p^2}P_L\left(\frac{a-1}{a^2}\frac{m^2}{p^2}\right)^2\left[-1+2\log\left(\frac{m^2}{-p^2}\right)+\mathcal{O}\left(\frac{m^2}{p^2}\right)\right].$$

By adding these contributions one has again (7.14). This proves that the fermion propagator can (only) be derived if the exact photon propagator is used from the start.

Diagram 7.2e needs special care. The mass m had to be introduced to regularize integrals which contain no mass. This is explained in Appendix B. In fact, I pointed out in sec.2 that the fermion propagator necessitates a (IR) wavefunction renormalization if it shall become free in the short-distance limit without an infinite constant. Therefore one had to expect that in perturbation theory some renormalization has to be done as well. This observation has another interesting implication. Since the IR-problem is clearly connected to the massless (ghost-) scalar coupling to the fermion, the same problem has to be expected if perturbation theory is applied to the physical propagator of the CSM in the GIF, (5.10). Doing perturbation theory means that we presuppose the existence of fermionic states. The IR-problem signals that we are working with a wrong assumption about the spectrum, and the renormalization changes the physical state space so as to eliminate the fermionic states. This argument is taken from Schroer's paper, ref.[21], where an analogous situation is studied.

Perturbation Theoretic Derivation of the Fermion Propagator in the Anomalous Formulation

As we know by now that even in the Landau gauge the exact photon propagator has to be used to derive the fermion propagator, the nonexistence of a free unitary gauge photon propagator poses no problem. The first step is again to expand the exact fermion propagator (1.37) to second order in $(a-1)^{-1}$ and do a Fourier transformation: the result is ($\frac{m^2}{p^2} \ll 1$, $\bar\gamma \equiv \gamma - \log(4\pi)$, and μ is an arbitrary constant with the dimension of a mass.)

$$\tilde G_L(p) \simeq \tilde G_L^{free}(p)\left\{1 + \frac{1}{a-1}\left(\frac{2}{\epsilon}+\bar\gamma+\log\left(\frac{-p^2}{\mu^2}\right)\right)\right.$$
$$+\frac{1}{(a-1)^2}\left(\left(\log\left(\frac{-p^2}{\mu^2}\right)\right)^2+\bar\gamma^2-2\frac{\bar\gamma}{\epsilon}-2\left(\bar\gamma+\frac{1}{\epsilon}\right)\log\left(\frac{-p^2}{\mu^2}\right)\right.$$
$$\left.\left.-\frac{2}{\epsilon^2}-\frac{1}{2}\frac{\pi^2}{6}\right)+\mathcal{O}\left(\frac{m^2}{p^2}\right)\right\}. \tag{7.16}$$

If the same is done for the renormalized fermion propagator one arrives at

$$\tilde G_L^{ren}(p) \simeq \tilde G_L^{free}(p)\left\{1+\frac{1}{a-1}\log\left(\frac{-p^2}{m^2}\right)+\frac{1}{2!(a-1)^2}\left(\log\left(\frac{-p^2}{m^2}\right)\right)^2+\mathcal{O}\left(\frac{m^2}{p^2}\right)\right\}. \tag{7.17}$$

Note that the second order term in $(a-1)^{-1}$ of (7.17) also appears in (7.16), but with a different coefficient.

From these approximations one can infer that (cp.(B.10))

$$\tilde G_L(p) = \tilde G_L^{free}(p)\exp\left[-\frac{4\pi i}{a-1}\int\frac{d^2k}{(2\pi)^2}\frac{1}{k^2+p^2-m^2}\right], \tag{7.18}$$

respectively

$$\tilde G_L^{ren}(p) = \tilde G_L^{free}(p)\exp\left[-\frac{4\pi i}{a-1}\int\frac{d^2k}{(2\pi)^2}\left(\frac{1}{k^2+p^2-m^2}-\frac{1}{k^2-m^2}\right)\right]. \tag{7.19}$$

If $\tilde G_L(p)$ is not evaluated with the help of dimensional regularization but by introducing a cutoff mass Λ, it becomes a free propagator for $-p^2 \to \infty$. Although this cannot be interpreted physically it shows that the anomalous dimension can be cancelled just by manipulating the renormalization constant.

The perturbation theoretic derivation is done with the photon propagator (1.28); one finds that only the following part gives a contribution:

$$\tilde P^{\mu\nu}(k) = \frac{i}{e^2(a-1)}\frac{k^\mu k^\nu}{k^2-m^2}. \tag{7.20}$$

This has been the part which contributed in the path integral calculation of the fermion propagator. It is just the propagator which is derived by the incorrect effective action (7.8). For the AF expression (7.8) and the correct form are the same, as can be seen when A_μ is expressed by basic fields (4.4). The result for the diagrams 7.2a-7.2e coincides exactly with (7.16); that is, perturbation theory gives the nonrenormalized fermion propagator. The procedure to derive the renormalized one is obvious. The wavefunction renormalization constant is equal to the vertex renormalization constant [23], and with $\psi_L^{ren} = \mathcal{Z}^{-\frac{1}{2}}\psi_L$ we get for the Lagrangian $\mathcal{L} = \bar\psi(i\not\partial + e\sqrt{\pi}\not A(1-\gamma^5))\psi$:

$$\mathcal{L}(\psi,\bar\psi,A_\mu) = \mathcal{L}(\psi^{ren},\bar\psi^{ren},A_\mu) + (\mathcal{Z}-1)\mathcal{L}(\psi^{ren},\bar\psi^{ren},A_\mu). \tag{7.21}$$

The diagrams to be added to Fig.7.2 are given in Fig.7.3. The total result is expression (7.17), while (7.16) is the result for the diagrams in Fig.7.2.

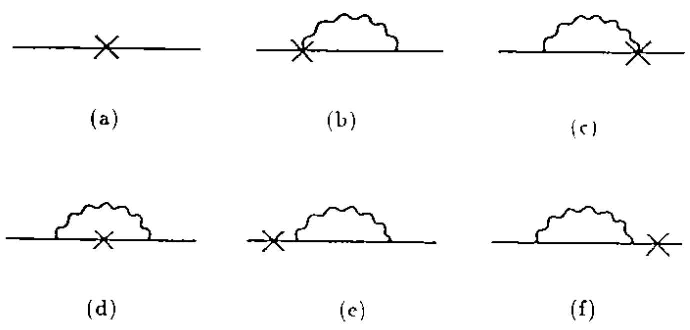

FIGURE 7.3: Diagrams generated by the counter term.

It is illuminating to look at the renormalized propagator from a different point of view. I use G_L^{ren} (cp. (1.37)) instead of (7.19), since the latter hides the physical picture behind the AF fermion propagator. The exponential function in G_L^{ren} is expanded to second order and Fourier transformed by convolutions:

$$\tilde{G}_L^{ren}(p) \simeq \tilde{G}_L^{free}(p) * \left[\delta(p) - \frac{4\pi i}{a-1}\frac{1}{p^2-m^2} - \frac{16\pi^2}{2(a-1)^2}\frac{1}{p^2-m^2} * \frac{1}{p^2-m^2} + \dots\right]. \quad (7.22)$$

One has

$$\frac{1}{p^2-m^2} * \frac{1}{p^2-m^2} = \frac{i}{2\pi p^2\sqrt{1-4\frac{m^2}{p^2}}}\log\left(\frac{1+\sqrt{1-4\frac{m^2}{p^2}}}{1-\sqrt{1+4\frac{m^2}{p^2}}}\right). \quad (7.23)$$

This shows explicitly the singularity structure of $\tilde{G}_L^{ren}$: a pole at $p^2=0$, and a branch point at $p^2=4m^2$. Therefore (7.21) seems to describe a system consisting of a free fermion and a cloud of massive scalars, connected by a coupling constant "1" instead of ϵ.

In sec.1 we realized that one can use a massive scalar instead of the gauge field to derive the fermion propagator. This feature also appears here; with the AF gauge field A_μ (4.4) and the Lagrangian (7.1) one finds that exactly the same calculation has to be done if the diagrams in Fig.7.2 and Fig.7.3 are read as referring to a scalar field $\Lambda = -\frac{1}{ea}f$ and vertices $\sqrt{\pi}\not{k}(1-\gamma^5)$. Moreover, when this attitute is employed it becomes reasonable to consider the $m^2=0$ limit, because there is no longer a photon propagator with a e^{-2}-pole. (Details of the calculation are presented in Appendix B.)

It is clear from the perturbation theory that the picture of a free fermion interacting with a gauge field is inadequate for the AF.

To make the last comment more concrete, the Dirac equation which describes the interaction between the fermion and the gauge boson is studied. It is known from sec.4 that the left-handed Landau gauge fermion ψ_L satisfies the Dirac equation

$$i\not{\partial}\psi_L(x) + 2e\sqrt{\pi} : (\not{A}\psi_L)(x) := 0. \quad (7.24)$$

How about the fermion in the anomalous formulation? The renormalized fermion Ψ_L^{ren} is written as (cp. (4.24))

$$\Psi_L^{ren}(x) =: \exp\left\{i\sqrt{\pi}\frac{2}{ea}f(x)\right\} : \psi_L^{free}(x). \quad (7.25)$$

From (4.4) one has

$$\not{A}(1-\gamma^5) = \frac{1}{e^2a}\not{\partial}f(1-\gamma^5), \quad (7.26)$$

i.e.

$$\begin{aligned} 2e\sqrt{\pi} : (\not{A}\Psi_L^{ren})(x) :=\ & \frac{\sqrt{\pi}}{ea}\gamma^\mu \lim_{\epsilon\to 0}\Big(\partial_\mu f(x+\epsilon) : \exp\left\{i\sqrt{\pi}\frac{2}{ea}f(x)\right\} : \\ & + : \exp\left\{i\sqrt{\pi}\frac{2}{ea}f(x)\right\} : \partial_\mu f(x-\epsilon)\Big)\psi_L^{free}(x). \end{aligned} \quad (7.27)$$

This is recast with the help of the relations [26]

$$\begin{aligned} A : e^B & :=: (A + [A^{(+)}, B^{(-)}])e^B :, \\ [A, e^B] & = [A, B]e^B. \end{aligned} \quad (7.28)$$

The result is

$$\begin{aligned} 2e\sqrt{\pi} : (\not{A}\Psi_L^{ren})(x) & := -i\not{\partial}\Psi_L^{ren}(x) + \frac{2\pi i}{e^2a^2}\lim_{\epsilon\to 0}G(x;\epsilon)\Psi_L^{ren}(x), \\ G(x;\epsilon) & = [\not{\partial}f^{(+)}(x+\epsilon), f^{(-)}(x)] - [\not{\partial}f^{(-)}(x-\epsilon), f^{(+)}(x)]. \end{aligned} \quad (7.29)$$

The $G(x;\epsilon)$-term prevents that Ψ_L^{ren} fulfills the Dirac equation (7.24). However, introducing the renormalization constant $\hat{Z}^{1/2}$ in a smeared form [37]

$$\Psi_L(x) = \lim_{\epsilon\to 0}\Psi_L(x;\epsilon) \equiv \lim_{\epsilon\to 0}\exp\left\{-\frac{2\pi}{e^2a^2}[f^{(+)}(x+\epsilon), f^{(-)}(x)]\right\}\Psi_L^{ren}(x), \quad (7.30)$$

one finds that $i\not{\partial}\Psi_L$ produces an extra term:

$$\begin{aligned} i\not{\partial}\Psi_L(x) & = i\hat{Z}^{\frac{1}{2}}\not{\partial}\Psi_L^{ren}(x) - \frac{2\pi i}{e^2a^2}\lim_{\epsilon\to 0}\not{\partial}[f^{(+)}(x+\epsilon), f^{(-)}(x)]\Psi_L(x) \\ & = i\hat{Z}^{\frac{1}{2}}\not{\partial}\Psi_L^{ren}(x) - \frac{2\pi i}{e^2a^2}\lim_{\epsilon\to 0}G(x;\epsilon)\Psi_L(x) = -2e\sqrt{\pi} : (\not{A}\Psi_L)(x) :. \end{aligned} \quad (7.31)$$

For the last step $i\not{\partial}\Psi_L^{ren}$ from (7.29) has been substituted. Therefore, the nonrenormalized AF fermion operator Ψ_L satisfies the Dirac equation (7.24).

As already noticed by Wightman [44], the extra term appearing in the Dirac equation for the renormalized fermion operator signals a lack of kinematical independence, which means that $[\psi_L(x), A(y)]_{ET} \neq 0$. One can infer from (4.4) and (4.24) that this relation is not fulfilled for Ψ_L^{ren} due to the A_0-component [45]. However, using Ψ_L in its smeared form (7.30) one has

$$[\Psi_L(x), A_0(y)]_{ET} = \lim_{\epsilon_1 \to 0} \left(\frac{2i\sqrt{\pi}}{a-1}\right) \delta(x_1 - y_1)[\Psi_L(x;\epsilon)]_{\epsilon_0 = 0} = 0, \tag{7.32}$$

because, while $\lim_{\epsilon \to 0} \exp[\frac{2\pi i}{a-1}\Delta^{(+)}(\epsilon; m^2)]$ is ill-defined, we have

$$\lim_{\epsilon_1 \to 0} \exp\left[\frac{2\pi i}{a-1}\Delta^{(+)}(\epsilon; m^2)\right]_{\epsilon_0=0}$$
$$= \lim_{\epsilon_1 \to 0} \exp\left[\frac{2\pi}{a-1}\frac{1}{4\pi}\left(\log(m^2|\epsilon^2|) - \frac{i}{4}\text{sign}(\epsilon_0)\Theta(\epsilon^2)\right)\right]_{\epsilon_0=0} = 0. \tag{7.33}$$

It should also be mentioned that contrary to Ψ_L^{ren} the nonrenormalized operator Ψ_L has the same anticommutator as the Landau gauge fermion : $\{\Psi_L(x), \Psi_L^+(0)\}_{ET} = \delta(x_1)$.

Here it has to be emphasized that the noncanonical features of the renormalized AF of the CSM are nothing unusual. On the contrary, any model where the fermion has an anomalous dimension has this defect: The Thirring model, the Schroer model, the VSM with BGS (sec.3), and also the (3+1)-dimensional QED when the (unitary) Coulomb gauge is chosen [46].

Conclusions on the Short-Distance Limit

Nonwithstanding the ϵ^{-2}-pole in the AF photon propagator (1.27), it has been shown that a perturbation theoretic derivation of the propagator (1.37) can be done. The short-distance behaviour necessitates a renormalization, which is however trivial in the present case. This triviality is underlined by the observation that all one has to do to delete the anomalous features is to introduce a "smearing" of the renormalization constant.

A decisive difference between the Landau gauge case and the AF is that the exact photon propagator can be derived perturbatively [40]. The AF really needs the exact solution of the model before perturbation theory can be done.

The main physical implication is the same as stated in the preceeding sections for the long-distance region: a physical scalar couples to the fermion in the AF, and this causes the short-distance problems. The problem is shared by many models in their unitary gauges, including the GIF of the CSM (cp.(5.10)). Therefore it is important to have a formulation of the model which allows to fix a gauge which decouples fermion and gauge field, like the Landau gauge. Stated differently, it is decisive to employ the GIF. While in (1+1) dimensions the short-distance problems can be controlled, it is likely to be fatal in (3+1) dimensions

(see ref.[47], chap.12), if the anomalous theory is not reduced to an effective model without fermions and without the need for renormalizability.

THE CHIRAL QUANTUM GRAVITY

Soon after the important physical implications of anomalies in global currents of gauge theories had been realized [1], a similar phenomenon was detected in gravitational theories [48] (see also ref.[49]). And when Jackiw and Rajaraman addressed the problem of anomalies in local currents by proposing the CSM as a consistent, though anomalous model, Li [50] tried to construct a consistent chiral quantum gravity (CQG) in (1+1) dimensions. He used an Einstein action derived by Jackiw [51] with the help of dimensional reduction from a (2+1)-dimensional action and an effective action for chiral Weyl fermions on a surface derived by Leutwyler [52]. His positive result for the consistency of the model was later disputed; using a more transparent quantization scheme Fukuyama and Kamimura [59] showed that the theory is inconsistent. However, as we will see both investigations are incomplete.

The topics anomaly and gravity already appeared in the approach of Polyakov to string theory [53]. For non-critical space-time dimensions a Weyl anomaly was shown to give rise to an effective action which is in the conformal gauge the Liouville action. This may be seen as an action for gravity in (1+1) dimensions, which is induced by the anomaly. Since then many attempts have been made to solve the quantum Liouville model (see e.g. ref.[54]). However, only recently a solution has been found for space-time dimensions smaller that one [55].

The decisive point for studying the possibility of a CQG is to take into account the Weyl anomaly of the gravitational sector, i.e. induced gravity. For the model building intended here it is not necessary to bother about the exponential term of the Liouville action; one may put the coefficient to zero and is left with the kinetic term of the Weyl scalar. In refs.[56,57,58] it has been shown by I. Tsutsui and myself that the resulting model is consistent if only a limited number of chiral fermions couples. Moreover, we detected many similarities to the CSM. Besides the CSM and its generalization (sec.3), the CQG is the second exactly solvable anomalous consistent model. Therefore it is particularly interesting to compare the fermonic sector with the result derived for the CSM.

The plan for secs.8 to 10 is the following:

In sec.8 the CQG is introduced in the conformal gauge. A perturbation theoretic derivation of the effective action of chiral fermions on a surface is presented, where a regularization parameter dependent term arises naturally and needs not be introduced by hand. Together with the gravitational action which is induced by the Weyl anomaly it constitutes the model. Its consistency is shown if not more than 24 fermions of either chirality couple. The similarity to the CSM is stressed.

Section 9 contains a reexamination of the previous work, refs.[50,59]. There the consistency of the model is checked by considering the algebra of surface deformation generators. In the corrected scheme with Weyl anomaly, I will show that the model is consistent if the above mentioned constraint on the number of chiral fermions is respected.

In sec.10 I turn to the main topic of the present work, the fermion. The fermion operator of the CQG is derived exactly and by perturbation theory, to show explicitly how the renormalization works. Here I capitalize on sec.7, since CQG and CSM are quite similar.

Some details of the calculations in sec.10 are contained in Appendix B. In Appendix C I present the consistency proof of the CQG in the case where besides the induced gravity Jackiw's ansatz for (1+1)-dimensional gravity [51] is employed. The result can be directly compared to refs.[50,59].

8. The Chiral Quantum Gravity as an Anomalous Gauge Theory

A new model is presented, the chiral quantum gravity (CQG). The effective action for chiral fermions on a surface is derived by perturbation theory: together with the Weyl anomaly induced gravitational action it constitutes the model, which is shown to parallel closely the CSM.

The Effective Action of the Fermions

Fermions on a (1+1)-dimensional surface are described by the action [49]

$$W_F = \int d^2x\sqrt{-g}\, e_a^{\mu}\frac{i}{2}(\bar\psi\gamma^a\bar\partial_\mu\psi), \tag{8.1}$$

where $g = \det g_{\mu\nu}$, e_a^{μ} is the zweibein and ψ denotes a set of n_R right-handed and n_L left-handed fermions. Usually one should expect a different derivative in eq.(8.1). $D_\mu = \partial_\mu + \omega_\mu$, with the spin connection $\omega_\mu = \epsilon^{ab}e_a^{\nu}\nabla_\mu e_{\nu b}$. However, this contribution vanishes if D_μ is substituted for ∂_μ. D_μ and ∇_μ are covariant derivatives connected by the zweibein [49].

The metric is given by the lapse and shift functions η_0 and η_1, together with the Weyl variable ϕ:

$$g^{\mu\nu} = e^{\phi}\begin{pmatrix} \eta_0^2-\eta_1^2 & -\eta_1 \\ -\eta_1 & -1 \end{pmatrix}. \tag{8.2}$$

The zweibein is parametrized with the help of ϕ and the Lorentz variable F:

$$e^a_{\ \mu} = \begin{pmatrix} \cosh\frac{F}{2} & -\sinh\frac{F}{2} \\ -\sinh\frac{F}{2} & \cosh\frac{F}{2} \end{pmatrix} e^{\frac{\phi}{2}} \begin{pmatrix} \eta_0 & \eta_1 \\ 0 & 1 \end{pmatrix}_{\mu c} = \begin{pmatrix} \cosh\frac{F}{2} & -\sinh\frac{F}{2} \\ -\sinh\frac{F}{2} & \cosh\frac{F}{2} \end{pmatrix} \tilde e^c_{\ \mu}. \tag{8.3}$$

Then the spin connection reads

$$\begin{aligned} \omega_\mu &= \epsilon^{ab}e_a^{\nu}\nabla_\mu e_{\nu b} = \partial_\mu F - \epsilon^{ab}\tilde e_a^{\nu}\nabla_\mu \tilde e_{\nu b} \\ &\equiv \partial_\mu F + \tilde\omega_\mu, \\ \tilde\omega_1 &= \eta_0^{-1}(\dot\phi - 2\eta_1' - \eta_1\phi'), \\ \tilde\omega_0 &= \eta_1\tilde\omega_1 + 2\eta_0' - \phi'\eta_0. \end{aligned} \tag{8.4}$$

Dots mean time-derivative and primes space-derivative. As a first step towards a solution one has to derive the effective action W_{eff} related to W_F. Classically, W_F is invariant both under Lorentz- and general coordinate transformations, i.e. one is free to choose any coordinate system at any space-time point. This is no longer true on the quantum level, and one is forced to give up one of the two symmetries. I keep general coordinate invariance, so the zweibein e_μ^a depends on F and ϕ.

In the present section I use the conformal gauge:

$$\eta_0 = 1, \quad \eta_1 = 0. \tag{8.5}$$

This gives $g_{\mu\nu} = e^{\phi}\eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the flat space metric. The spin connection is now

$$\omega_\mu = \partial_\mu F + \epsilon_{\mu\nu}\partial^\nu\phi, \tag{8.6}$$

and the fermion action simplifies to

$$W_F^{R,L} = \int d^2x e^{\frac{1}{2}(\phi\mp F)}\frac{i}{2}\left(\bar\psi_{R,L}\bar{\not\partial}\psi_{R,L}\right). \tag{8.7}$$

To first order in $(\phi\mp F)$ we have

$$W_{F(1)}^{R,L} = \int d^2x\frac{i}{8}(\phi\mp F)(\bar\psi\bar{\not\partial}\psi)(1\pm\gamma^5). \tag{8.8}$$

This has to be compared to the CSM case (7.1): The sign between $\bar\psi(\not\partial\psi)$ and $(\not\partial\psi)\bar\psi$ is different, and instead of $-\sqrt{\pi}$ one has a coefficient $\frac{i}{8}$. Because of the different sign, the momentum structure of the vertices is changed: $2k^\mu - p^\mu$ instead of p^μ.

In principle, the effective action can be calculated as in sec.7. In the present case there is no proof at hand to exclude contributions from diagrams with more than two external boson lines, since the derivative coupling in (8.8) makes them quadratically divergent. However, the form of the Lorentz anomaly, which is know to be proportional to the Riemann curvature, demands to consider only Fig.7.1b. (In the conformal gauge one has $\sqrt{-g}\,R = -\Box\phi$.) For the time being I take only a right-handed fermion. The effective action is written

$$W[F,\phi] = \int d^2p\int\frac{d^2k}{(2\pi)^2}\Pi(p,k) = \frac{1}{64}\int d^2p\,(I_0^{\mu\sigma\nu\rho}(p) + I_{-1}^{\mu\sigma\nu\rho}(p))\Gamma_{\mu\sigma\nu\rho}(p). \tag{8.9}$$

The "polarization scalar" is defined by

$$\Pi = -\left(\frac{i}{8}\right)^2\frac{(2k^\mu - p^\mu)(p^\sigma - k^\sigma)(2k^\nu - p^\nu)k^\rho}{k^2(p-k)^2}\Gamma_{\mu\sigma\nu\rho}. \tag{8.10}$$

With the vertices written down in sec.7, (7.9), and $\sigma\to F$, $\rho\to\phi$ and $(1-\bar\gamma^5)\to(1+\bar\gamma^5)$ one has

$$\Gamma_{\mu\sigma\nu\rho}(p) = \mathrm{Tr}\left[\mathcal{V}_\mu^R(F,\phi)\gamma_\sigma\mathcal{V}_\nu^R(F,\phi)\gamma_\rho\right]. \tag{8.11}$$

The result for the k-integration reads

$$\begin{aligned} I_0^{\mu\sigma\nu\rho}(p) = -\frac{i}{36\pi}\Big[& 4p^2(g^{\mu\sigma}g^{\nu\rho} + g^{\sigma\nu}g^{\mu\rho} + g^{\mu\nu}g^{\sigma\rho}) \\ & -13(p^\nu p^\rho g^{\mu\sigma} + p^\sigma p^\rho g^{\mu\nu}) \\ & -9p^\sigma p^\nu g^{\mu\rho} + 23p^\mu p^\rho g^{\nu\sigma} + 5p^\mu p^\sigma g^{\nu\rho} \\ & +9p^\mu p^\nu g^{\sigma\rho} + 3p^{-2}p^\mu p^\sigma p^\nu p^\rho\Big](1+\mathcal{O}(\epsilon)) \end{aligned} \tag{8.12}$$

and

$$I_{-1}^{\mu\sigma\nu\rho} = -\frac{i}{12\pi}\Big[-p^2(g^{\mu\sigma}g^{\nu\rho} + g^{\sigma\nu}g^{\mu\rho} + g^{\mu\nu}g^{\sigma\rho}) + 4(p^\nu p^\rho g^{\mu\sigma} + p^\sigma p^\rho g^{\mu\nu}) + 10p^\sigma p^\nu g^{\mu\rho} - 8p^\mu p^\rho g^{\nu\sigma} - 2p^\mu p^\sigma g^{\nu\rho} - 5p^\mu p^\nu g^{\sigma\rho}\Big]\left(\epsilon^{-1} + \mathcal{O}(\epsilon^0)\right). \quad (8.13)$$

The second expression again necessitates the derivation of the trace in d dimensions because of the ϵ^{-1} singularity. The calculation goes exactly as in the CSM case; the only difference is that $I_0^{\mu\sigma\nu\rho}$ and $I_{-1}^{\mu\sigma\nu\rho}$ are here far more complex. The final result has nevertheless the same structure as eq.(7.10):

$$W[F,\phi] = -\frac{i}{6\cdot 64\pi}\int d^2p\, p^2\left[(r+s)^2(F-\phi)^2 - (s^2-r^2)(F^2-\phi^2)\right]. \quad (8.14)$$

The conformal gauge effective action can now be written down ($a \equiv (s^2 - r^2)$):

$$W[F,\phi] = -\frac{1}{192\pi^2}\int d^2x\frac{1}{2}[-(F-\phi)\Box(F-\phi) + a(F\Box F - \phi\Box\phi)] \quad (8.15)$$

The introduction of n_R right-handed and n_L left-handed fermions ($\alpha = n_R + n_L$, $\beta = n_R - n_L$) gives

$$W[F,\phi] = \frac{1}{192\pi}\int d^2x\left[-\frac{1}{2}(a-1)\alpha F\Box F - \beta F\Box\phi + \frac{1}{2}(a+1)\alpha\phi\Box\phi\right] \quad (8.16)$$

This effective action is noninvariant under both Lorentz- and Weyl-transformations, as can be read off from the currents

$$\frac{\delta W}{\delta F} = -\frac{1}{192\pi}\int d^2x\,[(a-1)\alpha\Box F - \beta\Box\phi],$$
$$\frac{\delta W}{\delta\phi} = -\frac{1}{192\pi}\int d^2x\,[\beta\Box F - (a+1)\alpha\Box\phi]. \quad (8.17)$$

For Dirac fermions, $\beta = 0$, we can get rid of one of the two anomalies by choosing a appropriatly, but never both.

If I redefine a by $a \to a' = \frac{1}{4}(a-1)\alpha$ and rename a' by a, the result coincides with formula (2.14) of ref.[56], where the action has been derived by integrating the Lorentz anomaly known from the index theorem.

In arbitrary coordinates, one has

$$\phi = -\frac{1}{\sqrt{-g}\,\Box}\sqrt{-g}\,R, \quad F = \frac{1}{\sqrt{-g}\,\Box}\partial_\mu\sqrt{-g}\,g^{\mu\nu}\omega_\nu, \quad (8.18)$$

so that for n_R right-handed and n_L left-handed fermions one gets the action

$$W_{eff} = \frac{1}{192\pi}\int d^2x\left[\sqrt{-g}\,R\frac{1}{\sqrt{-g}\,\Box}(\alpha\sqrt{-g}\,R - \beta\partial_\mu\sqrt{-g}\,g^{\mu\nu}\omega_\nu) + \frac{a}{2}\sqrt{-g}\,g^{\mu\nu}\omega_\mu\omega_\nu\right]. \quad (8.19)$$

One possible general coordinate invariant term can be added which is not derived by the perturbation theory used above: $\sqrt{-g}\,\mu$. Since in conformal gauge it reads μe^ϕ I suspect that it is linked to (all) diagrams with many scalars coupling at one vertex to the fermion. In the present work I will use $\mu = 0$, although this convention is necessary only in sec.10.

The actions (8.1) and (8.19) describe the interaction between chiral fermions and gravity. However, one should think that a gravitational action also has to be included.

The Model

The action for (1+1)-dimensional gravity reads

$$W_G = -\frac{1}{16\pi G}\int d^2x\sqrt{-g}(R + 2\Lambda), \quad (8.20)$$

where G is the gravitational constant and Λ the cosmological constant. The first term of the action, however, is a surface term. The other is of the same form as the μ-term mentioned above. There had been a proposal to define (1+1)-dimensional gravity by "dimensional reduction" [51]. This is not necessary, since we already have a gravitational action in the theory! When the effective action (8.19) is quantized by use of the partition function

$$Z = \int\frac{de^a_\mu}{V_{diff}}e^{iW_{eff}}, \quad (8.21)$$

(V_{diff} is the diffeomorphism volume), and the conformal gauge is fixed, the measure can be written as (see Distler and Kawai, ref.[55])

$$\frac{de^a_\mu}{V_{diff}} = dF\,d\phi\,e^{i(26-2)W_L}. \quad (8.22)$$

Here

$$W_L = \frac{1}{48\pi}\int d^2x\left[-\frac{1}{2}\phi\Box\phi + \mu^2 e^\phi\right] \quad (8.23)$$

is the Liouville action, which is generated by the Weyl anomaly of the gravitational sector. The contribution -2 in (8.22) comes from the two scalars F and ϕ. The quantization of the Liouville action is problematic, and only recently some progress has been made [55]. I can confine myself to the case $\mu = 0$, so that the Liouville action becomes the action of a free scalar.

The Liouville action comes in thanks to gauge fixing, i.e. it is due to the ghost sector. Only if this contribution is taken into account we get a correctly quantized chiral gravity.

With $F \to \tilde{F} = F - \frac{\beta}{\alpha(a-1)}\phi$ I get from eqs.(8.16) and (8.23)

$$\begin{aligned} W_T[\tilde{F},\phi] &= W[\tilde{F},\phi] + 24W_L[\phi] \\ &= \frac{1}{192\pi}\int d^2x \frac{1}{2}\left[\frac{1}{a-1}\left(\alpha(a^2-1) - \frac{\beta^2}{\alpha} - 96(a-1)\right)\phi\Box\phi - (a-1)\alpha\tilde{F}\Box\tilde{F}\right]. \end{aligned} \tag{8.24}$$

Here $\tilde{F}$ and ϕ are dynamical variables, so that the condition on a for consistency of the model reads

$$1 \le a \le \frac{48}{\alpha} - \sqrt{\left(\frac{48}{\alpha}\right)^2 + 1 - \frac{\beta^2}{\alpha^2} - \frac{96}{\alpha}}\ , \tag{8.25}$$

and this results into the following condition on $n_{R,L}$:

$$n_{R,L} \le 24. \tag{8.26}$$

The model given by $W_T[\tilde{F},\phi]$ is the conformal gauge version of the model henceforth called the "chiral quantum gravity" (CQG). It is consistent if relation (8.26) is fulfilled.

Comparision with the Generalized Chiral Schwinger Model

To compare the result derived up to now with the GCSM (which includes the CSM), I use the effective action for n_R right-handed and n_L left-handed fermions writt.. down in sec.3 with $g_{L_i} = g_{R_j} = e$, and $s_{L_i} = r_{R_j} = \frac{a}{2}$:

$$\begin{aligned} W[A] = \int d^2x \Bigg[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} &+ n_L\, e^2\left(\frac{a}{2}A^2 - \frac{1}{2}A^-_\mu \frac{\partial^\mu\partial^\nu}{\Box} A^-_\nu\right) \\ &- n_R\, e^2\left(\frac{a}{2}A^2 - \frac{1}{2}A^+_\mu \frac{\partial^\mu\partial^\nu}{\Box} A^+_\nu\right)\Bigg]. \end{aligned} \tag{8.27}$$

With $A_\mu = \frac{1}{e}(\partial_\mu\sigma + \epsilon_{\mu\nu}\partial^\nu\rho)$ and

$$\begin{aligned} A^\pm_\mu \frac{\partial^\mu\partial^\nu}{\Box} A^\pm_\nu &= -\frac{1}{e^2}(\sigma\Box\sigma \pm 2\sigma\Box\rho - \rho\Box\rho), \\ A^2 &= -\frac{1}{e^2}(\sigma\Box\sigma - \rho\Box\rho), \\ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} &= \frac{1}{e^2}\rho\Box^2\rho, \end{aligned} \tag{8.28}$$

one has

$$W[\sigma,\rho] = \int d^2x \left[\frac{1}{e^2}\rho\Box^2\rho - \frac{1}{2}(a-1)\alpha\sigma\Box\sigma - \frac{1}{2}(a-1)\alpha\rho\Box\rho - \beta\sigma\Box\rho\right] \tag{8.29}$$

Diagonalizing this by $\sigma \to \tilde{\sigma} = \sigma - \frac{\beta}{(a-1)\alpha}\rho$ leads to

$$W[\tilde{\sigma},\rho] = \int d^2x \left[\frac{1}{e^2}\rho\Box^2\rho - \frac{1}{2}(a-1)\alpha\tilde{\sigma}\Box\tilde{\sigma} + \frac{m^2}{e^2}\rho\Box\rho\right], \tag{8.30}$$

where

$$\frac{m^2}{e^2} = \frac{1}{a-1}\left[\alpha(a^2-1) + \frac{\beta^2}{\alpha}\right], \tag{8.31}$$

which has already been found for the mass of A_μ in sec.3.

It is clear that the term $\frac{1}{e^2}\rho\Box^2\rho$ is needed to guarantee consistency of the GCSM; since $m^2 > 0$, the ρ particle would be a ghost if σ is physical and the kinetic term of A_μ was not existing. Therefore this term plays the role of the Liouville action in CQG. The most important practical difference between the CQG and the GCSM is that in the latter any number of chiral fermions can couple consistently.

9. The Algebra of Surface Deformations

In this section the invariance under general coordinate transformations is checked after the quantization is done.

Up to now I employed the conformal gauge, which leads to a drastic simplification of the model. It is therefore interesting to check the consistency of the CQG without fixing a special gauge. Nevertheless, one has to take into account that quantization always necessitates gauge fixing; this effect is accounted for by adding to the effective action (8.19) the general coordinate invariant Liouville action,

$$W_L = -\frac{1}{96\pi}\int d^2x\sqrt{-g}R\frac{1}{\sqrt{-g}\Box}\sqrt{-g}R, \tag{9.1}$$

multiplied by 26. The Weyl anomaly of the matter sector, -2, is also respected. The nonlocal term in W_{eff} is eliminated by introducing a scalar φ:

$$W_{loc} = \int d^2x\sqrt{-g}\left\{ \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \varphi(cR + bg^{\mu\nu}\nabla_\mu\omega_\nu) + \frac{1}{2}a'g^{\mu\nu}\omega_\mu\omega_\nu\right\} \tag{9.2}$$

The parameters a', b, and c are deduced by comparing $W_{eff} + 24W_L$ with the effective action derived from W_{loc} [50]:

$$b^2 = -\frac{1}{96\pi}(m_A - m_G), \quad c^2 = -\frac{1}{96\pi}(m_A + m_G), \quad a' = a + b^2, \tag{9.3}$$

with the algebraic and geometric means of $(24 - n_R)$ and $(24 - n_L)$:

$$m_A = \frac{(24 - n_R) + (24 - n_L)}{2}, \quad m_G = \sqrt{(24 - n_R)(24 - n_L)}. \tag{9.4}$$

If b and c are imaginary, φ is replaced by $i\varphi$, so that W_{loc} remains real. The effective action can be given explicitly by (a' is renamed a)

$$\begin{aligned} W_{loc} = \int d^2x\Big\{ & -c\eta_0^{-1}(\dot\phi - 2\eta_1' - \phi'\eta_1)(\dot\varphi - \eta_1\varphi') + c(2\eta_0' + \phi'\eta_0)\varphi' + 2\eta_0\Lambda e^\phi \\ & + \frac{1}{2}\eta_0^{-1}\left[(\dot\varphi - \varphi'\eta_1)^2 - \varphi'^2\eta_0^2\right] \\ & + b\eta_0^{-1}\left[-(\dot\varphi - \eta_1\varphi')(\omega_0 - \eta_1\omega_1) + \eta_0^2\omega_1\varphi'\right] \\ & + \frac{1}{2}a\eta_0^{-1}\left[\omega_0 - \omega_1\eta_1)^2 - \eta_0^2\omega_1^2\right]\Big\}, \end{aligned} \tag{9.5}$$

which can be recast into the form

$$W_{loc} = \int d^2x\left\{\dot\phi P_\phi + \dot\varphi P\varphi + \dot F P_F - (\eta_0 H + \eta_1 T)\right\}, \tag{9.6}$$

with

$$\begin{aligned} H = \; & -b\varphi' F' + 2c\varphi'' - c\varphi'\phi' - P_F\phi' + 2P_F' \\ & + \frac{1}{2}aF'^2 + \frac{1}{2}P_\varphi^2 + \frac{1}{2}\varphi'^2 + \frac{1}{2(a - b^2)}(P_F + bP_\varphi)^2 - 2\Lambda e^\phi \\ & - \frac{1}{2}\frac{a - b^2}{a(a - b^2 + c^2)}\left(-b\varphi' + aF' + \frac{cb}{a - b^2}P_F + \frac{ca}{a - b^2}P_\varphi + P_\phi\right)^2, \\ T = \; & P_\phi\phi' - 2P_\phi' + P_F F' + P_\varphi\varphi'. \end{aligned} \tag{9.7}$$

H and T form a closed algebra under Poisson brackets. This shows that the model is at the classical level general coordinate invariant, as it has to be. To quantize the system I follow a procedure due to D'Hoker and Jackiw [54,59]. First the constraints H and T are brought into a appropriate form by a transformation with the generating functional

$$\begin{aligned} \mathcal{W} = \; & \tilde P_\varphi\varphi + (\tilde P_F - b\tilde P_\varphi)F \\ & + \left(\tilde P_\phi - \frac{cb}{a - b^2}\tilde P_F - c\tilde P_\varphi - aF' + b\varphi'\right)\phi. \end{aligned} \tag{9.8}$$

After a rescaling of the new variables $\tilde F$ and $\tilde P_F$,

$$\tilde F \to \sqrt{a - b^2}^{\,-1}\tilde F, \quad \tilde P_F \to \sqrt{a - b^2}\tilde P_F, \tag{9.9}$$

and $\tilde\phi$, $\tilde P_\phi$,

$$\tilde\phi \to k^{-1}\tilde\phi, \quad \tilde P_\phi \to k\tilde P_\phi, \quad k \equiv \sqrt{-\frac{a(a - b^2 + c^2)}{a - b^2}}. \tag{9.10}$$

the quantum generators $H_\pm = H \pm T$ read (all tildes are omitted)

$$\begin{aligned} H_\pm = \; & \frac{1}{2} : (P_F \pm F')^2 : + \frac{1}{2} : (P_\phi \pm \phi')^2 : + \frac{1}{2} : (P_\varphi \pm \varphi')^2 : -2\Lambda : e^{p\phi} : \\ & \pm 2(c \mp b + \gamma_\varphi)(P_\varphi \pm \varphi')' + 2\left(\frac{a - b^2 \pm cb}{\sqrt{a - b^2}} + \gamma_F\right)(P_F \pm F')' \\ & -2(k + \gamma_\phi)(\pm P_\phi + \phi')'. \end{aligned} \tag{9.11}$$

Here the γ-parameters and p take account of possible quantum corrections. Colons do not necessarily mean normal ordering, but only that the coincident-point limit can be taken in terms of the following prescription [54]. Starting with the quantum expression $A \equiv \Pi \pm \Phi'$ for a generic canonical conjugate pair Π and Φ, one defines the product $A(x)A(y)$ by

$$A(x)A(y) + A(y)A(x) = : A(x)A(y) : + : A(y)A(x) : -\frac{2\zeta_1}{\pi(x - y)^2}, \tag{9.12}$$

where ζ_1 is an arbitrary constant, as only the form of the singularity can be inferred from a dimensional analysis. Another relation which we need is

$$\begin{aligned} m^2\left[A(x)e^{\beta\sigma(y)} + e^{\beta\sigma(y)}A(x)\right] = \; & M^2\left[: A(x)e^{\beta\sigma(y)} : + : e^{\beta\sigma(y)}A(x) :\right] \\ & - \frac{\beta\zeta_2}{\pi(x - y)}M^2 : e^{\beta\sigma(y)} : . \end{aligned} \tag{9.13}$$

Again, ζ_2 is an arbitrary constant and M a renormalized mass.

The generator of surface deformations should fulfill the following algebra.

$$\begin{aligned} \frac{1}{i}\left[H_\pm(x), H_\pm(y)\right]_{ET} &= \pm 2\left(H_\pm(x) + H_\pm(y)\right)\delta'(x-y) \pm C\,\delta'''(x-y), \\ \frac{1}{i}\left[H_+(x), H_-(y)\right]_{ET} &= 0. \end{aligned} \tag{9.14}$$

with $C = 0$. Thanks to the quantum corrections which I allowed in (9.11), it is very easy to get rid of this central charge, without any constraint on the parameters a, b or c. However, besides the general coordinate invariance the kinetic terms of the dynamical variables F and ϕ have to have the correct sign. This gives the relations

$$a - b^2 \geq 0, \quad k^2 \geq 0, \tag{9.15}$$

which results in

$$n_R \leq 24, \quad n_L \leq 24. \tag{9.16}$$

It coincides with the result found in sec.8.

If we break the Weyl invariance by a gauge fixing condition without altering the γ-parameters, we get a nonvanishing central charge $C = 4k^2$. If only Dirac fermions couple ($b^2 = 0$, $n_R = n_L = n$), the F-field does not appear ($a = 0$)[4] and we are left with $C = \frac{1}{12\pi}(26 - n)$, which is the familiar result [53].

In the present case the demand for a closed algebra of the generators of surface deformations did not give rise to any constraint. The situation changes if a gravitational action constructed by dimensional reduction is introduced in addition to the one induced by the Weyl anomaly. Such an analysis, which corresponds to what has been done in refs.[50,59], is contained in Appendix C.

[4] It has to be realized that the parameter a of the present section is related to the one of sec.8 by $a_{sec.9} = \frac{1}{4}(a_{sec.8} - 1)\alpha + b^2$.

10. The Fermion in the Chiral Quantum Gravity

The fermion operator is derived exactly and by using perturbation theory. The renormalization is done explicitly up to the two loop level.

The Derivation of the Fermion Propagator

If we look at the conformal gauge action for the fermions (8.7) and confine ourselves to right-handed fermions, the transformation

$$\psi \to \psi' = e^{\frac{1}{4}(\phi - F)}\psi \tag{10.1}$$

seems to lead to a free fermion Lagrangian. However, an effective action arises due to the noninvariance of the fermionic measure. The derivation of the fermion propagator can be done as in sec.1 for the CSM, with the result

$$G_R(x-y) = G_R^{free}(x-y)\int d\phi dF e^{-\frac{1}{4}(\phi-F)(x)} e^{-\frac{1}{4}(\phi-F)(y)} e^{iW[F,\phi]}. \tag{10.2}$$

Now the Liouville action is added to get the total effective action (8.24). We have two free scalar fields, ϕ and $\tilde{F}$:

$$W_T[\tilde{F},\phi] = \frac{1}{2}\int d^2x [r\phi\Box\phi + s\tilde{F}\Box\tilde{F}], \tag{10.3}$$

with

$$\begin{aligned} r &= \frac{1}{192\pi}\frac{1}{a-1}\left[\alpha(a^2-1) + \frac{\beta^2}{\alpha} - 96(a-1)\right], \\ s &= -\frac{1}{192\pi}(a-1)\alpha. \end{aligned} \tag{10.4}$$

I take $\bar{\phi} = \sqrt{r}\phi$, and $\bar{F} = \sqrt{s}\tilde{F}$; then one has

$$\begin{aligned} \frac{1}{4}(\phi - F) &= c_1\bar{\phi} + c_2\bar{F}, \\ c_1 &= \frac{1}{4}\sqrt{\frac{1}{r}}\left(1 - \frac{\beta}{\alpha(a-1)}\right), \\ c_2 &= -\frac{1}{4}\sqrt{\frac{1}{s}}. \end{aligned} \tag{10.5}$$

Therefore the contributions of $\bar{F}$ and $\bar{\phi}$ add up to one of a fictitious scalar Λ with the propagator

$$\bar{P}_\Lambda(k) = \frac{1}{192\pi}\frac{48 - \alpha + \beta}{8rs}\frac{i}{k^2}. \tag{10.6}$$

Following the procedure of sec.1, I get

$$\begin{aligned} G_R(x-y) = G_R^{free}(x-y) \int d\Lambda \exp\left\{\frac{i}{2}\int\frac{d^2k}{(2\pi)^2}\left[\tilde{\Lambda}(k)(i\bar{P})^{-1}(k)\tilde{\Lambda}(k)\right.\right. \\ \left.\left. + \left(e^{ikx} + e^{iky}\right)\tilde{\Lambda}(k) + \tilde{\Lambda}(k)\left(e^{-ikx} + e^{-iky}\right)\right]\right\} \end{aligned} \tag{10.7}$$

The fermion propagator then reads explicitly

$$G_R(x-y) = G_R^{free}(x-y)\exp\left\{4\pi i\delta \int \frac{d^2k}{(2\pi)^2}\left(e^{-ik(x-y)}+1\right)\frac{1}{k^2+i0}\right\}, \tag{10.8}$$

with the anomalous dimension

$$\delta = 6\frac{48-\alpha+\beta}{\alpha^2(\alpha^2-1)+\beta^2-96\alpha(\alpha-1)}. \tag{10.9}$$

The demand for unitarity of the model leads to $\delta \geq 0$. With $\mathcal{Z} = \exp\{4\pi i\delta\int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2}\}$ the renormalized fermion propagator is by use of (A.13)

$$G_R^{ren}(x-y) = \mathcal{Z}^{-1}G_R(x-y) = G_R^{free}(x-y)\left[-\mu^2(x-y)^2\right]^{-\delta}, \tag{10.10}$$

where μ is a constant with the dimension of a mass. This is similar to the propagator of the massless Thirring model [22].

I want to show how it can be derived by perturbation theory. As in sec.8 for the CSM, I first expand the exact result. Dimensional regularization is somewhat problematically in the present massless case, e.g. we have $\int d^2k k^{-2} = 0$, i.e. $\mathcal{Z} = 1$. Since I want to apply this procedure nevertheless, my renormalized fermion propagator reads

$$\hat{G}_R^{ren}(x-y) = G_R^{free}(x-y)\exp\left\{4\pi i\delta\int\frac{d^2k}{(2\pi)^2}\left(e^{-ik(x-y)}\frac{1}{k^2+i0}+\frac{1}{k^2-\mu^2+i0}\right)\right\}. \tag{10.11}$$

Expansion to second order in δ and Fourier transformation leads to

$$\bar{\tilde{G}}_L^{ren}(p) \simeq \tilde{G}_L^{free}(p)\left\{1+\delta\log\left(\frac{-p^2}{\mu^2}\right)+\frac{1}{2!}\delta^2\left(\log\left(\frac{-p^2}{\mu^2}\right)\right)^2\right\}. \tag{10.12}$$

This can obviously be summed to give

$$\bar{\tilde{G}}_L^{ren}(p) = \tilde{G}_L^{free}(p)\left[\frac{\mu^2}{-p^2}\right]^{-\delta}. \tag{10.13}$$

Now I turn to perturbation theory. With $\hat{\mathcal{Z}} = \exp\{4\pi i\delta\int\frac{d^2k}{(2\pi)^2}\frac{1}{k^2-\mu^2}\}$ the renormalized Lagrangian is (cp.(8.7))

$$\mathcal{L}(\psi,\bar{\psi},\phi-F) = \mathcal{L}(\psi^{ren},\bar{\psi}^{ren},\phi-F)+(\hat{\mathcal{Z}}-1)\mathcal{L}(\psi^{ren},\bar{\psi}^{ren},\phi-F). \tag{10.14}$$

The vertices needed are shown in Fig.10.1:

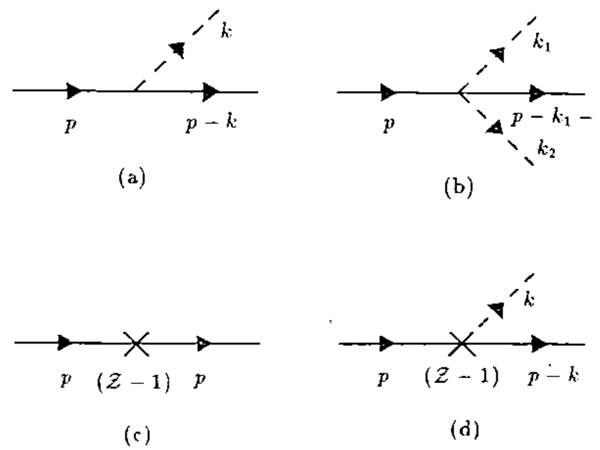

FIGURE 10.1: The vertices appearing in the CQG to order δ^2.
a: $i(2p-k)^\mu P_R$, b: $2i(2p-k_1-k_2)^\mu P_R$, c: $(\hat{\mathcal{Z}}-1)$ and d: $(\hat{\mathcal{Z}}-1)i(2p-k)^\mu P_R$.

Apart from the effect of the counter terms, this closely parallels scalar electrodynamics [47]; the factor 2 for vertex 10.1b is due to the necessity to take into account also the diagram with k_1 and k_2 interchanged. The Feynman diagrams that contribute are depiced in Fig.10.2. The nontrivial integrals are written down in the Appendix B. As one can use massless integrals, the calculation is simple compared with the one of sec.7. The result exactly coincides with (10.12). If the contributions from the counter term are disregarded, the result is

$$\begin{aligned}\tilde{G}_L(p) \simeq \tilde{G}_L^{free}(p)\Bigg\{1 &+ \delta\left(\frac{2}{\epsilon}+\bar{\gamma}+\log\left(\frac{-p^2}{\mu^2}\right)\right)\\ &+\delta^2\left(\left(\log\left(\frac{-p^2}{\mu^2}\right)\right)^2+\bar{\gamma}^2+2\frac{\bar{\gamma}}{\epsilon}+2\left(\bar{\gamma}+\frac{1}{\epsilon}\right)\log\left(\frac{-p^2}{\mu^2}\right)\right.\\ &\left.+\frac{2}{\epsilon^2}-\frac{1}{2}\frac{\pi^2}{6}\right)\Bigg\},\end{aligned} \tag{10.15}$$

which is just of the form (7.16), the fermion propagator in the AF of the CSM, with $m^2 = 0$.

What happens for Dirac fermions, i.e. $n_R = n_L = n$? It is reasonable to demand $a = 1$, since the Lorentz variable was introduced to deal with chirality. If this is substituted in (10.4), a straightforward calculation gives

$$\delta = \frac{3}{4(25-n)}. \tag{10.16}$$

The 25 replaces 24 because F no longer contributes to the Weyl anomaly. Only if the Weyl symmetry is restored, which happens for $n = 26$, the anomalous dimension becomes singular.

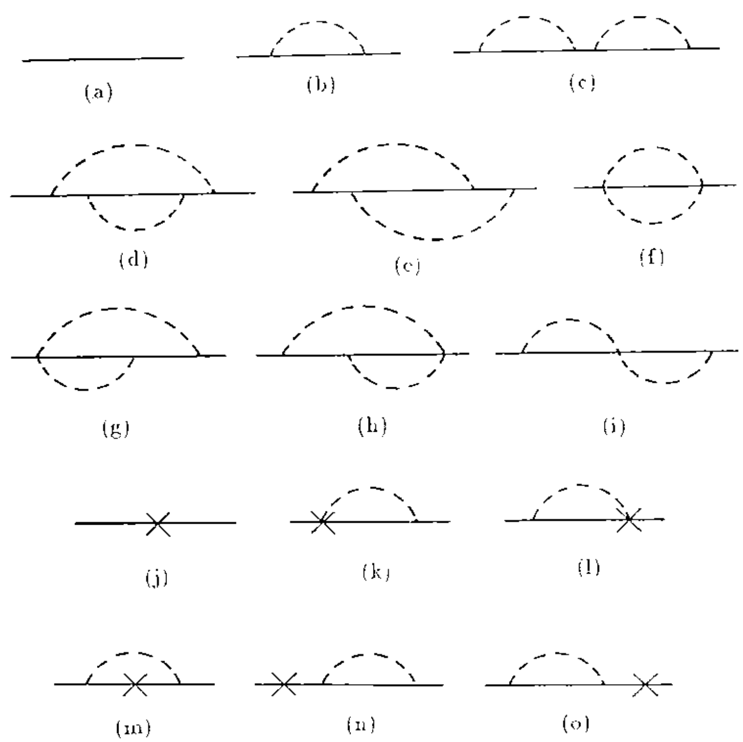

FIGURE 10.2: All Feynman diagrams contributing to the fermion propagator to $\mathcal{O}(\delta^2)$.

The fermion propagator of the CQG exhibits a close similarity to the one of the AF of the CSM in the region where perturbation theory can be applied ($m^2 \to 0$), although we have seen that the perturbative derivation is quite different.

In the CQG the interaction beween the fermion and scalar fields is totally due to anomalies. The same interpretation can be adopted for the AF of the CSM. It is interesting to compare the vector quantum gravity defined by $n_R = n_L$ and $a = 1$ with the vector quantum gravity with artificially broken Lorentz symmetry, i.e. $a \neq 1$. In the latter case the longitudinal component of the spin connection couples (see (10.3) and (10.4) and remember that $\tilde{F} = F$ for $\beta = 0$), just like the longitudinal component of the gauge field coupled in the VSM with BGS (sec.3).

SUMMARY AND CONCLUSIONS

In this work I investigated two (1+1)-dimensional consistent anomalous chiral theories: The chiral Schwinger model and the chiral quantum gravity.

SUMMARY FOR THE CHIRAL SCHWINGER MODEL

In the GIF there is a uniquely defined physical fermion operator which does not give rise to fermionic single particle states. This discriminates it from the AF in the standard interpretation [17].

The formal definition of gauge invariant fermion operators by some composite objects like $\exp\{2i\sqrt{\pi}\theta\}\psi_L$, and the inclusion of the AF as a special gauge of the GIF lead to an ambiguity in the interpretation of the physical fermionic content of the GIF. In the bosonized CSM there is no ambiguity concerning the physical sector, and the AF is the unitary gauge of the GIF [14].

The consideration of the vector Schwinger model with broken gauge symmetry motivated an attractive interpretation for the specific form of the fermion operator in the AF of the CSM; just as the artificial breakdown of the gauge symmetry in the vector Schwinger model couples a physical massless scalar (the longitudinal component of the gauge field) to the fermion operator, the anomaly of the CSM in the AF couples a massive scalar (the left-handed component of the gauge field) to a *free* fermion operator. This interpretation of the fermion operator in the AF indicates that there is no deconfining property of the anomaly, as has been suggested in refs.[8,23].

The short-distance behaviour of the CSM in the AF is problematic and needs UV-renormalization. This is easily done, but the reason for this defect, the coupling of a new physical scalar to the fermion, is likely to rule out this formulation for anomalous chiral gauge theories in (3+1) dimensions. The GIF allows to choose a gauge where fermion and gauge boson decouple in the short-distance limit.

SUMMARY FOR THE CHIRAL QUANTUM GRAVITY

The effective action of chiral fermions on a surface can be derived by using perturbation theory, including one ambiguous term whose coefficient is a regularization parameter.

Due to the Weyl anomaly of the gravitational sector, a limited number of chiral fermions can couple consistently to gravity in (1+1) dimensions. This result does not depend on a special gauge.

In the conformal gauge the fermion propagator has been derived exactly and in perturbation theory. The fermion operator does not give rise to an asymptotic fermionic field. The same is true for the vector quantum gravity, which is here defined analogously to the Schwinger model case. A vector quantum gravity with broken Lorentz symmetry has been introduced and compared to the vector Schwinger model with broken gauge symmetry. The main difference between the fermion operators in the vector quantum gravity with and without broken Lorentz symmetry consists in the coupling of the longitudinal component of the spin connection in the former, which parallels the vector Schwinger model case.

The fermion operator of the CQG and of the vector quantum gravity aquires an anomalous dimension.

Conclusions

The prominent aim of the present work has been to investigate the fermionic sector of anomalous (1+1)-dimensional models with respect to possible application of the knowledge to realistic (3+1)-dimensional models.

The CSM was found to be particularly unsuited for this purpose; as soon as a promising formulation, the GIF, is chosen, the scope defined by the original AF is left and a model is considered whose fermionic content has more similarity with the asymmetric Schwinger model than with the CSM in the AF. Although the AF of the CSM is interesting in its own right, special features as the existence of an interacting fermion operator which gives rise to an asymptotic fermion field are not even typical of other (1+1)-dimensional anomalous models (like the generalized chiral Schwinger model and the CQG), let alone realistic models.

The GIF of anomalous Schwinger models like the CSM seem to owe their gauge invariance indirectly to anomaly cancelling new fermions, which is an old measure to tackle the problem. Therefore I doubt that hopes are justified to solve a fundamental anomalous (3+1)-dimensional theory with chiral fermion content, where e.g. gauge fields aquire a mass by the anomaly.

However, the investigations on the exactly solvable CSM enlarged the understanding of anomalous gauge theories by demonstrating explicitly that the breakdown of the gauge symmetry need not rule out a model as inconsistent and nonunitary. This feature may well survive in (3+1) dimensions for effective models describing the low-energy domain of some fundamental theory.

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APPENDICES

A. Some Boson and Fermion Propagators

Here I study in detail some of the fermion propagators appearing in the main part of the work.

Two different invariant functions of the massive and the massless scalars are used in the text:

$$\begin{aligned} \left[\phi_m^{(+)}(x), \phi_m^{(-)}(0)\right] &= \Delta^{(+)}(x), \\ i\langle T(\phi_m(x)\phi_m(0))\rangle &= \Delta_F(x), \end{aligned} \tag{A.1}$$

where ϕ_m is massive and

$$\begin{aligned} \left[\phi^{(+)}(x), \phi^{(-)}(0)\right] &= D^{(+)}(x), \\ i\langle T(\phi(x)\phi(0))\rangle &= D_F(x). \end{aligned} \tag{A.2}$$

They are related by

$$i\Delta_F(x) = \Delta^{(+)}(x) + \Delta^{(-)}(x), \tag{A.3}$$

and for D accordingly. We have

$$\begin{aligned} \Delta^{(+)}(x) &= \frac{1}{2\pi} K_0\left(m\sqrt{-x^2 + i0x^0}\right), \\ \Delta_F(x) &= \frac{i}{2\pi} K_0\left(m\sqrt{-x^2 - i0}\right), \\ D^{(+)}(x) &= -\frac{1}{2\pi}\log\left(\mu\sqrt{-x^2 - i0x^0}\right), \\ D_F(x) &= -\frac{i}{2\pi}\log\left(\mu\sqrt{-x^2 - i0}\right). \end{aligned} \tag{A.4}$$

Here μ is an IR-regularization parameter. This is necessary to handle the well-known problem to define a two-point function of a massless scalar in (1+1) dimensions [44]. The formulas in (A.4) for the massless scalar have to be seen as definitions. For more details, see ref.[60].

In the following I discuss some fermion propagators appearing in the text.

i) AF fermion propagator:

The exponent of the propagator (1.37) in sec.1 can be given more explicitly [61]. By a Wick rotation ($k_0 = ik_4$, $\tau = it$, $d^2k = idk_4\,dk_1$) one has with $z = x - y$:

$$G_L(z) = G_L^{free}(z)\exp\left[-\frac{2}{a-1}\left(K_0(m|z|) + \log(\tfrac{1}{2}m|z|) + \gamma + \sum_{n=1}^{\infty}\frac{(-1)^n\left(\frac{1}{2}\Lambda|z|\right)^{2n}}{2n(n!)^2}\right)\right]. \tag{A.5}$$

Here Λ is an UV-cutoff mass. The sum has the interesting property that $\lim_{x\to\infty}\sum(x) = -\log(x) - \gamma$. So I can safely conclude that the exponent in (A.5) is a constant in the long-distance region, and the propagator becomes free. The limit $|z| \to 0$ also exists, however, since an UV-cutoff has been introduced, it cannot be interpreted physically.

The perturbation theory gave us a hint how to Fourier transform (A.5):

$$\tilde{G}_L(p) = \tilde{G}_L^{free}(p)\exp\left[\frac{1}{a-1}\log\left(\frac{-p^2 + m^2}{\Lambda}\right)\right]. \tag{A.6}$$

This has a pole at $p^2 = 0$, as expected. UV-renormalization leads to

$$G_L^{ren}(z) = G_L^{free}(z)\exp\left[\frac{2}{a-1}K_0(m|z|)\right], \tag{A.7}$$

and Fourier transformed,

$$\tilde{G}_L^{ren}(p) = \tilde{G}_L^{free}(p)\exp\left[\frac{1}{a-1}\log\left(\frac{-p^2 + m^2}{m^2}\right)\right]. \tag{A.8}$$

This propagator can be interpreted in all momentum regions.

ii) Landau gauge fermion propagator:

In sec.2 I presented the renormalized fermion propagator in the configuration space (2.16). It can be given explicitly:

$$G_L(z) = G_L^{free}(z)\exp\left[2\frac{a-1}{a^2}\left(K_0(m|z|) + \log(\tfrac{1}{2}m|z|) + \gamma\right)\right]. \tag{A.9}$$

The result for the exponent is composed of

$$\int\frac{d^2k}{(2\pi)^2}\frac{1}{k^2 - m^2}\left(e^{ikz} - 1\right) = -\frac{i}{2\pi}\left[K_0(m|z|) + \log(\tfrac{1}{2}m|z|) + \gamma + \sum_{n=1}^{\infty}\frac{(-1)^n\left(\frac{1}{2}\Lambda|z|\right)^{2n}}{2n(n!)^2}\right]. \tag{A.10}$$

and

$$\int\frac{d^2k}{(2\pi)^2}\frac{1}{k^2}\left(e^{-ikz} - 1\right) = \frac{i}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\left(\frac{1}{2}\Lambda|z|\right)^{2n}}{2n(n!)^2}. \tag{A.11}$$

In (A.10) an UV-problem is introduced by the "-1"; this divergence is regularized by Λ and then removed by subtracting (A.11). If we look at

$$\int\frac{d^2k}{(2\pi)^2}\frac{1}{k^2 - m^2}e^{-ikz} = -\frac{i}{2\pi}K_0(m|z|) \tag{A.12}$$

and

$$\int\frac{d^2k}{(2\pi)^2}\frac{1}{k^2}e^{-ikz} = \frac{i}{2\pi}\left[\log(\tfrac{1}{2}\mu|z|) + \gamma - \sum_{n=1}^{\infty}\frac{(-1)^n\left(\frac{1}{2}\mu|z|\right)^{2n}}{2n(n!)^2}\right] \tag{A.13}$$

it is obvious that the effect of the "-1" in the Landau gauge propagator (2.16) is removing an IR-cutoff. If the "-1" would not be there, one had to renormalize the expression

$$\int \frac{d^2k}{(2\pi)^2}\frac{m^2}{k^2(k^2-m^2)}e^{ikz} = -\frac{i}{2\pi}\left[K_0(m|z|)+\log(\tfrac{1}{2}\mu|z|)-\gamma+\sum_{n=1}^{\infty}\frac{(-1)^n\left(\frac{1}{2}\mu|z|\right)^{2n}}{2n(n!)^2}\right] \quad (A.14)$$

which corresponds to the momentum space expression

$$\log\left(\frac{-p^2+m^2}{m^2}\right)+\log\left(\frac{\mu^2}{-p^2+\mu^2}\right)=\log\left(\frac{-p^2+m^2}{-p^2+\mu^2}\right)+\log\left(\frac{\mu^2}{m^2}\right). \quad (A.15)$$

Now it is clear how to proceed: the $\log(\frac{\mu^2}{m^2})$-term has to be removed (not by $\mu \equiv m$!) and the limit $\mu \to 0$ can be taken.

iii) AF fermion propagator for $m^2 = 0$:

The $m^2 = 0$-case of the AF fermion propagator is given by (A.6) and (A.8) with m^2 substituted by μ^2. The large-momentum behaviour is recognized to be exactly the same as in case i). This time, however, a renormalization has to be done as in case ii). For this reason a physical mass parameter "M" has to be fixed (see Johnson, ref.[22]), and the renormalized fermion propagator is

$$\tilde{G}_L^{ren}(p) = \mathcal{Z}\,\tilde{G}_L^{free}(p)\exp\left[\frac{1}{a-1}\log\left(\frac{-p^2+\mu^2}{\mu^2}\right)\right]. \quad (A.16)$$

with

$$\mathcal{Z} = \exp\left[\frac{1}{a-1}\log\left(\frac{\mu^2}{M^2}\right)\right]. \quad (A.17)$$

The limit $\mu \to 0$ can be taken and we see that there is no pole for $p^2 = 0$ in (A.16).

B. Dimensional Regularization

In this section I present some of the tools used to do the pertubation theory in secs.7 and 10.

General Formulas

The integrals over d space-time dimensions in the present paper can be deduced from (see [47] and references therein)

$$\int \frac{d^2k}{(2\pi)^2}\frac{(k^2)^r}{(k^2-m^2)^s} = \frac{i(-1)^{r-s}}{(4\pi)^{\frac{d}{2}}(m^2)^{r-s+\frac{d}{2}}}\frac{\Gamma(r+\frac{d}{2})\Gamma(s-r-\frac{d}{2})}{\Gamma(\frac{d}{2})\Gamma(s)}. \quad (B.1)$$

Feynman parameters have been used according to the following formulas:

$$\begin{aligned}\frac{1}{A^\alpha B^\beta} &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 dx\frac{x^{\alpha-1}(1-x)^{\beta-1}}{[xA+(1-x)B]^{\alpha+\beta}},\\ \frac{1}{A^\alpha B^\beta C^\gamma} &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\int_0^1 dx\, x\int_0^1 dy\frac{(xy)^{\alpha-1}[x(1-y)]^{\beta-1}(1-x)^{\gamma-1}}{[xyA+x(1-y)B+(1-x)C]^{\alpha+\beta+\gamma}}.\end{aligned} \quad (B.2)$$

The momentum integrals used instead of the master formula (B.1) are

$$\begin{aligned}\int\frac{d^dk}{(2\pi)^d}\frac{1}{[k^2-2p\cdot k-m^2]^\alpha} &= \left(\frac{i}{4\pi}\right)(-1)^\alpha\frac{\Gamma(\alpha-\frac{d}{2})}{\Gamma(\alpha)}\frac{1}{[p^2+m^2]^{\alpha-\frac{d}{2}}}\\ \int\frac{d^dk}{(2\pi)^d}\frac{k^\mu}{[k^2-2p\cdot k-m^2]^\alpha} &= \left(\frac{i}{4\pi}\right)(-1)^\alpha\frac{\Gamma(\alpha-\frac{d}{2})}{\Gamma(\alpha)}\frac{p^\mu}{[p^2+m^2]^{\alpha-\frac{d}{2}}}\\ \int\frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu}{[k^2-2p\cdot k-m^2]^\alpha} &= \left(\frac{i}{4\pi}\right)(-1)^\alpha\frac{1}{\Gamma(\alpha)[p^2+m^2]^{\alpha-\frac{d}{2}}}\\ &\times\left[\Gamma(\alpha-\tfrac{d}{2})p_\mu p_\nu-\frac{1}{2}\Gamma(\alpha-\tfrac{d}{2}-1)g_{\mu\nu}(p^2+m^2)\right].\end{aligned} \quad (B.3)$$

Here are some examples:

$$\begin{aligned}\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m^2)[(p-k)^2]^\alpha} &= \left(\frac{i}{4\pi}\right)^{\frac{d}{2}}(-1)^{\alpha+1}\frac{\Gamma(\alpha+1-\frac{d}{2})}{\Gamma(\alpha)}(p^2)^{\frac{d}{2}-\alpha-1}\left(\frac{m^2-p^2}{p^2}\right)^{\frac{d}{2}-\alpha-1}\\ &\times B(\alpha,\tfrac{d}{2}-\alpha)\;{}_2F_1\left(\alpha+1-\tfrac{d}{2};\tfrac{d}{2}-\alpha;\tfrac{d}{2};\frac{-p^2}{-p^2+m^2}\right).\end{aligned} \quad (B.4)$$

The Beta-function is defined by $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

The hypergeometric function is transformed and then expanded in m^2 [61]:

$$
\begin{aligned}
&{}_2F_1\left(\alpha+1-\tfrac{d}{2},\tfrac{d}{2}-\alpha;\tfrac{d}{2};\frac{-p^2}{-p^2+m^2}\right) \\
&= \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)}{\Gamma(d+\alpha-1)\Gamma(\alpha)}\left(1-\frac{\left(\alpha+1-\frac{d}{2}\right)\left(\frac{d}{2}-\alpha\right)}{2-\frac{d}{2}}\frac{m^2}{-p^2+m^2}+\dots\right) \\
&+\left(\frac{m^2}{-p^2+m^2}\right)^{\frac{d}{2}-1}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)}{\Gamma\left(\alpha+1-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-\alpha\right)}\left(1+\frac{d-\alpha-1}{\frac{d}{2}}\frac{m^2}{-p^2+m^2}+\dots\right)
\end{aligned}
\tag{B.5}
$$

With the help of two Feynman parameters I get

$$
\begin{aligned}
\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2(k^2-m^2)[(p-k)^2]^\alpha} &= \left(\frac{i}{4\pi}\right)^{\frac{d}{2}}(-1)^{\alpha+1}\frac{\Gamma(\alpha+1-\frac{d}{2})}{\Gamma(\alpha)}m^{-2}\left(p^2\right)^{\frac{d}{2}-\alpha-1} \\
&\left\{\left(\frac{m^2-p^2}{p^2}\right)^{\frac{d}{2}-\alpha-1}B(\alpha,\tfrac{d}{2}-\alpha)\ {}_2F_1\left(\alpha+1-\tfrac{d}{2},\tfrac{d}{2}-\alpha;\tfrac{d}{2};\frac{-p^2}{-p^2+m^2}\right)\right. \\
&\left.+(-1)^{\frac{d}{2}-\alpha}B(\tfrac{d}{2}-1,\tfrac{d}{2}-\alpha)\right\}.
\end{aligned}
\tag{B.6}
$$

To (B.4) corresponds

$$
\begin{aligned}
\int\frac{d^dk}{(2\pi)^d}\frac{p\cdot k}{(k^2-m^2)[(p-k)^2]^\alpha} &= \left(\frac{i}{4\pi}\right)^{\frac{d}{2}}(-1)^{\alpha+1}\frac{\Gamma(\alpha+1-\frac{d}{2})}{\Gamma(\alpha)}\left(p^2\right)^{\frac{d}{2}-\alpha}\left(\frac{m^2-p^2}{p^2}\right)^{\frac{d}{2}-\alpha-1} \\
&\times B(\alpha+1,\tfrac{d}{2}-\alpha)\ {}_2F_1\left(\alpha+1-\tfrac{d}{2},\tfrac{d}{2}-\alpha;\tfrac{d}{2}+1;\frac{-p^2}{-p^2+m^2}\right).
\end{aligned}
\tag{B.7}
$$

and to (B.6)

$$
\begin{aligned}
\int\frac{d^dk}{(2\pi)^d}\frac{p\cdot k}{k^2(k^2-m^2)[(p-k)^2]^\alpha} &= \left(\frac{i}{4\pi}\right)^{\frac{d}{2}}(-1)^{\alpha+1}\frac{\Gamma(\alpha+1-\frac{d}{2})}{\Gamma(\alpha)}m^{-2}\left(p^2\right)^{\frac{d}{2}-\alpha-1} \\
&\left\{\left(\frac{m^2-p^2}{p^2}\right)^{\frac{d}{2}-\alpha-1}B(\alpha+1,\tfrac{d}{2}-\alpha)\ {}_2F_1\left(\alpha+1-\tfrac{d}{2},\tfrac{d}{2}-\alpha;\tfrac{d}{2}+1;\frac{-p^2}{-p^2+m^2}\right)\right. \\
&\left.+(-1)^{\frac{d}{2}-\alpha}B(\tfrac{d}{2},\tfrac{d}{2}-\alpha)\right\}.
\end{aligned}
\tag{B.8}
$$

The decisive difference between (B.4) and (B.7), respectively (B.6) and (B.8) is, that the former has no continuous $m^2\to 0$ limit, in contrast to the latter.

As special cases I have

$$
\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2(k^2-m^2)} = \left(\frac{i}{4\pi}\right)^{\frac{d}{2}}(m^2)^{\frac{d}{2}-2}\frac{\Gamma(\frac{d}{2}-1)\Gamma(2-\frac{d}{2})}{\Gamma(\frac{d}{2})} = -\left(\frac{i}{4\pi}\right)^{\frac{d}{2}}(m^2)^{\frac{d}{2}-2}\Gamma(1-\tfrac{d}{2}),
\tag{B.9}
$$

and

$$
\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m^2} = -\left(\frac{i}{4\pi}\right)^{\frac{d}{2}}(m^2)^{\frac{d}{2}-1}\Gamma(1-\tfrac{d}{2}).
\tag{B.10}
$$

From here it follows as a rule

$$
\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2} = 0.
\tag{B.11}
$$

The gamma functions are expanded as follows [61]:

$$
\begin{aligned}
\Gamma(1+\epsilon) &= 1+c_1\epsilon+c_2\epsilon^2+\dots, \\
\Gamma^{-1}(1+\epsilon) &= 1+d_1\epsilon+d_2\epsilon^2+\dots, \\
c_1 = -\gamma,\ \ c_2 &= \frac{1}{2}\left(\gamma^2+\frac{\pi^2}{6}\right), \\
d_1 = \gamma,\ \ d_2 &= \frac{1}{2}\left(\gamma^2-\frac{\pi^2}{6}\right).
\end{aligned}
\tag{B.12}
$$

Higher orders in ϵ are not necessary throughout this paper; γ is Euler's constant.

At last I write down the formulas used to handle the γ -matrices in d dimensions:

$$
\begin{aligned}
\gamma_\mu\gamma^\mu &= d, \\
\gamma_\mu\gamma_\nu\gamma^\mu &= (2-d)\gamma_\nu, \\
\gamma_\mu\gamma_\nu\gamma_\alpha\gamma^\mu &= 4g_{\nu\alpha}+(d-4)\gamma_\nu\gamma_\alpha, \\
\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta\gamma^\mu &= -2\gamma_\beta\gamma_\alpha\gamma_\nu-(d-4)\gamma_\nu\gamma_\alpha\gamma_\beta.
\end{aligned}
\tag{B.13}
$$

Details on the Perturbative Derivatation of thr Landau Gauge Fermion Propagator

At first the expansion of the exact propagator is looked at. For the free propagator one has

$$
G_L^{free}(x-y) = i\frac{(x-y)_\mu\gamma^\mu}{(x-y)^2}P_L \to \tilde{G}_L^{free}(p) = i\frac{\not{p}}{p^2}P_L.
\tag{B.14}
$$

The Fourier transformation of $F(x-y)$ is also easily done:

$$
F(x-y)\to\tilde{F}(p) = -4\pi i\frac{a-1}{a^2}\left[\delta(p)\int d^2k\frac{m^2}{k^2(k^2-m^2)}-\frac{m^2}{p^2(p^2-m^2)}\right].
\tag{B.15}
$$

The convolution formulas needed are

$$
\begin{aligned}
f(p)*g(p) &= \int\frac{d^2k}{(2\pi)^2}f(p-k)g(k), \\
f(p)*g(p)*k(p) &= \int\frac{d^2k}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}f(p-k-q)g(k)h(q).
\end{aligned}
\tag{B.16}
$$

In first order of $\frac{a-1}{a^2}$ I get

$$G_L^{free}(x-y)F(x-y) \to \tilde{G}_L^{free}(p) * \tilde{F}_1(p)$$

$$= 4\pi i \frac{a-1}{a^2}\left[\int \frac{d^2k}{(2\pi)^2}\frac{\not{p}-\not{k}}{(p-k)^2}\delta(k)\int\frac{d^2q}{(2\pi)^2}\frac{m^2}{q^2(q^2-m^2)} - \int\frac{d^2k}{(2\pi)^2}\frac{m^2(\not{p}-\not{k})}{k^2(k^2-m^2)(p-k)^2}\right]P_L$$

$$= i\frac{\not{p}}{p^2}\frac{a-1}{a^2}P_L\left[-\frac{m^2}{p^2}-\frac{1}{2}\frac{m^4}{p^4}+\mathcal{O}\left(\frac{m^6}{p^6}\right)\right]. \tag{B.17}$$

The calculation for the second order in $\frac{a-1}{a^2}$ is more tedious. The integrals to be calculated are

$$\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{\not{p}-\not{k}_1-\not{k}_2}{k_1^2(k_1^2-m^2)k_2^2(k_2^2-m^2)(p-k_1-k_2)^2},$$

$$\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\left(\delta(k_1)\int\frac{d^2q}{(2\pi)^2}\frac{1}{q^2(q^2-m^2)}\right)\frac{\not{p}-\not{k}_1-\not{k}_2}{k_2^2(k_2^2-m^2)(p-k_1-k_2)^2}, \tag{B.18}$$

$$\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\delta(k_1)\delta(k_2)\left(\int\frac{d^2q}{(2\pi)^2}\frac{1}{q^2(q^2-m^2)}\right)^2\frac{\not{p}-\not{k}_1-\not{k}_2}{(p-k_1-k_2)^2}.$$

Their calculation is straightforward and the result is given in (7.13).

Now we look at the perturbation theory: I write down the integrals for the calculation of the integrals refering to diagrams 7.2d and 7.2e. They are presented once more in Fig. B.1:

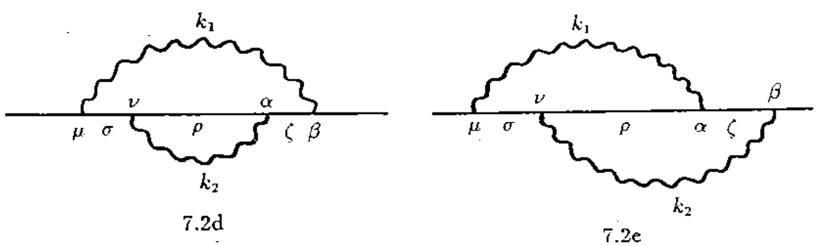

FIGURE B.1: Two diagrams from Fig.7.2 in more detail.

For diagram 7.2d one can exploit the result for 7.2b; with $\bar{p} \equiv p - k_1$, the inner k_2-integral reads

$$I_{3d}(k_1) = \int\frac{d^dk_2}{(2\pi)^2}\frac{(\bar{p}-k_2)_\rho\left(g_{\nu\alpha}-\frac{k_{2\nu}k_{2\alpha}}{k_2^2}\right)}{(k_2^2-m^2)(\bar{p}-k_2)^2}\gamma^\nu\gamma^\rho\gamma^\alpha = \int\frac{d^dk_2}{(2\pi)^2}\frac{\not{p}N(k_2)}{(k_2^2-m^2)(\bar{p}-k_2)^2}. \tag{B.19}$$

The nominator is reformulated, to avoid integrals with three k_2:

$$\not{p}N(k_2) = \frac{\not{p}}{\bar{p}^2}\left[(3-d)\bar{p}^2+\epsilon(\bar{p}\cdot k_2)+\frac{1}{k_2^2}(\bar{p}\cdot k_2)(\bar{p}-k_2)^2-\frac{\bar{p}^2}{k_2^2}(\bar{p}\cdot k_2)\right]. \tag{B.20}$$

The k_2-integration is performed and the result is expanded in $\frac{m^2}{\bar{p}^2}$. The nominator for the remaining k_1 interation is now

$$\not{p}N(k_1) = \gamma_\nu\gamma_\sigma(\not{p}-\not{k}_1)\gamma_\zeta\gamma_\mu(p-k_1)_\sigma(p-k_1)_\zeta$$
$$+\frac{1}{k_1^2}\left[-\not{k}_1\not{p}(\not{p}-\not{k}_1)\not{p}\not{k}_1+\not{k}_1\not{p}(\not{p}-\not{k}_1)k_1^2+k_1^2(\not{p}-\not{k}_1)\not{p}\not{k}_1-k_1^2(\not{p}-\not{k}_1)k_1^2\right]. \tag{B.21}$$

This is reformulated into

$$\not{p}N(k_1) = \not{p}\left[\tfrac{1}{2}(3-d)(p-k_1)^2+\tfrac{1}{2}\epsilon\tfrac{k_1^2}{p^2}(p-k_1)^2+\tfrac{1}{2}(3-d)(p-k_1)^4\right.$$
$$\left.-\tfrac{1}{2}\tfrac{p^2}{k_1^2}(p-k_1)^2+\tfrac{1}{k_1^2}(p-k_1)^4-\tfrac{1}{2}\tfrac{1}{p^2}k_1^2(p-k_1)^6\right]. \tag{B.22}$$

The denominators of the k_1 integrals to be performed are $(k_1^2-m^2)[(p-k_1)^2]^\alpha$, with $\alpha = 3$, 4 and $5-\frac{d}{2}$. All Γ-functions are expanded in ϵ, and the hypergeometric functions etc. in $\frac{m^2}{p^2}$. The multiplication of the numerous polynomials has been done with *Reduce* [62].

For diagram 7.2e the calculation is far more involved. The integral to be computed is

$$I_{3e} = \int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{(p-k_1)_\sigma(p-k_1-k_2)_\rho(p-k_2)_\zeta\left(g_{\mu\alpha}-\frac{k_{1\mu}k_{1\alpha}}{k_1^2}\right)\left(g_{\nu\beta}-\frac{k_{2\nu}k_{2\beta}}{k_2^2}\right)}{k_1^2(k_1^2-m^2)k_2^2(k_2^2-m^2)(p-k_1)^2(p-k_2)^2(p-k_1-k_2)^2}$$
$$\times\gamma_\mu\gamma_\sigma\gamma_\nu\gamma_\rho\gamma_\alpha\gamma_\zeta\gamma_\beta. \tag{B.23}$$

The reformulation of the nominator has again been done with *Reduce*. One gets 25 different terms of the form $\not{p}[k_1^2]^a[k_2^2]^b[(p-k_1)^2]^c[(p-k_2)^2]^d[(p-k_1-k_2)^2]^e$. It is decisive that there is no terms $\not{p}p^2$, $\not{p}k_1^2$, $\not{p}k_2^2$ or $\not{p}p^{-2}k_1^2k_2^2$. This would have led to integrals which cannot be evaluated by standard methods. Special care is necessary for integrations without a mass. In fact, the integral

$$\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{1}{(k_1^2-m^2)(p-k_2)^2(p-k_1-k_2)^2} \tag{B.24}$$

gives zero if the k_1-integral is done first, and a nonzero result if k_2 is first integrated. If the nominator leading to (B.24) is rewritten, $k_1^2k_2^4(p-k_1)^2 = k_1^2k_2^2(p-k_1)^2[(p-k_2)^2-p^2+2p\cdot k_2]$, one has to evaluate two integrals which contain a mass, and the result is unambigously zero. This gives a prescription to handle the case where one integration does not contain a mass. For the integral

$$\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{1}{(p-k_1)^2(p-k_2)^2(p-k_1-k_2)^2}, \tag{B.25}$$

however, only an ad-hoc procedure is successful: at first, the integration over k_1 is done; then the result is multiplied by $\frac{k_2^2-m^2}{k_2^2-m^2}$ and the k_2 integral done. The order $(m^2)^0$ result is different from the result one would have got without this trick, and it would lead to a wrong contribution to diagram 7.2e. It is not obvious how this trick can be replaced by a correct renormalization prescription. However, for the investigation done in this work it is enough to note that an IR-problem exists also in the perturbative derivation of the fermion propagator.

The other (double-) integrals can be evaluated straighforwardly. All results are expanded in ϵ and $\frac{m^2}{p^2}$, and the multiplication of the polynomials has been done with *Reduce*.

THE FERMION PROPAGATOR IN THE ANOMALOUS FORMULATION

In this section some details are presented relevant for the perturbative derivation of the AF fermion propagator. For the Landau gauge fermion propagator I had to confine myself to some technical points relevant for the derivation, since explicit formulas would have covered several pages. Here, however, it is very well possible to be explicit, at least in the limit $m^2 = 0$.

The integrals represented by the diagrams in Fig.7.2 are

$$I_b = \int \frac{d^dk}{(2\pi)^d} \frac{\not{k}(\not{p}-\not{k})\not{k}}{k^2(p-k)^2}$$

$$I_c = i\frac{\not{p}}{p^2}(I_b)^2 \qquad \text{(B.26)}$$

$$I_d = \int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d} \frac{\not{k}(\not{p}-\not{k}_1)\not{k}_2(\not{p}-\not{k}_1-\not{k}_2)\not{k}_2(\not{p}-\not{k}_1)\not{k}_1}{k_1^2k_2^2(p-k_1)^4(p-k_1-k_2)^2}$$

$$I_e = \int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d} \frac{\not{k}_1(\not{p}-\not{k}_1)\not{k}_2(\not{p}-\not{k}_1-\not{k}_2)\not{k}_1(\not{p}-\not{k}_2)\not{k}_2}{k_1^2k_2^2(p-k_1)^2(p-k_2)^2(p-k_1-k_2)^2}$$

Together with the anomalous dimension $\frac{1}{a-1}$, an "i" for every propagator and every vertex and $(i\frac{\not{p}}{p^2})^2$ for the external legs one has

$$D_b = \frac{1}{a-1}\left(-\frac{i}{4\pi}\right)^{\frac{d}{2}} \frac{\not{p}}{p^2} P_L(-p^2)^{\frac{\epsilon}{2}} \frac{1}{2}\frac{\Gamma^2(\frac{d}{2}-1)\Gamma(2-\frac{d}{2})}{\Gamma(d-2)},$$

$$D_d = \frac{1}{(a-1)^2}\left(-\frac{i}{4\pi}\right)^{d}(-i)\frac{\not{p}}{p^2}P_L(-p^2)^{\epsilon}\frac{1}{6}\frac{\Gamma^3(\frac{d}{2}-1)\Gamma(3-d)}{\Gamma(\frac{3}{2}d-3)}. \qquad \text{(B.27)}$$

$$D_e = \frac{1}{(a-1)^2}\left(-\frac{i}{4\pi}\right)^{d}(-i)\frac{\not{p}}{p^2}P_L(-p^2)^{\epsilon}\left(-\frac{1}{4}\right)\left(\frac{\Gamma^2(\frac{d}{2}-1)\Gamma(2-\frac{d}{2})}{\Gamma(d-2)}\right)^2.$$

This leads to eq.(7.16) with $m^2 = 0$.

The diagrams originating from the counter term in (7.21) can be directly deduced from the ones given above.

THE FERMION PROPAGATOR OF THE CHIRAL QUANTUM GRAVITY

The integrals to be calculated have complicated nominators, due to the vertices in the diagrams of Fig.10.2. However, they are brought into a manageable form by *Reduce*, and the integrals to be performed are simple. The experience with the CSM in the AF indicates that there is no need to bother about IR-problems. The integrals for the nonrenormalized propagator are:

$$I_b = \int \frac{d^dk}{(2\pi)^d} \frac{(2\not{p}-\not{k})(\not{p}-\not{k})(2\not{p}-\not{k})}{k^2(p-k)^2}$$

$$I_c = i\not{p}(I_b)^2$$

$$I_d = \int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d} \times \frac{(2\not{p}-\not{k}_1)(\not{p}-\not{k}_1)(2\not{p}-\not{k}_1-\not{k}_2)(\not{p}-\not{k}_1-\not{k}_2)(2\not{p}-\not{k}_1-2\not{k}_2)(\not{p}-\not{k}_2)(2\not{p}-\not{k}_2)}{k_1^2k_2^2(p-k_1)^2(p-k_2)^2(p-k_1-k_2)^2}$$

$$I_e = \int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d} \times \frac{(2\not{p}-\not{k}_1)(\not{p}-\not{k}_1)(2\not{p}-2\not{k}_1-\not{k}_2)(\not{p}-\not{k}_1-\not{k}_2)(2\not{p}-2\not{k}_1-\not{k}_2)(\not{p}-\not{k}_1)(2\not{p}-\not{k}_1)}{k_1^2k_2^2(p-k_1)^4(p-k_1-k_2)^2} \qquad \text{(B.28)}$$

$$I_f = \int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d} \frac{(2\not{p}-\not{k}_1-\not{k}_2)(\not{p}-\not{k}_1-\not{k}_2)(2\not{p}-2\not{k}_1-\not{k}_2)(\not{p}-\not{k}_1)(2\not{p}-\not{k}_1)}{k_1^2k_2^2(p-k_1)^2(p-k_1-k_2)^2}$$

$$I_g = \int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d} \frac{(2\not{p}-\not{k}_1)(\not{p}-\not{k}_1)(2\not{p}-2\not{k}_1-\not{k}_2)(\not{p}-\not{k}_1-\not{k}_2)(2\not{p}-\not{k}_1-\not{k}_2)}{k_1^2k_2^2(p-k_2)^2(p-k_1-k_2)^2}$$

$$I_h = \int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d} \frac{(2\not{p}-\not{k}_1-\not{k}_2)(\not{p}-\not{k}_1-\not{k}_2)(2\not{p}-\not{k}_1-\not{k}_2)}{k_1^2k_2^2(p-k_1-k_2)^2}$$

$$I_i = \int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d} \frac{(2\not{p}-\not{k}_1)(\not{p}-\not{k}_1)(2\not{p}-\not{k}_1-\not{k}_2)(\not{p}-\not{k}_2)(2\not{p}-\not{k}_2)}{k_1^2k_2^2(p-k_1)^2(p-k_2)^2}$$

Together with the anomalous dimension δ, an "i" for every propagator and every vertex and $(i\frac{\not{p}}{p^2})^2$ for the external legs one has

$$D_b = \delta\left(-\frac{i}{4\pi}\right)^{\frac{d}{2}} \frac{\not{p}}{p^2}P_L(-p^2)^{\frac{\epsilon}{2}}\frac{1}{2}\frac{\Gamma^2(\frac{d}{2}-1)\Gamma(2-\frac{d}{2})}{\Gamma(d-2)}.$$

$$D_c = \delta^2\left(-\frac{i}{4\pi}\right)^{d}(-i)\frac{\not{p}}{p^2}P_L(-p^2)^{\epsilon}\left(\frac{1}{2}\frac{\Gamma^2(\frac{d}{2}-1)\Gamma(2-\frac{d}{2})}{\Gamma(d-2)}\right)^2.$$

$$D_d = \delta^2\left(-\frac{i}{4\pi}\right)^d(-i)\frac{\not p}{p^2}P_L(-p^2)^\epsilon\left[\frac{2}{3}\frac{\Gamma^3(\frac{d}{2}-1)\Gamma(3-d)}{\Gamma(\frac{3}{2}d-3)}-\left(\frac{1}{2}\frac{\Gamma^2(\frac{d}{2}-1)\Gamma(2-\frac{d}{2})}{\Gamma(d-2)}\right)^2\right],$$

$$D_e = \delta^2\left(-\frac{i}{4\pi}\right)^d(-i)\frac{\not p}{p^2}P_L(-p^2)^\epsilon\frac{5}{6}\frac{\Gamma^3(\frac{d}{2}-1)\Gamma(3-d)}{\Gamma(\frac{3}{2}d-3)},$$

$$D_f = \delta^2\left(-\frac{i}{4\pi}\right)^d(+i)\frac{\not p}{p^2}P_L(-p^2)^\epsilon\frac{1}{2}\frac{\Gamma^3(\frac{d}{2}-1)\Gamma(3-d)}{\Gamma(\frac{3}{2}d-3)},$$

$$D_g = I_f.$$

$$D_h = \delta^2\left(-\frac{i}{4\pi}\right)^d(-i)\frac{\not p}{p^2}P_L(-p^2)^\epsilon\frac{1}{3}\frac{\Gamma^3(\frac{d}{2}-1)\Gamma(3-d)}{\Gamma(\frac{3}{2}d-3)},$$

$$D_i = 0. \tag{B.29}$$

If these contributions are added one ends up with the nonrenormalized fermion propagator (10.15).

C. The Algebra of Generators of Surface Deformations with Jackiw's Gravity

The analysis done in sec.9 is repeated for the chiral quantum gravity of refs.[50,59]. The demand for a vanishing central charge of the algebra of generators of surface deformations leads to a constraint on the number of chiral fermions allowed to couple.

The first approach towards a CQG [50,59] relied on Jackiw's gravitational action [51], which was derived by dimensional reduction from (2+1) dimensions:

$$W_G = -\frac{1}{16\pi G}\int d^2x\sqrt{-g}N(R+2\Lambda). \tag{C.1}$$

Here a auxiliary scalar N is introduced. The most important effect is that the cosmological term cannot be dropped as I did in sec.8-10. Such a measure would change the models consistency structure. Apart from (C.1), however, one nevertheless has to take into account the Liouville action (with $\mu = 0$). The combined system $W = W_{loc} + W_G$ reads

$$\begin{aligned} W = \int dx\Big\{ & \eta_0^{-1}(\dot\phi - 2\eta_1' - \phi'\eta_1)(\dot{\hat N} - \eta_1\hat N') - (2\eta_0' + \phi'\eta_0)\hat N' + 2\eta_0\hat N\Lambda e^\phi \\ & +\frac{1}{2}\eta_0^{-1}\left[(\dot\varphi - \varphi'\eta_1)^2 - {}'^2\eta_0^2\right] \\ & +b\eta_0^{-1}\left[-(\dot\varphi - \eta_1\varphi')(\omega_0 - \eta_1\omega_1) + \eta_0^2\omega_1\varphi'\right] \\ & +\frac{1}{2}a\eta_0^{-1}\left[\omega_0 - \omega_1\eta_1)^2 - \eta_0^2\omega_1^2\right]\Big\} \end{aligned} \tag{C.2}$$

where we defined $\hat N \equiv N - c\varphi$, and the factor $\frac{1}{16\pi G}$ has been absorbed in N. With the canonical momenta of the variables φ, $\hat N$, ϕ and F, this may be recast into

$$I = \int dx\left\{\dot{\hat N}P_{\hat N} + \dot\phi P_\phi + \dot\varphi P_\varphi + \dot F P_F - (\eta_0 H + \eta_1 T)\right\}, \tag{C.3}$$

where

$$\begin{aligned} H = & -b\varphi'F' - 2\hat N'' + \hat N'\phi' + (P_\phi - b\varphi')P_{\hat N} - 2(\hat N + c\varphi)\Lambda e^\phi \\ & -P_F\phi' + 2P_F' + \frac{1}{2}P_\varphi^2 + \frac{1}{2}\varphi'^2 + \frac{1}{2(a-b^2)}(P_F + bP_\varphi)^2 \\ & +\frac{1}{2}a(P_{\hat N} + F')^2, \end{aligned} \tag{C.4}$$

$$T = P_\phi\phi' - 2P_\phi' + P_{\hat N}\hat N' + P_F F' + P_\varphi\varphi'.$$

H and T are generators of surface deformations and satisfy closed algebras classically. Just as in sec.9 H and T are transformed canonically into a form where afterwards the singularities connected to the quantum nature of the fields can easily be dealt with. By means of the generating functional

$$W = (\tilde{P}_\varphi - \sqrt{a}\tilde{P}_{\dot{N}})\varphi + (\tilde{P}_F - b\tilde{P}_\phi - \sqrt{a}\,b\tilde{P}_{\dot{N}})F + (\tilde{P}_\phi - \sqrt{a}\tilde{P}_\varphi - a\tilde{P}_{\dot{N}} - aF' + b\varphi')\phi + \tilde{P}_{\dot{N}}\dot{N} \tag{C.5}$$

and the rescaling

$$\tilde{F} \to \sqrt{a - b^2}^{\,-1}\tilde{F}, \quad \tilde{P}_F \to \sqrt{a - b^2}\tilde{P}_F, \tag{C.6}$$

we arrive at (omitting all tildes)

$$\begin{aligned} H_\pm = \; & \frac{1}{2}(P_F \pm F')^2 + \frac{1}{2}(P_\varphi \pm \varphi')^2 \\ & \pm 2\sqrt{a - b^2}\,(P_F \pm F')' + 2(-b \pm \sqrt{a})(P_\varphi \pm \varphi')' \\ & +(P_\phi \pm \dot{N}')(P_{\dot{N}} \pm \phi') \mp 2(P_\phi \pm \dot{N}')' \\ & \pm 2\sqrt{a}(\sqrt{a} \mp b)(\pm P_N - \phi')' \\ & -2\left[\dot{N} - (c - \sqrt{a})\varphi - c\sqrt{a}\,\phi + \frac{cb}{\sqrt{a - b^2}}F\right]\Lambda e^{\phi}. \end{aligned} \tag{C.7}$$

The relation $a - b^2 \geq 0$ is required for the rescaling of F and P_F in order to ensure that the kinetic term of F has the correct sign.

Defining

$$\sigma_\pm = \frac{1}{\sqrt{2}}(\phi \pm \dot{N}), \quad \pi_\pm = \frac{1}{\sqrt{2}}(P_\phi \pm P_{\dot{N}}), \tag{C.8}$$

we get the quantum expression

$$\begin{aligned} H_\pm = \; & \frac{1}{2} :(P_F \pm F')^2: - \frac{1}{2} :(P_\varphi \pm \varphi')^2: + \frac{1}{\beta_F}(P_F \pm F')' + \frac{1}{\beta_\varphi}(P_\varphi \pm \varphi')' \\ & + \frac{1}{2} :(\pi_\pm \pm \sigma'_\pm)^2: - \frac{1}{2} :(\pi_\mp \mp \sigma'_\mp)^2: + \frac{1}{\beta_L^\mp}(\pi_\pm \pm \sigma'_\pm)' + \frac{1}{\beta_M^\mp}(\pi_\mp \mp \sigma'_\mp)' \\ & - 2 : \left[\frac{1}{\sqrt{2}}(c_+\sigma_+ - c_-\sigma_-) + (c - \sqrt{a})\varphi + \frac{cb}{\sqrt{a - b^2}}F\right]\Lambda e^{\sqrt{2}(p_+\sigma_+ + p_-\sigma_-)} : \end{aligned} \tag{C.9}$$

with

$$\begin{aligned} \frac{1}{\beta_L^\mp} = \; & \pm\sqrt{2}\left[1 + \sqrt{a}(\pm b - \sqrt{a})\right], \\ \frac{1}{\beta_M^\mp} = \; & \pm\sqrt{2}\left[1 - \sqrt{a}(\pm b - \sqrt{a})\right], \\ \frac{1}{\beta_\varphi} = \; & 2(-b \pm \sqrt{a}) + \gamma_\varphi, \\ \frac{1}{\beta_F} = \; & 2\sqrt{a - b^2} + \gamma_F, \\ c_\pm = \; & 1 \pm c\sqrt{a}, \\ p_\pm : \; & \text{arbitrary parameters}. \end{aligned} \tag{C.10}$$

γ_φ and γ_F were introduced to account for quantum ambiguities. For $\sigma_\pm$ this ambiguity is just given by the second terms in the expressions for $\beta_{L,M}^{-1}$.

In contrast to sec.9, this time the demand for a closed algebra of the generators of surface deformations (9.14) gives the constraint $a = 0$. The fate of the conformal variable ϕ is obscured in the present case, since it is mixed with the auxiliary field N. However, for F we get the demand $-b^2 \geq 0$. This leads to

$$n_R \leq 23, \quad n_L \leq 23. \tag{C.11}$$

The difference to the constraint (9.16) results from the Weyl anomaly of the N-field, which was not present in sec.9.

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