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Renormalized Loop Expansion for the Gauge Covariant Constraint Effective Higgs Potential on the Lattice

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Abstract

Knowledge of the constraint effective potential for a Higgs-field permits to determine the phase structure of the theory and to extract physical parameters (masses, coupling constants). Here, we compute the constraint effective potential by means of the renormalized loop expansion on a lattice.

We consider first the single component ϕ^4-theory with spontaneous symmetry breaking. In this model, close to the critical line, finite size effects can be computed by renormalized loop expansions if the renormalized mass and coupling constant are known. In order to do so, the constraint effective potential is computed to two loop order. Using this expression, we are able to extract results for renormalized masses and coupling constants from Multigrid Monte Carlo data of Mack and Meyer for the constraint effective potential close to the critical line, and compare them with the analytical results of Lüscher and Weisz. Perfect agreement is found.

Next, the standard lattice $SU(2)$-Higgs model is studied. We generalize the definition of the constraint effective potential to theories with non-Abelian gauge fields. It depends on a gauge covariant magnetization. We develop the formalism for the loop expansion. The constraint effective potential is computed up to one loop in the scalar field and order g^2 in the gauge coupling constant. From its shape, the critical surface is found. It is shown that in the infinite volume and large cutoff limit, the critical surface corresponds to a weak first-order phase transition. We find also a lower bound for the Higgs mass. Our results in this limit agree with the analysis of A. and P. Hasenfratz for the conventional effective potential performed in a particular gauge (the Landau gauge).

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Contents

1 Introduction

The constraint effective potential (CEP) contains a great deal of information about a lattice field theory. It was introduced by Fukuda and Kyriakoloulos [1] and further analysed by O'Raifertaigh, Wipf and Yoneyama [2] for theories without dynamical gauge fields. It is a not necessarily convex function of the magnetization. It determines the free energy as a function of the magnetization, and becomes equal to it in the large volume limit, provided one divides by the volume. Knowledge of the constraint effective potential locates phase transitions and determines their order, it also determines renormalized masses and coupling constants [3]. The β functions also can be derived from the CEP [4]. Therefore it is important to have methods to compute and analyse the constraint effective potential for various models.

In the first part of this paper, single component ϕ^4-theory is considered with bare mass m_0^2 and bare coupling constant λ_0 on a lattice whose lattice spacing is set equal to 1. This model has a critical line where the renormalized mass vanishes.

It is known [5] that the renormalized coupling constant λ_R is always rather small near the critical line, whatever the values of the bare parameters are, with a maximal value of about $2/3$ of the tree unitarity bound. This is true whenever the UV-cutoff is bigger than twice the renormalized mass (scaling region). Also, since the Callan Symanzik β-function is positive, the renormalization group equation drives λ_R to smaller values as m_R decreases. Therefore the renormalized perturbation theory becomes an even better approximation as the system goes to the critical line. In conclusion, the renormalized loop expansion can be expected to describe very accurately the low energy properties of the theory, at least in the scaling region where the UV-cutoff is high enough.

This observation does not amount to a complete solution of the model near the critical line yet. The renormalized loop expansion requires knowledge of the renormalized parameters (mass and coupling constant), and there remains the highly nontrivial problem of finding the renormalized parameters as functions of the bare parameters m_0^2 and λ_0. Lüscher and Weisz [5] were able to solve this problem by an ingenious combination of high temperature expansions and standard perturbation theory. They published also estimates of errors of their calculation. These errors are generally very small.

In this paper, a comparison of this calculation with numerical results for constraint effective potentials is reported. The numerical results were obtained by Mack and Meyer [6] using a Multigrid Monte Carlo method (MGMC). They computed the constraint effective potential on a 12^4 lattice for fixed bare coupling constant and 3 values of the bare mass very near to the critical line and on both sides of it. In this way the crossover of the constraint effective potential from single well shape to a double well behaviour in the vicinity of the second order phase transition could be studied.

The multigrid Monte Carlo method was designed to beat critical slowing down and this was indeed essential for the success of the numerical calculation.

Several methods have been proposed to beat critical slowing down [7]. The multigrid method of Meyer and Mack has the additional advantage that it permits very convenient calculation

of the constraint effective potential [8] (see Appendix A).

Ordinarily, physical results can be extracted from numerical data only if the lattice size is large compared to the correlation length. This condition was not fulfilled in the computation of Mack and Meyer. However, because the renormalized loop expansion is expected to be accurate near criticality, one can compute the constraint effective potential for arbitrary finite volume as a function of the renormalized coupling constants on an infinite lattice, and fit the result to the numerical data for the constraint effective potential. In this way the renormalized mass and coupling constant for the infinite volume theory can be extracted from the numerical data on a finite volume. Comparing with the result of Lüscher and Weisz gives a meaningful test of that part of their calculation that transcends perturbation theory.

Perfect agreement is found. The "numerical" values of the renormalized parameters agree with those of Lüscher and Weisz within the errors quoted by these authors (see table in section 2.6). The accuracy of our calculation is confirmed by the fact that the two loop corrections are negligibly small.

The first part of this paper is organized as follows: subsections 2.1 - 2.5 give a detailed description of the expansion for the constraint effective potential as a function of the renormalized quantities m_R and λ_R on an infinite lattice. The results are presented and conclusions are discussed in subsection 2.6.

In the second part of this paper (section 3.1), a gauge invariant definition of the constraint effective potential for lattice theories which include dynamical non-Abelian gauge fields is proposed. It is a natural extension of the definition for scalar theories. It involves a gauge covariant generalization of the magnetization. In the $g^2 = 0$ limit (absence of gauge field), the gauge covariant magnetization becomes the ordinary magnetization.
The gauge covariant magnetization Φ is obtained by the projection of the scalar field upon the eigenspace which is associated with the smallest eigenvalue of the gauge covariant lattice Laplace operator.
One single out a point $z = 0$ on the lattice. Under gauge transformations, Φ transforms like a Higgs field at the point z. More explicitly, the gauge covariant magnetization is defined as a gauge covariant average of the Higgs field [‡]

$$\Phi = \int_{x\in\Lambda} C(z)\,\phi(z) = C\phi \tag{1.1}$$

The averaging kernel $C(z)$ depends on the lattice gauge field A. We consider a Higgs doublet and a $SU(2)$ gauge field. If a 2 x 2 matrix representation of the Higgs field is used, the $C(z)$ is also a 2 x 2 matrix. It is a solution of the equation

$$-\Delta_U C(z) = \epsilon_0 C(z) \tag{1.2}$$

where $-\Delta_U$ is minus the gauge covariant lattice Laplacian, and ϵ_0 is its lowest eigenvalue and depends on the gauge field U. The arbitrariness in the solution is removed by demanding

$$\int_z C(z)C^\dagger(z) = 1\,, \quad C(0) = r\,\mathbb{1} \tag{1.3}$$

[‡]We use lattice notations as follows. On a lattice Λ of lattice spacing a, we write e_μ for the (lattice) vector of length a in μ-direction: $e_\mu = -e_{-\mu}$. In d dimensions $\int_{x\in\Lambda} = a^d \sum_{x\in\Lambda}$

for some positive real constant r. Actually, the last condition does not affect the constraint effective potential as we will show in subsection 3.1.3. The constraint effective potential is therefore independent of the choice of the point z.

The result of this paper shows that this definition of a gauge covariant magnetization is suitable for perturbative computations, in spite of the fact that it looks complicated at first sight.

The constraint effective potential $U_V(\Phi)$ in volume V in the presence of gauge fields is still defined in the customary way

$$\exp\left(-V\,U_V(\Phi)\right) = \int \mathcal{D}\mathcal{U}\,\mathcal{D}\phi\,\delta(\Phi - C\phi)\exp(-S\,[\phi, A]) \tag{1.4}$$

where $\mathcal{D}\mathcal{U}$ is the invariant Haar-measure, and S is the action.
Since Φ is a a real multiple of a $SU(2)$ matrix, this involves a δ-function in 4 real variables.

To perform a perturbative computation of the integral (1.4), one starts with a gauge covariant shift of the field ϕ, so that the shifted field ξ has gauge covariant average 0

$$\phi(z) = \mathcal{A}(z)\,\Phi + \xi(z) \tag{1.5}$$

with $\mathcal{A}(z) = C^\dagger(z)$. The normalization condition (1.3) implies that $C\mathcal{A} = 1$. Therefore $C\xi = 0$. Inserting the covariant shift of eqn. (1.5) in the action, it turns out that the action can be written as

$$\begin{aligned} S[\phi, A] &= S[\mathcal{A}\Phi, A] + \xi - \text{dep. terms} \\ &= \tfrac{1}{4}\,F^{-}_{\mu\nu}\,F_{\mu\nu} + V[\tfrac{1}{2}(\epsilon_0^{(2)} + m_0^2) < \Phi\,|\,\Phi > + \tfrac{\lambda_0}{4!} < \Phi\,|\,\Phi >^2] \\ &\quad + O(g^3) + \xi - \text{dep. terms} \end{aligned} \tag{1.6}$$

where $\frac{1}{4}\,F^{-}_{\mu\nu}\,F_{\mu\nu}$ is the Wilson action for the gauge field and $\epsilon_0^{(2)}$ is the contribution of second order in the gauge field A to the eigenvalue of minus the covariant Laplacian. The vector boson mass $m_W(\Phi)$ comes precisely from this contribution

$$m_W^2(\Phi) = \frac{1}{4}\,g^2\,\Phi^2 \tag{1.7}$$

where Φ is the abolute value of the covariant magnetization $\Phi = \|\Phi\|$.

In order to investigate the phase structure of the model, the necessary formalism for a lattice loop expansion is developed and an explicit computation is performed to one loop order in the scalar field and to order g^2 in the gauge coupling constant. It turns out that only a second order expansion in the gauge coupling constant g for the operator C is necessary to this order. In the infinite lattice volume limit, the one loop potential agrees with the expression obtained by Montvay [9] for the conventional effective potential in the Landau gauge (the conventional effective potential is the Legendre transform of the Schwinger functions).

According to the philosophy based on the triviality of the lattice $SU(2)$-Higgs theory, the scalar and gauge coupling constants λ_R and g_R are small near the critical surface, whatever

the values of the bare parameters are. Moreover, the renormalized coupling constant λ_R and g_R are driven by the Callan Symanzik equation to smaller values as the cutoff increases. Therefore, this one loop expression for the CEP is expected to be reliable in the scaling region [10].

In the infinite volume and large cutoff limit, our result for the constraint effective potential reproduces the known results of Anna and Peter Hasenfratz [11] for the conventional effective potential in the Landau gauge. It is to be noted, however, that the conventional effective potential is gauge dependent, whereas the constraint effective potential is gauge invariant. It appears that the Landau gauge was a fortunate choice in the earlier computation of the conventional effective potential.

From the analysis of the one loop expression, it follows that the critical line is localized in the region where $\lambda_R \sim O(g_R^4)$. This is within the domain of validity of the loop expansion. It turns out that the CEP has more than one relative minimum and the absolute minimum of the constraint effective potential jumps from zero to $\Phi_{min} \neq 0$ when one varies the parameters across the critical surface. This shows that the phase transition from the symmetric to the Higgs phase is of first order (see figure 3 of section 3.5). Using the stability condition of the minimum, a lower bound for the Higgs mass is obtained.

In summary, in this region, the Coleman Weinberg picture [12] is confirmed: the radiative corrections produce spontaneous symmetry breakdown although the tree level does not indicate such breakdown.

This analysis can also be extended to other regions of the parameter space and for finite volume in order to find the position and order of the phase transition. Masses, coupling constants, etc., can also be obtained, provides one knows the expression for the wave function renormalization constant. (Its one loop expression in the Landau gauge can be found in ref. [9]).

From the analysis of one loop renormalized constraint effective potential for finite volume, finite size effects numerical simulations of this model for larger bare coupling constants, can be determinated in the same maner as for the ϕ^4-theory. It would therefore be very interesting to have accurate numerical data for the CEP near the critical surface. It is not necessary to make the lattice very large.

Finally, we remark that if the lattice $SU(2)$-Higgs theory is in fact a non-interacting field theory in the continuum [13] (there is a lot of numerical and analytical evidences that support this view), then it is desirable to have expressions such as the result of this paper for lattices with finite lattice spacing.

The computation of the CEP proceeds in several steps: in section 3.1, the lattice $SU(2)$-Higgs model is introduced and some comments on the definition of block spins for Higgs fields for the theories with dynamical non-Abelian gauge fields are given. Using perturbation theory, the averaging and interpolating operators C and $\mathcal{A}$ are computed to order g^2 in section 3.2. In section 3.3, the general idea for the scalar loop expansion is explained. This expansion amounts to performing the integration over the scalar fields. Section 3.4 is devoted to explain the gauge and ghost loop expansion and the notion of naive continuum limit. Finally, results

are presented and discussed in section (3.5).

2 Constraint effective potential for the lattice ϕ^4-theory

We begin with the discussion of the single-component ϕ^4-theory without gauge fields.

We recall the definition of the constraint effective potential introduced by Fukuda and Kyriakopoulos:

$$\exp\left(-\frac{V}{\hbar}U_V(\Phi)\right) = \int \mathcal{D}[\phi]\,\delta(\Phi - C\,\phi)\exp\left(-\frac{S}{\hbar}[\phi]\right) \tag{2.1}$$

where V is the total volume of the system, $C\phi = \frac{1}{V}\int_V \phi$ is the average of ϕ in V and $S[\phi]$ is the action of the system.
Since the delta function introduced into the functional integral in (2.1) constrains the average value of the field ϕ to be Φ, one calls $U_V(\Phi)$ the constraint effective potential (CEP). It is not necessarily convex for finite V, but becomes convex as V goes to infinity.

Also the conventional effective potential $\Gamma_V(\Phi)$ used to study the radiative corrections in a quantum field theory [12] can be recovered as its convex hull [2], so that no information is lost. In fact, one knows the generating functional $W[J]$ (Schwinger functional for constant external source J) is given through the integral:

$$\exp\left(\frac{V}{\hbar}W[J]\right) = \int \mathcal{D}[\Phi]\,\exp\left\{\frac{V}{\hbar}\left(J\Phi - U_V(\Phi)\right)\right\} \tag{2.2}$$

Through its Legendre transformation, one gets the effective potential

$$\Gamma_V(\Phi) = \sup_J\left(J\Phi - W[J]\right) \tag{2.3}$$

which is always convex and whose expansion generates the one-particle irreducible n-points vertex functions.

We will now consider the lattice action for the ϕ^4- theory defined by:

$$S[\phi] = \sum_z \left\{\frac{1}{2}\sum_{\mu=0}^{3}\left(D_\mu\phi(z)\right)^2 + \frac{1}{2}m_0^2\,\phi^2(z) + \frac{\lambda_o}{4!}\,\phi^4(z)\right\} \tag{2.4}$$

where $\phi(z)$ is a real valued field defined on the hypercubic lattice with $z \in \Lambda$, m_0, λ_0 are the bare mass and coupling constant respectively and $D_\mu\,\phi(z)$ is the lattice derivative.

The lattice ϕ^4-theory is known to exist in two phases, one where the reflection symmetry $\phi \to -\phi$ is spontaneously broken and the other where it is not. Therefore, in order to expand around the true minimum of the theory, we will shift the fundamental field ϕ:

$$\phi(z) \to \Phi + \xi(z) \tag{2.5}$$

and the field $\xi(z)$ is required to satisfy the requirement that its zero momentum Fourier component $\tilde{\xi}(\hat{k}=0)=0$ vanishes:

$$\int_V \xi = 0 \quad \Rightarrow \quad \tilde{\xi}(\hat{k}=0)=0 \tag{2.6}$$

2.1 General loop expansion

Now we substitute the action (2.5) into the eq.(2.1) and, expressing the integrand as function of the field ξ, we get:

$$\exp\left(-\tfrac{V}{\hbar}U_V(\Phi)\right) = \exp\left(-\tfrac{V}{\hbar}U_{cl}(\Phi)\right)\int \mathcal{D}[\xi]\,\exp\left(-\tfrac{1}{\hbar}\sum_{zw}\xi(z)M(z,w)\xi(w)\right)\times \exp\left(-\tfrac{\lambda_0}{4!\hbar}\sum_z\left(4\Phi\xi^3(z)+\xi^4(z)\right)\right)\ , \tag{2.7}$$

where

$$U_{cl}(\Phi) = \frac{m_0^2}{2!}\Phi^2 + \frac{\lambda_0}{4!}\Phi^4 \tag{2.8}$$

is the classical potential and

$$S_0(\xi) = \sum_{zw}\xi(z)\frac{1}{2}\left\{D^\mu D_\mu + m_0^2 + \frac{\lambda_0}{2}\Phi^2\right\}\xi(w)\,\delta_{z,w} = \sum_{zw}\xi(z)\,M(z,w)\,\xi(w) \tag{2.9}$$

is the whole quadratic contribution to the action.
We make a Taylor expansion of the last term of Eq.(2.7):

$$\exp\left(-\tfrac{V}{\hbar}U_V(\Phi)\right) = \exp\left(-\tfrac{V}{\hbar}U_{cl}(\Phi)\right)\int \mathcal{D}[\xi]\,\exp\left(-\tfrac{1}{\hbar}\sum_{zw}\xi(z)M(z,w)\xi(w)\right)\left\{1-\tfrac{\lambda_0}{4!\hbar}\sum_z\left(4\xi^3(z)\Phi+\xi^4(z)\right)+\ldots\right\} \tag{2.10}$$

Using the formula:

$$\int \mathcal{D}[\phi]\exp\left(\frac{1}{\hbar}\sum_{zw}\phi(z)M(z,w)\phi(w)\right)\sum_z\phi^{2n}(z) = \frac{(2n-1)!!}{2^n}\left(\det(\frac{M}{\pi\hbar})\right)^{-1/2}\hbar^n \tag{2.11}$$

we conclude that one can factorize the contributions to $U_V(\Phi)$ in terms of growing powers of Planck's constant $\hbar$.This produces the loop expansion [14].
Thus, we can write the expansion (2.10) as:

$$U_V(\Phi) = U_{cl}(\Phi) + \hbar\,U_V^{(1)}(\Phi) + \hbar^2 U_V^{(2)}(\Phi) + \ldots \tag{2.12}$$

2.2 One loop contribution

We need to evaluate the Gaussian integral of the first term in Eq.(2.10). Using the trace representation for the determinant of a linear operator, we get:

$$\exp\left(-\frac{V}{\hbar}U_V^{(1)}(\Phi)\right) = (\det M/\pi\hbar)^{-1/2} = (\text{const})\,\exp\left(-\frac{1}{2}Tr(lnM)\right) \tag{2.13}$$

We can express the kernel of the operator M of Eq.(2.9) in the Fourier space as:

$$M(\hat{k}) = \left(\hat{k}^2 + m_0^2 + \frac{\lambda_0}{2}\Phi^2\right) \tag{2.14}$$

which corresponds to the inverse of the propagator in the real space:

$$M^{-1}_{zw}(\Phi) = \Delta_{zw}(\Phi) \equiv <\,\xi(z)\,\xi(w)\,> = \frac{1}{V}\sum_{\hat{k}^2\neq 0}\exp(-ik(z-w))\,\frac{\hbar}{\hat{k}^2+m^2+\frac{\lambda}{2}\Phi^2}\ , \tag{2.15}$$

where the lattice momentum vector $\hat{k}_\mu$ is given by:

$$\hat{k}_\mu \ = \ 2\sin(k_\mu/2).$$

so that

$$\hat{k}^2 = 4\sum_{\mu=1}^{4}\sin^2(k_\mu/2) \tag{2.16}$$

The momentum components k_μ are constrained to be in the Brillouin zone $k_\mu \in [-\pi,\pi]$, and because of the finite volume, they are constraint to take the discrete values

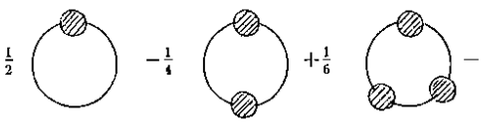
$$k_\mu = \frac{2\pi}{L}n_\mu \qquad -\frac{L}{2} < n_\mu \leq \frac{L}{2} \tag{2.17}$$

Substituting (2.14) into Eq.(2.13), we get up to an irrelevant constant

$$U_V^{(1)}(\Phi) = \frac{1}{2V}\sum_{\hat{k}^2\neq 0} ln\left[\left(\hat{k}^2 + m_0^2 + \frac{\lambda_0}{2}\Phi^2\right)\right] := \frac{1}{2V}\sum_{\hat{k}^2\neq 0} ln(\hat{k}^2 + m_\Phi^2) \tag{2.18}$$

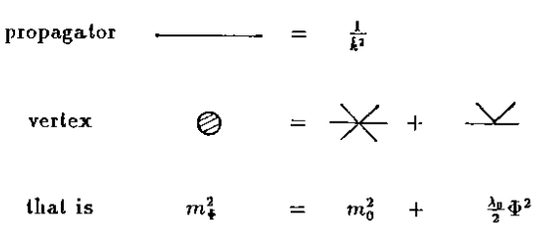
This agrees with reference [15] for the 1 component case $N=1$.

Expanding $ln(1+z)$ in a Taylor series, we obtain a sum of the graphs:



(2.19)

where we used the following Feynman rules:



(2.20)

Up to a constant, the diagrammatic representation of $U_V^{(1)}(\Phi)$ of Eq.(2.19) is given by the sum of the one-particle irreducible diagrams at zero external momentum, where the factors $1/2n$ are combinatorial factors, which can be computed from symmetry considerations.

In fact, from the diagrammatic expansion, we see that there are $(2n)!$ ways to distribute $2n$ particles to the external lines of a n-vertex diagram, but from them $2^n n$ are dependent, connected through reflections, rotations of n vertices on the ring and interchange of any two external legs at a given vertex. So the combinatorial factor is $(2n)!/2^n 2n\,(2n)! = 1/\,2n\,2^n$, where $1/\,2^n$ is responsible for the factor $1/2$ in the numerator of the argument of logarithm, and $1/2n$ is its Taylor's coefficient [16].

2.3 Two-loop contribution

Keeping the terms proportional to $\hbar$ in the expansion (2.10), one obtains:

$$\int \mathcal{D}[\xi] \exp\left\{-\frac{1}{\hbar} S_0(\xi)\right\} \left[-\frac{\lambda}{4!\,\hbar}\sum_z \xi^4(z) + \frac{\lambda^2\Phi^2}{72\,\hbar^2}\sum_{zw}\xi^3(z)\,\xi^3(w)\right] \tag{2.21}$$

Using Wick's theorem, we obtain:

$$< -\frac{\lambda}{4!\hbar}\sum_z \xi^4(z) >= -\frac{\hbar\lambda}{8}\sum_z \Delta_{zz}^2(\Phi) = -\frac{\lambda\hbar}{8V}\sum_{\hat{k}_1^2,\hat{k}_2^2\neq 0} \tilde{\Delta}_{k_1}(\Phi)\,\tilde{\Delta}_{k_2}(\Phi) \tag{2.22}$$

$$\begin{aligned} < \tfrac{\lambda^2\hat{\Phi}^2}{72\hbar^2}\sum_{zw}\xi^3(z)\,\xi^3(w) >= \; & \tfrac{\lambda^2\hbar\hat{\Phi}^2}{12}\sum_{zw}\{3\Delta_{zz}(\Phi)\,\Delta_{zw}(\Phi)\,\Delta_{ww}(\Phi) + 3!\Delta_{zw}^3(\Phi)\} \\ = \; & \tfrac{\lambda^2\hbar\hat{\Phi}^2}{12V^3}\sum_{\hat{k}_1^2,\hat{k}_2^2,\hat{k}_3^2\neq 0}\left\{V^2\delta_{k_2,0} + V\delta_{k_1+k_2+k_3,0}\right\}\tilde{\Delta}_{k_1}(\Phi)\,\tilde{\Delta}_{k_2}(\Phi)\,\tilde{\Delta}_{k_3}(\Phi) \end{aligned} \tag{2.23}$$

where $\tilde{\Delta}_k(\Phi)$ is the Fourier transformation of the propagator $\Delta_{zw}(\Phi)$, and the expectation value is defined as usual

$$< O(\xi) >= \frac{\int \mathcal{D}[\xi]\exp\left(-\frac{1}{\hbar}S_0(\xi)\right) O(\xi)}{\int \mathcal{D}[\xi]\exp\left(-\frac{1}{\hbar}S_0(\xi)\right)} \tag{2.24}$$

The two loop contributions of Eq.(2.21), (2.22) and (2.23) correspond to the Feynman graphs a), b) and c) of Fig. 1 respectively.

From (2.22) we see explicitly that the term corresponding to the diagram b) drops out because the zero momentum component of ξ_F vanishes according to Eq.(2.6).
Consequently, we obtain up to two loops:

$$\begin{aligned} U_V(\Phi) = \; & U_{cl}(\Phi) + \tfrac{\hbar}{2V}\sum_{\hat{k}^2\neq 0} ln(\hat{k}^2 + U''_{cl}(\Phi)) + \tfrac{\hbar^2\lambda}{8V^2}\sum_{\hat{k}_1^2,\hat{k}_2^2\neq 0}(\hat{k}_1^2 + U''_{cl}(\Phi))^{-1}(\hat{k}_2^2 + U''_{cl}(\Phi))^{-1} \\ & -\tfrac{\lambda\hat{\Phi}^2\hbar^2}{12V^2}\sum_{\hat{k}_1^2,\hat{k}_2^2,\hat{k}_3^2\neq 0}\delta_{k_1+k_2+k_3,0}(\hat{k}_1^2 + U''_{cl}(\Phi))^{-1}\,(\hat{k}_2^2 + U''_{cl}(\Phi))^{-1}\,(\hat{k}_3^2 + U''_{cl}(\Phi))^{-1} \end{aligned} \tag{2.25}$$

In the infinite volume limit, this expression agrees with the two loop computation of ref. [14] for the conventional effective potential (compare section 2 eqn.(2.3) for instance).

2.4 Expansion strategy

Because the Monte Carlo simulation was performed in a region of the bare parameter (m_0^2, λ_0) where the loop expansion in bare parameters is not reliable (the valid region is given by $0 < m_0^2 + \frac{\lambda_0}{2}\Phi^2$ in lattice units), we have to look for a reliable expansion in terms of running coupling constants.

In fact, it can be proved that the constraint effective potential becomes complex for some values of ϕ if the classical potential is non-convex, but this apparent complexity is due to the failure of the loop expansion. This was pointed out by Fujimoto et al. [17]. We observe from Eq.(2.10) that the gaussian integral is well defined only if the matrix $M(z,w)$ is positive definite or equivalently, if the second derivative of the classical potential is positive. Therefore, the loop expansion make sense as an asymptotic series, only up to the inflection points of the classical potential.

The fundamental observation is this: due to the proximity of the system to the critical line $m_{0c}^2(\lambda_0)$, the renormalized mass m_R and the self coupling λ_R are small and go to zero as the system approaches the critical line (triviality of the lattice ϕ^4-theory.)

Therefore it is a good idea to expand in λ_R and m_R. This can be done by manipulating the bare expansion. Instead of adding counterterms per hand to the tree level, we will define the renormalized quantities through the n-points vertex functions and compute them up to n loop consistent with the n loop expansion of $U_V(\Phi)$. Then, inverting this relations, we will obtain the bare quantities as a function of the renormalized ones and shall finally substitute them into the expansion of the constraint effective potential to obtain the desired expansion in terms of λ_R, m_R.

The fundamental philosophy of the renormalization theory implies that one shall obtain a finite expression without ultraviolet divergences in the limit when the cutoff is removed. This is the renormalized n loop expansion of $U_V(\Phi)$.
It is expected that one obtains in this way an expression for physical quantities like the

constraint effective potential which is accurate when m_R and λ_R are small, in spite of the fact that the expressions for the bare parameters (and other cutoff dependent quantities) are not trustworthy. For a discussion of related issues, see Lüscher and Weisz [5].

2.5 Renormalization

The definition of the renormalized quantities m_R^2, λ_R and Z_ϕ is to some extent arbitrary. (Some different definitions are related through the renormalization group equations). We will use the same definition as Lüscher and Weisz to make it possible to compare our numerical results with them. From here on $\hbar$ will be set to 1.

It is in principle possible to define these renormalized coupling constants on a finite lattice and to obtain a UV-finite renormalized loop expansion for the constraint effective potential following the idea of subsection 2.4.

It is however convenient to define the renormalized coupling constants on an infinite lattice. This has the advantage that the constraint effective potential on finite volume is parametrized by the infinite volume renormalized coupling constants and hence no finite size scaling analysis of coupling constants is necessary.
For an alternative but equivalent treatment, see refs. [18] - [20].

2.5.1 Symmetric phase

We introduce a wave function renormalization constant Z_ϕ, a renormalized mass parameter m_R and a renormalized coupling λ_R through the two and four-point vertex functions in infinite volume

$$\Gamma^{(2,0)}(\hat{p},-\hat{p}) = -Z_\phi^{-1}\left\{m_R^2 + \hat{p}^2 + O(\hat{p}^4)\right\} \tag{2.26}$$

$$\Gamma^{(4,0)}(0,0,0,0) = -Z_\phi^{-2}\,\lambda_R \tag{2.27}$$

Note that m_R and λ_R are not exactly equal to the physical mass m and coupling λ, which are defined as the pole of the propagator in the complex energy-plane and the on-shell four-point vertex functions respectively.
Corresponding to the lattice action (2.4), the one loop two and four-point vertex functions are given by

$$\begin{cases} \Gamma^{(2,0)}(\hat{p},-\hat{p}) & = -(m_0^2+\hat{p}^2) - \frac{\lambda_0}{2}\,I_1(m_0) + O(\hat{p}^4) \\ \Gamma^{(4,0)}(\hat{p}_1,\ldots,\hat{p}_4) & = -\lambda_0 + \frac{1}{2}\lambda_0^2\left\{A(\hat{p}_1+\hat{p}_2;m_0) + A(\hat{p}_1+\hat{p}_3;m_0) + A(\hat{p}_1+\hat{p}_4;m_0)\right\} \end{cases} \tag{2.28}$$

where

$$\begin{cases} I_p(m) & = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4}\,(m^2+\hat{k}^2)^{-p} \\ A(q;m) & = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4}\,\frac{1}{m^2+\hat{k}^2}\,\frac{1}{m^2+(\hat{k+q})^2} \end{cases} \tag{2.29}$$

are lattice 1-loop integrals.

Thus, after some algebra, we get

$$\begin{aligned} Z_\phi &= 1 + O(\lambda_0^2) \\ \lambda_R &= \lambda_0\left[1 - \tfrac{3}{2}\lambda_0\,I_2(m_0)\right] + O(\lambda_0^3) \\ m_R^2 &= m_0^2\left[1 + \tfrac{\lambda_0}{2m_0^2}\,I_1(m_0)\right] + O(\lambda_0^2) \end{aligned} \tag{2.30}$$

And therefore we obtain the renormalized 1-loop expansion of the constraint effective potential in a finite volume by inserting eqns. (2.30) into the 1-loop expression for the constraint effective potential of eqn. (2.18). The renormalized magnetization is related to the bare one by

$$\Phi_R \;=\; Z_\phi^{-1/2}\Phi$$

and we get

$$U_V(\Phi_R) = U_{cl}^{SR}(\Phi_R) + \frac{1}{2V}\sum_{\hat{k}^2\neq 0} ln\left\{1 + \frac{\lambda_R\Phi_R/2}{\hat{k}^2+m_R^2}\right\} - \frac{\lambda_R}{4}\,\Phi_R^2\,I_1(m_R) + \frac{1}{16}\,\lambda_R^2\,\Phi_R^4\,I_2(m_R) \tag{2.31}$$

where

$$U_{cl}^{SR}(\Phi_R) = \frac{1}{2}\,m_R^2\,\Phi_R^2 + \frac{\lambda_R}{4!}\,\Phi_R^4 \tag{2.32}$$

is the renormalized classical potential in the symmetric phase.
Eqn. (2.31) is in fact finite when the UV-cutoff $\Lambda = 1/a$ is removed ($\Lambda \to \infty$).

2.5.2 Broken phase

We define the renormalized mass m_R and the wave function renormalization constant Z_ϕ through the limit $\hat{p}^2 \to 0$ of the two points vertex function in the same way as in the symmetric phase:

$$\Gamma^{(2,0)}(\hat{p},-\hat{p}) = -Z_\phi^{-1}\left\{m_R^2 + \hat{p}^2 + O(\hat{p}^4)\right\} \tag{2.33}$$

The self-coupling constant λ_R is now defined through the tree level relation:

$$\lambda_R = 3\,m_R^2/v_R^2 \tag{2.34}$$

where v_R is the vacuum expectation value (V.E.V.) of the field $\phi_R = Z_\phi^{-1/2}\phi$.
This definition implies that the four-point vertex at zero momentum is a non-trivial function of λ_R and only approximately equal to λ_R.
One finds in an analogous way up to one loop:

$$\begin{aligned} Z_\phi &= 1 + \tfrac{\lambda_0 m_0^2}{16}[I_2(m_0) - (8+m_0^2)I_3(m_0)] \\ m_R^2 &= m_0^2\left\{1 - \tfrac{\lambda_0}{m_0^2}\,I_1(m_0) - \tfrac{3}{2}\lambda_0\,I_2(m_0) + \tfrac{\lambda_0 m_0^2}{16}[I_2(m_0) - (8+m_0^2)I_3(m_0)]\right\} \\ v_R &= \left(\tfrac{3m_0^2}{\lambda_0}\right)^{1/2}\left\{1 - \tfrac{\lambda_0}{2m_0^2}\,I_1(m_0) - \tfrac{\lambda_0 m_0^2}{32}[I_2(m_0) - (8+m_0^2)I_3(m_0)]\right\} \\ \lambda_R &= \lambda_0\left\{1 - \tfrac{3}{2}\lambda_0\,I_2(m_0) + \tfrac{\lambda_0 m_0^2}{8}[I_2(m_0) - (8+m_0^2)I_3(m_0)]\right\} \end{aligned} \tag{2.35}$$

After shifting the field to the true minimum (correct up to one loop) and inserting Eq.(2.35) into (2.18), we get:

$$U_V(\Phi_R) = U_{cl}^{BR}(\Phi_R) + \frac{1}{2V}\sum_{\hat{k}^2\neq 0} ln\left\{1+\frac{\lambda_R v_R \tilde{\Phi}_R + \lambda_R \tilde{\Phi}_R^2/2}{\hat{k}^2+m_R^2}\right\} - \frac{1}{2}\,v_R\,\lambda_R\,I_1(m_R)\Phi_R$$
$$- \left\{\frac{\lambda_R}{4}I_1(m_R) - \frac{v_R\lambda_R}{4}I_2(m_R)\right\}\Phi_R^2 + \frac{1}{4}\lambda_R^2\,v_R\,I_2(m_R)\Phi_R^3 + \frac{\lambda_R^2}{16}I_2(m_R)\Phi_R^4 \tag{2.36}$$

where

$$U_{cl}^{BR}(\Phi_R) = \frac{1}{2}\,m_R^2\,\Phi_R^2 + \frac{1}{3}\,v_R\,\lambda_R\,\Phi_R^3 + \frac{\lambda_R}{4!}\,\Phi_R^4 \tag{2.37}$$

is the renormalized classical potential in the broken phase.
Eqn.(2.36) is finite in the limit when we remove the UV cutoff ($\Lambda \to \infty$).

2.6 Results

We expect that the results of the loop expansion for physical quantities like the effective potential as function of renormalized parameters m_R, λ_R are accurate near the critical line where λ_R (and m_R) are small.
Lüscher and Weisz have shown that one can also compute the bare quantities m_0, λ_0, Z_ϕ with a great deal of work. Here we will use numerical data for the constraint effective potential instead, and compare the results with Lüscher and Weisz.

Corresponding to each value of the bare parameter chosen for the Monte Carlo simulation, we make a fit with the renormalized one loop expansion formula deduced in sections 2.2 - 2.5. We did not use the two-loop formula because the order of magnitude of the two-loop contribution is smaller than the error-bars of the Monte Carlo data. The smallness of the two-loop contribution confirms the accuracy of the expansion.
In the numerical work of ref. [6], the input values $\lambda_0 = 16.376$, $m_0^2 = -1.14, -1.15$ and -1.16 were chosen to study the crossover of the constraint effective potential from a single well shape to double well behaviour on a 12^4 lattice.

Therefore, the first two points must correspond to the symmetric phase in the infinite volume limit and the last to the broken phase. According to Lüscher and Weisz, the critical point associated with the second order phase transition lies at $m_0^2 = -1.157 \pm 0.006$ [5] for $\lambda_0 = 16.376$, in infinite volume.
The fits are shown in Fig. 2 (a), (b) and (c) for the values $\lambda_0 = 16.376$; $m_0^2 = -1.14, -1.15$ and -1.16 respectively. The error-bars on the numerical data were computed using the standard deviation formula for the available independent Monte Carlo simulation data set for each value of the bare parameters.

The errors for the fitted values were also estimated using a standard deviation formula.

From these fits, we can obtain immediately the renormalized quantities, as it was explained in section 2. The result is shown in the following table for the infinite volume limit. For comparison, the infinite volume results of Lüscher and Weisz are listed in the last two columns.

	$m_R(\infty)$	$\lambda_R(\infty)$	m_R^{LW}	λ_R^{LW}
$m_0^2 = -1.14$	0.079 ± 0.003	7.5 ± 0.6	0.08	7.4 ± 0.2
$m_0^2 = -1.15$	0.06 ± 0.005	7.1 ± 0.1	0.06	7.2 ± 0.2
$m_0^2 = -1.16$	0.13 ± 0.008	7.8 ± 0.1	0.13	7.6 ± 0.2

We see that they agree within the error quoted by Lüscher and Weisz.

3 Constraint effective potential for theories with non-Abelian gauge fields

3.1 Definitions and notations

3.1.1 Basic definitions

We consider a scalar doublet ϕ_z on the 4 dimensional hypercubic lattice Λ. We use the symbols $z, w, \ldots$ for sites in Λ. We shall use also the real components ϕ_0 and ϕ_r to represent the Higgs variable ϕ_z

$$\phi_z = \phi_{0z} + i\,\tau_r\,\phi_{rz} \tag{3.1}$$

where $\tau_r (r = 1, 2, 3)$ are the isospin Pauli matrices. (The summation convention over repeated isospin indices $r, s, \ldots$ is understood).

For the gauge variables, we introduce the Wilson link variable $U(z, \mu)$ on the link $(z + \hat{\mu}, z)$ from the point z to the neighbouring point $z + \hat{\mu}$. We use the 3 real components $a_{rz\mu}$ ($r = 1, 2, 3$; $\mu = 1, 2, 3, 4$) to parametrize the link variables $U(z, \mu)$ as in ref. [9]:

$$U(z, \mu) = 1 - a_{z\mu} + i\,\tau_r\,a_{rz\mu} \tag{3.2}$$

The $SU(2)$ condition $U^\dagger U = 1$ implies

$$a_{z\mu} = 1 \pm \sqrt{1 - a_{rz\mu}a_{rz\mu}} = \frac{1}{2}a_{rz\mu}a_{rz\mu} + \sum_{n\geq 2}\frac{(2n-3)!!}{2^n n!}(a_{rz\mu}a_{rz\mu})^n \tag{3.3}$$

In the expansion, we have chosen the minus sign, so that for $a_{rz\mu} \approx 0$, $U(z, \mu) \approx 1$ dominates, as is assumed in perturbation theory.

In terms of the ϕ_z and $U(z, \mu)$ variables, the $SU(2)$-Higgs model is described by the Euclidean action:

$$S = \beta\sum_p(1 - \frac{1}{2}\mathrm{Tr}U_p) + \sum_z\left\{\frac{1}{2} < \phi_z| - \Delta_U\phi_z > + \frac{1}{2}m_0^2 < \phi_z\,|\,\phi_z > + \frac{\lambda_0}{4!} < \phi_z\,|\,\phi_z >^2\right\} \tag{3.4}$$

We have adopted the Wilson action [21] for the pure gauge part of the lattice action S_g, $\beta = 4/g^2$ gives the bare gauge coupling, Σ_p stands for a summation over positively oriented plaquettes, U_p is the product of link variables around the plaquette, the symbol $< \phi_z | \phi_z >$ stands for the usual scalar product $< \phi_z \,|\, \phi_z > = \frac{1}{2}\,\mathrm{Tr}\,(\phi_z^\dagger \phi_z)$ and the gauge covariant Laplacian Δ_U is defined through:

$$-\Delta_U \phi_z := -\sum_{\mu=\pm 1}^{\pm 4} [U(z-\mu,\mu)\,\phi_{z-\mu} - \phi_z] \tag{3.5}$$

It is customary to express the model in terms of a completely equivalent parametrization, known as the lattice parametrization

$$\phi_z = \sqrt{2\kappa}\varphi_z \quad \lambda_0 = 6\lambda/\kappa^2 \quad m_0^2 = \frac{(1-2\lambda)}{\kappa} - 8 \tag{3.6}$$

with κ the hopping parameter and λ a re-scaled self coupling constant:

$$S_H = -2\kappa \sum_{z,\mu} < \varphi_z \,|\, U(z-\mu,\mu)\,\varphi_{z-\mu} > + \sum_z < \varphi_z | \varphi_z > + \lambda \sum_z [< \varphi_z | \varphi_z > - 1]^2 \tag{3.7}$$

This formulation of the $SU(2)$-Higgs theory is more convenient for Monte Carlo simulations and high temperature expansions [22].

3.1.2 Definition of the constraint effective potential (CEP)

In a natural way, we generalize the definition of constraint effective potential (CEP) $U_V(\Phi)$ given in section 2 to theories with (non-Abelian) gauge fields:

$$\exp\left(-V U_V\,(\Phi)\right) = \int \mathcal{D}\phi \mathcal{D}\mathcal{U}\; \delta(\Phi - C\phi)\; \exp(-S[\phi, a]) \tag{3.8}$$

where $S[\phi, a]$ is the action of the theory and the delta distribution constrains the gauge covariant magnetization $C\phi$ to have the given value Φ.

In order to give sense to this definition, we should face the highly non-trivial problem of defining the block spin transformation of scalar fields in a non-Abelian gauge theory. The gauge covariant magnetization is the special case of a block spin where the whole lattice is one block. Eventually, one may want to go beyond the computation of the constraint effective potential and compute effective actions. Because of this, we will make a degression to discuss gauge covariant block spins in general.

A proper definition of block spin transformation was used by Gawędzki and Kupiainen [23] to give rigorous proofs of the existence of continuum limits for some lattice field theories without gauge fields by means of renormalization group (RG) methods. Jaffe and Balaban introduced block spins for Higgs models with gauge fields which are suitable for superrenormalizable theories [24]. This proposal was later generalized to the renormalizable case and to pure gauge theories.

Mack [25] proposed an alternative and a very natural gauge covariant generalization of the above transformation to define block spins of Higgs fields in non-Abelian theories.

In order to explain Mack's proposal, it is appropriate to review briefly the block spin transformation used by Gawędzki and Kupiainen in the absence of gauge fields.

In one RG step, the original lattice Λ of sites z is covered by a block lattice Λ' whose sites x are blocks with side length $a' = L_B\, a$ ($L_B > 1, a$ is the lattice space) and containing L_B^d sites $z \in \Lambda$. We write $z \in x$ if z is in the block x.
The block spin Φ shall be a linear function of ϕ and it is given by a formula of the form

$$\Phi(x) = a^d \sum_{z \in \Lambda} C(x,z)\,\phi(z) \tag{3.9}$$

where $C(x,z)$ is the integral kernel of an operator C which maps a field on Λ into a field on the sublattice Λ'.

3.1.3 Definition of block spins for Higgs fields and the gauge covariant magnetization

For one-component scalar field, in the absence of a gauge field, the kernel C is given by

$$C(x,z) = (a')^{-d} \chi_x(z) \tag{3.10}$$

where χ_x is the characteristic function of x, i. e. $\chi_x(z) = 1$ if $z \in x$ and $\chi_x(z) = 0$ otherwise. This kernel implies a local definition of Φ as block average of ϕ and fulfils the normalization condition

$$CC^\dagger(x_1,x_2) = \int_z C(x_1,z) C^\dagger(z,x_2) = (a')^{-d}\delta_{x_1 x_2} = \delta(x_1 - x_2) \tag{3.11}$$

With a view towards generalization, eqn. (3.10) can be rewritten in a fancy way. It is convenient to consider the adjoint operator $C^\dagger$ of C where $\dagger$ denotes matrix conjugation (complex conjugation in the one-component case $C^\dagger(z,x) = C^*(x,z)$). The $C^\dagger(z,x)$ kernel can be defined as solution of the eigenvalue equation

$$(-\Delta_{N,x} C^\dagger)(z,x) = \epsilon_0(x) C^\dagger(z,x) \tag{3.12}$$

where $\Delta_{N,x}$ is a lattice substitute for the Laplacian with Neumann boundary conditions on the boundary of block x (when it has a boundary), and $\epsilon_0(x)$ is its lowest eigenvalue, equal to zero for all blocks in pure scalar theory. $\Delta_{N,x}$ acts on argument z. The solutions of (3.12) with $\epsilon_0(x) = 0$ are constants on the blocks, these are determined by the normalization condition (3.11).

Following [25] , this procedure can be re-interpreted as follows. Define a notion of (block-local) frequency which depends only on the behaviour of the function which is to be decomposed into frequency components on the chosen block x. Thus, define frequency (squared if one wants to) as eigenvalue of the Laplacian with Neumann boundary conditions. Define the block spin in two steps. First define the lowest frequency part $\psi(z)$ of $\phi(z)$ by projection

$$\psi(z) = \lim_{t\to\infty} \exp\left[-t(-\Delta_{N,x} - \epsilon_0(x))\right] \phi\,(z) \quad \text{for } z \in x \tag{3.13}$$

$\psi(z)$ will be a smooth function of z inside the block. In the second step the block spin is defined equal to the value of this smooth function at the block centre $z = \hat{x}$.

The Euclidean action of ϕ^4 theory is of the form $\mathcal{H} = \mathcal{H}_0 + V$, and $\mathcal{H}_0 = (\phi, h\phi)$ with "first quantized Hamiltonian" h in the space of wave functions ϕ equal to $h = -\Delta$. A local approximation of this which preserves the invariance under shifts by constant fields would be $h_x = -\Delta_{N,x}$. One sees that the block spin $\Phi(x)$ retains the component of ϕ associated with the lowest eigenvalue of h_x, in agreement with the philosophy of Brower, Rebbi and Vicari, ref. [26].

Mack [25] proposed to proceed for block spins of Higgs fields in non-Abelian gauge theories in exactly the same way, defining $C^\dagger$ as solution of eqn. (3.12). $C^\dagger(z,x)$ will be in the linear span[§] of the gauge group G, and $\Delta_{N,x}$ will be the lattice version of the covariant Laplacian with Neumann boundary conditions. It depends on the lattice gauge field U and is defined by

$$(\Delta_{N,x}\phi)(z) = \sum_{\substack{z' \text{ n.n. } z \\ z' \in x}} [U(z,z')\phi(z') - \phi(z)] \quad \text{for } z \in x \tag{3.14}$$

Summation is over next neighbours z' of z which lie in block x, and $U(z,z')$ is the gauge field attached to the link from z' to z. The idea is again that the definition of block spins involves dynamical information — think of $\int_{z\in x} \phi^*(z)(-\Delta_{N,x}\phi)(z)$ as the part of the kinetic energy which is associated with the inside of block x. Thus, what is called "low frequency" is actually determined by kinetic energy. In multigrid applications, one could also admit dielectric gauge fields [27] U, especially on coarser layers.

— Although we do not indicate notationally the explicit dependence of $\Delta_{N,x}$, $\epsilon_0(x)$, C and $C^\dagger$ on the gauge field U, it should be remembered in the following. —

Some properties of $\Delta_{N,x}$ are: $-\Delta_{N,x}$ is positive semidefinite so that its lowest eigenvalue $\epsilon_0(x)$ is always non-negative. $\epsilon_0(x)$ is only zero if U is a pure gauge within x. Under a gauge transformation the kernel of the covariant Laplacian Δ transforms according to $\Delta(z,z') \to g_z \Delta(z,z') g_{z'}^{-1}$. Eigenvectors of $-\Delta_{N,x}$ transform as (3.22) (modulo arbitrariness stated below), and the eigenvalues of $-\Delta_{N,x}$ are gauge invariant.

The normalization condition (3.11) does not specify C uniquely. One retains the freedom of a gauge transformation per block $C^\dagger(z,x) \to C^\dagger(z,x) g_x$, $g_x \in G$, under which eqn.(3.11) is invariant. Nevertheless, we prove in the next Lemma that this remaining freedom does not affect $U_V(\Phi)$ defined by eqn. (3.8).

Lemma

Consider the integral

$$\exp(-U_V(\Phi)) = \int \mathcal{D}\phi\, \mathcal{D}\mathcal{U} \delta(\Phi - C\phi) \exp(-S[\phi, U]) \tag{3.15}$$

The averaging kernel C is subject to the requirements

$$\begin{cases} -\Delta_U C^*(z) &= \epsilon_0 C^*(z) \\ C\, C^* &= 1 \end{cases} \tag{3.16}$$

[§] i. e. $C^\dagger(z,x)$ will be a real multiple of an element of $SU(2)$ if $G = SU(2)$, an arbitrary complex $N \times N$ matrix for $G = U(N)$, $N \geq 3$ etc, see ref. [27]

Eqn.(3.16) does not fix C uniquely, but the remaining freedom does not affect $U_V(\Phi)$, and

$$U_V(R\Phi) = U_V(\Phi) \quad \text{for all} \quad R \in SU(2) \tag{3.17}$$

Proof

Let us denote by C_R the kernel which is uniquely defined by the requirements (3.16) together with

$$C_R = r R^{-1}, \quad \text{with} \quad r > 0, \quad R \in SU(2),$$

and denote the corresponding potential by $U_{V,R}(\Phi)$. It follows from gauge invariance of $\mathcal{D}\phi\, \mathcal{D}\mathcal{U} \exp(-S[\phi,U])$ that

$$U_{V,1}(R\Phi) = U_{V,1}(\Phi). \tag{3.18}$$

It follows from invariance of eqn. (3.16) under $C(z) \to R\,C(z)$ that

$$C_R(z) = R^{-1}\, C_1(z). \tag{3.19}$$

Invariance of the δ-function, $\delta(R\psi) = \delta(\psi)$ implies now that

$$U_{V,R}(\Phi) = U_{V,1}(\Phi). \tag{3.20}$$

Eqns. (3.19) and (3.20) establish the desired result. q.e.d.

Without changing the CEP, one can eliminate this remaining freedom of a gauge transformation per block by fixing the block center $\hat{x}$ of x and demanding that $C(x,\hat{x})$ is a positive operator. In the case of gauge group $G = U(1)$ or $SU(2)$, this condition simplifies to

$$C(x,\hat{x}) = C^\dagger(\hat{x},x) = r(x)\,\mathbb{1} \quad , \quad r(x) \text{ a positive real number} \tag{3.21}$$

This ensures that C transforms as follows under gauge transformations

$$C^\dagger(z,x) \to g_z\, C^\dagger(z,x)\, g_{\hat{x}}^{-1}. \tag{3.22}$$

If U is pure gauge within a block x, $(a')^d C(x,z)$ is an element of G for all $z \in x$, equal to the path-ordered product of link variables along an arbitrary path $\mathcal{C}_{\hat{x}z}$ which stays entirely within x.

Now we turn to the interpolation kernel $\mathcal{A}$ which is used to recover a smooth field $\psi = \mathcal{A}\Phi$ on Λ from a block spin Φ on Λ'. In the work of Kupiainen and Gawędzki, the kernel $\mathcal{A}$ is defined so that $\psi = \mathcal{A}\Phi$ minimizes the kinetic energy $\mathcal{H}_0$ subject to the constraint that the block spin $\Phi = C\psi$ is prescribed. Given a field ϕ with block spin Φ, one regards $\psi = \mathcal{A}\Phi$ as the low frequency part of ϕ. One introduces the fluctuation field ζ, which is supposed to contain the high frequency components of ϕ, by $\phi = \psi + \zeta$. As a result

$$\begin{aligned} (\phi, -\Delta\phi) &= (\psi, -\Delta\psi) + (\zeta, -\Delta\zeta) + 2\,(\zeta, -\Delta\psi) \\ &= (\Phi, -\Delta_{eff}\Phi) + (\zeta, -\Delta\zeta) \end{aligned} \tag{3.23}$$

This is because the term $(\zeta, -\Delta\mathcal{A}\Phi)$ vanishes if $\psi = \mathcal{A}\,\Phi$ minimizes the kinetic energy. Δ_{eff} is an operator which acts on functions on Λ' and has kernel

$$\Delta_{eff}(x_1,x_2) = (\mathcal{A}^*\Delta\mathcal{A})(x_1,x_2) = \int_z \mathcal{A}(z,x_1)^*(\Delta\mathcal{A})(z,x_2) \tag{3.24}$$

The conditions on the interpolating operator $\mathcal{A}$ are fulfilled if

$$\begin{cases} -\Delta_z \mathcal{A}(z,x) &= \sum_{y\in\Lambda'} C^\dagger(z,y)\,\Upsilon^{-1}(y,x) \\ C\,\mathcal{A} &= \mathbb{1} \end{cases} \tag{3.25}$$

with some kernel $\Upsilon^{-1}(x,y)$. It follows that $\Delta_{eff} = \Upsilon^{-1}$. The solution of eqn. (3.25) is

$$\mathcal{A} = (-\Delta)^{-1} C^\dagger\,\Upsilon^{-1} \tag{3.26}$$

Υ is implicitly determined by the requirement that $C\,\mathcal{A} = 1$.

To compute the constraint effective potential, we need only the special case Λ' = a single point x. In this case, Υ^{-1} is a number and one can solve eqn. (3.25) by taking

$$\mathcal{A} = C^\dagger \tag{3.27}$$

and Υ^{-1} the smallest eigenvalue of minus the covariant Laplacian with periodic boundary conditions. The perturbative solution of this eigenvalue problem will be explained in section 3.2.

When defined by the covariant procedure explained above, the kernels $C(x,z)$ and $\mathcal{A}(z,x)$ transform under gauge transformations like parallel transporters from z to $\hat{x}$, the block center, i.e.

$$\begin{cases} C(x,z) \to g_{\hat{x}}\,C(x,z)\,g_z^{-1} \\ \mathcal{A}(z,x) \to g_z\,\mathcal{A}(z,x)\,g_{\hat{x}}^{-1} \end{cases} \tag{3.28}$$

where g_z are elements of the gauge group $G\,(=SU(2))$; $\hat{x}$ is the block centre of $x \in \Lambda'$.

We remark that the kernel C admits a representation of the form

$$C(x,z) = \sum_{\omega: z\to\hat{x}} \rho(\omega)\,U(\omega) \tag{3.29}$$

where $U(\omega)$ is the parallel transporter along the path ω from z to $\hat{x}$. This makes the gauge covariance properties manifest.

3.2 The averaging and interpolating operators C and $\mathcal{A}$ in perturbation theory

We consider the interpolating operator $\mathcal{A} = C^\dagger$ for the special case Λ' = a single point (only one block) introduced in the last section. The kernel $\mathcal{A}(z)$ is defined through the eigenvalue problem:

$$-\Delta_U\,\mathcal{A}(z) = \epsilon_0\,\mathcal{A}(z) \tag{3.30}$$

with periodic boundary conditions on the boundary of the block of volume $V = L^4$. We set here on the lattice spacing $a = 1$ from here on.
ϵ_0 is the smallest eigenvalue of the covariant lattice Laplacian

$$-\Delta_U\,\mathcal{A}(z) = -\sum_{\mu=\pm1}^{\pm4} \left[U\,(z-\mu,\mu)\,\mathcal{A}(z-\mu) - \mathcal{A}(z)\right] \tag{3.31}$$

The kernel $\mathcal{A}(z)$ is constrained to fulfil the normalization condition

$$C\,\mathcal{A} = \mathbb{1}, \quad \text{or, in terms of the kernels,} \quad V\sum_{z\in\Lambda} C(z)\,\mathcal{A}(z) = 1 \tag{3.32}$$

where C is the adjoint operator of $\mathcal{A}$, i.e. $C = \mathcal{A}^\dagger$.

The link variables $U(z,\mu)$ of section 3.1 depend on the gauge coupling constant g. Therefore, we can study the inclusion of non-Abelian gauge fields by using perturbation theory in g, the $SU(2)$-gauge coupling constant.
Assuming g weak, we seek the smallest eigenvalue $\epsilon_0(g)$, which exists because the negative covariant lattice Laplacian is a positive semi-definitive operator, its corresponding eigenfunction $\Psi(z)$, and the interpolating kernel $\mathcal{A}(z)$ of eqn. (3.30) in the form of a series

$$\begin{cases} \epsilon_0 = \epsilon_0^{(0)} + g\,\epsilon_0^{(1)} + g^2\,\epsilon_0^{(2)} + \dots \\ \Psi\,(z) = \Psi^{(0)}\,(z) + g\,\Psi^{(1)}\,(z) + g^2\Psi^{(2)} + \dots \\ \mathcal{A}(z) = \mathcal{A}^{(0)}\,(z) + g\mathcal{A}^{(1)}\,(z) + g^2\mathcal{A}^{(2)}\,(z) + \dots \end{cases} \tag{3.33}$$

The zeroth order eigenvalue $\epsilon_0^{(0)}$ vanishes.

3.2.1 Tree level and degeneracy

In zeroth order, one finds the set B of eigenfunctions of the Laplacian and their corresponding eigenvalues:

$$\begin{cases} B = & \{\ |\,\Psi_{n\alpha}^{(0)}> = |\,\psi_n> \otimes |\,e_\alpha> \quad |\,\psi_n\,(z) = \exp\left(\frac{2\pi i}{L} n z\right), -\frac{L}{2} < n_\mu \le \frac{L}{2}, \quad |\,e_\alpha> = \tau_\alpha\ \} \\ & \text{with } \tau_0 = \mathbb{1} \text{ the 2x2 matrix identity and } \tau_r\,(r = 1,\,2,\,3) \text{ the Pauli matrices} \\ \epsilon_n^{(0)} = & 4\sum_{\mu=1}^{4} \sin^2\left(\frac{\pi}{L} n_\mu\right) \equiv \hat{k}_n^2 \end{cases} \tag{3.34}$$

Inserting $\psi_n(z) \otimes |\,e_\alpha>$ into eqn. (3.30) for $g = 0$, one sees that

$$-\Delta\psi_n\,(z) \otimes |\,e_\alpha> = \hat{k}_n^2\ \psi_n(z) \otimes |\,e_\alpha> \tag{3.35}$$

with $\hat{k}_n^2$ given by eqn. (3.34).
With the following scalar product, defined on the Hilbert space H (generated by the complete set B)

$$<\phi_1\,|\,\phi_2> = \frac{1}{V}\sum_{z\in\Lambda} \frac{1}{2}\,\mathrm{Tr}\,(\phi_{1z}^\dagger\,\phi_{2z}) \tag{3.36}$$

the basis B turns out to be orthonormal:

$$<\Psi_{n\alpha}^{(0)}\,|\,\Psi_{m\beta}^{(0)}> = \delta_{\alpha\beta}\,\delta_{n,m} \tag{3.37}$$

For the smallest eigenvalue $\epsilon^{(0)} = 0$ there are four degenerate solutions. Imposing the normalization condition of eqn. (3.32), the degeneracy is not removed. This ambiguity remains in all orders in g, the $SU(2)$-gauge coupling constant. In fact, when eqn. (3.30) is regarded as an equation for a 2-component vector in place of a matrix, then if $(\phi_{1z}, \phi_{2z})^t$ is an eigenvector of $-\Delta_U$ corresponding to an eigenvalue ϵ, using the property that a multiple of an $SU(2)$ matrix M_{ij} fulfils $M_{11}^\dagger = M_{22}$, $M_{12}^\dagger = -M_{21}$, one proves that $(-\phi_{2z}^\dagger, \phi_{1z}^\dagger)^t$ is also an eigenvector of $-\Delta_U$ with the same eigenvalue ϵ.

The two independent 2-component solutions may be combined into a 2 x 2 matrix $\mathcal{A}(z)$. The freedom of taking linear combination reflects itself in the freedom of taking $\mathcal{A}(z) \to \mathcal{A}(z)\Omega$, where Ω in a unitary 2x2 matrix.

This freedom can be eliminated by imposing a condition of the form

$$\mathcal{A}(z=0) = a_\alpha \tau_\alpha \tag{3.38}$$

This condition is equivalent to rotating $\mathcal{A}(z=0)$ to lie in some given direction. Because of gauge invariance, this will not affect the CEP, as was shown by the Lemma in subsection 3.1.3.

The natural choice is $a_\alpha \tau_\alpha = 1$. But for technical reasons which we will explain below, we take a_α to depend on g. For the tree level, we choose $a_\alpha^{(0)} = a\,\delta_{\alpha 0}$ and therefore

$$\begin{cases} \mathcal{A}^{(0)}(z) &= \frac{1}{V}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \psi_0(z) \\ \\ C^{(0)}(z) &= \frac{1}{V}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \psi_0^*(z) \end{cases} \tag{3.39}$$

3.2.2 General expression and solution up to second order in g

Inserting the expansions for the eigenvector $\Psi_{n\alpha}(z)$, eigenvalue $\epsilon_0(g)$ and for the link variables $U(z,\mu)$ into eqn. (3.30), one obtains

$$\begin{aligned} -\sum_\mu \; & \Big[\Big(1 + \tfrac{i}{2} g\,\tau_r A_{rz-\mu\mu} - \tfrac{1}{8} g^2 A_{rz-\mu\mu} A_{rz-\mu\mu} + \ldots\Big) \Big(\Psi^{(0)}_{n\alpha}(z-\mu) + g\Psi^{(1)}_{n\alpha}(z-\mu) + \\ & g^2 \Psi^{(2)}_{n\alpha}(z-\mu) + \ldots \Big) - \Big(\Psi^{(0)}_{n\alpha}(z) + g\Psi^{(1)}_{n\alpha}(z) + g^2\Psi^{(2)}_{n\alpha}(z) + \ldots\Big) \Big] = \\ & (g\epsilon_0^{(1)} + g^2\epsilon_0^{(2)} + \ldots)\Big(\Psi^{(0)}_{n\alpha}(z) + g\Psi^{(1)}_{n\alpha}(z) + g^2\Psi^{(2)}_{n\alpha}(z) + \ldots\Big) \end{aligned} \tag{3.40}$$

Collecting terms of equal power in g

$$\begin{cases} -\sum_\mu & \Big[\Psi^{(0)}_{n\alpha}(z-\mu) - \Psi^{(0)}_{n\alpha}(z)\Big] = 0 \\ \\ -\sum_\mu & \Big[\Big[\Psi^{(1)}_{n\alpha}(z-\mu) - \Psi^{(1)}_{n\alpha}(z)\Big] + \tfrac{i}{2}\tau_r A_{rz-\mu\mu}\Psi^{(0)}_{n\alpha}(z-\mu)\Big] = \epsilon^{(1)}_{n\alpha}\Psi^{(0)}_{n\alpha}(z) \\ \\ -\sum_\mu & \Big[\Big[\Psi^{(2)}_{n\alpha}(z-\mu) - \Psi^{(2)}_{n\alpha}(z)\Big] + \tfrac{i}{2}\tau_r A_{rz-\mu\mu}\Psi^{(1)}_{n\alpha}(z-\mu) - \tfrac{1}{8}A_{rz-\mu\mu}A_{rz-\mu\mu}\Psi^{(0)}_{n\alpha}(z-\mu)\Big] \\ & = \epsilon^{(0)}_{n\alpha}\Psi^{(2)}_{n\alpha}(z) + \epsilon^{(1)}_{n\alpha}\Psi^{(1)}_{n\alpha}(z) + \epsilon^{(2)}_{n\alpha}\Psi^{(0)}_{n\alpha}(z) \\ \vdots & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \end{cases} \tag{3.41}$$

The first equation for $n = 0$ is the tree level discussed in subsection 3.2.1, i.e. $|\Psi^{(0)}_{n\alpha}> = |\psi_n> \otimes |e_\alpha>$. We set $n = 0$ and impose provisionally the normalization condition

$$< \Psi^{(0)}_{0\alpha} | \Psi^{(i)}_{n\alpha} > = 0 \qquad i \geq 1 \tag{3.42}$$

which means that the perturbative corrections to $|\Psi^{(0)}_{0\alpha}>$ lie in the orthogonal space $H_0^\perp$ of $H_0 = \{|\psi_0> \otimes |e_\alpha> / \alpha = 1,2,3,4\}$.
From the first order equation, we obtain for the eigenvalue $\epsilon^{(1)}_{0\alpha}$:

$$\epsilon^{(1)}_{0\alpha}\delta_{\alpha\beta} = -\frac{i}{2V}\sum_{z\in\Lambda}\sum_\mu \psi_0^*(z) A_{rz-\mu\mu}\psi_0(z-\mu)\frac{1}{2}\mathrm{Tr}(\tau_\alpha\tau_r\tau_\beta) \tag{3.43}$$

But the unitary condition $U(z-\mu,\mu) = U^\dagger(z-\mu)$ implies that $A_{rz-\mu\mu} = -A_{rz-\mu}$ and therefore the contributions to $\epsilon^{(1)}_{0\alpha}$ cancel in pairs

$$\epsilon^{(1)}_{0\alpha}\delta_{\alpha\beta} = \frac{-i}{2V}\sum_{z\in\Lambda}\sum_{\mu>0}(A_{rz-\mu\mu} + \mathcal{A}_{rz-\mu})\frac{1}{2}\mathrm{Tr}(\tau_\alpha\tau_r\tau_\beta) = 0 \qquad \alpha = 1,2,3,4 \tag{3.44}$$

where we have used that $\Psi_0(z) = 1$.

For the first order eigenvector, we obtain

$$\begin{aligned} \Psi^{(1)}_{0\alpha}(z) &= < z | \Psi^{(1)}_{0\alpha} > = \sum_{n\beta} < z | \psi_{n\beta} >< \psi_{n\beta} | \Psi^{(1)}_{0\alpha} > \\ &= \sum_{n\neq 0}\sum_\beta \psi_{n\beta}(z)\,\tfrac{1}{k_n^2}\,\tfrac{1}{V}\sum_{\bar{z}\mu}\psi_n^*(\bar{z})\,\tfrac{i}{2}\,A_{r\bar{z}-\mu\mu}\psi_0(\bar{z}-\mu)\,\tfrac{1}{2}\,\mathrm{Tr}(\tau_\beta\tau_r\tau_\alpha) + C^{(1)}_{\alpha r}\tau_r \end{aligned} \tag{3.45}$$

with $C^{(1)}_{\alpha r}$ some constants.

To simplify the algebra and without loss of generality, we choose $a^{(1)}_\alpha$ of eqn. (3.38) such that

$$\begin{cases} \mathcal{A}^{(1)}(z) &= \frac{1}{V}\sum_{n\neq 0}\frac{1}{2k_n^2}A_{rno}\,i\,\tau_r\,\psi_n(z) \\ \\ C^{(1)}(z) &= -\frac{1}{V}\sum_{n\neq 0}\frac{1}{2k_n^2}A^*_{rno}\,i\,\tau_r\,\psi_n^*(z) \end{cases} \tag{3.46}$$

where we have used the compact notation

$$A_{rno} = \frac{1}{V}\sum_{z\mu} \psi_n^*(z) A_{rz-\mu\mu} \psi_0(z-\mu) \tag{3.47}$$

The second order contribution to ϵ_0 comes out as

$$\epsilon_{0\alpha}^{(2)}\,\delta_{\alpha\beta} \equiv \left[\frac{1}{8}(A_r^2)_{oo} - \sum_{n\neq 0}\frac{1}{4\hat{k}_n^2}\sum_r |A_{rno}|^2\right]\delta_{\alpha\beta}\ , \tag{3.48}$$

and the second order eigenvector as

$$\begin{aligned}\Psi_{0\alpha}^{(2)}(z) \;\equiv & -\sum_{n,m\neq 0}\tfrac{1}{2\hat{k}_n^2}\tfrac{1}{2\hat{k}_m^2} A_{rnm}A_{smo}\,\psi_n(z)\,\tfrac{1}{2}\,\mathrm{Tr}\,(\tau_\beta\tau_r\tau_s)\,\tau_\beta \\ & -\tfrac{1}{8}\sum_{n\neq 0}\tfrac{1}{2\hat{k}_n^2}\,(A_r^2)_{n0}\,\psi_n(z)\,\tau_\alpha + C_{\alpha r}^{(2)}\tau_r\ ,\end{aligned} \tag{3.49}$$

where we have used the notations

$$\left\{\begin{array}{ll} (A_r^2)_{oo} & = \frac{1}{V}\sum_{z\mu}\psi_0^*(z) A_{rz-\mu\mu}A_{rz-\mu\mu}\psi_0(z-\mu) \\ (A_r^2)_{no} & = \frac{1}{V}\sum_{z\mu}\psi_n^*(z) A_{rz-\mu\mu}A_{rz-\mu\mu}\psi_0(z-\mu) \\ A_{rnm} & = \frac{1}{V}\sum_{z\mu}\psi_n^*(z) A_{rz-\mu\mu}\,\psi_m(z-\mu) \\ |A_{rno}|^2 & = A_{rno}A_{rno}^* \end{array}\right. \tag{3.50}$$

and $C_{\alpha r}^{(2)}$ are some constants.

Again, to remove the degeneracy of eqn. (3.48) we choose $a_\alpha^{(2)}$ in eqn. (3.38) such that the second order correction lies (except for the effect of normalization $C\mathcal{A}=1$) also in $H_0^\perp$. We obtain

$$\left\{\begin{array}{ll} \epsilon_0^{(2)} = & \frac{1}{8}(A_r^2)_{oo} - \sum_{n\neq 0}\frac{1}{4\hat{k}_n^2}\,\Sigma_r\,|A_{rno}|^2 \\ \mathcal{A}^{(2)}(z) = & -\frac{1}{V}\sum_{n\neq 0}\frac{1}{2\hat{k}_n^2}\left\{\sum_{m\neq 0}\frac{1}{2\hat{k}_m^2}A_{rnm}A_{smo} + \frac{1}{4}(A_r^2)_{no}\right\}\psi_n(z)\,\hat{\mathbb{1}} \\ & -\sum_{n,m\neq 0}\frac{1}{2\hat{k}_n^2}\frac{1}{2\hat{k}_m^2}A_{rnm}A_{smo}\,\epsilon_{rst}\psi_n(z) i\,\tau_t + N \end{array}\right. \tag{3.51}$$

where we have used the notation of eqn.(3.50). The last term N is determined by the normalization condition of eqn. (3.32):

$$N = -\frac{1}{2V}\sum_{n\neq 0}\frac{1}{(2\hat{k}_n^2)^2}\sum_r |A_{rno}|^2\,\psi_0(z)\hat{\mathbb{1}} \tag{3.52}$$

The averaging kernel $C^{(2)}(z)$ is equal to $\mathcal{A}^{(2)\dagger}(z)$.

3.3 Loop expansion for the scalar fields

As already explained in section 2.1 the power of $\hbar$ (Planck's constant) counts the number of loops in the expansion of the constraint effective potential. In fact, the loop expansion is a saddle-point expansion about the absolute minimum of the action (the classical field). Thus, if we want to obtain the one loop expansion in the scalar field, we have to shift it by the classical part, i.e. $\tilde{\phi} = \eta + \hbar^{1/2}\phi$, and perform the ϕ-integration retaining up to quadratic contributions in ϕ in the action. The covariant shift was mentioned in the introduction.

3.3.1 General idea of the scalar loop computation

We consider the scalar integration of eqn.(3.8).

$$\exp\left(-VU_V(\Phi,A)\right) = \int \mathcal{D}\phi\,\delta(\Phi - C\phi)\exp(-S_U[\phi,A]) \tag{3.53}$$

where the action S_U is given by

$$S_U[\phi,A] = \sum_z\left\{\frac{1}{2} < \phi_z\,|\,-\Delta_U\,\phi_z > + \frac{1}{2}\,m_0^2 < \phi_z\,|\,\phi_z > + \frac{\lambda_0}{4!} < \phi_z\,|\,\phi_z >^2\right\} \tag{3.54}$$

We use the notation and definitions explained in section 3.1, and introduce the gauge variable

$$A_{rz\mu} = \frac{2}{g}\,a_{rz\mu} \tag{3.55}$$

which is more appropriate for the continuum limit $g \to 0$.

To perform this integral, let us define the operator P acting on the Hilbert space $H = \{\phi : \Lambda \to G_\rho | G_\rho$ is the linear span of $SU(2)\}$. It is constructed from the interpolating and averaging operators $\mathcal{A}$ and C of the previous section as follows

$$P = \mathcal{A}C \quad \text{with kernel} \quad P(z_1,z_2) = \mathcal{A}(z_1)C(z_2)\,V \tag{3.56}$$

Obviously $P : H \to H$ satisfies the properties $P^2 = P$ (idempotent), $P^\dagger = P$ and $\|P\| < 1$ (P is a bounded linear operator). This means that P is a projection operator upon a subspace H_C, so that each vector $\phi \in H$ can be decomposed uniquely in the form with

$$\phi = \eta + \xi \quad \text{with} \quad \xi \in H_C^\perp \quad \text{and} \quad \eta \in H_C \tag{3.57}$$

$H_C^\perp$ in the orthogonal complement of H_C. ξ in eqn.(3.57) is the orthogonal projection of ϕ upon $H_C^\perp$. Using that $(H_C^\perp)^\perp = H_C$, we write

$$\phi = \eta + \xi := P\phi + (1-P)\phi \tag{3.58}$$

Therefore, to perform the ϕ integration of eqn.(3.53), we may instead integrate over the shifted field $\xi = \phi - \eta \in H_C^\perp$ with

$$\eta = P\phi = \mathcal{A}\Phi \ \text{ fixed , and } \ \Phi = C\phi \tag{3.59}$$

We integrate over all fields $\xi \in H_C^\perp$ (the high frequency fluctuation field) <u>without</u> further constraint (per construction the fluctuation field ξ satisfies automatically $C\xi = 0$).

Inserting the covariant shift of eqn.(3.58) and keeping up to the quadratic contribution in the fluctuation field, we obtain

$$\exp\left(-VU_V(\Phi,A)\right) = \int \mathcal{D}\xi \exp(-S_{sc}[\eta]) \exp\left(\sum_z \left\{-\tfrac{1}{2} < \xi_z | M_z\, \xi_z > + < \xi_z \,|\, J_z >\right\}\right) \times \{1 + 0(\xi_z^3)\ldots\} \tag{3.60}$$

where

$$\begin{cases} S_{sc}[\eta] &= \sum_z \left\{\frac{1}{2}(\epsilon_0 + m_0^2) < \eta_z \,|\, \eta_z > + \frac{\lambda_0}{4!} < \eta_z \,|\, \eta_z >^2\right\} \\ J_z^\alpha &= \frac{\lambda_0}{6} < \eta_z \,|\, \eta_z > \eta_z^\alpha \end{cases} \tag{3.61}$$

ϵ_0 is the smallest eigenvalue of the negative covariant Laplacian with p.b.c. discussed in the last section (see eqn.(3.30)). We note that

$$\sum_z < \eta_z | -\Delta_U \eta_z >= V\,\epsilon_0\,\Phi^2 \;\text{ and }\; \sum_z < \eta_z \,|\, \eta_z >= V\,\Phi^2 \tag{3.62}$$

because $\mathcal{A} = C^\dagger$ and $C\,C^\dagger = 1$. Using the standard formula for the Gaussian integral

$$\int \mathcal{D}\phi \exp\left\{-\sum_z \left\{\frac{1}{2} < \phi_z | M_z \phi_z > + < \phi_z | J_z >\right\}\right\} = (\det M)^{1/2} \exp\left(\frac{1}{2}\sum_z < J_z | M_z^{-1} J_z >\right) \tag{3.63}$$

we obtain from eqn.(3.60) up to one loop order

$$\exp\left(-VU_V(\Phi,A)\right) = \exp(-S_{sc}[\eta]) \exp\left(\frac{1}{2}\sum_z < J_z \,|\, M_z^{-1} J_z >\right) \exp\left(-\frac{1}{2}\,\mathbf{Tr}_{H_C^\perp}\{ln M\}\right) \tag{3.64}$$

where $H_C^\perp$ stands for the subspace $\{\xi \in H \,|\, C\xi = 0\}$, and $\mathbf{Tr}_{H_C^\perp}$ means the trace over the subspace $H_C^\perp$.

In the absence of gauge fields, it is easy to give an explicit description of H_C and $H_C^\perp$. In fact $H_{C^{(0)}}^\perp = \left\{\phi \in H \,|\, \tilde{\phi}_F(k=0) = 0\right\}$ and $H_{C^{(0)}} = \{\phi \in H \,|\, \phi \text{ constant }\}$.
Thus eqn.(3.64) is equivalent to eqn.(2.13) of chapter 1 in this case.

3.3.2 Explicit computation up to second order in g

We want to use the "master" formula eqn.(3.64) to compute the CEP up to one loop in the scalar fields and up to second order in the gauge coupling constant g.

From the action of eqn.(3.4), we obtain for the quadratic contribution

$$< \phi | M \phi >= \phi_\alpha [(-\Delta_U + m_0^2 + \frac{\lambda_0}{6} < \eta | \eta >)\, \delta_{\alpha\beta} + \frac{\lambda_0}{3} \eta_\alpha \eta_\beta]\, \phi_\beta =: \phi_\alpha M_{\alpha\beta}\, \phi_\beta \tag{3.65}$$

where we have used the isomorphism between the 2 x 2 matrices in the span $\rho SU(2)$ of $SU(2)$ and the $\mathbb{R}^4$ vector space given by

$$\eta = \eta_0 + i\tau_r \eta_r \in \rho SU(2) \;\Leftrightarrow\; (\eta)_\alpha \in \mathbb{R}^4 \tag{3.66}$$

$\eta = \mathcal{A}\Phi$ with $\mathcal{A}$ the interpolation operator. The covariant Laplacian $-\Delta_U$ acts on real 4-vectors and its expansion in power of g is computed in Appendix B up to second order in g.

To compute the trace of eqn.(3.64), we need M only on $H_C^\perp$. In order to avoid an infinite constant factor coming from the logarithm, we change M on H_C:

$$M_C := [-\Delta_U + m^2(\eta)]_C = \begin{cases} [-\Delta_U + m^2(\eta)] & on\;\; Hc^\perp \\ \mathbb{1} & on\;\; H_C \end{cases} \tag{3.67}$$

where the matrix $m^2(\eta)$ is defined by

$$m^2(\eta) = (m_0^2 + \frac{\lambda_0}{6} < \eta \,|\, \eta >)\, \delta_{\alpha\beta} + \frac{\lambda_0}{3}\, \eta_\alpha \eta_\beta \tag{3.68}$$

Thus we can write eqn.(3.64) as

$$U_V(\Phi,A) = U_V[\eta] + \frac{1}{2V}\,\mathbf{Tr}\left\{ln[-\Delta_U + m^2(\eta)]_C\right\} \tag{3.69}$$

where

$$U_V[\eta] = \frac{1}{V} S_{sc}[\eta] + \frac{1}{2V}\sum_z < J_z \,|\, M_z^{-1} J_z > \tag{3.70}$$

where the trace is now taken over the whole space H. Taking the trace involves summation over the matrix indices $\alpha, \beta, \ldots$ and over the space-time points $z, w, \ldots \in \Lambda$.

To simplify the computation of $U_V(\Phi, A)$, we decompose the trace in two terms, the scalar contribution coming from the $O(4)$ scalar theory and a gauge field dependent contribution

$$\begin{aligned} \mathbf{Tr} ln[-\Delta_U + m^2(\eta)]_C &= \mathbf{Tr} ln[-\Delta_U + m^2(\eta)]_C - \mathbf{Tr} ln[-\Delta + m^2(\Phi)]_{C^{(0)}} \\ &\quad + \mathbf{Tr} ln[-\Delta + m^2(\Phi)]_{C^{(0)}} - \mathbf{Tr} ln[-\Delta + m_0^2] + \text{constant} \\ &= \int_0^g dg' \tfrac{\partial}{\partial_{g'}} \mathbf{Tr} ln[-\Delta_U + m^2(\eta)]_C + ln(\det \Gamma^{(0)}) + \text{constant} \end{aligned} \tag{3.71}$$

where $C^{(0)}$ is the averaging operator of subsection 3.2 in the absence of gauge field, $\eta_z^{(0)} = \mathcal{A}^{(0)}(z)\Phi = \Phi$, see eqn. (3.39) and $ln(\det \Gamma^{(0)})$ is the contribution to the CEP to zeroth order in g which is obtained from consideration of the scalar theory without gauge fields.

The second term of eqn.(3.69) can be rewritten in a more convenient form

$$\begin{aligned} \tfrac{\partial}{\partial g} \mathbf{Tr}\left\{ln[-\Delta_U + m^2(\eta)]_C\right\} &= \mathbf{Tr}\left\{\tfrac{\partial}{\partial g} ln([-\Delta_U + m^2(\eta)]\,(1-P) + P)\right\} \\ &= \mathbf{Tr}\left\{\left(\tfrac{\partial}{\partial g}[-\Delta_U + m^2(\eta)]\,(1-P) + \right.\right. \\ &\quad \left.\left. [-\Delta_U + m^2(\eta)]\left(-\tfrac{\partial}{\partial g} P\right) + \left(\tfrac{\partial}{\partial g} P\right)\right)(\Gamma + P)\right\} \\ &= \mathbf{Tr}\left\{\left(\tfrac{\partial}{\partial g}[-\Delta_U + m^2(\eta)]\Gamma \right.\right. \\ &\quad \left.\left. -[-\Delta_U + m^2(\eta)]\,\left(\tfrac{\partial}{\partial g} P\right)(\Gamma + P) + \left(\tfrac{\partial}{\partial g} P\right) P\right\} \end{aligned} \tag{3.72}$$

where we have used that the inverse of $[-\Delta_U + m^2(\eta)]_C$ equals $[-\Delta_U + m^2(\eta)]^{-1}$ on $(1-P)H$ and $\mathbb{1}$ on the orthogonal complement H_C. Γ (the propagator for the high frequency field ξ) equals $[-\Delta_U + m^2(\eta)]^{-1}$ on $(1-P)H$ and zero on H_C. This implies $\Gamma = [-\Delta_U + m^2(\eta)]_C^{-1} - P$.

Finally, observing that

$$\begin{cases} \mathbf{Tr}\left\{\left(\frac{\partial}{\partial g}P\right)\Gamma\right\} = \mathbf{Tr}\left\{(\partial_g P)\,P\,\Gamma + P\,(\partial_g P)\,\Gamma\right\} = \mathbf{Tr}\left\{(\partial_g P)\,\Gamma\,P\right\} = 0\ , \\ (\Gamma + P)\,[-\Delta_U + m^2(\eta)]\,\partial_g P = [(1-P) + P[\epsilon_0 + m^2(\eta)]]\ \partial_g P \end{cases} \tag{3.73}$$

and after some algebra, we obtain for the gauge field-dependent potential

$$U_V(\mathbf{\Phi}, A) = U_V[\eta] + \frac{1}{2V}\int_0^g dg'\,\mathbf{Tr}\left\{\partial_{g'}[-\Delta_U + m^2(\eta)]\ \Gamma - P[\epsilon_0(g') + m^2(\eta)]\partial_{g'}P\right\} \tag{3.74}$$

where $\epsilon_0(g')$ is the smallest eigenvalue of the (negative) covariant Laplacian with periodic boundary condition of section 3.2, Γ is the ξ-propagator computed in Appendix B.

Therefore, to compute $U_V(\mathbf{\Phi}, A)$, we have to expand all the quantities involved in expression (3.74) in the gauge coupling constant and collect them consistently in each order. All these technical details are explained in Appendix B.

After a cumbersome, but straightforward computation, one obtains:

$$U_V[\eta] = V\left\{\frac{1}{2}\,(\epsilon_0^{(2)} + m_0^2)\,\Phi^2 + \frac{\lambda_0}{4!}\,\Phi^4\right\} + O(g^3) \tag{3.75}$$

and

$$U_V(\Phi, A) = U_V^{(0)}(\Phi) + g^2\,U_V^{(2)}(\Phi, A) \tag{3.76}$$

where

$$\begin{aligned} U_V^{(0)}(\Phi) = \quad & U_{cl}(\Phi) + \tfrac{1}{2V}{\sum_n}' ln\left(\hat{k}_n^2 + U'_{cl}(\Phi)\right) + \tfrac{3}{2V}{\sum_n}' ln(\hat{k}_n^2 + U'_{cl}(\Phi)/\,\Phi) \\ U_V^{(2)}(\Phi, A) = \quad & \tfrac{1}{2}\Phi^2\epsilon_0^{(2)} + \tfrac{1}{4V}\,\{\,(U'_{cl}(\Phi)/\Phi){\sum_n}'\tfrac{1}{(\hat{k}_n^2)^2}\sum_r |A_{rno}|^2 \\ & + \tfrac{1}{4V}{\sum_n}'(\Gamma_{n\phi_0}(\Phi) + 3\Gamma_{n\phi_r}(\Phi))\sum_r (A^2_{rnn} + \tfrac{1}{\hat{k}_n^2}|A_{rno}|^2) \\ & - \tfrac{1}{4V^2}{\sum_{n,m}}'(2\Gamma_{n\phi_r}(\Phi)\Gamma_{m\phi_r}(\Phi) + \Gamma_{n\phi_0}(\Phi)\Gamma_{m\phi_r}(\Phi) + \Gamma_{n\phi_r}(\Phi)\Gamma_{m\phi_0}(\Phi))\sum_r |A_{rnm}|^2 \\ & + \tfrac{\lambda_0}{12}\Phi^2{\sum_n}'(4\Gamma_{n\phi_r}(\Phi) + \Gamma_{n\phi_0}(\Phi))\tfrac{1}{(\hat{k}_n^2)^2}\sum_r |A_{rno}|^2\,\} + O(g^4) \end{aligned} \tag{3.77}$$

with

$$\begin{cases} \Phi = \|\mathbf{\Phi}\|, \quad U_{cl}(\Phi) = \frac{1}{2}\,m_0^2\,\Phi^2 + \frac{\lambda_0}{4!}\Phi^4 \qquad \hat{k}_n^2 = 4\sum_\mu \sin^2(\frac{\pi}{L}n_\mu) \\ \Gamma_{n\phi_0}(\Phi) = (\hat{k}_n^2 + U'_{cl}(\Phi))^{-1} \qquad \Gamma_{n\phi_r}(\Phi) = (\hat{k}_n^2 + U'_{cl}(\Phi)/\Phi)^{-1} \end{cases} \tag{3.78}$$

and $A_{rno}, A^2_{rnn}, A_{rnm}$ are defined in eqns. (3.46) and (3.50) and the eigenvalue $\epsilon_0^{(2)}$ is given

by eqn. (3.48). U'_{cl} and U''_{cl} are the first and second derivative of U_{cl}, and prime $'$ on the sum means that the $n = 0$ term is omitted.
We see explicitly that the expression (3.78) depends only on the absolute value $\Phi = \|\mathbf{\Phi}\|$ of the covariant magnetization $\mathbf{\Phi}$ according to eqn.(3.17).

3.4 Integration of the gauge and ghost fields

In the Wilson formulation of lattice gauge theory [21], it is in principle not necessary to fix the gauge because the gauge orbits are themselves compact. In the usual continuum formulations, the gauge fixing term eliminates a divergence from integrating over all gauges. This is related to the gauge invariance of the theory and is reflected in the fact that the quadratic operator $k_{\mu\nu} = \hat{k}^2\delta_{\mu\nu} - \hat{k}_\mu\hat{k}_\nu$ of the gauge action is idempotent, i.e. $k_{\mu\lambda}k_{\lambda\nu} \propto k_{\mu\nu}$ (the operator k projects out the transverse degrees of freedom of the gauge field) and therefore does not have an inverse. For a nice intuitive discussion of the problem, see chapter 9 of Cheng and Li [16]. We will however need to introduce a gauge fixing condition in order to formulate perturbation theory in the gauge coupling constant. This perturbation expansion is obtained performing a saddle point expansion about the absolute minimum of the total action, which includes the gauge fixing term S_{gf} and the Faddeev-Popov ghost action S_{FP}.

For physical observables which are gauge invariant, this makes no difference. As we will see, the gauge covariant effective potential permits to analyse the phase structure and questions of "symmetry breaking" in a gauge invariant way without contradicting Elitzur's theorem [28].

3.4.1 Gauge fixing condition and Faddeev-Popov determinant

In this subsection, we will fix the gauge. We discuss this issue and compute the Faddeev-Popov determinant following the treatment by Lüscher [29]. The reader is referred to this paper for a general and rigorous discussion of gauge fixing and Faddeev-Popov determinant on lattices with boundary. For a nice and geometric intuitive presentation of this subject see reference [30].

We first notice that the integral appearing in the definition of the covariant constraint effective potential of eqn. (3.8) is invariant under the subgroup $\mathcal{G}$ of $SU(2)$ defined by:

$$\mathcal{G} = \{G(z) \in SU(2),\ z \in \Lambda\ \text{the lattice}\ /\ G(0) = 1\} \tag{3.79}$$

In fact, the action and the measure are invariant under the gauge transformation

$$\begin{cases} U(z,\mu) \ \longrightarrow U^G(z,\mu) = G(z+\mu)\,U(z,\mu)\,G^{-1}(z) \\ \phi_z \ \longrightarrow G(z)\,\phi_z \qquad\qquad\qquad\qquad G(z) \in SU(2) \end{cases} \tag{3.80}$$

and using the transformation law of the kernel $C(z)$ of eqn.(3.28), we see that the delta constraint of eqn.(3.8) is invariant under gauge transformations, which are the identity at the origin. Only the restricted set $\mathcal{G}$ of gauge transformations will be fixed.

$\mathcal{G}$ is a compact Lie group which acts in a differentiable manner on the field manifold. We will use a gauge fixing condition which does not involve the Higgs field. We will write $\mathcal{F}$ for the manifold of gauge fields. Thus

$$G \in \mathcal{G}, U \in \mathcal{F} \Longrightarrow G \cdot U = U^G \in \mathcal{F} \tag{3.81}$$

Suppose that the gauge fixing function F of the gauge field U has the following properties for a sufficiently small neighborhood $\mathcal{N}$ of $U_0 = 1$ independent of g

F1. F is a differentiable mapping from an open neighborhood $\mathcal{N} \subset \mathcal{F}$ of the vacuum configuration U_0 to the Lie algebra $\mathcal{L}_\mathcal{G}$ of $\mathcal{G}$.

F2. For any given $U \in [\mathcal{N}]$, there exists a unique gauge transformation g such that $g \cdot U \in \mathcal{N}$ and $F(g \cdot U) = 0$. Furthermore, $F(U_0) = 0$.

F3. For all $U \in \mathcal{N}$, we have $\det L(U) \neq 0$, where $L(U) : \mathcal{L}_\mathcal{G} \rightarrow \mathcal{L}_\mathcal{G}$ is the linear operator defined through

$$L(U) \cdot X = \delta_X F(U) \quad \text{for all} \quad X \in \mathcal{L}_\mathcal{G} \ , \tag{3.82}$$

$\mathcal{N}$ is the union of all gauge orbits passing through $\mathcal{N}$

$$[\mathcal{N}] = \{U \in \mathcal{F} \mid G \cdot U \in \mathcal{N} \text{ for some } G \in \mathcal{G}\} \ , \tag{3.83}$$

and $\mathcal{L}_\mathcal{G}$ is the Lie algebra of G, with $X = i\, X_r\, T^r$, $T^r = \frac{1}{2}\tau_r$ a general element of $\mathcal{L}_\mathcal{G}$ and the first order differential operator $\delta_X F(U)$ is defined by

$$\delta_X F(U) = X_a \left\{ \frac{\partial}{\partial Y_a} \, F(e^{-Y} \cdot U) \right\}_{Y} = 0 \tag{3.84}$$

It is shown in ref. [29] that this will imply that

$$\int_\mathcal{F} \mathcal{D}\mathcal{U}\, f(U) = K_\epsilon \int_{N_\epsilon} \mathcal{D}\mathcal{U} f(U) \det L(U) \exp(-S_{gf}[U]) \tag{3.85}$$

where K_ϵ is a constant independent of f, $\mathcal{D}U$ the invariant $SU(2)$ measure, and

$$N_\epsilon = \{U \in \mathcal{N} \mid (F(U), F(U)) < \epsilon\} \tag{3.86}$$

(X, Y) is a positive definite and gauge invariant scalar product on the Lie algebra $\mathcal{L}_\mathcal{G}$, and

$$S_{gf}[U] = \frac{2}{\alpha\, g^2} \, (F(U), F(U)) \tag{3.87}$$

The gauge parameter α is introduced here for later convenience. Equation (3.85) is the desired expression, where $\det L(U)$ is the Faddeev-Popov determinant, which can be written in the exponent by introducing Faddeev-Popov ghosts c and $c^\dagger$ as usual

$$\int dc\, dc^\dagger \exp(-(c^\dagger, L(U) \cdot c)) = \det L(U) \tag{3.88}$$

We choose the gauge fixing function $F(U)$ to be

$$F(U)(z) = \frac{g}{2} \sum_{\mu > 0} D^*_\mu A_{r z \mu} \tag{3.89}$$

where:

$$D^*_\mu f(z) = f(z) - f(z - \mu) \tag{3.90}$$

In a sufficiently small neighborhood of the vacuum configuration $U_0(z, \mu) = 1$ the link variable $U(z, \mu)$ will be represented according to eqn.(3.2). Now we will show that F has the properties F1. - F3. This implies that the solutions to the equation $F(U) = 0, U \in \mathcal{N}$, form a smooth submanifold which meets any gauge orbit passing through N at exactly one point.

It is clear that F has the property F1. To show F2., we pick out a gauge field in the infinitesimal neighborhood of U_0. Now if $G(z) = \exp(-i\, \omega^a(z)\, T^a), \omega^a$ infinitesimal, one finds

$$F(gU) = 0 \ \Rightarrow \ \sum_{\mu > 0} D^*_\mu (1 - i D_\mu\, \omega^a(z)\, T^a + \ldots) = 0 \ . \tag{3.91}$$

The unique solution (due to the periodic boundary conditions) is a constant $\omega^a(z) = \omega^a$. But the Lie algebra $\mathcal{L}_\mathcal{G}$ contains all the elements $X(z) = i\, X^a(z)\, T^a$ which vanish at the origin

$$X^a(0) = 0 \tag{3.92}$$

This implies that $\omega^a(z) = 0$ and F2. follows. We can extend this argument for a sufficiently small neighborhood of U_0 by applying recursively this reasoning on each elemental step of a (large) series of infinitesimal transformations (the exponentiation of the Lie algebra gives the Lie group). To show that F3. is also fullfilled, we have first to compute $L(U)$. It is straightforward to show that the gauge field transforms under an infinitesimal gauge transformation according to

$$\delta A^\omega_{t z \mu} = -\frac{1}{g}(1 - a_{z\mu})\, D_\mu \omega^t(z) + \epsilon_{trs}[w^r(z) + \frac{1}{2}\, D_\mu \omega^r]\, A_{s z \mu} \tag{3.93}$$

therefore the ghost action is given by

$$\begin{aligned} S_{FP}[A, c^\dagger, c] \ &= (c^\dagger, L(U)\, c) = \sum_{z, z'} c^\dagger_{rz}\, D^*_\mu \left(\tfrac{\delta A_{r z \mu}}{\delta \omega^s(z')} \right) \, c_{s z'} \\ &= \sum_{z \mu} \Big\{ (c^\dagger_{r z+\mu} - c^\dagger_{rz})(1 - a_{rz})(c_{rz+\mu} - c_{rz}) - \epsilon_{rst} a_{rz\mu} (c^\dagger_{sz+\mu} + c^\dagger_{sz})(c_{tz+\mu} - c_{tz}) \Big\} \end{aligned} \tag{3.94}$$

To verify that $F(U)$ of eqn.(3.89) has the property F3., it is sufficient to consider the case of $A_{rz\mu}$ infinitesimal. From eqn. (3.94) we see that in the leading order contribution the linear operator L is the Laplacian and thus if

$$\sum_\mu D^*_\mu\, D_\mu\, X = 0 \quad \text{for} \quad X \in \mathcal{L}_\mathcal{G} \tag{3.95}$$

then $X = 0$ because of the periodic boundary condition and the condition of eqn.(3.92).

Finally, using arguments of continuity in the gauge field, one can extend this results and show that $\det L(U) \neq 0$ for a sufficiently small neighborhood of U_0.

3.4.2 One loop lattice Constraint Effective Potential

Having fixed the gauge using eqn.(3.89), we can use the expression (3.85) and rewrite the covariant constraint effective potential for the Higgs field of eqn.(3.8) as

$$\exp\left(-V U_V(\Phi)\right) = \int \mathcal{D}\phi_z\, \mathcal{D}\mathcal{U}\, \mathcal{D}c^\dagger_{rz}\, \mathcal{D}c_{rz}\, \delta(\Phi - C\phi) \exp(-S_{tot}[\phi, A, c^\dagger, c]) \tag{3.96}$$

where the total action $S_{\rm tot}$ contains the original action S of eqn.(3.4), the gauge fixing term S_{gf} and the Faddeev-Popov ghost term S_{FP}

$$S_{\rm tot}\,[\phi, A, c^\dagger, c] = S[\phi, A] + S_{gf}[A] + S_{FP}[A, c^\dagger, c] \tag{3.97}$$

where

$$\begin{cases} S_{gf}[A] &= \frac{1}{2\alpha}\sum_x \sum_{\mu\nu} D^*_\mu\, A_{x\mu}\, D^*_\nu\, A_{x\nu} \\ \\ S_{FP} &= \sum_{x\mu} \left\{ (c^\dagger_{x+\mu} - c^\dagger_x)(1 - a_{x\mu})(c_{x+\mu} - c_x) - \epsilon_{xst} a_{x\mu} (c^\dagger_{sx+\mu} + c^\dagger_{sx})(c_{tx+\mu} - c_{tx}) \right\} \end{cases} \tag{3.98}$$

For the invariant $SU(2)$ Haar-measure, we have

$$\mathcal{D}U \propto \prod_{x\mu} (1 - a_{x\mu} a_{x\mu})^{-1/2}\, d\, a_{x\mu} \propto \prod_{x\mu} \exp\left(\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} (a_{x\mu} a_{x\mu})^n \right)\, d\, a_{x\mu} \tag{3.99}$$

According to the result of Appendix B.5, B.6 and B.7, expression (3.76) for $U_V(\Phi, A)$ has no contribution of order g. Therefore the terms of order g in S_{FP} will not affect the constraint effective potential. The same is true for the terms of order g^2 because after the a-integration, the $c, c^\dagger$-dependent terms in the action are Φ-independent. Therefore, we can ignore the a-dependent terms in S_{FP}, since $a_{x\mu} = 0(g), a_x = 0(g^2)$. Thus, keeping in eqn.(3.98) terms up to second order in the gauge and ghost fields, one obtains after the scalar integration

$$\begin{aligned} \exp\left(-V U_V(\Phi)\right) = \; & \int \mathcal{D}A_{x\mu}\, \mathcal{D}c^\dagger_x\, \mathcal{D}c_x \exp\left(-U_V(\Phi, A)\right) \exp\left(-S^{(q)}_g[A]\right) \\ & \exp\left(-S^{(q)}_{FP}[c^\dagger, c]\right) \{1 + \ldots\} \end{aligned} \tag{3.100}$$

where the dots represent the Taylor expansion of the higher order terms in the gauge and ghost fields, which contribute to the higher loop orders. Observe that the quadratic contribution coming from the measure (the piece with $n = 1$ in eqn.(3.95) will be left as a two-point vertex [9] and [31]. This corresponds to the freedom of shifting the squared masses appearing in the propagators according to $m^2 \to m^2 + \delta m^2$. In order to compensate the shift δm^2, one has to take into account also two-point vertices in perturbation theory proportional to $-\delta m^2$. The convergence properties of the bare perturbation theory depends on the choice of δm^2. This freedom can be used to get a better approximation. $S^{(q)}_g$ and $S^{(q)}_{FP}$ are the whole gauge and ghost quadratic contributions to the action and $U_V(\Phi, A)$ is given by eqn.(3.76),

$$\begin{cases} S^{(q)}_g[\Phi, A] &= U^{(2)}_V(\Phi, A) + S^{(q)}_W[A] + S_{gf}[A] \\ \\ S^q_{FP}[c^\dagger, c] &= \sum_{x\mu} D_\mu\, c^\dagger_x\, D_\mu\, c_x \end{cases} \tag{3.101}$$

where $S^q_W[A]$ is the quadratic contribution in the gauge field $A_{x\mu}$ coming from the Wilson action and $U^{(2)}_V(\Phi, A)$ is given by eqn.(3.77).

It is easy to see that the ghost fields decouple in this order.

Since the integrand of eqn.(3.100) is quadratic in A, the A-integration can be performed right away with the result

$$U_V(\Phi) = U^{(0)}_V(\Phi) + \frac{1}{2V}\, \mathrm{Tr}\, \left[ln \left\{ \frac{1}{2} \frac{\delta}{\delta A_{x\mu}} \frac{\delta}{\delta A_{x\nu}} \left(S^{(q)}_g[\Phi, A] \right) \right\} \right]_{A=0} \tag{3.102}$$

The trace in eqn.(3.102) involves summation over the discrete $O(4)$-matrix indices and a sum over space-time points on a lattice Λ (see Appendix B).

Eqn.(3.102) together with eqn.(3.76) and eqn.(3.77) is our final result for the one loop Constraint Effective Potential for arbitrary volume.

3.4.3 Large volume limit expansion

We study the expression (3.102) in the large volume limit ($V \to \infty$). From eqn.(3.77), we see that with the exception of $\epsilon^{(2)}_0$, the contributions from $U^{(2)}_V(\Phi, A)$ to the CEP are of order $1/V$ smaller when compared with the other terms of $S^q_g[\Phi, A]$ of eqn.(3.101).

Introducing the Fourier transformed variables

$$A_{xk\mu} = \sum_{x\in\Lambda} \exp(-i(k, z) - \tfrac{i}{2} k_\mu) A_{x\mu}\ , \quad A_{x\mu} = \tfrac{1}{V} \sum_k \exp(i(k, z) + \tfrac{i}{2} k_\mu) A_{xk\mu}\ , \tag{3.103}$$

one can work out the leading contribution to the one loop CEP

$$\begin{aligned} S^{(2)}_g &= \beta \sum (1 - \tfrac{1}{2}\, \mathrm{Tr}\, U_p)^{(2)} \\ &= \frac{\beta}{2} \sum_{\mu\nu}^{P} \sum_x \{ (a_{x\mu} + a_{x+\mu\nu} + a_{x+\nu\mu} + a_{x\nu})^2 + [a_{x\mu} a_{x+\mu\nu} - a_{x\mu} a_{x+\nu\mu} - a_{x\mu} a_{x\nu} \\ &\quad - a_{x+\mu\nu} a_{x+\nu\mu} - a_{x+\mu\nu} a_{x\nu} + a_{x+\nu\mu} a_{x\nu}] \} \\ &= \frac{\beta}{2} \sum_{\mu\nu} \sum_x \{ a_{x\mu} a_{x\mu} + a_{x\nu} a_{x\nu} \} + \frac{\beta}{2V} \sum_{\mu\nu} \sum_k 2 \cos(\tfrac{k_\mu + k_\nu}{2}) a_{xk\mu} a^*_{xk\nu} \\ &\quad - \frac{\beta}{2V} \sum_{\mu\nu} \sum_k (\exp(-ik_\nu) a_{xk\mu} a^*_{xk\mu} + \exp(ik_\mu) a_{xk\nu} a^*_{xk\nu}) - \frac{\beta}{2V} \sum_{\mu\nu} \sum_k 2 \cos(\tfrac{k_\mu - k_\nu}{2}) a_{xk\mu} a^*_{xk\nu} \\ S^{(2)}_g &= \tfrac{1}{2V} \textstyle\sum_{\mu\nu} \sum_k \left\{ \hat{k}^2 \delta_{\mu\nu} - \hat{k}_\mu \hat{k}_\nu \right\} A_{xk\mu} A_{xk\nu} \end{aligned} \tag{3.104}$$

Anagolously

$$\tfrac{1}{2\alpha} \sum_x \sum_{\mu\nu} (A_{x\mu} - A_{x-\mu\mu})(A_{x\nu} - A_{x-\nu\nu}) = \frac{1}{2\alpha} \frac{1}{V} \sum_{\mu\nu} \sum_k \hat{k}_\mu \hat{k}_\nu\, A_{xk\mu} A^*_{xk\nu} \tag{3.105}$$

Finally from (3.51)

$$\epsilon^{(2)}_0 = \frac{1}{8} (A^2_x)_{00} - \sum_{n\neq 0} \frac{1}{4\hat{k}^2_n} \sum_x |A_{xn0}|^2 \tag{3.106}$$

with

$$V(A_r^2)_{00} = \sum_{z\mu} A_{rz-\mu\mu}A_{rz-\mu\mu} = \sum_{z,\mu>0}\left(A_{rz-\mu\mu}A_{rz-\mu\mu} + A_{rz+\mu-\mu}A_{rz+\mu-\mu}\right)$$

$$= \sum_{z\mu>0} 2A_{rz\mu}A_{rz\mu} = \frac{2}{V}\sum_{\mu>0}\sum_{k_1,k_2}\left(\frac{1}{V}\sum_z \exp(i(k_1,z)+\frac{i}{2}k_{1\mu}\right) A_{rk_1\mu}\exp\left(-i(k_2,z)-\frac{i}{2}k_{2\mu}\right)\left(A^*_{rk_2\mu}\right) \qquad (3.107)$$

$$= \frac{2}{V}\sum_{\mu>0}\sum_k A_{rk\mu}A^*_{rk\mu}$$

and

$$I_2 = \frac{1}{V}{\sum_n}' \beta_n \sum_r |A_{rno}|^2$$

$$= -\frac{1}{V}{\sum_n}'\beta_n \sum_r \sum_{z\mu}\Psi^*_n(z)\,A_{rz-\mu\mu}\sum_{\bar{z}\nu}\Psi_n(\bar{z})A_{r\bar{z}-\nu\nu}$$

$$= -\frac{1}{V}{\sum_n}'\beta_n \sum_r \sum_{\mu,\nu>0}\sum_{k_1,k_2}\left(\frac{1}{V}\sum_z e^{-ik_n z}\,e^{i(k_1,z)}(-2i)\sin(k_{1\mu}/2)\right) \qquad (3.108)$$

$$\left(\frac{1}{V}\sum_{\bar{z}} e^{i(k_n\bar{z})}\,e^{-i(k_2\bar{z})}\,2i\sin(k_{2\nu}/2)\right) A_{rk_1\mu}A^*_{rk_2\nu}$$

$$= -\frac{1}{V}{\sum_n}'\beta_n \sum_r \sum_{\mu,\nu>0}\hat{k}_{n\mu}\hat{k}_{n\nu}\,A_{rk\mu}\,A^*_{rk\nu}$$

Putting all the contributions together

$$S_g^q[\Phi,A] = \frac{1}{2V}\sum_k\sum_{\mu\nu}\delta_{rs}\left\{\delta_{\mu\nu}[\hat{k}^2+m_w^2(\Phi)] - \left(1-\frac{1}{a}+\frac{m_W^2(\Phi)}{\hat{k}^2}(\Phi)\Theta\left(\hat{k}^2\right)\right)\hat{k}_\mu\,\hat{k}_\nu\right\}A_{rk\mu}A^*_{sk\nu}$$

$$+O(1/V) \qquad (3.109)$$

where the vector mass $m_W(\Phi)$ is given by

$$m_W^2(\Phi) = \frac{1}{4}\,g^2\,\Phi^2 \qquad (3.110)$$

and $\Theta(\hat{k}^2)$ is equal to one for $\hat{k}^2 \neq 0$ and the interpolation to continuous $\hat{k}^2$ vanishes faster than $\hat{k}^2$ for $\hat{k}^2 \to 0$.

Thus, keeping the leading order $O(1)$ in the volume and using that in the infinite volume limit, the momentum sums go into integral in the first Brillouin zone

$$\frac{1}{V}\sum_k f(\hat{k}) \quad \stackrel{V\to\infty}{\longrightarrow} \quad \int_{-\pi/a}^{\pi/a}\frac{d^4k}{(2\pi)^4}\,f(\hat{k}) \qquad (3.111)$$

where the lattice momentum is defined by $\hat{k}_\mu = 2\,a\,\sin(k_\mu/2\,a)$, with a the lattice spacing,

we obtain

$$U_V(\Phi) = U_{cl}(\Phi) + \frac{1}{2}\int_{-\pi/a}^{\pi/a}\frac{d^4k}{(2\,\pi)^4}\,ln\left(\hat{k}^2+U''_{cl}(\Phi)\right) + \frac{3}{2}\int_{-\pi/a}^{\pi/a}\frac{d^4k}{(2\,\pi)^4}\,ln\left(\hat{k}^2+\frac{U'_{cl}(\Phi)}{\Phi}\right)$$

$$+\frac{9}{2}\int_{-\pi/a}^{\pi/a}\frac{d^4k}{(2\,\pi)^4}\,ln\left(\hat{k}^2+m_W^2(\Phi)\right)+O(\frac{1}{V}) \qquad (3.112)$$

with $m_W^2(\Phi)$ given by eqn. (3.110). We see explicitly that this expression for the one-loop CEP is independent of the gauge parametrization. Eqn.(3.112) also agrees with the expression obtained by Montway [9] for the conventional effective potential in the Landau gauge $\alpha = 0$.

3.4.4 Naive continuum limit of the CEP

In this subsection, we consider the infinite volume and the continuum limit of the effective potential of eqn. (3.112). This will enable us to study the well-known mechanism proposed by Coleman and Weinberg[12] that radiative corrections may produce spontaneous symmetry breakdown even if the semiclassical approximation does not indicate such breakdown.

In order to study the shape of the constraint effective potential near the critical surface, we have to renormalize the expression of eqn. (3.112) (the large cutoff effects are important in this region).

We can adopt a provisional renormalization scheme and fix the counterterms in the continuum, such that the effective potential has no UV-divergences in the limit when the cutoff Λ is removed.

If we set $\Lambda = \pi/a$ and consider the large cutoff limit ($a \to 0$), we obtain the standard cutoff regularized Euclidean momentum integrals.

$$\int_{-\pi/a}^{\pi/a}\frac{d^4k}{(2\pi)^4}\,f(\hat{k}) \quad \stackrel{a\to 0}{\longrightarrow} \quad \int^{\Lambda}\frac{d^4k}{(2\pi)^4}\,f(k) \qquad (3.113)$$

We consider a typical term of eqn. (3.112) in the infinite volume and large cutoff limit. After a straightforward computation, one obtains

$$\frac{1}{V}\sum_{\hat{k}} ln\,(\hat{k}^2+M) \quad \stackrel{V,\Lambda\to\infty}{\longrightarrow} \quad \frac{1}{32\pi^2}\left\{\Lambda^4(ln\,\Lambda-\frac{1}{4})+\Lambda^2 M + M^2(ln\frac{M}{\Lambda^2}-\frac{1}{2})+O\left(\frac{1}{\Lambda^2}\right)\right\} \qquad (3.114)$$

Therefore, we can obtain an UV-finite expression for the effective potential by adding the counterterms in the action, so that they cancel the UV-divergent part of the expression (3.112) (see ref. [32]). The induced change of the constraint effective potential is

$$\delta\,U_V(\Phi) = \frac{1}{64\pi^2}\left\{\left[U''_{cl}(\Phi)+3\frac{U'_{cl}(\Phi)}{\Phi}+9\mu_W^2(\Phi)\right]\Lambda^2\right.$$

$$\left.-2\left[(U''_{cl}(\Phi))^2+3\left(\frac{U'_{cl}(\Phi)}{\Phi}\right)^2+9\,\mu_W^4(\Phi)\right]\,ln\,\Lambda\right\} \qquad (3.115)$$

Inserting the counterterms into the 1-loop effective potential for the infinite volume and continuum limit, we obtain up to an irrelevant constant

$$U_{eff}(\Phi) = U_{cl}(\Phi) + \frac{1}{64\pi^2}\left\{(U''_{cl}(\Phi))^2\left[ln\left(U''_{cl}(\Phi)\right) - \frac{1}{2}\right]\right. \\ \left. +3\left(\frac{U'_{cl}}{\Phi}(\Phi)\right)^2\left[ln\left(\frac{U'_{cl}(\Phi)}{\Phi}\right) - \frac{1}{2}\right] + 9\mu_W^4(\Phi)\left[ln\left(\mu_W^2(\Phi)\right) - \frac{1}{2}\right]\right\} \tag{3.116}$$

with $U_{cl}(\Phi) = \frac{1}{2}m_r^2\Phi_r^2 + \frac{1}{4!}\lambda_r\Phi_r^4$ and $\mu_W^2(\Phi) = \frac{1}{4}g_r^2\Phi^2$.

The expression (3.116) is manifest gauge invariant and it agrees with the known result of Anna Hasenfratz and Peter Hasenfratz [11] for the conventional effective potential performed in the Landau gauge.

After the renormalization, the parameters appearing in the classical potential are the renormalized coupling constant m_r^2, λ_r and g_r. For small field Φ, the constraint effective potential of eqn. (3.116) will describe accurately the behaviour of the system near the critical surface if the parameters m_r^2, λ_r and g_r are chosen appropriately.

3.5 Results and discussion

We discuss the expression of eqn.(3.116) corresponding to the constraint effective potential in the infinite volume and large cutoff limit. We study its behaviour near the critical surface.

In principle, in the large cutoff region, where the radiative correction may influence the behaviour of the system, the higher loop correction become important and the one loop expression need not be reliable. But if the running coupling constants are fixed in an appropriate way, for example by means of renormalization group considerations, then the dangerous logarithms of eqn. (3.116) are under control [11] (become small), at least in the small field region, say for $\Phi \leq \Phi_{\min}$, where $\Phi_{\min}$ is the non-trivial (relative) minimum of the effective potential, if it exists.

Under these conditions, if we denote by m_r, λ_r and g_r the running coupling constants (corresponding to our intermediate renormalization), the radiative corrections to the effective potential $U_{eff}(\Phi; m_r^2, \lambda_r, g_r)$ are small and this expression should be reliable to study the region near the phase transition of the system (see A. and P.Hasenfratz [11]).

The condition can be met in the region $\lambda_r \sim O(g_r^4)$ and $m_r^2 \sim O(\lambda_r / g_r^2)$. Here, the scalar loop contributions are negligible compared to the other terms, and therefore we can drop them. From the condition for a non-trivial minimum at $\Phi_{\min}$, we find

$$U'_{eff}(\Phi_{\min}) = 0 \Longrightarrow \begin{cases} \Phi_{\min} &= \frac{2}{g_r} + O(1) \\ m_r^2 &= -\frac{2}{3}\frac{\lambda_r}{g_r^2} + O(\lambda_r) \end{cases} \tag{3.117}$$

This minimum is not illusory in contrast in the pure scalar case, where the minimum lies far outside the expected range of validity of the one-loop approximation and must be rejected as

an artifact of the one-loop approximation. The condition for the system to be in the Higgs phase is given by

$$U_{eff}(0) \geq U_{eff}(\Phi_{\min}) \implies \lambda_r \geq -\frac{27 g_r^4}{256\pi^2} \tag{3.118}$$

Expression (3.118) agrees with the corresponding one of ref. [11] in the Landau gauge $\alpha = 0$. The critical line is now given by

$$\lambda_r = -\frac{27}{256\pi^2} g_r^4 \tag{3.119}$$

Fig. [3] shows the crossover between the Higgs and the confinent phase. The gauge coupling was fixed to be $g_r^2 = 0.4$, the renormalized mass m_r^2 is given by eqn. (3.117) and the values for the scalar self coupling constant were chosen in both sides of the critical line (solid line). We observe that the minimum jumps from zero to $\Phi_{\min}$ when the parameters are varied from the symmetric to the Higgs phase. This means that the system has a weak first order transition. This is in agreement with the result of numerical studies [22].

Finally, we can also obtain a lower bound for the Higgs mass. In fact, from the second derivative of the effective potential at the minimum, one finds for the ratio

$$R = \frac{m_H^2(\Phi_{\min})}{m_W^2(\Phi_{\min})} \geq \frac{9}{64\pi^2} g_r^2 \tag{3.120}$$

We note that in this region, the infrared problems are under control. This is because the dimensionless mass drops to zero less fast than in a pure gauge theory, and thus perturbation theory remains applicable.
In conclusion, we can confirm the predictions of ref. [11] by means of the renormalized loop expansion of the constraint effective potential in the infinite volume and large cutoff limit.

There is a net gain by considering the constraint effective potential. Our computation yields an expression which is valid also on a finite lattice. Moreover, this quantity is numerically easier to compute than the conventional effective potential [6]. The CEP is not necessarily a convex function of the magnetization in contrast with the conventional effective potential. Therefore, one can search for phase transitions and determine its order by looking for relative minima.

In particular, one can study the influence of the gauge fields in other regions of the parameters space, starting for instance from the numerical analysis of the constraint effective potential for the pure scalar ϕ^4-theory without gauge fields made by Kuti et al. [32].

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A Appendix

In this appendix, we recall the Multigrid Monte Carlo method of Meyer and Mack [6] and present one criterium for its performance to beat critical slowing down (CSD).

A.1 Multigrid definitions

In order to define the algorithm, we must specify the following elements.

i) Multigrid

A multigrid $\Lambda = \Lambda_0 + \Lambda_1 + \ldots + \Lambda_N$ is introduced which consists of a sequence of coarse-grids Λ_j, called layers, of decreasing lattice spacing $a_j = L^{N-j} a_N$ where L is some integer ≥ 2.

Λ_N (called base) is the fundamental lattice of spacing a_N, where the field theory is defined.The points $x \in \Lambda_j$ of the j^{th} layer may be identified with hypercubes of side length a_j in the base space. If z is a point of the hypercube $y \in \Lambda_j$, then we write $z \in y$. Also if the point $x \in \Lambda_j$ is contained in the block $y \in \Lambda_k$ with $k < j$, then we write $x \in y$.

ii) Block spin fields

A sequence of block spins $\Phi^j(x)$ is defined with $x \in \Lambda_j$ as the block averages:

$$\Phi^j(x) = av_{z \in x}\, \phi(z) = C^j \phi(x) \qquad for \;\; z \in \Lambda_N \tag{A.1}$$

This definition of the block spins is particularly simple because the average of an average is again an average:

$$av_{x \in x'}\, \phi(x) = av_{x'' \in x'}\, av_{x \in x''}\, \phi(x) \tag{A.2}$$

This is very convenient for an iterative procedure (Multigrid).

iii) Linear decomposition

The fundamental field $\phi(z)$ is split into a term called "background field", which is determined by the block spin Φ^j and a "fluctuation field" ζ^j, which has vanishing block averages:

$$\phi(z) = a_j^d \sum_{x \in \Lambda_j} \mathcal{A}^j(z,x)\, \Phi^j(x) + \zeta^j(z) \tag{A.3}$$

where d is the dimension of the base space Λ_N.
If one requires that

$$av_{z \in y}\, \mathcal{A}^j(z,x) = a_j^{-d}\, \delta_{xy} \quad for \;\; x, y \in \Lambda_j \tag{A.4}$$

then the decomposition (A.3) is consistent with the block spin choice (A.1).

Following the renormalization group interpretation of the multigrid transform, we iterate the split (A.3) in each layer Λ_j:

$$\Phi^j(x) = \int_{x' \in \Lambda_{j-1}} \mathcal{A}^j(x,x')\, \Phi^{j-1}(x') + \varphi^j(x) \tag{A.5}$$

with

$$av_{x \in x'} \varphi^k(x) = 0 \;\; \forall \;\; x' \in \Lambda_{k-1}, k = 1, \ldots, N \tag{A.6}$$

Inserting this into Eq. (A.3), one finds:

$$\begin{aligned} \phi(z) = &\int_{x \in \Lambda_j} \mathcal{A}(z,x)\, \Phi^j(x) + \zeta^j(z) \\ \zeta^j = &\sum_{k=j+1}^{N} \int_{x \in \Lambda_k} \mathcal{A}(z,x) \varphi^k(x) \end{aligned} \tag{A.7}$$

The kernel $\mathcal{A}(z,x)$ for $x \in \Lambda_j$ is defined by induction as the convolution of $\mathcal{A}(z,y)$ with the kernel $\mathcal{A}(y,x)$ with $y \in \Lambda_{j+1}$, which appeared in the decomposition of Φ^{j+1}:

$$\mathcal{A}(z,x) = \int_{y \in \Lambda_{j+1}} \mathcal{A}^{j+1}(z,y)\, A(y,x) \qquad x \in \Lambda_j, z \in \Lambda_N \tag{A.8}$$

The kernel obey to the condition (A.4) which ensures the validity of (A.1).
One introduces a multigrid field $\varphi \in \Lambda$ by setting:

$$\begin{aligned} \varphi^0 = &\quad \Phi^0 \\ \varphi(x) = &\quad \varphi^j(x) \quad if \quad x \in \Lambda_j, j = 0, \ldots, N \end{aligned} \tag{A.9}$$

In terms of this field, one can re-express ϕ as:

$$\phi(z) = \int_{x \in \Lambda} \mathcal{A}(z,x)\, \varphi(x) \equiv \sum_{j=0}^{N} a_j^d \sum_{x \in \Lambda_j} \mathcal{A}(z,x)\, \varphi(x) \tag{A.10}$$

The field φ obeys to the constraints (A.6) for all layers except $j = 0$. These constraints determine $\varphi(x)$ at block centers $x = x_c \in \Lambda j$ in terms of the other variables (for simplicity, one chooses L odd).

Then, there are as many independent variables $\varphi(x)$ as $\phi(z)$, or equivalently φ is uniquely determined by ϕ. Thus, its probability distribution is determined by the probability distribution of ϕ. The idea is to map a (nearly) critical lattice field theory on base with hamiltonian $H(\phi)$ into a theory on Λ with hamiltonian $H(\varphi)$ which is hopefully not critical. To reach this aim, we have to solve the important problem how to choose the kernel $\mathcal{A}$ (coarse-grid operator) properly.

iv) Coarse-grid operator

The fundamental issue to beat critical slowing down is to choose $\mathcal{A}$ in such way that the different layers Λ_j decouple as much as possible. In fact, if the kernel $\mathcal{A}^j(z,x)$ are chosen to decay exponentially in the distance $|\, z - x \,|$ of z from the hypercube x, with decay length a_j, they introduce a non-locality in the horizontal direction over distances a_j only. But the

number of point $y \in \Lambda_k$ within the hypercube $x \in \Lambda_j$ increases exponentially with $k - j$ like $L^{d|k-j|}$. Therefore the coupling between layers Λ_j and Λ_k $(k > j)$ could induce correlations in Λ_k over distances of order L^{k-j} lattice spacings.

This shows that non criticality of the multigrid theory can only be achieved if the correlation between the low frequency field φ^j and the high frequency fields φ^k decay fast enough with $k - j$.

For weakly coupled models without spontaneous symmetry breaking, K. Gawedzlei and A. Kupianien [23] showed that the polymer representation of the theory on the multigrid admits convergent cluster expansions, and therefore is essentially non critical if one chooses the $\mathcal{A}$ kernel so that:

$$-\Delta_z\, \mathcal{A}^j(z,x) = const. \tag{A.11}$$

as a function of z on cubes $x \in \Lambda_j$.In fact, with these selections for $\mathcal{A}^j(z,x)$ it can be proved that [33]:

$$< \varphi^j(x)\, \varphi^k(y) >_0 = \nu^j(x,z)\, \delta_{jk} \qquad for \;\; x \in \Lambda_j, y \in \Lambda_k \tag{A.12}$$

where ν^j is the propagator on the layer Λ_j. The expectation value is defined according to the free theory:

$$< O(\varphi) >_0 = \int d\mu_v(\varphi) \exp\left(-H_o(\varphi)\right)\, O(\varphi) / \int d\mu_0(\varphi)\, \exp\left(-H_0(\varphi)\right) \tag{A.13}$$

That is, there is no correlation between different layers $\Lambda_j \neq \Lambda_k$ in the free theory.

Therefore, for weak coupling constant, one may follow the proposal of this rigorous analytical work based on renormalization group and cluster expansions and choose $\mathcal{A}^j(z,x)$ as:

$$\mathcal{A}^j(z,x) = v\, {C^j}^{\dagger}\, U_j^{-1} \tag{A.14}$$

where v is the free propagator on Λ_N
U_j the block spin propagator
${C^j}^{\dagger}$ the adjoint operator of C^j

The decomposition of the fundamental field ϕ induces a decomposition of the free propagator $v(z,z_2) = < \phi(z_1)\, \phi(z_2) >_0$. According to (A.7), it takes the form:

$$v(z_1\, z_2) = \sum_{j=0}^{N} \int \int_{x_1, x_2 \in \Lambda_j} \mathcal{A}(z_1, x_j)\, \nu^j(x_j, y_j)\, \mathcal{A}(z_2, y_j)\;\; z_1, z_2 \in \Lambda_N \tag{A.15}$$

These kernels $\mathcal{A}^j(z,x)$ have in fact the properties of exponential decay in $|\, z - x \,|$ over distances a_j and satisfies the constraints (A.4).

v) Effective Hamiltonian

Inserting the decomposition (A.7) into the free action $H_0 = \frac{1}{2}(\phi, -\Delta\phi)$, one obtains:

$$\begin{aligned} H_0 = &\; -\tfrac{1}{2} \int_{z \in \Lambda_N} \int_{x_j \in \Lambda_j} \int_{y_j \in \Lambda_j} \mathcal{A}(z,x_j) \Phi^j(x_j) \Delta_z \mathcal{A}(z,y_j) \Phi^j(y_j) \\ &-\tfrac{1}{2} \sum_{k=j+1}^{N} \sum_{l=j+1}^{N} \int_{z \in \Lambda_N} \int_{x_k \in \Lambda_k} \int_{y_l \in \Lambda_l} \mathcal{A}(z,x_k) \varphi_k(x_k) \Delta_z \mathcal{A}(z,y_l) \varphi_l(y_l) \\ &-\sum_{k=j+1}^{N} \int_{z \in \Lambda_N} \int_{y_l \in \Lambda_j} \int_{x_k \in \Lambda_k} \mathcal{A}(z,y_j) \Phi^j(y_j) \Delta_z \mathcal{A}(z,x_k) \varphi_k(x_k) \quad j < N \end{aligned} \tag{A.16}$$

And using (A.4) and (A.6), we find:

$$H_0 = \frac{1}{2}(\Phi^j, -\Delta^j_{eff}\,\Phi^j) + \sum_{k=j+1}^{N}\frac{1}{2}(\varphi^k, -\Delta^k_{eff}\,\varphi^k) = \frac{1}{2}\sum_{k=0}^{N}(\varphi^k, -\Delta^k_{eff}\varphi^k) \qquad (A.17)$$

where $\Delta^j_{eff}(x_j, y_j) = \int_{z\in\Lambda_N} \mathcal{A}(z, x_j)\Delta_z\mathcal{A}(z, y_j) \quad x_j, y_j \in \Lambda_j$.

By definition, the effective Hamiltonian at length scale a_j is a function of the block spin field $\Phi^j(x) \in \Lambda_j$ and is given by:

$$exp\left(H^j(\Phi^j)\right) = \int D\phi[\Pi_{x_j\in\Lambda^j}\delta\left(\Phi^j(x_j) - av_{z\in x_j}\phi(z)\right)]exp\left(-H(\phi)\right) \qquad (A.18)$$

Inserting the decomposition (A.7) into the Hamilton function, $H(\phi) = H_0(\phi) + V(\phi)$ with $V(\phi) = \int_z \nu\left(\phi(z)\right)$, we get:

$$H^j(\Phi) = H^j_0(\Phi) + \nu^j(\Phi) \qquad (A.19)$$

with:

$$\begin{cases} H^j_0(\Phi) = \frac{1}{2}(\Phi^j, -\Delta^{(j)}_{eff}\Phi^j) = -\frac{1}{2}\int\int_{x_jy_j\in\Lambda_j}\Phi^j(x_j)\Delta^j_{eff}(x_j, y_j)\Phi^j(y_j) \\ \exp\left(-\nu^j(\Phi_j)\right) = N_j\int\left[\Pi'_{x\in\Lambda_{\geq j+1}}d\varphi(x)\right]\exp\left(-\sum_{k=j+1}^{N}H^k_0(\varphi_k)\right)\times \\ \exp\left[-\int_{z\in\Lambda_N}\nu\left(\int_{y_j\in\Lambda_j}\mathcal{A}(z, y_j)\Phi^j(y_j) + \sum_{k=j+1}^{N}\int_{y_k\in\Lambda_k}\mathcal{A}(z, y_k)\varphi(y_k)\right)\right] \end{cases} \qquad (A.20)$$

N_j is a convenient constant normalization factor, $\Lambda_{\geq j} = \Lambda_j + \Lambda_{j+1} + \ldots + \Lambda_N$ and the prime on $\Pi' d\varphi(x)$ stands to indicate that the fields φ satisfy a constraint (A.6). Therefore, one should integrate over variables attached to sites $x \in \Lambda_k$ which are not centers of blocks $x' \in \Lambda_{k-1}$.
Using this notation the partition function and expectation value for the full theory can be expressed as:

$$\begin{aligned} z = &\quad \int\left[\Pi'_{x\in\Lambda}\,d\varphi(x)\right]\exp\left(-H(A\varphi)\right) \\ < O > = &\quad z^{-1}\int\left[\Pi'_{x\in\Lambda}\,d\varphi(x)\right]\,O(\mathcal{A}\varphi)\exp\left(-H(\mathcal{A}\varphi)\right) \end{aligned} \qquad (A.21)$$

vi) Updating

The aim of this procedure is to equilibrate the multigrid field φ^j through the sweeps on Λ_j. Due to the exponential decay of the kernel $\mathcal{A}^j(z, x)$, the contribution of the field φ^j to ϕ shall represent the contribution from a certain frequency range. Therefore, the conditional probability distribution $\exp[-H(\varphi)]d\varphi^j$ for φ^j, given the fields φ^k on the other layers $k \neq j$ is given by an auxiliary theory on Λ_j, with both an ultraviolet cutoff a_j^{-1} and an infrared cutoff a_{j-1}^{-1} for $j \neq 0$. This auxiliary theory shall have correlation length of order a_{j-1} and therefore shall be non singular. This affirmation is confirmed by the renormalization group interpretation of the multigrid transformation. For the full theory to be non critical, it is necessary that the coupling between φ^j and the fields φ^k with $k > j$ (which represent higher frequency ranges) decays fast enough with $k - j$. In fact, for $k > j$ the number of

cubes $y \in \Lambda_k$ contained in cube x or within horizontal distance of order a_j (horizontal non locality introduced by $\mathcal{A}^j(z, x)$) increases exponentially with $k - j$ like $L^{d(k-j)}$. Therefore, the coupling between layers Λ_j and Λ_k with $k > j$ could induce correlations in Λ_k over distances of order L^{k-j} lattice spacings.

In order to equilibrate φ^j through the sweeps on Λ_j, one can use then the fact that the auxiliary system described by the multigrid field φ is not singular and therefore, instead of update φ^j (which satisfies the constraint (A.6)), one updates the block spin $\Phi^j(x)$ keeping the fluctuation field $\zeta^j(z)$ fixed. In this way, the dominant and desired effect of update Φ^j is to change φ^j, which is the highest frequency part of Φ^j, while the lower frequency contributions φ^k with $k < j$ will change just a little.

From the above mentioned discussion, if the multigrid method beats critical slowing down completely, then the coupling between the layers must be small enough so that the updating in Λ_j would be nearly non-affected by the layers Λ_k with $k \gg j$.
This implies that the acceptance probability in Λ_j given by:

$$H^j_{,x}(\Phi, \zeta) = \frac{\delta H}{\delta\Phi^j(x)}\,(\Phi, \zeta) \qquad (A.22)$$

should not depend much on the high frequency contributions coming from ζ^k with $k > j$. Therefore, the variance of $H^j_{,x}$:

$$\sigma(H_{,x}) = < H^2_{,x}(\Phi, \zeta) >_\zeta - < H_{,x}(\Phi, \zeta) >^2_\zeta \qquad (A.23)$$

should have an UV limit or at least should not depend strongly on the UV cutoff.
The notation is as follows:

$$< O(\Phi, \zeta) >_\zeta = \frac{\int d\mu_\Gamma(\zeta)\,O(\Phi, \zeta)\,\exp\left(-V(\mathcal{A}\Phi + \zeta)\right)}{\int d\mu_\Gamma(\zeta)\,\exp\left(-V(\mathcal{A}\Phi + \zeta)\right)} \qquad (A.24)$$

is the average on the high frequency components of the fundamental field ϕ

If one computes $\sigma(H_{,x})$ by low order perturbation theory, one finds a quadratically UV-divergent expression to lowest order [34]. This would indicate that critical slowing down is not completely eliminated. It is not clear how reliable is perturbation theory. Therefore it would be desirable to compute the variance numerically for different lattice sizes.

A.2 Computation of the CEP and variance from probability considerations

We recall how to extract the effective action $H_{eff}(\Phi^0) := V\,U_V(\Phi^0)$ and its derivatives from the numerical data and explain how to compute $\sigma(H_{,x})$ on the last layer Λ_0, which is a single point. Let us define fluctuation coupling constants $g_n(\zeta^0)$ by:

$$H(\phi) = H(\Phi^0 + \zeta^0) = \sum_{n=0}^{4}\frac{1}{n!}g_n(\zeta^0)\,(\Phi^0)^n \qquad (A.25)$$

We regard the original Hamiltionian H as a function of the magnetization Φ^0 and fluctuation field $\zeta^0(z)$:

$$\Phi^0 = a v_{z\in\Lambda_N}\,\phi(z), \qquad \zeta^0(z) = \phi(z) - \Phi^0 \tag{A.26}$$

The Monte Carlo procedure produces a sequence of N_s configurations ϕ_a and, using (A.25), one obtains a sequence of fluctuating fields ζ_a.

For each of these configurations, one evaluates the fluctuating coupling constants $g_{na} = g_n(\zeta_a)$ using (A.25). With these data, one can compute the power series expansion of $H_{eff}(\Phi^0)$ around any arbitrary point Φ up to any desired order in $\delta\Phi$:

$$\delta\Phi = \Phi^0 - \hat{\Phi} \tag{A.27}$$

In fact, without loss of generality, we assume the normalization $\int D\phi \exp(-H(\phi)) = 1$. Then, the probability distribution $\exp(-H(\phi))\,D\phi$ specifies the probability distribution for Φ^0 viz. $\exp(-H_{eff}(\Phi^0))\,d\Phi^0$ and a probability distribution $dp(\zeta^0)$ of ζ^0.The probability of Φ^0 is the probability of ζ^0 times the conditional probability of Φ^0 given ζ^0 integrated over ζ^0.

$$\exp\left(-H_{eff}(\hat{\Phi}+\delta\Phi)\right) = \int dp(\zeta^0) \times z(\zeta^0)^{-1} \exp\left(-H(\hat{\Phi}+\delta\Phi+\zeta^0)\right) \tag{A.28}$$

where: $z(\zeta^0) = \int d\Phi^0 \exp(-H(\Phi^0+\zeta^0))$.
The Monte Carlo configurations ζ_a are distributed with probability $dp(\zeta)$ so that $\int dp(\zeta^0) f(\zeta^0) = \frac{1}{N_s}\sum_{a=1}^{N_s} f(\zeta_a)$.
Therefore:

$$\begin{aligned}\exp\left(-H(\hat{\Phi}+\delta\Phi)\right) &= N_s^{-1}\left\{\sum_{a=1}^{N_s} z(\zeta_a)^{-1}\exp\left(-H(\hat{\Phi}+\delta\Phi+\zeta_a)\right)\right\}\\ &= N_s^{-1}\left\{\sum_{a=1}^{N_s} z_a^{-1}\exp\left(-\sum_{n=0}^{4}\tfrac{1}{n!}g'_{n,a}(\hat{\Phi})(\delta\Phi)^n\right)\right\}\end{aligned} \tag{A.29}$$

where ζ_a and $g'_{n,a}$ are defined by:

$$\begin{aligned} z_a &= \int d\Phi \exp\left(-\sum_{n=0}^{4}\tfrac{1}{n!}g_{n,a}\Phi^n\right)\\ \sum_{n=0}^{4}\tfrac{1}{n!}g_{n,a}\Phi^n &= \sum_{n=0}^{4}\tfrac{1}{n!}g'_{n,a}(\hat{\Phi})(\Phi-\hat{\Phi})^n\end{aligned} \tag{A.30}$$

Expanding both sides of Eq.(A.29) in a power series in $\delta\Phi$, the effective action and its derivatives can be expressed in terms of the fluctuating coupling constant.

Analogously, we can compute the variance $\sigma(H^j_{,x})$ of Eq.(A.23). The general expression is found to be:

$$< O(\Phi,\zeta) >_\zeta = \frac{\sum_{a=1}^{N_s} Z^{-1}(\zeta_a^0)\,O(\Phi^0,\zeta_a^0)\exp\left(-\sum_{n=0}^{4}\frac{1}{n!}g_n(\zeta_a^0)(\Phi^0)^n\right)}{\sum_{a=1}^{n_s} Z^{-1}(\zeta_a^0)\exp\left(-\sum_{n=0}^{4}\frac{1}{n!}g_n(\zeta_a^0)(\Phi^0)^n\right)} \tag{A.31}$$

The criteria to know the performance of this Multigrid procedure to beat critical slowing down is thus given by Eq.(A.23) which can be numerically evaluated using Eq.(A.31). If the variance $\sigma(H_{,x})$ does not increase with the UV-cutoff, then C.S.D. is eliminated. Otherwise there may be a net gain to reduce the autocorrelation time, but C.S.D remains.

B Appendix

In this appendix, we want to explain the details of the computation of the two expressions $\mathrm{Tr}\{\partial_g[-\Delta_U + m^2(\eta)]\}$ and $\mathrm{Tr}\{P[\lambda_0(g) + m^2(\eta)]\partial_g P\}$ up to first order in the gauge coupling constant g. These are to be inserted in eqn.(3.74) to obtain the gauge field dependent effective potential $U_V(\Phi, A)$ up to second order in g and up to one loop in the scalar fields.

B.1 Vector representation

The scalar fields in the fundamental lattice-action eqn.(3.4) are represented as 2 x 2 complex matrices in the span $\rho SU(2)$ of the gauge group $SU(2)$. We can also represent them as real 4-vectors using the isomorphism between the groups $SU(2)$ and $O(4)$.

The correspondence between both representation is given by

$$\phi = \phi_0 + i\tau_r\phi_r \in \rho SU(2) \Longleftrightarrow \phi_\alpha = (\phi_0,\phi_r) \in \mathbb{R}^4 \tag{B.1}$$

for vectors and

$$M = M_0 + i\tau_r M_r \Longleftrightarrow M_{\alpha\beta} = \begin{bmatrix} M_0 & -M_1 & -M_2 & -M_3\\ M_1 & M_0 & M_3 & -M_2\\ M_2 & -M_3 & M_0 & M_1\\ M_3 & M_2 & -M_1 & M_0\end{bmatrix}_{\alpha\beta} \tag{B.2}$$

for operators. eqn.(B.2) comes from the following identity

$$M\phi = (M_0 + i\tau_r M_r)(\phi_0 + i\tau_s\phi_s) = M_0\phi_0 - M_r\phi_r + i\tau_t(M_0\phi_t + M_t\phi_0 - \epsilon_{rst}M_r\phi_s) \tag{B.3}$$

B.2 $O(4)$-rotation and fluctuation-propagator $\Gamma^{(1)}$

Inserting the split of eqn.(3.58) into the action of eqn.(3.4), we collect all the quadratic contributions in the fluctuation field ξ in the matrix M:

$$\xi_\alpha\, M_{\alpha\beta}\,\xi_\beta = \xi_\alpha\left[(-\Delta_U + m_0^2 + \frac{\lambda_0}{6}\|\eta\|^2)\,\delta_{\alpha\beta} + \frac{\lambda_0}{3}\eta_\alpha\eta_\beta\right]\xi_\beta \tag{B.4}$$

where the classical field η is obtained by interpolating the block spin field Φ, i.e. $\eta_\alpha = (\mathcal{A}\Phi)_\alpha$, with $\mathcal{A}$ as given in section 3.2.

The scalar CEP is invariant under $O(4)$-rotations. In fact, from the Lemma of page 19

$$U_V(R\Phi) = U_V(\Phi) \tag{B.5}$$

Therefore, and without loss of generality, we can (gauge)rotate the block spin field Φ to lie in the 0-direction

$$(\Phi)_\alpha = \|\Phi\|\,\delta_{\alpha 0}, \qquad \eta_\alpha = \mathcal{A}_\alpha\Phi \quad with \quad \Phi = \|\Phi\| \tag{B.6}$$

Then, the operator M in the absence of gauge fields can be written as:

$$M_{\alpha\beta} = \left(-\Delta + U''_{cl}(\Phi)\right)\,\delta_{a0}\,\delta_{\beta 0} + \left(-\Delta + U'_{cl}(\Phi)/\,\Phi\right)\,\delta_{ij} \tag{B.7}$$

where the classical potential $U_{cl}(\Phi)$ is given by eqn.(3.78).

In order to obtain the fluctuation propagator $\Gamma^{(0)}$, we have to invert the quadratic operator M in the subspace $H_{C^{(0)}}$ of the Hilbert space H. In terms of the base of subsection 3.2

$$B = \{\,|\,\Psi_n > \otimes |\,e_a > \;|\; \Psi_n(z) = \exp(2\pi\, inz/\,L) \text{ with } \eta_\mu \in I, z \in \Lambda \text{ and } |\,e_\alpha >= \tau_\alpha\,\}$$

where τ_α as defined in eqn. (3.34). The subspace $H_{C^{(0)}}$ is given by

$$H_{C^{(0)}} = \{\,|\,\Psi_n > \otimes |\,e_a > \;|\; \Psi_n(z) = \exp(2\pi\, inz/\,L) \text{ with } \eta_\mu \in I, z \in \Lambda \text{ and } n \neq \vec{0}\,\}.$$

i.e. the zero mode of the Laplace operator is taken out.
Therefore

$$\begin{aligned} M^{(0)} &= M^{(0)}\mathbb{1}_{H_{C^{(0)}}} = M\left(\sum_a |\,e_a >< e_a\,| \otimes {\sum_n}'|\,\Psi_n >< \Psi_n\,|\right) \\ &= {\sum_n}'\left\{\left(\hat{k}_n^2 + U''_{cl}(\Phi)\right)\,|\,e_0 >< e_0\,| + \left(\hat{k}_n^2 + U'_{cl}(\Phi)/\,\Phi\right)\right) \sum_r |\,e_r >< e_r\,|\Big\} \\ &\quad \otimes |\,\Psi_n >< \Psi_n\,| \end{aligned} \tag{B.8}$$

where $\sum'_n$ means the sum over the Brillouin zone with the constraint $n \neq 0$ and $\mathbb{1}_{H_{C^{(0)}}}$ the identity on $H_{C^{(0)}} = (1 - P^{(0)})\,H$.
Inverting M on the subspace $H_{C^{(0)}}$

$$\Gamma^{(0)} = \frac{1}{V}{\sum_n}'\begin{bmatrix} \Gamma_{n\phi_0}(\Phi) & 0 & 0 & 0 \\ 0 & \Gamma_{n\phi_r}(\Phi) & 0 & 0 \\ 0 & 0 & \Gamma_{n\phi_r}(\Phi) & 0 \\ 0 & 0 & 0 & \Gamma_{n\phi_r}(\Phi) \end{bmatrix} \otimes |\,\Psi_n >< \Psi_n\,| \tag{B.9}$$

where we have used the notation of section 3.3 to define $\Gamma_{n\phi_0}(\Phi)$and $\Gamma_{n\phi_r}(\Phi)$ (see (3.78)).

B.3 Gauge coupling expansion for the projector P

As explained in section 3.3, we choose the projection operator P as a product of the interpolating and the averaging operator, i.e. $P = \mathcal{A}C$. We want to write down explicitly P up to second order in g.

From section 3.2, we have for the $\mathcal{A}$ and C operators to zeroth order

$$\begin{cases} \mathcal{A}^{(0)} = \frac{1}{V}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \otimes |\,\Psi_0 > \\ C^{(0)} = \frac{1}{V}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \otimes < \Psi_0\,| \end{cases} \tag{B.10}$$

and the first order contribution

$$\begin{cases} \mathcal{A}^{(1)} = \frac{1}{V}{\sum_n}'\frac{1}{2\hat{k}_n^2}\begin{bmatrix} 0 & -A_{1no} & -A_{2no} & -A_{3no} \\ A_{1no} & 0 & A_{3no} & -A_{2no} \\ A_{2no} & -A_{3no} & 0 & A_{1no} \\ A_{3no} & A_{2no} & -A_{1no} & 0 \end{bmatrix} \otimes |\,\Psi_n > \\ C^{(1)} = \frac{1}{V}{\sum_n}'\frac{1}{2\hat{k}_n^2}\begin{bmatrix} 0 & A^*_{1no} & A^*_{2no} & A^*_{3no} \\ -A^*_{1no} & 0 & -A^*_{3no} & A^*_{2no} \\ -A^*_{2no} & A^*_{3no} & 0 & -A^*_{1no} \\ -A^*_{3no} & -A^*_{2no} & A^*_{1no} & 0 \end{bmatrix} \otimes < \Psi_n\,| \end{cases} \tag{B.11}$$

And therefore for the projection operator P

$$P^{(0)} = \frac{1}{V}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \otimes |\,\Psi_0 >< \Psi_0\,| \tag{B.12}$$

$$\begin{aligned} P^{(1)} = \frac{1}{V}{\sum_n}'\frac{1}{2\hat{k}_n^2}\{ &\begin{bmatrix} 0 & -A_{1no} & -A_{2no} & -A_{3no} \\ A_{1no} & 0 & A_{3no} & -A_{2no} \\ A_{2no} & -A_{3no} & 0 & A_{1no} \\ A_{3no} & A_{2no} & -A_{1no} & 0 \end{bmatrix} \otimes |\,\Psi_n >< \Psi_0\,| + \\ &\begin{bmatrix} 0 & A^*_{1no} & A^*_{2no} & A^*_{3no} \\ -A^*_{1no} & 0 & -A^*_{3no} & A^*_{2no} \\ -A^*_{2no} & A^*_{3no} & 0 & -A^*_{1no} \\ -A^*_{3no} & -A^*_{2no} & A^*_{1no} & 0 \end{bmatrix} \otimes |\,\Psi_0 >< \Psi_n\,|\,\} \end{aligned} \tag{B.13}$$

From the second order, we need only the spacialy constant contribution ($n = 0$). From

section 3.2, we have:

$$(\mathcal{A}^{(2)})_{const} = \frac{-1}{8V}\sum_n{}' \frac{1}{(\hat{k}_n^2)^2} \sum_r |A_{rno}|^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \otimes |\Psi_0> \tag{B.14}$$

and therefore

$$(P^{(2)})_{const} = \frac{1}{4V}\sum_n{}' \frac{1}{(\hat{k}_n^2)^2} \sum_r |A_{rno}|^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \otimes |\Psi_0><\Psi_0| \tag{B.15}$$

where we used the same notation for the matrix element $A_{rno} = \sum_{z\mu} \Psi^*(z) A_{rz-\mu\mu} \Psi_0(z-\mu)$.

B.4 The fluctuation operator Γ and the quadratic operator M in perturbation theory

The fluctuation operator Γ is defined through the equation

$$M\,\Gamma := \left[-\Delta_U + m^2(\eta)\right]\,\Gamma = (1-P) \tag{B.16}$$

where we have used the same notation as in section 3.3. Thus, the matrix $m^2(\eta)$ is

$$m^2(\eta) = (m_0^2 + \frac{\lambda_0}{6} <\eta|\eta>)\,\delta_{\alpha\beta} + \frac{\lambda_0}{3}\,\eta_\alpha\,\eta_\beta \tag{B.17}$$

Eqn.(B.16) means that Γ restricted to the subspace $H_C^\perp$ is the inverse of the quadratic operator M and vanishes on H_C .

Expanding eqn.(B.16) in power of the gauge coupling constant g and equating the terms of the same power, one gets

$$\begin{cases} \Gamma^{(1)} = -\Gamma^{(0)}\,P^{(1)} - \Gamma^{(0)}\,M^{(1)}\,\Gamma^{(0)} \\ \Gamma^{(2)} = -\Gamma^{(0)}\,P^{(2)} + \Gamma^{(0)}\,M^{(1)}\,\Gamma^{(0)}\,P^{(1)} + \Gamma^{(0)}\,M^{(1)}\,\Gamma^{(1)}\,M^{(1)}\,\Gamma^{(0)} - \Gamma^{(0)}\,M^{(2)}\,\Gamma^{(0)} \\ \dots \end{cases} \tag{B.18}$$

where the tree level propagator $\Gamma^{(0)}$ is given in eqn.(B.9).

To expand M as a series in g, we consider first the covariant Laplacian $-\Delta_U$

$$\begin{aligned} (-\Delta_U\phi)_{(z)} &= -\sum_\mu \left[U(z-\mu,\mu)\,\phi(z-\mu) - \phi(z)\right] \\ &= -\sum_\mu \left[[\phi(z-\mu)-\phi(z)] + \tfrac{i}{2}\,g\,\tau_r A_{rz-\mu\mu}\phi(z-\mu) + \right. \\ &\quad \left. \tfrac{i^2}{8}\,g^2 A_{rz-\mu\mu}A_{rz-\mu\mu}\phi(z-\mu) + \dots\right] \end{aligned} \tag{B.19}$$

therefore

$$\begin{cases} (-\Delta_U^{(0)}\,\phi)_{(z)} &= -\sum_\mu \left[\phi(z-\mu) - \phi(z)\right] \\ (-\Delta_U^{(1)}\,\phi)_{(z)} &= -\tfrac{i}{2}\tau_r \sum_\mu A_{rz-\mu\mu}\phi(z-\mu) \\ (-\Delta_U^{(2)}\,\phi)_{(z)} &= \tfrac{1}{8}\sum_\mu A_{rz-\mu\mu}A_{rz-\mu\mu}\phi(z-\mu) \end{cases} \tag{B.20}$$

In the other representation, as an operator acting on $\mathbb{R}^4$- vectors

$$(-\Delta_U) = (-\Delta_0) + i\tau_r(-\Delta)_r$$

$$\iff (-\Delta)_{\alpha\beta} = \begin{bmatrix} (-\Delta)_0 & -(-\Delta)_1 & -(-\Delta)_2 & -(-\Delta)_3 \\ (-\Delta)_1 & (-\Delta)_0 & (-\Delta)_3 & -(-\Delta)_2 \\ (-\Delta)_2 & -(-\Delta)_3 & (-\Delta)_0 & (-\Delta)_1 \\ (-\Delta)_3 & (-\Delta)_2 & -(-\Delta)_1 & (-\Delta)_0 \end{bmatrix} \tag{B.21}$$

Finally, the classical field $\eta = \mathcal{A}\Phi$ must be expanded in powers of g also because $\mathcal{A}$ depends on g.

$$\begin{aligned} \eta_\alpha\eta_\beta = \;& \Phi^2\delta_{\alpha 0}\delta_{\beta 0} + g\,\Phi^2(\mathcal{A}_\alpha^{(1)}\,\mathcal{A}_\beta^{(0)} + \mathcal{A}_\alpha^{(0)}\,\mathcal{A}_\beta^{(1)}) + \\ & g^2\Phi^2(\mathcal{A}_\alpha^{(0)}\,\mathcal{A}_\beta^{(2)} + \mathcal{A}_\alpha^{(1)}\,\mathcal{A}_\beta^{(1)} + \mathcal{A}_\alpha^{(2)}\,\mathcal{A}_\beta^{(0)}) \\ & + \dots \end{aligned} \tag{B.22}$$

We indicate the order in g by a super script in brackets, as usual. We deduce

$$(\eta(z)_\alpha\,\eta(z_1)_\beta)^{(0)} = \Phi^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Psi_0(z)\,\Psi_0(z_1) \tag{B.23}$$

$$(\eta(z)_\alpha\,\eta(z_1)_\beta)^{(1)} = \Phi^2 \sum_n{}' \tfrac{1}{2\hat{k}_n^2}$$

$$\left\{ \begin{bmatrix} 0 & A_{1no} & A_{2no} & A_{3no} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Psi_0(z)\,\Psi_n(z_1) + \begin{bmatrix} 0 & 0 & 0 & 0 \\ A_{1no} & 0 & 0 & 0 \\ A_{2no} & 0 & 0 & 0 \\ A_{3no} & 0 & 0 & 0 \end{bmatrix} \Psi_n(z)\,\Psi_0(z_1) \right\} \tag{B.24}$$

$$(\eta(z)_\alpha\,\eta(z_1)_\beta)^{(2)}_{const} = -\frac{1}{4}\sum_n{}' \frac{1}{(\hat{k}_n^2)^2} \sum_r |A_{rno}|^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Psi_0(z)\,\Psi_0(z_1) \tag{B.25}$$

where $\Psi_0(z) = 1$ in fact does not depend on the point z.

B.5 Computation of $\mathbf{Tr}\{P[\lambda_0(g)+m^2(\eta)]\partial_g P\}$ up to first order in g

Let us introduce the following notation

$$\begin{aligned} T &= \mathbf{Tr}\,\{P[\lambda_0(g)+m^2(\eta)]\partial_g P\} \\ &= \mathbf{Tr}\,\Big\{P^{(0)}[\lambda_0(g)+m^2(\eta)]^{(0)}P^{(1)} + \Big(P^{(1)}[\lambda_0(g)+m^2(\eta)]^{(0)}P^{(1)} + \\ &\quad P^{(0)}[\lambda_0(g)+m^2(\eta)]^{(1)}P^{(1)} + 2P^{(0)}[\lambda_0(g)+m^2(\eta)]^{(0)}P^{(2)}\Big)g + \dots\Big\} \\ &:= T^{(0)} + g(T_1^{(1)} + T_2^{(1)} + 2T_3^{(1)}) + \dots \end{aligned} \tag{B.26}$$

We notice first that the zeroth order in g vanishes.

$$\begin{aligned} T^{(0)} &= \mathbf{Tr}\,\Big\{P^{(0)}[\lambda_0(g)+m^2(\eta)]^{(0)}P^{(1)}\Big\} = \\ &\mathbf{Tr}\sum_{z\,z_1\,z_2}\frac{1}{V}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\Psi(z)\,\Psi_0^*(z_1) \\ &\begin{bmatrix}U''_{cl}(\Phi)&0&0&0\\0&U'_{cl}(\Phi)/\Phi&0&0\\0&0&U'_{cl}(\Phi)/\Phi&0\\0&0&0&U'_{cl}(\Phi)/\Phi\end{bmatrix}\Psi_0(z_1)\,\Psi_0^*(z_2) \\ &\frac{1}{V}{\sum_n}'\frac{1}{2\hat{k}_n^2}\Bigg\{\begin{bmatrix}0&-A_{1no}&-A_{2no}&-A_{3no}\\A_{1no}&0&A_{3no}&-A_{2no}\\A_{2no}&-A_{3no}&0&A_{1no}\\A_{3no}&A_{2no}&-A_{1no}&0\end{bmatrix}\Psi_n^*(z_2)\,\Psi_0(z) + \\ &\begin{bmatrix}0&A_{1no}^*&A_{2no}^*&A_{3no}^*\\-A_{1no}^*&0&-A_{3no}^*&A_{2no}^*\\-A_{2no}^*&A_{3no}^*&0&-A_{1no}^*\\-A_{3no}^*&-A_{2no}^*&A_{1no}^*&0\end{bmatrix}\Psi_0^*(z_2)\,\Psi_n(z)\Bigg\} = 0 \end{aligned} \tag{B.27}$$

where the trace is over the discrete matrix indices. This expression vanishes because of the orthogonality of the basis $\{\Psi_n \,|\, n\in I\}$, i.e. $<\Psi_n\,|\,\Psi_m> = \delta_{nm}$.

Now, let us consider the first order in g.

$$\begin{aligned} T_1^{(1)} &= \mathbf{Tr}\Big\{P^{(1)}[\lambda_0(g)+m^2(\eta)]^{(0)}P^{(1)}\Big\} = \mathbf{Tr}\Big\{[m^2(\eta)]^{(0)}P^{(1)}P^{(1)}\Big\} \\ &= \mathbf{Tr}\sum_{z\,z_1\,z_2}\begin{bmatrix}U''_{cl}(\Phi)&0&0&0\\0&U'_{cl}(\Phi)/\Phi&0&0\\0&0&U'_{cl}(\Phi)/\Phi&0\\0&0&0&U'_{cl}(\Phi)/\Phi\end{bmatrix} \\ &\qquad \frac{1}{V}\Psi_0(z)\,\Psi_0^*(z_1)\,P^{(1)}(z_1,z_2)\,P^{(1)}(z_2,z) \end{aligned}$$

and we see from eqn.(3.42) that only the constant part of the product $P^{(1)}\,P^{(1)}$ contributes:

$$\begin{aligned} T_1^{(1)} &= \mathbf{Tr}\sum_{z\,z_1\,z_2}\begin{bmatrix}U''_{cl}(\Phi)&0&0&0\\0&U'_{cl}(\Phi)/\Phi&0&0\\0&0&U'_{cl}(\Phi)/\Phi&0\\0&0&0&U'_{cl}(\Phi)/\Phi\end{bmatrix}\frac{1}{V}\Psi_0(z)\,\Psi_0^*(z_1) \\ &\quad \frac{1}{V}{\sum_n}'\frac{1}{2\hat{k}_n^2}\begin{bmatrix}0&A_{1no}^*&A_{2no}^*&A_{3no}^*\\-A_{1no}^*&0&-A_{3no}^*&A_{2no}^*\\-A_{2no}^*&A_{3no}^*&0&-A_{1no}^*\\-A_{3no}^*&-A_{2no}^*&A_{1no}^*&0\end{bmatrix}\Psi_0(z_1)\,\Psi_n^*(z_1) \\ &\quad \frac{1}{V}{\sum_m}'\frac{1}{2\hat{k}_m^2}\begin{bmatrix}0&-A_{1mo}&-A_{2mo}&-A_{3mo}\\A_{1mo}&0&A_{3mo}&-A_{2mo}\\A_{2mo}&A_{3mo}&0&A_{1mo}\\A_{3mo}&A_{2mo}&-A_{1mo}&0\end{bmatrix}\Psi_m(z_2)\Psi_0^*(z) \\ &= {\sum_n}'\frac{1}{(2\hat{k}_n^2)^2}\mathbf{Tr}\begin{bmatrix}U''_{cl}(\Phi)&0&0&0\\0&U'_{cl}(\Phi)/\Phi&0&0\\0&0&U'_{cl}(\Phi)/\Phi&0\\0&0&0&U'_{cl}(\Phi)/\Phi\end{bmatrix}\sum_r|A_{rno}|^2\,\mathbb{1} \end{aligned}$$

where we have used that $\sum_n' \frac{1}{(2\hat{k}_n^2)^2}A_{rno}\,A_{rno}^*$ is real, so that

$$T_1^{(1)} = \frac{1}{4}\,(U''_{cl}(\Phi)+3U'_{cl}(\Phi)/\Phi)\,{\sum_n}'\frac{1}{(2\hat{k}_n^2)^2}\sum_r|A_{rno}|^2 \tag{B.28}$$

Analogously

$$T_2^{(1)} = \mathbf{Tr}\Big\{P^{(0)}[m^2(\eta)]^{(1)}\,P^{(1)}\Big\} = \mathbf{Tr}\Big\{[m^2(\eta)]^{(1)}\,P^{(1)}P^{(0)}\Big\}$$

Because of the orthogonality relation, only the first term of $P^{(1)}$ contributes to the product

$P^{(1)}P^{(0)}$ (see eqn. (B.24):

$$
\begin{aligned}
T_2^{(1)} = \ \mathbf{Tr} \ \sum_{x\, x_1\, x_2} \tfrac{\lambda_0}{3}\Phi^2 \sum_n{}' \tfrac{1}{2\hat{k}_n^2} \Big\{ & \begin{bmatrix} 0 & 0 & 0 & 0 \\ A_{1no} & 0 & 0 & 0 \\ A_{2no} & 0 & 0 & 0 \\ A_{3no} & 0 & 0 & 0 \end{bmatrix} \Psi_n(x)\,\Psi_0^*(x_1) + \\
& \begin{bmatrix} 0 & A_{1no} & A_{2no} & A_{3no} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Psi_0(x)\,\Psi_n(x_1) \Big\} \\
& \tfrac{1}{V}\sum_m{}' \tfrac{1}{2\hat{k}_m^2} \begin{bmatrix} 0 & -A_{1mo} & -A_{2mo} & -A_{3mo} \\ A_{1mo} & 0 & A_{3mo} & -A_{2mo} \\ A_{2mo} & A_{3mo} & 0 & A_{1mo} \\ A_{3mo} & A_{2mo} & -A_{1mo} & 0 \end{bmatrix} \Psi_m(x_1)\,\Psi_0^*(x_2) \\
& \tfrac{1}{V} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Psi_0(x_2)\,\Psi_0^*(x)
\end{aligned}
\qquad \text{(B.29)}
$$

$$
T_2^{(1)} = \tfrac{\lambda_0}{12}\Psi^2 \sum_n{}' \tfrac{1}{(2\hat{k}_n^2)^2} \sum_r |A_{rno}|^2
$$

Finally

$$
T_3^{(1)} = \mathbf{Tr}\left\{P^{(0)}[m^2(\eta)]^{(1)}\,P^{(1)}\right\} = \mathbf{Tr}\left\{[m^2(\eta)]^{(1)}P^{(1)}P^{(0)}\right\}
$$

We need only the constant part of $P^{(2)}$, given by eqn.(3.52)

$$
\begin{aligned}
T_3^{(1)} \ = \ \mathbf{Tr} \ \sum_{x\, x_1\, x_2} & \begin{bmatrix} U''_{cl}(\Phi) & 0 & 0 & 0 \\ 0 & U'_{cl}(\Phi)/\,\Phi & 0 & 0 \\ 0 & 0 & U'_{cl}(\Phi)/\,\Phi & 0 \\ 0 & 0 & 0 & U'_{cl}(\Phi)/\,\Phi \end{bmatrix} \tfrac{1}{V}\Psi_0(x)\,\Psi_0^*(x_1) \\
& (-1)\,\tfrac{1}{4V}\sum_n{}' \tfrac{1}{(\hat{k}_n^2)^2}\sum_r |A_{rno}|^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Psi_0(x_1)\,\Psi_0^*(x_2) \qquad \text{(B.30)} \\
& \tfrac{1}{V} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Psi_0(x_2)\,\Psi_0^*(x) \\
= \ -\tfrac{1}{4}\left(U''_{cl}(\Phi) + 3U'_{cl}(\Phi)/\,\Phi\right) & \sum_n{}' \tfrac{1}{(\hat{k}_n^2)^2}\sum_r |A_{rno}|^2
\end{aligned}
$$

Putting together all the contributions of eqns. (B.26)-(B.30), we obtain:

$$
\mathbf{Tr}\left\{P[\lambda_0(g) + m^2(\eta)]\,\frac{\partial}{\partial g}\,P\right\}^{(1)} = -\left(U'_{cl}(\Phi)/\,\Phi\right)\sum_n{}' \frac{1}{(\hat{k}_n^2)^2}\sum_r |A_{rno}|^2 \qquad \text{(B.31)}
$$

B.6 Computation of $\mathbf{Tr}\{\partial_g[\lambda_0(g) + m^2(\eta)]\,\Gamma\}$ up to first order in g

As in the last subsection, we introduce the notation

$$
\begin{aligned}
\mathbf{Tr}\,\{\partial_g[\lambda_0(g) + m^2(\eta)]\Gamma\} = \ & \mathbf{Tr}\,\{\partial_g[-\Delta_U]\Gamma\} + \mathbf{Tr}\,\{\partial_g[m^2(\eta)]\Gamma\} := \\
& \tau + \sigma
\end{aligned}
\qquad \text{(B.32)}
$$

with

$$
\begin{cases}
\tau \ = \ \mathbf{Tr}\,\{\partial_g[-\Delta_U]\Gamma\} = \mathbf{Tr}\left\{[-\Delta_U]^{(1)}\Gamma^{(0)} + g\left([-\Delta_U]^{(1)}\Gamma^{(1)} + 2[-\Delta_U]^{(2)}\Gamma^{(0)}\right) + \ldots\right\} \\
\qquad\quad \equiv \tau^{(0)} + g\left(\tau_1^{(1)} + 2\tau_2^{(1)}\right) + \ldots \\
\sigma \ = \ \mathbf{Tr}\,\{\partial_g[m^2(\eta)]\,\Gamma\} = \mathbf{Tr}\left\{[m^2(\eta)]^{(1)}\Gamma^{(0)} + g\left([m^2(\eta)]^{(1)}\Gamma^{(1)} + 2[m^2(\eta)]^{(2)}\Gamma^{(0)}\right) + \ldots\right\} \\
\qquad\quad \equiv \sigma^{(0)} + g\left(\sigma_1^{(1)} + 2\sigma_2^{(1)}\right) + \ldots
\end{cases}
\qquad \text{(B.33)}
$$

We show first that the zero order contributions vanishes

$$\begin{aligned}\tau^{(0)} &= \mathbf{Tr}\ \left\{[-\Delta_U]^{(1)}\Gamma^{(0)}\right\} \\ &= \mathbf{Tr}\ \sum_{z\mu}\begin{bmatrix} 0 & A_{1z-\mu\mu} & A_{2z-\mu\mu} & A_{3z-\mu\mu} \\ -A_{1z-\mu\mu} & 0 & -A_{3z-\mu\mu} & A_{2z-\mu\mu} \\ A_{2z-\mu\mu} & A_{3z-\mu\mu} & 0 & -A_{1z-\mu\mu} \\ -A_{3z-\mu\mu} & -A_{2z-\mu\mu} & -A_{1z-\mu\mu} & 0 \end{bmatrix} \\ &\quad \sum_n{}' \begin{bmatrix} \Gamma_{n\phi_0}(\Phi) & 0 & 0 & 0 \\ 0 & \Gamma_{n\phi_r}(\Phi) & 0 & 0 \\ 0 & 0 & \Gamma_{n\phi_r}(\Phi) & 0 \\ 0 & 0 & 0 & \Gamma_{n\phi_r}(\Phi) \end{bmatrix} \Psi_n(z-\mu)\,\Psi_n^*(z) = 0 \end{aligned} \tag{B.34}$$

because the trace over the discrete indices vanishes, and

$$\begin{aligned}\sigma &= \mathbf{Tr}\ \left\{[m^2(\eta)]^{(1)}\Gamma^{(0)}\right\} \\ &= \mathbf{Tr}\ \sum_{z\,z_1}\tfrac{\lambda_0}{3}\Phi^2 \\ &\quad \sum_n{}' \tfrac{1}{2k_n^2}\begin{bmatrix} 0 & -A_{1no} & -A_{2no} & -A_{3no} \\ A_{1no} & 0 & A_{3no} & -A_{2no} \\ A_{2no} & A_{3no} & 0 & A_{1no} \\ A_{3no} & -A_{2no} & -A_{1no} & 0 \end{bmatrix} (\Psi_n(z)\,\Psi_0^*(z_1) + \Psi_0(z)\,\Psi_n(z_1)) \\ &\quad \sum_m{}' \begin{bmatrix} \Gamma_{m\phi_0}(\Phi) & 0 & 0 & 0 \\ 0 & \Gamma_{m\phi_r}(\Phi) & 0 & 0 \\ 0 & 0 & \Gamma_{m\phi_r}(\Phi) & 0 \\ 0 & 0 & 0 & \Gamma_{m\phi_r}(\Phi) \end{bmatrix} \Psi_m(z_1)\,\Psi_m^*(z) = 0 \end{aligned} \tag{B.35}$$

because of the orthogonality relations.

Now, let us consider the first order contributions. We start with $(\tau_1)_a$:

$$\begin{aligned}\tau_1^{(i)} = \mathbf{Tr}\left\{[\lambda_0(g)+m^2(\eta)]^{(1)}\Gamma^{(1)}\right\} &= -\mathbf{Tr}\left\{[\lambda_0(g)+m^2(\eta)]^{(1)}\left(\Gamma^{(0)}\,P^{(1)}+\Gamma^{(0)}\,M^{(1)}\,\Gamma^{(0)}\right)\right\} \\ &= -(\tau_1^{(1)})^I - (\tau_1^{(1)})^{II}\end{aligned}$$

where we have used eqn. (B.18) to express $\Gamma^{(1)}$

$$\begin{aligned}(\tau_1^{(1)})^I &= \mathbf{Tr}\ \left\{[-\Delta_U]^{(1)}\Gamma^{(0)}P^{(1)}\right\} \\ &= \mathbf{Tr}\ \sum_{z\mu\,z_2} \\ &\quad \begin{bmatrix} 0 & A_{1z-\mu\mu} & A_{2z-\mu\mu} & A_{3z-\mu\mu} \\ -A_{1z-\mu\mu} & 0 & -A_{3z-\mu\mu} & A_{2z-\mu\mu} \\ A_{2z-\mu\mu} & A_{3z-\mu\mu} & 0 & -A_{1z-\mu\mu} \\ -A_{3z-\mu\mu} & -A_{2z-\mu\mu} & -A_{1z-\mu\mu} & 0 \end{bmatrix} \\ &\quad \tfrac{1}{V}\sum_n{}' \begin{bmatrix} \Gamma_{n\phi_0}(\Phi) & 0 & 0 & 0 \\ 0 & \Gamma_{n\phi_r}(\Phi) & 0 & 0 \\ 0 & 0 & \Gamma_{n\phi_r}(\Phi) & 0 \\ 0 & 0 & 0 & \Gamma_{n\phi_r}(\Phi) \end{bmatrix} \Psi_n(z-\mu)\,\Psi_n^*(z_2) \\ &\quad \tfrac{1}{V}\sum_m \tfrac{1}{2k_m^2}\begin{bmatrix} 0 & -A_{1mo} & -A_{2mo} & -A_{3mo} \\ A_{1mo} & 0 & A_{3mo} & -A_{2mo} \\ A_{2mo} & A_{3mo} & 0 & A_{1mo} \\ A_{3mo} & A_{2mo} & -A_{1mo} & 0 \end{bmatrix} \Psi_m(z_2)\,\Psi_0^*(z) \end{aligned} \tag{B.36}$$

$$\begin{aligned}&= \sum_n{}' \tfrac{1}{4k_n^2}\tfrac{1}{V}\sum_{z\mu} A_{rz-\mu\mu}\,(3\Gamma_{n\phi_r}(\Phi)+\Gamma_{n\phi_0})\ \Psi_n(z-\mu)A_{rno}\Psi_0^*(z) \\ &= \tfrac{1}{4V}\sum_n{}' \tfrac{1}{k_n^2}\,(3\Gamma_{n\phi_r}(\Phi)+\Gamma_{n\phi_0}(\Phi))\ \sum_r |A_{rno}|^2\end{aligned}$$

$$(\tau_1^{(1)})^{II} = \mathbf{Tr}\left\{[-\Delta_U]^{(1)}\Gamma^{(0)}\,M^{(1)}\,\Gamma^{(0)}\right\} = \mathbf{Tr}\left\{[-\Delta_U]^{(1)}\,\Gamma^{(0)}\left([-\Delta_U]^{(1)}+[m^2(\eta)]^{(1)}\right)\Gamma^{(0)}\right\} \tag{B.37}$$

$$\mathbf{Tr}\left\{[-\Delta_U]^{(1)}\Gamma^{(0)}[-\Delta_U]^{(1)}\Gamma^{(0)}\right\} =$$

$$\mathbf{Tr}\ \sum_{z\mu\,\bar{z}\nu}\begin{bmatrix} 0 & A_{1z-\mu\mu} & A_{2z-\mu\mu} & A_{3z-\mu\mu} \\ -A_{1z-\mu\mu} & 0 & -A_{3z-\mu\mu} & A_{2z-\mu\mu} \\ A_{2z-\mu\mu} & A_{3z-\mu\mu} & 0 & -A_{1z-\mu\mu} \\ -A_{3z-\mu\mu} & -A_{2z-\mu\mu} & -A_{1z-\mu\mu} & 0 \end{bmatrix}$$

$$\tfrac{1}{V}\sum_n{}' \begin{bmatrix} \Gamma_{n\phi_0}(\Phi) & 0 & 0 & 0 \\ 0 & \Gamma_{n\phi_r}(\Phi) & 0 & 0 \\ 0 & 0 & \Gamma_{n\phi_r}(\Phi) & 0 \\ 0 & 0 & 0 & \Gamma_{n\phi_r}(\Phi) \end{bmatrix} \Psi_n(z-\mu)\,\Psi_n^*(\bar{z})$$

$$\begin{bmatrix} 0 & A_{1\bar{z}-\nu\nu} & A_{2\bar{z}-\nu\nu} & A_{3\bar{z}-\nu\nu} \\ -A_{1\bar{z}-\nu\nu} & 0 & -A_{3\bar{z}-\nu\nu} & A_{2\bar{z}-\nu\nu} \\ -A_{2\bar{z}-\nu\nu} & A_{3\bar{z}-\nu\nu} & 0 & -A_{1\bar{z}-\nu\nu} \\ -A_{3\bar{z}-\nu\nu} & -A_{2\bar{z}\nu\nu} & A_{1\bar{z}-\nu\nu} & 0 \end{bmatrix}$$

$$\tfrac{1}{V}\sum_m{}' \begin{bmatrix} \Gamma_{m\phi_0}(\Phi) & 0 & 0 & 0 \\ 0 & \Gamma_{m\phi_r}(\Phi) & 0 & 0 \\ 0 & 0 & \Gamma_{m\phi_r}(\Phi) & 0 \\ 0 & 0 & 0 & \Gamma_{m\phi_r}(\Phi) \end{bmatrix} \Psi_m(\bar{z}-\nu)\,\Psi_m^*(z)$$

$$= \ -\tfrac{1}{4V^2}\sum_{n,m}{}'\sum_{z\mu\,\bar{z}\nu}\left(2\Gamma_{n\phi_r}(\Phi)\,\Gamma_{m\phi_r}(\Phi)+\Gamma_{n\phi_r}(\Phi)\,\Gamma_{m\phi_0}(\Phi)+\Gamma_{n\phi_0}(\Phi)\,\Gamma_{m\phi_r}(\Phi)\right)$$

$$A_{rz-\mu\mu}\Psi_n(z-\mu)\Psi_n^*(\bar{z})\,A_{r\bar{z}-\nu\nu}\,\Psi_m(\bar{z}-\nu)\,\Psi_m^*(z)$$

$$= \ -\tfrac{1}{4V^2}\sum_{n,m}{}'\left(2\Gamma_{n\phi_r}(\Phi)\,\Gamma_{m\phi_r}(\Phi)+\Gamma_{n\phi_r}(\Phi)\,\Gamma_{m\phi_0}(\Phi)+\Gamma_{n\phi_0}(\Phi)\,\Gamma_{m\phi_r}(\Phi)\right)A_{rmn}\,A_{rnm} \qquad \text{(B.38)}$$

$$\mathbf{Tr}\left\{[-\Delta_U]^{(1)}\Gamma^{(0)}[m^2(\eta)]^{(1)}\Gamma^{(0)}\right\} = \mathbf{Tr}\left\{[-\Delta_U]^{(1)}\Gamma^{(0)}\tfrac{\lambda_0}{3}(\eta\,\eta)^{(1)}\Gamma^{(0)}\right\}$$

$$= \mathbf{Tr}\ \sum_{z\mu}\sum_{z_1\,z_2}\begin{bmatrix} 0 & A_{1z-\mu\mu} & A_{2z-\mu\mu} & A_{3z-\mu\mu} \\ -A_{1z-\mu\mu} & 0 & -A_{3z-\mu\mu} & A_{2z-\mu\mu} \\ A_{2z-\mu\mu} & A_{3z-\mu\mu} & 0 & -A_{1z-\mu\mu} \\ -A_{3z-\mu\mu} & -A_{2z-\mu\mu} & -A_{1z-\mu\mu} & 0 \end{bmatrix}$$

$$\tfrac{1}{V}\sum_n{}' \begin{bmatrix} \Gamma_{n\phi_0}(\Phi) & 0 & 0 & 0 \\ 0 & \Gamma_{n\phi_r}(\Phi) & 0 & 0 \\ 0 & 0 & \Gamma_{n\phi_r}(\Phi) & 0 \\ 0 & 0 & 0 & \Gamma_{n\phi_r}(\Phi) \end{bmatrix} \Psi_n(z-\mu)\Psi_n^*(z_1)\tfrac{\lambda_0}{3}\,\Phi^2$$

$$\sum_m{}' \tfrac{1}{2\hat{k}_{1n}^2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ A_{1mo} & 0 & 0 & 0 \\ A_{2mo} & 0 & 0 & 0 \\ A_{3mo} & 0 & 0 & 0 \end{bmatrix} \Psi_m(z_1)\Psi_0(z_2) \qquad \text{(B.39)}$$

$$\tfrac{1}{V}\sum_p{}' \begin{bmatrix} \Gamma_{p\phi_0}(\Phi) & 0 & 0 & 0 \\ 0 & \Gamma_{p\phi_r}(\Phi) & 0 & 0 \\ 0 & 0 & \Gamma_{p\phi_r}(\Phi) & 0 \\ 0 & 0 & 0 & \Gamma_{p\phi_r}(\Phi) \end{bmatrix} \Psi_P(z_1)\,\Psi_P^*(z)$$

$$= 0$$

Thus from eqns. (B.38) and (B.39)

$$(\tau_1^{(1)})^{II} \ = \ -\tfrac{1}{4V^2}\sum_{n,m}{}'\left(2\Gamma_{n\phi_r}(\Phi)\,\Gamma_{m\phi_r}(\Phi)+\Gamma_{n\phi_r}(\Phi)\,\Gamma_{m\phi_0}(\Phi)+\Gamma_{n\phi_0}(\Phi)\,\Gamma_{m\phi_r}(\Phi)\right) \qquad \text{(B.40)}$$

$$A_{rnm}\,A_{rmn}$$

and

$$\tau_2^{(1)} \ = \ \mathbf{Tr}\ \left\{[-\Delta_U]^{(2)}\Gamma^{(0)}\right\}$$

$$= \ \mathbf{Tr}\ \sum_{z\mu}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tfrac{1}{8}A_{rz-\mu\mu}A_{rz-\mu\mu}$$

$$\tfrac{1}{V}\sum_n{}' \begin{bmatrix} \Gamma_{n\phi_0}(\Phi) & 0 & 0 & 0 \\ 0 & \Gamma_{n\phi_r}(\Phi) & 0 & 0 \\ 0 & 0 & \Gamma_{n\phi_r}(\Phi) & 0 \\ 0 & 0 & 0 & \Gamma_{n\phi_r}(\Phi) \end{bmatrix} \Psi_n(z-\mu)\Psi_n^*(z) \qquad \text{(B.41)}$$

$$= \ \tfrac{1}{8V}\sum_n{}'\left(\Gamma_{n\phi_0}(\Phi)+3\Gamma_{n\phi_r}(\Phi)\right)\sum_r\sum_{z\mu}\Psi_n^*(z)A_{rz-\mu\mu}A_{rz-\mu\mu}\Psi_n(z-\mu)$$

Finally, wee need to compute σ_1 and σ_2.

$$\sigma_1^{(1)} = \mathbf{Tr}\left\{[m^2(\eta)]^{(1)}\Gamma^{(1)}\right\} = -\mathbf{Tr}\left\{[m^2(\eta)]^{(1)}\left(\Gamma^{(0)}P^{(1)} + \Gamma^{(0)}M^{(1)}\Gamma^{(0)}\right)\right\}$$

$$= -(\sigma_1^{(1)})^I - (\sigma_1^{(1)})^{II}$$

$$(\sigma_1^{(1)})^{II} = \mathbf{Tr}\left\{[m^2(\eta)]^{(1)}\Gamma^{(0)}P^{(1)}\right\} = \mathbf{Tr}\left\{\tfrac{\lambda_0}{3}(\eta\eta)^{(1)}\Gamma^{(0)}P^{(1)}\right\}$$

$$= \mathbf{Tr}\ \tfrac{\lambda_0}{3}\sum_{z\,z_1\,z_2}\Phi^2$$

$$\sum_n{}' \tfrac{1}{2\tilde{k}_n^2}\begin{bmatrix} 0 & A_{1no} & A_{2no} & A_{3no} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\Psi_0(z)\,\Psi_n(z_1)$$

$$\tfrac{1}{V}\sum_m{}'\begin{bmatrix} \Gamma_{m\phi_0}(\Phi) & 0 & 0 & 0 \\ 0 & \Gamma_{m\phi_r}(\Phi) & 0 & 0 \\ 0 & 0 & \Gamma_{m\phi_r}(\Phi) & 0 \\ 0 & 0 & 0 & \Gamma_{m\phi_r}(\Phi) \end{bmatrix}\Psi_m(z_1)\,\Psi_m^*(z_2) \qquad \text{(B.42)}$$

$$\tfrac{1}{V}\sum_p{}' \tfrac{1}{2\tilde{k}_p^2}\begin{bmatrix} 0 & -A_{1po} & -A_{2po} & -A_{3po} \\ A_{1po} & 0 & A_{3po} & -A_{2po} \\ A_{2po} & A_{3po} & 0 & A_{1po} \\ A_{3po} & A_{2po} & -A_{1po} & 0 \end{bmatrix}\Psi_p(z_2)\,\Psi_0^*(z)$$

$$= \tfrac{\lambda_0}{12}\Phi^2\Phi\sum_n{}' \tfrac{1}{2(\tilde{k}_n^2)^2}\Gamma_{n\phi_r}(\Phi)\sum_r|A_{rno}|^2$$

$$(\sigma_1^{(1)})^{II} = \mathbf{Tr}\left\{\tfrac{\lambda_0}{3}(\eta\,\eta)^{(1)}\Gamma^{(0)}M^{(1)}\,\Gamma^{(0)}\right\} = \tfrac{\lambda_0}{3}\mathbf{Tr}\left\{(\eta\,\eta)^{(1)}\Gamma^{(0)}\left([-\Delta_U]^{(1)} + [m^2(\eta)]^{(1)}\right)\Gamma^{(0)}\right\}$$

$$= \tfrac{\lambda_0}{3}\mathbf{Tr}\left\{(\eta\,\eta)^{(1)}\Gamma^{(0)}[m^2(\eta)]^{(1)}\Gamma^{(0)}\right\}$$

This is true because $\mathbf{Tr}\left\{(\eta\,\eta)^{(1)}\Gamma^{(0)}[-\Delta_U]^{(1)}\Gamma^{(0)}\right\} = \mathbf{Tr}\left\{[-\Delta_U]^{(1)}\Gamma^{(0)}(\eta\,\eta)^{(1)}\Gamma^{(0)}\right\} = 0$, see eqn. (B.39).

Therefore

$$(\sigma_1^{(1)})^{II} = \mathbf{Tr}\ (\tfrac{\lambda_0}{3})^2\sum_{z\,z_1} z_2\,z_3\Phi^2$$

$$\sum_n{}' \tfrac{1}{2\tilde{k}_n^2}\begin{bmatrix} 0 & A_{1no} & A_{2no} & A_{3n0} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\Psi_0(z)\,\Psi_n(z_1)$$

$$\tfrac{1}{V}\sum_m{}'\begin{bmatrix} \Gamma_{m\phi_0}(\Phi) & 0 & 0 & 0 \\ 0 & \Gamma_{m\phi_r}(\Phi) & 0 & 0 \\ 0 & 0 & \Gamma_{m\phi_r}(\Phi) & 0 \\ 0 & 0 & 0 & \Gamma_{m\phi_r}(\Phi) \end{bmatrix}\Psi_m(z_1)\,\Psi_m^*(z_2) \qquad \text{(B.43)}$$

$$\Phi^2\sum_p{}' \tfrac{1}{2\tilde{k}_p^2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ A_{1po} & 0 & 0 & 0 \\ A_{2po} & 0 & 0 & 0 \\ A_{3po} & 0 & 0 & 0 \end{bmatrix}\Psi_p(z_2)\Psi_0(z_3)$$

$$\tfrac{1}{V}\sum_q{}'\begin{bmatrix} \Gamma_{q\phi_0}(\Phi) & 0 & 0 & 0 \\ 0 & \Gamma_{q\phi_r}(\Phi) & 0 & 0 \\ 0 & 0 & \Gamma_{q\phi_r}(\Phi) & 0 \\ 0 & 0 & 0 & \Gamma_{q\phi_r}(\Phi) \end{bmatrix}\Psi_q(z_3)\,\Psi_q^*(z)$$

$$= 0$$

where we have used again the orthogonality relations for the set $\{\Psi_n \,|\, n \in I\}$.

For the last term, we need the diagonal elements of $[m^2(\eta)]^{(2)}$ because $\Gamma^{(0)}$ is diagonal.

$$\sigma_2^{(1)} = \mathbf{Tr}\left\{[m^2(\eta)]^{(2)}\Gamma^{(0)}\right\} = \mathbf{Tr}\left\{[\tfrac{\lambda_0}{6} < \eta\,|\,\eta >^{(2)}\,\delta_{\alpha\beta} + \tfrac{\lambda_0}{3}(\eta^{(1)}\,\eta^{(1)})]\,\Gamma^{(0)}\right\}$$

$$(\sigma_2^{(1)})^I = \mathbf{Tr}\left\{\tfrac{\lambda_0}{6} < \eta\,|\,\eta >^{(2)}\,\mathbb{1}\,\Gamma^{(0)}\right\}$$

$$\tfrac{\lambda_0}{6}\Phi^2\sum z\,z_1\sum_n{}' \tfrac{1}{2\tilde{k}_n^2}A_{rno}^*\Psi_n^*(z)\sum_m{}' \tfrac{1}{2\tilde{k}_n^2}A_{rmo}\Psi_m(z_1)\tfrac{1}{V}\sum_p{}'\left(3\Gamma_{p\phi_r}(\Phi) + \Gamma_{p\phi_0}(\Phi)\right)\Psi_p(z_1)\,\Psi_p^*(z)$$

$$= \tfrac{\lambda_0}{24}\Phi^2\Phi\sum_n{}' \tfrac{1}{(\tilde{k}_n^2)^2}\left(3\Gamma_{n\phi_r}(\Phi) + \Gamma_{n\phi_0}(\Phi)\right)\sum_r|A_{rno}|^2 \qquad \text{(B.44)}$$

Analogously, we have

$$
\begin{aligned}
(\sigma_2^{(1)})^{II} &= \mathbf{Tr}\left\{\tfrac{\lambda_0}{3}\,\eta^{(1)}\,\eta^{(1)}\,\Gamma^{(0)}\right\} \\
&= \mathbf{Tr}\sum_{z\,z_1}\frac{\lambda_0}{12} \\
&\quad \sum_{n,m}{}' \frac{\frac{1}{V^2}}{\tilde{k}_n^2\tilde{k}_m^2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{1no}A_{1mo}^{*} & 0 & 0 \\ 0 & 0 & A_{2no}A_{2mo}^{*} & 0 \\ 0 & 0 & 0 & A_{3no}A_{3mo}^{*} \end{bmatrix} \Psi_n(z)\,\Psi_m^{*}(z_1) \qquad \text{(B.45)} \\
&\quad \frac{1}{V}\sum_{p}{}'\begin{bmatrix} \Gamma_{p\phi_0}(\Phi) & 0 & 0 & 0 \\ 0 & \Gamma_{p\phi_r}(\Phi) & 0 & 0 \\ 0 & 0 & \Gamma_{p\phi_r}(\Phi) & 0 \\ 0 & 0 & 0 & \Gamma_{p\phi_r}(\Phi) \end{bmatrix} \Psi_p(z_1)\,\Psi_p^{*}(z) \\
&= \frac{\lambda_0}{12}\Phi^2\Phi\sum_{n}{}'\frac{1}{(\tilde{k}_n^2)^2}\Gamma_{n\phi_r}(\Phi)\sum_{r}|A_{rno}|^2
\end{aligned}
$$

Inserting the expressions of eqns.(B.34 - B.45) into eqn. (B.32), we get

$$
\begin{aligned}
\mathbf{Tr}\ \{\partial_g[\lambda_0(g)+m^2(\eta)]\Gamma\}^{(1)} &= -\tfrac{1}{4V}\sum_{n}{}'\tfrac{1}{\tilde{k}_n^2}\,(3\Gamma_{n\phi_r}(\Phi)+\Gamma_{n\phi_0}(\Phi))\ \sum_{r}|A_{rno}|^2 \\
&+\tfrac{1}{4V^2}\sum_{n,m}{}'(2\Gamma_{n\phi_r}(\Phi)\,\Gamma_{m\phi_r}(\Phi)+\Gamma_{n\phi_0}(\Phi)\,\Gamma_{m\phi_r}(\Phi)+\Gamma_{n\phi_r}(\Phi)\,\Gamma_{m\phi_0}(\Phi))\ \sum_{r}|A_{rnm}|^2 \\
&+\tfrac{1}{4V}\sum_{n}{}'(3\Gamma_{n\phi_r}(\Phi)+\Gamma_{n\phi_0}(\Phi))\ \sum_{r}|A_{rnn}^2| + \tfrac{\lambda_0}{12}\Phi^2\Phi\sum_{n}{}'\tfrac{1}{(\tilde{k}_n^2)2}\,(4\Gamma_{n\phi_r}(\Phi)+\Gamma_{n\phi_0}(\Phi))\sum_{r}|A_{rno}|^2
\end{aligned}
\qquad \text{(B.46)}
$$

B.7 Computation of $U_V[\eta]$

According to eqn.(3.70), we have

$$U_V[\eta] = S_{sc}[\eta] + \frac{1}{2V}\sum < J_z, M_z^{-1}\,J_z > \qquad \text{(B.47)}$$

where

$$J_z = \frac{\lambda_0}{6}\ <\eta_z\,|\,\eta_z>\ \eta_z^{\alpha} \qquad \text{(B.48)}$$

and $\eta_z = \mathcal{A}(z)\,\Phi$. The term $< J_z\,|\,M_z^{-1}\,J_z >$ originated from the linear term $< J_z\,|\,\xi_z >$ in the exponent of eqn.(3.60) upon integration of the ξ-field. We show first that

$$S_{\rm mix}[\xi,A] \equiv \sum_{z} < J_z\,|\,\xi_z >= 0 \qquad \text{(B.49)}$$

up to order g^2. It follows that

$$U_V[\eta] = S_{sc}[\eta]\ . \qquad \text{(B.50)}$$

By definition

$$S_{\rm mix}[\xi,A] = \frac{\lambda_0}{6}\sum_{z} < \eta_z\,|\,\eta_z >< \eta_z^{\alpha}\,|\,\xi_z > \qquad \text{(B.51)}$$

The tree level contribution vanishes identically, in fact using eqn.(B.23)

$$S_{\rm mix}^{(0)}[\xi,A] = \frac{\lambda_0}{6}\sum_{z}(< \eta_z\,|\,\eta_z >)^{(0)}\,(< \eta_z\,|\,\xi_z >)^{(0)}\ = \Phi^3\sum_{z}\xi_z^0 = 0 \qquad \text{(B.52)}$$

where we used that $\xi \in H_C^{\perp}$.

The first order contribution also vanishes because of eqn.(3.23)

$$(< \eta_z\,|\,\eta_z >)^{(1)} = 0 \qquad \text{(B.53)}$$

Finally, for the second order, we have

$$
\begin{aligned}
S_{\rm mix}^{(2)}[\xi,A] = \ \tfrac{\lambda_0}{6}\sum_{z}\{\,&(< \eta_z\,|\,\eta_z >)^{(0)}\,(< \eta_z\,|\,\xi_z >)^{(2)} + (< \eta_z\,|\,\eta_z >)^{(1)}(< \eta_z\,|\,\xi_z >)^{(1)} \\
&+(< \eta_z\,|\,\eta_z >)^{(2)} > (< \eta_z\,|\,\xi_z >)^{(0)}\,\} = 0
\end{aligned}
\qquad \text{(B.54)}
$$

where we have used eqn.(B.24) and the condition that the fluctuation field ξ has covariant average zero $C\xi = 0$.

Now we compute the term $S_{sc}[\eta]$ defined in eqn.(3.61):

$$S_{sc}[\eta] = \sum_{z}\left\{\frac{1}{2}\ <\eta_z\,|\,-\Delta_U\ \eta_z> + \frac{1}{2}\ m_0^2\ <\eta_z\,|\,\eta_z> + \frac{\lambda_0}{4!}\ <\eta_z\,|\,\eta_z>^2\right\} \qquad \text{(B.55)}$$

First, we observe that

$$\sum_{z} <\eta_z\,|\,-\Delta_U\eta_z> = \sum_{z} <\eta_z\,|\,-\Delta_U\,\mathcal{A}\,\Phi> = \epsilon_0\sum_{z} <\eta_z\,|\,\eta_z> = \epsilon_0^{(2)}\,V\,\Phi^2 + O(g^3) \qquad \text{(B.56)}$$

This is because $\epsilon_0^{(2)}$ is the first non-trivial contribution to the eigenvalue of the lattice Laplacian.

$$
\begin{aligned}
\sum_{z} <\eta_z\,|\,\eta_z> \ &= \sum_{z}\left\{(<\eta_z\,|\,\eta_z>)^{(0)} + g(<\eta_z\,|\,\eta_z>)^{(1)} + g^2(<\eta_z\,|\,\eta_z>)^{(2)} + \ldots\right\} \qquad \text{(B.57)} \\
&= V\,\Phi^2 + g^2\sum_{z}\left\{2\ {}^{(0)}<\eta_z\,|\,\eta_z>^{(2)} + {}^{(1)}<\eta_z\,|\,\eta_z^{(1)}>\right\} + \ldots
\end{aligned}
$$

where we have use that $(<\eta_z\,|\,\eta_z>)^{(1)} = 0$, which follows from eqn. (B.24).
We denote by ${}^{(0)}<\eta_z|$ the bra vector to the ket vector $|\,\eta_z>^{(0)}$, thus

$$
\begin{aligned}
&\sum_{z}\left\{2\ {}^{(0)}<\eta_z\,|\,\eta_z>^{(2)} + {}^{(1)}<\eta_z\,|\,\eta_z^{(1)}>\right\} = \qquad \text{(B.58)} \\
&\sum_{z} <\Phi\,|\,(C^{(0)}(z)\,\mathcal{A}^{(2)}(z) + C^{(2)}(z)\,\mathcal{A}^{(0)}(z) + C^{(1)}(z)\mathcal{A}^{(1)}(z))\,|\,\Phi> = 0
\end{aligned}
$$

because of the normalization condition of eqn.(3.32).

Analogously

$$\sum_x (<\eta_x|\eta_x>)^2 = \sum_z \left\{(<\eta_z|\eta_z>^{(0)})^2 + 2g^2\,\Phi^2(<\eta_z|\eta_z>)^{(2)} + \ldots\right\} = \Phi^4 + O(g^3) \quad \text{(B.59)}$$

Finally, putting together eqns.(B.50)-(B.59), one obtains

$$U_V[\eta] = S_{\mathrm{eff}}[\eta] = V\left\{\frac{1}{2}\,(\epsilon_0^{(2)} + m_0^2)\;\Phi^2 + \frac{\lambda_0}{4!}\,\Phi^4\right\} + O(g^3) \qquad \text{(B.60)}$$

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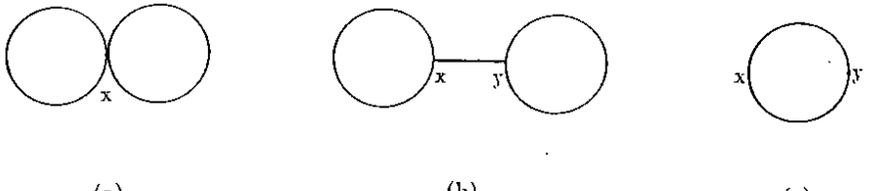

Fig. 1 The three connected diagrams to order $\hbar^2$ in the expression of $U_V(\Phi)$. The diagram (b) drops out because the internal line connecting the bubbles must carry zero momentum due to momentum conservation at vertices and $\bar{\xi}$ has no zero momentum component according to eqn.(2.6).

Fig. 3

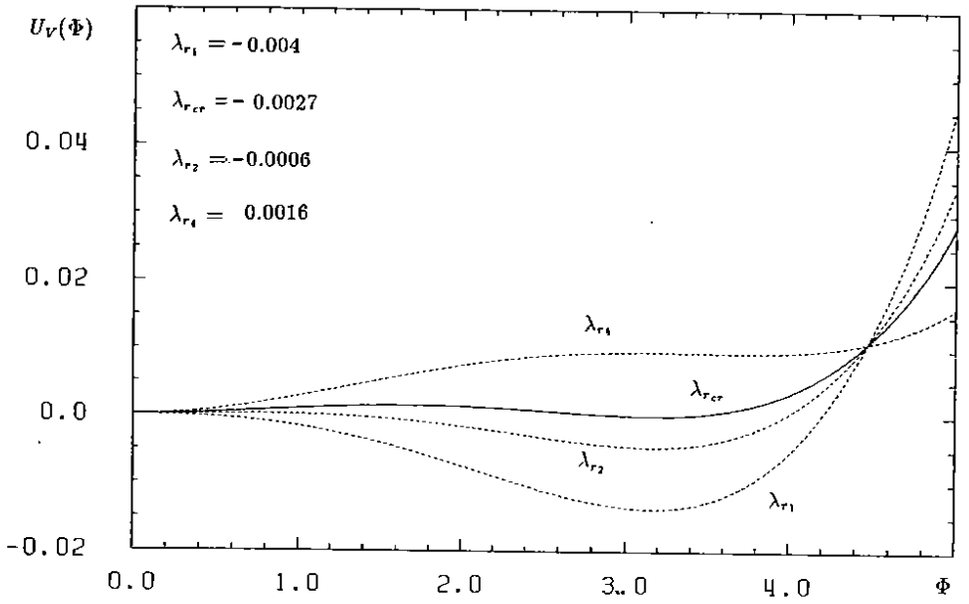

Fig. 3. The constraint effective potential in the presence of gauge fields in the infinite volume and large cutoff limit is plotted. The gauge coupling was fixed to be $g_r^2 = 0.4$, the renormalized mass m_r^2 is given by eqn. (3.117) and the scalar self coupling constant was choosen in both sides of the critical line (solide line). We observe that the minimum jumps from zero to Φ_{min} when the parameters are varied from the symmetric to the Higgs phase. This means that the system has a weak first order transition.

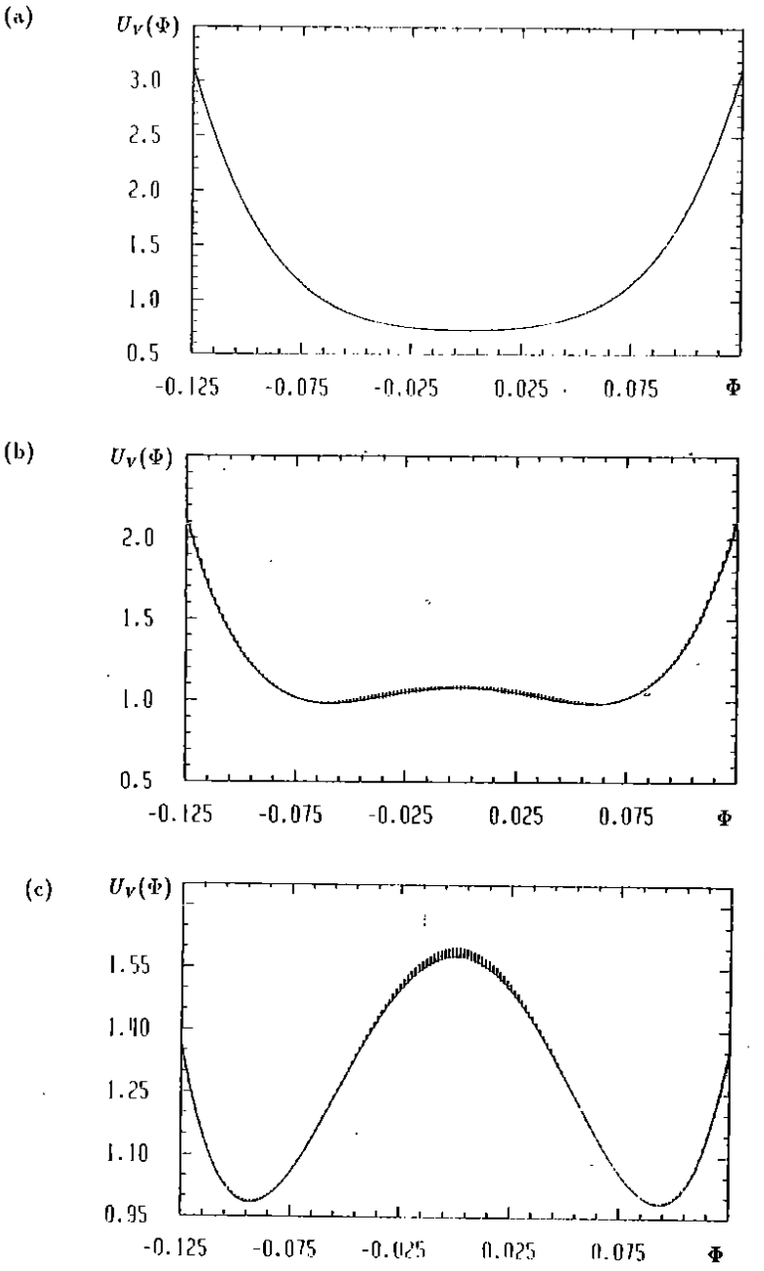

Fig. 2 Multigrid Monte Carlo data with error-bars as a function of the magnetization (in the lattice units) for $\lambda_0 = 16.376$ and fitting with the one loop formula of the constraint effective potential for the values of the bare mass: $a) m_0^2 = -1.14$ $b) m_0^2 = -1.15$ $c) m_0^2 = -1.16$