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Renormalisation Group Flow in QED *

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Abstract

I investigate the renormalisation group flow in QED using a set of truncated Schwinger-Dyson equations which include the effects of vacuum polarisation. I find a second order chiral phase transition, with mean field critical exponents, and with $\alpha_r = 0$ at the critical point. The electromagnetic β-function is positive, and does not have a zero at the phase transition. When a four-fermion interaction is added the only fixed points present are the free theory and the Nambu-Jona-Lasinio theory. These results suggest that QED is a trivial theory.

1 Results for Pure QED

In this talk I am going to discuss the results I have found by solving a set of truncated Schwinger-Dyson equations [1]. The truncation is made by using the bare fermion-photon vertex instead of the dressed one. The effect of this approximation is minimised by using the Landau gauge. The resulting integral equations and the methods for solving them can be found in [1]. The difference between these integral equations and those introduced by Miransky [2] is that the photon propagator includes the effect of fermion loops. This is a very important feature in discussing the possibility of triviality in QED, because it is the effects of fermion loops which lead to a positive β-function, the running of α and problems such as the Landau pole [3] at high momenta.

These equations lead to a breaking of chiral symmetry when the bare coupling is strong enough. The critical coupling is $\alpha_c = 2.25$ for the single flavour case (which is the case I concentrate on). Above this critical value $\langle\bar{\psi}\psi\rangle$ and m_r are finite as $m_0 \to 0$. (Note that the chiral phase transition is not a confinement-deconfinement transition - the theory has free fermions on both sides of the critical coupling.)

As well as measuring m_r and $\langle\bar{\psi}\psi\rangle$ I can find the renormalised coupling from the zero momentum limit of the photon propagator. Calculating the interactions of the theory is central to the question of triviality. Figure 1 shows the relationship between the renormalised charge and the renormalised mass. The renormalised β-function can be found from the slope of the lines in this figure.

$$\beta_r \equiv (m_r/\Lambda) \left.\frac{\partial \alpha_r}{\partial(m_r/\Lambda)}\right)_{\alpha_0 \,\text{fixed}} = -\alpha_r^2 \,(m_r/\Lambda) \left.\frac{\partial(1/\alpha_r)}{\partial(m_r/\Lambda)}\right)_{\alpha_0\,\text{fixed}} \tag{1.1}$$

(Λ is the ultra-violet cut-off). The fact that all the lines have negative slope tells us that the β-function is positive, both below and above α_c. In fact the slope is very close to that given by the 1 loop β-function. This allows us to write an expression for the renormalised charge when α_r and m_r are small.

$$1/\alpha_r = f(\alpha_0) - \frac{N_f}{3\pi}\log(m_r^2/\Lambda^2) \tag{1.2}$$

The dotted lines (for values of α_0 below α_c) and the dashed lines (for values of α_0 above α_c) are very similar, the only difference being that below α_c the lines extend to $m_r = 0$, $\alpha_r = 0$, while above α_c the lines end at finite m_r and α_r. The empty region, (the lower left-hand portion of the graph), shows that for any finite value of α_r there is a finite minimum m_r value. Only if $\alpha_r = 0$ can the correlation length of the fermion field be infinite. In this figure I plot the results for correlation lengths less than $10^5/\Lambda$ (so that the results in the broken phase can be clearly seen). In [1] I show that the same trends occur up to correlation lengths of $10^{12}/\Lambda$.

Figure 2 shows the bare β-function, defined by

$$\beta_0 \equiv -\,(m_r/\Lambda) \left.\frac{\partial \alpha_0}{\partial(m_r/\Lambda)}\right)_{\alpha_r\,\text{fixed}} \tag{1.3}$$

(This bare β-function is the one that tells us how the bare couplings run as the cut-off is changed). We see that the bare β-function is also positive on both sides of the phase transition, just as the renormalised β-function was.

Notice that both definitions of the β-function ((1.1) and (1.3)) lead immediately to the conclusion that the β-function is identically zero in the quenched theory (which has $\alpha_r = \alpha_0$). Claims of a negative β-function in this case [7,9] are based on taking the derivative at constant m_0 instead of constant bare or renormalised α, which does not give any of the text-book β-functions.

The chiral condensate $\langle\bar{\psi}\psi\rangle$ and the renormalised fermion mass m_r are non-zero when $m_0 \to 0$ for $\alpha_0 > \alpha_c$. The behaviour of these two order parameters is not at all like that found in the quenched planar case [2], which has an essential singularity (non-power law behaviour). Instead I find a modified form of mean field behaviour. The equation of state for the renormalised mass is (again at small m_r and α_r)

$$m_0 = m_r \left(\frac{\alpha_r}{\alpha_0}\right)^{\frac{9}{4N_f}} u(\alpha_0) + \frac{m_r^3}{\Lambda^2}\left(\frac{\alpha_r}{\alpha_0}\right)^{-\frac{9}{4N_f}} v_1(\alpha_0) + \frac{m_r^3}{\Lambda^2}\left(\frac{\alpha_r}{\alpha_0}\right)^{\frac{27}{4N_f}} v_2(\alpha_0) \tag{1.4}$$

Powers of α_r/α_0 are typical of the solutions of renormalisation group equations. The function $u(\alpha_0)$ changes sign at $\alpha_0 = \alpha_c$, so that above the critical coupling it is possible to have a non-zero m_r at $m_0 = 0$. Assuming that $u(\alpha_0)$ has a simple zero at α_c leads to the critical behaviour

$$m_r \propto \Lambda\,(\alpha_0 - \alpha_c)^{\frac{1}{2}} \left(\frac{\alpha_r}{\alpha_0}\right)^{\frac{9}{4N_f}} \tag{1.5}$$

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The equation of state for the chiral condensate $\sigma \equiv \langle\bar{\psi}\psi\rangle$ is

$$m_0 = \frac{\sigma}{\Lambda^2}a(\alpha_0) + \frac{\sigma^3}{\Lambda^8}\left(\frac{\alpha_r}{\alpha_0}\right)^{-\frac{6}{N_f}} b_1(\alpha_0) + \frac{\sigma^3}{\Lambda^8}\left(\frac{\alpha_r}{\alpha_0}\right)^{-\frac{27}{4N_f}} b_2(\alpha_0) + \frac{\sigma^3}{\Lambda^8}\, b_3(\alpha_0) \tag{1.6}$$

leading to a critical behaviour

$$\sigma \propto \Lambda^3\,(\alpha_0 - \alpha_c)^{\frac{1}{2}}\left(\frac{\alpha_r}{\alpha_0}\right)^{\frac{9}{2N_f}} \tag{1.7}$$

Figure 3 shows $\langle\bar{\psi}\psi\rangle$ compared with the asymptotic behaviour of equation (1.7) and the simple $(\alpha_0 - \alpha_c)^{\frac{1}{2}}$. It can be seen that although the critical behaviour of equations (1.5) and (1.7) leads to mean field critical exponents the scaling region is incredibly small. Unless you can look at a very large range in $\alpha_0 - \alpha_c$ and get very near to the critical coupling both m_r and $\langle\bar{\psi}\psi\rangle$ will appear to have a power law scaling with a critical exponent different from the mean field. (This misled me at first, see [1], and has probably caused difficulties for other authors as well [4]) There are now a number of authors who find mean field critical exponents when the effects of fermion loops are included in the Schwinger-Dyson equations [5].

To find out whether the theory is trivial or not we need to look at the interactions between the particles, and use these to find the renormalisation group flow. I start by considering the photon. The dotted lines in figure 4 are the curves of constant renormalised α_r in the bare coupling plane. The main characteristics of the flow are quite easy to explain. At very large bare masses fermion loops are suppressed, and the renormalised and bare couplings are the same. This gives the vertical portion of the trajectory at the top of figure 4 . As the bare mass is reduced fermion loops cause more efficient shielding of the charge and the renormalised α is less than the bare, giving a sloping trajectory at intermediate values of m_0. This running of α eventually stops at large α_0 because the trajectory moves into the phase where the chiral symmetry is broken. There the mass of the fermion comes mainly from the spontaneous breaking of chiral symmetry, and reducing m_0 no longer leads to a significant reduction in the fermion mass or improvement in the shielding and the renormalisation group trajectory becomes vertical. All the lines of finite α_r end at $m_0 = 0$, $\alpha_0 > \alpha_c$ with $m_r/\Lambda > 0$. Only if $\alpha_r = 0$ can the cut-off Λ be taken to infinity.

In addition to the lines of constant α_r we can consider lines of constant fermion-antifermion scattering amplitude. These are the solid lines in figure 3. In the symmetric phase the two sets of trajectories agree very well, showing a scattering amplitude that is a function of α_r, as would be naively expected. However in the broken phase the two sets of trajectories are completely different [1]. The lines of constant scattering all flow into the critical point at $\alpha_0 = \alpha_c$, $m_0 = 0$. The scattering amplitude is not a function of α_r. What this discrepancy shows is that QED is no longer renormalisable near the critical point. It is impossible to compensate for a change in the ultra-violet cut-off by making changes in the bare couplings. We have recently found that this is also the case on the lattice [6].

2 Results with a 4 Fermi coupling

We saw in the last section that in pure QED there are no lines of constant physics near the critical point, because on a line of constant α_r the scattering amplitude varies, while if the scattering amplitude is kept constant the renormalised charge varies. The only way to

have true renormalisation group trajectories is to add a new term to the action. Following the example of Leung, Love and Bardeen [7] I have looked at what happens when the chirally invariant Nambu-Jona-Lasinio [8] 4-fermi interaction

$$\frac{G_0}{2}\left((\bar{\psi}\psi)(\bar{\psi}\psi) - (\bar{\psi}\gamma_5\psi)(\bar{\psi}\gamma_5\psi)\right) \tag{2.1}$$

is added. The chiral phase boundary now separates a chirally symmetric region at low G_0 and α_0 from a region at high G_0 or α_0 where chiral symmetry is broken. This phase boundary is shown as the solid line in figure 6. It has a shape somewhat similar to that found in the quenched case [7].

In figure 5 I show some trajectories for the extended theory, which now has 3 free parameters. α_0, G_0 and m_0. Instead of lines of constant α_r we now have surfaces. The surface shown in the figure has $\alpha_r = 2.0$. Trajectories can be defined by keeping both α_r and the fermion-antifermion scattering amplitude (measured at some particular momentum) constant.

With the new interaction included these are now genuine scaling trajectories. It is not just the quantities which I fixed which are constant along these trajectories. The fermion and photon propagators at all values of p^2/m_r^2 scale along these lines, as do the scattering amplitudes. Now we have the signature of a renormalisable theory, that knowing a small number of parameters allows the calculation of an infinite number of observables.

However the extended theory, although it can now be renormalised, still does not give a continuum limit for finite α_r. This is because the fermion mass m_r/Λ is finite everywhere in a surface of constant α_r. The surface never touches the phase boundary, (where m_r/Λ is zero), all the trajectories go to $G_0 = -\infty$ with the correlation length still finite. With the 4-fermi interaction added QED is defined only as a cut-off theory.

With two interactions there are now two β-functions, one for the electromagnetic coupling, and one for the 4-fermi interaction. By inspecting the lines of constant physics in figure 5 we can immediately see that the electrodynamic β-function is always positive, while the β-function for $G_0\Lambda^2$ has a change of sign.

In figure 6 I show the $m_0 = 0$ plane. The dashed lines are trajectories with constant α_r, the solid line is the phase boundary, which is a line of constant physics with $\alpha_r = 0$. There are only two fixed points in this plane, the free-field one at the origin, and the Nambu-Jona-Lasinio fixed point at $\alpha_0 = 0$, $G_0\Lambda^2 = 4\pi^2$. The physics on the phase boundary is the same as that at the Nambu-Jona-Lasinio fixed point. The Nambu-Jona-Lasinio theory is generally believed to be a trivial theory, with mean field critical exponents. This flow pattern would seem to rule out the last possibility for a non-trivial theory of QED (namely that the fermions and bound states might be interacting at the phase transition, even though the photon has decoupled).

We are now ready to examine the argument given in [9] that "collapse of the wave function" in strongly coupled QED will lead to a decrease of α_0 as the ultra-violet cutoff Λ is increased. The argument says that at high α_0 a small pseudo-scalar bound state is formed, with a size set by the cutoff rather than by m_r, and that this will make 4-fermi interactions relevant, because of the high probability of the fermion and anti-fermion colliding at the bound state's centre of mass. This part of the argument is confirmed by the Schwinger-Dyson approach. If the ultra-violet cutoff is increased the attractive force between the fermion and antifermion must be decreased to keep the pseudo-scalar's properties fixed. Again the results presented here and in [1] confirm [9]. However in [9] it is assumed that this reduction in the attractive force can be achieved by decreasing the electromagnetic charge. This is not correct, because that would lead to changes in the photon and electron propagators, and in particular to changes in the renormalised charge α_r. The decrease in attraction must be achieved by decreasing $G_0\Lambda^2$, not by decreasing α_0. If physics is to be independent of the ultra-violet cutoff collapse of the wave function leads to a negative contribution to $\beta_{G_0\Lambda^2}$, and not to β_{α_0}. The electromagnetic β-function, β_{α_0}, is zero in the quenched case, and positive when fermion loops are included.

3 Conclusion

I have investigated both pure QED and the gauged Nambu-Jona-Lasinio model, and in both cases see no sign that the theory is interacting when vacuum polarisation is taken into account. The photon decouples at the phase transition, so it is not possible to have an interacting photon in the continuum limit. (If α_r is finite so is m_r/Λ, and so taking the ultra-violet cut-off to infinity removes the fermion from the spectrum.) The critical exponents for $\langle\bar{\psi}\psi\rangle$ and m_r appear to be mean field.

I reach the rather surprising conclusion that pure QED is not renormalisable for strong coupling, and that a four-fermi interaction is needed before the theory can even be defined as a cut-off theory.

When a four-fermi interaction is added the theory is still forced to be non-interacting when the cut-off is taken to infinity. The physics at all points on the phase boundary, (including that at the phase transition of pure QED) is that of the pure Nambu-Jona-Lasinio model.

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Figure Captions

Fig. 1:
The relationship between renormalised charge and renormalised mass. The solid line has $m_0 = 0$, and forms a limit, the smallest value of m_r/Λ possible for a given α_r. Only at $\alpha_r = 0$ can the fermion correlation length become infinite. The dotted lines are for fixed $\alpha_0 < \alpha_c$, the dashed for $\alpha_0 > \alpha_c$. The fact that the slope of these curves does not change sign at α_c shows that the β-function does not have a zero at α_c.

Fig. 2:
The bare β-function as a function of α_0 for various values of m_0. The β-function is positive on both sides of α_c (shown by the star). The 1-loop β-function is shown by the dashed line.

Fig. 3:
The critical behaviour of $\langle\bar{\psi}\psi\rangle$ (solid curve) compared with simple mean field behaviour (the straight dotted line) and with equation (1.7) (the dashed curve).

Fig. 4:
A comparison between lines of constant α_r (dotted curves) and lines of constant pseudo-scalar scattering amplitude (solid curves). The two ways of defining renormalisation group trajectories agree well at low α_0 but diverge for strong couplings, showing non-renormalisability for pure QED.

Fig. 5:
The surface $\alpha_r = 2$ and paths within this surface that keep both α_r and the pseudo-scalar scattering amplitude constant. The lines of constant physics begin at $\alpha_0 = \alpha_r$, $m_0 = \infty$ and end at $G_0 = -\infty$.

Fig. 6:
Lines of constant physics in the $m_0 = 0$ plane. The solid line is the phase boundary with $\alpha_r = 0$. The dashed lines have finite α_r. Physics on the phase boundary is the same as at the Nambu-Jona-Lasinio fixed point.

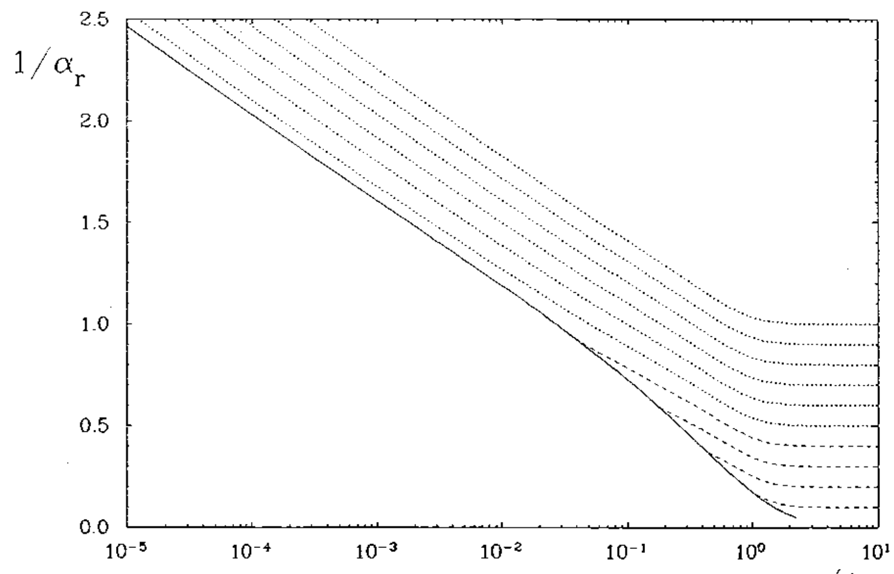

Figure 1

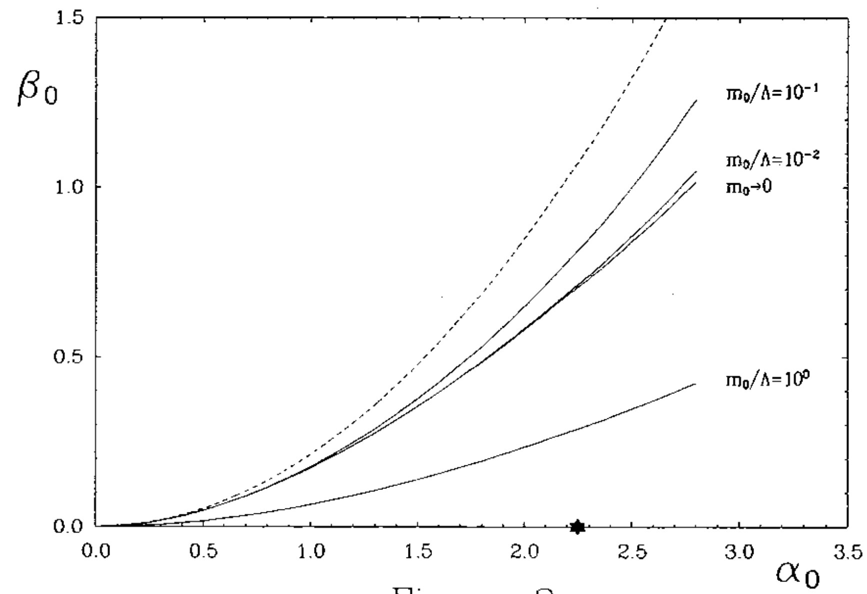

Figure 2

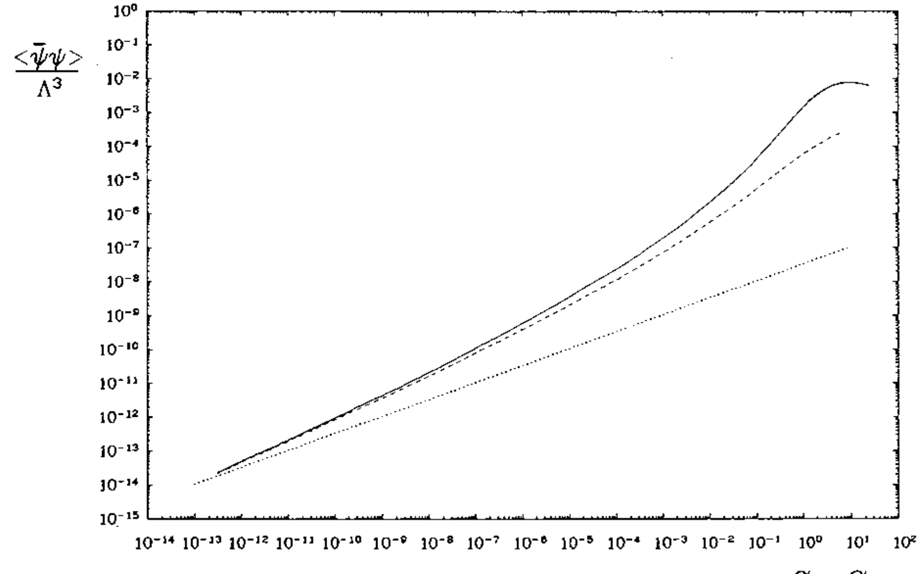

Figure 3

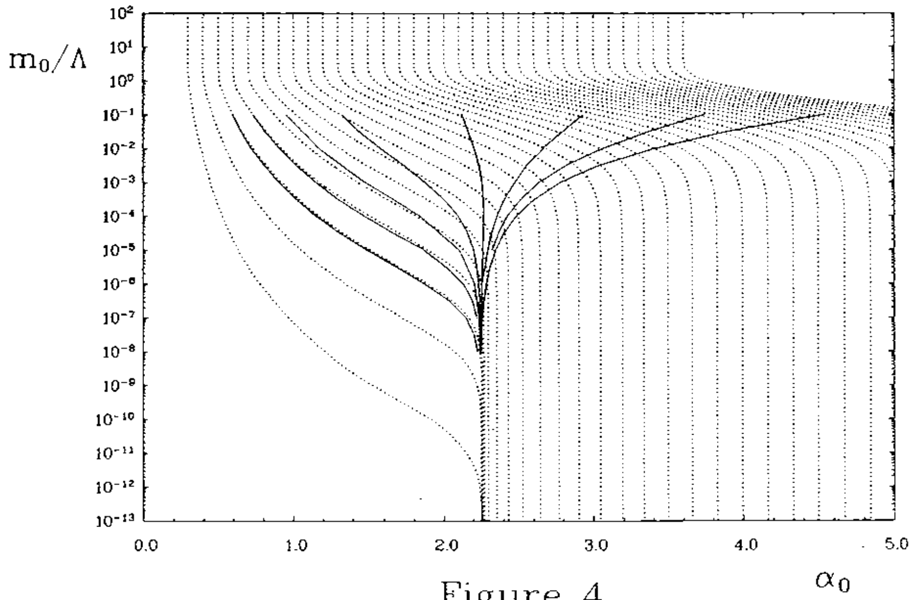

Figure 4

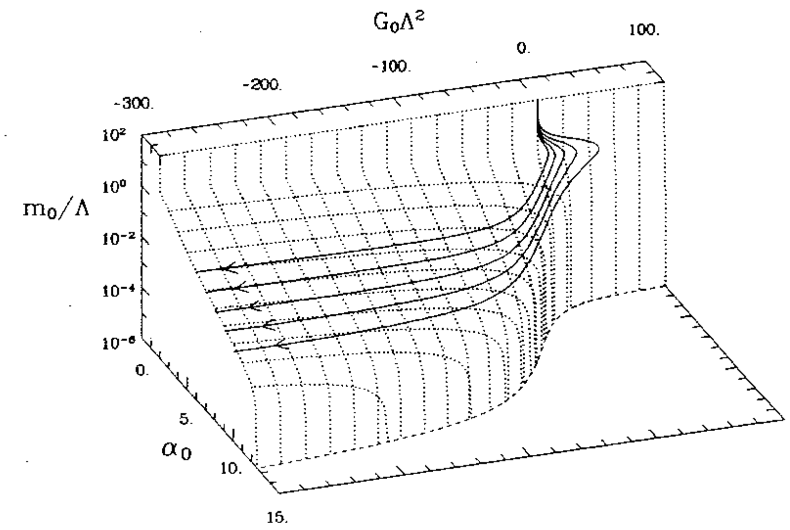

Figure 5

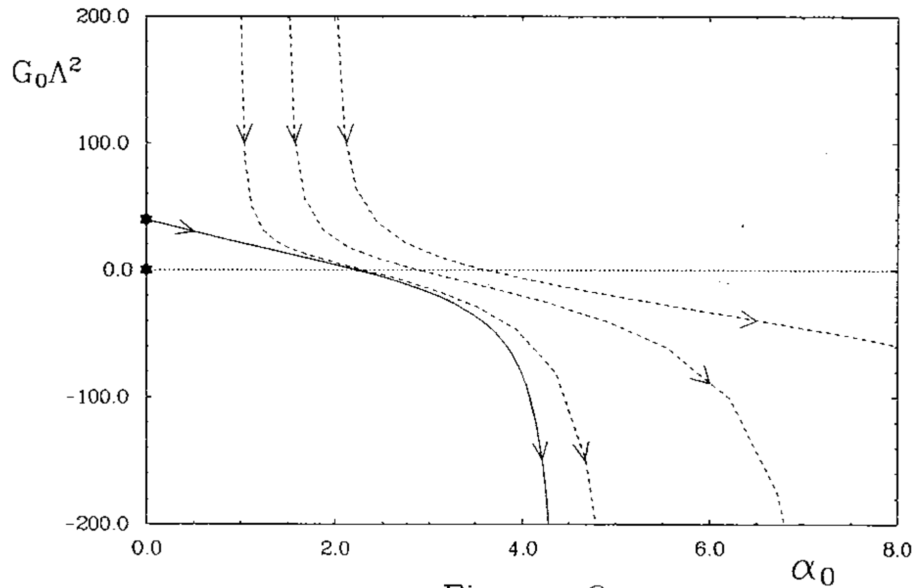

Figure 6