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Abstract

The interaction of the beam with the metallic surroundings (cavities, kicker tanks etc.) in an accelerator can be described in terms of an effective impedance. A fast computational method is needed to evaluate the effective impedance to study coupled bunch instabilities under realistic conditions. An analytic expression for the effective impedance is given which is the basis for a computer program to investigate coupled bunch instabilities.

Contents

1 Introduction

Currently many accelerators, for example B-Factories and SSC, with a high average current in multiple bunches are being designed and there are some machines coming into operation like HERA which require a high average current stored in many bunches to achieve the desired luminosity.
The current in this machines is limited by coupled bunch instabilities and it is of importance to calculate the threshold currents beforehand.
The purpose of this paper is to give some easy to use formulae for calculating the stability of a machine with a high number of bunches.
Coupled bunch motion is normally described in terms of normal modes, the frequencies of which are calculated by applying the well known formulae [1]:

$$\text{longitudinal:}\ (\omega - m\,\Omega_s) = \qquad (1)$$
$$m\,i\,\frac{\Omega_s I_0\omega_0}{hU\cos\phi_s}\frac{2\pi}{C_{\|}}\sum_{\mu=-\infty}^{\infty}\frac{Z^{\|}(\mu N\omega_0+\Omega)}{\mu N\omega_0+\Omega}\,|h_m(\mu N\omega_0+\Omega)|^2$$
$$\text{transverse:}\ (\omega - (\omega_\beta + m\,\Omega_s)) = \qquad (2)$$
$$-\,i\,\frac{\overline{\beta}I_0\omega_0}{4\pi E/e}\frac{2\pi}{C_\perp}\sum_{\mu=-\infty}^{\infty}Z^{\perp}(\mu N\omega_0+\Omega)\,|h_m(\mu N\omega_0+\Omega+\Omega_r)|^2$$

with

$$\Omega_r = -\frac{\xi}{\alpha}\omega_0 \ \text{and}\ \Omega = \begin{cases} k\omega_0 + m\Omega_s & \text{longitudinal} \\ k\omega_0 + \omega_\beta + m\Omega_s) & \text{transverse} \end{cases} . \qquad (3)$$

ω_0: $2\pi\cdot$ revolution frequency

ν_s: synchrotron tune

Ω_s: $= \omega_0\,\nu_s$

k: normal mode number

m: internal bunch mode (longitudinal: $m = 1$ rigid dipole oscillations, $m = 2$ quadrupole oscillations etc., transverse: $m = 0$ rigid dipole oscillations, $m = 1$ oscillation of first head tail mode)

E: energy

$\overline{\beta}$: amplitude function at resonant object like cavity.

N: number of bunches

U: rf voltage

ϕ_s: synchrotron phase angle; $\cos\phi_s \begin{cases} \geq 0 & \text{above transition} \\ \leq 0 & \text{below transition} \end{cases}$

h: harmonic number

I_0: total current

ξ: chromaticity

α: momentum compaction factor

$h_m(\omega)$ denotes the Fourier transform of the line charge density $\lambda_m(\tau)$:

$$h_m(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau\,\lambda_m(\tau)e^{i\omega\tau}\,. \qquad (4)$$

The constants $C_{\|}$ and $C_\perp$ are defined in such a way that the scalar product between the radial function of the m-th mode and the adjoint function is normalized to unity (for details see [2]):

$$< R^+_{mn}\,,\,R_{ml} >= \int_0^{\infty} dr r\,R^+_{mn}\,R_{ml} = \delta_{nl} \qquad (5)$$

where the adjoint function is given by

$$W(r)\,R^+_{mn}(r) = R_{mn} \qquad (6)$$
$$W(r) = \begin{cases} -C_{\|}\frac{1}{r}\frac{dg_0(r)}{dr} & \text{longitudinal} \\ C_\perp g_0(r) & \text{transverse} \end{cases}$$

The stationary distribution $g_0(r)$ is normalized as follows

$$\int_0^{\infty} dr r\,g_0(r) = \frac{1}{2\pi} \qquad (7)$$

As a direct consequence of the Panofsky-Wenzel relation [3] the transverse impedance $Z_\perp$ is associated with an equivalent longitudinal impedance $Z_{\|\perp}$ by [4]:

$$Z_\perp(\omega) = \frac{c}{\omega}\,Z_{\|\perp}(\omega)\,. \qquad (8)$$

where c denotes the velocity of light.
With the help of this relation the frequency of the $k - th$ normal coupled bunch mode for both the longitudinal and transverse direction can be written in the following form

$$(\omega - \omega_m) = -\,i\,K\sum_{\mu=-\infty}^{\infty}\omega_0\frac{Z^{\|}(\mu N\omega_0+\Omega)}{\mu N\omega_0+\Omega}\,|h_m(\mu N\omega_0+\Omega+\Omega_r)|^2\,. \qquad (9)$$

The problem is to evaluate the sum. This is called the effective impedance. We do this in section 2. In section 3 a concrete example is given for a Gaussian bunch and the results are applied for both PETRA and HERA.

2 Computation of the effective impedance

The effective impedance (9) can be translated into the following form

$$Z_{eff} = \sum_{\mu=-\infty}^{\infty}\omega_0\int_{-\infty}^{\infty} d\omega'\,\delta(\omega' - (\mu\,N\omega_0 + \Omega))\frac{Z(\omega')}{\omega'}\,|h_m(\omega'+\Omega_r)|^2\,. \qquad (10)$$

Inserting the Fourier series of the periodic delta function

$$\sum_{\mu=-\infty}^{\infty}\delta(\omega' - (\mu\,N\omega_0+\Omega)) = \frac{1}{N\omega_0}\sum_{k=-\infty}^{\infty}\exp(i\,k\frac{T}{N}(\omega'-\Omega)) \qquad (11)$$

with $T = 2\pi/\omega_0$ yields

$$Z_{eff} = \frac{1}{N} \sum_{k=-\infty}^{\infty} u(\frac{k}{N}T)\, \exp(-i\,\frac{k}{N}\Omega\, T) \tag{12}$$

$$u(t) = \int_{-\infty}^{\infty} d\omega'\, \frac{Z(\omega')}{\omega'}\, |h_m(\omega' + \Omega_r)|^2 \exp(i\omega' t)\,. \tag{13}$$

The parasitic modes of the resonant objects are described by impedances of parallel circuits thus

$$Z(\omega) = \sum_{j=1}^{N_{par}} \frac{R_{sj}}{1\, +\, iQ_j\,(\frac{\omega}{\omega_{rj}}\, -\, \frac{\omega_{rj}}{\omega})} \tag{14}$$

where each parasitic mode is characterized by

R_{sj}: shunt impedance

Q_j: Q factor

ω_{rj}: $2\pi\cdot$ resonance frequency.

Without loss of generality we consider only one parasitic mode and drop the index j. The impedance of this mode can be represented by partial fractions

$$\frac{Z(\omega)}{\omega} = -\frac{i\,R_s}{\sqrt{4Q^2\, -\, 1}} \left(\frac{1}{\omega\, -\, \omega_+} - \frac{1}{\omega\, -\, \omega_-} \right) \tag{15}$$

$$\omega_\pm = i\frac{\omega_r}{2Q} \pm \omega_r \sqrt{1\, -\, \frac{1}{4Q^2}}\,.$$

Using this expression, the function u (13) can be transformed into

$$u(t) = -\frac{iR_s}{\sqrt{4Q^2\, -\, 1}}\{G_+(t)\, -\, G_-(t)\} \tag{16}$$

$$G_\pm(t) = \int_{-\infty}^{\infty} d\omega'\, \frac{\exp(i\omega' t)}{\omega'\, -\, \omega_\pm}\, |h_m(\omega' + \Omega_r)|^2. \tag{17}$$

The function $G_\pm$ satisfies the following differential equation

$$e^{i\omega_\pm t}\, \frac{d}{dt} \left(e^{-i\omega_\pm t}\, G_\pm(t) \right) = i \int_{-\infty}^{\infty} d\omega' e^{i\omega t} |h_m(\omega' + \Omega_r)|^2 \tag{18}$$

which is solved by

$$G_\pm(t) = i\, e^{i\omega_\pm t} \int_{-\infty}^{t} d\tau \int_{-\infty}^{\infty} d\omega' e^{i(\omega\, -\, \omega_\pm)\tau} |h_m(\omega' + \Omega_r)|^2. \tag{19}$$

To proceed, it is useful to consider the properties of

$$g(\tau) = \int_{-\infty}^{\infty} d\omega'\, |h_m(\omega' + \Omega_r)|^2\, \exp(i\omega'\,\tau)\,. \tag{20}$$

Since $h_m(\omega)$ is the Fourier transform of the charge density (4), $\lambda_m(\tau)$, the function $g(\tau)$ is proportional to the correlation function of the charge density

$$g(\tau) \sim \int_{-\infty}^{\infty} dt\, \lambda(\tau - t)\, \lambda(t)\,. \tag{21}$$

In the case of bunched beams the correlation length is about twice the bunch length and this former is at the most comparable to the rf-wavelength. Since the rf-wavelength is smaller than the bunch spacing T/N one has

$$g(\tau) = \begin{cases} g(\tau) & \text{for} \quad -\frac{T}{N} \le \tau \le \frac{T}{N} \\ 0 & \text{else} \end{cases}\,. \tag{22}$$

Thus one gets

$$1)\ k \le -1 \tag{23}$$

$$G_\pm(kT/N) \;=\; 0$$

$$2)\ k = 0 \tag{24}$$

$$G_\pm(0) \;=\; i \int_0^{\infty} d\tau\, g(\tau) \exp(i(\omega_r + \omega_\pm)\tau)$$

$$3)\ k \ge 1 \tag{25}$$

$$\begin{aligned} G_\pm(kT/N) \;&=\; i\exp(i\omega_\pm kT/N) \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\omega'\, |h_m(\omega' + \Omega_r)|^2\, \exp(i(\omega'\, -\, \omega_\pm)\tau) \\ &=\; 2\pi\, i\, |h_m(\omega' + \Omega_r)|^2\, \exp(i\,\omega_\pm kT/N)\,. \end{aligned}$$

Introducing the latter result into the formula of the effective impedance one obtains

$$\begin{aligned} Z_{eff} = &\frac{2\pi\, R_s}{N\,\sqrt{4Q^2\, -\, 1}} * \\ &\sum_{k=1}^{\infty} \exp(-i\,k\Omega T/N)\, \left[\exp(i\,k\omega_+ T/N) |h_m(\omega_+ + \Omega_r)|^2\, -\, \exp(i\,k\omega_- T/N) |h_m(\omega_- + \Omega_r)|^2 \right] \\ &+ \frac{2i\,R_s}{N\sqrt{4Q^2\, -\, 1}} \int_0^{\infty} d\tau\, \exp(i\,\omega_r\tau)\, \exp(-\Gamma\tau)\, \sin(\omega_r'\tau)\, g(\tau) \end{aligned} \tag{26}$$

with

$$\Gamma = \frac{\omega_r}{2Q} \;\text{ and }\; \omega_r' = \omega_r \sqrt{1\, -\, \frac{1}{4Q^2}}\,. \tag{27}$$

Summing up the geometric series yields

$$\begin{aligned} Z_{eff} = &\frac{2\pi\, R_s}{N\sqrt{4Q^2\, -\, 1}} * \\ &\left\{ |h_m(\omega_+ + \Omega_r)|^2\, \mathcal{Z}_+\, -\, |h_m(\omega_- + \Omega_r)|^2\, \mathcal{Z}_-\, -\, |h_m(\omega_+ + \Omega_r)|^2\, +\, |h_m(\omega_- + \Omega_r)|^2 \right\} \\ &+ \frac{2i\,R_s}{N\sqrt{4Q^2\, -\, 1}} \int_0^{\infty} d\tau\, \exp(i\,\omega_r\tau)\, \exp(-\Gamma\tau)\, \sin(\omega_r'\tau)\, g(\tau) \end{aligned} \tag{28}$$

where the functions $\mathcal{Z}_\pm$ are defined as follows

$$\mathcal{Z}_\pm = \frac{\exp(i(\omega_\pm\, -\, \Omega)T/N)}{1\, -\, \exp(i(\omega_\pm\, -\, \Omega)T/N)} + 1 \tag{29}$$

thus

$$\mathcal{Z}_\pm = \mathcal{R}_\pm\, +\, i\mathcal{I}_\pm \tag{30}$$

$$\mathcal{R}_\pm \;=\; \frac{1\, -\, e^{-\Gamma T/N}\, \cos((\pm\omega_r'\, -\, \Omega)T/N)}{(1\, -\, e^{-\Gamma T/N} \cos((\pm\omega_r'\, -\, \Omega)T/N))^2\, +\, e^{-2\Gamma T/N} \sin^2((\pm\omega_r'\, -\, \Omega)T/N)} \tag{31}$$

$$\mathcal{I}_\pm \;=\; \frac{e^{-\Gamma T/N}\, \sin((\pm\omega_r'\, -\, \Omega)T/N)}{(1\, -\, e^{-\Gamma T/N} \cos((\pm\omega_r'\, -\, \Omega)T/N))^2\, +\, e^{-2\Gamma T/N} \sin^2((\pm\omega_r'\, -\, \Omega)T/N)}\,. \tag{32}$$

This is the general result. In many cases, however, it is not necessary to use the formulae in this general form, because they can be simplified on the basis of the following assumptions.

1. The integral in (28) obviously describes a single bunch effect. The real part of the effective impedance determines the imaginary part of the mode frequency which is related to the growth rate or damping rate respectively. Thus the integral term produces only a growth or damping effect if $\omega_r \neq 0$, i.e. for non vanishing chromaticity. This is the well known head tail effect which can be avoided by correcting the chromaticity. Therefore we assume the chromaticity to be corrected so that $\omega_r = 0$.

2. Assuming that the single bunch current is smaller than any single bunch threshold current, we neglect single bunch effects. This means that we can drop the integral.

3. We only consider narrow band objects i.e. $\Gamma T/N \leq 1$, concentrating on coupled bunch effects. In this case Γ is small compared to ω_r' and we neglect the contribution of the imaginary part in the argument of the spectral density:

$$|h_m(\omega_\pm)|^2 = |h_m(\omega_r')|^2 .$$

Using the approximations mentioned above we can write

$$Z_{eff} = \frac{\pi R_s}{NQ} |h_m(\omega_r)|^2 \, (\mathcal{Z}_+ - \mathcal{Z}_-) . \tag{33}$$

The coherent frequency shift is proportional to the imaginary part of the effective impedance and the growth rate is proportional to the real part. Since we are mainly interested in the growth rate we concentrate on that and give the formulae for both the longitudinal and transverse planes:

longitudinal: (34)

$$Im(\omega - m\Omega_s) = \frac{\Omega_s I_0}{U\cos\phi_s} \frac{\pi R_s^{\parallel}}{NhQ} \frac{2\pi}{C_{\parallel}} |h_m(\omega_r')|^2 \left(\mathcal{R}_+^{\parallel} - \mathcal{R}_-^{\parallel}\right)$$

$$\mathcal{R}_\pm^{\parallel} = \frac{1 - e^{-\Gamma T/N}\cos((\pm\omega_r' - \Omega^{\parallel})T/N)}{(1 - e^{-\Gamma T/N}\cos((\pm\omega_r' - \Omega^{\parallel})T/N))^2 + e^{-2\Gamma T/N}\sin^2((\pm\omega_r' - \Omega^{\parallel})T/N)}$$

$$\Omega^{\parallel} = k\omega_0 + m\Omega_s$$

transverse: (35)

$$Im(\omega - (\omega_\beta + m\Omega_s)) = -\frac{c\overline{\beta} I_0}{4\pi E/e} \frac{\pi R_s^{\parallel\perp}}{NQ} \frac{2\pi}{C_\perp} |h_m(\omega_r')|^2 \left(\mathcal{R}_+^{\perp} - \mathcal{R}_-^{\perp}\right)$$

$$\mathcal{R}_\pm^{\perp} = \frac{1 - e^{-\Gamma T/N}\cos((\pm\omega_r' - \Omega^{\perp})T/N)}{(1 - e^{-\Gamma T/N}\cos((\pm\omega_r' - \Omega^{\perp})T/N))^2 + e^{-2\Gamma T/N}\sin^2((\pm\omega_r' - \Omega^{\perp})T/N)}$$

$$\Omega^{\perp} = k\omega_0 + \omega_\beta + m\Omega_s .$$

Usually the transverse impedance is defined by

$$Z_\perp(\omega) = \frac{R_s^{-}}{\omega/\omega_r\,(1 + iQ(\omega/\omega_r - \omega_r/\omega))} \tag{36}$$

According to equation (8) the transverse shunt impedance is connected with the equivalent longitudinal shunt impedance by

$$R_s^{\perp} = \frac{c}{\omega_r} R_s^{\parallel\perp} . \tag{37}$$

Thus one gets the following alternative expression for the growth rate:

transverse: (38)

$$Im(\omega - (\omega_\beta + m\Omega_s)) = -\frac{\overline{\beta} I_0}{4\pi E/e} \frac{\pi\omega_r R_s^{\perp}}{NQ} \frac{2\pi}{C_\perp} |h_m(\omega_r')|^2 \left(\mathcal{R}_+^{\perp} - \mathcal{R}_-^{\perp}\right)$$

3 Results

To calculate the eigenfrequencies of the k-th coupled bunch mode requires the knowledge of the mode spectrum $h_m(\omega)$ and this in turn depends on the exact eigenfunctions $R_{mn}(r)$ of Sacherer's integral equation. Following Sacherer we do not try to find the exact functions but we instead look for a good approximation. In appendix A, model functions are derived in the case of a Gaussian charge distribution

$$g_0(r) = \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \tag{39}$$

$$\sigma = \frac{\sigma_z}{R} \quad ; R \text{ mean radius of accelerator}$$

It is shown that the spectrum and the constants $C_{\parallel}$ and $C_\perp$ are given by

longitudinal (40)

$$|h_{mk}(\omega)|^2 = \frac{(p_r\sigma)^{2m+4k}}{2^{m+2k} k!(m+k)!} e^{-p_r\sigma^2} \quad m \geq 1$$

$$C = 2\pi\sigma^2$$

transverse (41)

$$|h_{mk}(\omega)|^2 = \frac{(p_r\sigma)^{2m+4k}}{2^{m+2k} k!(m+k)!} e^{-p_r\sigma^2} \quad m \geq 0$$

$$C = 2\pi$$

with

$$p_r = \frac{\omega_r}{\omega_0} .$$

The index k characterizes internal radial motion. But that can be hardly excited and we therefore consider only $k = 0$. Thus we obtain as a final result:

longitudinal: (42)

$$Im(\omega - m\Omega_s) = \frac{\Omega_s I_0}{2U\cos\phi_s} \frac{\pi R_s^{\parallel}}{NhQ} p_r^2 \frac{(p_r\sigma)^{2(m-1)}}{2^{m-1}(m-1)!} e^{-(p_r\sigma)^2} \left(\mathcal{R}_+^{\parallel} - \mathcal{R}_-^{\parallel}\right)$$

$$\mathcal{R}_\pm = \frac{1 - e^{-\Gamma T/N}\cos((\pm\omega_r' - \Omega^{\parallel})T/N)}{(1 - e^{-\Gamma T/N}\cos((\pm\omega_r' - \Omega^{\parallel})T/N))^2 + e^{-2\Gamma T/N}\sin^2((\pm\omega_r' - \Omega^{\parallel})T/N)}$$

transverse: (43)

$$Im(\omega - (\omega_\beta + m\Omega_s)) = -\frac{c\beta I_0}{4\pi E/e}\,\frac{\pi R_s^{\parallel\perp}}{NQ}\,\frac{(p_r\sigma)^{2m}}{2^m m!}\,e^{-(p_r\sigma)^2}\left(\mathcal{R}_+^\perp - \mathcal{R}_-^\perp\right)$$

$$\mathcal{R}_\pm = \frac{1 - e^{-\Gamma T/N}\cos((\pm\omega_r' - \Omega^\perp)T/N)}{(1 - e^{-\Gamma T/N}\cos((\pm\omega_r' - \Omega^\perp)T/N))^2 + e^{-2\Gamma T/N}\sin^2((\pm\omega_r' - \Omega^\perp)T/N)}$$

It should be noted that one has to sum over all parasitic modes but this is only a slight modification.
The data on the parasitic modes can be obtained by computation and measurement. Due to fabrication tolerances and the various working conditions the frequencies of the parasitic resonances are randomly distributed around a mean value. Therefore the impedance of a machine is a random quantity. As a consequence the stability of a machine can only be predicted statistically.
To evaluate the formulae (42) and (43) a computer code has been developed which determines impedance by a Monte Carlo method [5]. This computer code has been used to calculate the threshold current of both PETRA and HERA. The measured threshold current in PETRA agrees well with the computed threshold current.

4 Appendix

The model modes are found as follows.
The condition imposed on the mode functions is

$$< R_{mn}^+, R_{ml} >= \delta_{kl} \tag{44}$$

In the case of a Gaussian bunch we have

$$g_0(r) = \frac{1}{2\pi\sigma^2}\,e^{-\frac{r^2}{2\sigma^2}} \tag{45}$$

which is normalized according to (5).
In the longitudinal case the scalar product reads

$$-C_\parallel \int_0^\infty drr\,\frac{1}{r}\frac{dg_0(r)}{dr}\,R_{mn}^+(r)R_{ml}^+(r) = \delta_{kl}$$
$$\Rightarrow \frac{C_\parallel}{2\pi\sigma^2}\int_0^\infty drr\,e^{-\frac{r^2}{2\sigma^2}}\,R_{mn}^+(r)R_{ml}^+(r) = \delta_{kl}$$

In the following we consider only non-negative m. A similar analysis can be done for negative m.
Substitution of

$$y = \frac{r^2}{2\sigma^2}$$

into the last equation and using the definitions

$$C_\parallel = 2\pi\sigma^2 \text{ and } f_n^m(y) = R_{mn}^+(r) \tag{46}$$

yields

$$\int_0^\infty dy\,e^{-y} f_n^m(y) f_l^m(y) = \delta_{nl} \tag{47}$$

An analogous consideration can be performed for the transverse plane and one obtains also (47) but with a different constant

$$C_\perp = 2\pi\,. \tag{48}$$

Since the generalized Laguerre polynomials

$$L_n^m(y) = \frac{1}{k}\,y^{-m}\,e^y\,\frac{d^n}{dy^n}\left(e^{-y}y^{m+n}\right) \tag{49}$$

are orthogonal with respect to the weight function

$$e^{-y}y^m$$

[6] the functions $f_n^m(y)$ satisfying (47) are given by

$$f_n^m(y) = \sqrt{\frac{k!}{(m+k)!}}\,y^{\frac{m}{2}}\,L_n^m(y)\,. \tag{50}$$

The mode spectrum is defined by (4) and the charge density is given by

$$\lambda(\tau) = \int_0^\infty drr \int_0^{2\pi} d\theta\,\delta(\tau - \frac{r}{\omega_0}\cos\theta)\,R_{mn}(r)\,e^{-im\theta} \tag{51}$$

thus

$$h_{mn}(\omega) = i^m \int_0^\infty drr\,R_{mn}(r)\,J_m(r\frac{\omega}{\omega_0})\,. \tag{52}$$

Inserting the definition of the adjoint function (7) into (52) one obtains for the longitudinal case

$$h_{mn}(\omega) = \frac{i^m}{\sigma^2}\int_0^\infty drr\,\frac{1}{r}\frac{dg_0(r)}{dr}\,R_{mn}^+(r)J_m(r\frac{\omega}{\omega_0})\,.$$

Substituting

$$y = \frac{r^2}{2\sigma^2}$$

one has

$$h_{mn}(\omega) = i^m \int_0^\infty dy\,f_n^m(y)J_m(a\sqrt{y}) \tag{53}$$

with

$$a = \sqrt{2}\sigma\,\frac{\omega}{\omega_0}\,.$$

Exactly the same equation is obtained for the transverse plane.
Inserting the function $f_n^m(y)$ (50) yields

$$\int_0^\infty dy\,f_n^m(y)J_m(a\sqrt{y}) \tag{54}$$
$$= \frac{1}{\sqrt{k!(m+k)!}}\int_0^\infty dy J_m(a\sqrt{y})y^{-\frac{m}{2}}\,\frac{d^n}{dy^n}\left(e^{-y}y^{m+n}\right)\,.$$

Since

$$\frac{d}{dy}\left(\sqrt{y}^{-n}\,J_n(a\sqrt{y})\right) = -\frac{a}{2}\,\sqrt{y}^{-(n+1)}\,J_{n+1}(a\sqrt{y})$$

k-fold partial integration of the last integral leads to

$$\int_0^\infty dy\, f_n^m(y) J_m(a\sqrt{y}) \tag{55}$$
$$= \frac{1}{\sqrt{k!(m+k)!}} \left(\frac{a}{2}\right)^k \int_0^\infty dy\, e^{-y}\, J_{m+k}(a\sqrt{y}) y^{\frac{m+k}{2}} .$$

With the replacement

$$y = v^2$$

one has [7]

$$\int_0^\infty dy\, e^{-y}\, J_{m+k}(a\sqrt{y}) y^{\frac{m+k}{2}} = \tag{56}$$

$$2\int_0^\infty dv\, v^{m+k+1}\, e^{-v^2}\, J_{m+k}(av) = \frac{a^{m+k}}{2^{m+k}}\, e^{-\frac{a^2}{4}} . \tag{57}$$

Putting (56) into (55) and inserting the result into (53) finally gives

$$h_{mn}(\omega) = \frac{i^m}{\sqrt{2}^{m+2k}\sqrt{k!(m+k)!}} (p_r\sigma)^{m+2k}\; e^{-(p_r\sigma)^2/2} \tag{58}$$

and thus

$$|h_{mn}(\omega)|^2 = \frac{1}{2^{m+2k} k!(m+k)!} (p_r\sigma)^{2m+4k}\; e^{-(p_r\sigma)^2} \tag{59}$$

with $pr = \omega_r/\omega_0$. Equation (59) is true for both the longitudinal and transverse case. The only difference is caused by the two different constants $C_{||}$ (46) and $C_{\perp}$ (48).
The mode spectrum for charge densities other than Gaussian can be determined in a similar way.

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