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A Canonical Eight-Dimensional Formalism for Linear and Non-Linear Classical Spin-Orbit Motion in Storage Rings

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Contents

1 Introduction

In recent years methods for obtaining spin polarized beams of antiprotons have been extensively studied [6,7,8,9]. One of these schemes, the Spin Splitter scheme [8], involves the use of Stern-Gerlach (SG) forces in the gradients of the magnetic fields in storage rings to drive the particle motion in phase with the betatron motion as in a driven oscillator. This would lead to the build up of coherent betatron oscillations. Of course, the need for a well defined relative phase means that the machine would have to run on or close to a resonance between spin and orbit natural oscillation frequencies, a so called spin-orbit resonance [10,11]. The initially unpolarized beam can be considered to consist of an equal mixture of spin states with opposite values of the third spin component, say up and down or left and right etc. Thus the two spin ensembles, with their opposite sign of SG forces would separate spatially and oscillate coherently in antiphase. After some time depending on the ring layout and energy [8] the betatron amplitudes of the two ensembles would be so big that the two ensembles could be separated. For example, one ensemble could be removed using a beam scraper, leaving the other ensemble in an almost pure spin state. Naturally, given the smallness of the SG forces the separation time is typically of the order of hours [8]. Furthermore they are small compared to other more familiar sources of orbit disturbance such as wake fields and noise. More details and explicit examples of ring layouts and optics can be found in the references. In this paper, these details are of no immediate concern. Instead we are more interested in establishing a Hamiltonian description at the single particle level.

Recently, Derbenev [1] has studied the Spin Splitter proposal again and has noted some difficulties in realizing it in its original form. His arguments are general and are based on the use of conservation laws that arise when two quantum oscillators, the spin and the orbit, are coupled. However, he is able to suggest ways to overcome these difficulties and also to suggest further improvements.

The aim of this and another paper is to rework Derbenev's picture, but this time using a purely classical picture of spin motion, and to investigate the extent to which his conclusions are to be expected on purely classical grounds by analogy with instability phenomena that appear in other branches of storage ring optics.

That the original Spin Splitter concept may need modification is already clear at the classical level when we realize that it utilizes only the SG forces in its modification of the spin orbit motion and does not consider the details of the effect of orbit motion on the spin. In the language of Hamiltonian mechanics this means that the formulation might not be symplectic. Thus it is of no great surprise that the orbit amplitude appears to grow: This could just be a manifestation of orbital antidamping of the kind that can be expected when transfer matrices are not symplectic. We are also not surprised about Derbenev's demonstration that an increase of betatron amplitude can cause spin flip and thus a subsequent decrease in the orbit amplitude, a process which would continue ad infinitum: For the Spin Splitter to work as prescibed, one would have to sit at a spin-orbit resonance, a condition which in other branches of spin physics in storage rings is well known to cause depolarization and spin flip [12].

In this paper we set up a consistent classical spin orbit formalism based on the semiclassical spin orbit Hamiltonian of Derbenev and Kondratenko [1,13] and which was also used by Yokoya [14]. From this Hamiltonian we derive, following some notions of Yokoya, a symplectic spin-orbit formalism which is correct at semiclassical order, i.e. which contains no terms in $\hbar$ above first order. This paper then forms the basis of another work in which we study the interaction of the spin and orbit motion using methods familiar from other parts of storage ring physics.

This formalism in fact represents a natural generalization of the earlier work of Refs. [2,3] where we presented an analytical technique for investigating linear and non-linear coupled synchro - betatron oscillations which handles the combined external magnetic and electric forces in a consistent canonical manner and which includes consistently and canonically the synchrotron oscillations in the electric fields of the accelerating cavities. The motion was described by using the canonical variables $\hat{x}$, $\hat{p}_x$, $\hat{z}$, $\hat{p}_z$, $\hat{\sigma}$, $\hat{p}_\sigma$ of the fully six-dimensional formalism. The equations derived in those papers provide the basis of a symplectic, non-linear, 6-dimensional tracking program.

In this report we extend these investigations by including the spin motion using two new spin variables ($\hat{\alpha}$, $\hat{\beta}$) which are canonical and which uniquely parametrize the classical spin over (almost) the whole 'spin sphere' and which behave in the small spin tilt limit like those used by Chao [10] in the SLIM formalism. Furthermore, the spin part of the Hamiltonian when written in terms of these variables takes a form which can be expanded into a power series in an economic way, leading to various orders of approximation of the canonical spin equations. It is this property which distinguishes our canonical coordinates $\hat{\alpha}$ and $\hat{\beta}$ from others occuring in the literature [14,15].

With the complete set $\hat{x}$, $\hat{p}_x$, $\hat{z}$, $\hat{p}_z$, $\hat{\sigma}$, $\hat{p}_\sigma$, $\hat{\alpha}$, $\hat{\beta}$ we are then in a position to develop, in the framework of this 8-dimensional formalism, a symplectic treatment of the combined orbital and spin motion in storage rings.

The equations so derived can serve to develop a non-linear, 8-dimensional (symplectic) tracking program and modern methods such as Lie algebra, normal forms and differential algebra which are well-known from orbital motion could also be used. Such a program may be used to study (in addition to orbital problems) chaotic behaviour of spin motion when spin-orbit resonances are wide and overlap and to investigate the influence of Stern-Gerlach forces. Furthermore, since our formalism automatically includes provision for describing skew quadrupoles and solenoids it is well suited for working with Spin Splitters since these usually rely on special arrangements of such elements to obtain the required spin orientation at the strong quadrupoles used to generate sufficiently strong SG forces. Derbenev restricted his discussion to one mode of uncoupled betatron motion.

In detail, our considerations are organized as follows :

Starting in Chapter 2 from the Hamiltonian of a charged particle for spin-orbit motion in an electromagnetic field, described in a fixed Cartesian coordinate system, in Chapter 3 we use a canonical transformation to arrive at the symplectic formalism for spin-orbit motion expressed in machine coordinates, taking into account all kinds of coupling induced by skew quadrupoles and solenoids (coupling of betatron motion), by a non-vanishing dispersion in the cavities (synchro-betatron coupling) and by Stern-Gerlach forces.

The vector potentials we need to describe the electro-magnetic field are calculated in Appendix A.

In Chapter 4 the arc length of the design orbit as independent variable (instead of the time t) is introduced and new (small and oscillating) variables σ, p_σ are defined which describe the longitudinal oscillations.

Spin motion in terms of the dreibein $(\vec{e}_s, \vec{e}_x, \vec{e}_z)$ is investigated in Chapter 5 and the corresponding Hamiltonian is derived by applying a transformation similar to that used by Yokoya.

Then in Chapter 6 and, with the help of Appendix B, we define an 8-dimensional closed orbit which we introduce as a new reference orbit for spin-orbit motion. The Hamiltonian with respect to the closed orbit is again obtained by using canonical transformations, whereby the canonical variables $\hat{\alpha}$ and $\hat{\beta}$ are introduced to describe the spin motion.

A summary is finally presented in Chapter 6.

2 Spin-Orbit Motion in a Fixed Coordinate System

2.1 The Starting Hamiltonian

The starting point of our description of classical spin-orbit motion will be the classical Hamiltonian, $\mathcal{H}$:

$$\mathcal{H}(\vec{r}, \psi; \vec{P}, J; t) = \mathcal{H}_{orb}(\vec{r}, \vec{P}, t) + \vec{\Omega}_0(\vec{r}, \vec{P}, t) \cdot \vec{\xi} \tag{2.1}$$

with

$$\mathcal{H}_{orb}(\vec{r}, \vec{P}, t) = c \cdot \left\{\vec{\pi}^2 + m_0^2 c^2\right\}^{1/2} + e\phi \tag{2.2}$$

and

$$\vec{\Omega}_0 = -\frac{e}{m_0 c}\left[\left(\frac{1}{\gamma} + a\right) \cdot \vec{B} - \frac{a\,(\vec{\pi} \cdot \vec{B})}{\gamma(\gamma+1)m_0^2 c^2} \cdot \vec{\pi} - \frac{1}{m_0 c \gamma}\left(a + \frac{1}{1+\gamma}\right) \vec{\pi} \times \vec{\varepsilon}\right] \tag{2.3}$$

where $\vec{r}$ and $\vec{P}$ are canonical orbital position and momentum variables, $\vec{\xi}$ is a classical spin vector of length $\hbar/2$ and where $\vec{\pi}$ and γ are given by:

$$\vec{\pi} = \vec{P} - \frac{e}{c}\vec{A} \quad \text{(kinetic momentum vector)}\ ; \tag{2.4}$$

$$\gamma = \frac{1}{m_0 c} \cdot \sqrt{m_0^2 c^2 + \vec{\pi}^2} \quad \text{(Lorentz factor)}\ . \tag{2.5}$$

The following abbreviations have been used:

- e = charge of the particle ;
- m_0 = rest mass of the particle ;
- c = velocity of light;
- $\vec{\varepsilon}$ = electric field;
- $\vec{B}$ = magnetic field ;
- $\vec{r}$ = radius vector of the particle.

- $\vec{\xi}$ = classical spin angular momentum vector in the rest frame of the particle of length $\hbar/2$;
- a = $(g-2)/2$ (0.00116 for electrons, 1.793 for protons) and quantifies the anomalous spin g factor ;
- $2\pi\hbar$ = Planck's constant.

The quantities $\vec{A}$ and ϕ appearing in eqn. (2.7) are the vector and scalar potentials from which the electric field $\vec{\epsilon}$ and the magnetic field $\vec{B}$ are derived as

$$\vec{\epsilon} = -\text{grad}\ \phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\ ; \tag{2.6a}$$

$$\vec{B} = \text{curl}\ \vec{A}\ . \tag{2.6b}$$

Our starting Hamiltonian (2.1) is that which is often used for describing the spin-orbit dynamics in accelerators [1,13,14,16,17] and is the classical reinterpretation of the effective quantum mechanical Hamiltonian derived by a unitary transformation of the Dirac Hamiltonian and by working in the semiclassical limit. This latter is valid when the external electromagnetic field is weak and it neglects bremsstrahlung effects [18].

In terms of the three unit cartesian coordinate vectors in the fixed laboratory frame, $\vec{e}_1,\ \vec{e}_2,\ \vec{e}_3$ we can write $\vec{r}$, $\vec{P}$ and $\vec{\xi}$ as:

$$\vec{r} = X_1\cdot\vec{e}_1 + X_2\cdot\vec{e}_2 + X_3\cdot\vec{e}_3\ ; \tag{2.7a}$$

$$\vec{P} = P_1\cdot\vec{e}_1 + P_2\cdot\vec{e}_2 + P_3\cdot\vec{e}_3\ ; \tag{2.7b}$$

$$\vec{\xi} = \xi_1\cdot\vec{e}_1 + \xi_2\cdot\vec{e}_2 + \xi_3\cdot\vec{e}_3\ ; \tag{2.7c}$$

Furthermore, we write the components of $\vec{\xi}$ in the form:

$$\begin{cases} \xi_1 = \sqrt{\xi^2 - J^2}\cdot\cos\psi\ ; \\ \xi_2 = \sqrt{\xi^2 - J^2}\cdot\sin\psi\ ; \\ \xi_3 = J\ ; \end{cases} \tag{2.8}$$

with

$$\xi^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 = \frac{\hbar^2}{4}\ .$$

We will treat ψ and J as canonical spin variables [14,16] to be used on an equal basis with $\vec{r}$ and $\vec{P}$. $\vec{\xi}$ is of constant length since it obeys a precession equation. See below.

With (2.1) and (2.8) we have the Hamiltonian for the canonical variables $\vec{r}$, $\vec{P}$, ψ, J.

One of the aims of this paper is to transform from the canonical variables $(\vec{r},\ \vec{P},\ \psi,\ J)$ to the new set of canonical variables $(\hat{x},\ \hat{z},\ \hat{\sigma},\ \hat{\alpha};\ \hat{p}_x,\ \hat{p}_z,\ \hat{p}_\sigma,\ \hat{\beta})$ (see eqn. (6.39)).

2.2 The Equations of Motion

2.2.1 Orbital Motion

With this Hamiltonian (2.1) the orbital equations of motion are:

$$\frac{d}{dt} X_k = +\frac{\partial \mathcal{H}_{orb}}{\partial P_k} + \frac{\partial \vec{\Omega}_0}{\partial P_k} \cdot \vec{\xi} ; \quad (2.9a)$$

$$\frac{d}{dt} P_k = -\frac{\partial \mathcal{H}_{orb}}{\partial X_k} - \frac{\partial \vec{\Omega}_0}{\partial X_k} \cdot \vec{\xi} ; \quad (2.9b)$$

$$(k = 1, 2, 3) .$$

The first terms on the rhs of (2.9) are the Lorentz terms and the second terms describe the Stern-Gerlach force [19]. Thus our Hamiltonian includes the SG force automatically. Note that here we deal with the relativistic generalization of the SG effect. It is clear (see eqn. 2.3) that our SG terms reduce to the usual non-relativistic forms in the limit that γ becomes unity and that for $\gamma > 1$ the factor $g/2$ in the expression for the SG force in field gradients in Ref. [8] should be replaced by $(g-2)/2 + 1/\gamma)$. Thus if γ is increased from 1 up to a large value, the SG force is reduced by the factor $(g-2)/g$. For protons $(g/2 = 2.793)$ this gives a 36% reduction . But for electrons $(g/2 = 1.00116)$ the reduction factor is large.

The discussion in this paper covers both relativistic and non-relativistic motion.

2.2.2 Spin Motion

Using (2.8) and treating J, ψ as canonical variables, we can easily show that [14]:

$$\{\xi_1, \xi_2\}_{\psi,J} = \xi_3 ; \quad (2.10a)$$

$$\{\xi_2, \xi_3\}_{\psi,J} = \xi_1 ; \quad (2.10b)$$

$$\{\xi_3, \xi_1\}_{\psi,J} = \xi_2 . \quad (2.10c)$$

These Poisson bracket relations for spin, which do not contain $\hbar$ on the rhs, are the classical analogues of the commutation relation among Pauli spin operators. Using these relations together with the canonical equations of the spin motion:

$$\frac{d}{dt} \psi = +\frac{\partial}{\partial J} \mathcal{H}_{spin} ; \quad (2.11a)$$

$$\frac{d}{dt} J = -\frac{\partial}{\partial \psi} \mathcal{H}_{spin} \quad (2.11b)$$

where

$$\mathcal{H}_{spin} = \vec{\Omega}_0 \cdot \vec{\xi} \quad (2.12)$$

and

$$\vec{\Omega}_0 = \Omega_{01} \cdot \vec{e}_1 + \Omega_{02} \cdot \vec{e}_2 + \Omega_{03} \cdot \vec{e}_3 \quad (2.13)$$

so that

$$\begin{aligned} \vec{\Omega}_0 \cdot \vec{\xi} &= \Omega_{01} \cdot \xi_1 + \Omega_{02} \cdot \xi_2 + \Omega_{03} \cdot \xi_3 \\ &= \sqrt{\xi^2 - J^2} \cdot [\Omega_{01} \cdot \cos\psi + \Omega_{02} \cdot \sin\psi] + \Omega_{03} \cdot J \end{aligned} \quad (2.14)$$

we find

$$\frac{d}{dt}\,\vec{\xi} \;=\; \vec{\Omega}_0 \times \vec{\xi} \tag{2.15}$$

Thus this Hamiltonian formalism reproduces the Thomas-BMT equation [20,21]. The result (2.15) can also be obtained by using the equation of motion:

$$\frac{d}{dt}\,\vec{\xi} \;=\; \left\{\vec{\xi},\,\mathcal{H}_{spin}\right\}_{\psi,J} \;\equiv\; \frac{\partial\vec{\xi}}{\partial\psi}\cdot\frac{\partial\mathcal{H}_{spin}}{\partial J} - \frac{\partial\vec{\xi}}{\partial J}\cdot\frac{\partial\mathcal{H}_{spin}}{\partial\psi}\;. \tag{2.16}$$

2.2.3 The combined Form of the Spin-Orbit Equations

The combined equations of spin-orbit motion can be written in the form:

$$\frac{d}{dt}\,X_k \;=\; +\frac{\partial\mathcal{H}}{\partial P_k}\;; \tag{2.17a}$$

$$\frac{d}{dt}\,P_k \;=\; -\frac{\partial\mathcal{H}}{\partial X_k} \tag{2.17b}$$

(k=1, 2, 3, 4)

with

$$X_4 \;\equiv\; \psi\;; \tag{2.18a}$$

$$P_4 \;\equiv\; J\;. \tag{2.18b}$$

3 Introduction of Machine Coordinates via a Canonical Transformation

3.1 Reference Trajectory and Coordinate Frame

The position vector $\vec{r}$ in eqn. (2.1) refers to a fixed coordinate system with the coordinates X_1, X_2 and X_3. However, in accelerator physics, it is useful to introduce the natural coordinates x, z, s in a suitable curvilinear coordinate system. With this in mind we assume that an ideal closed design orbit exists which describes the path of a particle of constant energy E_0, i.e. we neglect energy variations due to cavities and to radiation loss. In addition we assume that there are no field errors or correction magnets. We also require that the design orbit comprises piecewise flat curves which lie either in the horizontal or vertical plane so that it has (piecewise) no torsion. The design orbit which will be used as the reference system will, in the following, be described by the vector $\vec{r}_0(s)$ where s is the length along the design orbit. An arbitrary particle orbit $\vec{r}(s)$ is then described by the deviation $\delta\vec{r}(s)$ of the particle orbit $\vec{r}(s)$ from the design orbit $\vec{r}_0(s)$:

$$\vec{r}(s) = \vec{r}_0(s) + \delta\vec{r}(s)\;. \tag{3.1}$$

The vector $\delta\vec{r}$ can as usual [22] be described using an orthogonal coordinate system ("dreibein") accompanying the particle which travels along the design orbit and comprises

$$\text{the unit tangent vector} \quad \vec{e}_s(s) = \frac{d}{ds}\vec{r}_0(s) \equiv \vec{r}_0{}'(s) \; ;$$
$$\text{a unit vector} \quad \vec{e}_x(s)$$

which lies perpendicular to $\vec{e}_s$ in the horizontal plane [11]

$$\text{and the unit vector} \quad \vec{e}_z(s) = \vec{e}_s(s) \times \vec{e}_x(s) \; .$$

In this natural coordinate system we may represent $\delta\vec{r}(s)$ as:

$$\delta\vec{r}(s) = (\delta\vec{r} \cdot \vec{e}_x) \cdot \vec{e}_x + (\delta\vec{r} \cdot \vec{e}_z) \cdot \vec{e}_z$$

(since the "dreibein" accompanies the particle, the $\vec{e}_s$- component of $\delta\vec{r}$ is always zero by definition).

Thus, the orbit-vector $\vec{r}(s)$ can be written in the form

$$\vec{r}(x,z,s) = \vec{r}_0(s) + x(s) \cdot \vec{e}_x(s) + z(s) \cdot \vec{e}_z(s) \tag{3.2}$$

and the Serret-Fresnet formulae for the dreibein $(\vec{e}_s, \vec{e}_x, \vec{e}_z)$ read as:

$$\frac{d}{ds}\vec{e}_x(s) = +K_x(s) \cdot \vec{e}_s(s) \; ; \tag{3.3a}$$
$$\frac{d}{ds}\vec{e}_z(s) = +K_z(s) \cdot \vec{e}_s(s) \; ; \tag{3.3b}$$
$$\frac{d}{ds}\vec{e}_s(s) = -K_x(s) \cdot \vec{e}_x(s) - K_z(s) \cdot \vec{e}_z(s) \tag{3.3c}$$

where we assume that

$$K_x(s) \cdot K_z(s) = 0 \tag{3.4}$$

(piecewise no torsion) and where $K_x(s), K_z(s)$ designate the curvatures in the x-direction and in the z-direction respectively.

Note that the sign of $K_x(s)$ and $K_z(s)$ is fixed by eqns. (3.3).

3.2 Introduction of the Natural Coordinates x, z, s via a Canonical Transformation

Writing:

$$\begin{aligned} \vec{r} &= X_1 \cdot \vec{e}_1 + X_2 \cdot \vec{e}_2 + X_3 \cdot \vec{e}_3 = \vec{r}_0(s) + x \cdot \vec{e}_x(s) + z \cdot \vec{e}_z(s) \; ; \\ \vec{P} &= P_1 \cdot \vec{e}_1 + P_2 \cdot \vec{e}_2 + P_3 \cdot \vec{e}_3 \end{aligned}$$

we can obtain the canonical transformation:

$$X_1, X_2, X_3, P_1, P_2, P_3 \longrightarrow x, z, s, p_x, p_z, p_s \; ;$$
$$(\psi, \; J \; \text{unchanged})$$

by introducing the generating function [23] :

$$F_3(x,z,s,\psi';P_1,P_2,P_3,J;t) = -\left[\vec{r}_0(s) + x\cdot\vec{e}_x(s) + z\cdot\vec{e}_z(s)\right]\cdot\vec{P}\ ; \qquad (3.5)$$

This leads to the transformation equations :

$$\begin{aligned}
X_1 &= -\frac{\partial F_3}{\partial P_1} = [\vec{r}_0(s) + x(s)\cdot\vec{e}_x(s) + z(s)\cdot\vec{e}_z(s)]\cdot\vec{e}_1 = \vec{r}\cdot\vec{e}_1\ ; && (3.6a)\\
X_2 &= -\frac{\partial F_3}{\partial P_2} = [\vec{r}_0(s) + x(s)\cdot\vec{e}_x(s) + z(s)\cdot\vec{e}_z(s)]\cdot\vec{e}_2 = \vec{r}\cdot\vec{e}_2\ ; && (3.6b)\\
X_3 &= -\frac{\partial F_3}{\partial P_3} = [\vec{r}_0(s) + x(s)\cdot\vec{e}_x(s) + z(s)\cdot\vec{e}_z(s)]\cdot\vec{e}_3 = \vec{r}\cdot\vec{e}_3\ ; && (3.6c)\\
p_x &= -\frac{\partial F_3}{\partial x} = \vec{e}_x(s)\cdot\vec{P}\ ; && (3.6d)\\
p_z &= -\frac{\partial F_3}{\partial z} = \vec{e}_z(s)\cdot\vec{P}\ ; && (3.6e)\\
p_s &= -\frac{\partial F_3}{\partial s} = [1 + K_x\cdot x + K_z\cdot z]\cdot\vec{e}_s\cdot\vec{P}\ . && (3.6f)
\end{aligned}$$

Note, that eqns. (3.6a - c) reproduce the defining equation (2.1a) for the variables X_1, X_2 and X_3 and that eqns. (3.6d - f) determine the new momentum variables p_x, p_z and p_s. The spin variables ψ and J remain unchanged.

Because

$$\frac{\partial F_3}{\partial t} = 0,$$

the Hamiltonian is transformed to :

$$\begin{aligned}
\mathcal{H} \quad\longrightarrow\quad \mathcal{H} + \frac{\partial F_3}{\partial t} &= \mathcal{H}\\
&= \mathcal{H}_{orb} + \vec{\Omega}_0\cdot\vec{\xi}\ . && (3.7)
\end{aligned}$$

In order to obtain $\mathcal{H}$ in terms of the new variables $x, p_x,\ z, p_z,\ s, p_s$, we write:

$$\vec{\pi} = \pi_x\cdot\vec{e}_x + \pi_z\cdot\vec{e}_z + \pi_s\cdot\vec{e}_s \qquad (3.8)$$

with

$$\begin{aligned}
\pi_x &= \vec{\pi}\cdot\vec{e}_x \equiv \left(\vec{P} - \frac{e}{c}\vec{A}\right)\cdot\vec{e}_x = p_x - \frac{e}{c}A_x\ ; && (3.9a)\\
\pi_z &= \vec{\pi}\cdot\vec{e}_z \equiv \left(\vec{P} - \frac{e}{c}\vec{A}\right)\cdot\vec{e}_z = p_z - \frac{e}{c}A_z\ ; && (3.9b)\\
\pi_s &= \vec{\pi}\cdot\vec{e}_s \equiv \left(\vec{P} - \frac{e}{c}\vec{A}\right)\cdot\vec{e}_s = \frac{p_s}{[1 + K_x\cdot x + K_z\cdot z]} - \frac{e}{c}A_s && (3.9c)
\end{aligned}$$

whereby

$$\begin{aligned}
\vec{A} &= A_x\cdot\vec{e}_x + A_z\cdot\vec{e}_z + A_s\cdot\vec{e}_s\\
&= (\vec{A}\cdot\vec{e}_x)\cdot\vec{e}_x + (\vec{A}\cdot\vec{e}_z)\cdot\vec{e}_z + (\vec{A}\cdot\vec{e}_s)\cdot\vec{e}_s\ . && (3.10)
\end{aligned}$$

Thus we obtain:

$$\mathcal{H}_{orb} = e\phi +$$
$$c \cdot \left\{ \left(p_x - \frac{e}{c}A_x\right)^2 + \left(p_z - \frac{e}{c}A_z\right)^2 + \left(\frac{p_s}{[1 + K_x \cdot x + K_z \cdot z]} - \frac{e}{c}A_s\right)^2 + m_0^2c^2 \right\}^{1/2} \quad (3.11)$$

and

$$\vec{\Omega}_0 \quad = \quad \Omega_{0s} \cdot \vec{e}_s + \Omega_{0x} \cdot \vec{e}_x + \Omega_{0z} \cdot \vec{e}_z \quad (3.12)$$

with

$$\Omega_{0s} = -\frac{e}{m_0c}\left[\left(\frac{1}{\gamma} + a\right) \cdot B_s - \frac{a\ (\pi_s \cdot B_s + \pi_x \cdot B_x + \pi_z \cdot B_z)}{\gamma(\gamma+1) \cdot m_0^2c^2} \cdot \pi_s - \frac{1}{m_0c\gamma}\left(a + \frac{1}{1+\gamma}\right)(\pi_x\varepsilon_z - \pi_z\varepsilon_x)\right] ; \quad (3.13a)$$

$$\Omega_{0x} = -\frac{e}{m_0c}\left[\left(\frac{1}{\gamma} + a\right) \cdot B_x - \frac{a\ (\pi_s \cdot B_s + \pi_x \cdot B_x + \pi_z \cdot B_z)}{\gamma(\gamma+1) \cdot m_0^2c^2} \cdot \pi_x - \frac{1}{m_0c\gamma}\left(a + \frac{1}{1+\gamma}\right)(\pi_z\varepsilon_s - \pi_s\varepsilon_z)\right] ; \quad (3.13b)$$

$$\Omega_{0z} = -\frac{e}{m_0c}\left[\left(\frac{1}{\gamma} + a\right) \cdot B_z - \frac{a\ (\pi_s \cdot B_s + \pi_x \cdot B_x + \pi_z \cdot B_z)}{\gamma(\gamma+1) \cdot m_0^2c^2} \cdot \pi_z - \frac{1}{m_0c\gamma}\left(a + \frac{1}{1+\gamma}\right)(\pi_s\varepsilon_x - \pi_x\varepsilon_s)\right] \quad (3.13c)$$

whereby

$$\gamma \quad = \quad \frac{1}{m_0c}\sqrt{m_0^2c^2 + \vec{\pi}^2} \quad \doteq \quad \frac{\mathcal{H}_{orb} - e\phi}{m_0c^2}$$

and $\vec{B}$ and $\vec{\varepsilon}$ have to be written as functions of $s,\ x,\ z,\ t$.

With (3.7), (3.11), and (3.13) we have the Hamiltonian for the canonical variables

$$x,\ z,\ s,\ \psi;\ p_x,\ p_z,\ p_s,\ J.$$

Remark:

Equation (3.5) is an example of a point transformation

$$q_k \longrightarrow q_k' \quad (3.14)$$

which may be written in the most general form as:

$$q_k = f_k(q_l', t)\ . \quad (3.15)$$

This transformation can be obtained as a canonical transformation

$$q_k,\ p_k \longrightarrow q_k',\ p_k' \tag{3.16}$$

by the generating function

$$F_3(q_l', p_l, t) = -\sum_n p_n \cdot f_n(q_l', t). \tag{3.17}$$

The corresponding transformation equations read as:

$$q_k = -\frac{\partial F_3}{\partial p_k} = f_k(q_l', t); \tag{3.18a}$$

$$p_k' = -\frac{\partial F_3}{\partial q_k'} = \sum_n p_n \cdot \frac{\partial}{\partial q_k'} f_n(q_l', t); \tag{3.18b}$$

$$\mathcal{H}' = \mathcal{H} + \frac{\partial F_3}{\partial t}. \tag{3.18c}$$

Here the relation (3.18a) coincides with eqn. (3.15) defining the new variables q_k' and eqn. (3.18b) determines the new momenta p_k' corresponding to the variables q_k', whereas the new Hamiltonian $\mathcal{H}'$ is given by (3.18c) which has to be written in terms of q_k' and p_k'.

3.3 The Equations of Motion

3.3.1 Orbital Motion

In the new orbital coordinates the equations of orbital motion are :

$$\frac{d}{dt}x = +\frac{\partial \mathcal{H}_{orb}}{\partial p_x} + \frac{\partial \vec{\Omega}_0}{\partial p_x}\cdot\vec{\xi}\,; \quad \frac{d}{dt}p_x = -\frac{\partial \mathcal{H}_{orb}}{\partial x} - \frac{\partial \vec{\Omega}_0}{\partial x}\cdot\vec{\xi}\,; \tag{3.19a}$$

$$\frac{d}{dt}z = +\frac{\partial \mathcal{H}_{orb}}{\partial p_z} + \frac{\partial \vec{\Omega}_0}{\partial p_z}\cdot\vec{\xi}\,; \quad \frac{d}{dt}p_z = -\frac{\partial \mathcal{H}_{orb}}{\partial z} - \frac{\partial \vec{\Omega}_0}{\partial z}\cdot\vec{\xi}\,; \tag{3.19b}$$

$$\frac{d}{dt}s = +\frac{\partial \mathcal{H}_{orb}}{\partial p_s} + \frac{\partial \vec{\Omega}_0}{\partial p_s}\cdot\vec{\xi}\,; \quad \frac{d}{dt}p_s = -\frac{\partial \mathcal{H}_{orb}}{\partial s} - \frac{\partial \vec{\Omega}_0}{\partial s}\cdot\vec{\xi}\,. \tag{3.19c}$$

3.3.2 Spin Motion

Although we have not yet written the spin, $\vec{\xi}$, in terms of $\vec{e}_x,\ \vec{e}_z,\ \vec{e}_s$ the equations of spin motion are as before :

$$\frac{d}{dt}\vec{\xi} = \vec{\Omega}_0 \times \vec{\xi} \tag{3.20}$$

or

$$\frac{d}{dt}\psi = +\frac{\partial}{\partial J}\left[\vec{\Omega}_0\cdot\vec{\xi}\right] = +\frac{\partial}{\partial J}\mathcal{H}\,; \tag{3.21a}$$

$$\frac{d}{dt}J = -\frac{\partial}{\partial \psi}\left[\vec{\Omega}_0\cdot\vec{\xi}\right] = -\frac{\partial}{\partial \psi}\mathcal{H}\,. \tag{3.21b}$$

3.3.3 The combined Form of the Spin-Orbit Equations

The combined equations of spin-orbit motion are :

$$\frac{d}{dt}\,x = +\frac{\partial\mathcal{H}}{\partial p_x}\,; \quad \frac{d}{dt}\,p_x = -\frac{\partial\mathcal{H}}{\partial x}\,; \tag{3.22a}$$

$$\frac{d}{dt}\,z = +\frac{\partial\mathcal{H}}{\partial p_z}\,; \quad \frac{d}{dt}\,p_z = -\frac{\partial\mathcal{H}}{\partial z}\,; \tag{3.22b}$$

$$\frac{d}{dt}\,s = +\frac{\partial\mathcal{H}}{\partial p_s}\,; \quad \frac{d}{dt}\,p_s = -\frac{\partial\mathcal{H}}{\partial s}\,; \tag{3.22c}$$

$$\frac{d}{dt}\,\psi = +\frac{\partial\mathcal{H}}{\partial J}\,; \quad \frac{d}{dt}\,J = -\frac{\partial\mathcal{H}}{\partial \psi}\,; \tag{3.22d}$$

4 The Arc Length of the Design Orbit as Independent Variable

In eqn. (3.22) the time t appeared as independent variable. In order, as usual in accelerator physics, to introduce the arc length s of the design orbit as independent variable we recall that eqn. (3.22) is equivalent to a version of Hamilton's principle [24]:

$$\delta\int_{t_1}^{t_2} dt\cdot\left\{\dot{x}\cdot p_x+\dot{z}\cdot p_z+\dot{s}\cdot p_s+\dot{\psi}\cdot J-\mathcal{H}(x,z,s,\psi;p_x,p_z,p_s,J;t)\right\}=0 \tag{4.1a}$$

with

$$\begin{cases} \delta x(t_1)=\delta z(t_1)=\delta s(t_1)=\delta\psi(t_1)=0\;; & \delta p_x(t_1)=\delta p_z(t_1)=\delta p_s(t_1)=\delta J(t_1)=0; \\ \delta x(t_2)=\delta z(t_2)=\delta s(t_2)=\delta\psi(t_2)=0\;; & \delta p_x(t_2)=\delta p_z(t_2)=\delta p_s(t_2)=\delta J(t_2)=0; \\ \delta t_1=\delta t_2=0\;, & \end{cases} \tag{4.1b}$$

where the variables $x,\ z,\ s,\ \psi,\ p_x,\ p_z,\ p_s,\ J,\ t$ are varied independently of each other and are held constant at the end points. (For the usual derivation of the Hamiltonian equations (3.22) from the variational principle (4.1) the variation of the time t is actually not needed. However, in order to be able to carry out the derivation of eqn. (4.1) it is useful, nevertheless, to allow t to vary.)

Eqn. (4.1) can now be rewritten using

$$dt=\frac{dt}{ds}\cdot ds$$

as:

$$\delta\int_{s_1}^{s_2} ds\cdot\{x'\cdot p_x+z'\cdot p_z+\psi'\cdot J+t'\cdot(-\mathcal{H})+p_s(x,z,t,\psi;p_x,p_z,-\mathcal{H},J;s)\}=0 \tag{4.2a}$$

with

$$\begin{cases} \delta x(s_1)=\delta z(s_1)=\delta t(s_1)=\delta\psi(s_1)=0; \\ \delta p_x(s_1)=\delta p_z(s_1)=\delta\mathcal{H}(s_1)=\delta J(s_1)=0; \\ \delta x(s_2)=\delta z(s_2)=\delta t(s_2)=\delta\psi(s_2)=0; \\ \delta p_x(s_2)=\delta p_z(s_2)=\delta\mathcal{H}(s_2)=\delta J(s_2)=0; \\ \delta s_1=\delta s_2=0 \end{cases} \tag{4.2b}$$

and

$$y' \equiv \frac{dy}{ds} \; ; \quad (y \equiv x, z, t, \psi)$$

(where we make independent variations of the variables $x,\ z,\ t,\ \psi,\ p_x,\ p_z,\ -\mathcal{H},\ J,\ s$ and where s is the independent variable).

The required equations with s as independent variable are then obtained from the Euler equations of the variational problem (4.2):

$$\frac{d}{ds}\, x = +\frac{\partial \mathcal{K}}{\partial p_x} \; ; \qquad \frac{d}{ds}\, p_x = -\frac{\partial \mathcal{K}}{\partial x} \; ; \tag{4.3a}$$

$$\frac{d}{ds}\, z = +\frac{\partial \mathcal{K}}{\partial p_z} \; ; \qquad \frac{d}{ds}\, p_z = -\frac{\partial \mathcal{K}}{\partial z} \; ; \tag{4.3b}$$

$$\frac{d}{ds}\, t = +\frac{\partial \mathcal{K}}{\partial(-\mathcal{H})} \; ; \qquad \frac{d}{ds}\, (-\mathcal{H}) = -\frac{\partial \mathcal{K}}{\partial t} \; ; \tag{4.3c}$$

$$\frac{d}{ds}\, \psi = +\frac{\partial \mathcal{K}}{\partial J} \; ; \qquad \frac{d}{ds}\, J = -\frac{\partial \mathcal{K}}{\partial \psi} \tag{4.3d}$$

with

$$\mathcal{K} \equiv -p_s. \tag{4.4}$$

So eqn. (3.7) must be solved for p_s. To come to that, we recall that in storage rings the total energy is very much greater than the energy due to SG forces and that our Hamiltonian (2.1) is based on semiclassical quantum mechanics where terms in $\hbar$ above first order are ignored. Thus here we also only keep zeroth and first order terms in $\hbar$ and make a perturbation calculation with respect to $\hbar$ [14]. Starting with zeroth order in $\hbar$, the term $\vec{\Omega}_0 \cdot \vec{\xi}$ in (3.7) vanishes and $\mathcal{H} = \mathcal{H}_{orb}$. Solving for p_s and using (3.11):

$$\begin{aligned} p_{s0} \;=\; & [1 + K_x \cdot x + K_z \cdot z] \cdot \left\{ \frac{(\mathcal{H} - e\phi)^2}{c^2} - \left(p_x - \frac{e}{c}A_x\right)^2 - \left(p_z - \frac{e}{c}A_z\right)^2 - m_0^2 c^2 \right\}^{1/2} \\ & + [1 + K_x \cdot x + K_z \cdot z] \cdot \frac{e}{c} A_s \; . \end{aligned} \tag{4.5}$$

Since we are interested only in terms up to order $\hbar$ [14], we make the ansatz :

$$p_s = p_{s0} + \hbar \cdot R_s, \tag{4.6}$$

where R_s is a function of $x,\ z,\ t,\ \psi,\ p_x,\ p_z,\ -\mathcal{H},\ J$ to be determined.

Because $\vec{\xi} \cdot \vec{\Omega}_0$ is already $O(\hbar)$, we can, in the argument of $\vec{\Omega}_0$, make the approximation:

$$p_s \implies p_{s0} \; . \tag{4.7}$$

This simplifies the problem because p_s now only appears in the orbital part of $\mathcal{H}$ (in $\vec{\Omega}_0 \cdot \vec{\xi}$ the term p_s can be replaced by p_{s0}, i.e. by the known function (4.5) of $x,\ z,\ t,\ p_x,\ p_z\ ,\ -\mathcal{H},\ s$). Hence eqn. (3.7) becomes an equation quadratic in p_s and we obtain:

$$\begin{aligned} & p_s(x, z, t, \psi, p_x, p_z, -\mathcal{H}, J) = [1 + K_x \cdot x + K_z \cdot z] \cdot \frac{e}{c} A_s \\ & \quad + [1 + K_x \cdot x + K_z \cdot z] \\ & \quad \times \left\{ \frac{(\mathcal{H} - e\phi - \vec{\Omega}_0 \cdot \vec{\xi})^2}{c^2} - \left(p_x - \frac{e}{c}A_x\right)^2 - \left(p_z - \frac{e}{c}A_z\right)^2 - m_0^2 c^2 \right\}^{1/2} . \end{aligned} \tag{4.8}$$

This can be simplified again by neglecting terms of $O(\hbar^2)$:

$$p_s \; = \; p_{s0} - \frac{1}{c^2} \cdot [1 + K_x \cdot x + K_z \cdot z]^2 \cdot \frac{(\mathcal{H} - e\phi)}{p_{s0} - [1 + K_x \cdot x + K_z \cdot z] \cdot \frac{e}{c} A_s} \cdot [\vec{\Omega}_0 \cdot \vec{\xi}] \; . \quad (4.9)$$

The second term in (4.9) is just $\hbar \cdot R_s$ and p_s is a well defined function of $x,\; z,\; t,\; \psi,\, p_x,$ $p_z,\, -\mathcal{H},\, J,\, s.$

For the new Hamiltonian $\mathcal{K}$ we obtain from (4.4), (4.5) and (4.9):

$$\mathcal{K}(x,\; z,\; t,\; \psi;\; p_x,\; p_z,\; -\mathcal{H},\; J;\; s) = \mathcal{K}_{orb} + \mathcal{K}_{spin} \qquad (4.10)$$

with

$$\begin{aligned} \mathcal{K}_{orb} \;&\equiv\; -p_{s0} \\ &= \; -[1 + K_x \cdot x + K_z \cdot z] \times \\ &\qquad \left\{ \frac{(\mathcal{H} - e\phi)^2}{c^2} - \left(p_x - \frac{e}{c}A_x\right)^2 - \left(p_z - \frac{e}{c}A_z\right)^2 - m_0^2 c^2 \right\}^{1/2} \\ &\quad - [1 + K_x \cdot x + K_z \cdot z] \cdot \frac{e}{c} A_s \; ; \end{aligned} \qquad (4.11a)$$

$$\begin{aligned} \mathcal{K}_{spin} \;&=\; [\vec{\Omega}_0 \cdot \vec{\xi}] \cdot \frac{1}{c^2} [1 + K_x \cdot x + K_z \cdot z] \cdot \frac{\mathcal{H} - e\phi}{\left(\frac{p_{s0}}{[1 + K_x \cdot x + K_z \cdot z]} - \frac{e}{c} A_s \right)} \\ &= \; [\vec{\Omega}_0 \cdot \vec{\xi}] \cdot \frac{[1 + K_x \cdot x + K_z \cdot z] \cdot (\mathcal{H} - e\phi)}{c^2 \cdot \left\{ \frac{(\mathcal{H} - e\phi)^2}{c^2} - \left(p_x - \frac{e}{c}A_x\right)^2 - \left(p_z - \frac{e}{c}A_z\right)^2 - m_0^2 c^2 \right\}^{1/2}} \; . \end{aligned} \qquad (4.11b)$$

Note, that the factor after the quantity $[\vec{\Omega}_0 \cdot \vec{\xi}]$ in eqn. (4.11b) is, apart from terms which only contribute to $O(\hbar^2)$, just $(1/\dot{s})$, since we obtain from eqn. (3.11):

$$\begin{aligned} \dot{s} \; &= \; \frac{\partial \mathcal{H}}{\partial p_s} \; = \; \frac{\partial \mathcal{H}_{orb}}{\partial p_s} + O(\hbar) \\ &= \; \frac{c^2}{(\mathcal{H} - e\phi)} \cdot \frac{1}{[1 + K_x \cdot x + K_z \cdot z]} \cdot \left(\frac{p_s}{[1 + K_x \cdot x + K_z \cdot z]} - \frac{e}{c} A_s \right) + O(\hbar) \; . \end{aligned} \qquad (4.12)$$

This result could also have been obtained by much simpler means as follows:

$$\frac{d}{dt}\,\vec{\xi} \; = \; \vec{\Omega}_0 \times \vec{\xi} \quad \Longrightarrow \quad \frac{d}{ds}\,\vec{\xi} \; = \; \frac{1}{\dot{s}}\,\vec{\Omega}_0 \times \vec{\xi} \qquad (4.13)$$

but we wanted to obtain it within a Hamiltonian formalism.

Thus, setting

$$p_t \equiv -\mathcal{H}$$

we have with (4.10 - 11) the Hamiltonian for the canonical variables

$$x,\; z,\; t,\; \psi;\; p_x,\; p_z,\; p_t,\; J$$

and the arc length, s, of the design orbit acts as the independent variable.

We repeat that in a semiclassical treatment it is sufficient to evaluate $\vec{\Omega}_0$ using the substitution (4.7).

In the remaining part of this chapter we perform some further canonical transformations of the variables $x, z, t, \psi; p_x, p_z, p_t, J$ in order to prepare for the next chapters.

In the following we choose a gauge in which

$$\phi = 0 \ .$$

Then from eqn. (2.4) and (3.11) we obtain:

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_{orb} + O(\hbar) \\ &= \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} + O(\hbar) \\ &= E + O(\hbar) \ ; \end{aligned}$$

$$(E \equiv \mathcal{H}_{orb} = \text{the orbital energy of the particle})$$

and thus

$$p_t + E = O(\hbar) \ .$$

(Note that $v^2 \equiv \vec{v} \cdot \vec{v}$ and $\vec{v} \equiv \frac{d\vec{r}}{dt}$.)

In order to describe the energy oscillations we use the design energy, E_0, to introduce the (small) quantity

$$\tilde{p}_t = p_t + E_0 \equiv -(E - E_0) + O(\hbar) \equiv -\Delta E + O(\hbar) \tag{4.14}$$

as a new (canonical) variable:

$$t, \ p_t \quad \longrightarrow \quad \tilde{t}, \ \tilde{p}_t \ . \tag{4.15}$$

This transformation can be obtained using the generating function

$$F_2(t, \tilde{p}_t) \ = \ t \cdot (\tilde{p}_t - E_0). \tag{4.16}$$

The transformation equations read as:

$$p_t \ = \ \frac{\partial F_2}{\partial t} \ = \ \tilde{p}_t - E_0 \ ; \tag{4.17a}$$

$$\tilde{t} \ = \ \frac{\partial F_2}{\partial \tilde{p}_t} \ = \ t \ ; \tag{4.17b}$$

$$\mathcal{K} \quad \longrightarrow \quad \mathcal{K} + \frac{\partial F_2}{\partial s} = \mathcal{K} \tag{4.17c}$$

whereby (4.17a) reproduces the defining equation (4.14) for $\tilde{p}_t$.

Finally, since the variable t increases without limit, it is more useful to introduce the variable

$$\sigma = s - v_0 \cdot t \tag{4.18}$$

with

$$v_0 \;=\; \text{design speed} \;=\; c\beta_0 \;;\;\; \beta_0 \;=\; \sqrt{1 - \left(\frac{m_0 c^2}{E_0}\right)^2}$$

which describes the delay in arrival time at position s of a particle :

$$t,\ \tilde{p}_t \quad \longrightarrow \quad \sigma,\ p_\sigma \tag{4.19}$$

(σ describes the longitudinal separation of the particle from the centre of the bunch.)

This point transformation can also be made by a canonical transformation (see section 3.2). The generating function is

$$F_3(\tilde{p}_t, \sigma; s) = -\frac{1}{v_0}\ \tilde{p}_t \cdot (s - \sigma)\ . \tag{4.20}$$

From this follows:

$$t \;=\; -\frac{\partial F_3}{\partial \tilde{p}_t} \;=\; \frac{1}{v_0} \cdot (s - \sigma)\ ; \tag{4.21a}$$

$$p_\sigma \;=\; -\frac{\partial F_3}{\partial \sigma} \;=\; -\frac{1}{v_0}\ \tilde{p}_t \;\equiv\; \frac{E - E_0}{v_0} + O(\hbar) \;=\; \frac{\Delta E}{v_0} + O(\hbar) \tag{4.21b}$$

and

$$\begin{aligned} \mathcal{K} \quad \longrightarrow \quad \overline{\mathcal{K}}(x, z, \sigma, \psi; p_x, p_z, p_\sigma, J; s) \;&=\; \mathcal{K} + \frac{\partial F_3}{\partial s} \\ &=\; \mathcal{K} + p_\sigma \\ &=\; \overline{\mathcal{K}}_{orb} + \bar{\vec{\Omega}}_0 \cdot \vec{\xi} \end{aligned} \tag{4.22}$$

with

$$\begin{aligned} \overline{\mathcal{K}}_{orb} \;=\;& p_\sigma - [1 + K_x \cdot x + K_z \cdot z] \times \\ & \left\{\beta_0^2 \cdot \left(p_\sigma + \frac{E_0}{v_0}\right)^2 - \left(p_x - \frac{e}{c}A_x\right)^2 - \left(p_z - \frac{e}{c}A_z\right)^2 - m_0^2 c^2\right\}^{1/2} \\ & - [1 + K_x \cdot x + K_z \cdot z] \cdot \frac{e}{c}A_s\ ; \end{aligned} \tag{4.23a}$$

$$\bar{\vec{\Omega}}_0 \;=\; \vec{\Omega}_0 \cdot \frac{[1 + K_x \cdot x + K_z \cdot z] \cdot \beta_0 \left(p_\sigma + \dfrac{E_0}{v_0}\right)}{c \cdot \left\{\beta_0^2 \cdot \left(p_\sigma + \dfrac{E_0}{v_0}\right)^2 - \left(p_x - \dfrac{e}{c}A_x\right)^2 - \left(p_z - \dfrac{e}{c}A_z\right)^2 - m_0^2 c^2\right\}^{1/2}}\ . \tag{4.23b}$$

With (4.22 - 23) we have the Hamiltonian for the canonical variables

$$x,\ z,\ \sigma,\ \psi;\ p_x,\ p_z,\ p_\sigma,\ J.$$

In order to utilize the new Hamiltonian (4.22), the magnetic field $\vec{B}$ and the corresponding vector potential,

$$\vec{A} = \vec{A}(x, y, \sigma; s), \tag{4.24}$$

for commonly occurring types of accelerator magnet and for cavities must be given. Once $\vec{A}$ is known, the fields $\vec{\varepsilon}$ and $\vec{B}$ can be found using eqn. (2.8a, b). In the variables x, z, s, σ these become (with $\phi = 0$):

$$\vec{\varepsilon} = \beta_0 \cdot \frac{\partial}{\partial \sigma}\ \vec{A} \tag{4.25}$$

and

$$B_x = \frac{1}{(1 + K_x \cdot x + K_z \cdot z)} \cdot \left\{ \frac{\partial}{\partial z}[(1 + K_x \cdot x + K_z \cdot z) \cdot A_s] - \frac{\partial}{\partial s} A_z \right\} \ ; \tag{4.26a}$$

$$B_z = \frac{1}{(1 + K_x \cdot x + K_z \cdot z)} \cdot \left\{ \frac{\partial}{\partial s} A_x - \frac{\partial}{\partial x}[(1 + K_x \cdot x + K_z \cdot z) \cdot A_s] \right\} \ ; \tag{4.26b}$$

$$B_s = \frac{\partial}{\partial x} A_z - \frac{\partial}{\partial z} A_x \ . \tag{4.26c}$$

In Appendix A the vector potential $\vec{A}$ is calculated for various types of lenses.

In the following we assume that the ring consists of bending magnets, quadrupoles, skew quadrupoles, solenoids, cavities and dipoles. Then the vector potential $\vec{A}$ can be written as (see Appendix A) :

$$\begin{aligned} \frac{e}{E_0} A_s = & -\frac{1}{2}\beta_0 \cdot (1 + K_x \cdot x + K_z \cdot z) + \frac{1}{2} g \cdot \beta_0 \cdot (z^2 - x^2) + N \cdot \beta_0 \cdot xz \\ & -\frac{1}{\beta_0} \cdot \frac{L}{2\pi \cdot h} \cdot \frac{eV(s)}{E_0} \cdot \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] \\ & + \frac{e}{E_0}[\Delta B_x \cdot z - \Delta B_z \cdot x] \end{aligned} \tag{4.27}$$

(h=harmonic number) with

$$\Delta B_x = \ = \sum_\mu \Delta \hat{B}_x^{(\mu)} \cdot \delta(s - s_\mu) \ ; \tag{4.28a}$$

$$\Delta B_z = \ = \sum_\mu \Delta \hat{B}_z^{(\mu)} \cdot \delta(s - s_\mu) \tag{4.28b}$$

(dipole field in x- and z-direction)

and

$$\frac{e}{E_0} A_x = -\beta_0 \cdot H \cdot z \ ; \quad \frac{e}{E_0} A_z = +\beta_0 \cdot H \cdot x \tag{4.29}$$

whereby the following abbreviations have been used :

$$g = \frac{e}{p_0 \cdot c} \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} ; \tag{4.30a}$$

$$N = \frac{1}{2} \cdot \frac{e}{p_0 \cdot c} \cdot \left(\frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z}\right)_{x=z=0} ; \tag{4.30b}$$

$$H = \frac{1}{2} \cdot \frac{e}{p_0 \cdot c} \cdot B_s(0,0,s) ; \tag{4.30c}$$

$$K_x = +\frac{e}{p_0 \cdot c} \cdot B_z(0,0,s) ; \quad K_z = -\frac{e}{p_0 \cdot c} \cdot B_x(0,0,s) \tag{4.30d}$$

(p_0 = momentum corresponding to energy E_0).

In detail, one has:

a)	$g \neq 0$;	$N = K_x = K_z = H = V = 0$:	quadrupole;
b)	$N \neq 0$;	$g = K_x = K_z = H = V = 0$:	skew quadrupole;
c)	$K_x^2 + K_z^2 \neq 0$;	$g = N = H = V = 0$:	bending magnet;
d)	$H \neq 0$;	$g = N = K_x = K_z = V = 0$:	solenoid;
e)	$V \neq 0$;	$g = K_x = K_z = N = H = 0$:	cavity.

Furthermore, for the magnetic field $\vec{B}$ we get (see Appendix A) :

$$\frac{e}{E_0} B_x = \beta_0 \left[-K_z + \frac{e}{p_0 \cdot c} \Delta B_x + (N - H') \cdot x + g \cdot z\right] ; \tag{4.31a}$$

$$\frac{e}{E_0} B_z = \beta_0 \left[+K_x + \frac{e}{p_0 \cdot c} \Delta B_z - (N + H') \cdot z + g \cdot x\right] ; \tag{4.31b}$$

$$\frac{e}{E_0} B_s = \beta_0 \cdot 2H \tag{4.31c}$$

and for the electric field $\vec{\epsilon}$ we have:

$$\begin{aligned} \epsilon_s &= V(s) \sin\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] \\ &= V(s) \sin\varphi + \sigma(s) \cdot h \cdot \frac{2\pi}{L} \cdot V(s) \cos\varphi + \cdots ; \end{aligned} \tag{4.32a}$$

$$\epsilon_x = \epsilon_z = 0 . \tag{4.32b}$$

Although x, z, σ, ψ; p_x, p_z, p_σ, J are canonical variables, it is still useful to introduce the new quantities

$$\eta = \frac{v_0}{E_0} \cdot p_\sigma \equiv \frac{\Delta E}{E_0} + O(\hbar) \tag{4.33}$$

and

$$\begin{aligned} \hat{\eta} &= \frac{1}{\beta_0} \sqrt{(1+\eta)^2 - (\frac{m_0 c^2}{E_0})^2} - 1 \\ &= \frac{1}{\beta_0} \cdot \frac{p \cdot c}{E_0} - 1 + O(\hbar) = \frac{p}{p_0} - 1 + O(\hbar) = \frac{\Delta p}{p_0} + O(\hbar) \end{aligned} \tag{4.34}$$

where

$$p \equiv \frac{1}{c}\sqrt{E^2 - m_0^2c^4} \quad = \text{ momentum corresponding to energy } E \;;$$

$$p_0 \equiv \frac{1}{c}\sqrt{E_0^2 - m_0^2c^4} \quad = \text{ momentum corresponding to energy } E_0 \;;$$

$$\Delta p \equiv p - p_0 \,.$$

Then for the term

$$W \equiv \left\{\beta_0^2 \cdot \left(p_\sigma + \frac{E_0}{v_0}\right)^2 - \left(p_x - \frac{e}{c}A_x\right)^2 - \left(p_z - \frac{e}{c}A_z\right)^2 - m_0^2c^2\right\}^{1/2} \tag{4.35a}$$

appearing in eqn. (4.23) we have:

$$\begin{aligned}
W &= \sqrt{\beta_0^2 \cdot \left(p_\sigma + \frac{E_0}{v_0}\right)^2 - m_0^2c^2} \cdot \left\{1 - \frac{\left(p_x - \frac{e}{c}A_x\right)^2 + \left(p_z - \frac{e}{c}A_z\right)^2}{\beta_0^2 \cdot \left(p_\sigma + \frac{E_0}{v_0}\right)^2 - m_0^2c^2}\right\}^{1/2} \\
&= \beta_0 \cdot \frac{E_0}{v_0} \cdot \sqrt{\left(\frac{v_0}{E_0}p_\sigma + 1\right)^2 - m_0^2c^2 \cdot \frac{v_0^2}{E_0^2} \cdot \frac{1}{\beta_0^2}} \\
&\quad \times \left\{1 - \left(\frac{E_0}{v_0}\right)^2 \cdot \frac{\left(\frac{v_0}{E_0}p_x - \frac{e}{E_0}\beta_0 \cdot A_x\right)^2 + \left(\frac{v_0}{E_0}p_z - \frac{e}{E_0}\beta_0 \cdot A_z\right)^2}{\beta_0^2 \cdot \left(p_\sigma + \frac{E_0}{v_0}\right)^2 - m_0^2c^2}\right\}^{1/2} \\
&= \frac{E_0}{c} \cdot \sqrt{(1+\eta)^2 - \left(\frac{m_0c^2}{E_0}\right)^2} \\
&\quad \times \left\{1 - \left(\frac{E_0}{v_0}\right)^2 \cdot \frac{\left(\frac{v_0}{E_0}p_x - \frac{e}{E_0}\beta_0 \cdot A_x\right)^2 + \left(\frac{v_0}{E_0}p_z - \frac{e}{E_0}\beta_0 \cdot A_z\right)^2}{\beta_0^2 \cdot \left(p_\sigma + \frac{E_0}{v_0}\right)^2 - m_0^2c^2}\right\}^{1/2} \\
&= \frac{E_0}{c} \cdot \beta_0 \cdot (1 + \hat{\eta}) \\
&\quad \times \left\{1 - \left(\frac{E_0}{v_0}\right)^2 \cdot \frac{\left(\frac{v_0}{E_0}p_x + \beta_0^2 \cdot Hz\right)^2 + \left(\frac{v_0}{E_0}p_z - \beta_0^2 \cdot Hx\right)^2}{\left[\frac{E_0}{c} \cdot \beta_0 \cdot (1+\hat{\eta})\right]^2}\right\}^{1/2} \\
&= \frac{E_0}{v_0} \cdot \beta_0^2 \cdot (1 + \hat{\eta}) \\
&\quad \times \left\{1 - \frac{\left(\frac{v_0}{E_0}p_x + \beta_0^2 \cdot Hz\right)^2 + \left(\frac{v_0}{E_0}p_z - \beta_0^2 \cdot Hx\right)^2}{\beta_0^4 \cdot (1+\hat{\eta})^2}\right\}^{1/2} .
\end{aligned} \tag{4.35b}$$

Thus, putting (4.27), (4.29), and (4.35b) into (4.23a), we obtain for the orbital part $\overline{K}_{orb}$ of the Hamiltonian:

$$\overline{K}_{orb} = \frac{E_0}{v_0} \cdot \frac{v_0}{E_0}p_\sigma - \frac{E_0}{v_0} \cdot \beta_0^2 \cdot (1+\hat{\eta}) \cdot [1 + K_x \cdot x + K_z \cdot z] \times$$

$$\left\{1 - \frac{\left(\frac{v_0}{E_0}p_x + \beta_0^2 \cdot Hz\right)^2 + \left(\frac{v_0}{E_0}p_z - \beta_0^2 \cdot Hx\right)^2}{\beta_0^4 \cdot (1+\hat{\eta})^2}\right\}^{1/2}$$

$$- [1 + K_x \cdot x + K_z \cdot z] \cdot \frac{E_0}{c} \times$$

$$\left\{-\frac{1}{2}\beta_0 \cdot (1 + K_x \cdot x + K_z \cdot z) + \frac{1}{2} g \cdot \beta_0 \cdot (z^2 - x^2) + N \cdot \beta_0 \cdot xz \right.$$

$$- \frac{1}{\beta_0} \cdot \frac{L}{2\pi \cdot h} \cdot \frac{eV(s)}{E_0} \cdot \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right]$$

$$\left. + \frac{e}{p_0 \cdot c} \cdot \frac{p_0 \cdot c}{E_0} \cdot [\Delta B_x \cdot z - \Delta B_z \cdot x]\right\}$$

or

$$\begin{aligned}
\frac{v_0}{E_0} \cdot \overline{\mathcal{K}}_{orb} \quad &= \quad \eta - \beta_0^2 \cdot (1+\hat{\eta}) \cdot [1 + K_x \cdot x + K_z \cdot z] \times \\
&\left\{1 - \frac{\left(\frac{v_0}{E_0}p_x + \beta_0^2 \cdot Hz\right)^2 + \left(\frac{v_0}{E_0}p_z - \beta_0^2 \cdot Hx\right)^2}{\beta_0^4 \cdot (1+\hat{\eta})^2}\right\}^{1/2} \\
&- [1 + K_x \cdot x + K_z \cdot z] \cdot \beta_0^2 \times \\
&\left\{-\frac{1}{2} \cdot (1 + K_x \cdot x + K_z \cdot z) + \frac{1}{2} g \cdot (z^2 - x^2) + N \cdot xz \right. \\
&- \frac{1}{\beta_0^2} \cdot \frac{L}{2\pi \cdot h} \cdot \frac{eV(s)}{E_0} \cdot \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] \\
&\left. + \frac{e}{p_0 \cdot c} \cdot [\Delta B_x \cdot z - \Delta B_z \cdot x]\right\} . \qquad (4.36)
\end{aligned}$$

The vector $\vec{\Omega}_0$ in (4.23b):

$$\vec{\Omega}_0 \quad = \quad \Omega_{0s} \cdot \vec{e}_s + \Omega_{0x} \cdot \vec{e}_x + \Omega_{0z} \cdot \vec{e}_z$$

(see eqns. (3.12) and (3.13)) as a function of the variables x, z, σ, p_x, p_z, p_σ, s now takes the form:

$$\begin{aligned}
\frac{1}{c} \cdot \Omega_{0s} \quad &= \quad -\frac{E_0}{m_0 c^2}\left[\left(\frac{1}{\gamma} + a\right) \cdot \frac{e}{E_0} B_s \right. \\
&\left. - \frac{aE_0^2}{\gamma(\gamma+1) \cdot m_0^2 c^4 \cdot \beta_0^2}\left(\frac{v_0}{E_0}\pi_s \cdot \frac{e}{E_0}B_s + \frac{v_0}{E_0}\pi_x \cdot \frac{e}{E_0}B_x + \frac{v_0}{E_0}\pi_z \cdot \frac{e}{E_0}B_z\right)\frac{v_0}{E_0}\pi_s\right] \\
&= \quad -\gamma_0\left[\left(\frac{1}{\gamma} + a\right) \cdot \frac{e}{E_0} B_s \right. \\
&\left. - \frac{a\gamma_0^2}{\gamma(\gamma+1)} \cdot \frac{1}{\beta_0^2}\left(\frac{v_0}{E_0}\pi_s \cdot \frac{e}{E_0}B_s + \frac{v_0}{E_0}\pi_x \cdot \frac{e}{E_0}B_x + \frac{v_0}{E_0}\pi_z \cdot \frac{e}{E_0}B_z\right)\frac{v_0}{E_0}\pi_s\right] ; \quad (4.37a)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{c} \cdot \Omega_{0x} \quad &= \quad -\gamma_0\left[\left(\frac{1}{\gamma} + a\right) \cdot \frac{e}{E_0} B_x \right. \\
&- \frac{a\gamma_0^2}{\gamma(\gamma+1)} \cdot \frac{1}{\beta_0^2}\left(\frac{v_0}{E_0}\pi_s \cdot \frac{e}{E_0}B_s + \frac{v_0}{E_0}\pi_x \cdot \frac{e}{E_0}B_x + \frac{v_0}{E_0}\pi_z \cdot \frac{e}{E_0}B_z\right)\frac{v_0}{E_0}\pi_x
\end{aligned}$$

$$-\frac{\gamma_0}{\gamma}\cdot\frac{1}{\beta_0}\left(a+\frac{1}{1+\gamma}\right)\cdot\frac{v_0}{E_0}\pi_z\cdot\frac{e}{E_0}\varepsilon_s\Big]\,; \tag{4.37b}$$

$$\begin{aligned}\frac{1}{c}\cdot\Omega_{0z} \;=\; & -\gamma_0\Big[\left(\frac{1}{\gamma}+a\right)\cdot\frac{e}{E_0}B_z \\ & -\frac{a\gamma_0^2}{\gamma(\gamma+1)}\cdot\frac{1}{\beta_0^2}\left(\frac{v_0}{E_0}\pi_s\cdot\frac{e}{E_0}B_s+\frac{v_0}{E_0}\pi_x\cdot\frac{e}{E_0}B_x+\frac{v_0}{E_0}\pi_z\cdot\frac{e}{E_0}B_z\right)\frac{v_0}{E_0}\pi_z \\ & +\frac{\gamma_0}{\gamma}\cdot\frac{1}{\beta_0}\left(a+\frac{1}{1+\gamma}\right)\cdot\frac{v_0}{E_0}\pi_x\cdot\frac{e}{E_0}\varepsilon_s\Big]\,,\end{aligned} \tag{4.37c}$$

whereby the fields B_s, B_x, B_z and ϵ_s are taken from eqns. (4.31) and (4.32) and the term γ_0 is defined by

$$\gamma_0 \;=\; \frac{E_0}{m_0c^2}\,. \tag{4.38}$$

For the Lorentz factor γ appearing in (4.37) one has:

$$\begin{aligned}\gamma \;&=\; \frac{E}{m_0c^2} \\ &=\; \frac{E_0}{m_0c^2}\cdot\frac{v_0}{E_0}\cdot\left[p_\sigma+\frac{E_0}{v_0}\right]+O(\hbar) \\ &=\; \gamma_0\cdot(1+\eta)+O(\hbar)\end{aligned} \tag{4.39}$$

and for the quantity π_s we have ((3.9c), (4.5) and (4.6)) :

$$\begin{aligned}\pi_s \;&=\; \left\{\beta_0^2\cdot\left(p_\sigma+\frac{E_0}{v_0}\right)^2-\left(p_x-\frac{e}{c}A_x\right)^2-\left(p_z-\frac{e}{c}A_z\right)^2-m_0^2c^2\right\}^{1/2} \;\equiv\; W \\ &=\; \frac{E_0}{v_0}\cdot\beta_0^2\cdot(1+\hat\eta) \\ &\quad\times\left\{1-\frac{\left(\frac{v_0}{E_0}p_x+\beta_0^2\cdot Hz\right)^2+\left(\frac{v_0}{E_0}p_z-\beta_0^2\cdot Hx\right)^2}{\beta_0^4\cdot(1+\hat\eta)^2}\right\}^{1/2} \\ &=\; \frac{E_0}{v_0}\cdot\beta_0^2\cdot(1+\hat\eta) \\ &\quad\times\left\{1-\frac{\left(\frac{v_0}{E_0}\pi_x\right)^2+\left(\frac{v_0}{E_0}\pi_z\right)^2}{\beta_0^4\cdot(1+\hat\eta)^2}\right\}^{1/2}\end{aligned} \tag{4.40}$$

(see eqn. (4.35b)) with

$$\frac{v_0}{E_0}\pi_x \;=\; \frac{v_0}{E_0}p_x+\beta_0^2\cdot Hz\;; \tag{4.41a}$$

$$\frac{v_0}{E_0}\pi_z \;=\; \frac{v_0}{E_0}p_z-\beta_0^2\cdot Hx\;. \tag{4.41b}$$

With (4.23b), (4.36 - 37), (4.39 - 41) we have rewritten the Hamiltonian for the canonical variables

$$x,\; z,\; \sigma,\; \psi;\; p_x,\; p_z,\; p_\sigma,\; J$$

in a more convenient form by replacing the terms in p_σ by the equivalent quantities η, $\hat{\eta}$.

Remark:

Eqn. (4.36) is valid only for protons. For electrons one needs the extra-term in the Hamiltonian

$$\mathcal{K}_{rad} = \frac{E_0}{v_0} \cdot C_1 \cdot [K_x^2 + K_z^2] \cdot \sigma \tag{4.42}$$

$$\left(\text{where} \;\; C_1 = \frac{2}{3} e^2 \frac{\gamma_0^4}{E_0} \right)$$

(for $v_0 \approx c$) in order to describe the energy loss by radiation in the bending magnets [11,25]. In this case, the cavity phase φ in (4.33) is determined by the need to replace the energy radiated in the bending magnets. Thus:

$$\underbrace{\int_{s_0}^{s_0+L} ds \cdot eV(s) \cdot \sin\varphi}_{\text{average energy uptake in the cavities ;}} \quad = \quad \underbrace{\int_{s_0}^{s_0+L} ds \cdot E_0 \cdot C_1 \cdot [K_x^2 + K_z^2] \; .}_{\text{average energy loss due to radiation}} \tag{4.43}$$

Note, that the $\mathcal{K}_{rad}$ term only accounts for the average energy loss. Deviations from this average due to stochastic radiation effects and damping introduce non-symplectic terms into the equation of motion.

For proton storage rings, where radiation effects can be neglected, one has:

$$\sin\varphi = 0 \quad \Longrightarrow \quad \varphi = 0, \; \pi \tag{4.44}$$

(no average energy gain in the cavities) and the choice for φ is determined by the stability condition for synchrotron motion [3]

$$\begin{cases} \varphi = 0 \;\; \text{above "transition" ;} \\ \\ \varphi = \pi \;\; \text{below "transition" .} \end{cases}$$

5 Spin Motion in Terms of the Dreibein $(\vec{e}_s, \; \vec{e}_x, \; \vec{e}_z)$; Canonical Spin Transformation

In this chapter we show how to describe the motion of the spin with respect to the $\vec{e}_x$, $\vec{e}_z$, $\vec{e}_s$ basis. The variables x, z, σ; p_x, p_z, p_σ need no further transformation.

5.1 A New Spin Hamiltonian

The transformation of the spin from the $(\vec{e}_1, \; \vec{e}_2, \; \vec{e}_3)$ - basis to the $(\vec{e}_s, \; \vec{e}_x, \; \vec{e}_z)$ - basis

$$\xi_1, \xi_2, \xi_3 \quad \Longrightarrow \quad \xi_s, \xi_x, \xi_z \; . \tag{5.1}$$

is merely a rotation and is defined by:

$$\begin{aligned}\vec{\xi} &= \xi_1 \cdot \vec{e}_1 + \xi_2 \cdot \vec{e}_2 + \xi_3 \cdot \vec{e}_3 \\ &= \xi_s \cdot \vec{e}_s + \xi_x \cdot \vec{e}_x + \xi_z \cdot \vec{e}_z \ . \end{aligned} \tag{5.2}$$

If, by analogy to eqn. (2.2), we introduce canonical variables ψ', J' for ξ_s, ξ_x, ξ_z:

$$\begin{cases} \xi_s = \sqrt{\xi^2 - J'^2} \cdot \cos\psi' \ ; \\ \xi_x = \sqrt{\xi^2 - J'^2} \cdot \sin\psi' \ ; \\ \xi_z = J' \ , \end{cases} \tag{5.3}$$

then (5.2) becomes a canonical transformation:

$$x, z, \sigma, \psi, p_x, p_z, p_\sigma, J \quad \Longrightarrow \quad x' = x, z' = z, \sigma' = \sigma, \psi', p_x' = p_x, p_z' = p_z, p_\sigma' = p_\sigma, J' \ . \tag{5.4}$$

Following Yokoya, who uses a Lie transform, the new Hamiltonian $\overline{\overline{\mathcal{K}}}$ is [14] :

$$\begin{aligned}\overline{\overline{\mathcal{K}}}(x, z, \sigma, \psi'; p_x, p_z, p_\sigma, J'; s) &= \overline{\mathcal{K}}_{orb}(x, z, \sigma; p_x, p_z, p_\sigma; s) \\ &+ \sum_{\nu=1}^{3} \left[\tilde{\vec{\Omega}}_0(x, z, \sigma; p_x, p_z, p_\sigma; s) - \vec{U}(x, z, \sigma; p_x, p_z, p_\sigma; s)\right] \cdot \vec{u}_\nu(s)\xi_\nu' \end{aligned} \tag{5.5}$$

where

$$\vec{u}_1 \equiv \vec{e}_s \ ; \ \ \xi_1' \equiv \xi_s \ ; \tag{5.6}$$

$$\vec{u}_2 \equiv \vec{e}_x \ ; \ \ \xi_2' \equiv \xi_x \ ; \tag{5.7}$$

$$\vec{u}_3 \equiv \vec{e}_z \ ; \ \ \xi_3' \equiv \xi_z \tag{5.8}$$

and

$$\vec{U} = \frac{1}{2}\sum_{\nu=1}^{3} \vec{u}_\nu \times \frac{d\vec{u}_\nu}{ds} \ . \tag{5.9}$$

From eqn. (3.3) we have:

$$\frac{d}{ds}\,\vec{u}_1(s) = -K_x(s) \cdot \vec{u}_2(s) - K_z(s) \cdot \vec{u}_3(s) \ ; \tag{5.10a}$$

$$\frac{d}{ds}\,\vec{u}_2(s) = +K_x(s) \cdot \vec{u}_1(s) \ ; \tag{5.10b}$$

$$\frac{d}{ds}\,\vec{u}_3(s) = +K_z(s) \cdot \vec{u}_1(s) \ . \tag{5.10c}$$

Putting (5.10) into (5.9) we obtain:

$$\begin{aligned}\vec{U} &= \frac{1}{2}\{\vec{e}_s \times [-K_x(s) \cdot \vec{e}_x(s) - K_z(s) \cdot \vec{e}_z(s)] + \vec{e}_x \times (K_x \vec{e}_s) + \vec{e}_z \times (K_z \vec{e}_s)\} \\ &= -K_x \cdot \vec{e}_z + K_z \cdot \vec{e}_x \ , \end{aligned}$$

and it follows that:

$$\overline{\overline{\mathcal{K}}}(x, z, \sigma, \psi'; p_x, p_z, p_\sigma, J'; s)$$

$$= \overline{\overline{\mathcal{K}}}_{orb}(x, z, \sigma; p_x, p_z, p_\sigma; s) + \vec{\Omega}(x, z, \sigma; p_x, p_z, p_\sigma; s) \cdot (\xi_s \cdot \vec{e}_s + \xi_x \cdot \vec{e}_x + \xi_z \cdot \vec{e}_z) \quad (5.11)$$

with

$$\begin{aligned}
\frac{v_0}{E_0} \cdot \overline{\overline{\mathcal{K}}}_{orb} &= \frac{v_0}{E_0} \cdot \overline{\mathcal{K}}_{orb} \\
&= \eta - \beta_0^2 \cdot (1 + \hat{\eta}) \cdot [1 + K_x \cdot x + K_z \cdot z] \times \\
&\qquad \left\{ 1 - \frac{\left(\frac{v_0}{E_0} p_x + \beta_0^2 \cdot H z\right)^2 + \left(\frac{v_0}{E_0} p_z - \beta_0^2 \cdot H x\right)^2}{\beta_0^4 \cdot (1 + \hat{\eta})^2} \right\}^{1/2} \\
&\quad - [1 + K_x \cdot x + K_z \cdot z] \cdot \beta_0^2 \times \\
&\qquad \left\{ -\frac{1}{2} \cdot (1 + K_x \cdot x + K_z \cdot z) + \frac{1}{2} g \cdot (z^2 - x^2) + N \cdot xz \right. \\
&\qquad - \frac{1}{\beta_0^2} \cdot \frac{L}{2\pi \cdot h} \cdot \frac{eV(s)}{E_0} \cdot \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] \\
&\qquad \left. + \frac{e}{p_0 \cdot c} \cdot [\Delta B_x \cdot z - \Delta B_z \cdot x] \right\} \\
&\quad + \frac{v_0}{E_0} \mathcal{K}_{rad} \qquad (5.12)
\end{aligned}$$

and

$$\vec{\Omega}(x, z, \sigma; p_x, p_z, p_\sigma; s)$$

$$\begin{aligned}
&= \tilde{\vec{\Omega}} + K_x \cdot \vec{e}_z - K_z \cdot \vec{e}_x \\
&= \frac{[1 + K_x \cdot x + K_z \cdot z] \cdot \beta_0 \left(p_\sigma + \frac{E_0}{v_0}\right)}{c \cdot \left\{ \beta_0^2 \cdot \left(p_\sigma + \frac{E_0}{v_0}\right)^2 - \left(p_x - \frac{e}{c} A_x\right)^2 - \left(p_z - \frac{e}{c} A_z\right)^2 - m_0^2 c^2 \right\}^{1/2}} \cdot \vec{\Omega}_0 \\
&\quad + K_x \cdot \vec{e}_z - K_z \cdot \vec{e}_x \\
&= [1 + K_x \cdot x + K_z \cdot z] \cdot \frac{(1 + \eta)}{\beta_0 (1 + \hat{\eta})} \\
&\quad \times \left\{ 1 - \frac{\left(\frac{v_0}{E_0} p_x + \beta_0^2 \cdot H z\right)^2 + \left(\frac{v_0}{E_0} p_z - \beta_0^2 \cdot H x\right)^2}{\beta_0^4 \cdot (1 + \hat{\eta})^2} \right\}^{-1/2} \cdot \frac{1}{c} \vec{\Omega}_0 \\
&\quad + K_x \cdot \vec{e}_z - K_z \cdot \vec{e}_x \,. \qquad (5.13)
\end{aligned}$$

With (5.11 - 13) we have the Hamiltonian for the canonical variables

$$x,\ z,\ \sigma,\ \psi';\ p_x,\ p_z,\ p_\sigma,\ J'.$$

Remark:

The equation for spin motion corresponding to the Hamiltonian (4.22) reads as:

$$\frac{d}{ds}\,\vec{\xi} \;=\; \vec{\tilde{\Omega}}_0 \times \vec{\xi}\,. \tag{5.14}$$

Representing the spin vector $\vec{\xi}$ in the form

$$\vec{\xi} = \xi_s \cdot \vec{e}_s + \xi_x \cdot \vec{e}_x + \xi_z \cdot \vec{e}_z \tag{5.15}$$

and using eqn. (3.3) we have:

$$\begin{aligned}
\frac{d}{ds}\,\vec{\xi} &= \xi_s' \cdot \vec{e}_s + \xi_x' \cdot \vec{e}_x + \xi_z' \cdot \vec{e}_z + \xi_x \cdot \frac{d}{ds}\,\vec{e}_x + \xi_s \cdot \frac{d}{ds}\,\vec{e}_s + \xi_z \cdot \frac{d}{ds}\,\vec{e}_z \\
&= \xi_s' \cdot \vec{e}_s + \xi_x' \cdot \vec{e}_x + \xi_z' \cdot \vec{e}_z - \xi_s \cdot (K_x \cdot \vec{e}_x + K_x \cdot \vec{e}_z) + \xi_x \cdot K_x \vec{e}_s + \xi_z \cdot K_z \vec{e}_s \\
&= \xi_s' \cdot \vec{e}_s + \xi_x' \cdot \vec{e}_x + \xi_z' \cdot \vec{e}_z - \vec{\xi} \times (K_z \cdot \vec{e}_x - K_x \cdot \vec{e}_z)\,.
\end{aligned} \tag{5.16}$$

Thus eqn. (5.14) can be rewritten as:

$$\vec{e}_s \cdot \frac{d}{ds}\,\xi_s + \vec{e}_x \cdot \frac{d}{ds}\,\xi_x + \vec{e}_z \cdot \frac{d}{ds}\,\xi_z \;=\; \vec{\Omega} \times \vec{\xi} \tag{5.17}$$

with $\vec{\Omega}$ given by (5.13) which confirms the validity of the spin part $\vec{\Omega} \cdot \vec{\xi}$ in the Hamiltonian $\overline{\overline{\mathcal{K}}}(x, z, \sigma, \psi'; p_x, p_z, p_\sigma, J'; s)$.

If the new spin basis had been an explicit function of the canonical orbital variables, then even at first order in $\hbar$ the orbital variables and the orbital Hamiltonian would have been modified by the canonical transformation (see Ref. [14], eqns. (3.16), (3.17), (3.24)). However, at this stage in our treatment, the azimuthal variable, s, is the independent parameter, not a canonical variable. Therefore the variables $x, z, \sigma, p_x, p_z, p_\sigma$ remain unmodified by the transformation and $\overline{\overline{\mathcal{K}}}_{orb}$ and $\overline{\mathcal{K}}_{orb}$ are identical. Furthermore, since $\overline{\overline{\mathcal{K}}}$ and $\overline{\mathcal{K}}$ differ only by the term $(K_x \cdot \vec{e}_z - K_z \cdot \vec{e}_x) \cdot \vec{\xi}$ which is independent of the variables $x, z, \sigma, p_x, p_z, p_\sigma$, the Hamiltonians $\overline{\overline{\mathcal{K}}}$ and $\overline{\mathcal{K}}$ lead to the same equations of orbital motion.

5.2 Series Expansion of the Hamiltonian

Since

$$\left| \frac{\left(\frac{v_0}{E_0} p_x + \beta_0^2 \cdot H z\right)^2 + \left(\frac{v_0}{E_0} p_z - \beta_0^2 \cdot H x\right)^2}{\beta_0^4 \cdot (1+\hat{\eta})^2} \right| \;\ll\; 1$$

the square root

$$\left\{ 1 - \frac{\left(\frac{v_0}{E_0} p_x + \beta_0^2 \cdot H z\right)^2 + \left(\frac{v_0}{E_0} p_z - \beta_0^2 \cdot H x\right)^2}{\beta_0^4 \cdot (1+\hat{\eta})^2} \right\}^{1/2}$$

in (5.12) and (5.13) may be expanded in a series :

$$\left\{1 - \frac{\left(\frac{v_0}{E_0}p_x + \beta_0^2 \cdot Hz\right)^2 + \left(\frac{v_0}{E_0}p_z - \beta_0^2 \cdot Hx\right)^2}{\beta_0^4 \cdot (1+\hat{\eta})^2}\right\}^{1/2}$$
$$= 1 - \frac{1}{2} \cdot \frac{\left(\frac{v_0}{E_0}p_x + \beta_0^2 \cdot Hz\right)^2}{\beta_0^4 \cdot (1+\hat{\eta})^2} - \frac{1}{2} \cdot \frac{\left(\frac{v_0}{E_0}p_z - \beta_0^2 \cdot Hx\right)^2}{\beta_0^4 \cdot (1+\hat{\eta})^2} + \cdots \quad (5.18)$$

and the same can be done with the term

$$\frac{L}{2\pi \cdot h} \cdot \frac{eV(s)}{E_0} \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right]$$

resulting from the cavity field :

$$\begin{aligned} \frac{L}{2\pi \cdot h} \cdot \frac{eV(s)}{E_0} \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] &= \frac{L}{2\pi \cdot h} \cdot \frac{eV(s)}{E_0} \cos\varphi \\ &- \sigma \cdot \frac{eV(s)}{E_0} \sin\varphi \\ &- \frac{1}{2}\sigma^2 \cdot h \cdot \frac{2\pi}{L} \cdot \frac{eV(s)}{E_0} \cos\varphi + \cdots . \end{aligned} \quad (5.19)$$

Furthermore, for the term

$$\frac{1}{1+\gamma}$$

appearing in eqn. (4.37) we may write :

$$\begin{aligned} \frac{1}{1+\gamma} &= \frac{1}{(1+\gamma_0) + \gamma_0 \cdot \eta} + O(\hbar) \\ &= \frac{1}{1+\gamma_0} \cdot \left[1 - \frac{\gamma_0}{1+\gamma_0} \cdot \eta\right] + \cdots + O(\hbar) \end{aligned} \quad (5.20a)$$

and for the quantity

$$\hat{\eta} \equiv f(\eta)$$

one obtains from eqn. (4.34) :

$$\begin{aligned} \hat{\eta} &\equiv f(\eta) \\ &= f(0) + f'(0) \cdot \eta + f''(0) \cdot \frac{1}{2}\eta^2 + \cdots \\ &= \frac{1}{\beta_0^2} \cdot \eta - \frac{1}{\beta_0^4 \cdot \gamma_0^2} \cdot \frac{1}{2}\eta^2 + \cdots \end{aligned} \quad (5.20b)$$

so that in practice the spin - orbit motion can be conveniently calculated to various orders of approximation in the orbit variables.

If we wish to obtain a symplectic linearised treatment of synchro - betatron motion (including SG effects) we expand the Hamiltonian up to second order in the orbit variables. Then from (5.12) and (5.13) we obtain:

a) For the orbital part $\overline{\overline{\mathcal{K}}}_{orb}$ of the Hamiltonian :

$$\overline{\overline{\mathcal{K}}}_{orb} = \mathcal{K}_0 + \mathcal{K}_1 \tag{5.21}$$

where $\mathcal{K}_0$ and $\mathcal{K}_1$ are given by:

$$\begin{aligned} \frac{v_0}{E_0}\cdot\mathcal{K}_0 &= \frac{1}{2}\cdot\frac{1}{\gamma_0^2-1}\cdot\eta^2 - [K_x\cdot x + K_z\cdot z]\cdot\eta \\ &+\frac{1}{2\beta_0^2}\cdot\left\{\left[\frac{v_0}{E_0}p_x + \beta_0^2 H\cdot z\right]^2 + \left[\frac{v_0}{E_0}p_z - \beta_0^2 H\cdot x\right]^2\right\} \\ &+\frac{1}{2}\beta_0^2\cdot\left\{(K_x^2+g)\cdot x^2 + (K_z^2-g)\cdot z^2 - 2N\cdot xz\right\} \\ &-\frac{1}{2}\sigma^2\cdot\frac{eV(s)}{E_0}\cdot h\cdot\frac{2\pi}{L}\cdot\cos\varphi\ ; \end{aligned} \tag{5.22a}$$

$$\begin{aligned} \frac{v_0}{E_0}\cdot\mathcal{K}_1 &= -\sigma\cdot\left[\frac{eV}{E_0}\sin\varphi - C_1\cdot(K_x^2+K_z^2)\right] \\ &-\beta_0^2\cdot\frac{e}{p_0\cdot c}\left[\Delta B_x\cdot z - \Delta B_z\cdot x\right] \end{aligned} \tag{5.22b}$$

(constant terms, $(L/2\pi h)\cdot(eV/E_0)\cdot\cos\varphi$ and $(-\beta_0^2/2)$, in the Hamiltonian, which have no influence on the motion have been dropped) .

b) For the spin part $\overline{\overline{\mathcal{K}}}_{spin}$ of the Hamiltonian :

$$\overline{\overline{\mathcal{K}}}_{spin} = \vec{\Omega}\cdot\vec{\xi} \tag{5.23}$$

with

$$\begin{aligned} \Omega_s &= -2H\cdot(1+a) \\ &+2H\cdot(1+a)\cdot\frac{1}{\beta_0^2}\cdot\eta \\ &-\frac{v_0}{E_0}p_x\cdot\frac{a\gamma_0^2}{1+\gamma_0}\left[K_z - \frac{e}{p_0\cdot c}\cdot\Delta B_x\right] \\ &+\frac{v_0}{E_0}p_z\cdot\frac{a\gamma_0^2}{1+\gamma_0}\left[K_x + \frac{e}{p_0\cdot c}\cdot\Delta B_z\right]\ ; \end{aligned} \tag{5.24a}$$

$$\Omega_x = K_z\cdot a\gamma_0 - (1+a\gamma_0)\cdot\frac{e}{p_0\cdot c}\Delta B_x$$

$$
\begin{aligned}
&-(1+a\gamma_0)\cdot[(N-H')\cdot x-(K_z^2-g)\cdot z] \\
&+\frac{a\gamma_0^2}{1+\gamma_0}\cdot 2H\cdot\left[\frac{v_0}{E_0}p_x+\beta_0^2\cdot Hz\right] \\
&+\frac{1}{\beta_0^2}\cdot\left[a\gamma_0+\frac{\gamma_0}{1+\gamma_0}\right]\cdot\frac{eV(s)}{E_0}\sin\varphi\cdot\frac{v_0}{E_0}p_z \\
&-\left[1+\frac{1+a\gamma_0}{\gamma_0^2-1}\right]\cdot\left(K_z-\frac{e}{p_0\cdot c}\Delta B_x\right)\cdot\eta\ ;
\end{aligned}
\tag{5.24b}
$$

$$
\begin{aligned}
\Omega_z \;=\;& -K_x\cdot a\gamma_0-(1+a\gamma_0)\cdot\frac{e}{p_0\cdot c}\Delta B_z \\
&+(1+a\gamma_0)\cdot[(N+H')\cdot z-(K_x^2+g)\cdot x] \\
&+\frac{a\gamma_0^2}{1+\gamma_0}\cdot 2H\cdot\left[\frac{v_0}{E_0}p_z-\beta_0^2\cdot Hx\right] \\
&-\frac{1}{\beta_0^2}\cdot\left[a\gamma_0+\frac{\gamma_0}{1+\gamma_0}\right]\cdot\frac{eV(s)}{E_0}\sin\varphi\cdot\frac{v_0}{E_0}p_x \\
&+\left[1+\frac{1+a\gamma_0}{\gamma_0^2-1}\right]\cdot\left(K_x+\frac{e}{p_0\cdot c}\Delta B_z\right)\cdot\eta
\end{aligned}
\tag{5.24c}
$$

(no solenoid field in the bending magnets and in the cavities $\Longrightarrow$ $K_x\cdot H = K_z\cdot H = 0\ ;\ V\cdot H = 0$).

6 Introduction of an Eight-Dimensional Closed Orbit and a New Pair of Canonical Variables for Spin

As can be seen from (5.21 - 5.24), the series expansion for $\overline{\overline{K}}_{orb}$ contains terms linear in the orbital coordinates and $\vec{\Omega}$ contains terms independent of the orbital coordinates. These and the linear terms can be eliminated by introducing a new 8-dimensional reference orbit. This orbit can then be used to construct a new reference frame for the spin motion and, as we show below, it is then possible to introduce new variables to describe the spin which are canonical and are related to the spin variables used by Chao [10].

6.1 Definition of the Eight-Dimensional Closed Orbit

We begin by defining the 8-dimensional closed orbit:

$$(\vec{y}_0(s), J_0(s), \psi_0(s))$$

containing a periodic orbital part

$$\vec{y}_0^{\,T} = (x_0,\ p_{x0};\ z_0,\ p_{z0};\ \sigma_0,\ p_{\sigma 0}),$$

with

$$\vec{y}_0(s+L) \;=\; \vec{y}_0(s) \tag{6.1a}$$

and a spin part $J_0(s), \psi_0(s)$ which defines (see eqn. (5.3)) a periodic spin vector

$$\vec{\xi}_0(s) \;=\; \xi_{0s} \cdot \vec{e}_s + \xi_{0x} \cdot \vec{e}_x + \xi_{0z} \cdot \vec{e}_z$$

with

$$\vec{\xi}_0(s+L) \;=\; \vec{\xi}_0(s) \tag{6.1b}$$

whereby the equations of motion read as:

$$\frac{d}{ds}\,\vec{y}_0 \;=\; -\underline{S} \cdot \frac{\partial}{\partial \vec{y}_0}\,\overline{\overline{\mathcal{K}}}(\vec{y}_0; \psi_0, J_0; s)\;; \tag{6.2a}$$

$$\vec{e}_s \cdot \frac{d}{ds}\,\xi_{0s} + \vec{e}_x \cdot \frac{d}{ds}\,\xi_{0x} + \vec{e}_z \cdot \frac{d}{ds}\,\xi_{0z} \;=\; \vec{\Omega}^{(0)} \times \vec{\xi}_0 \tag{6.2b}$$

with

$$\vec{\Omega}^{(0)} \equiv \vec{\Omega}(\vec{y}_0, s) \tag{6.3}$$

and

$$\underline{S} = \begin{pmatrix} \underline{S}_2 & \underline{0} & \underline{0} \\ \underline{0} & \underline{S}_2 & \underline{0} \\ \underline{0} & \underline{0} & \underline{S}_2 \end{pmatrix} \;;\quad \underline{S}_2 = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}\,, \tag{6.4}$$

i.e. $(\vec{y}_0(s), J_0(s), \psi_0(s))$ is a periodic solution of the combined equations of motion.

Using $\vec{\xi}_0$ we can now construct a periodic spin frame $\left(\vec{n}_0, \vec{m}, \vec{l}\right)$ along the closed orbit (see Appendix B) :

$$\left[\vec{n}_0(s+L),\ \vec{m}(s+L),\ \vec{l}(s+L)\right] = \left[\vec{n}_0(s),\ \vec{m}(s),\ \vec{l}(s)\right]$$

with

$$\vec{n}_0 \;=\; \vec{\xi}_0/|\vec{\xi}_0|\;; \tag{6.5a}$$

$$\vec{n}_0(s) \;\perp\; \vec{m}(s) \;\perp\; \vec{l}(s)\;; \tag{6.5b}$$

$$\vec{n}_0(s) \;=\; \vec{m}(s) \times \vec{l}(s)\;; \tag{6.5c}$$

$$|\vec{n}_0(s)| \;=\; |\vec{m}(s)| \;=\; |\vec{l}(s)| \;=\; 1 \tag{6.5d}$$

and

$$\frac{d}{ds}\,\vec{n}_0(s) \;=\; \vec{\Omega}^{(0)} \times \vec{n}_0(s)\;; \tag{6.6a}$$

$$\frac{d}{ds}\,\vec{m}(s) \;=\; \vec{\Omega}^{(0)} \times \vec{m}(s) + \vec{l}(s) \cdot \frac{d}{ds}\,\psi_{spin}(s)\;; \tag{6.6b}$$

$$\frac{d}{ds}\,\vec{l}(s) \;=\; \vec{\Omega}^{(0)} \times \vec{l}(s) - \vec{m}(s) \cdot \frac{d}{ds}\,\psi_{spin}(s)\;; \tag{6.6c}$$

$$\psi_{spin}(s+L) - \psi_{spin}(s) = 2\pi \cdot Q_{spin}\;. \tag{6.7}$$

6.2 Canonical Transformations

The 8-dimensional closed orbit together with $\vec{l}(s)$, $\vec{m}(s)$ will now be used to construct new canonical spin - orbit variables. The canonical transformation for orbit and spin will be carried out separately.

6.2.1 Canonical Transformation for the Spin Variables

To derive the new spin-Hamiltonian, we proceed in two steps:

1) Canonical spin-transformation:

Firstly we follow the method of section (5.1) to transform from the $\vec{e}_x$, $\vec{e}_z$, $\vec{e}_s$ basis to the $\vec{n}_0$, $\vec{m}$, $\vec{l}$ basis:

$$\xi_s,\ \xi_x,\ \xi_z \quad \Longrightarrow \quad \xi_n,\ \xi_m,\ \xi_l\ . \tag{6.8}$$

with

$$\begin{aligned}\vec{\xi} &= \xi_s \cdot \vec{e}_s + \xi_x \cdot \vec{e}_x + \xi_z \cdot \vec{e}_z \\ &= \xi_n \cdot \vec{n}_0 + \xi_m \cdot \vec{m} + \xi_l \cdot \vec{l}\ .\end{aligned} \tag{6.9}$$

Introducing for $\xi_n,\ \xi_m,\ \xi_l$ canonical variables ψ'', J'' :

$$\begin{cases} \xi_m = \sqrt{\xi^2 - (J'')^2} \cdot \cos\psi''\ ; \\ \xi_l = \sqrt{\xi^2 - (J'')^2} \cdot \sin\psi''\ ; \\ \xi_n = J''\ , \end{cases} \tag{6.10}$$

eqn. (6.9) becomes a canonical transformation:

$$\psi', J' \quad \Longrightarrow \quad \psi'', J'' \tag{6.11}$$

and the new Hamiltonian $\tilde{\mathcal{K}}$ reads as :

$$\tilde{\mathcal{K}}(x, z, \sigma, \psi'; p_x, p_z, p_\sigma, J'; s) = \overline{\overline{\mathcal{K}}}_{orb}(x, z, \sigma; p_x, p_z, p_\sigma; s) + \tilde{\mathcal{K}}_{spin} \tag{6.12}$$

with

$$\begin{aligned}&\tilde{\mathcal{K}}_{spin}(x, z, \sigma, \psi''; p_x, p_z, p_\sigma, J''; s) \\ &= \left\{\vec{\Omega}(x, z, \sigma; p_x, p_z, p_\sigma; s) - \vec{U}'(x, z, \sigma; p_x, p_z, p_\sigma; s)\right\} \cdot \left(\ \xi_n \cdot \vec{n}_0 + \xi_m \cdot \vec{m} + \xi_l \cdot \vec{l}\,\right)\end{aligned} \tag{6.13}$$

and

$$\vec{U}' = \frac{1}{2}\left[\vec{n}_0 \times \frac{d\vec{n}_0}{ds} + \vec{m} \times \frac{d\vec{m}}{ds} + \vec{l} \times \frac{d\vec{l}}{ds}\right]$$

$$= \frac{1}{2}\left[\vec{n}_0 \times \left(\vec{\Omega}^{(0)} \times \vec{n}_0\right) + \vec{m} \times \left(\vec{\Omega}^{(0)} \times \vec{m} + \vec{l} \cdot \frac{d}{ds}\,\psi_{spin}(s)\right) + \vec{l} \times \left(\vec{\Omega}^{(0)} \times \vec{l} - \vec{m} \cdot \frac{d}{ds}\,\psi_{spin}(s)\right)\right]$$

$$= \frac{1}{2}\left[3\vec{\Omega}^{(0)} - \vec{n}_0 \cdot \left(\vec{\Omega}^{(0)} \cdot \vec{n}_0\right) - \vec{m} \cdot \left(\vec{\Omega}^{(0)} \cdot \vec{m}\right) - \vec{l} \cdot \left(\vec{\Omega}^{(0)} \cdot \vec{l}\right) + (\vec{n}_0 + \vec{n}_0) \cdot \frac{d}{ds}\,\psi_{spin}(s)\right]$$

$$= \frac{1}{2}\left[3\vec{\Omega}^{(0)} - \vec{\Omega}^{(0)} + 2\vec{n}_0 \cdot \frac{d}{ds}\,\psi_{spin}(s)\right]$$

$$= \vec{\Omega}^{(0)} + \vec{n}_0 \cdot \frac{d}{ds}\,\psi_{spin}(s)\ . \tag{6.14}$$

Thus we find:

$$\tilde{\mathcal{K}}_{spin}(x, z, \sigma, \psi''; p_x, p_z, p_\sigma, J''; s)$$

$$= \left\{\vec{\Omega} - \vec{\Omega}^{(0)} - \vec{n}_0 \cdot \frac{d}{ds}\,\psi_{spin}(s)\right\} \cdot \left(\ \xi_n \cdot \vec{n}_0 + \xi_m \cdot \vec{m} + \xi_l \cdot \vec{l}\ \right)$$

$$= \vec{\omega}(x, z, \sigma; p_x, p_z, p_\sigma; s) \cdot \left[\xi_n \cdot \vec{n}_0 + \xi_m \cdot \vec{m} + \xi_l \cdot \vec{l}\right] - \xi_n \cdot \frac{d}{ds}\,\psi_{spin}(s)$$

$$= \left[\xi_n \cdot (\vec{n}_0 \cdot \vec{\omega}) + \xi_m \cdot (\vec{m} \cdot \vec{\omega}) + \xi_l \cdot (\vec{l} \cdot \vec{\omega})\right] - \xi_n \cdot \frac{d}{ds}\,\psi_{spin}(s)$$

$$= (\xi_n,\ \xi_m,\ \xi_l) \cdot \begin{pmatrix} n_{os}(s) & n_{0x}(s) & n_{oz}(s) \\ m_s(s) & m_x(s) & m_z(s) \\ l_s(s) & l_x(s) & l_z(s) \end{pmatrix} \cdot \begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix} - \xi_n \cdot \frac{d}{ds}\,\psi_{spin}(s) \tag{6.15}$$

where we have introduced for abbreviation the vector

$$\vec{\omega} \ = \ \vec{\Omega} - \vec{\Omega}^{(0)}\ . \tag{6.16}$$

This is equivalent to the form for the spin Hamiltonian given by Derbenev [1].

Writing

$$\vec{\omega} \ = \ \omega_s \cdot \vec{e}_s + \omega_x \cdot \vec{e}_x + \omega_z \cdot \vec{e}_z \tag{6.17a}$$

and

$$\tilde{\vec{y}} = \vec{y} - \vec{y}_0 \tag{6.17b}$$

we obtain from eqns. (5.24) for the linearised components ω_s, ω_x, ω_z, of the vector $\vec{\omega}$:

$$\begin{aligned} \omega_s \ = \ & +2H \cdot (1 + a) \cdot \frac{1}{\beta_0^2} \cdot \tilde{\eta} \\ & -\frac{v_0}{E_0}\tilde{p}_x \cdot \frac{a\gamma_0^2}{1 + \gamma_0}\left[K_z - \frac{e}{p_0 \cdot c} \cdot \Delta B_x\right] \\ & +\frac{v_0}{E_0}\tilde{p}_z \cdot \frac{a\gamma_0^2}{1 + \gamma_0}\left[K_x + \frac{e}{p_0 \cdot c} \cdot \Delta B_z\right]\ ; \end{aligned} \tag{6.18a}$$

$$
\begin{aligned}
\omega_x &= -(1+a\gamma_0)\cdot[(N-H')\cdot\tilde{x}-(K_z^2-g)\cdot\tilde{z}] \\
&+\frac{a\gamma_0^2}{1+\gamma_0}\cdot 2H\cdot\left[\frac{v_0}{E_0}\tilde{p}_x+\beta_0^2\cdot H\tilde{z}\right] \\
&+\frac{1}{\beta_0^2}\cdot\left[a\gamma_0+\frac{\gamma_0}{1+\gamma_0}\right]\cdot\frac{eV(s)}{E_0}\sin\varphi\cdot\frac{v_0}{E_0}\tilde{p}_z \\
&-\left[1+\frac{1+a\gamma_0}{\gamma_0^2-1}\right]\cdot\left(K_z-\frac{e}{p_0\cdot c}\Delta B_x\right)\cdot\tilde{\eta}\ ; \qquad (6.18b)
\end{aligned}
$$

$$
\begin{aligned}
\omega_z &= +(1+a\gamma_0)\cdot[(N+H')\cdot\tilde{z}-(K_x^2+g)\cdot\tilde{x}] \\
&+\frac{a\gamma_0^2}{1+\gamma_0}\cdot 2H\cdot\left[\frac{v_0}{E_0}\tilde{p}_z-\beta_0^2\cdot H\tilde{x}\right] \\
&-\frac{1}{\beta_0^2}\cdot\left[a\gamma_0+\frac{\gamma_0}{1+\gamma_0}\right]\cdot\frac{eV(s)}{E_0}\sin\varphi\cdot\frac{v_0}{E_0}\tilde{p}_x \\
&+\left[1+\frac{1+a\gamma_0}{\gamma_0^2-1}\right]\cdot\left(K_x+\frac{e}{p_0\cdot c}\Delta B_z\right)\cdot\tilde{\eta} \qquad (6.18c)
\end{aligned}
$$

with (see eqn. (4.33))

$$\tilde{\eta}=\frac{v_0}{E_0}\cdot\tilde{p}_\sigma\ . \qquad (6.19)$$

With (5.12), (6.12), (6.15) we have the Hamiltonian (up to second order in the orbital variables) for the canonical variables

$$x,\ z,\ \sigma,\ \psi'';\ p_x,\ p_z,\ p_\sigma,\ J''.$$

2) Introduction of a new pair of canonical spin variables:

We now introduce the spin variables (α,β) defined by:

$$
\begin{aligned}
\alpha &= \sqrt{2\cdot(\xi-J'')}\cdot\cos\psi''\ ; \qquad (6.20a)\\
\beta &= \sqrt{2\cdot(\xi-J'')}\cdot\sin\psi''\ . \qquad (6.20b)
\end{aligned}
$$

From this definition we have:

$$
\begin{aligned}
\frac{\beta}{\alpha} &= \tan\psi''\ ; \qquad (6.21a)\\
J'' &= \xi-\frac{1}{2}\left(\alpha^2+\beta^2\right) \qquad (6.21b)
\end{aligned}
$$

and

$$\xi_n = \xi - \frac{1}{2}\left(\alpha^2+\beta^2\right) \; ; \tag{6.22a}$$

$$\xi_m = \frac{1}{\sqrt{2}}\cdot\alpha\cdot\sqrt{2\xi-\frac{1}{2}(\alpha^2+\beta^2)} \; ; \tag{6.22b}$$

$$\xi_l = \frac{1}{\sqrt{2}}\cdot\beta\cdot\sqrt{2\xi-\frac{1}{2}(\alpha^2+\beta^2)} \; . \tag{6.22c}$$

The latter can be inverted to give:

$$\alpha = +\sqrt{\frac{2}{\xi+\xi_n}}\cdot\xi_m \; ; \tag{6.23a}$$

$$\beta = +\sqrt{\frac{2}{\xi+\xi_n}}\cdot\xi_l \; . \tag{6.23b}$$

The transformation

$$\psi'',\ J'' \Longrightarrow \quad \alpha,\ \beta$$

can be obtained from the generating function

$$F_1(\alpha,\ \psi'') = \frac{1}{2}\alpha^2\cdot\tan\psi''-\xi\cdot\psi'' : \tag{6.24}$$

$$+\frac{\partial F_1}{\partial\alpha} = \alpha\cdot\tan\psi'' = \beta \; ; \tag{6.25a}$$

$$\begin{aligned} -\frac{\partial F_1}{\partial\psi''} &= -\frac{1}{2}\alpha^2\cdot\left(1+\tan^2\psi''\right) \\ &= -\frac{1}{2}\alpha^2\cdot\left(1+\frac{\beta^2}{\alpha^2}\right) \\ &= -\frac{1}{2}\left(\alpha^2+\beta^2\right)+\xi \\ &= J'' \; ; \end{aligned} \tag{6.25b}$$

$$\tilde{\mathcal{K}} \longrightarrow \tilde{\mathcal{K}}+\frac{\partial F_1}{\partial s} = \tilde{\mathcal{K}} = \overline{\overline{\mathcal{K}}}_{orb}+\tilde{\mathcal{K}}_{spin}. \tag{6.25c}$$

Thus α, β are canonical variables.

From (6.15) and (6.22) we obtain:

$$\begin{aligned} \tilde{\mathcal{K}}_{spin} = &\left(\xi-\frac{1}{2}\left(\alpha^2+\beta^2\right),\ \frac{1}{\sqrt{2}}\cdot\alpha\cdot\sqrt{2\xi-\frac{1}{2}(\alpha^2+\beta^2)},\ \frac{1}{\sqrt{2}}\cdot\beta\cdot\sqrt{2\xi-\frac{1}{2}(\alpha^2+\beta^2)}\right) \\ &\times\begin{pmatrix} n_{os}(s) & n_{0x}(s) & n_{oz}(s) \\ m_s(s) & m_x(s) & m_z(s) \\ l_s(s) & l_x(s) & l_z(s) \end{pmatrix}\begin{pmatrix}\omega_s \\ \omega_x \\ \omega_z\end{pmatrix} \\ &-\left[\xi-\frac{1}{2}\left(\alpha^2+\beta^2\right)\right]\cdot\frac{d}{ds}\,\psi_{spin}(s) \; . \end{aligned} \tag{6.26}$$

With (5.12), (6.25c), (6.26) we have the Hamiltonian for the canonical variables

$$x,\ z,\ \sigma,\ \alpha;\ p_x,\ p_z,\ p_\sigma,\ \beta.$$

Remarks:

1) The values of α and β are restricted by the condition :

$$\alpha^2 + \beta^2 \le 4\xi \quad \Longrightarrow \quad \xi \ \ge\ \xi_n\ \ge\ -\xi\ .$$

2) For

$$\alpha^2 + \beta^2 < 4\xi$$

the correspondence between α, β and $\xi_n,\ \xi_m,\ \xi_l$ is one-one.

3) For

$$\left|\frac{\alpha}{\sqrt{\xi}}\right| \ll 1\ ; \quad \left|\frac{\beta}{\sqrt{\xi}}\right| \ll 1$$

we have:

$$\begin{aligned} \xi_m &\approx \alpha \cdot \sqrt{\xi}\ ; \\ \xi_l &\approx \beta \cdot \sqrt{\xi}\ . \end{aligned}$$

and in this case our canonical α and β behave like the spin-coordinates introduced by Chao in the SLIM-program [10].

4) For the Poisson-bracket

$$\{\alpha, \beta\}_{\psi'', J''} \ \equiv\ \frac{\partial \alpha}{\partial \psi''} \cdot \frac{\partial \beta}{\partial J''} - \frac{\partial \alpha}{\partial J''} \cdot \frac{\partial \beta}{\partial \psi''}$$

we obtain from (6.20) :

$$\begin{aligned} \{\alpha, \beta\}_{\psi'', J''} &= \left[-\sqrt{2 \cdot (\xi - J'')} \cdot \sin \psi''\right] \cdot \frac{-2}{2 \cdot \sqrt{2 \cdot (\xi - J'')}} \cdot \sin \psi'' \\ &- \frac{-2}{2 \cdot \sqrt{2 \cdot (\xi - J'')}} \cdot \cos \psi'' \cdot \left[+\sqrt{2 \cdot (\xi - J'')} \cdot \cos \psi''\right] \\ &= 1\ . \end{aligned}$$

This relation demonstrates again that α and β are canonical variables.

5) The variables α and β could already have been introduced at the beginning in the starting Hamiltonian (2.1). They completely replace ψ and J.

6.2.2 Transformation of the orbital Variables

The orbit vector $\vec{y}(s)$ can be separated into two components (see eqn. (6.17b)) :

$$\vec{y}(s) = \vec{y}_0(s) + \tilde{\vec{y}}(s) \ , \tag{6.27}$$

where the vector $\tilde{\vec{y}}(s)$ describes the synchro-betatron oscillations about the new closed equilibrium trajectory $\vec{y}_0(s)$.

The transformation

$$\vec{y} \ ; \ \alpha, \ \beta \quad \Longrightarrow \quad \tilde{\vec{y}} \ ; \ \tilde{\alpha} = \alpha, \ \tilde{\beta} = \beta \ . \tag{6.28}$$

can be obtained from the generating function

$$\begin{aligned} F_2(x, \tilde{p}_x; z, \tilde{p}_z; \sigma, \tilde{p}_\sigma; \alpha, \tilde{\beta}; s) &= (x - x_0) \cdot (\tilde{p}_x + p_{x0}) + (z - z_0) \cdot (\tilde{p}_z + p_{z0}) \\ &\quad + (\sigma - \sigma_\sigma) \cdot (\tilde{p}_\sigma + p_{\sigma 0}) + \alpha \cdot \tilde{\beta} + f(s) \end{aligned} \tag{6.29}$$

with an arbitrary function $f(s)$. The transformation equations read as:

$$p_x = \frac{\partial F_2}{\partial x} = \tilde{p}_x + p_{x0} \ ; \quad \tilde{x} = \frac{\partial F_2}{\partial \tilde{p}_x} = x - x_0 \ ; \tag{6.30a}$$

$$p_z = \frac{\partial F_2}{\partial z} = \tilde{p}_z + p_{z0} \ ; \quad \tilde{x} = \frac{\partial F_2}{\partial \tilde{p}_x} = x - x_0 \ ; \tag{6.30b}$$

$$p_\sigma = \frac{\partial F_2}{\partial \sigma} = \tilde{p}_\sigma + p_{\sigma 0} \ ; \quad \tilde{\sigma} = \frac{\partial F_2}{\partial \tilde{p}_\sigma} = \sigma - \sigma_0 \tag{6.30c}$$

which reproduce the defining equation (6.27) for $\tilde{\vec{y}}$.

Furthermore we have $\left(\text{with } \frac{d}{ds} f(s) = x_0(s) \cdot \frac{d}{ds} p_{x0}(s)\right)$:

$$\begin{aligned} \frac{\partial F_2}{\partial s} &= -\frac{dx_0}{ds} \cdot p_x + \frac{dp_{x0}}{ds} \cdot x - \frac{dz_0}{ds} \cdot p_z + \frac{dp_{z0}}{ds} \cdot z - \frac{d\sigma_0}{ds} \cdot p_\sigma + \frac{dp_{\sigma 0}}{ds} \cdot \sigma \\ &= -p_x \cdot \left(\frac{\partial \tilde{\mathcal{K}}}{\partial p_x}\right)_{\vec{y}=\vec{y}_0 \ ; \ \alpha=\beta=0} - x \cdot \left(\frac{\partial \tilde{\mathcal{K}}}{\partial x}\right)_{\vec{y}=\vec{y}_0 \ ; \ \alpha=\beta=0} \\ &\quad - p_z \cdot \left(\frac{\partial \tilde{\mathcal{K}}}{\partial p_z}\right)_{\vec{y}=\vec{y}_0 \ ; \ \alpha=\beta=0} - z \cdot \left(\frac{\partial \tilde{\mathcal{K}}}{\partial z}\right)_{\vec{y}=\vec{y}_0 \ ; \ \alpha=\beta=0} \\ &\quad - p_\sigma \cdot \left(\frac{\partial \tilde{\mathcal{K}}}{\partial p_\sigma}\right)_{\vec{y}=\vec{y}_0 \ ; \ \alpha=\beta=0} - \sigma \cdot \left(\frac{\partial \tilde{\mathcal{K}}}{\partial \sigma}\right)_{\vec{y}=\vec{y}_0 \ ; \ \alpha=\beta=0} \\ &= -\vec{y} \cdot \left(\frac{\partial \tilde{\mathcal{K}}}{\partial \vec{y}}\right)_{\vec{y}=\vec{y}_0 \ ; \ \alpha=\beta=0} \end{aligned} \tag{6.31}$$

and therefore

$$\tilde{\mathcal{K}} \equiv \overline{\overline{\mathcal{K}}}_{orb} + \tilde{\mathcal{K}}_{spin}, \quad \longrightarrow \quad \hat{\mathcal{K}} = \tilde{\mathcal{K}} - \vec{y} \cdot \left(\frac{\partial \tilde{\mathcal{K}}}{\partial \vec{y}}\right)_{\vec{y}=\vec{y}_0 \ ; \ \alpha=\beta=0}$$

$$
\begin{aligned}
&= \overline{\overline{\mathcal{K}}}_{orb} - \vec{y}\cdot\left(\frac{\partial\overline{\overline{\mathcal{K}}}_{orb}}{\partial\vec{y}}\right)_{\vec{y}=\vec{y}_0\ ;\ \alpha=\beta=0} \\
&+ \tilde{\mathcal{K}}_{spin} - \vec{y}\cdot\left(\frac{\partial\tilde{\mathcal{K}}_{spin}}{\partial\vec{y}}\right)_{\vec{y}=\vec{y}_0\ ;\ \alpha=\beta=0} \\
&= \hat{\mathcal{K}}_{orbit} + \hat{\mathcal{K}}_{spin}
\end{aligned}
\tag{6.32}
$$

with

$$
\hat{\mathcal{K}}_{orbit} = \overline{\overline{\mathcal{K}}}_{orb} - \vec{y}\cdot\left(\frac{\partial\overline{\overline{\mathcal{K}}}_{orb}}{\partial\vec{y}}\right)_{\vec{y}=\vec{y}_0\ ;\ \alpha=\beta=0} ; \tag{6.33a}
$$

$$
\hat{\mathcal{K}}_{spin} = \tilde{\mathcal{K}}_{spin} - \vec{y}\cdot\left(\frac{\partial\tilde{\mathcal{K}}_{spin}}{\partial\vec{y}}\right)_{\vec{y}=\vec{y}_0\ ;\ \alpha=\beta=0} \tag{6.33b}
$$

For the linearised form of $\vec{\omega}$ (see eqn. (6.18)), eqns. (6.26) and (6.32) lead to:

$$
\begin{aligned}
&\hat{\mathcal{K}}_{spin}(\tilde{x},\tilde{z},\tilde{\sigma},\alpha;\tilde{p}_x,\tilde{p}_z,\tilde{p}_\sigma,\beta;s) \\
&= \left(\xi - \frac{1}{2}\left(\alpha^2+\beta^2\right),\ \frac{1}{\sqrt{2}}\cdot\alpha\cdot\sqrt{2\xi-\frac{1}{2}(\alpha^2+\beta^2)},\ \frac{1}{\sqrt{2}}\cdot\beta\cdot\sqrt{2\xi-\frac{1}{2}(\alpha^2+\beta^2)}\right) \\
&\quad\times\begin{pmatrix} n_{os}(s) & n_{0x}(s) & n_{oz}(s) \\ m_s(s) & m_x(s) & m_z(s) \\ l_s(s) & l_x(s) & l_z(s)\end{pmatrix}\begin{pmatrix}\omega_s\\ \omega_x\\ \omega_z\end{pmatrix} \\
&\quad+\frac{1}{2}\left(\alpha^2+\beta^2\right)\cdot\frac{d}{ds}\,\psi_{spin}(s) \\
&- (\xi,\ 0,\ 0)\begin{pmatrix} n_{os}(s) & n_{0x}(s) & n_{oz}(s) \\ m_s(s) & m_x(s) & m_z(s) \\ l_s(s) & l_x(s) & l_z(s)\end{pmatrix}\begin{pmatrix}\omega_s\\ \omega_x\\ \omega_z\end{pmatrix} \\
&= \left(-\frac{1}{2}\left(\alpha^2+\beta^2\right),\ \frac{1}{\sqrt{2}}\cdot\alpha\cdot\sqrt{2\xi-\frac{1}{2}(\alpha^2+\beta^2)},\ \frac{1}{\sqrt{2}}\cdot\beta\cdot\sqrt{2\xi-\frac{1}{2}(\alpha^2+\beta^2)}\right) \\
&\quad\times\begin{pmatrix} n_{os}(s) & n_{0x}(s) & n_{oz}(s) \\ m_s(s) & m_x(s) & m_z(s) \\ l_s(s) & l_x(s) & l_z(s)\end{pmatrix}\begin{pmatrix}\omega_s\\ \omega_x\\ \omega_z\end{pmatrix} \\
&\quad+\frac{1}{2}\left(\alpha^2+\beta^2\right)\cdot\frac{d}{ds}\,\psi_{spin}(s) \\
&= \xi\cdot\left(-\frac{1}{2}\left[\left(\frac{\alpha}{\sqrt{\xi}}\right)^2+\left(\frac{\beta}{\sqrt{\xi}}\right)^2\right],\ \frac{\alpha}{\sqrt{\xi}}\cdot\sqrt{1-\frac{1}{4}\left[\left(\frac{\alpha}{\sqrt{\xi}}\right)^2+\left(\frac{\beta}{\sqrt{\xi}}\right)^2\right]},\right. \\
&\qquad\left.\frac{\beta}{\sqrt{\xi}}\cdot\sqrt{1-\frac{1}{4}\left[\left(\frac{\alpha}{\sqrt{\xi}}\right)^2+\left(\frac{\beta}{\sqrt{\xi}}\right)^2\right]}\right)
\end{aligned}
$$

$$\times \begin{pmatrix} n_{os}(s) & n_{0x}(s) & n_{oz}(s) \\ m_s(s) & m_x(s) & m_z(s) \\ l_s(s) & l_x(s) & l_z(s) \end{pmatrix} \begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix}$$

$$+\frac{\xi}{2}\left[\left(\frac{\alpha}{\sqrt{\xi}}\right)^2 + \left(\frac{\beta}{\sqrt{\xi}}\right)^2\right] \cdot \frac{d}{ds}\ \psi_{spin}(s) \tag{6.34a}$$

and at second order the orbital Hamiltonian $\hat{\mathcal{K}}_{orb}$ takes the form, using eqns. (5.21) and (5.22) :

$$\begin{aligned}
&\frac{v_0}{E_0} \cdot \hat{\mathcal{K}}_{orb}(\tilde{x}, \tilde{z}, \tilde{\sigma}; \tilde{p}_x, \tilde{p}_z, \tilde{p}_\sigma; s) \\
&\quad = \frac{1}{2} \cdot \frac{1}{\gamma_0^2 - 1} \cdot \tilde{\eta}^2 - [K_x \cdot \tilde{x} + K_z \cdot \tilde{z}] \cdot \tilde{\eta} \\
&\quad + \frac{1}{2\beta_0^2} \cdot \left\{ \left[\frac{v_0}{E_0}\tilde{p}_x + \beta_0^2 H \cdot \tilde{z}\right]^2 + \left[\frac{v_0}{E_0}\tilde{p}_z - \beta_0^2 H \cdot \tilde{x}\right]^2 \right\} \\
&\quad + \frac{1}{2}\beta_0^2 \cdot \left\{ (K_x^2 + g) \cdot \tilde{x}^2 + (K_z^2 - g) \cdot \tilde{z}^2 - 2N \cdot \tilde{x}\tilde{z} \right\} \\
&\quad - \frac{1}{2}\tilde{\sigma}^2 \cdot \frac{eV(s)}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \cos\varphi
\end{aligned} \tag{6.34b}$$

(the constant terms

$$\left(-\xi \cdot \frac{d}{ds}\ \psi_{spin}(s)\right) \quad \text{and} \quad \overline{\overline{\mathcal{K}}}_{orb}(x_0, z_0, \sigma_0; p_{x0}, p_{z0}, p_{\sigma 0}; s)$$

in the Hamiltonian (6.34), which have no influence on the motion, have been neglected).

With (6.32), (6.34) we have the Hamiltonian for the canonical variables

$$\tilde{x},\ \tilde{z},\ \tilde{\sigma},\ \alpha;\ \tilde{p}_x,\ \tilde{p}_z,\ \tilde{p}_\sigma,\ \beta.$$

and the canonical equations for spin-orbit motion are:

$$\frac{d}{ds}\ \tilde{x} = +\frac{\partial \hat{\mathcal{K}}}{\partial \tilde{p}_x}; \quad \frac{d}{ds}\ \tilde{p}_x = -\frac{\partial \hat{\mathcal{K}}}{\partial \tilde{x}}; \tag{6.35a}$$

$$\frac{d}{ds}\ \tilde{z} = +\frac{\partial \hat{\mathcal{K}}}{\partial \tilde{p}_z}; \quad \frac{d}{ds}\ \tilde{p}_z = -\frac{\partial \hat{\mathcal{K}}}{\partial \tilde{z}}; \tag{6.35b}$$

$$\frac{d}{ds}\ \tilde{\sigma} = +\frac{\partial \hat{\mathcal{K}}}{\partial \tilde{p}_\sigma}; \quad \frac{d}{ds}\ \tilde{p}_\sigma = -\frac{\partial \hat{\mathcal{K}}}{\partial \tilde{\sigma}}; \tag{6.35c}$$

$$\frac{d}{ds}\ \alpha = +\frac{\partial \hat{\mathcal{K}}}{\partial \beta}; \quad \frac{d}{ds}\ \beta = -\frac{\partial \hat{\mathcal{K}}}{\partial \alpha}. \tag{6.35d}$$

As in eqn. (5.18) for the orbital motion, we can expand the square root

$$\sqrt{1 - \frac{1}{4}\left[\left(\frac{\alpha}{\sqrt{\xi}}\right)^2 + \left(\frac{\beta}{\sqrt{\xi}}\right)^2\right]}$$

appearing in the spin-Hamiltonian (6.34a) in a series :

$$\sqrt{1-\frac{1}{4}\left[\left(\frac{\alpha}{\sqrt{\xi}}\right)^2+\left(\frac{\beta}{\sqrt{\xi}}\right)^2\right]} \;=\; 1-\frac{1}{8}\left[\left(\frac{\alpha}{\sqrt{\xi}}\right)^2+\left(\frac{\beta}{\sqrt{\xi}}\right)^2\right]+\cdots \qquad (6.36)$$

so that the spin motion can be conveniently calculated to various orders of approximation.

If $\vec{\xi}$ is sufficiently parallel to $\vec{n}_0$ an expression to linear order suffices and the Hamiltonian (6.34a) becomes:

$$\begin{aligned}
&\hat{\mathcal{K}}_{spin}(\tilde{x},\tilde{z},\tilde{\sigma},\alpha;\tilde{p}_x,\tilde{p}_z,\tilde{p}_\sigma,\beta;s)\\
&= \sqrt{\xi}\cdot(\alpha,\ \beta)\cdot\begin{pmatrix} m_s(s) & m_x(s) & m_z(s)\\ l_s(s) & l_x(s) & l_z(s)\end{pmatrix}\begin{pmatrix}\omega_s\\ \omega_x\\ \omega_z\end{pmatrix}\\
&\quad +\frac{\xi}{2}\left[\left(\frac{\alpha}{\sqrt{\xi}}\right)^2+\left(\frac{\beta}{\sqrt{\xi}}\right)^2\right]\cdot\frac{d}{ds}\,\psi_{spin}(s) \qquad (6.37)
\end{aligned}$$

and the corresponding canonical equations for α and β read :

$$\begin{aligned}
\frac{d}{ds}\left(\frac{\alpha}{\sqrt{\xi}}\right) &= +\begin{pmatrix} l_s(s), & l_x(s), & l_z(s)\end{pmatrix}\begin{pmatrix}\omega_s\\ \omega_x\\ \omega_z\end{pmatrix}\\
&\quad +\left(\frac{\beta}{\sqrt{\xi}}\right)\cdot\frac{d}{ds}\,\psi_{spin}(s)\ ; \qquad (6.38a)\\
\frac{d}{ds}\left(\frac{\beta}{\sqrt{\xi}}\right) &= -\begin{pmatrix} m_s(s) & m_x(s) & m_z(s)\end{pmatrix}\begin{pmatrix}\omega_s\\ \omega_x\\ \omega_z\end{pmatrix}\\
&\quad -\left(\frac{\alpha}{\sqrt{\xi}}\right)\cdot\frac{d}{ds}\,\psi_{spin}(s)\ . \qquad (6.38b)
\end{aligned}$$

In this form the relations (6.38) are the basic equations for spin motion used in the computer program SLIM [10,11]. We have thus derived the SLIM-formalism from canonical equations based on a polynomial expansion of a spin Hamiltonian.

6.2.3 Scale Transformation

In order to eliminate the factors (v_0/E_0) and β_0^2 appearing in the Hamiltonian (6.34), we define new relative variables :

$$\hat{x} \equiv \tilde{x}\ ; \quad \hat{p}_x \equiv \frac{1}{\beta_0^2}\frac{v_0}{E_0}\cdot\tilde{p}_x = \frac{\tilde{p}_x}{p_0}\ ; \qquad (6.39a)$$

$$\hat{z} \equiv \tilde{z}\ ; \quad \hat{p}_z \equiv \frac{1}{\beta_0^2}\frac{v_0}{E_0}\cdot\tilde{p}_z = \frac{\tilde{p}_z}{p_0}\ ; \qquad (6.39b)$$

$$\hat{\sigma} \equiv \tilde{\sigma}\ ; \quad \hat{p}_\sigma \equiv \frac{1}{\beta_0^2}\frac{v_0}{E_0}\cdot\tilde{p}_\sigma \equiv \frac{1}{\beta_0^2}\tilde{\eta}\ ; \qquad (6.39c)$$

$$\hat{\alpha} \equiv \frac{1}{\beta_0}\sqrt{\frac{v_0}{E_0}}\cdot\alpha\ ; \quad \hat{\beta} \equiv \frac{1}{\beta_0}\sqrt{\frac{v_0}{E_0}}\cdot\beta \qquad (6.39d)$$

($\tilde{x}$, $\tilde{z}$, $\tilde{\sigma}$ unchanged).

Note that (6.39) is a combination of a scale transformation (using the scale factor $\frac{1}{\beta_0^2}\frac{v_0}{E_0}$) and a canonical (point-) transformation (involving α, β only).

Furthermore, the linearised vector (6.18)

$$\begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix}$$

in the spin-Hamiltonian (6.34a) can be written as:

$$\begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix} = \underline{F}_{(3\times 6)} \cdot \begin{pmatrix} \hat{x} \\ \hat{p}_x \\ \hat{z} \\ \hat{p}_z \\ \hat{\sigma} \\ \hat{p}_\sigma \end{pmatrix} \tag{6.40}$$

with

$$\begin{aligned}
F_{12} &= -a(\gamma_0 - 1) \cdot \left[K_z - \frac{e}{p_0 \cdot c} \cdot \Delta B_x\right] \ ; \\
F_{14} &= +a(\gamma_0 - 1) \cdot \left[K_x + \frac{e}{p_0 \cdot c} \cdot \Delta B_z\right] \ ; \\
F_{16} &= +2H \cdot (1 + a) \ ; \\
F_{21} &= -(1 + a\gamma_0) \cdot (N - H') \ ; \\
F_{22} &= +a(\gamma_0 - 1) \cdot 2H \ ; \\
F_{23} &= +(1 + a\gamma_0) \cdot (K_z^2 - g) + \frac{a\gamma_0^2}{1 + \gamma_0} \cdot 2\beta_0^2 \cdot H^2 \ ; \\
F_{24} &= \left[a\gamma_0 + \frac{\gamma_0}{1 + \gamma_0}\right] \cdot \frac{e}{E_0} V(s) \sin\varphi \ ; \\
F_{26} &= -\left[1 + \frac{a}{\gamma_0}\right] \cdot \left(K_z - \frac{e}{p_0 \cdot c} \Delta B_x\right) \ ; \\
F_{31} &= -(1 + a\gamma_0) \cdot (K_x^2 + g) - \frac{a\gamma_0^2}{1 + \gamma_0} \cdot 2\beta_0^2 \cdot H^2 \ ; \\
F_{32} &= -\left[a\gamma_0 + \frac{\gamma_0}{1 + \gamma_0}\right] \cdot \frac{e}{E_0} V(s) \sin\varphi \ ; \\
F_{33} &= +(1 + a\gamma_0) \cdot (N + H') \ ;
\end{aligned}$$

$$F_{34} = +a(\gamma_0 - 1) \cdot 2H \ ;$$

$$F_{36} = +\left[1 + \frac{a}{\gamma_0}\right] \cdot \left(K_x + \frac{\epsilon}{p_0 \cdot c}\Delta B_z\right) \ ;$$

$$F_{ik} = 0 \quad \text{otherwise} \ . \tag{6.41}$$

Introducing now

$$\begin{aligned} \hat{\mathcal{H}} &= \frac{1}{\beta_0^2}\frac{v_0}{E_0} \cdot \hat{\mathcal{K}} \\ &= \hat{\mathcal{H}}_{orb} + \hat{\mathcal{H}}_{spin} \end{aligned} \tag{6.42}$$

with

$$\hat{\mathcal{H}}_{orb} = \frac{1}{\beta_0^2}\frac{v_0}{E_0} \cdot \hat{\mathcal{K}}_{orb} \ ; \tag{6.43a}$$

$$\hat{\mathcal{H}}_{spin} = \frac{1}{\beta_0^2}\frac{v_0}{E_0} \cdot \hat{\mathcal{K}}_{spin} \ , \tag{6.43b}$$

we can rewrite the canonical equations (6.35) in the form:

$$\frac{d}{ds}\,\hat{x} = +\frac{\partial \hat{\mathcal{H}}}{\partial \hat{p}_x} \ ; \quad \frac{d}{ds}\,\hat{p}_x = -\frac{\partial \hat{\mathcal{H}}}{\partial \hat{x}} \ ; \tag{6.44a}$$

$$\frac{d}{ds}\,\hat{z} = +\frac{\partial \hat{\mathcal{H}}}{\partial \hat{p}_z} \ ; \quad \frac{d}{ds}\,\hat{p}_z = -\frac{\partial \hat{\mathcal{H}}}{\partial \hat{z}} \ ; \tag{6.44b}$$

$$\frac{d}{ds}\,\hat{\sigma} = +\frac{\partial \hat{\mathcal{H}}}{\partial \hat{p}_\sigma} \ ; \quad \frac{d}{ds}\,\hat{p}_\sigma = -\frac{\partial \hat{\mathcal{H}}}{\partial \hat{\sigma}} \ ; \tag{6.44c}$$

$$\frac{d}{ds}\,\hat{\alpha} = +\frac{\partial \hat{\mathcal{H}}}{\partial \hat{\beta}} \ ; \quad \frac{d}{ds}\,\hat{\beta} = -\frac{\partial \hat{\mathcal{H}}}{\partial \hat{\alpha}} \tag{6.44d}$$

so that $\hat{\mathcal{H}}$ is the Hamiltonian for the canonical variables

$$\hat{x},\ \hat{z},\ \hat{\sigma},\ \hat{\alpha};\ \hat{p}_x,\ \hat{p}_z,\ \hat{p}_\sigma,\ \hat{\beta}.$$

By expanding the Hamiltonian $\hat{\mathcal{H}}$ in a power series in these variables, we can calculate spin-orbit motion in the required order of approximation and be sure that the equations of motion are symplectic.

To obtain linearised equations of motion we use (6.34b) and (6.37) :

$$\hat{\mathcal{H}}_{orb}(\hat{x}, \hat{z}, \hat{\sigma}; \hat{p}_x, \hat{p}_z, \hat{p}_\sigma; s)$$

$$
\begin{aligned}
&= \frac{1}{2}\cdot\frac{1}{\gamma_0^2}\cdot\hat{p}_\sigma^2 - [K_x\cdot\hat{x} + K_z\cdot\hat{z}]\cdot\hat{p}_\sigma \\
&+\frac{1}{2}\cdot\left\{[\hat{p}_x + H\cdot\hat{z}]^2 + [\hat{p}_z - H\cdot\hat{x}]^2\right\} \\
&+\frac{1}{2}\cdot\left\{G_1\cdot\hat{x}^2 + G_2\cdot\hat{z}^2 - 2N\cdot\hat{x}\hat{z}\right\} \\
&-\frac{1}{2}\hat{\sigma}^2\cdot\frac{1}{\beta_0^2}\cdot\frac{eV(s)}{E_0}\cdot h\cdot\frac{2\pi}{L}\cdot\cos\varphi\ ;
\end{aligned}
\tag{6.45a}
$$

$$
\begin{aligned}
&\hat{\mathcal{H}}_{spin}(\hat{x},\hat{z},\hat{\sigma};\hat{p}_x,\hat{p}_z,\hat{p}_\sigma;s) \\
&= \sqrt{\xi}\cdot\frac{1}{\beta_0}\cdot\sqrt{\frac{v_0}{E_0}}\cdot\left(\hat{\alpha},\ \hat{\beta}\right)\cdot\begin{pmatrix} m_s(s) & m_x(s) & m_z(s) \\ l_s(s) & l_x(s) & l_z(s)\end{pmatrix}\begin{pmatrix}\omega_s\\ \omega_x\\ \omega_z\end{pmatrix} \\
&\qquad +\frac{1}{2}\cdot\left[\hat{\alpha}^2+\hat{\beta}^2\right]\cdot\frac{d}{ds}\ \psi_{spin}(s) \\
&= \sqrt{\xi}\cdot\frac{1}{\beta_0}\sqrt{\frac{v_0}{E_0}}\cdot(\omega_s,\ \omega_x,\ \omega_z)\cdot\begin{pmatrix} m_s(s) & l_s(s)\\ m_x(s) & l_x(s)\\ m_z(s) & l_z(s)\end{pmatrix}\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix} \\
&\qquad +\frac{1}{2}\left[\hat{\alpha}^2+\hat{\beta}^2\right]\cdot\frac{d}{ds}\ \psi_{spin}(s)\ ,
\end{aligned}
\tag{6.45b}
$$

where we have written for abbreviation:

$$G_1 = K_x^2 + g\ ; \tag{6.46a}$$

$$G_2 = K_z^2 - g\ . \tag{6.46b}$$

The corresponding canonical equations take the form :

$$
\begin{aligned}
\frac{d}{ds}\ \hat{x} &= \hat{p}_x + H\cdot\hat{z} \\
&\quad +\sqrt{\xi}\cdot\frac{1}{\beta_0}\cdot\sqrt{\frac{v_0}{E_0}}\cdot(F_{12},\ F_{22},\ F_{32})\begin{pmatrix} m_s(s) & l_s(s)\\ m_x(s) & l_x(s)\\ m_z(s) & l_z(s)\end{pmatrix}\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix}\ ;
\end{aligned}
\tag{6.47a}
$$

$$
\begin{aligned}
\frac{d}{ds}\ \hat{p}_x &= K_x\cdot\hat{p}_\sigma + [\hat{p}_z - H\cdot\hat{x}]\cdot H - G_1\cdot\hat{x} + N\cdot\hat{z} \\
&\quad -\sqrt{\xi}\cdot\frac{1}{\beta_0}\cdot\sqrt{\frac{v_0}{E_0}}\cdot(F_{11},\ F_{21},\ F_{31})\begin{pmatrix} m_s(s) & l_s(s)\\ m_x(s) & l_x(s)\\ m_z(s) & l_z(s)\end{pmatrix}\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix}\ ;
\end{aligned}
\tag{6.47b}
$$

$$
\begin{aligned}
\frac{d}{ds}\ \hat{z} &= \hat{p}_z - H\cdot\hat{x} \\
&\quad +\sqrt{\xi}\cdot\frac{1}{\beta_0}\cdot\sqrt{\frac{v_0}{E_0}}\cdot(F_{14},\ F_{24},\ F_{34})\begin{pmatrix} m_s(s) & l_s(s)\\ m_x(s) & l_x(s)\\ m_z(s) & l_z(s)\end{pmatrix}\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix}\ ;
\end{aligned}
\tag{6.47c}
$$

$$
\frac{d}{ds}\ \hat{p}_z = K_z\cdot\hat{p}_\sigma - [\hat{p}_x + H\cdot\hat{z}]\cdot H - G_2\cdot\hat{z} + N\cdot\hat{x}
$$

$$-\sqrt{\xi}\cdot\frac{1}{\beta_0}\cdot\sqrt{\frac{v_0}{E_0}}\cdot(F_{13},\ F_{23},\ F_{33})\begin{pmatrix} m_s(s) & l_s(s)\\ m_x(s) & l_x(s)\\ m_z(s) & l_z(s)\end{pmatrix}\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix}\ ; \qquad (6.47d)$$

$$\frac{d}{ds}\,\hat{\sigma} = \frac{1}{\gamma_0^2}\cdot\hat{p}_\sigma - [K_x\cdot\hat{x} + K_z\cdot\hat{z}]$$
$$+\sqrt{\xi}\cdot\frac{1}{\beta_0}\cdot\sqrt{\frac{v_0}{E_0}}\cdot(F_{16},\ F_{26},\ F_{36})\begin{pmatrix} m_s(s) & l_s(s)\\ m_x(s) & l_x(s)\\ m_z(s) & l_z(s)\end{pmatrix}\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix}\ ; \qquad (6.47e)$$

$$\frac{d}{ds}\,\hat{p}_\sigma = \frac{1}{\beta_0^2}\cdot\frac{eV(s)}{E_0}\cdot h\cdot\frac{2\pi}{L}\cdot\cos\varphi\cdot\hat{\sigma}$$
$$-\sqrt{\xi}\cdot\frac{1}{\beta_0}\cdot\sqrt{\frac{v_0}{E_0}}\cdot(F_{15},\ F_{25},\ F_{35})\begin{pmatrix} m_s(s) & l_s(s)\\ m_x(s) & l_x(s)\\ m_z(s) & l_z(s)\end{pmatrix}\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix}\ ; \qquad (6.47f)$$

$$\frac{d}{ds}\,\hat{\alpha} = +\sqrt{\xi}\cdot\frac{1}{\beta_0}\cdot\sqrt{\frac{v_0}{E_0}}\cdot(0,\ 1)\begin{pmatrix} m_s(s) & m_x(s) & m_z(s)\\ l_s(s) & l_x(s) & l_z(s)\end{pmatrix}\underline{F}\cdot\hat{\vec{y}}$$
$$+\hat{\beta}\cdot\frac{d}{ds}\,\psi_{spin}(s)\ ; \qquad (6.47g)$$

$$\frac{d}{ds}\,\hat{\beta} = -\sqrt{\xi}\cdot\frac{1}{\beta_0}\cdot\sqrt{\frac{v_0}{E_0}}\cdot(1,\ 0)\begin{pmatrix} m_s(s) & m_x(s) & m_z(s)\\ l_s(s) & l_x(s) & l_z(s)\end{pmatrix}\underline{F}\cdot\hat{\vec{y}}$$
$$-\hat{\alpha}\cdot\frac{d}{ds}\,\psi_{spin}(s) \qquad (6.47h)$$

or in matrix-form:

$$\frac{d}{ds}\begin{pmatrix}\hat{\vec{y}}\\ \hat{\alpha}\\ \hat{\beta}\end{pmatrix} = \underline{A}\cdot\begin{pmatrix}\hat{\vec{y}}\\ \hat{\alpha}\\ \hat{\beta}\end{pmatrix} \qquad (6.48)$$

with

$$\underline{A}(s) = \begin{pmatrix}\underline{A}_{orb} & \underline{B}\\ \underline{C} & \underline{D}\end{pmatrix} \qquad (6.49)$$

and

$$\underline{A}_{orb}(s) = \begin{pmatrix} 0 & 1 & H & 0 & 0 & 0\\ -(G_1+H^2) & 0 & N & H & 0 & K_x\\ -H & 0 & 0 & 1 & 0 & 0\\ N & -H & -(G_2+H^2) & 0 & 0 & K_z\\ -K_x & 0 & -K_z & 0 & 0 & 1/\gamma_0^2\\ 0 & 0 & 0 & 0 & \frac{eV(s)}{E_0}\cdot\frac{1}{\beta_0^2}\cdot\frac{2\pi h}{L}\cos\varphi & 0\end{pmatrix}; \qquad (6.50a)$$

$$\underline{B}(s) = -\sqrt{\xi}\cdot\frac{1}{\beta_0}\cdot\sqrt{\frac{v_0}{E_0}}\cdot\underline{S}\cdot\underline{F}^T\cdot\begin{pmatrix} m_s(s) & l_s(s)\\ m_x(s) & l_x(s)\\ m_z(s) & l_z(s)\end{pmatrix}; \tag{6.50b}$$

$$\underline{C}(s) = \sqrt{\xi}\cdot\frac{1}{\beta_0}\cdot\sqrt{\frac{v_0}{E_0}}\cdot\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\cdot\begin{pmatrix} m_s(s) & m_x(s) & m_z(s)\\ l_s(s) & l_x(s) & l_z(s)\end{pmatrix}\underline{F}; \tag{6.50c}$$

$$\underline{D}(s) = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\cdot\frac{d}{ds}\,\psi_{spin}(s). \tag{6.50d}$$

Here the matrix $\underline{B}(s)$ describes the influence of Stern-Gerlach forces on the orbital motion and the matrix $\underline{C}(s)$ the influence of orbital motion on the spin motion. The matrices $\underline{A}(s)$ and $\underline{D}(s)$ correspond to the "unperturbed" spin-orbit motion. We emphasize again that the approximation in (6.45b) can only be used if the spin is almost parallel to $\vec{n}_0$.

Because the equations of motion (6.46) are linear and homogeneous, the solution can be written in the form:

$$\begin{pmatrix}\hat{\vec{y}}(s)\\ \hat{\alpha}(s)\\ \hat{\beta}(s)\end{pmatrix} = \hat{\underline{M}}(s,s_0)\cdot\begin{pmatrix}\hat{\vec{y}}(s_0)\\ \hat{\alpha}(s_0)\\ \hat{\beta}(s_0)\end{pmatrix}. \tag{6.51}$$

This defines the symplectic 8-dimensional transfer matrix $\hat{\underline{M}}(s,s_0)$ of linearised spin-orbit motion.

If the matrix $\underline{B}$ in (6.49) is retained but the matrix $\underline{C}$ is put to zero, i.e. if SG forces are included but the effect of orbital motion on spin is neglected, then $\hat{\underline{M}}$ will be non-symplectic and the orbital coordinates in (6.5) can, in principle, grow or shrink indefinitely —at least in this linearised description.

Another observation is that the matrices $\underline{B}$ and $\underline{C}$ serve to couple orbit and spin in a way analogous to the way that the off diagonal 2×2 blocks in solenoid and skew quadrupole matrices couple x and z motion. In the presence of orbital coupling and near resonance the x and z modes exchange energy and, depending on whether the system is at a sum or a difference resonance, the beam blows up or is stable as energy is exchanged between the modes indefinitely [26]. It will be interesting to see if analogous phenomena occur in the spin and orbit coordinates at spin orbit resonances. We will treat this case in another paper.

Remarks:

1) Neglecting the Stern-Gerlach terms coming from the component $\hat{\mathcal{H}}_{spin}$ the orbital part (eqns. (6.44a, b, c)) of the canonical equations (6.44) can be approximated as:

$$\frac{d}{ds}\,\hat{x} = +\frac{\partial\hat{\mathcal{H}}_{orb}}{\partial\hat{p}_x};\quad \frac{d}{ds}\,\hat{p}_x = -\frac{\partial\hat{\mathcal{H}}_{orb}}{\partial\hat{x}}; \tag{6.52a}$$

$$\frac{d}{ds}\,\hat{z} = +\frac{\partial\hat{\mathcal{H}}_{orb}}{\partial\hat{p}_z};\quad \frac{d}{ds}\,\hat{p}_z = -\frac{\partial\hat{\mathcal{H}}_{orb}}{\partial\hat{z}}; \tag{6.52b}$$

$$\frac{d}{ds}\,\hat{\sigma} \;=\; +\frac{\partial \hat{\mathcal{H}}_{orb}}{\partial \hat{p}_\sigma}\;;\quad \frac{d}{ds}\,\hat{p}_\sigma \;=\; -\frac{\partial \hat{\mathcal{H}}_{orb}}{\partial \hat{\sigma}}\,. \tag{6.52c}$$

This canonical system is then separate (and independent) from the spin motion and corresponds to the fully coupled 6-dimensional formalism [2,3].

If the orbit vector

$$\vec{\hat{y}}(s) \;=\; \begin{pmatrix} \hat{x} \\ \hat{p}_x \\ \hat{z} \\ \hat{p}_z \\ \hat{\sigma} \\ \hat{p}_\sigma \end{pmatrix}$$

is known, we can calculate the spin motion from the equations :

$$\frac{d}{ds}\,\hat{\alpha} \;=\; +\frac{\partial \hat{\mathcal{H}}_{spin}}{\partial \hat{\beta}}\;;\quad \frac{d}{ds}\,\hat{\beta} \;=\; -\frac{\partial \hat{\mathcal{H}}_{spin}}{\partial \hat{\alpha}} \tag{6.53}$$

or

$$\frac{d}{ds}\,\alpha \;=\; +\frac{\partial \hat{\mathcal{K}}_{spin}}{\partial \beta}\;;\quad \frac{d}{ds}\,\beta \;=\; -\frac{\partial \hat{\mathcal{K}}_{spin}}{\partial \alpha}\,, \tag{6.54}$$

where $\hat{\mathcal{K}}_{spin}$ is given by eqn. (6.34a). These spin-equations are again in canonical form and provide a method alternative to that in Ref. [27] for calculating the $\vec{n}$-axis, based on a canonical perturbation technique for investigating the "forced solution" of eqn. (6.53) or (6.54).

2) The perturbation of the orbit motion by SG forces is of $O(\hbar)\cdot(a\gamma+1)$ but the effect of the orbit on spin of order $(a\gamma+1)$. The fact that $\underline{B}$ and $\underline{C}$ are of similar order of magnitude is an artefact of the choice of canonical variables.

Note also that the $(\hat{x},\,\hat{p}_x)$, $(\hat{z},\,\hat{p}_z)$, $(\hat{\sigma},\,\hat{p}_\sigma)$ and $(\hat{\alpha},\,\hat{\beta})$ phase space areas all have the dimension of length.

3) The formalism presented here describes the effect of SG forces in all three (x, z, s) planes. In particular it automatically describes the effect of longitudinal field gradients on the transverse motion.

7 Summary

Following earlier works of Yokoya and Derbenev, we have used a classical Hamiltonian in a fixed Cartesian coordinate system for a spin 1/2 charged particle to investigate a canonical formalism of spin-orbit motion expressed in machine coordinates, taking into account all kinds of coupling induced by skew quadrupoles and solenoids (coupling of betatron motion), by a non-vanishing dispersion in the cavities (synchro-betatron coupling) and by Stern-Gerlach forces (spin-orbit coupling).

In addition to the well-known orbital variables $\hat{x}$, $\hat{p}_x$, $\hat{z}$, $\hat{p}_z$, $\hat{\sigma}$, $\hat{p}_\sigma$ of the fully coupled 6-dimensional formalism we introduce the canonical variables $\hat{\alpha}$ and $\hat{\beta}$ to describe the spin motion.

By expanding the Hamiltonian into a power series in these variables, one may obtain various orders of approximation for the canonical equations and the canonical structure of the formalism allows modern techniques such as Lie-algebra, normal forms and differential algebra to be included in a natural way. For example, the $\hat{\alpha}$ and $\hat{\beta}$ variables might simplify calculation of the $\vec{n}$-axis using normal forms [28,29].

The equations presented in this paper can serve to develop a non-linear, 8-dimensional (symplectic) tracking program for the combined spin-orbit system.

Such a program may be used to study (in addition to orbital problems) chaotic behaviour of spin motion and to investigate the influence of Stern-Gerlach forces.

In this paper we have treated motion in a storage ring, i.e. the average energy E_0 of the particles is constant. But it is easy to encompass acceleration by cavity fields in this formalism. For more details see Refs. [30,31].

Finally we remark that, starting from the variables $\hat{x}$, $\hat{p}_x$, $\hat{z}$, $\hat{p}_z$, $\hat{\sigma}$, $\hat{p}_\sigma$ $\hat{\alpha}$, $\hat{\beta}$ and using analytical techniques as described in Refs. [3,32,33] one can also develop an 8-dimensional dispersion formalism.

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Appendix A: Vector Potentials for various Lenses

Using the freedom to select a gauge, we can choose any vector potential which leads to the correct form of the fields. Suitable vector potentials are as follows and have been chosen for their simplicity [2].

A.1 Bending Magnet

Since the design orbit

$$x(s) = z(s) \equiv 0 \tag{A.1}$$

is a solution of the equations of motion for

$$\vec{\epsilon} = 0 \; ; \quad E \equiv E_0 \tag{A.2}$$

by definition, the magnetic bending field $B_x^{(0)}(s)$ and $B_z^{(0)}(s)$ is fixed by the curvatures K_x and K_z of the design orbit:

$$\frac{e}{p_0 \cdot c} \cdot B_x^{(0)} = -K_z \; ; \tag{A.3a}$$

$$\frac{e}{p_0 \cdot c} \cdot B_z^{(0)} = +K_x \; . \tag{A.3b}$$

The corresponding vector potential can be written as:

$$\frac{e}{p_0 \cdot c} \cdot A_s = -\frac{1}{2}(1 + K_x \cdot x + K_z \cdot z) \; ; \tag{A.4a}$$

$$A_x = A_z = 0 \; . \tag{A.4b}$$

A.2 Quadrupole

The quadrupole fields are

$$B_x = z \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} ; \quad \text{(A.5a)}$$

$$B_z = x \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} , \quad \text{(A.5b)}$$

so that we may use the vector potential

$$A_s = \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \cdot \frac{1}{2}\left(z^2 - x^2\right) ; \quad \text{(A.6a)}$$

$$A_x = A_z = 0 . \quad \text{(A.6b)}$$

We rewrite this as:

$$\frac{e}{p_0 \cdot c} A_s = \frac{1}{2} g \cdot (z^2 - x^2) \quad \text{(A.7a)}$$

with

$$g = \frac{e}{p_0 \cdot c} \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} . \quad \text{(A.7b)}$$

A.3 Skew Quadrupole

The fields are

$$B_x = -\frac{1}{2} \cdot \left(\frac{\partial B_z}{\partial z} - \frac{\partial B_x}{\partial x}\right)_{x=z=0} \cdot x ; \quad \text{(A.8a)}$$

$$B_z = +\frac{1}{2} \cdot \left(\frac{\partial B_z}{\partial z} - \frac{\partial B_x}{\partial x}\right)_{x=z=0} \cdot z . \quad \text{(A.8b)}$$

Thus we may use

$$A_s = -\frac{1}{2} \left(\frac{\partial B_z}{\partial z} - \frac{\partial B_x}{\partial x}\right)_{x=z=0} \cdot xz ; \quad \text{(A.9a)}$$

$$A_x = A_z = 0 , \quad \text{(A.9b)}$$

and we write :

$$\frac{e}{p_0 \cdot c} A_s = N \cdot xz ; \quad \text{(A.10a)}$$

$$N = \frac{1}{2} \cdot \frac{e}{p_0 \cdot c} \left(\frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z}\right)_{x=z=0} . \quad \text{(A.10b)}$$

A.4 Solenoid Fields

The field components in the current free region are given by [2,34]:

$$B_x(x,z,s) = x \cdot \sum_{\nu=0}^{\infty} b_{2\nu+1} \cdot (x^2+z^2)^\nu \; ; \tag{A.11a}$$

$$B_z(x,z,s) = z \cdot \sum_{\nu=0}^{\infty} b_{2\nu+1} \cdot (x^2+z^2)^\nu \; ; \tag{A.11b}$$

$$B_s(x,z,s) = \sum_{\nu=0}^{\infty} b_{2\nu} \cdot (x^2+z^2)^\nu \tag{A.11c}$$

where for consistency with Maxwell's equations the coefficients b_μ obey the recursion equations:

$$b_{2\nu+1}(s) = -\frac{1}{(2\nu+2)} \cdot b'_{2\nu}(s) \; ; \tag{A.12a}$$

$$b_{2\nu+2}(s) = +\frac{1}{(2\nu+2)} \cdot b'_{2\nu+1}(s) \; ; \tag{A.12b}$$

$$(\nu = 0, 1, 2, ...)$$

and where

$$b_0(s) \equiv B_s(0,0,s) \; . \tag{A.13}$$

The vector potential leading to the solenoid field of eqn. (A.11) is then:

$$A_x(x,z,s) = -z \cdot \sum_{\nu=0}^{\infty} \frac{1}{(2\nu+2)} \cdot b_{(2\nu)}(s) \cdot r^{2\nu} \; ; \tag{A.14a}$$

$$A_z(x,z,s) = +x \cdot \sum_{\nu=0}^{\infty} \frac{1}{(2\nu+2)} \cdot b_{(2\nu)}(s) \cdot r^{2\nu} \; ; \tag{A.14b}$$

$$A_s(x,z,s) = 0 \tag{A.14c}$$

with

$$r^2 = x^2 + z^2 \; .$$

Thus we can write :

$$\frac{e}{E_0} A_x = -\beta_0 \cdot H(s) \cdot z + \frac{1}{8}\beta_0 \cdot H''(s) \cdot (x^2+z^2) \cdot z + \cdots \; ; \tag{A.15a}$$

$$\frac{e}{E_0} A_z = +\beta_0 \cdot H(s) \cdot x - \frac{1}{8}\beta_0 \cdot H''(s) \cdot (x^2+z^2) \cdot x + \cdots \tag{A.15b}$$

with

$$\begin{aligned} H(s) &= \frac{1}{2} \cdot \frac{e}{p_0 \cdot c} \cdot b_0(s) \\ &\equiv \frac{1}{2} \cdot \frac{e}{p_0 \cdot c} \cdot B_s(0,0,s) \; . \end{aligned} \tag{A.16}$$

Note that the cyclotron radius for the longitudinal field (A.13) is given by

$$R = \frac{1}{2 \cdot H} \; .$$

A.5 Dipole

$$A_s = \Delta B_x \cdot z - \Delta B_z \cdot x$$

with

$$\Delta B_x = \; = \; \sum_\mu \Delta \hat{B}_x^{(\mu)} \cdot \delta(s - s_\mu) \; ; \tag{A.17a}$$

$$\Delta B_z = \; = \; \sum_\mu \Delta \hat{B}_z^{(\mu)} \cdot \delta(s - s_\mu) \tag{A.17b}$$

so that

$$\frac{e}{p_0 \cdot c} A_s = \frac{e}{p_0 \cdot c} \cdot \sum_\mu \delta(s - s_\mu) \cdot \left[\Delta \hat{B}_x^{(\mu)} \cdot z - \Delta \hat{B}_z^{(\mu)} \cdot x \right] \; . \tag{A.18}$$

A.6 Cavity Field

For a longitudinal electric field

$$\begin{aligned} \varepsilon_x &= 0 \; ; \\ \varepsilon_z &= 0 \; ; \\ \varepsilon_s &= \epsilon(s, \sigma) \end{aligned} \tag{A.19}$$

we write:

$$\begin{aligned} A_x &= 0 \; ; \\ A_z &= 0 \; ; \\ A_s &= \frac{1}{\beta_0} \cdot \int_{\sigma_0}^{\sigma} d\tilde{\sigma} \cdot \varepsilon(s, \tilde{\sigma}) \; , \end{aligned} \tag{A.20}$$

which by (4.25) immediately gives ε_s.

Now the cavity field may be represented by

$$\varepsilon(s, \sigma) = V(s) \sin \left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi \right] \tag{A.21}$$

and we obtain using (A.20):

$$A_s = -\frac{1}{\beta_0} \cdot \frac{L}{2\pi \cdot h} \cdot V(s) \cdot \cos \left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi \right] \; , \tag{A.22}$$

in which the phase φ is defined so that the average energy radiated away in the bending magnets is replaced by the cavities and h is the harmonic number.

Appendix B: The Periodic Spin Frame $(\vec{n}_0, \vec{m}, \vec{l})$ along the Closed Orbit

In order to define the periodic spin frame, we first introduce a compact matrix notation. Rewriting an arbitrary vector

$$\vec{A} = A_s \cdot \vec{e}_s + A_x \cdot \vec{e}_x + A_z \cdot \vec{e}_z$$

as a column vector with components A_s, A_x, A_z:

$$A_s \cdot \vec{e}_s + A_x \cdot \vec{e}_x + A_z \cdot \vec{e}_z = \begin{pmatrix} A_s \\ A_x \\ A_z \end{pmatrix}$$

and defining the derivative of a column vector with respect to the arc length s as the derivative of the corresponding components A_i but not of the unit vectors :

$$\frac{d}{ds} \begin{pmatrix} A_s \\ A_x \\ A_z \end{pmatrix} \equiv \vec{e}_s \cdot \frac{d}{ds} A_s + \vec{e}_x \cdot \frac{d}{ds} A_x + \vec{e}_z \cdot \frac{d}{ds} A_z$$

we get from (5.17) and (6.2b) :

$$\frac{d}{ds} \vec{\xi}^{(0)}(s) = \underline{\Omega}^{(0)} \cdot \vec{\xi}^{(0)}(s) \tag{B.1}$$

where we have set

$$\vec{\xi}^{(0)} = \begin{pmatrix} \xi_{0s} \\ \xi_{0x} \\ \xi_{0z} \end{pmatrix} \tag{B.2a}$$

and

$$\underline{\Omega}^{(0)}(s) = \begin{pmatrix} 0 & -\Omega_z^{(0)} & \Omega_x^{(0)} \\ \Omega_z^{(0)} & 0 & -\Omega_s^{(0)} \\ -\Omega_x^0 & \Omega_s^{(0)} & 0 \end{pmatrix} . \tag{B.2b}$$

The transfer matrix $\underline{M}_{(spin)}(s, s_0)$ for the spin motion defined by

$$\vec{\xi}^{(0)}(s) = \underline{M}_{(spin)}(s, s_0) \cdot \vec{\xi}^{(0)}(s_0)$$

satisfies the relationships:

$$\underline{M}^T_{(spin)}(s, s_0) \cdot \underline{M}_{(spin)}(s, s_0) \;=\; \underline{1} \; ; \tag{B.3a}$$

$$\det \left[\underline{M}_{(spin)}(s, s_0)\right] = 1 \tag{B.3b}$$

since (with eqn. (B.1))

$$\frac{d}{ds} \underline{M}_{(spin)}(s, s_0) \;=\; \underline{\Omega}^{(0)}(s) \cdot \underline{M}_{(spin)}(s, s_0) \; ;$$

$$\underline{M}_{(spin)}(s_0, s_0) \;=\; \underline{1}$$

and therefore (with $[\underline{\Omega}^{(0)}]^T = -\underline{\Omega}^{(0)}$)

$$\begin{aligned}
\frac{d}{ds}\left[\underline{M}^T_{(spin)}(s,s_0)\cdot\underline{M}_{(spin)}(s,s_0)\right] &= \left[\underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(spin)}(s,s_0)\right]^T\cdot\underline{M}_{(spin)}(s,s_0) \\
&\quad +\underline{M}^T_{(spin)}(s,s_0)\cdot\left[\underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(spin)}(s,s_0)\right] \\
&= \underline{M}_{(spin)}(s,s_0)^T\cdot\underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(spin)}(s,s_0) \\
&\quad +\underline{M}^T_{(spin)}(s,s_0)\cdot\underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(spin)}(s,s_0) \\
&= \underline{0} ;
\end{aligned}$$

$$\det M(s,s_0) = \det M(s_0,s_0) = 1 ,$$

i.e. $\underline{M}_{(spin)}(s,s_0)$ is an orthogonal matrix with determinant 1.

Let us now consider the eigenvalue problem for the revolution matrix $\underline{M}(s_0+L,s_0)$ with the eigenvalues α_μ and eigenvectors $\vec{r}_\mu(s_0)$:

$$\underline{M}(s_0+L,s_0)\ \vec{r}_\mu(s_0) = \alpha_\mu\cdot\vec{r}_\mu(s_0) ; \qquad (\mu = 1,\ 2,\ 3). \tag{B.4}$$

Because of (B.3a,b) we can write [23,35]:

$$\begin{aligned}
\alpha_1 &= 1 ; \\
\alpha_2 &= e^{i\cdot 2\pi\cdot Q_{spin}} ; \\
\alpha_3 &= e^{-i\cdot 2\pi\cdot Q_{spin}} ;
\end{aligned} \tag{B.5}$$

$$(Q_{spin} = \text{real number})$$

and

$$\vec{r}_1(s_0) = \vec{n}_0(s_0) ; \tag{B.6a}$$
$$\vec{r}_2(s_0) = \vec{m}_0(s_0) + i\cdot\vec{l}_0(s_0) ; \tag{B.6b}$$
$$\vec{r}_3(s_0) = \vec{m}_0(s_0) - i\cdot\vec{l}_0(s_0) ; \tag{B.6c}$$

$$(\vec{n}_0,\vec{m}_0,\vec{l}_0 = \text{real vectors}) .$$

If we require that

$$\vec{r}_1^{\,+}\cdot\vec{r}_1 = 1 ; \tag{B.7a}$$
$$\vec{r}_2^{\,+}\cdot\vec{r}_2 \equiv \vec{r}_3^{\,+}\cdot\vec{r}_3 = 2 ; \tag{B.7b}$$

(normalizing conditions)

we find, using also eqn. (B.3a) [35]:

$$|\vec{n}_0(s_0)| = |\vec{m}_0(s_0)| = |\vec{l}_0(s_0)| = 1 ; \tag{B.8a}$$

$$\vec{n}_0(s_0) \perp \vec{m}_0(s_0) \perp \vec{l}_0(s_0) . \tag{B.8b}$$

Thus the vectors $\vec{n}_0(s_0)$, $\vec{m}_0(s_0)$ and $\vec{l}_0(s_0)$ form an orthogonal system of unit vectors. Choosing the direction of $\vec{n}_0(s_0)$ such that

$$\vec{n}_0(s_0) \; = \; \vec{m}_0(s_0) \times \vec{l}_0(s_0) \tag{B.8c}$$

these vectors form a right-handed coordinate system.

In this way we have found a coordinate frame for the position $s = s_0$.

An orthogonal system of unit vectors at an arbitrary position s can be defined by applying the transfer matrix $\underline{M}_{(spin)}(s, s_0)$ to the vectors $\vec{n}_0(s_0)$, $\vec{m}_0(s_0)$ and $\vec{l}_0(s_0)$:

$$\vec{n}_0(s) \; = \; \underline{M}_{(spin)}(s, s_0)\; \vec{n}_0(s_0) \; ; \tag{B.9a}$$

$$\vec{m}_0(s) \; = \; \underline{M}_{(spin)}(s, s_0)\; \vec{m}_0(s_0) \; ; \tag{B.9b}$$

$$\vec{l}_0(s) \; = \; \underline{M}_{(spin)}(s, s_0)\; \vec{l}_0(s_0) \; . \qquad \text{(B.9b)}$$

Because of eqn. (B.3a,b) the orthogonality relations remain unchanged:

$$\vec{n}_0(s) \; = \; \vec{m}_0(s) \times \vec{l}_0(s) \tag{B.10a}$$

$$\vec{m}_0(s) \; \perp \; \vec{l}_0(s) \; ; \tag{B.10b}$$

$$|\vec{n}_0(s)| \; = \; |\vec{m}_0(s)| \; = \; |\vec{l}_0(s)| \; = \; 1 \; . \tag{B.10c}$$

The coordinate frame defined by $\vec{n}_0(s)$, $\vec{m}_0(s)$ and $\vec{l}_0(s)$ is not yet appropriate for a description of the spin motion, because it does not transform into itself after one revolution of the particles:

$$\begin{aligned}
\vec{m}_0(s_0 + L) + i\vec{l}_0(s_0 + L) \; &= \; \underline{M}_{(spin)}(s_0 + L, s_0)\left[\vec{m}_0(s_0) + i\vec{l}_0(s_0)\right] \\
&= \; e^{i \cdot 2\pi \cdot Q_{spin}} \cdot \left[\vec{m}_0(s_0) + i\vec{l}_0(s_0)\right] \\
&\neq \; \vec{m}_0(s_0) + i\vec{l}_0(s_0)
\end{aligned}$$

$$(\text{if } \; Q_{spin} \neq \text{integer}).$$

But by introducing a phase function $\psi_{spin}(s)$ and using another orthogonal matrix $\underline{D}(s, s_0)$:

$$\underline{D}(s, s_0) = \begin{pmatrix} \cos[\psi_{spin}(s) - \psi_{spin}(s_0)] & \sin[\psi_{spin}(s) - \psi_{spin}(s_0)] \\ -\sin[\psi_{spin}(s) - \psi_{spin}(s_0)] & \cos[\psi_{spin}(s) - \psi_{spin}(s_0)] \end{pmatrix} \tag{B.11}$$

with

$$\underline{D}^T(s, s_0) \cdot \underline{D}(s, s_0) \; = \; \underline{1} \; ; \tag{B.12a}$$

$$\det\, [\underline{D}(s, s_0)] = 1 \tag{B.12b}$$

we can construct a periodic orthogonal system of unit vectors from $\vec{n}_0(s)$, $\vec{m}_0(s)$ and $\vec{l}_0(s)$. Namely, if we put [35]:

$$\begin{pmatrix} \vec{m}(s) \\ \vec{l}(s) \end{pmatrix} = \underline{D}(s, s_0) \begin{pmatrix} \vec{m}(s_0) \\ \vec{l}(s_0) \end{pmatrix}$$

$$\Longrightarrow \quad \vec{m}(s) + i\vec{l}(s) = e^{-i \cdot [\psi_{spin}(s) - \psi_{spin}(s_0)]} \cdot \left[\vec{m}_0(s) + i\vec{l}_0(s)\right] \neq \vec{m}_0(s_0) + i\vec{l}_0(s_0) \tag{B.13}$$

we find, using eqns. (B.12a, b):

$$\vec{n}_0(s) = \vec{m}(s) \times \vec{l}(s) \tag{B.14a}$$

$$\vec{m}(s) \perp \vec{l}(s) \; ; \tag{B.14b}$$

$$|\vec{n}_0(s)| = |\vec{m}(s)| = |\vec{l}(s)| = 1 \; . \tag{B.14c}$$

Since

$$\vec{m}(s_0 + L) + i\vec{l}(s_0 + L) = e^{-i \cdot [\psi_{spin}(s_0 + L) - \psi_{spin}(s_0)]} \cdot \left[\vec{m}(s_0) + i\vec{l}(s_0)\right]$$

it follows that the condition of periodicity for $\vec{n}_0$, $\vec{m}$ and $\vec{l}$:

$$\left(\vec{n}_0, \; \vec{m}, \; \vec{l}\right)_{s=s_0+L} = \left(\vec{n}_0, \; \vec{m}, \; \vec{l}\right)_{s=s_0} \tag{B.15}$$

can indeed be fulfilled if the phase function $\psi_{spin}(s)$ satisfies the relationship:

$$\psi_{spin}(s_0 + L) - \psi_{spin}(s) = 2\pi \cdot Q_{spin} \; ; \tag{B.16a}$$

$$(Q_{spin} = \text{spin tune}).$$

For instance we can choose:

$$\psi_{spin}(s) = 2\pi \cdot Q_{spin} \cdot \frac{s}{L} \; . \tag{B.16b}$$

Taking the derivatives of $\vec{m}(s)$ and $\vec{l}(s)$ with respect to s, and taking into account eqns. (B.13), (B.9), and (B.1) we get

$$\frac{d}{ds}\, \vec{m}(s) = \underline{\Omega}^{(0)}(s)\; \vec{m}(s) + \psi\,'(s) \cdot \vec{l}(s) \; ; \tag{B.17a}$$

$$\frac{d}{ds}\, \vec{l}(s) = \underline{\Omega}^{(0)}(s)\; \vec{l}(s) - \psi\,'(s) \cdot \vec{m}(s) \tag{B.17b}$$

and $\vec{n}_0(s)$ satisfies (see (B.9a))

$$\frac{d}{ds}\, \vec{n}_0(s) = \underline{\Omega}^{(0)}(s)\; \vec{n}_0(s) \; . \tag{B.17c}$$

Finally the vectors

$$\vec{r}_1(s) = \vec{n}_0(s) \equiv \underline{M}_{(spin)}(s, s_0)\; \vec{r}_1(s_0) \; ; \tag{B.18a}$$

$$\vec{r}_2(s) = \vec{m}_0(s) + i\vec{l}_0(s) \equiv \underline{M}_{(spin)}(s, s_0)\; \vec{r}_2(s_0) \; ; \tag{B.18b}$$

$$\vec{r}_3(s) = \vec{m}_0(s) - i\vec{l}_0(s) \equiv \underline{M}_{(spin)}(s, s_0)\; \vec{r}_3(s_0) \tag{B.18c}$$

are eigenvectors of the revolution matrix $\underline{M}_{(spin)}$ with the same eigenvalues as in (B.5):

$$\underline{M}(s + L, s)\; \vec{r}_\mu(s) = \alpha_\mu \cdot \vec{r}_\mu(s) \; . \tag{B.19}$$

Thus, the eigenvalues α_μ and the quantity Q_{spin} defined by eqn. (B.5) are independent of the chosen initial position s_0.

Remark:

In order to solve eqn. (B.1), the 6-dimensional orbit vector $\vec{y}_0$ must be known (see eqn. (6.3)). This vector can be approximated by neglecting the Stern-Gerlach term

$$\frac{\partial}{\partial \vec{y}_0} \overline{\overline{\mathcal{K}}}_{spin}(\vec{y}_0; \psi_0, J_0; s)$$

in eqn. (6.2a), giving:

$$\frac{d}{ds}\vec{y}_0 = -\underline{S} \cdot \frac{\partial}{\partial \vec{y}_0} \overline{\overline{\mathcal{K}}}_{orb}(\vec{y}_0; s) \ . \tag{B.20}$$

The error in calculating $\vec{\xi}_0(s)$ is of order $\hbar^2$ which we can neglect at our semiclassical level of approximation. A solution of (B.1) may then be obtained by using the method of thin-lens approximation as described in Refs. [10,36].

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