DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY

DESY 90-097
August 1990

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ISSN 0418-9833

NOTKESTRASSE 85 · 2 HAMBURG 52

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Abstract

The length spectrum of closed geodesics on a compact Riemann surface corresponding to a regular octagon on the Poincaré disc is investigated. The general form of the elements of the "octagon group", a discrete subgroup of $SU(1,1)/\{\pm\mathbf{1}\}$, in terms of 2×2 matrices is derived, and the previously conjectured law (Physica **D32**(1988) 451) for the length of periodic orbits is proved analytically. An algorithm for the multiplicity of geodesics with a given length is developed, which leads to an efficient enumeration of the periodic orbits of this strongly chaotic system.

Submitted to Physica D

[1] Supported by Deutsche Forschungsgemeinschaft under Contract No. DFG-Ste 241/4-2
[2] On sabbatical leave from II.Institut für Theoretische Physik, Universität Hamburg
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I Introduction

The free motion on a compact two-dimensional surface of constant negative curvature is one of the simplest and best investigated ergodic models of classical mechanics (see e. g. ref. [1] and references therein). In ref. [2] some properties of periodic trajectories were investigated for one of such surfaces which corresponds to a regular octagon on the Poincaré disc with opposite sides being identified. (For a discussion of asymmetric octagons corresponding to different compact Riemann surfaces of genus 2, see ref. [3].)

It is known [1,2] that for such a system the periodic orbits are in one-to-one correspondence with the distinct hyperbolic conjugacy classes of fundamental group matrices. For the problem considered, i. e. the "octagon group" G, a discrete subgroup of $\mathrm{SU}(1,1)/\{\pm\mathbf{1}\}$, the latter can be represented as products of an arbitrary number of the following 4 generators ($k = 0,1,2,3$) and their inverses ($k = 4,5,6,7$) [1,2]:

$$b_k = \begin{pmatrix} 1+\sqrt{2} & e^{i\frac{k\pi}{4}}\sqrt{2}\sqrt{\sqrt{2}+1} \\ e^{-i\frac{k\pi}{4}}\sqrt{2}\sqrt{\sqrt{2}+1} & 1+\sqrt{2} \end{pmatrix} \quad . \tag{1}$$

However, if one would generate all group elements M successively, one would count the orbits repeatedly, since all conjugated matrices

$$M' = S\, M\, S^{-1} \quad , \tag{2}$$

where S is an arbitrary fundamental group matrix, correspond to the same orbit. Therefore, to enumerate the periodic orbits, it is necessary to know which fundamental group matrices are conjugated to each other, i. e. one has to generate the conjugacy classes $[M] = \{M' | M' = SMS^{-1}, S \in G\}$ instead of the group elements itself. In addition, there is the basic relation

$$b_0 b_1^{-1} b_2 b_3^{-1} b_0^{-1} b_1 b_2^{-1} b_3 = \mathbf{1} \tag{3}$$

which has to be used in its various equivalent forms in the process of enumerating the conjugacy classes.

In ref. [2] all products up to 11 generators were found and, using a particular algorithm for separating the conjugacy classes, the length spectrum of 206 796 242 primitive periodic orbits was calculated. The numerical results strongly suggested that there exists an exact formula for the lengths of primitive periodic orbits, and thus two of us were led to the following *conjecture* [2]:

$$\cosh\frac{l_n}{2} = m + n\sqrt{2}\,, \tag{4}$$

where l_n is the length of a periodic orbit with n being a natural number ($0 < l_1 < l_2 < \ldots$) and $m = m(n)$ an odd natural number, which is uniquely defined by the condition that the modulus of the difference

$$\Delta := |m(n) - n\sqrt{2}| \tag{5}$$

has a minimum value at given n.

The existence of such arithmetic relations in terms of algebraic numbers was not expected before for this ergodic system. In particular, from these relations it could be concluded [2] that the *mean multiplicity* $\widehat{g}(l)$ of periodic orbits for a given length l is unexpectedly large, i. e. $\widehat{g}(l) \sim 8\sqrt{2}e^{l/2}/l$, $l \to \infty$.

In section II of this note we shall study the fundamental group matrices for the regular octagon and shall find their general form which characterizes them in an explicit way as 2x2 matrices. From this we shall prove the conjectured law (4), (5) analytically.

In section III we shall derive a condition on the matrix elements which ensures that the invariant geodesic corresponding to a given matrix goes through the fundamental region.

In section IV we develop an algorithm for the calculation of the multiplicity of periodic orbits with a given length. The usual method of constructing the fundamental group matrices as products of a finite

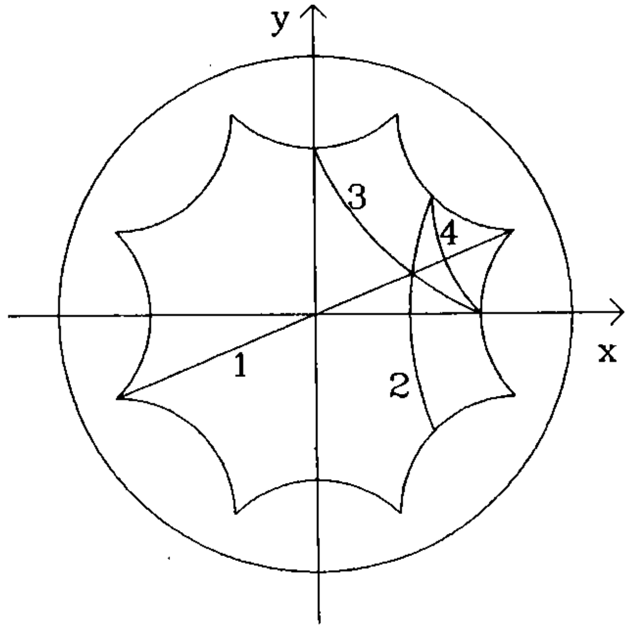

Figure 1: The regular octagon on the Poincaré disc. Numbers denote circles of inversion associated with the symmetries $T_1, \ldots, T_4$ discussed in the Appendix.

number of generators (which was used in ref. [2]) suffers from the drawback that products consisting of a large number of generators can give a periodic trajectory with a small period. This means that the multiplicity of periodic trajectories (even for small lengths) obtained by such a method, in general, will be underestimated due to the contribution from products with a larger number of generators. This fact restricts the applicability of such calculations, especially for checking the Selberg trace formula (periodic-orbit theory) for the system considered [4]. The method proposed in section IV permits us to find the exact multiplicity of periodic trajectories with a given length independent of the number of generators taken into account.

This paper is an extended and improved version of ref. [5].

II General form of fundamental group matrices for the regular octagon

An arbitrary fundamental group matrix of the "octagon group" corresponding to the regular octagon shown in fig.1 can be written as [2]:

$$M = \begin{pmatrix} A_1 + iA_2 & \sqrt{\sqrt{2}-1}(B_1 + iB_2) \\ \sqrt{\sqrt{2}-1}(B_1 - iB_2) & A_1 - iA_2 \end{pmatrix}, \tag{6}$$

where A_1, A_2, B_1, B_2 are special algebraic numbers of the form

$$m + n\sqrt{2} \tag{7}$$

with integers m, n, and $|\,\mathrm{Tr}\, M| = 2|A_1| > 2$.

(Note that we choose in the off-diagonal elements the factor $\sqrt{\sqrt{2}-1}$ instead of $\sqrt{\sqrt{2}+1}$ as in [2]. The reason for it will become clear below.)

The obvious property of (6) being an element of $G \subset \mathrm{SU}(1,1)$ is that its determinant must be equal to 1:

$$A_1^2 + A_2^2 - (\sqrt{2}-1)(B_1^2 + B_2^2) = 1\,. \tag{8}$$

This condition seems to be trivial, but we shall see in a moment that it is a key relation giving us a lot of information about the matrix elements.

First of all, we introduce a few definitions. Let us call the algebraic numbers of form (7) even or odd depending on whether the parity of m is even or odd, respectively. (Here the parity of an algebraic number $m + n\sqrt{2}$ is defined by $p(m+n\sqrt{2}) \equiv m(\mathrm{mod}2)$.) It is easy to show that A_1 must be odd, A_2 even, and B_1, B_2 must have the same parity (both even or odd). Among the algebraic numbers (7) which are defined by two independent integers m and n we shall be interested in particular subsets of these numbers for which $m = m(n)$ is uniquely connected with n by the requirement that the quantity

$$\Delta = |m - n\sqrt{2}| \tag{9}$$

acquires its minimum value for fixed n and for a given parity of m. We shall call the numbers with this property minimal numbers. There are two types of minimal numbers: even and odd depending on whether m is allowed to be even or odd in the minimization of (9). The necessary and sufficient condition that an algebraic number $C = m + n\sqrt{2}$ $(n \neq 0)$ belongs to the set of minimal numbers can be expressed in form of the inequality

$$|\tilde{C}| := |m - n\sqrt{2}| < 1 \; . \tag{10}$$

In table 1 we present the first 40 minimal numbers for the case $n > 0$. Minimal numbers have the interesting property that each class of minimal numbers is closed under multiplication. This means that if one multiplies two arbitrary minimal numbers with the same parity, the result will be again a minimal number with the same parity.

n	1	2	3	4	5	6	7	8	9	10
m even	2	2	4	6	8	8	10	12	12	14
m odd	1	3	5	5	7	9	9	11	13	15
n	11	12	13	14	15	16	17	18	19	20
m even	16	16	18	20	22	22	24	26	26	28
m odd	15	17	19	19	21	23	25	25	27	29

Table 1: First positive minimal numbers $m + n2^{1/2}$.

Let us consider condition (8) in detail. It is an algebraic relation for the numbers (7). It is clear that it will remain true if one changes the sign of $\sqrt{2}$ in all terms. This implies that if A_1, A_2, B_1, B_2 of form (7) obey (8), then their algebraic conjugates $\tilde{A}_1$ etc. will obey the following relation:

$$\tilde{A}_1^2 + \tilde{A}_2^2 + (\sqrt{2}+1)(\tilde{B}_1^2 + \tilde{B}_2^2) = 1 \; , \tag{11}$$

where $\tilde{A}_1 := m_1 - n_1\sqrt{2}$ etc.. But all terms in (11) are positive numbers, and therefore they are restricted by the following values

$$|\tilde{A}_i| < 1 \; , \; |\tilde{B}_i| < \sqrt{\sqrt{2}-1} < 1 \; , i = 1,2 \; . \tag{12}$$

These inequalities mean that all A_i and B_i belong to minimal numbers.

Taking into account the above-mentioned parity properties, we conclude that all fundamental group matrices for the regular octagon must have form (6) where:

$$\begin{array}{ll} A_1 & \text{is an odd minimal number} \\ A_2 & \text{is an even minimal number} \\ B_1 & \text{and } B_2 \text{ are minimal numbers of the same parity.} \end{array} \tag{13}$$

The length l of a periodic orbit corresponding to a fundamental group matrix M can be calculated from the relation [2]:

$$\cosh\frac{l}{2} = \frac{1}{2}|\mathrm{Tr}M| = |A_1| \,. \tag{14}$$

Combining this with (13), one obtains formulae (4), (5) which were the mathematical expression for the conjecture proposed previously in ref. [2].

Now we prove the reverse statement, i. e. that any matrix of form (6) with unit determinant and with A_i, B_i obeying (13) belongs to the fundamental group of the regular octagon. Our proof will be based on the theorem proven in ref. [6] and cited in ref. [1].

According to this theorem the group of *all* matrices of form (6) with unit determinant differs from the considered "octagon group" by the existence of an additional generator

$$R_\pi = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{15}$$

with the properties

$$R_\pi b_k R_\pi^{-1} = b_{k+4} = b_k^{-1} \ , \ R_\pi^2 = -1 \quad , \tag{16}$$

i. e. an arbitrary matrix (6) can be represented as a word constructed from the generators b_k and the additional matrix R_π. But according to (16) matrices with an even number of R_π's can be reduced to fundamental group matrices (without any R_π), and matrices with an odd number of R_π's can be reduced to matrices with just one R_π. Therefore, an arbitrary matrix (6) with algebraic elements A_1, A_2, B_1, B_2 belongs either to a fundamental group matrix or to a product of a fundamental group matrix with one R_π. It is not difficult to find a criterion which distinguishes these two cases. As was indicated above, A_1 must be an odd algebraic number for any fundamental group matrix and A_2 must be an even one. The application of R_π to a matrix (6) results in the following substitution: $A_1 \to -A_2$, $A_2 \to A_1$ and $B_1 \to -B_2$, $B_2 \to B_1$. Hence, for a product of R_π and a fundamental group matrix the A_1 element will be an even algebraic number and A_2 will be an odd one. (B_1 and B_2 will be, as before, numbers of the same parity). This means that the parity of A_1 uniquely discriminates between these two cases. If A_1 is an odd number, the matrix (6) belongs to the fundamental group, and if A_1 is an even number, the matrix (6) is a product of R_π and a fundamental group matrix.

Thus we have proved the following Theorem: The necessary and sufficient condition that a matrix (6) with unit determinant belongs to the fundamental group of the regular octagon is that the A_1 element is an odd minimal number and all other elements obey (13).

From this theorem it follows that to construct a fundamental group matrix it is enough to sort out minimal algebraic numbers obeying conditions (13) and select from them those obeying (8). Let us emphasize that the minimality condition (10), which in the end is a simple consequence of the unit determinant condition (8), is of very importance. As it will be shown below, due to this condition it will be enough to sort out only a finite number of minimal numbers in order to find all periodic trajectories with a given length. Let

$$\begin{aligned} A_1 &= m_1 + n_1\sqrt{2}, \quad & A_2 &= m_2 + n_2\sqrt{2}, \\ B_1 &= p_1 + q_1\sqrt{2}, \quad & B_2 &= p_2 + q_2\sqrt{2} \end{aligned} \tag{17}$$

be the representation of the matrix elements in terms of integers. As was noted above, m_1 is an odd integer, m_2 is an even integer, whereas p_1 and p_2 can be either even or odd, but must have the same parity.

We present here a few more parity properties. From eq. (8) one can deduce the following:

1) if n_1 is even, then n_2, p_1 and p_2 are even and q_1 and q_2 are of the same parity;

2) if n_1 is odd and p_1, p_2 are even, then n_2 is even and q_1 and q_2 are of different parity;

3) if n_1 is odd and p_1, p_2 are odd, then n_2 is even or odd depending on whether q_1 and q_2 have the same or opposite parity.

III Conjugacy classes and their associated periodic orbits

To enumerate the periodic orbits it is necessary to know which fundamental group matrices are conjugated to each other. (See the remarks after eq.(1).) If the matrices are given as products of fundamental group generators, then a pure algebraic algorithm exists [2], which solves this problem within a finite number of steps.

In the approach developed in this paper, we can construct any fundamental group matrix directly, but, a priori, we do not know its representation as a generator product, and the question of the separation of conjugacy classes has to be considered in detail.

Let us recall a few general facts [1]. Any geodesic on the Poincaré disc is a circle which is perpendicular to the boundary circle $|z| = 1$. Inside the fundamental region a closed geodesic (i.e. a periodic trajectory) consists of a set of segments of such circles connected with each other by the identification of the boundary arcs via the generators (1). An arbitrary fundamental group matrix of the form

$$M = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \tag{18}$$

with unit determinant defines the linear fractional transformation ($z = x + iy$)

$$z' = M(z) = \frac{\alpha z + \beta}{\beta^* z + \alpha^*} , \tag{19}$$

which leaves the circle $|z| = 1$ invariant.

Simultaneously, a matrix (18) defines a unique geodesic on the Poincaré disc which is not changed by the transformation (19). In Cartesian coordinates this invariant geodesic is given by the equation

$$x^2 + y^2 - \frac{2}{\alpha_2}(\beta_1 y - \beta_2 x) + 1 = 0 , \tag{20}$$

where α_1, α_2, β_1, β_2 are real and imaginary parts, respectively, of α and β:

$$\alpha = \alpha_1 + i\alpha_2 \ , \ \beta = \beta_1 + i\beta_2 \ . \tag{21}$$

If $\alpha_2 = 0$, then the invariant geodesic is the straight line

$$\beta_1 y = \beta_2 x \ . \tag{22}$$

It is not difficult to show that the conjugated matrix (2) is in geometrical terms the result of the translation of the circle (20) (corresponding to the matrix M) under a transformation of type (19) defined by the matrix S.

Let us assume that we know a fundamental group matrix and we want to construct the corresponding circle (20) on the Poincaré disc. Two variants are possible. Either the circle (20) goes through the fundamental domain or it entirely lies outside of it. Only the first case corresponds to an arc of a periodic trajectory of the free motion on the surface considered. The second case has to be considered as the result of a transformation of a geodesic under the action of a fundamental group matrix. This means that we have not to consider matrices for which the invariant circle (20) lies outside the fundamental region.

The necessary and sufficient condition that the circle (20) goes through the fundamental octagon is that the distance between the centre of the circle and a certain corner of the octagon is smaller or equal to the radius of (20). If the matrix M is written in the form (6), this condition is equivalent to the following inequality

$$|A_2| \leq (2 - \sqrt{2})(|B_1| + (\sqrt{2} - 1)|B_2|) , \tag{23}$$

where we assume that $|B_1| \geq |B_2|$ (this can always be achieved by rotations over $\pi/4$, see the Appendix).

Using eq. (8) one obtains the following condition on B_1, B_2 (assuming $B_1 \geq B_2 \geq 0$)

$$B_1^2 + 5B_2^2 - 4B_1B_2 \leq (1+\sqrt{2})^3(A_1^2 - 1)\,. \tag{24}$$

Therefore, if the length of the geodesic is fixed (i.e. A_1 is fixed, see eq.(14)), there exists only a finite number of matrices (6) which we have to consider. These and only these matrices correspond to invariant geodesics (20) which go through the fundamental region.

IV Determination of the multiplicities of the length spectrum

In this section, we describe our algorithm which determines the multiplicities g'_n of the length spectrum $\{l_n\}$ of the periodic orbits using only matrices obeying the properties (13) and, in addition, (24). (Here g'_n denotes the multiplicity which counts also the non-primitive periodic orbits.)

First the number n_1 is chosen, for which the multiplicity g'_{n_1} should be computed. This determines A_1 and therefore the r.h.s. of (24). In the next step, all pairs (B_1, B_2) with $B_1 \geq B_2 \geq 0$ are generated which obey (24) and are both odd in the first run and both even in the second. The modulus of A_2 is then computed by eq.(8). If this A_2 is an even number as required by (13), a valid group matrix is found which corresponds to a geodesic of length l_{n_1}.

By rotations over $\frac{\pi}{4}$ (see the Appendix), seven further group matrices can be found which also correspond to geodesics inside the octagon. In this way we could compute all geodesics with the same A_1 lying inside the octagon, but we would not know the number of the *periodic* orbits which are composed of the pieces of the former. In principle, one could compute the number of the periodic orbits from this set of matrices, if one could find out which matrices belong to a given periodic orbit. One has to select one matrix after the other of the set and then to conjugate the matrices which yield the matrices corresponding to the other geodesic pieces of the same periodic orbit. This group matrices have to be omitted in the set. The conjugation must be repeated as long as one arrives at the starting matrix. In this way, only group matrices belonging to different periodic orbits survive in the set and the multiplicity g'_{n_1} is determined. However, this method is uneconomical and we therefore use a more subtle algorithm.

First we concentrate on only those group matrices which obey $B_1 \geq B_2 \geq 0$. An array is used, in which all matrices corresponding to pieces of the periodic orbits are stored. At the program start, this array is empty, of course. Then the first valid group matrix is computed and stored in the array. Then this matrix $M^{(1)}$ is conjugated by the eight generators b_k, $k = 0, \ldots, 7$. Two of these new eight group matrices $b_k M^{(1)} b_k^{-1}$ correspond to geodesics which continue the original geodesic in the interior of the octagon along the periodic orbit. To find them, the eight matrices are rotated by an angle of $\frac{\pi}{4}$ until $B_1 \geq B_2 \geq 0$; then the two correct geodesics can be recognized by the inequality (24) which they have to fulfil. After the first conjugation one of these two matrices can be chosen arbitrarily as $M^{(2)}$ for the next conjugation step. However, after the n-th conjugation ($n > 1$) that one has to be chosen as $M^{(n+1)}$, which is different from the matrix $M^{(n-1)}$ obtained two steps earlier. Otherwise this would lead to an orbit segment which one has already obtained. As noted, for the application of (24), the matrix $M^{(n+1)}$ was rotated to obey $B_1 \geq B_2 \geq 0$. This rotated matrix $R^{(n+1)}$ is also stored in the array, the reason will be clear soon. The conjugation cycle is finished, when the new matrix $M^{(n+1)}$ is equal to the matrix $M^{(1)}$ at the starting-point. Then one has followed a whole traversal of the periodic orbit and one can turn to the next group matrix obeying (24). But before starting the conjugation cycle again, one has to ensure that the geodesic corresponding to this matrix is not a piece of a periodic orbit already computed. This can be checked, because all pieces considered already are stored in terms of their matrices $M^{(1)}$ and $R^{(n)}$ in the array. This is the reason why the rotated matrices $R^{(n)}$ have also to be stored.

After repeating this procedure for all allowed values (A_2, B_1, B_2), one gets distinct periodic orbits, all of which have the same length determined by A_1. However, one point was omitted until now. Because one considers only matrices $M^{(1)}$ obeying $B_1 \geq B_2 \geq 0$, one obtains only part of the periodic orbits. In general, a whole periodic orbit can be rotated over $\frac{\pi}{4}$, which yields a new periodic orbit. After eight of such rotations one arrives at the original periodic orbit. Thus in general, for each computed periodic orbit one rather has eight instead of one. However, there are exceptions to this rule. Sometimes a periodic orbit has a higher symmetry, such that one arrives already after 4, 2 or 1 rotations over $\frac{\pi}{4}$ at the original one (see fig.2 in [2]). Such a behaviour is betrayed by the sequence $R^{(n)}$. Take a conjugation cycle of length N. The condition $M^{(1)} = R^{(n+1)}$ is obeyed for a non-symmetric orbit only for $n = N$. However for a symmetrical one, one of the values $n = \frac{N}{2}, \frac{N}{4}, \frac{N}{8}$ is the lowest one obeying the condition. For $n = \frac{N}{8}$ one has only this single periodic orbit, for $n = \frac{N}{4}$ there are two and for $n = \frac{N}{2}$ four periodic orbits. In the case $n = N$, eight periodic orbits are to be taken into account as mentioned already. With this geodesic counting, one arrives at the correct multiplicities g_n'.

Finally, the multiplicities g_n corresponding to *primitive* periodic orbits are obtained by subtracting from g_n' the multiplicities of shorter orbits (if they exist) having length $l_n/k, k = 2, 3, \ldots$, which have also been taken into account in g_n'.

Using the described algorithm we have computed the complete length spectrum $\{l_n\}$ together with the associated multiplicities $\{g_n\}$ of the primitive periodic orbits of the regular octagon for $n = 1$ to 1500. Unfortunately, the computation time increases rapidly with n ($\sim n^2$), which sets a practical limit to the calculation of the longer orbits. For $n = 1500$ we have $A_1 = 2121 + 1500\sqrt{2}$, and our computed length spectrum covers the range from $l_1 = 3.057141839\ldots$ to $l_{1500} = 18.092025632\ldots$. There exists an independent check of our results carried out for $n = 1$ to 1099 by C. Matthies based on her own computer program. The two results agree exactly in their common range.

It turns out, that not all n–values correspond to primitive periodic orbits; indeed, for the n–values given by n =4, 24, 48, 72, 140, 160, 176, 184, 200, 224, 288, 432, 456, 472, 496, 704, 728, 816, 856, 880, 1024, 1088, 1112, 1128, 1136, 1152, 1360, 1384, 1400, 1424, 1488 there exist no primitive periodic orbits, which is equivalent to say that for these n–values we must set $g_n = 0$.

At this moment we are not aware of a closed formula which would reproduce the above n–values and, even more, would yield the n–values corresponding to the "forbidden" primitive periodic orbits having $g_n = 0$ for $n > 1500$.

Our results nicely confirm our previous computation [2] of the length spectrum. In table 1 of ref. [2] we presented all lengths up to $n = 82$ which were obtained by generating all fundamental group matrices constructed from products of up to 11 generators. We find that the length spectrum given in this table is almost complete; there are only two lengths for which the multiplicity is a bit too low, since products of 12 generators give a contribution. They correspond to $A_1 = 97 + 68\sqrt{2}$ and $A_1 = 97 + 69\sqrt{2}$ for which the multiplicities have to be equal to 48 and 576 instead of 40 and 560, respectively.

In fig.2 we show the staircase function

$$N(l) := \sum_{l_n \leq l} g_n \tag{25}$$

for $l \leq l_{1500}$, which counts the number of all primitive periodic orbits of length less than l. Noticing the logarithmic scale in fig.2 one immediately confirms the exponential proliferation of the number of periodic orbits with increasing length, a characteristic feature of chaotic systems. Notice, that there are more than 4.2 million primitive periodic orbits involved. In fig.2 we also show the asymptotic behaviour

$$N(l) \sim \mathrm{Ei}(l) := \int_{-\infty}^{l} dx \, \frac{e^x}{x} \quad , \quad l \to \infty \quad , \tag{26}$$

which is seen to describe the average behaviour of the true staircase function very well down to the shortest orbit. Fig.2 should be compared with the corresponding fig.3 of ref. [2], where the staircase function lies for $l > 14.5$ clearly below the prediction (26) since the conjugacy classes generated by products of more than 11 generators were not taken into account. Fig.3 shows the staircase in the small range $l = 17.9$ to $l = 18$ in order to allow a comparison with the asymptotic behaviour (26).

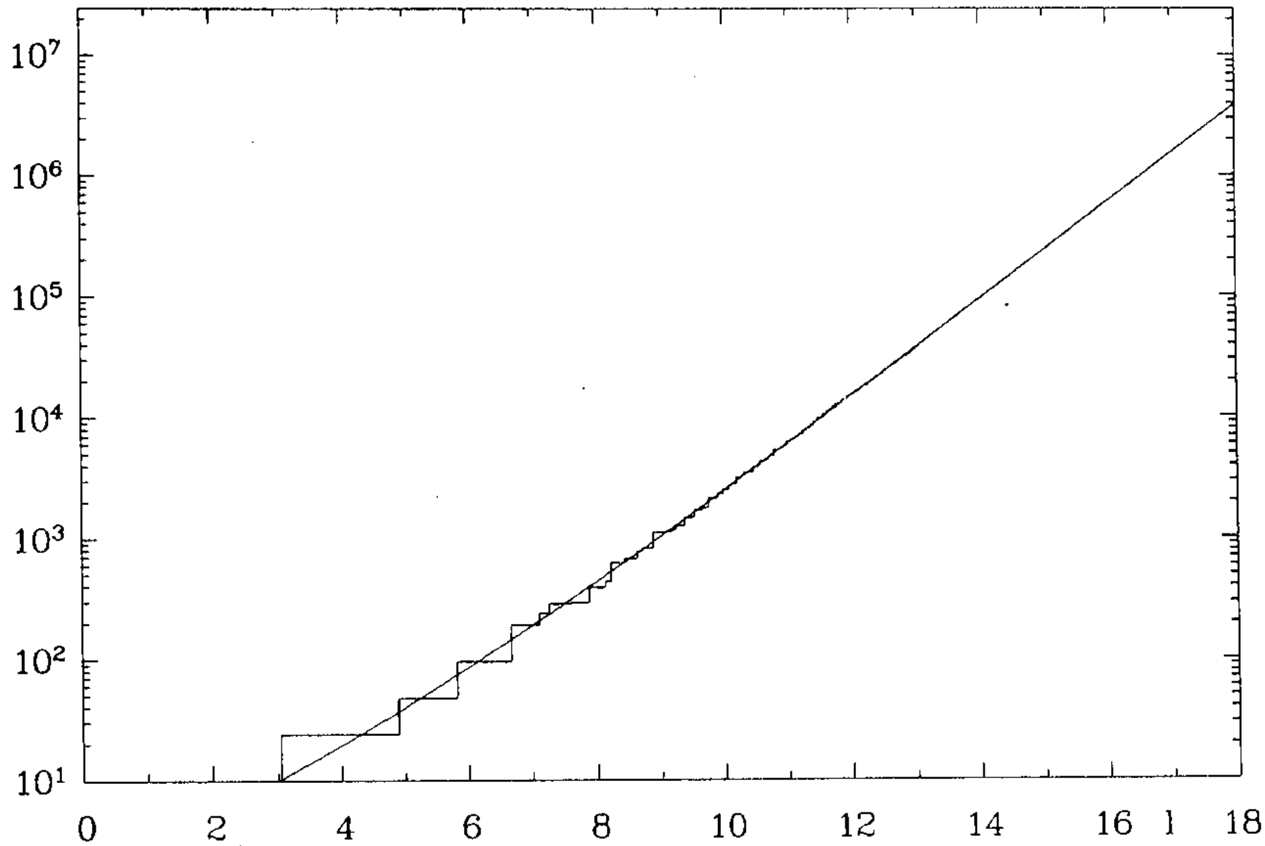

Figure 2: The figure shows the staircase $N(l)$ computed from the length spectrum in comparison with the asymptotic behaviour $N(l) \sim \mathrm{Ei}(l)$, $l \to \infty$.

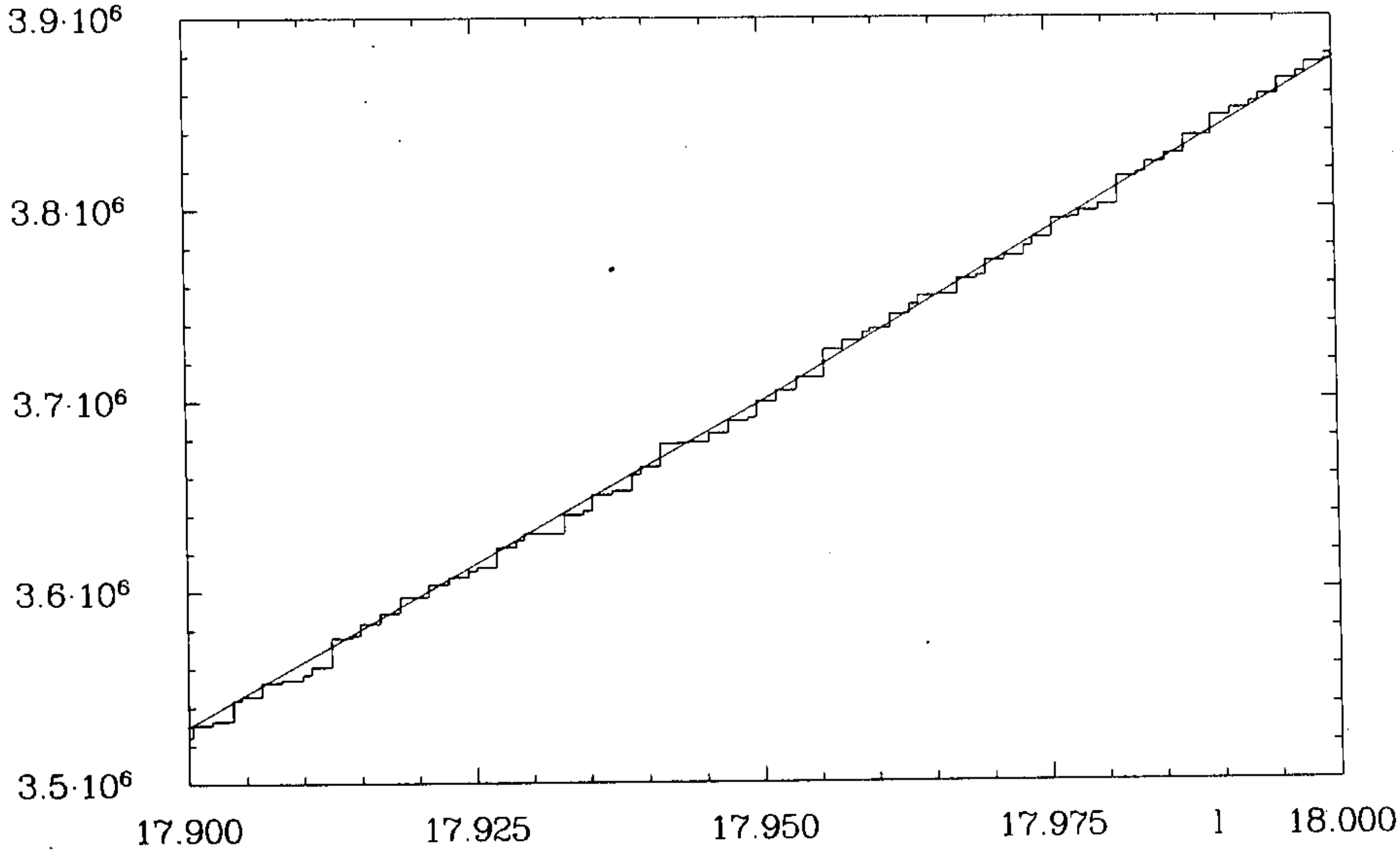

Figure 3: The staircase $N(l)$ is shown in the small interval $l = 17.9$ to $l = 18$ in comparison with the asymptotic behaviour $N(l) \sim \mathrm{Ei}(l)$, $l \to \infty$.

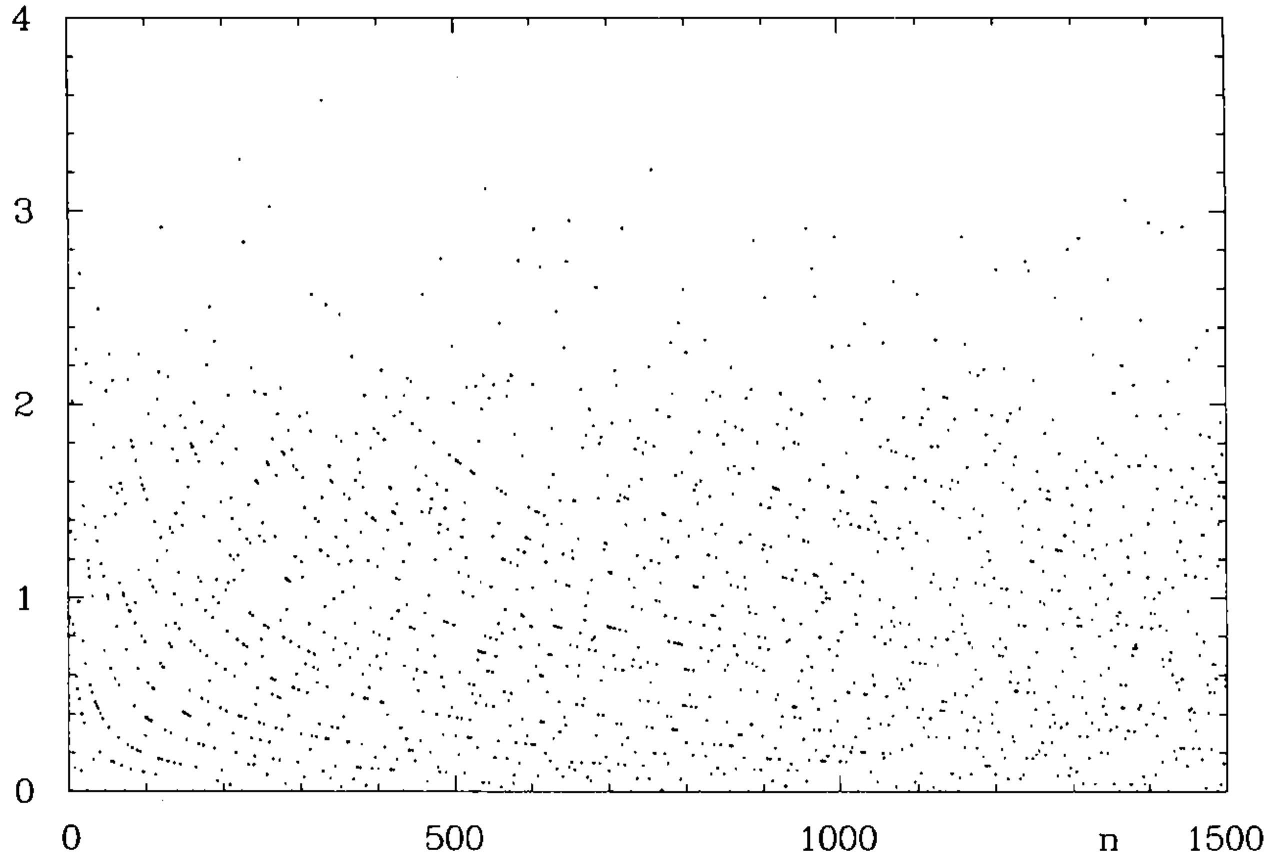

Figure 4: The strongly fluctuating behaviour of the normalized multiplicities g_n^N of the length spectrum of primitive periodic orbits is shown.

In ref. [2] the following asymptotic law for the average multiplicities $\widehat{g}(l)$ was derived on theoretical grounds

$$\widehat{g}(l) \sim 8\sqrt{2}\frac{e^{l/2}}{l} \quad , \quad l \to \infty \quad . \tag{27}$$

While the average law (27) was confirmed by the numbers computed in [2], it was found that the actual multiplicities g_n fluctuate wildly about the average (27). We are thus led to define *normalized* multiplicities g_n^N by

$$g_n^N := g_n \, \frac{l_n e^{-l_n/2}}{8\sqrt{2}} \tag{28}$$

which are expected to fluctuate around 1. In fig.4 we display the normalized multiplicities for $n = 1$ to 1500. It is seen, that for the first 1500 periodic orbit lengths g_n^N does not exceed 4 and that it is actually fluctuating mainly between 0 and 2, which is a striking confirmation of the theoretically predicted exponential increase (27). Furthermore, we present the distribution of the normalized multiplicities g_n^N in fig.5, which shows a maximum below the mean value 1.

V Summary

In this note it is shown that the fundamental group matrices for the regular octagon can be represented in the form (6), where the matrix elements are uniquely characterized as minimal algebraic numbers with well-defined parity, see (13). This representation immediately proves the conjecture (4), (5) put forward in [2] for the geodesic-length law. A geometrical method for the separation of distinct hyperbolic conjugacy classes is described, and a numerical algorithm for the determination of the multiplicities of the primitive periodic orbits is constructed. The proposed method is used for an

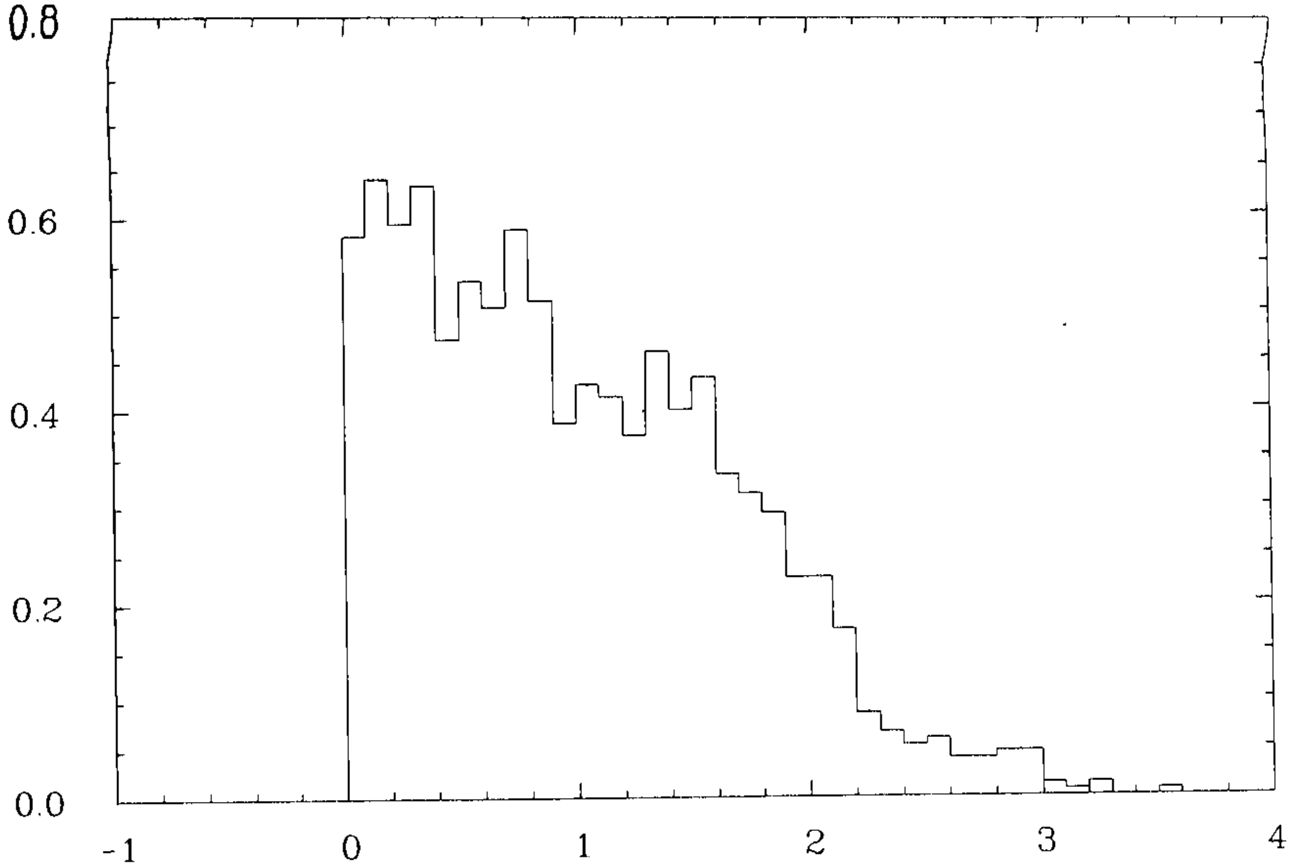

Figure 5: The distribution of the normalized multiplicities g_n^N is presented.

actual enumeration of the primitive periodic orbits of the regular octagon for $n = 1$ to 1500. The multiplicities g_n show strong fluctuations about the theoretical mean value (28).

Appendix (Symmetry transformations)

From fig.1 it is clear that the simplest symmetry of the regular octagon is a rotation over $\frac{\pi}{4}$. If A_1, A_2, B_1, B_2 define an admissible matrix (6), then A_1, A_2, B_1', B_2' with

$$\begin{aligned} B_1' &= (B_1 - B_2)/\sqrt{2} \\ B_2' &= (B_1 + B_2)/\sqrt{2} \end{aligned} \tag{29}$$

also give an admissible fundamental group matrix. Note that an inverse matrix corresponds to A_1, $-A_2$, $-B_1$, $-B_2$. The reflection over the coordinate axis is equivalent to the inversion with respect to the line which has the angle $\pi/8$ with the abscissa. (It is denoted by 1 in Fig1). This inversion corresponds to the following transformation:

$$T_1 : \begin{cases} A_1' = A_1 \\ A_2' = -A_2 \\ B_1' = (B_1 + B_2)/\sqrt{2} \\ B_2' = (B_1 - B_2)/\sqrt{2} \end{cases} , \; T_1^2 = 1 \, . \tag{30}$$

The inversion on circle 2 in fig.1 gives the transformation

$$T_2 : \begin{cases} A_1' = A_1 \\ A_2' = -(\sqrt{2}+1)A_2 - \sqrt{2}B_2 \\ B_2' = (2+\sqrt{2})A_2 + (\sqrt{2}+1)B_2 \\ B_1' = -B_1 \end{cases} , \; T_2^2 = 1 \, . \tag{31}$$

Analogously, the inversion on circle 3 in fig.1 corresponds to the transformation

$$T_3 : \begin{cases} A_1' = A_1 \\ A_2' = -(1+\sqrt{2})A_2 + B_1 - B_2 \\ B_1' = B_1/\sqrt{2} - (1+\sqrt{2})/\sqrt{2}B_2 - (1+\sqrt{2})A_2 \\ B_2' = -(1+\sqrt{2})/\sqrt{2}B_1 + B_2/\sqrt{2} + (1+\sqrt{2})A_2 \end{cases} , \; T_3^2 = 1 . \tag{32}$$

Note that

$$T_2T_1 = T_3T_2 \tag{33}$$

as it must be for the reflections on 3 lines having an angle of 60° with each other. And, finally, the inversion on circle 4 in fig.1 gives

$$T_4 : \begin{cases} A_1' = A_1 \\ A_2' = -(3+2\sqrt{2})A_2 + \sqrt{2}B_1 - (2+\sqrt{2})B_2 \\ B_1' = -(1+\sqrt{2})B_2 - (2+\sqrt{2})A_2 \\ B_2' = (4+3\sqrt{2})A_2 - (1+\sqrt{2})B_1 + (2+2\sqrt{2})B_2 \end{cases} , \; T_4^2 = 1 . \tag{34}$$

Two important properties of the transformations (29) - (34) are:

i) If A_1, A_2, B_1, B_2 obey Eq. (8), then A_1, A_2', B_1', B_2' also obey this relation.

ii) If A_1, A_2, B_1, B_2 are integer algebraic numbers of type (17) obeying (13), then A_1, A_2', B_1', B_2' will be also integer algebraic numbers obeying (13).

Any sequence of these transformations will be an admissible transformation. So, knowing a fundamental group matrix, one can construct another one with the same trace by the above symmetry transformations.

Acknowledgements

Two of the authors (E.B. and F.S.) are indebted to Professor R. Schrieffer for the hospitality of ITP of UCSB where the discussion on this subject began. One of the authors (E. B.) is also indebted to the II. Institut für Theoretische Physik of the University of Hamburg where the main results of this paper were obtained. R. A. and F. S. thank the Deutsche Forschungsgemeinschaft DFG for financial support. F. S. thanks Professor J. Dupont-Roc for the kind hospitality at the Laboratoire de Spectroscopie Hertzienne.

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