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Abstract

In the following report we investigate the stochastic particle motion in proton storage rings due to cavity noise in the framework of a Fokker-Planck treatment. The motion is decribed by using the canonical variables $\tilde{x}$, $\tilde{p}_x$, $\tilde{z}$, $\tilde{p}_z$, $\tilde{\sigma}$, $\tilde{p}_\sigma = \Delta E/E_0$ which are derived from the variables x, p_x, z, p_z, $\sigma = s - v_0 \cdot t$, $p_\sigma = \Delta E/E_0$ of the fully six-dimensional canonical formalism by introducing the dispersion formalism via a canonical transformation. Thus synchrotron- and betatron-oscillations are treated simultaneously taking into account localised cavities and all kinds of coupling (synchro-betatron coupling and coupling of the betatron oscillations by skew quadrupoles and solenoids). For the unperturbed system we assume that the dispersion vanishes in the cavities. Then in linear order the synchrotron oscillation is decoupled from the betatron motion. In order to set up the Fokker-Planck equation, action-angle variables of the linear coupled motion are introduced. The Fokker-Planck equation is solved for the case of phase noise and amplitude noise resulting from the cavities. The equations thus obtained are valid for arbitrary velocity of the protons (below and above transition energy).

Contents

1 Introduction

In this paper we study the influence of cavity noise on particle motion in proton storage rings. Investigations on this subject with respect to synchroton motion have already been made by several authors [1,2]. The aim of this report is to generalize these considerations by including the betatron oscillations, i.e. we investigate the combined system of longitudinal and transverse motion by a simultaneous treatment of synchrotron and betatron oscillations, taking into account all kinds of coupling (synchro-betatron coupling and coupling of the betatron oscillations by skew quadrupoles and solenoids).

The concept to be used in this report is well known from radiation theory [3].

In detail, the considerations are organized as follows:

The starting point of our investigations is the fully coupled 6-dimensional description of the particle motion with the coordinates $x,\ p_x,\ z,\ p_z,\ \sigma,\ p_\sigma = \Delta E/E_0$ which allows to handle the external magnetic forces in a consistent canonical manner and which includes consistently and canonically the synchrotron oscillations in the electric fields of the accelerating cavities. This description is summarized in chapter 2.1 and leads to a derivation of the stochastic equations of motion, taking into account the influence of cavity noise (phase noise as well as amplitude noise).

In chapter 3 the dispersion function is introduced by defining new variables $\tilde{x},\ \tilde{p}_x,\ \tilde{z},\ \tilde{p}_z,\ \tilde{\sigma},\ \tilde{p}_\sigma$. Since this is done via a canonical transformation, the symplectic structure of the equations of motion is completely preserved. In this formulation the coupling between betatron and synchrotron motion only appears in the cavities.

The linear equations of motion, neglecting the synchro-betatron coupling as well as stochastic terms induced by the cavity noise, (i.e. the unperturbed problem) are studied in chapter 4 by defining the 6-dimensional transfer matrix and by investigating the eigenvalue spectrum of the revolution matrix. Furthermore, action-angle variables for the coupled orbital motion are introduced.

The perturbed problem, taking into account the cavity noise and small coupling terms of synchro-betatron motion is investigated in chapter 5.

We are then prepared to derive the stochastic equations of motion in terms of the orbital action-angle variables (chapter 6) which are the basis for a Fokker-Planck treatment of stochastic particle motion.

The Fokker-Planck equation for orbital motion under the influence of cavity noise (phase noise and amplitude noise) is written down in chapter 7 and a solution of this equation is presented in chapter 8 by a separated consideration of the phase noise and the amplitude noise.

The equations so derived are valid for arbitrary velocity of the protons (below and above transition energy).

A summary of the results is finally presented in chapter 9.

2 Equations of Motion

Our investigation of cavity noise in proton storage rings begins with the derivation of the equations of motion. We will use the same variables as those in Refs. [4,5] :

$$x,\ z,\ \sigma = s - v_0 \cdot t \text{ and } \eta = \Delta E/E_0$$

(v_0 =design speed= $c\beta_0$) by introducing as usual [15]:

a) the closed design orbit (a piecewise flat path of a particle with constant Energy E_0) which will in the following be described by the vector $\vec{r}_0(s)$ where s is the length along this ideal orbit;

b) an orthogonal coordinate system ("dreibein") accompanying the particles which travels along the design orbit and comprises [8]:

$$\begin{array}{rl} \text{the unit tangent vector} & \vec{e}_s(s) = \frac{d}{ds}\vec{r}_0(s) \equiv \vec{r}_0{}'(s) \; ; \\ \text{a unit vector} & \vec{e}_x(s) \quad \text{perpendicular to } \vec{e}_s \text{ in the horizontal plane} \\ \text{and the unit vector} & \vec{e}_z(s) = \vec{e}_s(s) \times \vec{e}_x(s) \; . \end{array}$$

The Serret-Fresnet formulae for the dreibein $(\vec{e}_s, \vec{e}_x, \vec{e}_z)$ read as:

$$\frac{d}{ds}\vec{e}_x(s) = +K_x(s) \cdot \vec{e}_s(s) \; ; \tag{2.1a}$$

$$\frac{d}{ds}\vec{e}_z(s) = +K_z(s) \cdot \vec{e}_s(s) \; ; \tag{2.1b}$$

$$\frac{d}{ds}\,\vec{e}_s(s) = -K_x(s) \cdot \vec{e}_x(s) - K_z(s) \cdot \vec{e}_z(s) \tag{2.1c}$$

with the assumption that

$$K_x(s) \cdot K_z(s) = 0$$

(piecewise no torsion) and where $K_x(s), K_z(s)$ designate the curvatures in the x-direction and in the z-direction respectively.

In this natural coordinate system an arbitrary orbit-vector $\vec{r}(s)$ can be written in the form

$$\vec{r}(x,z,s) = \vec{r}_0(s) + x(s) \cdot \vec{e}_x(s) + z(s) \cdot \vec{e}_z(s) \; . \tag{2.2}$$

Note that the sign of $K_x(s)$ and $K_z(s)$ is fixed by eqns. (2.1).

Thus x and z describe the amplitude of transverse motion (betatron oscillations), while $\sigma = s - v_0 \cdot t$ and $\eta = \Delta E/E_0$ describe the longitudinal (synchrotron) oscillation. The quantity σ defines the longitudinal separation of particles from the centre of the bunch and η describes the energy deviation of the particle.

Starting then from the Lagrangian

$$\mathcal{L} = -m_0c^2 \cdot \sqrt{1 - \frac{v^2}{c^2}} + \frac{e}{c} \cdot (\dot{\vec{r}} \cdot \vec{A}) - e \cdot \phi$$

for the motion of a relativistic charged particle with the orbit-vector $\vec{r}$ in an electromagnetic field and introducing the length s along the design orbit as the independent variable (instead of the time t), one can construct the Hamiltonian of the orbit motion by a succession of canonical transformations. Choosing a gauge with $\phi = 0$ (e.g. Coulomb gauge), one thus obtains [4,5] :

$$\mathcal{H}(x,p_x,z,p_z,\sigma,p_\sigma;s) = p_\sigma - \beta_0 \cdot \sqrt{(1+p_\sigma)^2 - (\frac{m_0c^2}{E_0})^2} \cdot [1 + K_x \cdot x + K_z \cdot z] \times$$

$$\left\{1 - \frac{(p_x - \beta_0 \cdot \frac{\epsilon}{E_0} A_x)^2 + (p_z - \beta_0 \cdot \frac{\epsilon}{E_0} A_z)^2}{\beta_0^2 \cdot \left[(1+p_\sigma)^2 - (\frac{m_0 c^2}{E_0})^2\right]}\right\}^{1/2}$$

$$- \quad [1 + K_x \cdot x + K_z \cdot z] \cdot \beta_0 \cdot \frac{\epsilon}{E_0} A_s \ . \tag{2.3}$$

The corresponding canonical equations read as :

$$\frac{d}{ds}\, x \;=\; +\frac{\partial \mathcal{H}}{\partial p_x}\;; \quad \frac{d}{ds}\, p_x = -\frac{\partial \mathcal{H}}{\partial x}\;; \tag{2.4a}$$

$$\frac{d}{ds}\, z \;=\; +\frac{\partial \mathcal{H}}{\partial p_z}\;; \quad \frac{d}{ds}\, p_z = -\frac{\partial \mathcal{H}}{\partial z}\;; \tag{2.4b}$$

$$\frac{d}{ds}\, \sigma \;=\; +\frac{\partial \mathcal{H}}{\partial p_\sigma}\;; \quad \frac{d}{ds}\, p_\sigma = -\frac{\partial \mathcal{H}}{\partial \sigma} \tag{2.4c}$$

with

$$p_\sigma \equiv \eta \tag{2.5}$$

or, using a matrix form:

$$\frac{d}{ds}\, \vec{y} = -\underline{S} \cdot \frac{\partial \mathcal{H}}{\partial \vec{y}} \tag{2.6}$$

with

$$\vec{y}^{\,T} = (x,\ p_x,\ z,\ p_z,\ \sigma,\ p_\sigma) \tag{2.7}$$

where the matrix $\underline{S}$ is given by

$$\underline{S} = \begin{pmatrix} \underline{S}_2 & \underline{0} & \underline{0} \\ \underline{0} & \underline{S}_2 & \underline{0} \\ \underline{0} & \underline{0} & \underline{S}_2 \end{pmatrix} \;; \quad \underline{S}_2 = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} . \tag{2.8}$$

Since $\mathcal{H}$ contains the transverse coordinates $x,\ p_x,\ z,\ p_z$ as well as the longitudinal coordinates $\sigma,\ p_\sigma$ we are thus able to handle synchrotron oscillations (longitudinal motion) and betatron oscillations (transverse motion) simultaneously.

In order to utilize this Hamiltonian, the electric field $\vec{\epsilon}$ and the magnetic field $\vec{B}$ or the corresponding vector potential,

$$\vec{A} = \vec{A}(x, z, s), \tag{2.9}$$

for the cavities and for commonly occurring types of accelerator magnets must be given. Once $\vec{A}$ is known the fields $\vec{\epsilon}$ and $\vec{B}$ may be found using the relations:

$$\vec{\epsilon} \;=\; -\text{grad}\ \phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\;; \tag{2.10a}$$

$$\vec{B} \;=\; \text{curl}\ \vec{A}\ . \tag{2.10b}$$

Expressed in the variables $x,\ z,\ s,\ \sigma$, eqns. (2.10) become (with $\phi = 0$):

$$\vec{\epsilon} = \beta_0 \cdot \frac{\partial}{\partial \sigma}\ \vec{A} \tag{2.11}$$

and

$$B_x = \frac{1}{(1+K_x\cdot x+K_z\cdot z)}\cdot\left\{\frac{\partial}{\partial z}\left[(1+K_x\cdot x+K_z\cdot z)\cdot A_s\right]-\frac{\partial}{\partial s}A_z\right\}\ ; \quad (2.12a)$$

$$B_z = \frac{1}{(1+K_x\cdot x+K_z\cdot z)}\cdot\left\{\frac{\partial}{\partial s}A_x-\frac{\partial}{\partial x}\left[(1+K_x\cdot x+K_z\cdot z)\cdot A_s\right]\right\}\ ; \quad (2.12b)$$

$$B_s = \frac{\partial}{\partial x}A_z-\frac{\partial}{\partial z}A_x\ . \quad (2.12c)$$

We assume that the ring consists of bending magnets, quadrupoles, skew quadrupoles, solenoids and cavities. Then the vector potential $\vec{A}$ can be written as (see Appendix A) :

$$\begin{aligned}\frac{e}{E_0}A_s &= -\frac{1}{2}\beta_0\cdot[1+K_x\cdot x+K_z\cdot z]+\frac{1}{2}g\cdot\beta_0\cdot(z^2-x^2)+N\cdot\beta_0\cdot xz\\ &\quad -\frac{1}{\beta_0}\cdot\frac{L}{2\pi\cdot h}\cdot\frac{eV(s)}{E_0}\cdot\cos\left[h\cdot\frac{2\pi}{L}\cdot\sigma+\varphi\right]\ ; \quad (2.13a)\\ \frac{e}{E_0}A_x &= -\beta_0\cdot H\cdot z\ ;\quad \frac{e}{E_0}A_z = +\beta_0\cdot H\cdot x \quad (2.13b)\end{aligned}$$

(h=harmonic number) with the following abbreviations:

$$\begin{aligned} g &= \frac{e}{p_0\cdot c}\cdot\left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \quad (2.14a)\\ N &= \frac{1}{2}\cdot\frac{e}{p_0\cdot c}\cdot\left(\frac{\partial B_x}{\partial x}-\frac{\partial B_z}{\partial z}\right)_{x=z=0}\ ; \quad (2.14b)\\ H &= \frac{1}{2}\cdot\frac{e}{p_0\cdot c}\cdot B_s(0,0,s)\ ; \quad (2.14c)\\ K_x &= +\frac{e}{p_0\cdot c}\cdot B_z(0,0,s)\ ;\quad K_z = -\frac{e}{p_0\cdot c}\cdot B_x(0,0,s)\ . \quad (2.14d)\end{aligned}$$

In detail, one has:

a)	$g \neq 0$;	$N = K_x = K_z = H = V = 0$:	quadrupole;
b)	$N \neq 0$;	$g = K_x = K_z = H = V = 0$:	skew quadrupole;
c)	$K_x^2 + K_z^2 \neq 0$;	$g = N = H = V = 0$:	bending magnet;
d)	$H \neq 0$;	$g = N = K_x = K_z = V = 0$:	solenoid;
e)	$V \neq 0$;	$g = K_x = K_z = N = H = 0$:	cavity.

Thus the Hamiltonian takes the form :

$$\begin{aligned}\mathcal{H}(x,p_x,z,p_z,\sigma,p_\sigma;s) &= p_\sigma-\beta_0\cdot\sqrt{(1+p_\sigma)^2-(\frac{m_0c^2}{E_0})^2}\cdot(1+K_x\cdot x+K_z\cdot z)\times\\ &\quad \left[1-\frac{[p_x+\beta_0^2H\cdot z]^2+[p_z-\beta_0^2H\cdot x]^2}{\beta_0^2\cdot\left[(1+p_\sigma)^2-(\frac{m_0c^2}{E_0})^2\right]}\right]^{1/2}\\ &\quad +\frac{1}{2}\beta_0^2\cdot(1+K_x\cdot x+K_z\cdot z)^2-\frac{1}{2}g\cdot(z^2-x^2)-N\cdot xz\\ &\quad +\frac{L}{2\pi\cdot h}\cdot\frac{eV(s)}{E_0}\cdot\cos\left[h\cdot\frac{2\pi}{L}\cdot\sigma+\varphi\right]\ . \quad (2.15)\end{aligned}$$

Now, since

$$
\begin{aligned}
|p_x + \beta_0^2 H \cdot z| &\ll 1 \ ; \\
|p_z - \beta_0^2 H \cdot x| &\ll 1
\end{aligned}
$$

the square root

$$
\left[1 - \frac{[p_x + \beta_0^2 H \cdot z]^2 + [p_z - \beta_0^2 H \cdot x]^2}{\beta_0^2 \cdot \left[(1+p_\sigma)^2 - (\frac{m_0 c^2}{E_0})^2\right]}\right]^{1/2}
$$

in (2.14) may be expanded in a series :

$$
\begin{aligned}
&\left[1 - \frac{[p_x + \beta_0^2 H \cdot z]^2 + [p_z - \beta_0^2 H \cdot x]^2}{\beta_0^2 \cdot \left[(1+p_\sigma)^2 - (\frac{m_0 c^2}{E_0})^2\right]}\right]^{1/2} = \\
&\qquad 1 - \frac{1}{2} \cdot \frac{[p_x + \beta_0^2 H \cdot z]^2 + [p_z - \beta_0^2 H \cdot x]^2}{\beta_0^4 \cdot (1+\hat{\eta})^2} + \ldots
\end{aligned} \tag{2.16}
$$

with $\hat{\eta}$ defined by:

$$
(1+\hat{\eta}) = \frac{1}{\beta_0}\sqrt{(1+\eta)^2 - (\frac{m_0 c^2}{E_0})^2} = \frac{1}{\beta_0} \cdot \frac{p \cdot c}{E_0} = \frac{p}{p_0} \tag{2.17}
$$

($p = m_0 \gamma v$) so that in practice the particle motion can be conveniently calculated to various orders of approximation. (The subscript "0" refers to the synchronous particle.)

In the following we shall use a series expansion of the Hamiltonian up to second order in the variables x, p_x, z, p_z, σ, p_σ. Then we obtain from (2.14) :

$$
\begin{aligned}
\mathcal{H} \ =\ & p_\sigma - [1 + K_x \cdot x + K_z \cdot z] \cdot \beta_0^2 \cdot f(p_\sigma) \\
& + \frac{1}{2\beta_0^2} \cdot \left\{[p_x + \beta_0^2 H \cdot z]^2 + [p_z - \beta_0^2 H \cdot x]^2\right\} \\
& + \frac{1}{2}\beta_0^2 \cdot \left\{[K_x^2 + g] \cdot x^2 + [K_z^2 - g] \cdot z^2 - 2N \cdot xz\right\} \\
& - \frac{1}{2}\sigma^2 \cdot \frac{eV(s)}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \cos\varphi - \sigma \cdot \frac{eV(s)}{E_0} \cdot \sin\varphi
\end{aligned} \tag{2.18}
$$

with

$$
p_\sigma \equiv \eta(s)
$$

and

$$
\begin{aligned}
f(p_\sigma) &= \frac{1}{\beta_0}\sqrt{(1+p_\sigma)^2 - (\frac{m_0 c^2}{E_0})^2} - 1 \equiv \frac{\Delta p}{p_0} \\
&= f(0) + f'(0) \cdot p_\sigma + f''(0) \cdot \frac{1}{2} p_\sigma^2 \pm \cdots \\
&= \frac{1}{\beta_0^2} \cdot p_\sigma - \frac{1}{\beta_0^4 \cdot \gamma_0^2} \cdot \frac{1}{2} p_\sigma^2 \pm \cdots
\end{aligned} \tag{2.19}
$$

(constant terms in the Hamiltonian with no influence in the motion have been dropped).

Taking into account the cavity noise we write:

$$\varphi = \varphi_0 + \delta\varphi \; ; \tag{2.20a}$$

$$V = V_0 + \delta V \tag{2.20b}$$

where $\delta\varphi$ describes the phase noise:

$$\delta\varphi = \lambda_{PH} \cdot \xi(s) \tag{2.21}$$

and δV the amplitude-noise:

$$\delta V = V_0 \cdot \lambda_{AM} \cdot \xi(s) \; . \tag{2.22}$$

We assume, without loss of generality, Gaussian white noise, i.e. noise described by a stationary stochastic process with

$$< \xi(s) > \; = \; 0 \; ; \tag{2.23a}$$

$$< \xi(s) \cdot \xi(s') > \; = \; \delta(s - s') \; . \tag{2.23b}$$

One may visualize $\xi(s)$ as a random sequence of small positive and negative pulses.

Then the Hamiltonian becomes:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 \tag{2.24}$$

with

$$\begin{aligned} \mathcal{H}_0 = \; & p_\sigma - [1 + K_x \cdot x + K_z \cdot z] \cdot \beta_0^2 \cdot f(p_\sigma) \\ & + \frac{1}{2\beta_0^2} \cdot \left\{ [p_x + \beta_0^2 H \cdot z]^2 + [p_z - \beta_0^2 H \cdot x]^2 \right\} \\ & + \frac{1}{2}\beta_0^2 \cdot \left\{ (K_x^2 + g) \cdot x^2 + (K_z^2 - g) \cdot z^2 - 2N \cdot xz \right\} \\ & - \frac{1}{2}\sigma^2 \cdot \frac{eV_0(s)}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \cos\varphi_0 - \sigma \cdot \frac{eV_0(s)}{E_0} \cdot \sin\varphi_0 \; ; \end{aligned} \tag{2.25a}$$

$$\begin{aligned} \mathcal{H}_1 = \; & -\frac{1}{2}\sigma^2 \cdot \frac{\delta V}{V_0} \cdot \frac{eV_0(s)}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \cos\varphi_0 \\ & - \sigma \cdot \frac{eV_0(s)}{E_0} \cdot \cos\varphi_0 \cdot \delta\varphi \; . \end{aligned} \tag{2.25b}$$

Remark:

Eqn. (2.24a) is valid only for protons. For electrons one needs the extra-term in the Hamiltonian

$$\mathcal{H}_{rad} = C_1 \cdot [K_x^2 + K_z^2] \cdot \sigma \tag{2.26}$$

$$\left(\text{where} \;\; C_1 = \frac{2}{3} e^2 \frac{\gamma_0^4}{E_0} \right)$$

in order to describe the energy loss by radiation in the bending magnets [8,9]. In this case, the cavity phase φ_0 in (2.21) is determined by the need to replace the energy radiated in the bending magnets. Thus:

$$\underbrace{\int_{s_0}^{s_0+L} ds \cdot eV(s) \cdot \sin\varphi_0}_{\text{average energy uptake in the cavities ;}} = \underbrace{\int_{s_0}^{s_0+L} ds \cdot E_0 \cdot C_1 \cdot [K_x^2 + K_z^2] \, .}_{\text{average energy loss due to radiation}} \tag{2.27}$$

Note, that the $\mathcal{H}_{rad}$ term only accounts for the average energy loss. Deviations from this average due to stochastic radiation effects and damping introduce non-symplectic terms into the equation of motion.

For proton storage rings, where radiation effects can be neglected, one has:

$$\sin\varphi_0 = 0 \quad \Longrightarrow \quad \varphi_0 = 0, \ \pi \ \text{(for protons)} \tag{2.28}$$

(– no average energy gain in the cavities) and φ_0 is determined by the stability condition for synchrotron motion:

$$\begin{cases} \varphi_0 = 0 & \text{above "transition" ;} \\ \varphi_0 = \pi & \text{below "transition" ;} \end{cases}$$

(– see later, eqns. (4.37a,b)).

3 Introduction of the Dispersion via a Canonical Transformation

The Hamiltonian (2.24) now leads to the canonical equations :

$$\begin{aligned}
\frac{d}{ds}\, x &= \frac{1}{\beta_0^2} \cdot [p_x + \beta_0^2 H \cdot z] \, ; && (3.1a) \\
\frac{d}{ds}\, p_x &= -\beta_0^2 \cdot [K_x^2 + g] \cdot x + \beta_0^2 \cdot N \cdot z + \beta_0^2 \cdot K_x \cdot f(p_\sigma) \\
&\quad + [p_z - \beta_0^2 H \cdot x] \cdot H \, ; && (3.1b) \\
\frac{d}{ds}\, z &= \frac{1}{\beta_0^2} \cdot [p_z - \beta_0^2 H \cdot x] \, ; && (3.1c) \\
\frac{d}{ds}\, p_z &= -\beta_0^2 \cdot [K_z^2 - g] \cdot z + \beta_0^2 \cdot N \cdot x + \beta_0^2 \cdot K_z \cdot f(p_\sigma) \\
&\quad - [p_x + \beta_0^2 H \cdot z] \cdot H \, ; && (3.1d) \\
\frac{d}{ds}\, \sigma &= 1 - [K_x \cdot x + K_z \cdot z] \cdot \beta_0^2 \cdot f'(p_\sigma) \, ; && (3.1e) \\
\frac{d}{ds}\, p_\sigma &= \sigma \cdot \frac{eV_0(s)}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \cos\varphi_0 \\
&\quad + \sigma \cdot \frac{eV_0(s)}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \cos\varphi_0 \cdot \lambda_{AM} \cdot \xi(s) \\
&\quad + \frac{eV_0(s)}{E_0} \cdot \cos\varphi_0 \cdot \lambda_{PH} \cdot \xi(s) \, . && (3.1f)
\end{aligned}$$

which represent stochastic differential equations describing coupled synchro-betatron oscillations of protons under the influence of cavity noise (phase noise and amplitude noise).

Note that the linear (transverse) betatron oscillations (eqns. (3.1a - d)) and the longitudinal motion (eqns. (3.1e, f)) are coupled by the term

$$-[K_x \cdot x + K_z \cdot z] \cdot \beta_0^2 \cdot f'(p_\sigma) \tag{3.2}$$

appearing in (3.1e) which depends on the curvature of the orbit in the bending magnets.

In order to simplify these equations we now introduce dispersion (- see later: eqn. (3.9))

$$\vec{D}(s) = \begin{pmatrix} D_1(s) \\ D_2(s) \\ D_3(s) \\ D_4(s) \end{pmatrix} ; \tag{3.3a}$$

$$\vec{D}(s) = \vec{D}(s+L) \tag{3.3b}$$

and replace the quantities $x,\ p_x,\ z,\ p_z,\ \sigma,\ \eta \equiv p_\sigma$ by the new variables $\tilde{x},\ \tilde{p}_x,\ \tilde{z},\ \tilde{p}_z,\ \tilde{\sigma},\ \tilde{p}_\sigma$ which according to the definition of dispersion satisfy:

$$\tilde{x} = x - f(p_\sigma) \cdot D_1 ; \tag{3.4a}$$

$$\tilde{p}_x = p_x - f(p_\sigma) \cdot D_2 ; \tag{3.4b}$$

$$\tilde{z} = z - f(p_\sigma) \cdot D_3 ; \tag{3.4c}$$

$$\tilde{p}_z = p_z - f(p_\sigma) \cdot D_4 . \tag{3.4d}$$

This replacement

$$(x,\ p_x,\ z,\ p_z,\ \sigma,\ \eta \equiv p_\sigma) \longrightarrow (\tilde{x},\ \tilde{p}_x,\ \tilde{z},\ \tilde{p}_z,\ \tilde{\sigma},\ \tilde{p}_\sigma) \tag{3.5}$$

can be achieved using the generating function [6,7,5]:

$$\begin{aligned} F_2(x,\ z,\ \sigma,\ \tilde{p}_x,\ \tilde{p}_z,\ \tilde{p}_\sigma) &= \tilde{p}_x \cdot [x - f(\tilde{p}_\sigma) \cdot D_1] + f(\tilde{p}_\sigma) \cdot D_2 \cdot x \\ &+ \tilde{p}_z \cdot [z - f(\tilde{p}_\sigma) \cdot D_3] + f(\tilde{p}_\sigma) \cdot D_4 \cdot z \\ &- \frac{1}{2} \cdot [D_1 \cdot D_2 + D_3 \cdot D_4] \cdot f^2(\tilde{p}_\sigma) + \tilde{p}_\sigma \cdot \sigma \end{aligned} \tag{3.6}$$

with the result that :

$$\tilde{x} = \frac{\partial F_2}{\partial \tilde{p}_x} = x - f(\tilde{p}_\sigma) \cdot D_1 ; \tag{3.7a}$$

$$p_x = \frac{\partial F_2}{\partial x} = \tilde{p}_x + f(\tilde{p}_\sigma) \cdot D_2 ; \tag{3.7b}$$

$$\tilde{z} = \frac{\partial F_2}{\partial \tilde{p}_z} = z - f(\tilde{p}_\sigma) \cdot D_3 ; \tag{3.7c}$$

$$p_z = \frac{\partial F_2}{\partial z} = \tilde{p}_z + f(\tilde{p}_\sigma) \cdot D_4 . \tag{3.7d}$$

$$\begin{aligned} \tilde{\sigma} = \frac{\partial F_2}{\partial \tilde{p}_\sigma} &= \sigma + f'(\tilde{p}_\sigma) \cdot [-\tilde{p}_x \cdot D_1 + x \cdot D_2 - \tilde{p}_z \cdot D_3 + z \cdot D_4] \\ &- [D_1 \cdot D_2 + D_3 \cdot D_4] \cdot f(\tilde{p}_\sigma) \cdot f'(\tilde{p}_\sigma) \end{aligned}$$

$$
\begin{aligned}
&= \sigma + f'(\tilde{p}_\sigma)\cdot\{-\tilde{p}_x\cdot D_1 + [x - D_1\cdot f(\tilde{p}_\sigma)]\cdot D_2 \\
&\qquad -\tilde{p}_z\cdot D_3 + [z - D_3\cdot f(\tilde{p}_\sigma)]\cdot D_4\} \\
&= \sigma + f'(\tilde{p}_\sigma)\cdot[-\tilde{p}_x\cdot D_1 + \tilde{x}\cdot D_2 - \tilde{p}_z\cdot D_3 + \tilde{z}\cdot D_4] \\
&= \sigma + f'(\tilde{p}_\sigma)\cdot[-p_x\cdot D_1 + x\cdot D_2 - p_z\cdot D_3 + z\cdot D_4]\ ; \qquad (3.7e)
\end{aligned}
$$

$$
p_\sigma = \frac{\partial F_2}{\partial \sigma} = \tilde{p}_\sigma \qquad (3.7f)
$$

and

$$
\tilde{\mathcal{H}} = \mathcal{H} + \frac{\partial F_2}{\partial s}\ . \qquad (3.8)
$$

These in turn lead to eqn. (3.4).

On taking into account the defining equations for the dispersion in the general case of arbitrary velocity β_0 (see eqn. (3.1)):

$$
\begin{aligned}
\frac{d}{ds}D_1 &= \frac{1}{\beta_0^2}\cdot[D_2 + \beta_0^2 H\cdot D_3]\ ; && (3.9a)\\
\frac{d}{ds}D_2 &= +[D_4 - \beta_0^2 H\cdot D_1]\cdot H - \beta_0^2\cdot[K_x^2 + g]\cdot D_1 + \beta_0^2\cdot N\cdot D_3 + \beta_0^2\cdot K_x\ ; && (3.9b)\\
\frac{d}{ds}D_3 &= \frac{1}{\beta_0^2}\cdot[D_4 - \beta_0^2 H\cdot D_1]\ ; && (3.9c)\\
\frac{d}{ds}D_4 &= -[D_2 + \beta_0^2 H\cdot D_3]\cdot H + \beta_0^2\cdot N\cdot D_1 - \beta_0^2\cdot[K_z^2 - g]\cdot D_3 + \beta_0^2\cdot K_z && (3.9d)
\end{aligned}
$$

we have the new Hamiltonian (3.8):

$$
\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_0 + \tilde{\mathcal{H}}_1 \qquad (3.10)
$$

with

$$
\begin{aligned}
\tilde{\mathcal{H}}_0 &= \frac{1}{2\beta_0^2}\cdot\left\{[\tilde{p}_x + \beta_0^2 H\cdot\tilde{z}]^2 + [\tilde{p}_z - \beta_0^2 H\cdot\tilde{x}]^2\right\} \\
&\quad + \frac{1}{2}\beta_0^2\cdot\left\{[K_x^2 + g]\cdot\tilde{x}^2 + [K_z^2 - g]\cdot\tilde{z}^2 - 2N\cdot\tilde{x}\tilde{z}\right\} \\
&\quad - \frac{1}{2}\beta_0^2\cdot f^2(\tilde{p}_\sigma)\cdot[K_x\cdot D_1 + K_z\cdot D_3] + \tilde{p}_\sigma - \beta_0^2\cdot f(\tilde{p}_\sigma) \\
&\quad - \frac{1}{2}h\cdot\frac{2\pi}{L}\cdot\frac{eV_0}{E_0}\cos\varphi_0 \\
&\qquad \times\{\tilde{\sigma} + f'(\tilde{p}_\sigma)\cdot[\tilde{p}_x\cdot D_1 - \tilde{x}\cdot D_2 + \tilde{p}_z\cdot D_3 - \tilde{z}\cdot D_4]\}^2\ ; \qquad (3.11a)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathcal{H}}_1 &= -\{\tilde{\sigma} + f'(\tilde{p}_\sigma)\cdot[\tilde{p}_x\cdot D_1 - \tilde{x}\cdot D_2 + \tilde{p}_z\cdot D_3 - \tilde{z}\cdot D_4]\}\cdot\frac{eV_0(s)}{E_0}\cdot\cos\varphi_0\cdot\delta\varphi \\
&\quad - \frac{1}{2}\{\tilde{\sigma} + f'(\tilde{p}_\sigma)\cdot[\tilde{p}_x\cdot D_1 - \tilde{x}\cdot D_2 + \tilde{p}_z\cdot D_3 - \tilde{z}\cdot D_4]\}^2 \\
&\qquad \times\frac{eV_0(s)}{E_0}\cdot h\cdot\frac{2\pi}{L}\cdot\cos\varphi_0\cdot\frac{\delta V}{V_0}\ . \qquad (3.11b)
\end{aligned}
$$

In this Hamiltonian the coupling term (3.2) which arose from the orbit curvature no longer appears. Instead, there appears a term for the cavities :

$$-\frac{1}{2}\cdot h\cdot\frac{2\pi}{L}\cdot\frac{eV_0}{E_0}\cos\varphi_0\cdot\{\tilde{\sigma}+f'(\tilde{p}_\sigma)\cdot[\tilde{p}_x\cdot D_1-\tilde{x}\cdot D_2+\tilde{p}_z\cdot D_3-\tilde{z}\cdot D_4]\}^2 \tag{3.12}$$

representing a coupling between the longitudinal and transverse motion which disappears if

$$V(s)\cdot D_1 = V(s)\cdot D_3 = 0\ ; \tag{3.13a}$$

$$V(s)\cdot D_2 = V(s)\cdot D_4 = 0 \tag{3.13b}$$

(e.g. no dispersion in the cavities).

Taking into account eqn. (2.18):

$$\begin{aligned} f^2(\tilde{p}_\sigma) &= \frac{1}{\beta_0^4}\cdot\tilde{p}_\sigma^2 \pm\cdots ; \\ \tilde{p}_\sigma-\beta_0^2\cdot f(\tilde{p}_\sigma) &= \frac{1}{\beta_0^2\gamma_0^2}\cdot\frac{1}{2}\tilde{p}_\sigma^2\pm\cdots ; \\ f'(\tilde{p}_\sigma) &= \frac{1}{\beta_0^2}\pm\cdots \end{aligned}$$

and decomposing $\tilde{\mathcal{H}}_0$ and $\tilde{\mathcal{H}}_1$ into two components:

$$\tilde{\mathcal{H}}_0 = \tilde{\mathcal{H}}_{01}+\tilde{\mathcal{H}}_{02} \tag{3.14a}$$

$$\tilde{\mathcal{H}}_1 = \tilde{\mathcal{H}}_{11}+\tilde{\mathcal{H}}_{12} \tag{3.14b}$$

we now may write (linear approximation):

$$\begin{aligned} \tilde{\mathcal{H}}_{01} &= \frac{1}{2\beta_0^2}\cdot\left\{[\tilde{p}_x+\beta_0^2 H\cdot\tilde{z}]^2+[\tilde{p}_z-\beta_0^2 H\cdot\tilde{x}]^2\right\} \\ &\quad +\frac{1}{2}\beta_0^2\cdot\left\{[K_x^2+g]\cdot\tilde{x}^2+[K_z^2-g]\cdot\tilde{z}^2-N\cdot\tilde{x}\tilde{z}\right\} \\ &\quad -\frac{1}{2\beta_0^2}\cdot[(K_x\cdot D_1+K_z\cdot D_3)-1/\gamma_0^2]+\tilde{p}_\sigma-\beta_0^2\cdot f(\tilde{p}_\sigma) \\ &\quad -\frac{1}{2}h\cdot\frac{2\pi}{L}\cdot\frac{eV_0}{E_0}\cos\varphi_0\cdot\tilde{\sigma}^2\ ; \end{aligned} \tag{3.15a}$$

$$\begin{aligned} \tilde{\mathcal{H}}_{02} &= -\frac{1}{2}h\cdot\frac{2\pi}{L}\cdot\frac{eV_0}{E_0}\cdot\cos\varphi_0\cdot\frac{1}{\beta_0^2} \\ &\quad\times\{2\tilde{\sigma}\cdot[\tilde{p}_x\cdot D_1-\tilde{x}\cdot D_2+\tilde{p}_z\cdot D_3-\tilde{z}\cdot D_4] \\ &\quad +(1/\beta_0^2)\cdot[\tilde{p}_x\cdot D_1-\tilde{x}\cdot D_2+\tilde{p}_z\cdot D_3-\tilde{z}\cdot D_4]^2\right\} \end{aligned} \tag{3.15b}$$

and

$$\begin{aligned} \tilde{\mathcal{H}}_{11} &= -\left\{\tilde{\sigma}+(1/\beta_0^2)\cdot[\tilde{p}_x\cdot D_1-\tilde{x}\cdot D_2+\tilde{p}_z\cdot D_3-\tilde{z}\cdot D_4]\right\} \\ &\quad\times\frac{eV_0}{E_0}\cos\varphi_0\cdot\lambda_{PH}\cdot\xi(s)\ ; \end{aligned} \tag{3.16a}$$

$$\begin{aligned} \tilde{\mathcal{H}}_{12} &= -\frac{1}{2}\cdot\left\{\tilde{\sigma}+(1/\beta_0^2)\cdot[\tilde{p}_x\cdot D_1-\tilde{x}\cdot D_2+\tilde{p}_z\cdot D_3-\tilde{z}\cdot D_4]\right\}^2 \\ &\quad\times\frac{eV_0}{E_0}\cdot h\cdot\frac{2\pi}{L}\cdot\cos\varphi_0\cdot\lambda_{AM}\cdot\xi(s)\ . \end{aligned} \tag{3.16b}$$

In terms of the variables $\tilde{x}$, $\tilde{p}_x$, $\tilde{z}$, $\tilde{p}_z$, $\tilde{\sigma}$, $\tilde{p}_\sigma$ eqn. (3.1) then takes the form:

$$\frac{d}{ds}\,\vec{\tilde{y}} = \underline{\tilde{A}} \cdot \vec{\tilde{y}} + \delta\underline{\tilde{A}} \cdot \vec{\tilde{y}} + \delta\vec{c}_{PH} + \delta\vec{c}_{AM} \tag{3.17}$$

with

$$\begin{aligned}
\underline{\tilde{A}} \cdot \vec{\tilde{y}} &= -\underline{S} \cdot \frac{\partial\tilde{\mathcal{H}}_{01}}{\partial\vec{\tilde{y}}} \quad\Longrightarrow\quad \tilde{A}_{mn} = -S_{ml} \cdot \frac{\partial^2\tilde{\mathcal{H}}_{01}}{\partial y_l \partial y_n}\ ; && (3.18a)\\
\delta\underline{\tilde{A}} \cdot \vec{\tilde{y}} &= -\underline{S} \cdot \frac{\partial\tilde{\mathcal{H}}_{02}}{\partial\vec{\tilde{y}}} \quad\Longrightarrow\quad \delta\tilde{A}_{mn} = -S_{ml} \cdot \frac{\partial^2\tilde{\mathcal{H}}_{02}}{\partial y_l \partial y_n}\ ; && (3.18b)\\
\delta\vec{c}_{PH} &= -\underline{S} \cdot \frac{\partial\tilde{\mathcal{H}}_{11}}{\partial\vec{\tilde{y}}}\ ; && (3.18c)\\
\delta\vec{c}_{AM} &= -\underline{S} \cdot \frac{\partial\tilde{\mathcal{H}}_{12}}{\partial\vec{\tilde{y}}}\ . && (3.18d)
\end{aligned}$$

Here the quatities $\delta\vec{c}_{PH}$ and $\delta\vec{c}_{AM}$ describe the influence of cavity noise on the particle motion while the matrix $\delta\underline{\tilde{A}}$ results from the synchro-betatron coupling induced by the cavities which vanishes for a vanishing dispersion in the cavities (see eqn. (**3.13a**, b)).

In detail one obtains from eqns. (3.15a):

$$\underline{\tilde{A}}(s) = \begin{pmatrix} \underline{\tilde{A}}^{(\beta)}_{(4\times4)}(s) & \underline{0}_{(2\times2)} \\ \underline{0}_{(2\times4)} & \underline{\tilde{A}}^{(\sigma)}_{(2\times2)}(s) \end{pmatrix} \tag{3.19}$$

with

$$\underline{\tilde{A}}^{(\beta)}_{(4\times4)}(s) = \begin{pmatrix} 0 & 1/\beta_0^2 & H & 0 \\ -(K_x^2+g+H^2) & 0 & N & H \\ -H & 0 & 0 & 1/\beta_0^2 \\ N & -H & -(K_z^2-g+H^2) & 0 \end{pmatrix} \tag{3.20}$$

and

$$\underline{\tilde{A}}^{(\sigma)}_{(2\times2)}(s) = \begin{pmatrix} 0 & -(1/\beta_0^2)\cdot[(K_x\cdot D_1 + K_z\cdot D_3) - 1/\gamma_0^2] \\ \frac{eV(s)}{E_0}\cdot h\cdot\frac{2\pi}{L}\cos\varphi_0 & 0 \end{pmatrix} \tag{3.21}$$

which includes the main part of the motion and which we will consider in the following as representing the unperturbed problem. Furthermore, from (3.16a, b) one has:

$$\begin{aligned}
\delta\vec{c}_{PH} &= -\lambda_{PH}\cdot\xi(s)\cdot\frac{eV_0}{E_0}\cdot\cos\varphi_0\cdot\beta_0^2\cdot\vec{\tilde{D}} && (3.22a)\\
\delta\vec{c}_{AM} &= \lambda_{AM}\cdot\xi(s)\cdot\frac{eV_0}{E_0}\cdot\cos\varphi_0\cdot h\cdot\frac{2\pi}{L}\cdot\beta_0^2\cdot\vec{\tilde{D}}\\
&\quad\times\left\{\tilde{\sigma} + (1/\beta_0^2)\cdot[\tilde{p}_x\cdot D_1 - \tilde{x}\cdot D_2 + \tilde{p}_z\cdot D_3 - \tilde{z}\cdot D_4]\right\} && (3.22b)
\end{aligned}$$

where we have introduced in (3.22a, b) the vector

$$\vec{\tilde{D}} = \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ 0 \\ -1/\beta_0^2 \end{pmatrix} . \tag{3.23}$$

For later considerations we finally remark that from (3.18b) one gets the relation:

$$\delta\underline{\tilde{A}}^T(s) \cdot \underline{S} + \underline{S} \cdot \delta\underline{\tilde{A}}(s) = 0 . \tag{3.24}$$

4 The Unperturbed Problem

In order to investigate the particle's motion under the influence of noise it is reasonable to neglect in eqn. (3.17) in a first approximation the small terms $\delta\underline{\tilde{A}}$, $\delta\vec{c}_{PH}$ and $\delta\vec{c}_{AM}$ and to consider only the "unperturbed problem" :

$$\frac{d}{ds}\vec{\tilde{y}} = \underline{\tilde{A}} \cdot \vec{\tilde{y}} . \tag{4.1}$$

The (small) perturbations described by $\delta\underline{\tilde{A}}$, $\delta\vec{c}_{PH}$ and $\delta\vec{c}_{AM}$ will then be treated in a second step with perturbation theory.

Since eqn. (4.1) is linear and homogeneous, the solution can be written in the form:

$$\vec{\tilde{y}}(s) = \underline{M}(s, s_0) \cdot \vec{\tilde{y}}(s_0) \tag{4.2}$$

which defines the transfer matrix $\underline{M}(s, s_0)$ of the motion.

In order to obtain more information about the particle motion and to set up the Fokker-Planck equation we now look for the eigenvalue spectrum of the revolution matrix:

$$\begin{aligned} \underline{M}(s_0 + L, s_0)\ \vec{v}_\mu(s_0) &= \lambda_\mu \cdot \vec{v}_\mu(s_0) ; \\ (\mu &= 1, 2, 3, 4, 5, 6) . \end{aligned} \tag{4.3}$$

We require that the stability condition

$$|\lambda_\mu| \leq 1 \tag{4.4}$$

be satisfied.

Since the equations of motion can be written in canonical form (see eqn. (3.18a)), i.e. $\tilde{x}$, $\tilde{p}_x$, $\tilde{z}$, $\tilde{p}_z$, $\tilde{\sigma}$, $\tilde{p}_\sigma$ are canonical variables, the transfer matrix is symplectic [10]:

$$\underline{M}^T(s, s_0) \cdot \underline{S} \cdot \underline{M}(s, s_0) = \underline{S} \tag{4.5}$$

(with $\underline{S}$ given by eqn. (2.8)).

As a result, the eigenvectors $\vec{v}_\mu(s_0)$ occur in pairs

$$(\vec{v}_k,\ \vec{v}_{-k}) ;\ k = I,\ II,\ III \tag{4.6a}$$

with the reciprocal eigenvalues λ_k, λ_{-k} satisfying:

$$\lambda_k \cdot \lambda_{-k} = 1 \tag{4.6b}$$

($k = I,\ II,\ III$ refers to the three 2-dimensional subspaces of the full six dimensional phase space).

Thus the stability condition (4.4) can be written as [15]:

$$|\lambda_k| \quad = \quad |\lambda_{-k}| = 1 \ ; \quad \lambda_{-k} = \lambda_k^*$$

$$\Longrightarrow \quad \begin{cases} \lambda_k = e^{-i\cdot 2\pi Q_k} \ ; \quad Q_k \ \text{ real} \ ; \\ \\ \vec{v}_{-k}(s_0) = \vec{v}_k^{\,*}(s_0) \end{cases} \tag{4.7}$$

so that all eigenvalues must lie on a unit circle. With the normalization condition for the $\vec{v}_k(s_0)$:

$$\vec{v}_k^{\,+}(s_0) \cdot \underline{S} \cdot \vec{v}_{-k}(s_0) \quad = \quad i$$

we then obtain from (4.5) the relations:

$$\begin{cases} \vec{v}_k^{\,+}(s_0) \cdot \underline{S} \cdot \vec{v}_k(s_0) = -\vec{v}_{-k}^{\,+}(s_0) \cdot \underline{S} \cdot \vec{v}_{-k}(s_0) = i \ ; \\ \\ \vec{v}_\mu^{\,+}(s_0) \cdot \underline{S} \cdot \vec{v}_\nu(s_0) = 0 \quad \text{otherwise} \ ; \end{cases} \tag{4.8}$$

$$(k = I, II, III) \ .$$

The vectors $\vec{v}_{\pm k}(s_0)$ are the eigenvectors of the revolution matrix $\underline{M}(s_0 + L, s_0)$ with starting point s_0. The eigenvectors of the revolution matrix $\underline{M}(s + L, s)$ with starting point s can finally be obtained by operating with the transfer matrix $\underline{M}(s, s_0)$:

$$\vec{v}_{\pm k}(s) \quad = \quad \underline{M}(s, s_0)\ \vec{v}_{\pm k}(s_0) \ ; \tag{4.9a}$$

$$\underline{M}(s + L, s)\ \vec{v}_{\pm k}(s) \quad = \quad \lambda_{\pm k} \cdot \vec{v}_{\pm k}(s) \tag{4.9b}$$

whereby the eigenvalues remain unchanged:

$$\lambda_{\pm k}(s) = \lambda_{\pm k}(s_0) \equiv \lambda_{\pm k} \ . \tag{4.9c}$$

The vectors $\vec{v}_{\pm k}(s)$ defined by (4.9a) also fulfill the same orthogonality relations (4.8) as $\vec{v}_{\pm k}(s_0)$:

$$\begin{cases} \vec{v}_k^{\,+}(s) \cdot \underline{S} \cdot \vec{v}_k(s) = -\vec{v}_{-k}^{\,+}(s) \cdot \underline{S} \cdot \vec{v}_{-k}(s) = i \ ; \\ \\ \vec{v}_\mu^{\,+}(s) \cdot \underline{S} \cdot \vec{v}_\nu(s) = 0 \quad \text{otherwise} \ ; \end{cases} \tag{4.10}$$

$$(k = I, II, III) \ .$$

Putting

$$\vec{v}_\mu(s) = \hat{\vec{v}}_\mu(s) \cdot e^{-i\cdot 2\pi Q_\mu \cdot (s/L)} \tag{4.11}$$

the factor $\vec{\tilde{v}}_\mu(s)$

$$\vec{\tilde{v}}_\mu(s) = \vec{v}_\mu(s) \cdot e^{+i \cdot 2\pi Q_\mu \cdot (s/L)}$$

is seen to be a periodic function with period L:

$$\begin{aligned}
\vec{\tilde{v}}_\mu(s+L) &= \vec{v}_\mu(s+L) \cdot e^{+i \cdot 2\pi Q_\mu \cdot (s+L)/L} \\
&= \vec{v}_\mu(s) \cdot e^{-i \cdot 2\pi Q_\mu} \cdot e^{+i \cdot 2\pi Q_\mu \cdot (s+L)/L} \\
&= \vec{v}_\mu(s) \cdot e^{+i \cdot 2\pi Q_\mu \cdot s/L} \\
&= \vec{\tilde{v}}_\mu(s) \; . \qquad (4.12)
\end{aligned}$$

Eqn. (4.12) is a statement of the Floquet theorem : Vectors $\vec{v}_\mu(s)$ are special solutions of the equations of motion (4.1) which can be expressed as the product of a periodic function $\vec{\tilde{v}}_\mu(s)$ and a harmonic function

$$e^{-i \cdot 2\pi Q_\mu \cdot (s/L)} \; .$$

Note that the orthogonality relations (4.10) are also valid for the "Floquet-vectors" $\vec{\tilde{v}}_\mu(s)$:

$$\begin{cases}
\vec{\tilde{v}}_k^{\,+}(s) \cdot \underline{S} \cdot \vec{\tilde{v}}_k(s) = -\vec{\tilde{v}}_{-k}^{\,+}(s) \cdot \underline{S} \cdot \vec{\tilde{v}}_{-k}(s) = i \; ; \\
\vec{\tilde{v}}_\mu^{\,+}(s) \cdot \underline{S} \cdot \vec{\tilde{v}}_\nu(s) = 0 \quad \text{otherwise} \; .
\end{cases} \qquad (4.13)$$

The general solution of the equation of motion (4.1) is a linear combination of the special solutions (4.11) and can be therefore written in the form

$$\vec{y}(s) = \sum_{k=I,II,III} \left\{ A_k \cdot \vec{\tilde{v}}_k(s) e^{-i \cdot 2\pi Q_k \cdot (s/L)} + A_{-k} \cdot \vec{\tilde{v}}_{-k}(s) e^{+i \cdot 2\pi Q_{-k} \cdot (s/L)} \right\} . \qquad (4.14)$$

Using these results we are now able to introduce a new set of canonical variables which will be important for further investigations [3,11].

For this we write for the coefficients A_k, A_{-k} $(k = I, II, III)$ in eqn. (4.14) :

$$\begin{aligned}
A_k &= \sqrt{J_k} \cdot e^{-i[\Phi_k - 2\pi Q_k \cdot s/L]} \; ; \qquad (4.15a) \\
A_{-k} &= \sqrt{J_k} \cdot e^{+i[\Phi_k - 2\pi Q_k \cdot s/L]} \; . \qquad (4.15b)
\end{aligned}$$

Then eqn. (4.14) takes the form:

$$\vec{y}(s) = \sum_{k=I,II,III} \sqrt{J_k} \cdot \left\{ \vec{\tilde{v}}_k(s) \cdot e^{-i\Phi_k} + \vec{\tilde{v}}_{-k}(s) \cdot e^{+i\Phi_k} \right\} . \qquad (4.16)$$

From (4.16) we now have:

$$\begin{aligned}
\frac{\partial \vec{y}}{\partial \Phi_k} &= -i \cdot \sqrt{J_k} \cdot \left\{ \vec{\tilde{v}}_k(s) \cdot e^{-i\Phi_k} - \vec{\tilde{v}}_{-k}(s) \cdot e^{+i\Phi_k} \right\} \; ; \qquad (4.17a) \\
\frac{\partial \vec{y}}{\partial J_k} &= +\frac{1}{2\sqrt{J_k}} \cdot \left\{ \vec{\tilde{v}}_k(s) \cdot e^{-i\Phi_k} + \vec{\tilde{v}}_{-k}(s) \cdot e^{+i\Phi_k} \right\} . \qquad (4.17b)
\end{aligned}$$

Taking into account the relations (4.13) one obtains the equations:

$$\frac{\partial \vec{y}}{\partial J_k}^T \cdot \underline{S} \cdot \frac{\partial \vec{y}}{\partial \Phi_l} = -\frac{\partial \vec{y}}{\partial \Phi_l}^T \cdot \underline{S} \cdot \frac{\partial \vec{y}}{\partial J_k} = \delta_{kl} \; ; \tag{4.18a}$$

$$\frac{\partial \vec{y}}{\partial J_k}^T \cdot \underline{S} \cdot \frac{\partial \vec{y}}{\partial J_l} = \frac{\partial \vec{y}}{\partial \Phi_k}^T \cdot \underline{S} \cdot \frac{\partial \vec{y}}{\partial \Phi_l} = 0 \tag{4.18b}$$

which can be combined into the matrix form

$$\underline{\mathcal{J}}^T \cdot \underline{S} \cdot \underline{\mathcal{J}} = \underline{S} \tag{4.19}$$

with $\underline{\mathcal{J}}$ the Jacobian matrix

$$\underline{\mathcal{J}} = \left(\frac{\partial \vec{y}}{\partial \Phi_I}, \frac{\partial \vec{y}}{\partial J_I}, \frac{\partial \vec{y}}{\partial \Phi_{II}}, \frac{\partial \vec{y}}{\partial J_{II}}, \frac{\partial \vec{y}}{\partial \Phi_{III}}, \frac{\partial \vec{y}}{\partial J_{III}} \right) \tag{4.20}$$

being a 6×6-matrix just written as a row of column vectors $(\partial \vec{y} / \partial \Phi_I)$ etc.

Equation (4.19) proves that eqn. (4.16) represents a canonical transformation

$$\tilde{x},\ \tilde{p}_x,\ \tilde{z},\ \tilde{p}_z,\ \tilde{\sigma},\ \tilde{p}_\sigma \longrightarrow \Phi_I,\ J_I,\ \Phi_{II},\ J_{II},\ \Phi_{III},\ J_{III} \tag{4.21}$$

and that $\Phi_k,\ J_k\ (k = I, II, III)$ are indeed canonical variables which can now be interpreted as action-angle variables since

$$\frac{dJ_k}{ds} = 0 \; ; \tag{4.22a}$$

$$\frac{d\Phi_k}{ds} = \frac{2\pi}{L} Q_k \, . \tag{4.22b}$$

These variables may also be used to describe the orbital motion.

For later considerations we finally mention that the revolution matrix $\underline{M}(s+L,s)$ has according to eqn. (3.19) the simple block diagonal form:

$$\underline{\tilde{M}}(s+L,s) = \begin{pmatrix} \underline{M}^{(\beta)}_{(4\times4)}(s+L,s) & \underline{0}_{(4\times2)} \\ \underline{0}_{(2\times4)} & \underline{M}^{(\sigma)}_{(2\times2)}(s+L,s) \end{pmatrix} \tag{4.23}$$

where $\underline{M}^{(\beta)}_{(4\times4)}(s+L,s)$ corresponds to the (transverse) betatron motion and $\underline{M}^{(\sigma)}_{(2\times2)}(s+L,s)$ to the (longitudinal) synchrotron oscillations.

Furthermore, the 2-dimensional revolution matrix $\underline{M}^{(\sigma)}_{(2\times2)}(s+L,s)$ which is defined by the equations of synchrotron motion:

$$\frac{d}{ds}\,\tilde{\sigma} = -\frac{1}{\beta_0^2} \cdot \left[(K_x \cdot D_x + K_z \cdot D_z) - 1/\gamma_0^2 \right] \cdot \tilde{p}_\sigma \; ; \tag{4.24a}$$

$$\frac{d}{ds}\,\tilde{p}_\sigma = h \cdot \frac{2\pi}{L} \cdot \frac{eV(s)}{E_0} \cos\varphi_0 \cdot \tilde{\sigma} \tag{4.24b}$$

(see eqns. (3.11), (3.18) and (3.19)) can be represented in the form

$$\underline{M}^{(\sigma)}_{(2\times2)}(s+L,s) = \begin{pmatrix} \cos 2\pi Q_\sigma + \alpha_\sigma(s) \cdot \sin 2\pi Q_\sigma & \beta_\sigma(s) \cdot \sin 2\pi Q_\sigma \\ -\gamma_\sigma(s) \cdot \sin 2\pi Q_\sigma & \cos 2\pi Q_\sigma + \alpha_\sigma(s) \cdot \sin 2\pi Q_\sigma \end{pmatrix} \tag{4.25}$$

with

$$\beta_\sigma \cdot \gamma_\sigma = \alpha_\sigma^2 + 1 \; . \tag{4.26}$$

From these equations one sees that for the eigenvectors $\vec{v}_k(s)$ one can write:

$$\vec{v}_k = \begin{pmatrix} \vec{v}_k^{\,(\beta)} \\ \vec{0}_2 \end{pmatrix} \; ; \quad (k = I,\ II) \; ; \tag{4.27a}$$

$$\vec{v}_{III} = \begin{pmatrix} \vec{0}_4 \\ \vec{w}_\sigma \end{pmatrix} \; ; \quad \vec{w}_\sigma \equiv \begin{pmatrix} w_{\sigma 1} \\ w_{\sigma 2} \end{pmatrix} = \frac{1}{\sqrt{2\beta_\sigma(s)}} \cdot \begin{pmatrix} \beta_\sigma(s) \\ -[\alpha_\sigma(s) + i] \end{pmatrix} \cdot e^{-i\varphi_\sigma(s)} \tag{4.27b}$$

where, in the case that the betatron oscillations are decoupled:

$$\underline{M}^{(\beta)}_{(4\times 4)}(s+L,s) = \begin{pmatrix} \underline{M}^{(x)}_{(2\times 2)}(s+L,s) & \underline{0}_{(2\times 2)} \\ \underline{0}_{(2\times 2)} & \underline{M}^{(z)}_{(2\times 2)}(s+L,s) \end{pmatrix} ; \tag{4.28a}$$

$$\underline{M}^{(y)}_{(2\times 2)}(s+L,s) = \begin{pmatrix} \cos 2\pi Q_y + \alpha_y \sin 2\pi Q_y & \beta_y \sin 2\pi Q_y \\ -\gamma_y \sin 2\pi Q_y & \cos 2\pi Q_y + \alpha_y \sin 2\pi Q_y \end{pmatrix} ; \tag{4.28b}$$

$$\beta_y \cdot \gamma_y = \alpha_y^2 + 1 \; ; \quad (y \equiv x,\ z) \tag{4.28c}$$

the vectors $\vec{v}_I$ and $\vec{v}_{II}$ take a form similar to $\vec{v}_{III}$:

$$\vec{v}_I^{\,(\beta)} = \begin{pmatrix} \vec{w}_x \\ \vec{0}_2 \end{pmatrix} \; ; \quad \vec{v}_{II}^{\,(\beta)} = \begin{pmatrix} \vec{0}_2 \\ \vec{w}_z \end{pmatrix} \; ; \tag{4.29a}$$

$$\vec{w}_y \equiv \begin{pmatrix} w_{y1} \\ w_{y2} \end{pmatrix} = \frac{1}{\sqrt{2\beta_y(s)}} \cdot \begin{pmatrix} \beta_y(s) \\ -[\alpha_y(s) + i] \end{pmatrix} \cdot e^{-i\varphi_y(s)} \; ; \tag{4.29b}$$

$$(y \equiv x,\ z) \; .$$

Remarks:

1) An approximate form for the matrix $\underline{M}^{(\sigma)}_{(2\times 2)}(s+L,s)$ can be established in which in the equation of motion (4.24a, b) the coefficients of $\tilde{\sigma}$ and $\tilde{p}_\sigma$ are averaged over one turn (oscillator-model):

$$[K_x \cdot D_1 + K_z \cdot D_3] \longrightarrow \kappa = \frac{1}{L} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot [K_x(\tilde{s}) \cdot D_1(\tilde{s}) + K_z(\tilde{s}) \cdot D_3(\tilde{s})] \tag{4.30a}$$

(momentum compaction factor) ;

$$\begin{aligned} h \cdot \frac{2\pi}{L} \cdot \frac{eV(s)}{E_0} \cos\varphi_0 \quad \longrightarrow \quad & \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot h \cdot \frac{2\pi}{L} \cdot \frac{eV(\tilde{s})}{E_0} \cos\varphi_0 \\ = \; & \Omega^2 \cdot \frac{\beta_0^2}{(\kappa - 1/\gamma_0^2)} \end{aligned} \tag{4.30b}$$

with

$$\Omega^2 = \frac{(\kappa - 1/\gamma_0^2)}{\beta_0^2} \cdot h \cdot \frac{2\pi}{L} \cdot \cos\varphi_0 \cdot \frac{1}{L}\int_{s_0}^{s_0+L} d\bar{s} \cdot \frac{eV(\bar{s})}{E_0} \; . \tag{4.31}$$

Thus, eqn. (4.24) transforms to

$$\frac{d}{ds}\,\tilde{\sigma} = -\frac{(\kappa - 1/\gamma_0^2)}{\beta_0^2} \cdot \tilde{p}_\sigma \; ; \tag{4.32a}$$

$$\frac{d}{ds}\,\tilde{p}_\sigma = \Omega^2 \cdot \frac{\beta_0^2}{(\kappa - (1/\gamma_0^2)} \cdot \tilde{\sigma} \tag{4.32b}$$

or

$$\frac{d^2}{ds^2}\,\tilde{\sigma} = -\Omega^2 \cdot \tilde{\sigma} \; ; \tag{4.33a}$$

$$\frac{d^2}{ds^2}\,\tilde{p}_\sigma = \Omega^2 \cdot \tilde{p}_\sigma \tag{4.33b}$$

with the solution

$$\begin{pmatrix} \tilde{\sigma}(s) \\ \tilde{p}_\sigma(s) \end{pmatrix} = \begin{pmatrix} \cos\Omega(s-s_0) & -(\kappa/\Omega)\cdot\sin\Omega(s-s_0) \\ (\kappa/\Omega)\cdot\sin\Omega(s-s_0) & \cos\Omega(s-s_0) \end{pmatrix} \begin{pmatrix} \tilde{\sigma}(s_0) \\ \tilde{p}_\sigma(s_0) \end{pmatrix} . \tag{4.34}$$

Using this "oscillator-model", one obtains for the one turn matrix:

$$\underline{M}^{(\sigma)}_{(2\times2)}(s+L,s) = \begin{pmatrix} \cos\Omega L & -(\kappa/\Omega)\cdot\sin\Omega L \\ (\kappa/\Omega)\cdot\sin\Omega L & \cos\Omega L \end{pmatrix} \begin{pmatrix} \tilde{\sigma}(s_0) \\ \tilde{p}_\sigma(s_0) \end{pmatrix} \tag{4.35}$$

and by the comparison of (4.34) with (4.25) we find ($\beta_\sigma > 0$) :

$$2\pi Q_\sigma = -\Omega \cdot L \; ; \tag{4.36a}$$

$$\beta_\sigma = \frac{\kappa}{\Omega} \; ; \tag{4.36b}$$

$$\alpha_\sigma = 0 \; ; \tag{4.36c}$$

$$\gamma_\sigma = \frac{\Omega}{\kappa} = \frac{1}{\beta_\sigma} \tag{4.36d}$$

where the quantities Ω and κ are taken from (4.30a) and (4.31).

Note that with respect to eqn. (4.33a, b) the synchrotron oscillations are stable only if

$$\Omega^2 > 0 \; . \tag{4.37a}$$

For protons ($eV(s) > 0$) with $\sin\varphi_0 = 0$ (no energy uptake in the cavities) this corresponds to the usual conditions [5]:

$$\left\{ \begin{array}{l} \varphi_0 = 0 \;\; \text{for} \;\; \kappa > (1/\gamma_0^2) \;\; \text{above "transition"} \; ; \\ \\ \varphi_0 = \pi \;\; \text{for} \;\; \kappa < (1/\gamma_0^2) \;\; \text{below "transition"} \; . \end{array} \right. \tag{4.37b}$$

2) The phase function $\varphi_y(s)$ $(y \equiv x, z, \sigma)$ in eqns. (4.27) and (4.29) of an uncoupled mode is determined by the fact that the eigenvector $\vec{w}_y(s)$ must obey the equations of motion.

In order to find the analytic form of this function, we take the Hamiltonian for uncoupled synchro-betatron oscillations in its most general form:

$$\mathcal{H}_0 = \mathcal{H}_{0\tilde{x}} + \mathcal{H}_{0\tilde{z}} + \mathcal{H}_{0\tilde{\sigma}} \tag{4.38}$$

with

$$\mathcal{H}_{0y} = \frac{1}{2} F_y(s) \cdot p_y^2 + R_y \cdot y \cdot p_y + \frac{1}{2} G_y(s) \cdot y^2 \tag{4.39}$$

$(y \equiv \tilde{x}, \tilde{z}, \tilde{\sigma})$ from which result the corresponding canonical equations of motion:

$$\frac{d}{ds} \begin{pmatrix} y \\ p_y \end{pmatrix} = \underline{A}_y \cdot \begin{pmatrix} y \\ p_y \end{pmatrix} \tag{4.40a}$$

with

$$\underline{A}_y(s) = \begin{pmatrix} R_y & F_y \\ -G_y & -R_y \end{pmatrix} . \tag{4.40b}$$

Comparing eqn. (4.40a) with (3.21) one gets for example:

$$\begin{aligned} R_\sigma &= 0 ; \\ F_\sigma &= -(1/\beta_0^2) \cdot [(K_x \cdot D_1 + K_z \cdot D_3) - 1/\gamma_0^2] ; \\ G_\sigma &= -\frac{eV(s)}{E_0} \cdot h \cdot \frac{2\pi}{L} \cos \varphi_0 . \end{aligned}$$

Furthermore, we write with respect to eqn. (4.28b) for the revolution matrix

$$\underline{M}^{(y)}_{(2\times2)}(s+L, s)$$

of an uncoupled mode:

$$\underline{M}^{(y)}_{(2\times2)}(s+L, s) = \cos 2\pi Q_y \cdot \underline{1} + \sin 2\pi Q_y \cdot \underline{K}_y(s) \tag{4.41a}$$

with

$$\underline{K}_y(s) = \begin{pmatrix} \alpha_y(s) & \beta_y(s) \\ -\gamma_y(s) & -\alpha_y(s) \end{pmatrix} . \tag{4.41b}$$

From the condition:

$$\begin{aligned} \underline{M}^{(y)}_{(2\times2)}(s+L, s) &= \underline{M}^{(y)}_{(2\times2)}(s+L, s_0+L) \cdot \underline{M}^{(y)}_{(2\times2)}(s_0+L, s_0) \cdot \underline{M}^{(y)}_{(2\times2)}(s_0, s) \\ &= \underline{M}^{(y)}_{(2\times2)}(s, s_0) \cdot \underline{M}^{(y)}_{(2\times2)}(s_0+L, s_0) \cdot \left[\underline{M}^{(y)}_{(2\times2)}(s, s_0)\right]^{-1} \end{aligned} \tag{4.42}$$

one then obtains with respect to (4.41a):

$$\underline{K}_y(s) = \underline{M}^{(y)}_{(2\times2)}(s, s_0) \cdot \underline{K}_y(s_0) \cdot \left[\underline{M}^{(y)}_{(2\times2)}(s, s_0)\right]^{-1} . \tag{4.43}$$

For the derivative

$$\underline{K}'_y(s) \;\equiv\; \begin{pmatrix} \alpha'_y(s) & \beta'_y(s) \\ -\gamma'_y(s) & -\alpha'_y(s) \end{pmatrix} \tag{4.44}$$

one gets:

$$\begin{aligned}
\underline{K}'_y(s) &= \frac{1}{\Delta s}\cdot \lim_{\Delta s \to 0} \left\{\underline{K}_y(s+\Delta s) - \underline{K}_y(s)\right\} \\
&= \frac{1}{\Delta s}\cdot \lim_{\Delta s \to 0} \left\{\underline{m}_y(s+\Delta s, s)\cdot \underline{K}_y(s)\cdot \underline{m}_y^{-1}(s+\Delta s, s) - \underline{K}_y(s)\right\} \\
&= \frac{1}{\Delta s}\cdot \lim_{\Delta s \to 0} \left\{\left[\underline{1} + \Delta s\cdot \underline{A}_y(s)\right]\cdot \underline{K}_y(s)\cdot \left[\underline{1} - \Delta s\cdot \underline{A}_y(s)\right] - \underline{K}_y(s)\right\} \\
&= \underline{A}_y(s)\cdot \underline{K}_y(s) - \underline{K}_y(s)\cdot \underline{A}_y(s) \\
&= \begin{pmatrix} [-\gamma_y\cdot F_y + \beta_y(s)\cdot G_y] & 2\cdot[\beta_y\cdot R_y(s) - \alpha_y\cdot F_y] \\ 2\cdot[-\alpha_y\cdot G_y + \gamma_y\cdot R_y] & -[-\gamma_y\cdot F_y + \beta_y\cdot G_y] \end{pmatrix} .
\end{aligned} \tag{4.45}$$

By comparing (4.44) and (4.45) we then find that:

$$\alpha'_y(s) \;=\; -\gamma_y\cdot F_y + \beta_y\cdot G_y \;; \tag{4.46a}$$

$$\beta'_y(s) \;=\; 2\cdot[\beta_y\cdot R_y - \alpha_y\cdot F_y] \;; \tag{4.46b}$$

$$\gamma'_y(s) \;=\; 2\cdot[\alpha_y\cdot G_y - \gamma_y\cdot R_y] \;. \tag{4.46c}$$

Finally, using the fact that $\vec{w}_y(s)$ in (4.27b) and (4.29b) must be a solution of the equation of motion (4.40) we gain by taking into account eqn. (4.46):

$$\varphi'_y(s) \;=\; \frac{F_y(s)}{\beta_y(s)} \tag{4.47a}$$

$$\Longrightarrow \quad \varphi_y(s) \;=\; \int_0^s d\tilde{s}\cdot \frac{F_y(\tilde{s})}{\beta_y(\tilde{s})} + \varphi_{0y} \;. \tag{4.47b}$$

The Floquet-vector

$$\vec{\hat{w}}_y(s) \equiv \begin{pmatrix} \hat{w}_{y1} \\ \hat{w}_{y2} \end{pmatrix} = \vec{w}_y(s)\cdot e^{+i\,\cdot\,2\pi Q_y\,\cdot\,(s/L)} \tag{4.48}$$

as defined by eqn. (4.11) now reads :

$$\vec{\hat{w}}_y = \frac{1}{\sqrt{2\beta_y(s)}}\cdot \begin{pmatrix} \beta_y(s) \\ -[\alpha_y(s) + i] \end{pmatrix}\cdot \exp\left\{i\cdot \left[2\pi Q_y \frac{s}{L} - \int_0^s d\tilde{s}\cdot \frac{F_y(\tilde{s})}{\beta_y(\tilde{s})} - \varphi_{0y}\right]\right\} \tag{4.49}$$

and the "action-angle representation" (4.16) of the orbital motion with the action variable J_y and the angle variable Φ_y takes the form:

$$\begin{aligned}
\begin{pmatrix} y \\ p_y \end{pmatrix} &= \sqrt{J_y}\cdot \frac{1}{\sqrt{2\beta_y(s)}}\cdot \begin{pmatrix} \beta_y(s) \\ -[\alpha_y(s) + i] \end{pmatrix} \\
&\quad \times \exp\left\{-i\cdot \left[\Phi_y(s) + \varphi_{0y} + \int_0^s d\tilde{s}\cdot \frac{F_y(\tilde{s})}{\beta_y(\tilde{s})} - 2\pi Q_y \frac{s}{L}\right]\right\} \\
&\quad + compl.conj.
\end{aligned}$$

or, written in components:

$$y(s) = \sqrt{2J_y \cdot \beta_y} \cdot \cos\left[\Phi_y(s) + \varphi_{0y} + \int_0^s d\tilde{s} \cdot \frac{F_y(\tilde{s})}{\beta_y(\tilde{s})} - 2\pi Q_y \frac{s}{L}\right] \; ; \tag{4.50a}$$

$$p_y(s) = -\sqrt{\frac{2J_y}{\beta_y}} \cdot \left\{\sin\left[\Phi_y(s) + \varphi_{0y} + \int_0^s d\tilde{s} \cdot \frac{F_y(\tilde{s})}{\beta_y(\tilde{s})} - 2\pi Q_y \frac{s}{L}\right] + \alpha_y \cdot \cos\left[\Phi_y(s) + \varphi_{0y} + \int_0^s d\tilde{s} \cdot \frac{F_y(\tilde{s})}{\beta_y(\tilde{s})} - 2\pi Q_y \frac{s}{L}\right]\right\} \tag{4.50b}$$

which can be the starting point for a canonical perturbation treatment of coupled synchro-betatron oscillations as demonstrated in Ref. [16].

3) Equation (4.50) describes at a fixed point s an ellipse in the $(y - p_y)$-phase-plane. The area of the ellipse is given by:

$$F = 2\pi \cdot J_y \; . \tag{4.51}$$

Usually one writes:

$$F = \pi \cdot \epsilon_y \tag{4.52}$$

where ϵ_y significates the emittance of the uncoupled oscillation in y-direction [9].

Generalizing this equation for coupled motion, we may write:

$$\epsilon_k = 2J_k \tag{4.53}$$

defining ϵ_k as the emittance of the k-th (coupled) mode.

5 The Perturbed Problem

The general solution of the unperturbed equation of motion (4.1) can be written in the form

$$\vec{y}(s) = \sum_{k=I,II,III} \{A_k \cdot \vec{v}_k(s) + A_{-k} \cdot \vec{v}_{-k}(s)\}$$

with A_k, A_{-k} being constants of integration ($k = I,\ II,\ III$).

In order to solve the perturbed problem (3.17) we now make the following "ansatz" (variation of constants) :

$$\vec{y}(s) = \sum_{k=I,II,III} \{A_k(s) \cdot \vec{v}_k(s) + A_{-k}(s) \cdot \vec{v}_{-k}(s)\} \; . \tag{5.1}$$

Inserting (5.1) into (3.17) one obtains :

$$\sum_{k=I,II,III} \left\{A'_k(s) \cdot \vec{v}_k + A'_{-k}(s) \cdot \vec{v}_{-k}\right\} = \delta\underline{\tilde{A}} \cdot \sum_{k=I,II,III} \{A_k(s) \cdot \vec{v}_k + A_{-k}(s) \cdot \vec{v}_{-k}\} + \delta\vec{c}_{PH} + \delta\vec{c}_{AM} \; . \tag{5.2}$$

Using the orthogonality conditions (4.10) and the relations (3.22a, b) one gets from (5.2) for $k = I, II, III$:

$$\begin{aligned}
A'_k(s) &= X_k(s) + i \cdot \lambda_{PH} \cdot \xi(s) \cdot \frac{eV_0}{E_0} \cdot \cos\varphi_0 \cdot \beta_0^2 \cdot \vec{v}_k^{\,+}(s)\underline{S}\vec{\tilde{D}}(s) \\
&\quad - i \cdot \lambda_{AM} \cdot \xi(s) \cdot \frac{eV_0}{E_0} \cdot \cos\varphi_0 \cdot h \cdot \frac{2\pi}{L} \cdot \beta_0^2 \cdot \vec{v}_k^{\,+}(s)\underline{S}\vec{\tilde{D}}(s) \\
&\quad \times \left\{ \tilde{\sigma} + (1/\beta_0^2) \cdot [\tilde{p}_x \cdot D_1 - \tilde{x} \cdot D_2 + \tilde{p}_z \cdot D_3 - \tilde{z} \cdot D_4] \right\} ; \qquad (5.3a)
\end{aligned}$$

$$A'_{-k}(s) = [A'_k(s)]^* \qquad (5.3b)$$

with

$$\begin{aligned}
X_k(s) &= \sum_{l=I,II,III} A_l(s) \cdot (-i) \cdot \vec{v}_k^{\,+} \cdot \underline{S} \cdot \delta\underline{\tilde{A}} \cdot \vec{v}_l \\
&\quad + \sum_{l=I,II,III} A_{-l}(s) \cdot (-i) \cdot \vec{v}_k^{\,+} \cdot \underline{S} \cdot \delta\underline{\tilde{A}} \cdot \vec{v}_{-l} \ . \qquad (5.4)
\end{aligned}$$

Taking into account eqn. (2.7) and the defining equation (3.23) for $\vec{\tilde{D}}$ one can write for the term $\vec{v}_k^{\,+}\underline{S}\vec{\tilde{D}}$ appearing on the r.h.s. of (5.3a):

$$\vec{v}_k^{\,+}\underline{S}\vec{\tilde{D}} = v_{k2}^* \cdot D_1 - v_{k1}^* \cdot D_2 + v_{k4}^* \cdot D_3 - v_{k3}^* \cdot D_4 + (1/\beta_0^2) \cdot v_{k5}^*$$

or with respect to (4.27) :

$$\vec{v}_k^{\,+}\underline{S}\vec{\tilde{D}} = \begin{cases} v_{k2}^* \cdot D_1 - v_{k1}^* \cdot D_2 + v_{k4}^* \cdot D_3 - v_{k3}^* \cdot D_4 & \text{for } k = I,\ II\ ; \\ (1/\beta_0^2) \cdot w_{\sigma 1}^* \equiv (1/\beta_0^2) \cdot v_{k5}^* & \text{for } k = III\ . \end{cases} \qquad (5.5)$$

6 Stochastic Equations for the Action-Angle Variables $J_k(s)$ and $\Phi_k(s)$

Representing $A_k(s)$ in the form (4.15), we obtain for the derivative $A'_k(s)$ $(k = I,\ II,\ III)$:

$$\begin{aligned}
A'_k(s) &= \frac{1}{2} \cdot \frac{J'_k}{\sqrt{J_k}} \cdot e^{-i \cdot [\Phi_k - 2\pi Q_k \cdot s/L]} - i \cdot \left[\Phi'_k - \frac{2\pi}{L} Q_k\right] \cdot A_k \\
&= A_k \cdot \left\{ \frac{1}{2} \cdot \frac{J'_k}{J_k} - i \cdot \left[\Phi'_k - \frac{2\pi}{L} Q_k\right] \right\}
\end{aligned}$$

and it follows for the derivatives $J'_k(s)$ and $\Phi'_k(s)$ of the action - angle variables J_k and Φ_k [3]:

$$\begin{aligned}
J'_k(s) &= \frac{d}{ds}\,[A_k(s) \cdot A_{-k}(s)] \\
&= A'_k(s) \cdot A_{-k}(s) + A_k(s) \cdot A'_{-k}(s) \\
&= 2 \cdot \Re e\,\{A'_k(s) \cdot A_{-k}(s)\} \ ; \qquad (6.1a)
\end{aligned}$$

$$\begin{aligned}
\Phi'_k(s) - \frac{2\pi}{L} \cdot Q_k &= -\frac{A'_k(s) \cdot A_{-k}(s)}{i \cdot J_k(s)} + \frac{1}{2} \cdot \frac{A'_k(s) \cdot A_{-k}(s) + A_k(s) \cdot A'_{-k}(s)}{i \cdot J_k(s)} \\
&= -\frac{1}{2} \cdot \frac{A'_k(s) \cdot A_{-k}(s) - A_k(s) \cdot A'_{-k}(s)}{i \cdot J_k(s)} \\
&= -\frac{1}{J_k(s)} \cdot \Im m \left\{ A'_k(s) \cdot A_{-k}(s) \right\} \; . \qquad (6.1b)
\end{aligned}$$

Here the terms $(A'_k \cdot A_{-k})$ appearing in (6.1a,b) are given by:

$$A'_k(s) \cdot A_{-k}(s) = Y_k(s) + Z_k(s) \qquad (6.2a)$$

$$Z_k(s) = Z_k^{(1)}(s) + Z_k^{(2)}(s) \qquad (6.2b)$$

with

$$\begin{aligned}
Y_k(s) &= X_k(s) \cdot A_{-k}(s) \\
&= \sum_{l=I,II,III} \sqrt{J_l} \cdot \sqrt{J_k} \cdot (-i) \cdot \vec{\tilde{v}}_k^{\,+} \cdot \underline{S} \cdot \delta \underline{\tilde{A}} \cdot \vec{\tilde{v}}_l \cdot e^{i \cdot [\Phi_k - \Phi_l]} \\
&+ \sum_{l=I,II,III} \sqrt{J_l} \cdot \sqrt{J_k} \cdot (-i) \cdot \vec{\tilde{v}}_k^{\,+} \cdot \underline{S} \cdot \delta \underline{\tilde{A}} \cdot \vec{\tilde{v}}_{-l} \cdot e^{i \cdot [\Phi_k + \Phi_l]} \qquad (6.3)
\end{aligned}$$

and

$$\begin{aligned}
Z_k^{(1)}(s) &= i \cdot \lambda_{PH} \cdot \xi(s) \cdot \frac{eV_0}{E_0} \cdot \cos\varphi_0 \cdot \beta_0^2 \cdot \left[\vec{\tilde{v}}_k^{\,+}(s) \cdot A_{-k}(s) \right] \cdot \underline{S} \cdot \vec{\tilde{D}} \\
&= i \cdot \lambda_{PH} \cdot \xi(s) \cdot \frac{eV_0}{E_0} \cdot \cos\varphi_0 \cdot \beta_0^2 \cdot \sqrt{J_k} \cdot e^{i \cdot \Phi_k(s)} \cdot \vec{\tilde{v}}_k^{\,+}(s) \cdot \underline{S} \cdot \vec{\tilde{D}} \; ; \qquad (6.4a)
\end{aligned}$$

$$\begin{aligned}
Z_k^{(2)}(s) &= -i \cdot \lambda_{AM} \cdot \xi(s) \cdot \frac{eV_0}{E_0} \cdot \cos\varphi_0 \cdot h \cdot \frac{2\pi}{L} \cdot \beta_0^2 \cdot \left[\vec{\tilde{v}}_k^{\,+}(s) \cdot A_{-k}(s) \right] \cdot \underline{S} \cdot \vec{\tilde{D}} \\
&\quad \times \left\{ \tilde{\sigma} + (1/\beta_0^2) \cdot [\tilde{p}_x \cdot D_1 - \tilde{x} \cdot D_2 + \tilde{p}_z \cdot D_3 - \tilde{z} \cdot D_4] \right\} \\
&= -i \cdot \lambda_{AM} \cdot \xi(s) \cdot \frac{eV_0}{E_0} \cdot \cos\varphi_0 \cdot h \cdot \frac{2\pi}{L} \cdot \beta_0^2 \cdot \sqrt{J_k} \cdot e^{i \cdot \Phi_k(s)} \cdot \vec{\tilde{v}}_k^{\,+}(s) \cdot \underline{S} \cdot \vec{\tilde{D}} \\
&\quad \times \left\{ \tilde{\sigma} + (1/\beta_0^2) \cdot [\tilde{p}_x \cdot D_1 - \tilde{x} \cdot D_2 + \tilde{p}_z \cdot D_3 - \tilde{z} \cdot D_4] \right\} \; . \qquad (6.4b)
\end{aligned}$$

If we write then $J'_k(s)$ and $\Phi'_k(s)$ in the form:

$$J'_k(s) = K_J^{(k)}(\Phi_l, J_l) + Q_J^{(k)}(\Phi_l, J_l) \cdot \xi(s) \; ; \qquad (6.5a)$$

$$\Phi'_k(s) = K_\Phi^{(k)}(\Phi_l, J_l) + Q_\Phi^{(k)}(\Phi_l, J_l) \cdot \xi(s) \qquad (6.5b)$$

we obtain, using eqn. (5.5):

$$\begin{aligned}
K_J^{(k)} &= 2 \cdot \Re e \left\{ Y_k(s) \right\} \; ; \\
Q_J^{(k)} &= i \cdot \frac{eV_0}{E_0} \cdot \cos\varphi_0 \cdot \beta_0^2 \cdot \sqrt{J_k} \cdot \left\{ \vec{\tilde{v}}_k^{\,+} \underline{S} \vec{\tilde{D}} \cdot e^{i \cdot \Phi_k} - \left[\vec{\tilde{v}}_k^{\,+} \underline{S} \vec{\tilde{D}} \right]^* \cdot e^{-i \cdot \Phi_k} \right\} \\
&\times \left\{ \lambda_{PH} - \lambda_{AM} \cdot h \cdot \frac{2\pi}{L} \left[\tilde{\sigma} + \frac{1}{\beta_0^2} \cdot (\tilde{p}_x \cdot D_1 - \tilde{x} \cdot D_2 + \tilde{p}_z \cdot D_3 - \tilde{z} \cdot D_4) \right] \right\} ; \qquad (6.6a)
\end{aligned}$$

$$
\begin{aligned}
K_{\Phi}^{(k)} &= \frac{2\pi}{L} \cdot Q_k - \frac{1}{J_k(s)} \cdot \Im m \{Y_k(s)\} \ ; \\
Q_{\Phi}^{(k)} &= -\frac{eV_0}{E_0} \cdot \cos\varphi_0 \cdot \beta_0^2 \cdot \frac{1}{2\sqrt{J_k}} \cdot \left\{ \vec{\hat{v}}_k^{+} \underline{S} \vec{\hat{D}} \cdot e^{i \cdot \Phi_k} + \left[\vec{\hat{v}}_k^{+} \underline{S} \vec{\hat{D}} \right]^* \cdot e^{-i \cdot \Phi_k} \right\} \\
&\times \left\{ \lambda_{PH} - \lambda_{AM} \cdot h \cdot \frac{2\pi}{L} \left[\tilde{\sigma} + \frac{1}{\beta_0^2} \cdot (\tilde{p}_x \cdot D_1 - \tilde{x} \cdot D_2 + \tilde{p}_z \cdot D_3 - \tilde{z} \cdot D_4) \right] \right\} . \qquad (6.6b)
\end{aligned}
$$

The stochastic equations (6.4) for the variables J_k and Φ_k ($k = I, II, III,$) together with (6.6a,b) are now the basis for a Fokker-Planck treatment of stochastic particle motion in storage rings.

7 The Fokker-Planck Equation for Cavity Noise

The Fokker-Planck equation for the phase space density function W (in the Stratonowich-version) now reads as [12]:

$$
\begin{aligned}
\frac{\partial W}{\partial s} &= \sum_{k=I,II,III} \left\{ -\frac{\partial}{\partial J_k}[D_J^{(k)} \cdot W] - \frac{\partial}{\partial \Phi_k}[D_{\Phi}^{(k)} \cdot W] \right\} \qquad (7.1) \\
&+ \sum_{k,l=I,II,III} \left\{ \frac{1}{2} \frac{\partial^2}{\partial J_k \partial J_l}[Q_J^{(k)} \cdot Q_J^{(l)} \cdot W] + \frac{\partial^2}{\partial J_k \partial \Phi_l}[Q_J^{(k)} \cdot Q_{\Phi}^{(l)} \cdot W] \right. \\
&\qquad \left. + \frac{1}{2} \frac{\partial^2}{\partial \Phi_k \partial \Phi_l}[Q_{\Phi}^{(k)} \cdot Q_{\Phi}^{(l)} \cdot W] \right\}
\end{aligned}
$$

with

$$
\begin{aligned}
D_J^{(k)} &= K_J^{(k)} + \tilde{K}_J^{(k)} \ ; \qquad (7.2a) \\
D_{\Phi}^{(k)} &= K_{\Phi}^{(k)} + \tilde{K}_{\Phi}^{(k)} \qquad (7.2b)
\end{aligned}
$$

where the quantities $\tilde{K}_J^{(k)}$ and $\tilde{K}_{\Phi}^{(k)}$ contain the drift terms:

$$
\begin{aligned}
\tilde{K}_J^{(k)} &= \frac{1}{2} \sum_{l=I,II,III} \left\{ \frac{\partial Q_J^{(k)}}{\partial J_l} \cdot Q_J^{(l)} + \frac{\partial Q_J^{(k)}}{\partial \Phi_l} \cdot Q_{\Phi}^{(l)} \right\} \ ; \qquad (7.3a) \\
\tilde{K}_{\Phi}^{(k)} &= \frac{1}{2} \sum_{l=I,II,III} \left\{ \frac{\partial Q_{\Phi}^{(k)}}{\partial J_l} \cdot Q_J^{(l)} + \frac{\partial Q_{\Phi}^{(k)}}{\partial \Phi_l} \cdot Q_{\Phi}^{(l)} \right\} . \qquad (7.3b)
\end{aligned}
$$

Introducing the time t by the relation:

$$ds = v_0 \cdot dt$$

and using a long time scale [12] which is comparable with the oscillation time of synchro-betatron motion we make an average around several circumferences, which we indicate by a bracket $\langle \quad \rangle$, and then write the Fokker-Planck equation in the form:

$$
\begin{aligned}
\frac{1}{v_0} \frac{\partial W}{\partial t} &= \sum_{k=I,II,III} \left\{ -\frac{\partial}{\partial J_k}[\langle D_J^{(k)} \rangle \cdot W] - \frac{\partial}{\partial \Phi_k}[\langle D_{\Phi}^{(k)} \rangle \cdot W] \right\} \qquad (7.4) \\
&+ \sum_{k,l=I,II,III} \left\{ \frac{1}{2} \frac{\partial^2}{\partial J_k \partial J_l}[\langle Q_J^{(k)} \cdot Q_J^{(l)} \rangle \cdot W] + \frac{\partial^2}{\partial J_k \partial \Phi_l}[\langle Q_J^{(k)} \cdot Q_{\Phi}^{(l)} \rangle \cdot W] \right. \\
&\qquad \left. + \frac{1}{2} \frac{\partial^2}{\partial \Phi_k \partial \Phi_l}[\langle Q_{\Phi}^{(k)} \cdot Q_{\Phi}^{(l)} \rangle \cdot W] \right\}
\end{aligned}
$$

whereby oscillating terms of the integrand due to the (linear) s-dependence of the angle variables Φ_k:

$$\Phi_k(s) = \Phi_k(0) + (s - s_0) \cdot \frac{2\pi}{L} Q_k$$

(see eqn. (4.22b)) may be neglected since they are approximately averaged away by integration.

In order to calculate the coefficients $\langle D_J^{(k)} \rangle$, $\langle D_\Phi^{(k)} \rangle$, $\langle Q_J^{(k)} \cdot Q_J^{(l)} \rangle$, $\langle Q_J^{(k)} \cdot Q_\Phi^{(l)} \rangle$ and $\langle Q_\Phi^{(k)} \cdot Q_\Phi^{(l)} \rangle$ appearing in (7.4), we first introduce the following abbreviation:

$$\begin{aligned} \delta Q_k &= \frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} ds \cdot \vec{v}_{\hat{k}}^{\,+}(s) \cdot \underline{S} \cdot \delta \underline{\tilde{A}}(s) \cdot \vec{v}_{\hat{k}}(s) \\ &= \frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} ds \cdot \vec{v}_k^{\,+}(s) \cdot \underline{S} \cdot \delta \underline{\tilde{A}}(s) \cdot \vec{v}_k(s) \ ; \end{aligned} \tag{7.5}$$

$$(k = I,\ II,\ III)$$

so that one can write using (6.3a,.b) and (3.22c) :

$$\langle Y_k \rangle = J_k \cdot (-i) \cdot \frac{2\pi}{L} \delta Q_k \ . \tag{7.6}$$

Since

$$\begin{aligned} \delta Q_k^* &= \frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} ds \cdot \left\{ \vec{v}_k^{\,+}(s) \cdot \underline{S} \cdot \delta \underline{\tilde{A}}(s) \cdot \vec{v}_k(s) \right\}^+ \\ &= -\frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} ds \cdot \vec{v}_k^{\,+}(s) \cdot \delta \underline{\tilde{A}}^{\,T}(s) \cdot \underline{S} \cdot \vec{v}_k(s) \end{aligned}$$

it follows from eqn. (3.34) that δQ_k is a real number:

$$\Im m\{\delta Q_k\} = 0 \ . \tag{7.7}$$

In Appendix B it is shown that the quantity δQ_k appearing in (7.6) is just the Q-shift of the k-th oscillation mode ($k = I, II, III$) caused by the (linear) perturbation $\delta \underline{A}$.

Thus we obtain from (6.6) using (7.6) and (7.7):

$$\langle K_J^{(k)} \rangle = 0 \ ; \tag{7.8a}$$

$$\langle K_\Phi^{(k)} \rangle = \frac{2\pi}{L} \cdot \hat{Q}_k \tag{7.8b}$$

with

$$\hat{Q}_k = Q_k + \Re e\{\delta Q_k\} \ . \tag{7.9}$$

The further investigations shall be done for phase noise and cavity noise separately.

Remarks:

1) The reality of δQ_k:

$$\Re e\{\delta Q_k\} = \delta Q_k$$

results from the fact that the perturbation $\delta\underline{A}$ only contains symplectic terms (see eqns. (3.18b) and (3.24)). An imaginary part of δQ_k which would lead to a damping of the oscillation modes [3] only appears if the perturbation terms are nonsymplectic.

2) The quantity $\acute{Q}_k$ defined by eqn. (7.9) designates the whole Q-value in the presence of synchro-betatron coupling (3.15b) which results from non-vanishing dispersion in the cavities.

7.1 Phase Noise

From eqn. (6.6) one obtains the relations:

$$Q_J^{(k)} = i \cdot \frac{eV_0}{E_0} \cdot \cos\varphi_0 \cdot \beta_0^2 \cdot \sqrt{J_k} \left\{ \vec{\hat{v}}_k^{+} \underline{S} \vec{\hat{D}} \cdot e^{i \cdot \Phi_k} - \left[\vec{\hat{v}}_k^{+} \underline{S} \vec{\hat{D}} \right]^* \cdot e^{-i \cdot \Phi_k} \right\} \cdot \lambda_{PH} \; ; \quad (7.10a)$$

$$Q_\Phi^{(k)} = -\frac{eV_0}{E_0} \cdot \cos\varphi_0 \cdot \beta_0^2 \cdot \frac{1}{2\sqrt{J_k}} \left\{ \vec{\hat{v}}_k^{+} \underline{S} \vec{\hat{D}} \cdot e^{i \cdot \Phi_k} + \left[\vec{\hat{v}}_k^{+} \underline{S} \vec{\hat{D}} \right]^* \cdot e^{-i \cdot \Phi_k} \right\} \cdot \lambda_{PH} \quad (7.10b)$$

which lead to:

$$\langle (Q_J^{(k)})^2 \rangle = 2J_k \cdot \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot |\vec{\tilde{v}}_k^{+} \underline{S} \vec{\tilde{D}}|^2 \cdot \left[\frac{eV_0(\tilde{s})}{E_0} \cdot \beta_0^2 \cdot \lambda_{PH} \right]^2 \; ; \quad (7.11a)$$

$$\langle (Q_\Phi^{(k)})^2 \rangle = \frac{1}{2J_k} \cdot \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot |\vec{\tilde{v}}_k^{+} \underline{S} \vec{\tilde{D}}|^2 \cdot \left[\frac{eV_0(\tilde{s})}{E_0} \cdot \beta_0^2 \cdot \lambda_{PH} \right]^2 \quad (7.11b)$$

and

$$\langle Q_J^{(k)} \cdot Q_J^{(l)} \rangle = 0 \;\text{ for }\; k \neq l \; ; \quad (7.12a)$$

$$\langle Q_\Phi^{(k)} \cdot Q_\Phi^{(l)} \rangle = 0 \;\text{ for }\; k \neq l \; ; \quad (7.12b)$$

$$\langle Q_J^{(k)} \cdot Q_\Phi^{(l)} \rangle = 0 \; . \quad (7.12c)$$

Furthermore one has, taking into account (7.3a, b):

$$\tilde{K}_J^{(k)} = \left[\frac{eV_0(s)}{E_0} \cdot \beta_0^2 \cdot \lambda_{PH} \right]^2 \cdot |\vec{\hat{v}}_k^{+} \underline{S} \vec{\hat{D}}|^2$$

$$= \left[\frac{eV_0(s)}{E_0} \cdot \beta_0^2 \cdot \lambda_{PH} \right]^2 \cdot |\vec{v}_k^{+} \underline{S} \vec{D}|^2 \; ; \quad (7.13a)$$

$$\tilde{K}_\Phi^{(k)} = \frac{i}{4J_k} \cdot \left[\frac{eV_0(s)}{E_0} \cdot \beta_0^2 \cdot \lambda_{PH} \right]^2 \times \left\{ \left[\vec{\hat{v}}_k^{+} \underline{S} \vec{\hat{D}} \right]^2 e^{i \cdot 2\Phi_k} - \left[\left(\vec{\hat{v}}_k^{+} \underline{S} \vec{\hat{D}} \right)^* \right]^2 e^{-i \cdot 2\Phi_k} \right\} \quad (7.13b)$$

and thus

$$\langle \tilde{K}_J^{(k)} \rangle = \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot |\vec{v}_k^{+} \underline{S} \vec{D}|^2 \cdot \left[\frac{eV_0(\tilde{s})}{E_0} \cdot \beta_0^2 \cdot \lambda_{PH} \right]^2 \; ; \quad (7.14a)$$

$$\langle \tilde{K}_\Phi^{(k)} \rangle = 0 \; . \quad (7.14b)$$

From (7.2a, b), (7.8a, b) and (7.14a, b) we then get:

$$\langle D_J^{(k)}\rangle = \frac{1}{L}\int_{s_0}^{s_0+L} d\tilde{s}\cdot|\vec{v}_k^{+}\underline{S}\vec{\tilde{D}}|^2\cdot\left[\frac{eV_0(\tilde{s})}{E_0}\cdot\beta_0^2\cdot\lambda_{PH}\right]^2 ; \tag{7.15a}$$

$$\langle D_\Phi^{(k)}\rangle = \frac{2\pi}{L}\cdot\hat{Q}_k . \tag{7.15b}$$

Introducing the constants

$$M_k = v_0\cdot\frac{1}{L}\int_{s_0}^{s_0+L} d\tilde{s}\cdot|\vec{v}_k^{+}\underline{S}\vec{\tilde{D}}|^2\left[\frac{eV_0(\tilde{s})}{E_0}\cdot\beta_0^2\cdot\lambda_{PH}\right]^2 ; \tag{7.16a}$$

$$b_k = 2\pi\cdot\frac{v_0}{L}\cdot\hat{Q}_k \tag{7.16b}$$

we may finally write:

$$\begin{aligned}\frac{\partial W}{\partial t} &= \sum_{k=I,II,III}\left\{-\frac{\partial}{\partial J_k}[+M_k\cdot W]-\frac{\partial}{\partial\Phi_k}[b_k\cdot W]\right.\\ &\qquad\left.+\frac{1}{2}\frac{\partial^2}{\partial J_k^2}[2J_k\cdot M_k\cdot W]+\frac{1}{2}\frac{\partial^2}{\partial\Phi_k^2}\left[\frac{1}{2J_k}\cdot M_k\cdot W\right]\right\}\\ &= \sum_{k=I,II,III}\left\{-\frac{\partial}{\partial J_k}\left[+M_k\cdot W-\frac{\partial}{\partial J_k}(J_k\cdot M_k\cdot W)\right]\right.\\ &\qquad\left.-\frac{\partial}{\partial\Phi_k}\left[b_k\cdot W-\frac{1}{4J_k}\cdot M_k\cdot\frac{\partial}{\partial\Phi_k}W\right]\right\}\\ &= \sum_{k=I,II,III}\left\{-\frac{\partial}{\partial J_k}\left[-M_k\cdot J_k\cdot\frac{\partial W}{\partial J_k}\right]\right.\\ &\qquad\left.-\frac{\partial}{\partial\Phi_k}\left[b_k\cdot W-\frac{M_k}{4J_k}\cdot\frac{\partial W}{\partial\Phi_k}\right]\right\} .\end{aligned} \tag{7.17}$$

With the ansatz

$$W = w_I(J_I,\Phi_I)\cdot w_{II}(J_{II},\Phi_{II})\cdot w_{III}(J_{III},\Phi_{III}) \tag{7.18}$$

eqn. (7.17) simplifies to:

$$\frac{\partial w_k}{\partial t} = -\frac{\partial}{\partial J_k}\left[-M_k\cdot J_k\cdot\frac{\partial w_k}{\partial J_k}\right]-\frac{\partial}{\partial\Phi_k}\left[b_k\cdot w_k-\frac{M_k}{4J_k}\cdot\frac{\partial w_k}{\partial\Phi_k}\right] . \tag{7.19}$$

Remark:

For $k = III$ (synchrotron oscillation) one gets using (5.5) and (4.27b) for the coefficient M_k in (7.15a):

$$\begin{aligned}M_{III} &= v_0\cdot\frac{1}{L}\int_{s_0}^{s_0+L} d\tilde{s}\cdot(1/\beta_0^4)\cdot|\vec{v}_{III5}|^2\left[\frac{eV_0(\tilde{s})}{E_0}\cdot\beta_0^2\cdot\lambda_{PH}\right]^2\\ &= v_0\cdot\frac{1}{L}\int_{s_0}^{s_0+L} d\tilde{s}\cdot\frac{1}{2}\beta_\sigma(\tilde{s})\cdot\left[\frac{eV_0(\tilde{s})}{E_0}\cdot\lambda_{PH}\right]^2\end{aligned} \tag{7.20}$$

or, using the oscillator model (see eqns. (4.31) and (4.36b)):

$$M_{III} \;=\; \frac{1}{2} \cdot \lambda_{PH}^2 \cdot v_0 \cdot \beta_0^4 \cdot \frac{\kappa \cdot \Omega^3}{(\kappa - 1/\gamma_0^2)^2} \cdot \left(\frac{L}{2\pi k}\right)^2 . \tag{7.21}$$

7.2 Amplitude Noise

Using the smooth approximation [12]:

$$\left\{\begin{array}{l} x \approx \eta \cdot D_1 \; ; \\ p_x \approx \eta \cdot D_2 \; ; \\ z \approx \eta \cdot D_3 \; ; \\ p_z \approx \eta \cdot D_4 \; ; \end{array}\right. \quad \Longrightarrow \quad \left\{\begin{array}{l} \tilde{x} \approx 0 \; ; \\ \tilde{p}_x \approx 0 \; ; \\ \tilde{z} \approx 0 \; ; \\ \tilde{p}_z \approx 0 \end{array}\right. \tag{7.22}$$

one obtains from (6.6) the relations:

$$\begin{aligned} Q_J^{(k)} \;&=\; -i \cdot \frac{eV_0}{E_0} \cdot \cos\varphi_0 \cdot \beta_0^2 \cdot \sqrt{J_k} \cdot \left\{ \vec{\hat{v}}_k^{+} \underline{S} \vec{\hat{D}} \cdot e^{i \cdot \Phi_k} - \left[\vec{\hat{v}}_k^{+} \underline{S} \vec{\hat{D}} \right]^* \cdot e^{-i \cdot \Phi_k} \right\} \\ &\qquad \times h \cdot \frac{2\pi}{L} \cdot \tilde{\sigma} \cdot \lambda_{AM} \; ; \end{aligned} \tag{7.23a}$$

$$\begin{aligned} Q_\Phi^{(k)} \;&=\; +\frac{eV_0}{E_0} \cdot \cos\varphi_0 \cdot \beta_0^2 \cdot \frac{1}{2\sqrt{J_k}} \cdot \left\{ \vec{\hat{v}}_k^{+} \underline{S} \vec{\hat{D}} \cdot e^{i \cdot \Phi_k} + \left[\vec{\hat{v}}_k^{+} \underline{S} \vec{\hat{D}} \right]^* \cdot e^{-i \cdot \Phi_k} \right\} \\ &\qquad \times h \cdot \frac{2\pi}{L} \cdot \tilde{\sigma} \cdot \lambda_{AM} \end{aligned} \tag{7.23b}$$

with (see eqns. (4.16), (4.27) and (4.48))

$$\begin{aligned} \tilde{\sigma} \;&=\; \sqrt{J_k} \cdot \left\{ \hat{v}_{k5} \cdot e^{-i\Phi_k} + [\hat{v}_{k5}]^* \cdot e^{+i\Phi_k} \right\} \quad \text{for} \quad k = III \\ &=\; \sqrt{J_\sigma} \cdot \left\{ \hat{w}_{\sigma 1} \cdot e^{-i\Phi_\sigma} + [\hat{w}_{\sigma 1}]^* \cdot e^{+i\Phi_\sigma} \right\} \end{aligned} \tag{7.24}$$

where we have introduced the notation:

$$\Phi_\sigma \equiv \Phi_{III} \; ; \quad J_\sigma \equiv J_{III} \; .$$

These equations lead to:

$$\begin{aligned} Q_J^{(k)} \cdot Q_J^{(l)} \;&=\; -\left[\frac{eV_0}{E_0} \cdot \beta_0^2 \cdot h \cdot \frac{2\pi}{L} \cdot \lambda_{AM} \right]^2 \cdot \sqrt{J_k J_l} \cdot J_\sigma \\ &\qquad \times \left\{ \vec{\hat{v}}_k^{+} \underline{S} \vec{\hat{D}} \cdot e^{i \cdot \Phi_k} - \left[\vec{\hat{v}}_k^{+} \underline{S} \vec{\hat{D}} \right]^* \cdot e^{-i \cdot \Phi_k} \right\} \\ &\qquad \times \left\{ \vec{\hat{v}}_l^{+} \underline{S} \vec{\hat{D}} \cdot e^{i \cdot \Phi_l} - \left[\vec{\hat{v}}_k^{+} \underline{S} \vec{\hat{D}} \right]^* \cdot e^{-i \cdot \Phi_l} \right\} \\ &\qquad \times \left\{ \hat{w}_{\sigma 1} \cdot e^{-i\Phi_\sigma} + [\hat{w}_{\sigma 1}]^* \cdot e^{+i\Phi_\sigma} \right\}^2 \; ; \end{aligned} \tag{7.25a}$$

$$Q_\Phi^{(k)} \cdot Q_\Phi^{(l)} \;=\; +\left[\frac{eV_0}{E_0} \cdot \beta_0^2 \cdot h \cdot \frac{2\pi}{L} \cdot \lambda_{AM} \right]^2 \cdot \frac{1}{4} \frac{1}{\sqrt{J_k J_l}} \cdot J_\sigma$$

$$\times \left\{ \vec{\hat{v}}_k^{+} \underline{S} \vec{\hat{D}} \cdot e^{i \cdot \Phi_k} + \left[\vec{\hat{v}}_k^{+} \underline{S} \vec{\hat{D}} \right]^* \cdot e^{-i \cdot \Phi_k} \right\}$$
$$\times \left\{ \vec{\hat{v}}_l^{+} \underline{S} \vec{\hat{D}} \cdot e^{i \cdot \Phi_l} + \left[\vec{\hat{v}}_l^{+} \underline{S} \vec{\hat{D}} \right]^* \cdot e^{-i \cdot \Phi_l} \right\}$$
$$\times \left\{ \hat{w}_{\sigma 1} \cdot e^{-i\Phi_\sigma} + [\hat{w}_{\sigma 1}]^* \cdot e^{+i\Phi_\sigma} \right\}^2 \; ; \tag{7.25b}$$

$$Q_J^{(k)} \cdot Q_\Phi^{(l)} = -i \left[\frac{eV_0}{E_0} \cdot \beta_0^2 \cdot h \cdot \frac{2\pi}{L} \cdot \lambda_{AM} \right]^2 \cdot \frac{1}{2} \sqrt{\frac{J_k}{J_l}} \cdot J_\sigma$$
$$\times \left\{ \vec{\hat{v}}_k^{+} \underline{S} \vec{\hat{D}} \cdot e^{i \cdot \Phi_k} - \left[\vec{\hat{v}}_k^{+} \underline{S} \vec{\hat{D}} \right]^* \cdot e^{-i \cdot \Phi_k} \right\}$$
$$\times \left\{ \vec{\hat{v}}_l^{+} \underline{S} \vec{\hat{D}} \cdot e^{i \cdot \Phi_l} + \left[\vec{\hat{v}}_l^{+} \underline{S} \vec{\hat{D}} \right]^* \cdot e^{-i \cdot \Phi_l} \right\}$$
$$\times \left\{ \hat{w}_{\sigma 1} \cdot e^{-i\Phi_\sigma} + [\hat{w}_{\sigma 1}]^* \cdot e^{+i\Phi_\sigma} \right\}^2 \; . \tag{7.25c}$$

From (7.25a, b, c) one gets the equations:

$$\langle Q_J^{(k)} \cdot Q_J^{(l)} \rangle = \langle [Q_J^{(k)}]^2 \rangle \cdot \delta_{kl} \; ; \tag{7.26a}$$

$$\langle Q_\Phi^{(k)} \cdot Q_\Phi^{(l)} \rangle = \langle [Q_\Phi^{(k)}]^2 \rangle \cdot \delta_{kl} \; ; \tag{7.26b}$$

$$\langle Q_J^{(k)} \cdot Q_\Phi^{(l)} \rangle = 0 \tag{7.26c}$$

with

$$\langle [Q_J^{(III)}]^2 \rangle = J_\sigma^2 \cdot 2 \cdot \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot |w_{\sigma 1}|^4 \cdot \left[\frac{eV_0}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \lambda_{AM} \right]^2 \; ; \tag{7.27a}$$

$$\langle [Q_\Phi^{(III)}]^2 \rangle = \frac{3}{2} \cdot \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot |w_{\sigma 1}|^4 \cdot \left[\frac{eV_0}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \lambda_{AM} \right]^2 \tag{7.27b}$$

and

$$\langle [Q_J^{(k)}]^2 \rangle = 4 \cdot J_k \cdot J_\sigma \cdot \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot |w_{\sigma 1}|^2 \cdot |\vec{\hat{v}}_k^{+} \underline{S} \vec{\hat{D}}|^2 \cdot \left[\frac{eV_0}{E_0} \cdot \beta_0^2 \cdot h \cdot \frac{2\pi}{L} \cdot \lambda_{AM} \right]^2 \; ; \tag{7.28a}$$

$$\langle [Q_\Phi^{(k)}]^2 \rangle = \frac{J_\sigma}{J_k} \cdot \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot |w_{\sigma 1}|^2 \cdot |\vec{\hat{v}}_k^{+} \underline{S} \vec{\hat{D}}|^2 \cdot \left[\frac{eV_0}{E_0} \cdot \beta_0^2 \cdot h \cdot \frac{2\pi}{L} \cdot \lambda_{AM} \right]^2 \tag{7.28b}$$

for $k = I, \; II$.

Furthermore one has, taking into account (7.3a, b) :

a) For $k = I, \; II$:

$$\tilde{K}_J^{(k)} = \frac{1}{2} \cdot \left\{ \frac{\partial Q_J^{(k)}}{\partial J_k} \cdot Q_J^{(k)} + \frac{\partial Q_J^{(k)}}{\partial \Phi_k} \cdot Q_\Phi^{(k)} \right\} +$$
$$\frac{1}{2} \cdot \left\{ \frac{\partial Q_J^{(k)}}{\partial J_\sigma} \cdot Q_J^{(III)} + \frac{\partial Q_J^{(k)}}{\partial \Phi_\sigma} \cdot Q_\Phi^{(III)} \right\}$$

$$
\begin{aligned}
= \quad & -\frac{1}{4}\cdot\left[\frac{eV_0(s)}{E_0}\cdot\beta_0^2\cdot h\cdot\frac{2\pi}{L}\cdot\lambda_{AM}\right]^2\cdot J_\sigma \\
& \quad\times\left\{\vec{\hat{v}}_k^{+}\underline{S}\vec{\hat{D}}\cdot e^{i\cdot\Phi_k}-\left[\vec{\hat{v}}_k^{+}\underline{S}\vec{\hat{D}}\right]^*\cdot e^{-i\cdot\Phi_k}\right\}^2 \\
& \quad\times\left\{\hat{w}_{\sigma 1}\cdot e^{-i\Phi_\sigma}+[\hat{w}_{\sigma 1}]^*\cdot e^{+i\Phi_\sigma}\right\}^2 \\
& +\frac{1}{4}\cdot\left[\frac{eV_0(s)}{E_0}\cdot\beta_0^2\cdot h\cdot\frac{2\pi}{L}\cdot\lambda_{AM}\right]^2\cdot J_\sigma \\
& \quad\times\left\{\vec{\hat{v}}_k^{+}\underline{S}\vec{\hat{D}}\cdot e^{i\cdot\Phi_k}+\left[\vec{\hat{v}}_k^{+}\underline{S}\vec{\hat{D}}\right]^*\cdot e^{-i\cdot\Phi_k}\right\}^2 \\
& \quad\times\left\{\hat{w}_{\sigma 1}\cdot e^{-i\Phi_\sigma}+[\hat{w}_{\sigma 1}]^*\cdot e^{+i\Phi_\sigma}\right\}^2 \\
& -\frac{1}{4}\cdot\left[\frac{eV_0(s)}{E_0}\cdot\beta_0^2\cdot h\cdot\frac{2\pi}{L}\cdot\lambda_{AM}\right]^2\cdot\sqrt{J_k}\cdot\sqrt{J_\sigma} \\
& \quad\times\left\{\vec{\hat{v}}_k^{+}\underline{S}\vec{\hat{D}}\cdot e^{i\cdot\Phi_k}-\left[\vec{\hat{v}}_k^{+}\underline{S}\vec{\hat{D}}\right]^*\cdot e^{-i\cdot\Phi_k}\right\} \\
& \quad\times\left\{\hat{w}_{\sigma 1}\cdot e^{-i\Phi_\sigma}+[\hat{w}_{\sigma 1}]^*\cdot e^{+i\Phi_\sigma}\right\} \\
& \quad\times\left\{[(\hat{w}_{\sigma 1})^*]^2\cdot e^{2i\cdot\Phi_\sigma}-[\hat{w}_{\sigma 1}]^2\cdot e^{-2i\cdot\Phi_\sigma}\right\} \\
& -\frac{1}{4}\cdot\left[\frac{eV_0(s)}{E_0}\cdot\beta_0^2\cdot h\cdot\frac{2\pi}{L}\cdot\lambda_{AM}\right]^2\cdot\sqrt{J_k}\cdot\sqrt{J_\sigma} \\
& \quad\times\left\{\vec{\hat{v}}_k^{+}\underline{S}\vec{\hat{D}}\cdot e^{i\cdot\Phi_k}-\left[\vec{\hat{v}}_k^{+}\underline{S}\vec{\hat{D}}\right]^*\cdot e^{-i\cdot\Phi_k}\right\} \\
& \quad\times\left\{\hat{w}_{\sigma 1}\cdot e^{-i\Phi_\sigma}-[\hat{w}_{\sigma 1}]^*\cdot e^{+i\Phi_\sigma}\right\} \\
& \quad\times\left\{\hat{w}_{\sigma 1}\cdot e^{-i\Phi_\sigma}+[\hat{w}_{\sigma 1}]^*\cdot e^{+i\Phi_\sigma}\right\}^2 \quad ; \qquad (7.29a)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{K}_\Phi^{(k)} = \quad & \frac{1}{2}\cdot\left\{\frac{\partial Q_\Phi^{(k)}}{\partial J_k}\cdot Q_J^{(k)}+\frac{\partial Q_\Phi^{(k)}}{\partial\Phi_k}\cdot Q_\Phi^{(k)}\right\}+ \\
& \frac{1}{2}\cdot\left\{\frac{\partial Q_\Phi^{(k)}}{\partial J_\sigma}\cdot Q_J^{(III)}+\frac{\partial Q_\Phi^{(k)}}{\partial\Phi_\sigma}\cdot Q_\Phi^{(III)}\right\} \\
= \quad & +\frac{1}{8}\cdot\left[\frac{eV_0(s)}{E_0}\cdot\beta_0^2\cdot h\cdot\frac{2\pi}{L}\cdot\lambda_{AM}\right]^2\cdot\frac{J_\sigma}{J_k}\cdot i \\
& \quad\times\left\{\vec{\hat{v}}_k^{+}\underline{S}\vec{\hat{D}}\cdot e^{i\cdot\Phi_k}-\left[\vec{\hat{v}}_k^{+}\underline{S}\vec{\hat{D}}\right]^*\cdot e^{-i\cdot\Phi_k}\right\} \\
& \quad\times\left\{\vec{\hat{v}}_k^{+}\underline{S}\vec{\hat{D}}\cdot e^{i\cdot\Phi_k}+\left[\vec{\hat{v}}_k^{+}\underline{S}\vec{\hat{D}}\right]^*\cdot e^{-i\cdot\Phi_k}\right\} \\
& \quad\times\left\{\hat{w}_{\sigma 1}\cdot e^{-i\Phi_\sigma}+[\hat{w}_{\sigma 1}]^*\cdot e^{+i\Phi_\sigma}\right\}^2 \\
& +\frac{1}{8}\cdot\left[\frac{eV_0(s)}{E_0}\cdot\beta_0^2\cdot h\cdot\frac{2\pi}{L}\cdot\lambda_{AM}\right]^2\cdot\frac{J_\sigma}{J_k}\cdot i
\end{aligned}
$$

$$
\begin{aligned}
&\times \left\{ \vec{\hat{v}}_k^+ \underline{S} \vec{\hat{D}} \cdot e^{i \cdot \Phi_k} - \left[\vec{\hat{v}}_k^+ \underline{S} \vec{\hat{D}} \right]^* \cdot e^{-i \cdot \Phi_k} \right\} \\
&\times \left\{ \vec{\hat{v}}_k^+ \underline{S} \vec{\hat{D}} \cdot e^{i \cdot \Phi_k} + \left[\vec{\hat{v}}_k^+ \underline{S} \vec{\hat{D}} \right]^* \cdot e^{-i \cdot \Phi_k} \right\} \\
&\times \left\{ \hat{w}_{\sigma 1} \cdot e^{-i\Phi_\sigma} - [\hat{w}_{\sigma 1}]^* \cdot e^{+i\Phi_\sigma} \right\}^2 \\
-\frac{1}{8} \cdot &\left[\frac{eV_0(s)}{E_0} \cdot \beta_0^2 \cdot h \cdot \frac{2\pi}{L} \cdot \lambda_{AM} \right]^2 \cdot \beta_0^{-2} \cdot \sqrt{\frac{J_\sigma}{J_k}} \cdot i \\
&\times \left\{ \vec{\hat{v}}_k^+ \underline{S} \vec{\hat{D}} \cdot e^{i \cdot \Phi_k} + \left[\vec{\hat{v}}_k^+ \underline{S} \vec{\hat{D}} \right]^* \cdot e^{-i \cdot \Phi_k} \right\} \\
&\times \left\{ \hat{w}_{\sigma 1} \cdot e^{-i\Phi_\sigma} + [\hat{w}_{\sigma 1}]^* \cdot e^{+i\Phi_\sigma} \right\} \\
&\times \left\{ [(\hat{w}_{\sigma 1}^*]^2 \cdot e^{2i \cdot \Phi_\sigma} - [\hat{w}_{\sigma 1}]^2 \cdot e^{-2i \cdot \Phi_\sigma} \right\} \\
-\frac{1}{8} \cdot &\left[\frac{eV_0(s)}{E_0} \cdot \beta_0^2 \cdot h \cdot \frac{2\pi}{L} \cdot \lambda_{AM} \right]^2 \cdot \beta_0^{-2} \cdot \sqrt{\frac{J_\sigma}{J_k}} \cdot i \\
&\times \left\{ \vec{\hat{v}}_k^+ \underline{S} \vec{\hat{D}} \cdot e^{i \cdot \Phi_k} + \left[\vec{\hat{v}}_k^+ \underline{S} \vec{\hat{D}} \right]^* \cdot e^{-i \cdot \Phi_k} \right\} \\
&\times \left\{ \hat{w}_{\sigma 1} \cdot e^{-i\Phi_\sigma} - [\hat{v}_{\sigma 1}]^* \cdot e^{+i\Phi_\sigma} \right\} \\
&\times \left\{ \hat{w}_{\sigma 1} \cdot e^{-i\Phi_\sigma} + [\hat{v}_{\sigma 1}]^* \cdot e^{+i\Phi_\sigma} \right\}^2 \qquad (7.29b)
\end{aligned}
$$

and thus:

$$
\langle \tilde{K}_J^{(k)} \rangle = J_\sigma \cdot 2\beta_0^4 \cdot \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot |w_{\sigma 1}|^2 \cdot |\vec{v}_k \underline{S} \vec{\hat{D}}|^2 \cdot \left[\frac{eV_0(\tilde{s})}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \lambda_{AM} \right]^2 ; \qquad (7.30a)
$$

$$
\langle \tilde{K}_\Phi^{(k)} \rangle = 0 ; \qquad (7.30b)
$$

b) for $k = III$:

$$
\begin{aligned}
\tilde{K}_J^{(III)} &= \frac{1}{2} \cdot \left\{ \frac{\partial Q_J^{(III)}}{\partial J_\sigma} \cdot Q_J^{(III)} + \frac{\partial Q_J^{(III)}}{\partial \Phi_\sigma} \cdot Q_\Phi^{(III)} \right\} \\
&= -\frac{1}{2} \cdot \left[\frac{eV_0(s)}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \lambda_{AM} \right]^2 \cdot J_\sigma \\
&\qquad \times \left\{ [(\hat{w}_{\sigma 1})^*)]^2 \cdot e^{2i \cdot \Phi_\sigma} - [\hat{w}_{\sigma 1}]^2 \cdot e^{-2i \cdot \Phi_\sigma} \right\}^2 \\
&\quad +\frac{1}{2} \cdot \left[\frac{eV_0(s)}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \lambda_{AM} \right]^2 \cdot J_\sigma \\
&\qquad \times \left\{ [(\hat{w}_{\sigma 1})^*)]^2 \cdot e^{2i \cdot \Phi_\sigma} + [\hat{w}_{\sigma 1}]^2 \cdot e^{-2i \cdot \Phi_\sigma} \right\} \\
&\qquad \times \left\{ [\hat{w}_{\sigma 1}]^* \cdot e^{i \cdot \Phi_\sigma} + \hat{w}_{\sigma 1} \cdot e^{-i \cdot \Phi_\sigma} \right\}^2 ; \qquad (7.31a)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{K}_{\Phi}^{(III)} &= \frac{1}{2} \cdot \left\{ \frac{\partial Q_{\Phi}^{(III)}}{\partial J_\sigma} \cdot Q_J^{(III)} - \frac{\partial Q_{\Phi}^{(III)}}{\partial \Phi_\sigma} \cdot Q_{\Phi}^{(III)} \right\} \\
&= i \cdot \frac{1}{4} \cdot \left[\frac{eV_0(s)}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \lambda_{AM} \right]^2 \\
&\qquad \times \left\{ |(\hat{w}_{\sigma 1})^*|^2 \cdot e^{2i \cdot \Phi_\sigma} - |\hat{w}_{\sigma 1}|^2 \cdot e^{-2i \cdot \Phi_\sigma} \right\} \\
&\qquad \times \left\{ [\hat{w}_{\sigma 1}]^* \cdot e^{i \cdot \Phi_\sigma} + \hat{w}_{\sigma 1} \cdot e^{-i \cdot \Phi_\sigma} \right\} \qquad (7.31b)
\end{aligned}
$$

and thus:

$$
\langle \tilde{K}_J^{(III)} \rangle = J_\sigma \cdot 2 \cdot \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot |w_{\sigma 1}|^4 \cdot \left[\frac{eV_0(\tilde{s})}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \lambda_{AM} \right]^2 ; \qquad (7.32a)
$$

$$
\langle \tilde{K}_{\Phi}^{(III)} \rangle = 0 . \qquad (7.32b)
$$

From (7.2a, b), (7.8a, b), (7.30a, b) and (7.32a, b) we then get:

$$
\langle D_J^{(k)} \rangle = \begin{cases} J_\sigma \cdot 2\beta_0^4 \cdot \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot |w_{\sigma 1}|^2 | \cdot \vec{v}_k \underline{S} \vec{\tilde{D}}|^2 \cdot \left[\frac{eV_0(\tilde{s})}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \lambda_{AM} \right]^2 & \text{for } k = I,\ II ; \\ J_\sigma \cdot 2 \cdot \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot |w_{\sigma 1}|^4 \cdot \left[\frac{eV_0(\tilde{s})}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \lambda_{AM} \right]^2 & \text{for } k = III \end{cases} \qquad (7.33)
$$

and

$$
\langle D_{\Phi}^{(k)} \rangle = \frac{2\pi}{L} \cdot \hat{Q}_k \quad \text{for } k = I,\ II,\ III . \qquad (7.34)
$$

Introducing the constants:

$$
\begin{cases} \tilde{M}_k = v_0 \cdot \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot |w_{\sigma 1}|^2 | \cdot \vec{v}_k \underline{S} \vec{\tilde{D}}|^2 \cdot \left[\frac{eV_0(\tilde{s})}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \lambda_{AM} \right]^2 \\ \qquad \text{for } k = I,\ II ; \\ \tilde{M}_\sigma = v_0 \cdot \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot |w_{\sigma 1}|^4 \cdot \left[\frac{eV_0(\tilde{s})}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \lambda_{AM} \right]^2 \end{cases}
$$

and

$$
\begin{cases} b_k = 2\pi \cdot \frac{v_0}{L} \cdot \hat{Q}_k \quad \text{for } k = I,\ II ; \\ b_\sigma = 2\pi \cdot \frac{v_0}{L} \cdot \hat{Q}_{III} \end{cases} \qquad (7.35)
$$

we finally may write:

$$\frac{\partial W}{\partial t} = \sum_{k=I,II} \left\{ -\frac{\partial}{\partial J_k}[2\beta_0^4 \cdot J_\sigma \cdot \tilde{M}_k \cdot W] - \frac{\partial}{\partial \Phi_k}[b_k \cdot W] \right.$$

$$\left. + \frac{1}{2}\frac{\partial^2}{\partial J_k^2}[4\beta_0^4 \cdot J_\sigma \cdot J_k \cdot \tilde{M}_k \cdot W] + \frac{1}{2}\frac{\partial^2}{\partial \Phi_k^2}[(J_\sigma / J_k) \cdot \beta_0^4 \cdot \tilde{M}_k \cdot W] \right\}$$

$$- \frac{\partial}{\partial J_\sigma}\left[2J_\sigma \cdot \tilde{M}_\sigma \cdot W\right] - \frac{\partial}{\partial \Phi_\sigma}[b_\sigma \cdot W]$$

$$+ \frac{1}{2}\frac{\partial^2}{\partial J_\sigma^2}\left[J_\sigma^2 \cdot 2\tilde{M}_\sigma \cdot W\right] + \frac{1}{2}\frac{\partial^2}{\partial \Phi_\sigma^2}\left[\frac{3}{2} \cdot \tilde{M}_\sigma \cdot W\right] . \qquad (7.36)$$

Remark:

From eqn. (4.27b) one gets for the coefficient M_σ in (7.32a):

$$M_\sigma = v_0 \cdot \frac{1}{L}\int_{s_0}^{s_0+L} d\tilde{s} \cdot \frac{1}{4}\beta_\sigma^2(\tilde{s}) \cdot \left[\frac{eV_0(\tilde{s})}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \lambda_{AM}\right]^2 \qquad (7.37)$$

or, using the oscillator model (see eqns. (4.31) and (4.36b)):

$$M_\sigma = \frac{1}{4} \cdot \lambda_{AM}^2 \cdot v_0 \cdot \beta_0^4 \cdot \frac{\kappa^2 \cdot \Omega^2}{(\kappa - 1/\beta_0^2)^2} . \qquad (7.38)$$

8 Solution of the Fokker-Planck Equation

8.1 Phase Noise

From eqn. (7.19) we have ($k = I,\ II,\ III$):

$$\frac{\partial w_k}{\partial t} = -\frac{\partial}{\partial J_k}\left[-M_k \cdot J_k \cdot \frac{\partial w_k}{\partial J_k}\right] - \frac{\partial}{\partial \Phi_k}\left[b_k \cdot w_k - \frac{M_k}{4J_k} \cdot \frac{\partial w_k}{\partial \Phi_k}\right] . \qquad (8.1)$$

Since the phase diffusion term

$$\frac{M_k}{4J_k} \cdot \frac{\partial^2 w_k}{\partial^2 \Phi_k}$$

superposed on the phase advance term

$$-b_k \cdot \frac{\partial w_k}{\partial \Phi_k}$$

will lead to a uniform distribution of the phase Φ_k in $[0, 2\pi]$ [12], we make the ansatz:

$$w_k(J_k, \Phi_k) = \frac{1}{2\pi} \cdot \hat{w}_k(J_k) \qquad (8.2)$$

and obtain:

$$\frac{\partial \hat{w}_k}{\partial t} = -\frac{\partial}{\partial J_k}\left[-M_k \cdot J_k \cdot \frac{\partial \hat{w}_k}{\partial J_k}\right]$$

or for abbreviation:

$$\frac{\partial \hat{w}}{\partial t} = \frac{\partial}{\partial J}\left[M \cdot J \cdot \frac{\partial \hat{w}}{\partial J}\right] \tag{8.3}$$

with

$$\begin{cases} \hat{w} = \hat{w}_k ; \\ J = J_k ; \\ M = M_k ; \end{cases} \tag{8.4}$$

$$(k = I,\ II,\ III) .$$

Note that the constant b_k no longer appears in (8.3) and we are left only with the coefficient M_k.

The solution of eqn. (8.3) is given by [13]:

$$\hat{w}(J,t) = \int_0^\infty dJ_0 \cdot K(J,J_0;t) \cdot \rho(J_0) \tag{8.5}$$

($\rho(J_0)$ =arbitrary function of J_0) with

$$K(J,J_0;t) = \frac{1}{M \cdot t} \cdot \exp\left[-\frac{(J+J_0)}{M \cdot t}\right] \cdot I_0\left(\frac{2\sqrt{J \cdot J_0}}{M \cdot t}\right) \tag{8.6}$$

where $I_0(\ )$ is the zeroth-order modified Bessel function.

$K(J,J_0;t)$ is the fundamental solution of eqn. (8.3) and by definition satisfies the initial condition

$$\lim_{t \to 0+} K(J,J_0;t) = \delta(J - J_0) \tag{8.7}$$

($\delta(J - J_0)$ = Dirac delta function) . It then follows that $\rho(J_0)$ is just the initial distribution existing at $t = 0$:

$$\hat{w}(J,0) = \rho(J) . \tag{8.8}$$

The distribution $\hat{w}(J,t)$ can be characterized by the moments

$$m_n(t) = \int_0^\infty dJ \cdot J^n \cdot \hat{w}(J,t) . \tag{8.9}$$

Using the relation:

$$\int_0^\infty dx \cdot e^{-\alpha \cdot x} \cdot I_0(2\sqrt{\beta x}) = \frac{1}{\alpha} \cdot e^{\beta/\alpha}$$

and the integrals which may be obtained by differentiating both sides of (8.10) with respect to α, one is able to calculate the moments corresponding to the fundamental solution (8.6). For $m_1(t)$ and $m_2(t)$ one gets for example:

$$m_1(t) = J_0 + M \cdot t \tag{8.10a}$$

and

$$m_2(t) = J_0^2 + 4MJ_0 \cdot t + 2M^2 \cdot t^2 \tag{8.10b}$$

(see also Ref. [2]).

Thus the time development of the distribution function $w(J,t)$ and of the moments $m_n(t)$ is characterized by the coefficient $M \equiv M_k$ ($k = I, II, III$) which may be interpreted as the reciprocal of the rise time constant τ_k:

$$\tau_k = \frac{1}{M_k} \ . \tag{8.11}$$

8.2 Amplitude Noise

From eqn. (7.36) we obtain:

$$\begin{aligned}
\frac{\partial W}{\partial t} &= \sum_{k=I,II} \left\{ -\frac{\partial}{\partial J_k} \left[2\beta_0^4 \cdot J_\sigma \cdot \tilde{M}_k \cdot W + \frac{\partial}{\partial J_k} \left[2\beta_0^4 \cdot J_\sigma \cdot J_k \cdot \tilde{M}_k \cdot W \right] \right] \right. \\
&\qquad \left. - \frac{\partial}{\partial \Phi_k} [b_k \cdot W] + \frac{1}{2} \frac{\partial^2}{\partial \Phi_k^2} \left[(J_\sigma / J_k) \cdot \beta_0^4 \cdot \tilde{M}_k \cdot W \right] \right\} \\
&+ \frac{\partial}{\partial J_\sigma} \left[-2J_\sigma \cdot \tilde{M}_\sigma \cdot W + \frac{\partial}{\partial J_\sigma} \left[J_\sigma^2 \cdot \tilde{M}_\sigma \cdot W \right] \right] \\
&\qquad - \frac{\partial}{\partial \Phi_\sigma} [b_\sigma \cdot W] + \frac{3}{4} \frac{\partial^2}{\partial \Phi_\sigma^2} \left[\tilde{M}_\sigma \cdot W \right] \ .
\end{aligned} \tag{8.12}$$

As in eqn. (8.1) for the phase noise, we assume a uniform distribution in Φ_I, Φ_{II} and Φ_σ:

$$W(J, \Phi) = \left(\frac{1}{2\pi} \right)^3 \cdot \hat{W}(J_I, J_{II}, J_\sigma) \tag{8.13}$$

and obtain from (8.12):

$$\begin{aligned}
\frac{\partial \hat{W}}{\partial t} &= \sum_{k=I,II} \left\{ -\frac{\partial}{\partial J_k} \left[2\beta_0^4 \cdot J_k \cdot \tilde{M}_k \cdot \hat{W} + \frac{\partial}{\partial J_k} \left[2\beta_0^4 \cdot J_\sigma \cdot J_k \cdot \tilde{M}_k \cdot \hat{W} \right] \right] \right\} \\
&+ \frac{\partial}{\partial J_\sigma} \left[-2J_\sigma \cdot \tilde{M}_\sigma \cdot \hat{W} + \frac{\partial}{\partial J_\sigma} \left[J_\sigma^2 \cdot \tilde{M}_\sigma \cdot \hat{W} \right] \right] \ .
\end{aligned} \tag{8.14}$$

With the ansatz

$$\hat{W} = w(J_\sigma) \tag{8.15}$$

eqn. (8.14) simplifies to:

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\partial}{\partial J_\sigma} \left[-2J_\sigma \cdot \tilde{M}_\sigma \cdot \hat{W} + \frac{\partial}{\partial J_\sigma} \left[J_\sigma^2 \cdot \tilde{M}_\sigma \cdot \hat{W} \right] \right] \\
&= \tilde{M}_\sigma \cdot \frac{\partial}{\partial J_\sigma} \left[J_\sigma^2 \cdot \frac{\partial w}{\partial J_\sigma} \right] \ .
\end{aligned} \tag{8.16}$$

The solution of eqn. (8.16) reads as:

$$w(J_\sigma, t) = \int_0^\infty dJ_0 \cdot K(J_\sigma, J_0; t) \cdot w(J_0, 0) \tag{8.17}$$

with

$$K(J_\sigma, J_0; t) = \frac{1}{J_0 \cdot \sqrt{4\pi \tilde{M}_\sigma \cdot t}} \cdot \exp\left[-\frac{[\tilde{M}_\sigma \cdot t + \ln(J_\sigma / J_o)]^2}{4\tilde{M}_\sigma \cdot t}\right] \tag{8.18}$$

($K(J_\sigma, J_0; t)$= fundamental solution of eqn. (8.16)).

The moments corresponding to the fundamental solution (8.18) grow exponentially with time

$$m_n(t) = J_0^n \cdot e^{n(n+1)\tilde{M}_\sigma \cdot t} \tag{8.19}$$

(see also Ref. [2]).

The rise time constant due to amplitude noise is thus given by:

$$\tilde{\tau}_\sigma = \frac{1}{\tilde{M}_\sigma} . \tag{8.20}$$

9 Summary

We have investigated the influence of cavity noise on the motion of charged particles in storage rings by using the Fokker-Planck technique.

The motion was described in terms of a coupled six-dimensional dispersion formalism with the canonical variables $\tilde{x}$, $\tilde{p}_x$, $\tilde{z}$, $\tilde{p}_z$, $\tilde{\sigma}$, $\tilde{p}_\sigma = \Delta E / E_0$.

This set of variables can be obtained from the variables x, p_x, z, p_z, $\sigma = s - ct$, p_σ of the fully six-dimensional formalism via a canonical transformation.

With this new set of variables we were then in a position to treat the betatron and synchrotron oscillations simultaneously in a canonical manner, i.e. to provide an analytic technique which includes consistently and canonically the synchrotron oscillations in the electric fields of the accelerating cavities.

In order to derive the Fokker-Planck equation (canonical) action-angle variables were introduced taking into account a coupling of the betatron oscillations by skew quadrupoles and solenoids and the linear coupling between the betatron and the synchrotron oscillations induced by a non-vanishing dispersion in the cavities.

The Fokker-Planck equation was solved separately for phase noise and amplitude noise.

In this paper we have (for simplicity) neglected a shift of the six-dimensional closed orbit induced by magnetic dipole fields. A technique to handle this effect may for instance be found in Refs. [7,8]. The formalism developed remains valid. Furthermore, we have restricted our investigations to proton rings. But it is easy to extend these considerations to electron beams. In this case it is necessary to introduce cavity phase $\neq (0, \pi)$ determined by eqn. (2.27), and additional nonsymplectic terms due to stochastic radiation effects have to be taken into account in the equations of motion which lead to a damping of the oscillation modes.

Finally we mention that the influence of phase fluctuations alone could already be investigated within the framework of the fully six-dimensional formalism in the variables x, p_x, z, p_z, σ, p_σ of eqn. (2.3) (without introducing a dispersion formalism). This shall be demonstrated in our next report [17].

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Appendix A: Description of the Electromagnetic Field

Using the freedom to select a gauge, we can choose any vector potential in eqns. (2.10) and (2.11a, b, c) that leads to the correct form of the fields. Suitable vector potentials are as follows and have been chosen for their simplicity.

A.1 Bending Magnet

If the curvatures K_x and K_z of the design orbit are given, the magnetic bending fields on the design orbit, $B_x^{(0)}(s)$ and $B_z^{(0)}(s)$:

$$B_x^{(0)}(s) = B_x(0,0,s) \; ; \qquad \text{(A.1a)}$$

$$B_z^{(0)}(s) = B_z(0,0,s) \qquad \text{(A.1b)}$$

are determined by [3]:

$$\frac{e}{p_0 \cdot c} \cdot B_x^{(0)} = -K_z \; ; \qquad \text{(A.2a)}$$

$$\frac{e}{p_0 \cdot c} \cdot B_z^{(0)} = +K_x \; . \qquad \text{(A.2b)}$$

The corresponding vector potential can be written as

$$\frac{e}{p_0 \cdot c} \cdot A_s = -\frac{1}{2}(1 + K_x \cdot x + K_z \cdot z) \; ; \qquad \text{(A.3a)}$$

$$A_x = A_z = 0 \; . \qquad \text{(A.3b)}$$

A.2 Quadrupole

The quadrupole fields are

$$B_x = z \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \; ; \qquad \text{(A.4a)}$$

$$B_z = x \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \; , \qquad \text{(A.4b)}$$

so that we may use the vector potential

$$A_s = \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \cdot \frac{1}{2}\left(z^2 - x^2\right) \; ; \qquad \text{(A.5a)}$$

$$A_x = A_z = 0 \; . \qquad \text{(A.5b)}$$

In the following we rewrite the term $(e/E_0) \cdot A_s$ in (2.2) as

$$\frac{e}{E_0} A_s = \frac{1}{2} \cdot \frac{p_0 \cdot c}{E_0} \cdot g \cdot (z^2 - x^2) = \frac{1}{2}\beta_0 \cdot g \cdot (z^2 - x^2) \; ; \qquad \text{(A.6a)}$$

$$g = \frac{e}{p_0 \cdot c} \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \; . \qquad \text{(A.6b)}$$

A.3 Skew Quadrupole

The fields are

$$B_x = -\frac{1}{2} \cdot \left(\frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z} \right)_{x=z=0} \cdot x \; ; \qquad \text{(A.7a)}$$

$$B_z = -\frac{1}{2} \cdot \left(\frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z} \right)_{x=z=0} \cdot z \; . \qquad \text{(A.7b)}$$

Thus we may use

$$A_s = +\frac{1}{2} \left(\frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z} \right)_{x=z=0} \cdot xz \; ; \qquad \text{(A.8a)}$$

$$A_x = A_z = 0 \; , \qquad \text{(A.8b)}$$

and we write

$$\frac{e}{E_0} A_s = N \cdot \beta_0 \cdot xz \; ; \qquad \text{(A.9a)}$$

$$N = \frac{1}{2} \cdot \frac{e}{p_0 \cdot c} \left(\frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z} \right)_{x=z=0} \cdot xz \; . \qquad \text{(A.9b)}$$

A.4 Solenoid Fields

The field components in the current free region are given by [14,4]:

$$B_x(x,z,s) = x \cdot \sum_{\nu=0}^{\infty} b_{2\nu+1}(s) \cdot (x^2 + z^2)^{\nu} \; ; \qquad \text{(A.10a)}$$

$$B_z(x,z,s) = z \cdot \sum_{\nu=0}^{\infty} b_{2\nu+1}(s) \cdot (x^2 + z^2)^{\nu} \; ; \qquad \text{(A.10b)}$$

$$B_s(x,z,s) = \sum_{\nu=0}^{\infty} b_{2\nu}(s) \cdot (x^2 + z^2)^{\nu} \qquad \text{(A.10c)}$$

where for consistency with Maxwell's equations

$$\text{div}\, \vec{B} = 0 \implies \frac{1}{r} \cdot \frac{\partial}{\partial r} (r \cdot B_r) = -\frac{\partial}{\partial s} B_s \; ;$$

$$\text{curl}\, \vec{B} = 0 \implies \frac{\partial}{\partial s} B_r = -\frac{\partial}{\partial r} B_s \; ;$$

$$(r^2 = x^2 + z^2 \; ; \quad B_r^2 = B_x^2 + B_z^2)$$

the coefficients $b_{\mu}(s)$ obey the recursion equations:

$$b_{2\nu+1}(s) = -\frac{1}{(2\nu+2)} \cdot b'_{2\nu}(s) \; ; \qquad \text{(A.10a)}$$

$$b_{2\nu+2}(s) = +\frac{1}{(2\nu+2)} \cdot b'_{2\nu+1}(s) \; ; \qquad \text{(A.10b)}$$

$$(\nu = 0, 1, 2, \ldots)$$

and where

$$b_0(s) \equiv B_s(0,0,s) \; . \qquad \text{(A.11)}$$

The vector potential leading to the solenoid field of eqn. (A.10) is then:

$$A_x(x,z,s) = -z \cdot \sum_{\nu=0}^{\infty} \frac{1}{(2\nu+2)} \cdot b_{(2\nu)}(s) \cdot r^{2\nu} ; \tag{A.12a}$$

$$A_z(x,z,s) = -x \cdot \sum_{\nu=0}^{\infty} \frac{1}{(2\nu+2)} \cdot b_{(2\nu)}(s) \cdot r^{2\nu} ; \tag{A.12b}$$

$$A_s(x,z,s) = 0 . \tag{A.12c}$$

Thus we can write :

$$\frac{\epsilon}{E_0} A_x = -\beta_0 H(s) \cdot z + \frac{1}{8}\beta_0 H''(s) \cdot (x^2 + z^2) \cdot z + ... ; \tag{A.13a}$$

$$\frac{\epsilon}{E_0} A_z = +\beta_0 H(s) \cdot x - \frac{1}{8}\beta_0 H''(s) \cdot (x^2 + z^2) \cdot x + ... \tag{A.13b}$$

with

$$\begin{aligned} H(s) &= \frac{1}{2} \cdot \frac{e}{p_0 \cdot c} \cdot b_0(s) \\ &\equiv \frac{1}{2} \cdot \frac{e}{p_0 \cdot c} \cdot B_s(0,0,s) . \end{aligned} \tag{A.14}$$

Note that the cyclotron radius R according to the longitudinal field (A.11) is given by

$$R = \frac{1}{2 \cdot H} .$$

A.5 Cavity Field

For a longitudinal electric field

$$\begin{aligned} \varepsilon_x &= 0 ; \\ \varepsilon_z &= 0 ; \\ \varepsilon_s &= \epsilon(s,\sigma) \end{aligned} \tag{A.15}$$

we write:

$$\begin{aligned} A_x &= 0 ; \\ A_z &= 0 ; \\ A_s &= \frac{1}{\beta_0} \cdot \int_{\sigma_0}^{\sigma} d\tilde{\sigma} \cdot \varepsilon(s,\tilde{\sigma}) , \end{aligned} \tag{A.16}$$

which by (2.10) immediately gives ε_s.

Now the cavity field may be represented by:

$$\varepsilon(s,\sigma) = V(s) \sin\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] \tag{A.17}$$

and we obtain using (A.17):

$$A_s = -\frac{1}{\beta_0} \cdot \frac{L}{2\pi \cdot k} \cdot V(s) \cdot \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] , \tag{A.18}$$

in which the phase φ is defined so that the average energy radiated away in the bending magnets is replaced by the cavities (see eqn. (2.26)) and h is the harmonic number.

Appendix B: Calculation of the Tune Shift caused by a Perturbation $\delta\underline{A}$

In order to calculate the tune shift which is induced by a perturbation $\delta\underline{A}$ we investigate the eigenvalue spectrum of the revolution matrix

$$\underline{M}(s_0 + L, s_0) + \delta\underline{M}(s_0 + L, s_0)$$

of the perturbed problem

$$\frac{d}{ds}\vec{y} = (\underline{A} + \delta\underline{A}) \cdot \vec{y} \ . \tag{B.1}$$

B.1 The Perturbation Part of the Revolution Matrix

According to eqn. (B.1) the transfer matrix

$$\underline{M}(s, s_0) + \delta\underline{M}(s, s_0)$$

with the perturbation part $\delta\underline{M}(s_0 + L, s_0)$ obeys the equation:

$$\frac{d}{ds}[\underline{M}(s, s_0) + \delta\underline{M}(s, s_0)] = [\underline{A}(s) + \delta\underline{A}(s)] \cdot [\underline{M}(s, s_0) + \delta\underline{M}(s, s_0)] \ ; \tag{B.2a}$$

$$\underline{M}(s_0, s_0) + \delta\underline{M}(s_0, s_0) = \underline{1} \ . \tag{B.2b}$$

Taking into account the corresponding equations for the unperturbed transfer matrix $\underline{M}(s, s_0)$:

$$\begin{aligned} \frac{d}{ds}\underline{M}(s, s_0) &= \underline{A}(s) \cdot \underline{M}(s, s_0) \ ; \\ \underline{M}(s_0, s_0) &= \underline{1} \end{aligned}$$

we obtain from (B.2) in first order the differential equation for $\delta\underline{M}(s, s_0)$:

$$\frac{d}{ds}\delta\underline{M}(s, s_0)] = \underline{A}(s) \cdot \delta\underline{M}(s, s_0) + \delta\underline{A}(s) \cdot \underline{M}(s, s_0)$$

with the initial condition:

$$\delta\underline{M}(s_0, s_0) = \underline{0} \ .$$

The solution of this equation (and thus the first order solution of eqn. (B.2)) reads as:

$$\begin{aligned} \delta\underline{M}(s, s_0) &= \int_{s_0}^{s} d\tilde{s} \cdot \underline{M}(s, \tilde{s}) \cdot \delta\underline{A}(\tilde{s}) \cdot \underline{M}(\tilde{s}, s_0) \\ &= \underline{M}(s, s_0) \cdot \int_{s_0}^{s} d\tilde{s} \cdot \underline{M}^{-1}(\tilde{s}, s_0) \cdot \delta\underline{A}(\tilde{s}) \cdot \underline{M}(\tilde{s}, s_0) \ . \end{aligned}$$

For the perturbative part $\delta\underline{M}(s_0 + L, s_0)$ of the revolution matrix one therefore gets in first order the expression:

$$\begin{aligned} \delta\underline{M}(s_0 + L, s_0) &= \int_{s_0}^{s_0+L} d\tilde{s} \cdot \underline{M}(s_0 + L, \tilde{s}) \cdot \delta\underline{A}(\tilde{s}) \cdot \underline{M}(\tilde{s}, s_0) \\ &= \underline{M}(s_0 + L, s_0) \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \underline{M}^{-1}(\tilde{s}, s_0) \cdot \delta\underline{A}(\tilde{s}) \cdot \underline{M}(\tilde{s}, s_0) \end{aligned} \tag{B.3a}$$

and for $\delta\underline{M}(s + L, s)$ one thus may write:

$$\delta\underline{M}(s + L, s) = \underline{M}(s + L, s) \cdot \int_{s}^{s+L} d\tilde{s} \cdot \underline{M}^{-1}(\tilde{s}, s) \cdot \delta\underline{A}(\tilde{s}) \cdot \underline{M}(\tilde{s}, s) \ . \tag{B.3b}$$

B.2 The Tune Shift

Eqn. (B.3b) determines the perturbed part $\delta\underline{M}(s+L,s)$ of the revolution matrix if the (unperturbed) transfer matrix $\underline{M}(\tilde{s},s)$ and the perturbation $\delta\underline{A}(\tilde{s})$ are known. Using the eigenvalue equation

$$(\underline{M}+\delta\underline{M})\cdot(\vec{v}_\mu+\delta\vec{v}_\mu)=(\lambda_\mu+\delta\lambda_\mu)\cdot(\vec{v}_\mu+\delta\vec{v}_\mu)\ ; \qquad (\mu=\pm I,\ \pm II,\ \pm III)$$

or (since $\underline{M}\vec{v}_\mu=\lambda_\mu\vec{v}_\mu$)

$$\underline{M}\cdot\delta\vec{v}_\mu+\delta\underline{M}\cdot\vec{v}_\mu=\lambda_\mu\cdot\delta\vec{v}_\mu+\delta\lambda_k\cdot\vec{v}_\mu \tag{B.4}$$

we are now able to calculate the Q-shift

$$\delta Q_\kappa=\frac{i}{2\pi\cdot\lambda_\kappa}\cdot\delta\lambda_\kappa \tag{B.5}$$

caused by $\delta\underline{M}$.

For that purpose we expand $\delta\vec{v}_\mu$ in terms of the eigenvectors $\vec{v}_\mu$ of the unperturbed problem:

$$\delta\vec{v}_\mu=\sum_\nu a_{\mu\nu}\cdot\vec{v}_\nu \tag{B.6}$$

and by inserting (B.6) into (B.5) we get:

$$\sum_\mu a_{\mu\nu}\cdot\lambda_\nu\vec{v}_\nu+\delta\underline{M}\cdot\vec{v}_\mu=\lambda_\mu\cdot\sum_\mu a_{\mu\nu}\vec{v}_\nu+\delta\lambda_\mu\cdot\vec{v}_\mu\ . \tag{B.7}$$

Multiplying this equation from the left hand side with

$$\frac{1}{i}\cdot\vec{v}_\kappa^{+}\underline{S}$$

and taking into account eqn. (4.10) we obtain:

$$a_{\mu\kappa}\cdot\lambda_\kappa\cdot\frac{1}{i}\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa+\frac{1}{i}\vec{v}_\kappa^{+}\underline{S}\cdot\delta\underline{M}\cdot\vec{v}_\mu=\lambda_\mu\cdot a_{\mu\kappa}\cdot\frac{1}{i}\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa+\delta\lambda_\kappa\cdot\frac{1}{i}\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa\cdot\delta_{\mu\kappa} \tag{B.8}$$

with

$$\frac{1}{i}\cdot\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa=\begin{cases}+1 & \text{for } \kappa=I,\ II,\ III;\\ -1 & \text{for } \kappa=-I,-II,-III.\end{cases} \tag{B.9}$$

For $\mu=\kappa$ the first terms on both sides of eqn. (B.8) cancel and one obtains with (B.3) and (B.5) the following approximate expression for the Q-shift δQ_κ in linear order:

$$\begin{aligned}\delta Q_\kappa &= \left(\frac{1}{i}\cdot\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa\right)\cdot\frac{1}{2\pi\cdot\lambda_\kappa}\cdot\vec{v}_\kappa^{+}\underline{S}\cdot\delta\underline{M}(s+L,s)\cdot\vec{v}_\kappa(s)\\ &= \left(\frac{1}{i}\cdot\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa\right)\cdot\frac{1}{2\pi\cdot\lambda_\kappa}\cdot\vec{v}_\kappa^{+}\underline{S}\cdot\underline{M}(s+L,s)\times\\ &\qquad\int_s^{s+L}d\tilde{s}\cdot\underline{M}^{-1}(\tilde{s},s)\cdot\delta\underline{A}(\tilde{s})\cdot\underline{M}(\tilde{s},s)\cdot\vec{v}_\kappa(s)\ .\end{aligned}$$

Furthermore, using the symplectic condition of the transfer matrix $\underline{M}(s_1, s_2)$:

$$\underline{M}^T(s_1, s_2) \cdot \underline{S} \cdot \underline{M}(s_1, s_2) = \underline{S}$$

and the equation

$$\begin{aligned}
\vec{v}_\kappa^{+}(s) \cdot \underline{S} \cdot \underline{M}(s-L, s) &= \vec{v}_\kappa^{+}(s) \cdot \left[\underline{M}^{-1}(s-L, s)\right]^T \cdot \underline{S} \\
&= \left[\underline{M}^{-1}(s-L, s) \cdot \vec{v}_\kappa(s)\right]^+ \cdot \underline{S} \\
&= \left[\lambda_\kappa^{-1} \cdot \vec{v}_\kappa(s)\right]^+ \cdot \underline{S} \\
&= \lambda_\kappa \cdot \vec{v}_\kappa^{+}(s) \cdot \underline{M}^T(\tilde{s}, s) \cdot \underline{S} \cdot \underline{M}(\tilde{s}, s) \\
&\quad \left(\text{since } \ (\lambda_\kappa^{-1})^* = \lambda_\kappa \ \text{ and } \ \underline{S} = \underline{M}^T \cdot \underline{S} \cdot \underline{M}\right) \\
&= \lambda_\kappa \cdot \left[\underline{M}(\tilde{s}, s) \cdot \vec{v}_\kappa(s)\right]^+ \cdot \underline{S} \cdot \underline{M}(\tilde{s}, s) \\
&= \lambda_\kappa \cdot \vec{v}_\kappa^{+}(\tilde{s}) \cdot \underline{S} \cdot \underline{M}(\tilde{s}, s) \qquad \text{(B.10)}
\end{aligned}$$

we get:

$$\begin{aligned}
\delta Q_\kappa &= \left(\frac{1}{i} \cdot \vec{v}_\kappa^{+} \underline{S} \vec{v}_\kappa\right) \cdot \frac{1}{2\pi} \cdot \int_s^{s+L} d\tilde{s} \cdot \vec{v}_\kappa^{+}(\tilde{s}) \cdot \underline{S} \cdot \delta\underline{A}(\tilde{s}) \cdot \vec{v}_\kappa(\tilde{s}) \\
&= \left(\frac{1}{i} \cdot \vec{v}_\kappa^{+} \underline{S} \vec{v}_\kappa\right) \cdot \frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \vec{v}_\kappa^{+}(\tilde{s}) \cdot \underline{S} \cdot \delta\underline{A}(\tilde{s}) \cdot \vec{v}_\kappa(\tilde{s})
\end{aligned}$$

(in the last step we have used the fact that the integrand is a periodic function of period L; see eqns. (4.11, 12))
or for $\kappa = k$ and $\kappa = -k$ $\quad (k = \pm I, \pm II, \pm III)$:

$$\delta Q_k = \frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \vec{v}_k^{+}(\tilde{s}) \cdot \underline{S} \cdot \delta\underline{A}(\tilde{s}) \cdot \vec{v}_k(\tilde{s}) \ ; \qquad \text{(B.11a)}$$

$$\delta Q_{-k} = -\frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \vec{v}_{-k}^{+}(\tilde{s}) \cdot \underline{S} \cdot \delta\underline{A}(\tilde{s}) \cdot \vec{v}_{-k}(\tilde{s}) \ . \qquad \text{(B.11b)}$$

Taking into account:

$$\begin{aligned}
\delta Q_\kappa^* &= \left(\frac{1}{i} \cdot \vec{v}_\kappa^{+} \underline{S} \vec{v}_\kappa\right)^+ \cdot \frac{1}{2\pi} \int_{s_0}^{s_0+L} d\tilde{s} \cdot \left[\vec{v}_\kappa^{+}(\tilde{s}) \cdot \underline{S} \cdot \delta\underline{A}(\tilde{s}) \cdot \vec{v}_\kappa(\tilde{s})\right]^+ \\
&= \left(\frac{1}{i} \cdot \vec{v}_\kappa^{+} \underline{S} \vec{v}_\kappa\right) \cdot \frac{1}{2\pi} \int_{s_0}^{s_0+L} d\tilde{s} \cdot \left[-\vec{v}_\kappa^{+}(\tilde{s}) \cdot \delta\underline{A}^T(\tilde{s}) \cdot \underline{S} \cdot \vec{v}_\kappa(\tilde{s})\right]
\end{aligned}$$

as well as

$$\vec{v}_{-\kappa} = (\vec{v}_\kappa)^*$$

the following relations can be derived from (B.11a,b):

$$\begin{aligned}
\Re e\{\delta Q_k\} &= \frac{1}{4\pi} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \vec{v}_k^{+}(\tilde{s}) \cdot \left[\underline{S} \cdot \delta\underline{A}(\tilde{s}) - \delta\underline{A}^T(\tilde{s}) \cdot \underline{S}\right] \cdot \vec{v}_k(\tilde{s}) \\
&= -\Re e\{\delta Q_{-k}\} \ ; \qquad \text{(B.12a)}
\end{aligned}$$

$$\begin{aligned}
\Im m\{\delta Q_k\} &= -\frac{i}{4\pi} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \vec{v}_k^{+}(\tilde{s}) \cdot \left[\underline{S} \cdot \delta\underline{A}(\tilde{s}) + \delta\underline{A}^T(\tilde{s}) \cdot \underline{S}\right] \cdot \vec{v}_k(\tilde{s}) \\
&= +\Im m\{\delta Q_{-k}\} \ . \qquad \text{(B.12b)}
\end{aligned}$$

This means that in addition to a real Q-shift, there is in general also a complex Q-shift which leads to a damping (or antidamping) of the oscillation modes [3].

In our case we have (see eqn. (3.24)):

$$\underline{S} \cdot \delta \underline{A}(\tilde{s}) - \delta \underline{A}^T(\tilde{s}) \cdot \underline{S} = 0 \tag{B.13}$$

and therefore:

$$\Im m\{\delta Q_k\} = \Im m\{\delta Q_{-k}\} = 0 \Longrightarrow \delta Q_{\pm k} = \delta Q^*_{\pm k} \ (real) . \tag{B.14}$$

Note that eqn. (B.13) results from the symplectic structure of $\delta \underline{A}$.

Finally we mention that the case $\mu \neq \kappa$ in eqn. (B.8) would lead to an estimation of $\delta \vec{v}_\mu$ [11] which we do not need here.

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