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Abstract

Some misleading statements in a recent paper by Kleinert are pointed out and various aspects of the path integral formulation on curved manifolds in general and on the D-dimensional sphere in particular are clarified.

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In a recent paper Kleinert [1] claims to have been the first to solve unambiguously and successfully the path integral on the D-dimensional sphere S^{D-1} and on related group spaces (e.g.symmetric top). We do not contest that his results are correct, but in his paper he makes the following statements, namely:

i) that the DeWitt approach [2] for formulating path integrals on curved manifolds is of no use ("little help") for the path integration on spheres, and

ii) that there has been no successful attempts of actually calculating the path integral for the quantum motion on the D-dimensional sphere.

These two arguments are definitely wrong.
Let us discuss this in some detail.

i) In the classical work of DeWitt [2], generalizing Feynman's original work [3], the first consistent formulation of path integrals on curved manifolds has been given. Starting from the classical Lagrangian of the problem in question

$$\mathcal{L}_{Cl}(q,\dot q) = \frac{m}{2} g_{ab}(q)\dot q^a \dot q^b - V(q), \tag{1}$$

or alternatively from the quantum Hamiltonian

$$H = -\frac{\hbar^2}{2m} g^{-\frac{1}{2}}\partial_a g^{ab} g^{\frac{1}{2}}\partial_b + V(q) \tag{2}$$

with g_{ab} and g^{ab} the metric tensor and its inverse, $g = \det(g_{ab})$, the path integral is found to have the form ($T = t'' - t'$):

$$\begin{aligned} K(q'',q';T) &= \int_{DeW} \sqrt{g}\,\mathcal{D}q(t) \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2} g_{ab}(q)\dot q^a \dot q^b - V(q) + \frac{\hbar^2 \tilde R(q)}{6m}\right]dt\right\} \\ &:= \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon\hbar}\right)^{\frac{ND}{2}} \prod_{j=1}^{N-1}\int\sqrt{g(q^{(j)})}\,dq^{(j)} \\ &\quad\times \exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon} g_{ab}(q^{(j-1)})\Delta q^{a,(j)}\Delta q^{b,(j)} - \epsilon V(q^{(j)}) + \epsilon\frac{\hbar^2}{6m}\tilde R(q^{(j)})\right]\right\} \end{aligned} \tag{3}$$

[$\tilde R = g^{ab}(\Gamma^c_{ab,c} - \Gamma^d_{db,a} + \Gamma^d_{ab}\Gamma^c_{cd} - \Gamma^d_{cb}\Gamma^c_{ad})$: scalar curvature; Γ^a_{bc}: Christoffel symbols]. Here $q^{(j)} = q(t' + j\cdot\epsilon)$, $\epsilon = T/N$, $\Delta q^{(j)} = q^{(j)} - q^{(j-1)}$. The main result of the path integral quantization is therefore that in the action in the exponent of the path integral appears an additional **quantum potential** modifying the classical Lagrangian to an effective one:

$$\mathcal{L}_{eff}(q,\dot q) = \mathcal{L}_{Cl}(q,\dot q) - \Delta V_{DeW}(q) \tag{4}$$

with $\Delta V_{DeW} = -\hbar^2\tilde R/6m$. ΔV_{DeW} is essential in deriving the time-dependent Schrödinger equation from Eq.(3) by means of the time evolution-equation. The specific form of ΔV_{DeW} depends crucially on the lattice formulation of the path integral. Because $\tilde R$ is often a constant, i.e.$\tilde R(q^{(j)}) \equiv \tilde R$ (for all j), one could get the impression that the only effect of

ΔV_{DeW} is to add some constant to the Schrödinger energy, proportional to the curvature. However, changing the lattice formulation of Eq.(3) will change ΔV_{DeW} to another ΔV and thus spoils in general the property that ΔV is just a constant (in contrast to the line of reasoning in Ref.[1]). One well-known lattice formulation is the midpoint (mp) prescription of the path integral. It is based on the Weyl-ordering of position and momentum operators in the quantum Hamiltonian. Introducing

$$p_a = \frac{\hbar}{i}\left(\frac{\partial}{\partial q^a} + \frac{1}{2}\Gamma_a\right), \qquad \Gamma_a = \frac{\partial}{\partial q^a}\ln\sqrt{g}, \tag{5}$$

H is rewritten as

$$H = \frac{1}{8m}\left(g^{ab}p_a p_b + 2p_a g^{ab} p_b + p_a p_b g^{ab}\right) + V(q) + \Delta V_{Weyl}(q) \tag{6}$$

with the well-defined quantum potential

$$\Delta V_{Weyl} = \frac{\hbar^2}{8m}\left(g^{ab}\Gamma^d_{ac}\Gamma^c_{bd} - \tilde{R}\right) = \frac{\hbar^2}{8m}\left[g^{ab}\Gamma_a\Gamma_b + 2(g^{ab}\Gamma_a)_{,b} + g^{ab}{}_{,ab}\right]. \tag{7}$$

The path integral is given by [4,5]

$$\begin{aligned} K(q'',q';T) &= \left[g(q')g(q'')\right]^{-\frac{1}{4}} \int_{mp} \sqrt{g}\,\mathcal{D}q(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\mathcal{L}_{Cl}(q,\dot{q}) - \Delta V_{Weyl}(q)\right]dt\right\} \\ &:= \left[g(q')g(q'')\right]^{-\frac{1}{4}} \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon\hbar}\right)^{\frac{ND}{2}} \prod_{j=1}^{N-1}\int dq^{(j)} \\ &\quad\times \prod_{j=1}^{N}\sqrt{g(\bar{q}^{(j)})}\exp\left\{\frac{i}{\hbar}\left[\frac{m}{2\epsilon}g_{ab}(\bar{q}^{(j)})\Delta q^{a,(j)}\Delta q^{b,(j)} - \epsilon V(\bar{q}^{(j)}) - \epsilon\Delta V_{Weyl}(\bar{q}^{(j)})\right]\right\} \end{aligned} \tag{8}$$

$[\bar{q}^{(j)} = \frac{1}{2}(q^{(j)} + q^{(j-1)}]$. ΔV_{Weyl} is in general not a constant at all. There has been a lot of justification [4,5] and modification [6] of Eqs.(3) and (8). In all these references there are also other lattice definitions discussed, like prepoint, postpoint or something in between. Considerations connecting theses lattices with the appropriate quantum potentials can be found e.g. in [7] (see also appendix A of [5]). The Weyl-ordering and midpoint prescription, however, expresses most clearly the symmetries of the Lagrangian [8].

The purpose of this section was to emphasize that there exists a well-established approach to the path integral formulation on curved manifolds. The various quantum potentials may or may not be constant. From this point of view, the claim in Ref.[1] seems rather surprising and puzzling. Of course, the path integral formulations Eqs.(3) and (8) can be applied in a straightforward manner to the problem of the quantum motion on the sphere.

ii) There is a good deal of discussion of the quantum motion on spheres and on group spaces [9], also [5,10-12]. However, concerning the path integral formulation on spheres, the confusion arises because several authors started in D-dimensional Euclidean space and then introduced a δ-function in the measure in order to take into account that the quantum motion on the sphere is constrained to the surface of the sphere. But this causes a lot of problems, due to the fact that the quantization of a system with constraints is in general far from being trivial [13]. This simple procedure gives in the present case the wrong answer, since the energy spectrum comes out wrong and the actual incorporation of the correct measure via δ-functions is ad hoc (and a posteriori).

Let us discuss the path integral formulation of the quantum motion on the D-dimensional sphere along the lines of Eqs.(3) and (8). A detailed discussion was already given in Ref.[5]. The classical Lagrangian on the D-dimensional sphere with radius R reads in D-dimensional polar coordinates [14]

$$\mathcal{L}_{Cl}(\{\theta\},\{\dot{\theta}\}) = \frac{mR^2}{2}\left[\dot{\theta}_1^2 + \sin^2\theta_1\dot{\theta}_2^2 + \cdots + (\sin^2\theta_1\ldots\sin^2\theta_{D-2})\dot{\phi}^2\right], \tag{9}$$

the metric tensor is given by

$$(g_{ab}) = \mathrm{diag}(1, \sin^2\theta_1, \ldots, \sin^2\theta_1\ldots\sin^2\theta_{D-2}). \tag{10}$$

Therefore

$$\Delta V_{Weyl}(\{\theta\}) = -\frac{\hbar^2}{8mR^2}\left[(D-2)^2 + \frac{1}{\sin^2\theta_1} + \cdots + \frac{1}{\sin^2\theta_1\ldots\sin^2\theta_{D-2}}\right]. \tag{11}$$

Consequently

$$\begin{aligned} K^{S^{D-1}}(\{\theta''\},\{\theta'\};T) &= \left[g(\{\theta'\})g(\{\theta''\})\right]^{-\frac{1}{4}} \\ &\quad\times\int_{mp}\sqrt{g(\{\theta\})}\,\mathcal{D}\{\theta\}(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\mathcal{L}_{Cl}(\{\theta\},\{\dot{\theta}\}) - \Delta V_{Weyl}(\{\theta\})\right]dt\right\}. \end{aligned} \tag{12}$$

The path integral is, as it stands, not tractable. Also the singularities in ΔV_{Weyl} are problematic and must be regularized. This can be done e.g. by the functional measure introduced in [10] (and rediscovered in [15]). Another regularization scheme, used in [5], exploits the addition theorem in D dimensions:

$$\begin{aligned} \cos\psi^{(1,2)} &= \cos\theta_1^{(1)}\cos\theta_1^{(2)} \\ &\quad+\sum_{m=1}^{D-2}\cos\theta_{m+1}^{(1)}\cos\theta_{m+1}^{(2)}\prod_{n=1}^{m}\sin\theta_n^{(1)}\sin\theta_n^{(2)} + \sum_{n=1}^{D-1}\sin\theta_n^{(1)}\sin\theta_n^{(2)}, \end{aligned} \tag{13}$$

where $\cos\psi^{(1,2)} = \Omega^{(1)}\cdot\Omega^{(2)}$ and Ω denotes a unit vector on the S^{D-1}-sphere. This regularization produces again quantum potentials cancelling the singular terms in (12) yielding

$$K^{S^{D-1}}(\{\theta''\},\{\theta'\};T) = \int\mathcal{D}\Omega(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}R^2\dot{\Omega}^2 + \frac{(D-1)(D-3)}{8mR^2}\right]dt\right\}. \tag{14}$$

Using now the expansion

$$e^{z\Omega^{(1)}\cdot\Omega^{(2)}} = e^{z\cos\psi^{(1,2)}} = 2\pi\left(\frac{2\pi}{z}\right)^{\frac{D-2}{2}}\sum_{l=0}^{\infty}\sum_{\mu=1}^{M}S_l^{\mu}(\Omega^{(2)})S_l^{\mu*}(\Omega^{(1)})I_{l+\frac{D-2}{2}}(z), \tag{15}$$

where $M = (2l+D-2)(l+D-3)!/l!(D-2)!$ and S_l^μ are the hyperspherical surface harmonics on S^{D-1} [14], the path integral (14) can be explicitly evaluated yielding

$$K^{S^{D-1}}(\{\theta''\},\{\theta'\};T) = \sum_{l=0}^{\infty}\sum_{\mu=1}^{M} e^{-iTE_l/\hbar} S_l^\mu(\Omega'') S_l^{\mu\,*}(\Omega') \tag{16}$$

with the correct energy spectrum

$$E_l = \frac{\hbar^2}{2mR^2} l(l+D-2). \tag{17}$$

Of course, there is no need of any additional assumptions, constants, etc.

The same line of reasoning holds for the path integral of the symmetric top (formulated in terms of Euler angles) [11] and for the D-dimensional pseudosphere Λ^{D-1} [12].

We summarize our criticism of Kleinert [1]:

i) In contrast to the claim of Kleinert, the DeWitt approach to path integrals on curved manifolds is, of course, of valuable help.

ii) Using this well-developed formulation it is straightforward to evaluate the path integral for the quantum motion on the D-dimensional sphere, done already in an unambiguous and consistent way some years ago.

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