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Abstract

The rise times of bunch dimensions and energy spread due to intra-beam scattering in presence of linear coupling between horizontal and vertical betatron oscillations are calculated. The distribution of skew quadrupoles and solenoids along the ring is taken into account as well as the variation of amplitude functions and dispersions. The result is represented as a single integral.

1. Introduction

Intra-beam scattering is a multiple Coulomb scattering of charged particles in a bunched or unbunched beam. It leads to a continouos increase of bunch or beam dimensions and to a reduction of the beam lifetime when the particles hit the aperture. In proton or antiproton storage rings it causes, therefore, a faster decay of the luminosity, and in electron or positron storage rings with high particle densities for synchrotron light it reduces the brightness.

The exchange of energies between the three modes of oscillation, the horizontal and vertical betatron oscillation and the synchrotron oscillation, caused by intra-beam scattering, was first investigated in 1974 [1]). The rise times for all three modes of oscillation were calculated and it was found that below transition energy there is always a stable equilibrium distribution similar as for gas molecules in a closed box.

Whereas in the first investigation the derivatives of the amplitude function and the dispersion, β_x' and D_x', were neglected, they were introduced in 1977 into a CERN code[2]) by F. Sacherer and the author, and this code was used over several years for the calculation of rise times in different proton storage rings.

The same rise times were later (1983) also derived by J.D.Bjorken and S.K.Mtingwa[3]), who used a quantum field theory approach. They got an elegant and comprehensive representation of the intra-beam scattering theory.

The intra-beam scattering is different from the Touschek-Effect [4]) which is also caused by Coulomb scattering, but is given by large single scattering events for which only the energy transfer from transverse to longitudinal direction is considered. The intra-beam scattering, on the other side, is essentially a diffusion process in all three dimensions.

The agreement of the calculation with measurements in the ISR[5]), in the SPS[6,7]) and in the AA[8,9]) is very good for the beam width and for the momentum spread. Only for the beam height the agreement was sometimes unsatisfactory. For a perfect machine without vertical dispersion and coupling the calculation gives usually a very small negative rise time, i.e. a shrinking of the beam height, whereas a small increase of the beam height is generally observed. In [10]) is was shown that a small vertical dispersion, which is caused by machine imperfections and which is practically inevitable, will always enlarge the beam height, and a better agreement with measurements was obtained. However, there was still disagreement if the machine is operated near the coupling resonance [11]).

In the present paper a linear coupling between horizontal and vertical betatron oscillation is taken into account. Whereas in [12]) already the special case of 100% coupling had already been considered, we will here investigate the general case of coupling where the coupling elements are distributed arbitrarily around the ring. The coupling elements may be skew quadrupoles, or solenoids which need a somewhat different treatment. The same procedure as used in [1]) is then applied to the generalized emittances, and the result can be expressed in terms of the eigenvectors of the betatron oscillation as calculated in standard computer codes. The result can be written as a single integral and this representation is useful also in the uncoupled case including $\beta_{x,z}'$ and $D_{x,z}'$ since the calculation is faster than existing methods.

2. Invariants of the coupled oscillations

In presence of linear coupling between horizontal and vertical betatron oscillations it is convenient to use vectors with 4 components. The horizontal and vertical displacements $x(s)$ and $z(s)$, which refer to the design orbit, and the betatron displacements $x_\beta(s)$ and $z_\beta(s)$, which refer to the closed orbit for the momentum deviation Δp, and the corresponding derivatives with respect to the longitudinal coordinate s can be represented by

$$\vec{r}(s) = \begin{pmatrix} x(s) \\ x'(s) \\ z(s) \\ z'(s) \end{pmatrix} \quad (1), \qquad \vec{r}_\beta(s) = \begin{pmatrix} x_\beta(s) \\ x'_\beta(s) - R(s)\, z_\beta(s) \\ z_\beta(s) \\ z'_\beta(s) + R(s)\, x_\beta(s) \end{pmatrix} \quad (2)$$

R(s) describes the longitudinal field of a solenoid (see App. A1) and it appears, therefore, only inside a solenoid. In Eq.(2) R(s) provides a simple invariant of motion [13,14]. With these definitions we have the relation

$$\vec{r}(s) = \vec{r}_\beta(s) + \vec{D}\,\frac{\Delta p}{p} \quad (3)$$

where p is the mean particle momentum and Δp is the instantaneous deviation from the mean momentum. $\vec{D}$ is defined by

$$\vec{D}(s) = \begin{pmatrix} D_x(s) \\ D'_x(s) - R(s)\, D_z(s) \\ D_z(s) \\ D'_z(s) + R(s)\, D_x(s) \end{pmatrix} \quad (4)$$

where $D_{x,z}$ are the horizontal and vertical dispersion, respectively, and $D'_{x,z}$ are their derivatives with respect to the longitudinal coordinate s.

The transformation of the betatron coordinates for one revolution can then be written in the form

$$\vec{r}_\beta(s+L) = M(s+L,s)\,\vec{r}_\beta(s) \quad (5)$$

where $M(s+L,s)$ is a 4×4 matrix, which includes all coupling elements as skew quadrupoles or solenoids, and L is the length of the orbit.

The betatron motion of the particles can be described in terms of the four eigenvectors $\vec{e}_j(s)$ of the matrix $M(s+L,s)$

$$\vec{e}_j(s) = \begin{pmatrix} e_{j,1}(s) \\ e_{j,2}(s) \\ e_{j,3}(s) \\ e_{j,4}(s) \end{pmatrix} \quad (6)$$

and is given by:

$$\vec{r}_\beta(s) = C_1\,\vec{e}_1(s) + C_2\,\vec{e}_2(s) + C_1^*\,\vec{e}_1^{\,*}(s) + C_2^*\,\vec{e}_2^{\,*}(s) \quad (7)$$

where the constants C_k are determind by the initial conditions and * denotes the complex conjugate. We have used here the symplectic property of the matrix M which means that the

complex conjugate of an eigenvector is also an eigenvector. This is possible since we consider only stable oscillations and exclude integer resonances and sum resonances. With help of the matrix

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (8)$$

an orthogonality relation can be derived (see App. A1):

$$\vec{e}_j(s)\, S\, \vec{e}_k^{\,*}(s) = 0 \qquad \text{for } j \neq k \quad (9)$$

The eigenvectors can be normalized in the same way as in the case without coupling

$$\vec{e}_j(s)\, S\, \vec{e}_j^{\,*}(s) = -2i\,, \quad (10)$$

and Eq.(7) can then be solved for the C_k:

$$-2iC_k = \vec{r}_\beta\, S\, \vec{e}_k^{\,*} \quad (11)$$

Without coupling $|e_{1,1}^2|$ and $|e_{2,3}^2|$ become β_x and β_z, and $|e_{1,3}^2|$ and $|e_{2,1}^2|$ vanish. The mean values of x_β^2 and z_β^2 averaged over many revolutions then are $2|C_1^2|\beta_x$ and $2|C_2^2|\beta_z$ according to Eq.(7). On the other hand, in terms of the emittances ϵ_x and ϵ_z the mean values of x_β^2 and z_β^2 are $\epsilon_x\beta_x/2$ and $\epsilon_z\beta_z/2$, respectively. Therefore, we can write:

$$\epsilon_k = 4|C_k^2| = (\vec{r}_\beta\, S\, \vec{e}_k)(\vec{r}_\beta\, S\, \vec{e}_k^{\,*}) \quad (12)$$

With this definition ϵ_1 and ϵ_2 become ϵ_x and ϵ_z for uncoupled betatron oscillations. In order to describe the beam dimensions beam emittances are defined as in the uncoupled case ($\epsilon_{xb}=\sigma_x^2/\beta_x$, $\epsilon_{zb}=\sigma_z^2/\beta_z$) by

$$\langle x^2\rangle = \epsilon_{1b}\,|e_{1,1}^2| + \epsilon_{2b}\,|e_{2,1}^2| = \epsilon_{1b}\,\beta_{x1} + \epsilon_{2b}\,\beta_{x2} \quad (13)$$

$$\langle z^2\rangle = \epsilon_{1b}\,|e_{1,3}^2| + \epsilon_{2b}\,|e_{2,3}^2| = \epsilon_{1b}\,\beta_{z1} + \epsilon_{2b}\,\beta_{z2} \quad (14)$$

where the brackets denote the average over all particles and the new amplitude functions β_{xk} and β_{zk} are defined by $|e_{k,1}^2|$ and $|e_{k,3}^2|$, respectively.

A sudden change of the coordinates $\delta\vec{r}_\beta$ changes ϵ_k by

$$\delta\epsilon_k = (\delta\vec{r}_\beta\, S\, \vec{e}_k)(\vec{r}_\beta\, S\, \vec{e}_k^{\,*}) + (\vec{r}_\beta\, S\, \vec{e}_k)(\delta\vec{r}_\beta\, S\, \vec{e}_k^{\,*}) + (\delta\vec{r}_\beta\, S\, \vec{e}_k)(\delta\vec{r}_\beta\, S\, \vec{e}_k^{\,*}) \quad (15)$$

The synchrotron oscillation is not influenced by the coupling of the betatron oscillations and the amplitude invariant of the motion can be written in the form

$$H = \begin{cases} \dfrac{\Delta^2 p}{p^2} + \dfrac{1}{\Omega^2}\left(\dfrac{d}{dt}\dfrac{\Delta p}{p}\right)^2 & \text{for bunched beams} \\[2ex] \dfrac{\Delta^2 p}{p^2} & \text{for unbunched beams} \end{cases} \quad (16)$$

Ω is the synchrotron frequency and Δp is the momentum deviation varying with the synchrotron oscillation or being constant. A sudden change of H due to a scattering is, in both cases, given by:

$$\delta H = 2\,\frac{\Delta p}{p}\frac{\delta p}{p} + \frac{\delta^2 p}{p^2} \tag{17}$$

3. Change of the coordinates of two colliding particles

The momenta of two colliding particles before the collision are given by

$$\vec{p}_{1,2} = \begin{pmatrix} p_{s1,2} \\ p_{x1,2} \\ p_{z1,2} \end{pmatrix} \tag{18}$$

In order to determine the change of the two momenta due to the collision we transform the momenta into the center of mass system where they have opposite directions but the same absolute value. Due to the collision their directions are changed by the azimuthal angle $\bar{\psi}$ and by the polar angel $\bar{\phi}$ whereas their absolute values remain constant. The changed momenta can then be transformed back into the laboratory system. This calculation was done in [1]) and is repeated briefly in App. A2. The result can be written in the form

$$\delta\vec{p}_{1,2} = \pm\frac{p}{2}\begin{pmatrix} \gamma\chi\cos\bar{\phi}\sin\bar{\psi} + \gamma\xi(\cos\bar{\psi}-1) \\ \frac{1}{\chi}(\zeta\rho\sin\bar{\phi} - \theta\xi\cos\bar{\phi})\sin\bar{\psi} + \theta(\cos\bar{\psi}-1) \\ \frac{1}{\chi}(-\theta\rho\sin\bar{\phi} - \zeta\xi\cos\bar{\phi})\sin\bar{\psi} + \zeta(\cos\bar{\psi}-1) \end{pmatrix} \tag{19}$$

with

$$\xi = \frac{p_1 - p_2}{\gamma p}, \qquad \theta = \frac{p_{x1} - p_{x2}}{p} \qquad \zeta = \frac{p_{z1} - p_{z2}}{p}$$
$$\chi^2 = \theta^2 + \zeta^2, \qquad \rho^2 = \xi^2 + \theta^2 + \zeta^2$$

p is the mean particle momentum and γ is the mean particle energy divided by its rest energy. We have assumed in this calculation that the particle velocities are nonrelativistic in the center of mass system. This assumption leads to the condition (see App. A2):

$$\bar{v}^2 = \tfrac{1}{4}\,v^2\gamma^2\rho^2 << c^2$$

where v is the mean particle velocity in the laboratory system, $\bar{v}$ is the particle velocity in the center of mass system and c is the velocity of light. We have also assumed, as usual, that the relative momentum deviation and the betatron angles are small as compared to one.

The angles $x'_{1,2}$ and $z'_{1,2}$ are changed directly due to the change of the transverse momentum, namely by $\delta p_{x1,2}/p$ and $\delta p_{z1,2}/p$. The positions $x_{1,2}$ and $z_{1,2}$, however, cannot be changed in such a short time interval. Therefore the closed orbit and the betatron coordinates will change. With Eq.(3) one obtains

$$\delta\vec{r}_{\beta 1,2} = \delta\vec{r}_{1,2} - \vec{D}\,\frac{\delta p_{1,2}}{p} = \begin{pmatrix} 0 \\ \delta p_{x1,2}/p \\ 0 \\ \delta p_{z1,2}/p \end{pmatrix} - \vec{D}\,\frac{\delta p_{1,2}}{p} \tag{20}$$

where the last term represents a shift of the closed orbit due to the change of the momentum.

The probability for the scattering into a solid angle determined by $d\bar{\psi}$ and $d\bar{\phi}$ is given by the Rutherford cross-section in the center of mass system

$$d\bar{\sigma} = \left(\frac{r_p c^2}{4\bar{v}^2\sin^2\frac{\bar{\psi}}{2}}\right)^2 \sin\bar{\psi}\,d\bar{\psi}\,d\bar{\phi} \tag{21}$$

where r_p is the classical particle radius. The minimum scattering angle $\bar{\psi}_{min}$ is determined by the maximum impact parameter $\bar{d}$ in the center of mass system:

$$\tan\frac{\bar{\psi}_{min}}{2} = \frac{r_p c^2}{2\bar{d}\bar{v}^2} \tag{22}$$

$\bar{d}$ gives a cut-off angle for $\bar{\psi}$ which is rather arbitrary, but its influence on the result is weak. We assume that $\bar{d}$ is half the beam diameter.

Inserting $\delta\vec{p}_1$ (Eq.(19)) into $\delta\vec{r}_{\beta 1}$ (Eq.(20)), $\delta\vec{r}_1$ into $\delta\epsilon_k$ (Eq.(15)), multiplying $\delta\epsilon_k$ by $d\bar{\sigma}$ (Eq.(21)), and integrating with respect to $\bar{\psi}$ and $\bar{\phi}$ yields for the emittance change $\delta\epsilon_k$ of the particle with the momentum $\vec{p}_1$

$$\begin{aligned}\int_0^{2\pi}\int_{\bar{\psi}_{min}}^{\pi} \delta\epsilon_k\,d\bar{\sigma} = \frac{\pi r_p^2 c^4}{8\bar{v}^4}\Bigg(&\left(2\ln\left(\frac{2\bar{d}\bar{v}^2}{r_p c^2}\right) - 1\right)\Big(\frac{1}{\chi^2}\left|\gamma\chi^2\,\vec{D}S\vec{e}_k - \xi\theta e_{k,1} - \xi\zeta e_{k,3}\right|^2 \\ &+ \frac{\rho^2}{\chi^2}\left|\zeta e_{k,1} - \theta e_{k,3}\right|^2\Big) + 2\left|\gamma\xi\,\vec{D}S\vec{e}_k + \theta e_{k,1} + \zeta e_{k,3}\right|^2 \\ &+ 8\ln\left(\frac{2\bar{d}\bar{v}^2}{r_p c^2}\right) Re\{(\gamma\xi\,\vec{D}S\vec{e}_k + \theta e_{k,1} + \zeta e_{k,3})(\vec{r}_{\beta 1}S\vec{e}_k^{\,*})\}\Bigg)\end{aligned} \tag{23}$$

where $Re\{...\}$ denotes the real part. Here we have assumed that the minimum scattering angle $\bar{\psi}_{min}$ is small, or more precisely

$$\frac{4\bar{v}^4\bar{d}^2}{c^4 r_p^2} >> 1 \tag{24}$$

In a similar way one obtains for the average change of the synchrotron invariant with Eqs.(17) and (19) and with $\delta p \approx \delta p_s$:

$$\int_0^{2\pi}\int_{\bar{\psi}_{min}}^{\pi} \delta H\,d\bar{\sigma} = \frac{\pi r_p^2 c^4}{8\bar{v}^4}\left(2\ln\left(\frac{2\bar{d}\bar{v}^2}{r_p c^2}\right)\left(\gamma^2\chi^2 - 4\gamma\xi\frac{\Delta p}{p}\right) + \gamma^2(2\xi^2 - \chi^2)\right) \tag{25}$$

4. Rise times

We want to calculate the change of the dimensions of the whole beam and we shall average, therefore, the change of the coordinates of one colliding particle over all positions, angles and energy deviations of both colliding particles. For this purpose we assume an exponential distribution of the three invariants ϵ_1, ϵ_2 and H. In the case of electrons or positrons quantum fluctuation and damping yield exactly an exponential distribution of the invariants, also in presence of linear coupling [16,17]. This yields a Gaussian particle distribution, or a superposition of two Gaussian functions in presence of linear coupling. For protons, a Gaussian particle distribution is in good agreement with observations. Thus we assume for the distribution of the betatron coordinates:

$$P_\beta(x_\beta, x'_\beta, z_\beta, z'_\beta) = K \exp\left\{-\frac{\epsilon_1(x_\beta, x'_\beta, z_\beta, z'_\beta)}{2\epsilon_{1b}} - \frac{\epsilon_2(x_\beta, x'_\beta, z_\beta, z'_\beta)}{2\epsilon_{2b}}\right\} = K \exp\left\{-\frac{|\vec{r}_\beta S \vec{e}_1|^2}{2\epsilon_{1b}} - \frac{|\vec{r}_\beta S \vec{e}_2|^2}{2\epsilon_{2b}}\right\} \quad (26)$$

ϵ_1 and ϵ_2 are given by Eq.(12) and ϵ_{1b} and ϵ_{2b} are the generalized beam emittances (Eqs.(13) and (14)) determining the width of the distribution. They are, as in the uncoupled case, half as large as the mean emittances obtained by averaging over ϵ_1 and ϵ_2 with the distribution function P_β: $\langle\epsilon_1\rangle = 2\epsilon_{1b}$, $\langle\epsilon_2\rangle = 2\epsilon_{2b}$. The normalization constant K is given by:

$$K = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{\epsilon_1}{2\epsilon_{1b}} - \frac{\epsilon_2}{2\epsilon_{2b}}\right\} dx_\beta\, dx'_\beta\, dz_\beta\, dz'_\beta\right)^{-1} = \frac{1}{4\pi^2}\sqrt{\det\ M_e} \quad (27)$$

where the elements of the matrix M_e are given by

$$(M_e)_{m,n} = Re\left\{\frac{1}{\epsilon_{1b}}(S\vec{e}_1)_m(S\vec{e}_1^{\,*})_n + \frac{1}{\epsilon_{2b}}(S\vec{e}_2)_m(S\vec{e}_2^{\,*})_n\right\} \quad (28)$$

In the uncoupled case K becomes $1/(4\pi^2\epsilon_{xb}\epsilon_{zb})$. The distribution of the synchrotron coordinates Δs and Δp is given by

$$P_l(\Delta s, \Delta p) = \begin{cases} \dfrac{1}{2\pi\sigma_s p\sigma_p} \exp\left\{-\dfrac{1}{2\sigma_p^2}\dfrac{\Delta^2 p}{p^2} - \dfrac{\Delta^2 s}{2\sigma_s^2}\right\} & \text{for bunched beams} \\ \dfrac{1}{\sqrt{2\pi}Lp\sigma_p} \exp\left\{-\dfrac{1}{2\sigma_p^2}\dfrac{\Delta^2 p}{p^2}\right\} & \text{for unbunched beams} \end{cases} \quad (29)$$

where σ_p and σ_s determine the relative momentum spread and the bunch length, respectively. L is the length of the circumference. This distribution for Δp and Δs can be obtained from the exponential distribution of the invariant H (Eq.(16)) by using the well known relations between Δs, Ω and the time derivative of Δp.

The number of scattering events per unit time for a single particle in the center of mass system is given by $\bar{v}_{rel}\bar{P}_o d\bar{\sigma}$. The relative velocity $\bar{v}_{rel}$ between two colliding particles is $2\bar{v}$ and the spatial density $\bar{P}_o$ is P_o/γ, where P_o is the spatial density in the laboratory system. The number of scattering events for a single particle in the laboratory system is given by

$2\bar{v}P_o d\bar{\sigma}/\gamma^2$, since the transformation of the time gives another factor of γ. The change per unit time of the mean values of ϵ_k and H, averaged over all particles, is then determined by

$$\frac{d}{dt}\langle\epsilon_k\rangle = \left\langle\frac{2}{\gamma^2}\int \bar{v}\, P \int_0^{2\pi}\int_{\bar{\psi}_{min}}^{\pi} \delta\epsilon_k\, d\bar{\sigma}\, dV\right\rangle \quad (30)$$

$$\frac{d}{dt}\langle H\rangle = \left\langle\frac{2}{\gamma^2}\int \bar{v}\, P \int_0^{2\pi}\int_{\bar{\psi}_{min}}^{\pi} \delta H\, d\bar{\sigma}\, dV\right\rangle \quad (31)$$

where the large brackets denote the average over the whole circumference. P and dV are given by

$$P = N P_l(\Delta s_1, \Delta p_1)\, P_l(\Delta s_2, \Delta p_2)\, P_\beta(x_{\beta 1}, x'_{\beta 1}, z_{\beta 1}, z'_{\beta 1})\, P_\beta(x_{\beta 2}, x'_{\beta 2}, z_{\beta 2}, z'_{\beta 2})$$

$$dV = d\Delta s_1\, dx_{\beta 1}\, dz_{\beta 1}\, d\Delta p_1\, d\Delta p_2\, dx'_{\beta 1}\, dx'_{\beta 2}\, dz'_{\beta 1}\, dz'_{\beta 2}$$

N is the number of particles in the bunch or, in the unbunched case, in the beam. When averaging the position coordinates over the whole beam we assume that the two colliding particles have always the same position, i.e. we take into account the following conditions:

$$\Delta s_1 = \Delta s_2, \qquad x_{\beta 1} + D_x\frac{\Delta p_1}{p} = x_{\beta 2} + D_x\frac{\Delta p_2}{p}, \qquad z_{\beta 1} + D_z\frac{\Delta p_1}{p} = z_{\beta 2} + D_z\frac{\Delta p_2}{p}$$

Since the bunch dimensions are proportional to the square roots of the invariants (see Eqs.(13) and (14)) a relative change of the dimensions is one half of the relative change of the invariants. Thus the rise times of the transverse dimensions are given by

$$\frac{1}{\tau_x} = \frac{1}{2\langle x^2\rangle}\frac{d\langle x^2\rangle}{dt} = \frac{\beta_{x1}\,\epsilon_{1b}/\tau_1 + \beta_{x2}\,\epsilon_{2b}/\tau_2}{2(\beta_{x1}\,\epsilon_{1b} + \beta_{x2}\,\epsilon_{2b})} \quad (32)$$

and

$$\frac{1}{\tau_z} = \frac{1}{2\langle z^2\rangle}\frac{d\langle z^2\rangle}{dt} = \frac{\beta_{z1}\,\epsilon_{1b}/\tau_1 + \beta_{z2}\,\epsilon_{2b}/\tau_2}{2(\beta_{z1}\,\epsilon_{1b} + \beta_{z2}\,\epsilon_{2b})} \quad (33)$$

where β_{xk} and β_{zk} are defined by Eqs.(13) and (14) and the rise times τ_k for the generalized emittances are defined by

$$\frac{1}{\tau_k} = \frac{1}{\langle\epsilon_k\rangle}\frac{d}{dt}\langle\epsilon_k\rangle = \frac{1}{2\epsilon_{kb}}\frac{d}{dt}\langle\epsilon_k\rangle \quad (34)$$

Since β_{xk} and β_{zk} depend on s the horizontal and vertical rise times depend on the position in the ring where they are measured. τ_s is defined by

$$\frac{1}{\tau_s} = \frac{1}{2\langle H\rangle}\frac{d}{dt}\langle H\rangle = \frac{n}{4\sigma_p^2}\frac{d}{dt}\langle H\rangle \quad (35)$$

where n is 1 for bunched beams and 2 for unbunched beams according to Eq.(16). In Eqs.(30) and (31) seven of the nine integrations can be performed immediately (App. A3) and one obtains for τ_k (k=1,2) and τ_s:

$$\frac{1}{\tau_k} = \left\langle \frac{r_p^2 c^4 N K}{4\gamma^4 v^3 \sigma_s \sigma_p \epsilon_{kb}} I_k \right\rangle \quad (36) \qquad \frac{1}{\tau_s} = \left\langle \frac{n r_p^2 c^4 N K}{8\gamma^2 v^3 \sigma_s \sigma_p^3} I_s \right\rangle \quad (37)$$

where I_k and I_s are given by

$$I_{k,s} = \int_0^\pi \int_0^{2\pi} \ln\left(\frac{2\bar{d}\gamma^2 v^2}{g\, r_p c^2 C_E \sqrt{e}}\right) \frac{h_{k,s}}{g} \sin\mu \, d\nu \, d\mu \quad (38)$$

$C_E = \exp(0.5772..)$ is Euler's constant, e is the base of the natural logarithm and h_k, h_s and g are given by

$$h_k = \gamma^2 |\vec{D}S\vec{e}_k|^2 + |e_{k,1}^2| + |e_{k,3}^2| - 3|W_k^2|, \quad (39) \qquad h_s = 1 - 3\cos^2\mu \quad (40)$$

$$g = g(\nu,\mu) = \frac{\gamma^2 \cos^2\mu}{\sigma_p^2} + \frac{|W_1^2|}{\epsilon_{1b}} + \frac{|W_2^2|}{\epsilon_{2b}} \quad (41)$$

with

$$W_k = W_k(\nu,\mu) = \gamma \vec{D}S\vec{e}_k \cos\mu + (e_{k,1}\cos\nu + e_{k,3}\sin\nu)\sin\mu$$

For unbunched beams one has to replace σ_s by $L/\sqrt{2\pi}$ in Eqs.(36) and (37). The eigenvectors $\vec{e}_k$ are evaluated in several computer codes (see for example [17]) and the rise times can now be calculated. The douple integrals I_k and I_s can also be represented as single integrals which look more complicated but reduce the computer time.

5. Representation of I_k and I_s as single integrals

The functions h_k and g can be written in the form (see App. A3):

$$h_k(\nu) = A_{hk} + B_{hk}\cos\nu + C_{hk}\sin\nu + D_{hk}\cos^2\nu + E_{hk}\sin^2\nu + F_{hk}\cos\nu\sin\nu \quad (42)$$

$$g(\nu) = A_g + B_g\cos\nu + C_g\sin\nu + D_g\cos^2\nu + E_g\sin^2\nu + F_g\cos\nu\sin\nu \quad (43)$$

with k=1,2. The A_g ... F_{hk} are functions of $\cos\mu$ and $\sin\mu$. I_k and I_s are then given by (App. A4):

$$I_k = 2\pi \int_0^\pi \Big(L_{k0}\ln(l_0) + L_{k1}\ln(l_1) + L_{k2}\ln(l_2) - J_{k0}\arctan(j_0)\Big)\sin\mu\, d\mu \quad (44)$$

$$I_s = 2\pi \int_0^\pi \Big(L_{s1}\ln(l_1) + L_{s2}\ln(l_2) - J_{s0}\arctan(j_0)\Big)\sin\mu\, d\mu \quad (45)$$

with

$$L_{k0} = \frac{F_{hk}F_g + T_{hk}T_g}{F_g^2 + T_g^2}, \qquad L_{k1,2} = \frac{\bar{A}_{hk}D_{1,2}u_{1,2} \mp \bar{B}_{hk}(b_0 - c_{1,2}t_1) \mp \bar{C}_{hk}(c_0 + b_{1,2}t_1)}{a N_0 D_{1,2}}$$

$$l_0 = \frac{8\bar{d}\gamma^2 v^2}{r_p c^2 C_E \sqrt{e}a(1+D_1)(1+D_2)}, \qquad l_{1,2} = \frac{\bar{d}\gamma^2 v^2 (1+D_{1,2})^2}{r_p c^2 C_E \sqrt{e} a D_{1,2}^2 (1 - t_0 + D_1 D_2)}$$

$$J_{k0} = \frac{2}{aN_0}(\bar{A}_{hk}t_1 + \bar{B}_{hk}c_0 - \bar{C}_{hk}b_0), \qquad j_0 = \frac{t_1(t_0u_1 + t_0u_2 + u_1u_2 - t_1^2)}{t_0u_1u_2 - t_1^2(t_0 + u_1 + u_2)}$$

$$L_{s1,2} = \frac{u_{1,2}}{aN_0}(1 - 3\cos^2\mu) \qquad J_{s0} = \frac{2t_1}{aN_0}(1 - 3\cos^2\mu)$$

$$\bar{A}_{hk} = \tfrac{1}{2}\big(R_{hk} + S_{hk} - L_{k0}(R_g + S_g)\big), \qquad \bar{B}_{hk} = B_{hk} - L_{k0}B_g - M_{k0}C_g$$
$$\bar{C}_{hk} = C_{hk} - L_{k0}C_g + M_{k0}B_g, \qquad M_{0k} = (-F_{hk}T_g + T_{hk}F_g)/(2(F_g^2 + T_g^2))$$

with $R_{hk} = A_{hk} + D_{hk}$, $S_{hk} = A_{hk} + E_{hk}$, $T_{hk} = D_{hk} - E_{hk}$, $R_g = A_g + D_g$, $S_g = A_g + E_g$, $T_g = D_g - E_g$.

$$u_{1,2} = (d_{1,2}^2 - t_0)/D_{1,2}, \qquad N_0 = b_0^2 + c_0^2 - t_1^2, \qquad d_{1,2}^2 = b_{1,2}^2 + c_{1,2}^2$$
$$D_{1,2}^2 = 1 - d_{1,2}^2, \qquad t_0 = b_1b_2 + c_1c_2, \qquad t_1 = b_1c_2 - b_2c_1$$

$$b_{1,2} = \frac{1}{2}\left(\frac{B_g}{a} \pm b_0\right), \qquad c_{1,2} = \frac{1}{2}\left(\frac{C_g}{a} \pm c_0\right)$$

$$b_0 = \frac{1}{a}\sqrt{B_g^2 + 4a(a - R_g)}, \qquad c_0 = \pm\frac{1}{a}\sqrt{C_g^2 + 4a(a - S_g)}$$

The sign of c_0 must be chosen so that

$$b_0 c_0 a^2 = B_g C_g - 2aF_g.$$

a is determined by a cubic equation (Eq.(A4.3)). It has one solution for $q^2 + p^3 > 0$:

$$a = \frac{1}{3}(R_g + S_g) - \frac{q}{|q|}\left(\left(|q| + \sqrt{q^2 + p^3}\right)^{1/3} - p\left(|q| + \sqrt{q^2 + p^3}\right)^{-1/3}\right)$$

and three solutions for $q^2 + p^3 < 0$:

$$a = \frac{1}{3}(R_g + S_g) - 2\frac{q}{|q|}\sqrt{|p|}\cos\left(\frac{1}{3}\arccos\left(\frac{|q|}{\sqrt{|p|^3}}\right)\right)$$

$$a = \frac{1}{3}(R_g + S_g) + 2\frac{q}{|q|}\sqrt{|p|}\cos\left(\frac{\pi}{3} \mp \frac{1}{3}\arccos\left(\frac{|q|}{\sqrt{|p|^3}}\right)\right)$$

with

$$9p = -R_gS_g - T_g^2 + \tfrac{3}{4}(B_g^2 + C_g^2 - F_g^2)$$

$$8q = B_gC_gF_g - B_g^2S_g - C_g^2R_g + \tfrac{1}{3}(R_g + S_g)(4R_gS_g + B_g^2 + C_g^2 - F_g^2) - \tfrac{8}{27}(R_g + S_g)^3$$

In the last case a must be chosen from the three solutions so that $b_{1,2}$ and $c_{1,2}$ have an absolute value smaller than one.

For $F_g = T_g = 0$ I_k and I_s are given by

$$I_k = \frac{2\pi}{aD_1}\int_0^\pi \left(\frac{B_{hk}b_1 + C_{hk}c_1 - \tilde{T}_{hk}}{d_1^2}\Big(D_1\ln(l_0) - \ln(l_1)\Big) + \frac{R_{hk} + S_{hk} - \tilde{T}_{hk}}{2}\ln(l_1) - \frac{\tilde{T}_{hk}D_1}{1 + D_1}\right)\sin\mu\, d\mu \quad (46)$$

$$I_s = \frac{2\pi}{aD_1}\int_0^\pi \ln(l_1)(1 - 3\cos^2\mu)\sin\mu\, d\mu \quad (47)$$

with $t_0 = 0$, $D_2 = 1$ and

$$\tilde{T}_{hk} = \frac{b_1^2 - c_1^2}{d_1^2}T_{hk}$$

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Appendix A1

The two differential equations of the coupled linear betatron oscillations can be written in the form [13,14]:

$$x_\beta'' + (K_x^2 + k_q)\,x_\beta + (k_s - R')\,z_\beta - 2Rz_\beta' = 0 \qquad \text{(A1.1)}$$

$$z_\beta'' + (K_z^2 - k_q)\,z_\beta + (k_s + R')\,x_\beta + 2Rx_\beta' = 0 \qquad \text{(A1.2)}$$

K_x and K_z are the horizontal and vertical curvature, k_q is the strength of normal quadrupoles, k_s is the strength of skew quadrupoles and R is the solenoid strength, given by $R = eB_s/2p$ where e is the elementary charge and B_s is the longitudinal field strength. R' takes into account the transverse end fields of a solenoid. The solenoid is assumed to be rotationally symmetric.

The longitunal flux in a cylinder with the arbitrary radius r is $\pi r^2 B_l$ where B_l is the mean longitudinal magnetic field in this cylinder. At the end of the solenoid the longitudinal flux becomes a transverse flux given by $2\pi r d B_t$ where B_t is the mean transverse field and d is the mean width of the transverse field. Since the divergence is zero it follows $B_t = B_l\,r/2d$. If d is small as compared to the betatron wave length (including the longitudinal field of the solenoid) the transverse field changes the transverse momentum of a particle which enters the solenoid by

$$\delta x_\beta' = \frac{eB_l}{2p} z_\beta = R\,z_\beta \quad \text{(A1.3)}, \qquad \delta z_\beta' = -\frac{eB_l}{2p} x_\beta = -R\,x_\beta \qquad \text{(A1.4)}$$

This means that the quantities $x_\beta' - R\,z_\beta$ and $z_\beta' + R\,x_\beta$ remain constant when a particle passes the end fields of a solenoid.

But there is a more general invariant [13,15]. If $x_{\beta1}(s), z_{\beta1}(s)$ and $x_{\beta2}(s), z_{\beta2}(s)$ are two solutions of Eqs.(A1.1) and (A1.2) then

$$\vec{r}_{\beta1}(s)\,S\,\vec{r}_{\beta2}(s) = x_{\beta1}' x_{\beta2} - x_{\beta1} x_{\beta2}' + z_{\beta1}' z_{\beta2} - z_{\beta1} z_{\beta2}' + 2R\,(x_{\beta1} z_{\beta2} - x_{\beta2} z_{\beta1}) \qquad \text{(A1.5)}$$

is an invariant of motion where $\vec{r}_{\beta1,2}$ and S are defined by Eqs.(2) and (8). This can easily be verified by differentiating Gl.(A1.5) with respect to s and applying Eqs.(A1.1) and (A1.2):

$$\frac{d}{ds}\,\vec{r}_1\,S\,\vec{r}_2 = 0 \qquad \text{(A1.6)}$$

If we consider especially two eigenvectors $\vec{e}_j$ and $\vec{e}_k^{\,*}$ of the revolution matrix M with the eigenvalues λ_j and λ_k^* we get the relation

$$\vec{e}_j S \vec{e}_k^{\,*} = (M\vec{e}_j) S (M\vec{e}_k^{\,*}) = \lambda_j \lambda_k^* \,\vec{e}_j S \vec{e}_k^{\,*} \qquad \text{(A1.7)}$$

This means that either $\vec{e}_j S \vec{e}_k^{\,*}=0$ or $\lambda_j \lambda_k^*=1$. Since for stable motion the eigenvalues of two eigenvectors are different and have an absolute value of one we have obtained the orthogonal relation Eq.(9).

Appendix A2

We define a coordinate system with the axes j, k and l which are parallel to $\vec{p}_1 + \vec{p}_2$, $\vec{p}_1 \times \vec{p}_2$ and $(\vec{p}_1 + \vec{p}_2) \times (\vec{p}_1 \times \vec{p}_2)$, respectively. The two momenta are then given by

$$\vec{p}_{1,2} = p_{1,2} \begin{pmatrix} \cos\chi_{1,2} \\ 0 \\ \pm\sin\chi_{1,2} \end{pmatrix}_{j,k,l} \tag{A2.1}$$

where $\chi_{1,2}$ are the angles between the vector $\vec{p}_1 + \vec{p}_2$ and the vectors $\vec{p}_{1,2}$, respectively. If we apply a Lorentz transformation parallel to the j-axis we obtain for the transformed momenta

$$\vec{\bar{p}}_{1,2} = p_{1,2} \begin{pmatrix} \gamma_t(\cos\chi_{1,2} - v_t/v_{1,2}) \\ 0 \\ \pm\sin\chi_{1,2} \end{pmatrix}_{j,k,l} \tag{A2.2}$$

where v_t is the velocity of the c.o.m. system, γ_t is the Lorentz factor, $v_{1,2}$ are the velocities of the two particles and the bars denote all quantities in the c.o.m. system. v_t is determind by the condition that the sum of the two momenta vanishes in the c.o.m. system and is given by:

$$v_t = \frac{v_1\gamma_1\cos\chi_1 + v_2\gamma_2\cos\chi_2}{\gamma_1 + \gamma_2} \tag{A2.3}$$

We may now assume that the quantities $\Delta^2 p/p^2$, p_z^2/p^2 and p_x^2/p^2 are small as compared to one, which means also $\chi_1 \approx \chi_2 \approx \chi/2$. Eq.(A2.2) then simplifies to

$$\vec{\bar{p}}_{1,2} = \pm\frac{p}{2} \begin{pmatrix} \xi\sqrt{1+\gamma^2\chi^2/4} \\ 0 \\ \chi \end{pmatrix}_{j,k,l} \approx \pm\frac{p}{2} \begin{pmatrix} \xi \\ 0 \\ \chi \end{pmatrix}_{j,k,l} \tag{A2.4}$$

with

$$\xi = \frac{p_1 - p_2}{\gamma p}, \qquad \theta = \frac{p_{x1} - p_{x2}}{p} \qquad \zeta = \frac{p_{z1} - p_{z2}}{p}$$
$$\chi^2 = \theta^2 + \zeta^2, \qquad \rho^2 = \xi^2 + \theta^2 + \zeta^2$$

The last approximation is obtained if we also assume that $\gamma^2\chi^2/4$ is small as compared to one. With these approximations one obtains from $\bar{p} = p\rho/2$

$$\bar{v}^2 = \frac{v^2\gamma^2(\xi^2 + \chi^2)}{4 + \frac{v^2}{c^2}\gamma^2(\xi^2 + \chi^2)} \approx \frac{1}{4}v^2\gamma^2(\xi^2 + \chi^2) \tag{A2.5}$$

If $\gamma^2\chi^2/4$ is small as compared to one the particle velocity in the c.o.m. system is nonrelativistic since $\gamma^2\xi^2 = (p_1 - p_2)^2/p^2$ is also small as compared to one. After the collision the momenta are rotated by the polar angle $\bar{\psi}$ and by the azimuthal angle $\bar{\phi}$ and can be written in the form:

$$\vec{\bar{p}}'_{1,2} = \pm \begin{pmatrix} \bar{p}'_j \\ \bar{p}'_k \\ \bar{p}'_l \end{pmatrix}_{j,k,l} = \pm\frac{p}{2} \begin{pmatrix} \xi\cos\bar{\psi} + \chi\cos\bar{\phi}\sin\bar{\psi} \\ \rho\sin\bar{\phi}\sin\bar{\psi} \\ \chi\cos\bar{\psi} - \xi\cos\bar{\phi}\sin\bar{\psi} \end{pmatrix}_{j,k,l} \tag{A2.6}$$

The inverse Lorentz transformation gives the two rotated momenta in the laboratory system:

$$\vec{p}'_{1,2} = \begin{pmatrix} \gamma_t(\pm\bar{p}'_j + v_t\bar{p}/\bar{v}) \\ \pm\bar{p}'_k \\ \pm\bar{p}'_l \end{pmatrix}_{j,k,l} = \pm\frac{p}{2} \begin{pmatrix} \gamma(\xi\cos\bar{\psi} + \chi\cos\bar{\phi}\sin\bar{\psi}) \mp 2 \\ \rho\sin\bar{\phi}\sin\bar{\psi} \\ \chi\cos\bar{\psi} - \xi\cos\bar{\phi}\sin\bar{\psi} \end{pmatrix}_{j,k,l} \tag{A2.7}$$

or

$$\vec{p}'_{1,2} - \vec{p}_{1,2} = \pm\frac{p}{2} \begin{pmatrix} \gamma(\xi\cos\bar{\psi} + \chi\cos\bar{\phi}\sin\bar{\psi}) \\ \rho\sin\bar{\phi}\sin\bar{\psi} \\ \chi(\cos\bar{\psi} - 1) - \xi\cos\bar{\phi}\sin\bar{\psi} \end{pmatrix}_{j,k,l} \tag{A2.8}$$

since $\bar{p}' = \bar{p} = \rho\, p/2$ and $\bar{v} = v\gamma\rho/2$. In order to represent the change of the momenta in the initial {s,x,z} system we have to multiply it by the matrix

$$M_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta/\chi & \theta/\chi \\ 0 & -\theta/\chi & \zeta/\chi \end{pmatrix} \tag{A2.9}$$

and get Eq.(19). Here we have used the same approximations as before. The momenta $\vec{p}'_{1,2}$ cannot be transformed by the matrix M_i, since they have components of different orders of magnitude so that the approximations of M_i are not valid for this transformation, whereas the approximations are valid for the transformation of the change of the momenta with components of equal orders of magnitude.

Appendix A3

By substituting

$$\vec r_{\beta 1,2} = \vec r_{\beta o} \pm \tfrac{1}{2}\vec w\,, \qquad \Delta p_{1,2} = \Delta p_o \pm \tfrac{1}{2} p\gamma\xi$$

with

$$\vec w = \begin{pmatrix} -\gamma\xi D_x \\ \theta - \gamma\xi(D'_x - RD_z) \\ -\gamma\xi D_z \\ \zeta - \gamma\xi(D'_z + RD_x) \end{pmatrix} = \begin{pmatrix} 0 \\ \theta \\ 0 \\ \zeta \end{pmatrix} - \gamma\xi\vec D$$

and by integrating with respect to Δs_1, $x_{\beta o}$, $z_{\beta o}$, Δp_o, $x'_{\beta o}$ and $z'_{\beta o}$ (the Jacobian of the transformation is γp) one obtains from Eqs.(23) and (30):

$$\frac{d\langle\epsilon_k\rangle}{dt} = \frac{r_p^2 c^4 NK}{32\gamma\sigma_s\sigma_p}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\Big(-\frac{\gamma^2\xi^2}{4\sigma_p^2} - \frac{|\vec w S\vec e_1|^2}{4\epsilon_{1b}} - \frac{|\vec w S\vec e_2|^2}{4\epsilon_{2b}}\Big)\Big(\ln\Big(\frac{2\bar d\bar v^2}{r_p c^2}\Big) - \frac{1}{2}\Big)$$
$$* \Big(\frac{1}{\chi^2}\big|\gamma\chi^2\vec D S\vec e_k - \xi\theta e_{k,1} - \xi\zeta e_{k,3}\big|^2 + \frac{\rho^2}{\chi^2}\big|\zeta e_{k,1} - \theta e_{k,3}\big|^2 - 2\,|\vec w S\vec e_k|^2\Big)\frac{d\xi\, d\theta\, d\zeta}{\bar v^3} \qquad (A3.1)$$

Substituting $\xi = \sqrt{\rho}\cos\mu$, $\theta = \sqrt{\rho}\cos\nu\sin\mu$, $\zeta = \sqrt{\rho}\sin\nu\sin\mu$ and using Eq.(A2.5) yields

$$\frac{d\langle\epsilon_k\rangle}{dt} = \frac{r_p^2 c^4 NK}{8\gamma^4 v^3\sigma_s\sigma_p}\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi}\exp(-\tfrac{1}{4}g\rho)\ln\Big(\frac{\bar d\gamma^2 v^2\rho}{2r_p c^2\sqrt{e}}\Big)h_k\sin\mu\, d\nu\, d\mu\, d\rho \qquad (A3.2)$$

In a similar way one obtains

$$\frac{d\langle H\rangle}{dt} = \frac{r_p^2 c^4 NK}{8\gamma^2 v^3\sigma_s\sigma_p}\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi}\exp(-\tfrac{1}{4}g\rho)\ln\Big(\frac{\bar d\gamma^2 v^2\rho}{2r_p c^2\sqrt{e}}\Big)h_s\sin\mu\, d\nu\, d\mu\, d\rho \qquad (A3.3)$$

With

$$\int_0^{\infty}\exp(-Cx)\ln(x)\,dx = -\frac{1}{C}(\ln(C) + 0.5772\ldots)$$

one finally obtains Eq.(38). The functions h_k and g are given by Eqs.(39) and (41) but can also be written in the form:

$$h_k = \gamma^2|\vec D S\vec e_k|^2(1 - 3\cos^2\mu) + |e_{k,1}^2|(1 - 3\cos^2\nu\sin^\mu) + |e_{k,3}^2|(1 - 3\sin^2\nu\sin^2\mu)$$
$$- 6Re\{\gamma\vec D S\vec e_1(e_{k,1}^*\cos\nu + e_{k,3}^*\sin\nu)\cos\mu\sin\mu + e_{k,1}e_{k,3}^*\cos\nu\sin\nu\sin^2\mu\} \qquad (A3.4)$$

$$g = \Big(\frac{1}{\sigma_p^2} + \frac{|\vec D S\vec e_1|^2}{\epsilon_{1b}} + \frac{|\vec D S\vec e_2|^2}{\epsilon_{2b}}\Big)\gamma^2\cos^2\mu + \Big(\Big(\frac{|e_{1,1}|^2}{\epsilon_{1b}} + \frac{|e_{2,1}|^2}{\epsilon_{2b}}\Big)\cos^2\nu$$
$$+ \Big(\frac{|e_{1,3}|^2}{\epsilon_{1b}} + \frac{|e_{2,3}|^2}{\epsilon_{2b}}\Big)\sin^2\nu\Big)\sin^2\mu + 2\,Re\Big\{\gamma\Big(\frac{(\vec D S\vec e_1)(e_{1,1}^*\cos\nu + e_{1,3}^*\sin\nu)}{\epsilon_{1b}}$$
$$+ \frac{(\vec D S\vec e_2)(e_{2,1}^*\cos\nu + e_{2,3}^*\sin\nu)}{\epsilon_{2b}}\Big)\cos\mu + \Big(\frac{e_{1,1}e_{1,3}^*}{\epsilon_{1b}} + \frac{e_{2,1}e_{2,3}^*}{\epsilon_{2b}}\Big)\cos\nu\sin\nu\sin\mu\Big\}\sin\mu \qquad (A3.5)$$

Appendix A4

$g = g(\nu)$ can be written in the form:

$$g(\nu) = a\,(1 + r_1(\nu))\,(1 + r_2(\nu)) \qquad (A4.1)$$

with

$$r_{1,2}(\nu) = b_{1,2}\cos\nu + c_{1,2}\sin\nu \qquad (A4.2)$$

where $b_{1,2}$ and $c_{1,2}$ (see chapter 5) are found by comparing Eqs.(A4.1) and (42), and a is determined by the cubic equation

$$\Big(a - (R_g + S_g)/3\Big)^3 + 3\,p\Big(a - (R_g + S_g)/3\Big) + 2\,q = 0 \qquad (A4.3)$$

The function h_k can be decomposed as

$$h(\nu) = \bar h(\nu) + L_0 g(\nu) + M_0 g'(\nu) \qquad (A4.4)$$

with

$$\bar h(\nu) = \bar A_h + \bar B_h\cos\nu + \bar C_h\sin\nu$$

and

$$L_0 = \frac{F_h F_g + T_h T_g}{F_g^2 + T_g^2}, \qquad 2M_0 = \frac{-F_h T_g + T_h F_g}{F_g^2 + T_g^2}$$

where as in the whole of Appendix 4 we omitt the index k.

For $F_g^2 + T_g^2 \neq 0$ one obtains

$$\int_0^{2\pi}\frac{h}{g}\ln(g)\,d\nu = \int_0^{2\pi}\Big(L_0 + \frac{\bar h}{g}\Big)\ln(g)\,d\nu$$
$$= \int_0^{2\pi}\Big(L_0 + \frac{\bar h}{a(1 + r_1)(1 + r_2)}\Big)\Big(\ln(a) + \int_0^1\Big(\frac{r_1}{1 + \lambda r_1} + \frac{r_2}{1 + \lambda r_2}\Big)d\lambda\Big)d\nu$$
$$= 2\pi\Big(L_0\ln(a) + \frac{\ln(a)}{a}\Big(\frac{A_1}{D_1} + \frac{A_2}{D_2}\Big) + \int_0^1 I(\lambda)\,d\lambda\Big) \qquad (A4.5)$$

with $D_{1\lambda,2\lambda}^2 = 1 - \lambda^2 d_{1,2}^2$ and

$$I(\lambda) = \frac{L_0}{\lambda}\Big(2 - \frac{1}{D_{1\lambda}} - \frac{1}{D_{2\lambda}}\Big)$$
$$+ \frac{1}{a}\Big(\frac{2A_1}{D_1} + \frac{2A_2}{D_2} - \frac{A_1(1,\lambda)}{D_1} - \frac{A_2(1,\lambda)}{D_{2\lambda}} - \frac{A_1(\lambda,1)}{D_{1\lambda}} - \frac{A_2(\lambda,1)}{D_2}\Big)\frac{1}{\lambda - 1} \qquad (A4.6)$$

In Eq.(A4.5) we have used the relation:

$$\int_0^{2\pi}\frac{\bar h\, d\nu}{(1 + r_1)(1 + r_2)} = \int_0^{2\pi}\frac{A_0\big(r_1'(1 + r_2) - r_2'(1 + r_1)\big) + A_2(1 + r_1) + A_1(1 + r_2)}{(1 + r_1)(1 + r_2)}d\nu$$
$$= \int_0^{2\pi}\Big(\frac{A_1}{1 + r_1} + \frac{A_2}{1 + r_2}\Big)d\nu = \frac{2\pi A_1}{D_1} + \frac{2\pi A_2}{D_2} \qquad (A4.7)$$

The terms with A_0 vanish. The other quantities are given by

$$A_{1,2} = A_{1,2}(1,1) = \frac{1}{N_0}\Big(\bar{A}_h(d_{1,2}^2 - t_0) \mp \bar{B}_h(b_0 - c_{1,2}t_1) \mp \bar{C}_h(c_0 + b_{1,2}t_1)\Big)$$

$$\begin{aligned}
A_1(\lambda,1) &= \frac{1}{N_1}\Big(\bar{B}_h b_2 + \bar{C}_h c_2 - \lambda\big(\bar{A}_h t_0 + \bar{B}_h b_1 + \bar{C}_h c_1\big) + \lambda^2\big(\bar{A}_h d_1^2 + \bar{B}_h c_1 t_1 - \bar{C}_h b_1 t_1\big)\Big)\\
A_1(1,\lambda) &= \frac{1}{N_2}\Big(\bar{A}_h d_1^2 - \bar{B}_h b_1 - \bar{C}_h c_1 - \lambda\big(\bar{A}_h t_0 - \bar{B}_h(b_2 + c_1 t_1) - \bar{C}_h(c_2 - b_1 t_1)\big)\Big)\\
A_2(\lambda,1) &= \frac{1}{N_1}\Big(\bar{A}_h d_2^2 - \bar{B}_h b_2 - \bar{C}_h c_2 - \lambda\big(\bar{A}_h t_0 - \bar{B}_h(b_1 - c_2 t_1) - \bar{C}_h(c_1 + b_2 t_1)\big)\Big)\\
A_2(1,\lambda) &= \frac{1}{N_2}\Big(\bar{B}_h b_1 + \bar{C}_h c_1 - \lambda\big(\bar{A}_h t_0 + \bar{B}_h b_2 + \bar{C}_h c_2\big) + \lambda^2\big(\bar{A}_h d_2^2 - \bar{B}_h c_2 t_1 + \bar{C}_h b_2 t_1\big)\Big)
\end{aligned}$$

$$N_{1,2} = \lambda^2 M_{1,2} - 2\lambda t_0 + d_{2,1}^2 = M_{1,2}(\lambda - \lambda_{1,2})(\lambda - \lambda_{1,2}^*)$$

$$M_{1,2} = d_{1,2}^2 - t_1^2 = d_{1,2}^2 D_{2,1}^2 + t_0^2 \geq 0$$

$$\lambda_1 = \Big(t_0 \pm i t_1 D_2\Big)/M_1\,, \qquad \lambda_2 = \Big(t_0 \pm i t_1 D_1\Big)/M_2$$

with $|\lambda_{1,2}^2| = d_{2,1}^2/M_{1,2}$ and $|\lambda - \lambda_{1,2}|^2 = N_{1,2}/M_{1,2}$. The integration of $I(\lambda)$ (Eq.(A4.6)) can be performed by using the following integrals:

$$\begin{aligned}
\int \frac{A_1(1,\lambda)}{\lambda - 1}d\lambda &= \int \frac{U_1 + \lambda V_1}{N_2(\lambda - 1)}d\lambda\\
&= A_1 \ln\Big(\frac{1-\lambda}{\sqrt{N_2}}\Big) + \frac{A_1 u_1 D_1 - U_1}{t_1 D_1}\arctan\Big(\frac{\lambda M_2 - t_0}{t_1 D_1}\Big)
\end{aligned} \qquad (A4.8)$$

and

$$\begin{aligned}
\int \frac{A_1(\lambda,1)d\lambda}{(\lambda - 1)D_{1\lambda}} &= \int \frac{\bar{U}_1 + \lambda\bar{V}_1 + \lambda^2\bar{W}_1}{N_1(\lambda - 1)D_{1\lambda}}d\lambda\\
&= \frac{A_1}{D_1}\ln\Big(\frac{(1-\lambda)d_1}{1 - \lambda d_1^2 + D_1 D_{1\lambda}}\Big)\\
&\quad + 2Re\left\{\frac{M_1\sqrt{1 - \lambda_1^{*2}d_1^2}(\bar{U}_1 + \lambda_1\bar{V}_1 + \lambda_1^2\bar{W}_1)(1 - \lambda_1^*)}{N_0(d_1^2 - t_0^2)(\lambda_1^* - \lambda_1)}\right.\\
&\qquad\qquad \left. * \ln\Big(\frac{(\lambda_1 - \lambda)d_1}{1 - \lambda\lambda_1 d_1^2 + D_{1\lambda}\sqrt{1 - \lambda_1^2 d_1^2}}\Big)\right\}\\
&= \frac{A_1}{D_1}\ln\Big(\frac{1-\lambda}{1 - \lambda d_1^2 + D_1 D_{1\lambda}}\Big)\\
&\quad + \frac{A_1(M_1 - t_0 d_2^2) + \bar{U}_1 t_0 - \bar{W}_1}{D_2(d_1^2 - t_0^2)}\ln\big((1 - \lambda t_0 + D_{1\lambda}D_2)/\sqrt{N_1}\big)\\
&\quad + \frac{A_1(M_1 t_0 - d_1^2 d_2^2) + \bar{U}_1 d_1^2 - \bar{W}_1 t_0}{t_1(d_1^2 - t_0^2)}\arctan\Big(\frac{t_1 D_{1\lambda}}{\lambda d_1^2 - t_0}\Big)
\end{aligned} \qquad (A4.10)$$

The integral of $I(\lambda)$ has then the solution

$$\begin{aligned}
\int I(\lambda)\,d\lambda &= L_0 \ln\big((1 + D_{1\lambda})(1 + D_{2\lambda})\big)\\
&\quad + \frac{A_1}{aD_1}\ln\big((1 - \lambda d_1^2 + D_{1\lambda}D_1)\sqrt{N_2}\big) - \frac{A_1(d_1^2 - t_0) - U_1}{at_1 D_1^2}\arctan\Big(\frac{\lambda M_2 - t_0}{t_1 D_1}\Big)\\
&\quad + \frac{A_2}{aD_2}\ln\big((1 - \lambda d_2^2 + D_{2\lambda}D_2)\sqrt{N_1}\big) - \frac{A_2(d_2^2 - t_0) - U_2}{at_1 D_2^2}\arctan\Big(\frac{\lambda M_1 - t_0}{t_1 D_2}\Big)\\
&\quad - \frac{A_1(M_1 - t_0 d_2^2) + \bar{U}_1 t_0 - \bar{W}_1}{aD_2(d_1^2 - t_0^2)}\ln\big((1 - \lambda t_0 + D_{1\lambda}D_2)/\sqrt{N_1}\big)\\
&\quad - \frac{A_1(M_1 t_0 - d_1^2 d_2^2) + \bar{U}_1 d_1^2 - \bar{W}_1 t_0}{at_1(d_1^2 - t_0^2)}\arctan\Big(\frac{t_1 D_{1\lambda}}{\lambda d_1^2 - t_0}\Big)\\
&\quad - \frac{A_2(M_2 - t_0 d_1^2) + \bar{U}_2 t_0 - \bar{W}_2}{aD_1(d_2^2 - t_0^2)}\ln\big((1 - \lambda t_0 + D_{2\lambda}D_1)/\sqrt{N_2}\big)\\
&\quad - \frac{A_2(M_2 t_0 - d_1^2 d_2^2) + \bar{U}_2 d_2^2 - \bar{W}_2 t_0}{at_1(d_2^2 - t_0^2)}\arctan\Big(\frac{t_1 D_{2\lambda}}{\lambda d_2^2 - t_0}\Big)
\end{aligned} \qquad (A4.11)$$

which can be verified by differentiating with respect to λ. With $b_{1,2}t_1 = \mp(c_{1,2}t_0 - c_{2,1}d_{1,2}^2)$ and $c_{1,2}t_1 = \pm(b_{1,2}t_0 - b_{2,1}d_{1,2}^2)$ one obtains

$$\begin{aligned}
A_1 &= \big(A_2(M_2 - t_0 d_1^2) + \bar{U}_2 t_0 - \bar{W}_2)\big)/(t_0^2 - d_2^2)\\
&= \Big(\bar{A}_h(d_1^2 - t_0) - \bar{B}_h(b_0 - c_2 t_1) - \bar{C}_h(c_0 + b_2 t_1)\Big)/N_0\\
A_2 &= \big(A_1(M_1 - t_0 d_2^2) + \bar{U}_1 t_0 - \bar{W}_1)\big)/(t_0^2 - d_1^2)\\
&= \Big(\bar{A}_h(d_2^2 - t_0) + \bar{B}_h(b_0 - c_2 t_1) + \bar{C}_h(c_0 + b_2 t_1)\Big)/N_0
\end{aligned}$$

$$\begin{aligned}
\frac{A_1(d_1^2 - t_0) - U_1}{t_1 D_1^2} &= \frac{A_2(d_2^2 - t_0) - U_2}{t_1 D_2^2} = \frac{A_1(M_1 t_0 - d_1^2 d_2^2) + \bar{U}_1 - \bar{W}_1 t_0}{t_1(d_1^2 - t_0^2)}\\
&= \frac{A_2(M_2 t_0 - d_1^2 d_2^2) + \bar{U}_2 - \bar{W}_2 t_0}{t_1(d_2^2 - t_0^2)} = -\frac{1}{N_0}(\bar{A}_h t_1 + \bar{B}_h c_0 - \bar{C}_h b_0)
\end{aligned}$$

and Eq.(A4.11) simplifies to

$$\begin{aligned}
\int I(\lambda)\,d\lambda &= L_0 \ln\big((1 + D_{1\lambda})(1 + D_{2\lambda})\big)\\
&\quad + \frac{A_1}{aD_1}\ln\Big((1 - \lambda d_1^2 + D_{1\lambda}D_1)(1 - \lambda t_0 + D_{2\lambda}D_1)\Big)\\
&\quad + \frac{A_2}{aD_2}\ln\Big((1 - \lambda d_2^2 + D_{2\lambda}D_2)(1 - \lambda t_0 + D_{1\lambda}D_2)\Big)\\
&\quad + \frac{1}{aN_0}(\bar{A}_h t_1 + \bar{B}_h c_0 - \bar{C}_h b_0)\Big(\arctan\Big(\frac{\lambda M_2 - t_0}{t_1 D_1}\Big) + \arctan\Big(\frac{\lambda M_1 - t_0}{t_1 D_2}\Big)\\
&\qquad + \arctan\Big(\frac{t_1 D_{1\lambda}}{\lambda d_1^2 - t_0}\Big) + \arctan\Big(\frac{t_1 D_{2\lambda}}{\lambda d_2^2 - t_0}\Big)\Big)
\end{aligned} \qquad (A4.12)$$