80-6-1

BROWN-HET-414
February 1980

Loop Space Representation and the Large N Behavior of the One-Plaquette Kogut-Susskind Hamiltonian*

by

A. Jevicki
Department of Physics
Brown University
Providence, Rhode Island 02912

and

B. Sakita
Physics Department, The City College of C.U.N.Y.
New York, New York 10031

ABSTRACT

We study the large N behavior of the one-plaquette Kogut-Susskind guage theory Hamiltonian. Based on the quantum collective field method we derive an effective loop space Hamiltonian whose stationary points then determine the large N limit.

*Supported in part by the National Science Foundation Grant No. PHY-78-24888 and PSC-BHE-13085

I. INTRODUCTION

There is now a renewed effort to solve Quantum Chromodynamics in the large N limit. In this approach to hadron structure proposed originally by 't Hooft [1] one has to evaluate the sum of all planar diagrams. This represents a rather large set [2] of highly complex graphs and at present the summation could be achieved only in some simpler models [3,4].

Recently we have developed an approach to this problem in terms of the quantum collective field method [5,6]. In this method one derives an effective field theory for the collective field and then the large N behavior becomes in principle straightforward. It is simply given by the classical stationary solutions of the effective field equations. However, this "gap equation" can be rather complex and in Yang-Mills theory [5] it represents a nonlinear equation for a field defined in loop space(*).

In the present paper we solve in this approach the much simpler example of a one-plaquette Kogut-Susskind gauge theory. Here it is simple to parametrize the loops by an integer and through a Fourier transform the problem reduces to a singular, integral equation on the circle. Solving this equation we determine the large N behavior.

The content of this paper goes as follows: First in Section II we derive the loop space representation of the Hamiltonian. Then in Section III we consider the effective gap equation for the stationary collective field. Finally Section IV is reserved for the conclusions.

II. THE LOOP SPACE REPRESENTATION

The one-plaquette Kogut-Susskind gauge theory [8] Hamiltonian is given by

$$\hat{H} = \frac{g^2}{2a}\left\{\sum_{p=1}^{4} \hat{E}^{\alpha}(\ell)\,\hat{E}^{\alpha}(\ell) - \frac{2}{g^4}\,[\mathrm{Tr}\; U(1)U(2)U(3)U(4) + \mathrm{h.c.}]\right\} \tag{2.1}$$

We consider here for notational simplicity the case of a unitary group U(N) instead of SU(N). The basic degrees of freedom are the unitary matrices $U(\ell)$ ($\ell = 1,2,3,4$) and for the conjugate variables one has

$$[\hat{E}^{\alpha}(\ell),\, \hat{E}^{\beta}(\ell')] = i\,\delta_{\ell\ell'}\, f_{\alpha\beta\gamma}\, \hat{E}^{\gamma}(\ell) \tag{2.2}$$

$$\alpha,\ \beta,\ \gamma = 0,1,\cdots,N^2-1$$

An explicit representation for this electric field operators is given by

$$\hat{E}^{\alpha} = t^{\alpha}_{ab}\, U_{bc}\, \frac{\partial}{\partial U_{ac}} \equiv \mathrm{Tr}\left\{t^{\alpha}\, U\, \frac{\partial}{\partial U}\right\} \tag{2.3}$$

Here the N × N matrices t^{α} denote the generators of the U(N) group and in what follows we shall need the following identities:

$$\sum_{\alpha=0}^{N^2-1} (t^{\alpha})_{ab}\,(t^{\alpha})_{a'b'} = \frac{1}{2}\,\delta_{ab'}\,\delta_{ba'} \tag{2.4a}$$

$$\mathrm{Tr}(t^{\alpha}t^{\beta}) = \frac{1}{2}\,\delta_{\alpha\beta} \tag{2.4b}$$

The second important ingredient of the theory is the requirement that the physical states are gauge invariant. This means that the physical wave-functions obey the constraint:

$$\Psi[U'] = \Psi[U] \tag{2.5}$$

where U' denotes an arbitrary time-independent gauge transformation

$$U'(i,i+n) = V(i)\;U(i,i+n)\;V^{-1}(i+n) \tag{2.6}$$

This represents a realization of the Gauss law.

We now proceed to describe the collective field representation of the Hamiltonian (2.1). The basic idea is to reformulate the theory in terms of new variables such that:

(a) The constraint on the wavefunctions is automatically satisfied.

(b) There are no matrix-valued fields.

One is then naturally led to the overcomplete set of commuting gauge invariant operators given by the traces:

$$W_n = \mathrm{Tr}\left\{(U(1)U(2)U(3)U(4))^n\right\} \tag{2.7}$$

These are the "loop space" collective variables and the integer n represents the winding number of the Wilson loop.

The change of variables to this new set goes as follows: First, using the chain rule we rewrite the electric field term as

$$\sum_{\alpha=0}^{N^2-1} \hat{E}^{\alpha}(\ell)\,\hat{E}^{\alpha}(\ell) = \frac{1}{2}\sum_{n=-\infty}^{\infty} w(n;W)\,\frac{\partial}{\partial W_n} - \frac{1}{2}\sum_{n,n'} \Omega(n,n';W)\,\frac{\partial^2}{\partial W_n\,\partial W_{n'}} \tag{2.8}$$

where we have denoted

$$w(n;W) = 2(\hat{E}(\ell)^2 W_n) \tag{2.9a}$$

$$\Omega(n,n';W) = -2\sum_{\ell} (\hat{E}^{\alpha}\, W_n)(\hat{E}^{\alpha}\, W_{n'}) \tag{2.9b}$$

A direct computation based on relations (2.3) and (2.4) produces:

$$w(n\,;W) = n\left\{ NW_n + \sum_{n'=\varepsilon(n)}^{n-\varepsilon(n)} W_{n'}\, W_{n-n'} \right\} \tag{2.10a}$$

where $\varepsilon(n)$ denotes ±1 depending on the sign of n. Also

$$\Omega(n,n';W) = -\,nn'W_{n+n'} \tag{2.11b}$$

Now after a similarity transformation one obtains the effective Hamiltonian:

$$H_{eff} = 4\cdot\frac{g^2}{2a}\left\{ -\frac{1}{2}\sum_{n,n'} \frac{\partial}{\partial W_n}\,\Omega(n,n';W)\,\frac{\partial}{\partial W_{n'}} + \frac{1}{8}\sum_{n,n'} w(n)\Omega^{-1}w(n') - \frac{1}{2g^4}\left(W_1 + W_{-1}\right) \right\} \tag{2.11}$$

plus counterterms. We mention that it is not difficult to derive an analogous representation for the full three-dimensional Kogut-Susskind Hamiltonian. Instead of the integers n one now has arbitrary contours C and the collective field is the space-like Wilson loop W(C). Proceeding as above there follow the explicit expression for ω and Ω. First

$$w(C;W) = N\ell(C)W(C) + \sum_{(C_1,C_2)} P(C;C_1,C_2)\; W(C_1)W(C_2) \tag{2.12a}$$

where $\ell(C)$ denotes the total length of the contour C and the sum is over all possible rearrangements (C_1,C_2) of the contour C. $P(C;C_1C_2)$ is essentially the length of the pinch where the rearrangement has occurred: $0 = \pm\ell$ where the sign ± represents the relative direction of the two contours C_1 and C_2. Similarly:

$$\Omega(C_1,C_2;W) = \sum_{C_3} O(C_3;C_1,C_2)\; W(C_3) \tag{2.12b}$$

Here the sum is over all possible rearrangements of the two contours C_1 and C_2 along their overlaps into some single contour C_3. The overlap function is

essentially the lattice version of the integral:

$$-\frac{1}{2}\int_{0_1} d\vec{\ell}\cdot\int_{0_2} d\vec{\ell}\,'\;\delta(|\vec{x}-\vec{x}\,'|)\ .$$

Concerning the above expressions it is interesting to see that they are exactly equal in form to the continuum theory expressions derived in Ref. [5]. This happens even though the Kogut-Susskind electric field term differs in form from the continuum theory kinetic term $\delta^2/\delta A_i^a(x)^2$.

We now proceed with the one-plaquette theory. To determine the large N behavior one now has to solve for the stationary point of our effective Hamiltonian (2.11). This still looks intractable and the essential reason for the solvability of this problem will come from the fact that one is able to introduce a Fourier transform. Let us define:

$$\phi(\sigma) = \sum_{n=-\infty}^{\infty} \frac{e^{in\sigma}}{2\pi}\, W_n \tag{2.13}$$

where $\sigma\,\varepsilon(-\pi,\pi)$. Some calculation gives

$$\begin{aligned} \Omega(\sigma,\sigma';\phi) &= \partial_\sigma\partial_{\sigma'}\,[\delta(\sigma-\sigma')\phi(\sigma)] \\ w(\sigma;\phi) &= -\partial_\sigma[\phi(\sigma)\;G(\sigma;\phi)] \end{aligned} \tag{2.14}$$

with

$$G(\sigma;\phi) = \fint d\sigma'\;\mathrm{ctg}\,\frac{\sigma-\sigma'}{2}\;\phi(\sigma') \tag{2.15}$$

The expression for the effective Hamiltonian now becomes

$$H_{eff} = \frac{2g^2}{a}\int_{-\pi}^{\pi} d\sigma\left\{ \partial_\sigma\Pi\;\phi\;\partial_\sigma\Pi + \frac{1}{8}\,\phi(\sigma)\;G(\sigma;\phi)^2 - \frac{1}{g^4}\cos\sigma\;\phi(\sigma) \right\} \tag{2.16}$$

Concerning this form we comment that the first and the third term in this effective Hamiltonian represent the classical contributions while the second term is of purely quantum origin. Had we kept track of this, this term would be proportional to $\hbar$. In general it is this additional term which plays an important role in the large N limit.

III. THE LARGE N LIMIT

As we have argued before in the collective field representation the large N limit becomes especially simple. This comes about because in the effective Hamiltonian (2.16) one does not have any more the variables with matrix index and the only place where N appears is the constraint:

$$\int_{-\pi}^{\pi} d\sigma \phi(\sigma) = \mathrm{Tr}(1) = N \tag{3.1}$$

Consequently, we have the fact that the large N behavior is directly determined by the classical stationary points of the effective Hamiltonian:

$$H_{eff}\,[\phi] + \mu(N - \int d\sigma \phi(\sigma)) \tag{3.2}$$

where μ represents the Lagrange multiplier enforcing the constraint (3.1).

Let us now consider the ground state which is determined by a time-independent classical solution. The equation determining this configuration follows from the variation of (3.2) and reads

$$\left[\int d\sigma' \,\mathrm{ctg}\, \frac{\sigma'-\sigma}{2}\, \tilde{\phi}(\sigma') \right]^2 - 2 \int \mathrm{ctg}\, \frac{\sigma'-\sigma}{2}\, \tilde{\phi}(\sigma') \int \mathrm{ctg} \frac{\sigma''-\sigma'}{2}\, \phi(\sigma'') = \mu + \frac{8}{\lambda^2} \cos\sigma \tag{3.3}$$

Here we have rescaled the field and the coupling constant according to:

$$\phi = N\tilde{\phi} \quad ; \quad g^2 = N\lambda \tag{3.4}$$

This nonlinear, singular, integral equation is essentially of the same form as the equation for the U(N) symmetric quantum mechanical problem which we have considered earlier [6]. The only difference is that here we have a compact manifold, the circle and then instead of the Cauchy kernel we have the $\mathrm{ctg}(\frac{\sigma'-\sigma}{2})$ kernel. In fact we can explicitly establish a relation between these two equations by making a change of variables:

$$e^{i\sigma} = t \quad , \quad |t| = 1 \tag{3.5}$$

so that

$$d\sigma = \frac{1}{i}\frac{dt}{t} \quad ; \quad \mathrm{ctg}\, \frac{\sigma'-\sigma}{2} = i\, \frac{t'+t}{t'-t} \tag{3.6}$$

Then it follows that

$$\int_{\sigma_1}^{\sigma_2} d\sigma' \,\mathrm{ctg}\, \frac{\sigma'-\sigma}{2} \tilde{\phi}(\sigma') = 2\int_{t_1}^{t_2} dt' \,\frac{\rho(t')}{t'-t} - i \tag{3.7}$$

where we have identified $\tilde{\phi}(\sigma) = \rho(t)$ and we made use of the normalization condition (3.1). Substituting this into (3.3) we obtain for the left-hand side of this equation

$$4\left\{ \left(\int_{t_1}^{t_2} dt' \,\frac{\rho(t')}{t'-t}\right)^2 - 2\int_{t_1}^{t_2} \frac{\rho(t')}{t'-t} \int \frac{\rho(t'')}{t''-t'} \right\} + 1 + 8 \,\frac{dt'}{t'}\, \rho(t') \,\frac{\rho(t'')}{t''-t'} \tag{3.8}$$

Here the last two terms are constants so they only shift the Lagrange multiplier constant μ. Consequently Equation (3.3) becomes

$$\left(\int_{t_1}^{t_2} dt' \,\frac{\rho(t')}{t'-t}\right)^2 - 2\int_{t_1}^{t_2} dt' \frac{\rho(t')}{t'-t} \int \frac{\rho(t'')}{t''-t'} = \mu + \frac{2}{\lambda^2} \cos\sigma \tag{3.9}$$

In this form the left-hand side of this equation coincides with our equation for the matrix model. In our earlier work [6] we have only stated the final solution to such a nonlinear integral equation. However, it is possible to give a systematic derivation and at the end of this section we shall describe the general procedure for reaching the solution. Essentially one can show that the left-hand side of (3.9) equals to $\pi^2 \rho(t)^2$ and then the problem simplifies to the algebraic equation

$$\pi^2 \tilde{\phi}(\sigma)^2 = \mu + \frac{2}{\lambda^2} \cos\sigma \tag{3.10}$$

on some interval $(-\sigma_o,\sigma_o)$ where $\phi(\sigma)$ is assumed to be nonzero.

The solution for the classical stationary point is now simply given by

$$\phi(\sigma) = \begin{cases} \frac{N}{\pi}\left(e - \frac{4}{\lambda^2}\sin\frac{\sigma}{2}\right)^{1/2} & |\sigma| < \sigma_o \\ 0 & |\sigma| > \sigma_o \end{cases} \tag{3.11a}$$

where the constant e should be determined from the normalization condition (3.1). Concerning the range $(-\sigma_o,\sigma_o)$ one has to distinguish the following two different regimes:

(i) <u>Weak coupling regime</u>: $4/e\lambda^2 \geqslant 1$

The range is now given by $\sigma_o = 2\alpha$ where we have denoted

$$\frac{e\lambda^2}{4} = \sin^2\alpha \tag{3.12}$$

The normalization condition (3.1) now reads

$$1 = \frac{8}{\pi\lambda}\int_0^\alpha d\phi\,(\sin^2\alpha - \sin^2\phi)^{1/2} \tag{3.13}$$

(ii) <u>Strong coupling regime</u>: $4/e\lambda^2 < 1$

In this case the range is obviously given by the whole circle $(-\pi,\pi)$. Denoting

$$\frac{4}{e\lambda^2} = \sin^2\beta \tag{3.14}$$

the normalization condition now reads

$$1 = \frac{8}{\pi\lambda}\frac{1}{\sin\beta}\int_0^{\pi/2} d\phi\,(1 - \sin^2\beta\,\sin^2\phi)^{1/2} \tag{3.15}$$

From the above we see that the solution (3.11) exists a phase transition(**) at the critical coupling $\lambda_c = 8/\pi$. This phase transition of the $N \to \infty$ limiting theory is of the same type as the one discovered recently for the two-dimensional Wilson theory in Ref. [10] and studied further in Refs. [11,12].

To complete our discussion let us finally sketch the procedure for reducing the nonlinear, singular, integral equation (3.9) to the algebraic equation (3.10). Consider the function on the left-hand side

$$L(t) \equiv \left(⨍ dt'\,\frac{\rho(t')}{t'-t}\right)^2 - 2⨍\frac{\rho(t')}{t'-t}\int\frac{\rho(t'')}{t''-t'} \tag{3.16}$$

Here the function $\rho(t)$ is assumed to be nonvanishing on some contour $C = (t_1,t_2)$ where t_1,t_2 denote the end points. Define now the complex function:

$$G(z) = \int_{t_1}^{t_2} dt\,\frac{\rho(t)}{t-z} \tag{3.17}$$

which obviously satisfies the following properties:

(i) $G(z)$ is analytic everywhere except on the line $(t_1 t_2)$.

(ii) $G(z) \to 0$, $|z| \to \infty$

Furthermore by approaching the contour C from above and below one has the Plemelj formulas [13]

$$G_+(t) = \int_{t_1}^{t_2} \frac{\rho(t')}{t'-t} + i\,\pi\rho(t)$$
$$G_-(t) = \int_{t_1}^{t_2} \frac{\rho(t')}{t'-t} - i\,\pi\rho(t) \qquad (3.18)$$

Now, consider the analytic function:

$$K(z) = G(z)^2 - 2\int_{t_1}^{t_2} \frac{\rho(t)}{t-z} \int_{t_1}^{t_2} \frac{\rho(t')}{t'-t}\, dt' \qquad (3.19)$$

This is an entire function since one can easily show that it is continuous on the curve C

$$K_+(t) - K_-(t) = 0 \qquad (3.20)$$

Since in addition we also have the fact that it vanishes at infinity the conclusion is that

$$K(z) = 0 \qquad (3.21)$$

Now as a direct consequence of (3.21) we obtain the identity

$$\left(\int_{t_1}^{t_2} \frac{\rho(t')}{t'-t}\right)^2 - \pi^2\rho(t)^2 - 2\int\frac{\rho(t')}{t'-t}\int\frac{\rho(t'')}{t''-t} \qquad (3.22)$$

and the final statement

$$L(t) = \pi^2\rho^2(t) \qquad (3.23)$$

This identity then reduces the integral equation to an algebraic equation.

IV. CONCLUSIONS

This simple, one-plaquette theory illustrates in a very nice way how the collective field approach to the large N limit in Yang-Mills gauge theory works. Let us now stress the following basic facts which allowed us to solve explicitly the effective gap equation in the present case. First, it was possible to define a Fourier transform on the loop space which was here a transform from integers to a circle. Second, and equally important was the fact that we could reduce the singular effective potential term to a simpler algebraic form. It seems that in order to solve explicitly the real three-dimensional Yang-Mills problem it will be important to be able to generalize these two basic steps. At present, however not much is known about an integro-differential calculus in loop-space and one is hopeful that more progress in this direction can be made. On the other hand recently a new proposal was put forward [14,15] which may eventually reduce the problem to classical Yang-Mills equations in ordinary space.

ACKNOWLEDGMENT

This work was completed when one of us (B.S.) was visiting the Laboratoire de Physique Théorique de l'École Normale Supérieure whose hospitality is gratefully acknowledged.

FOOTNOTES

(*) For some other, different attempts to study the large N limit in Yang-Mills theory see Refs. [7,8].

(**) The phase transition behavior of this solution was pointed out to us by S. Wadia. He has independently studied the same model using a different method.

REFERENCES

[1] G. 't Hooft, Nucl. Phys. B72 (1974) 461.

[2] J. Koplik, A. Neveu and S. Nussinov, Nucl. Phys. B123 (1977) 109.

[3] G. 't Hooft, Nucl. Phys. B75 (1974) 461.

[4] E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, Comm. Math. Phys. 59, (1978) 35.

[5] B. Sakita, Phys. Rev. D. (to appear).

[6] A. Jevicki and B. Sakita, Nucl. Phys. B (to appear)

[7] C. Thorn, Phys. Rev. D19 (1979) 639.
I. Bars and F. Green, Phys. Rev. D20 (1979) 3311.
D. Foerster, Phys. Lett. 87B (1979) 87.
T. Eguchi, ibid. pp. 91; D. Weingarten, ibid. pp. 97 and Indiana preprint.
J. Engels and I. Montvay, preprint BI-TP 79/36.

[8] Yu. M. Makeenko and A. A. Migdal, Phys. Lett. 88B (1979) 135.
A. A. Migdal, preprint 1979.
D. Foerster, Saclay preprint DPh-T/170/79.

[9] J. Kogut and L. Susskind, Phys. Rev. D11 (1975) 395.

[10] D. Gross and E. Witten, Phys. Rev. D21 (1980) 446.

[11] S. Wadia, Chicago preprint EFI.

[12] Y. Goldschmidt, Saclay preprint 1979.

[13] B. Davis, "Integral Transforms and Their Applications", Springer-Verlag, New York.

[14] A. Jevicki and N. Papanicolaou, Nucl. Phys. B (to appear).

[15] A. Jevicki and H. Levine, Harvard preprint and Brown preprint HET-418 (1980).