DEUTSCHES ELEKTRONEN – SYNCHROTRON

DESY 92-036
March 1992

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ISSN 0418-9833

NOTKESTRASSE 85 · D-2000 HAMBURG 52

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ISSN 0418-9833

Self-stabilisation of Collective Instabilities

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Abstract

In an electron storage ring instabilities occur above a certain threshold value of the circulating current. Nevertheless it is often possible to store currents above this threshold value. In that case strong coherent oscillations of the beam , excited by the instability, are observed, but the beam is not lost. Until now this self-stabilisation effect had not been explained theoretically.

It will be derived here from the equation of motion and the collective behaviour of the beam will be quantitatively described. It will be shown, that the self-stabilisation mechanism relies on an increase in efficiency of the stabilizing effect due to Landau-damping with increasing amplitude of the stored oscillating electrons. From these considerations it is possible to predict the actual limiting value for the currents in electron storage rings.

1 The effect of self-stabilisation

An important problem in the operation of a storage ring is the interaction of the circulating bunch with unwanted electro-magnetic modes always present in cavitylike objects.

In fig (1) we see, as an illustration, a bunch traversing an axially-symmetric structure at a small distance from the axis. The bunch sees the transverse magnetic component of a disturbing mode which then deflects it further away from the axis.

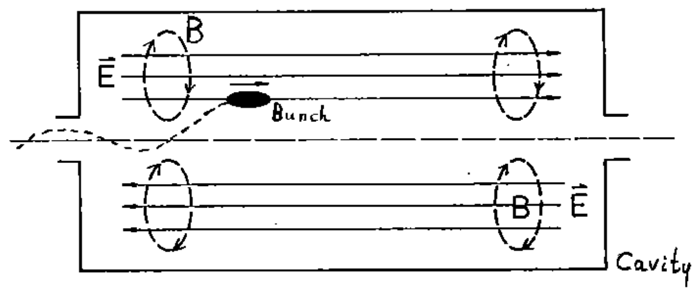


Figure 1: A bunch traversing a cavity at a small distance from the axis will be further deflected by the magnetic field of a parasitic mode.

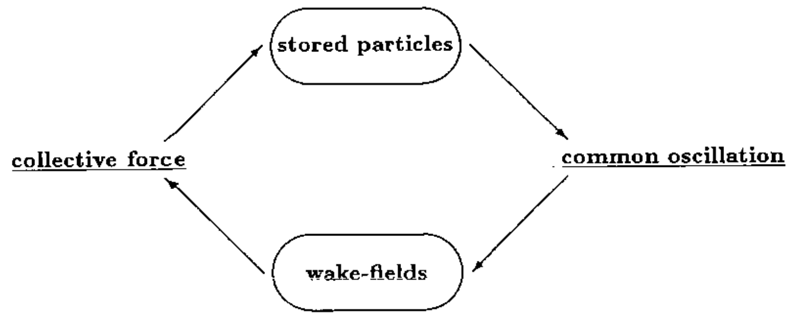
If these deflecting fields are generated by the oscillations of the bunch itself, they lead to a variety of intensity-dependent beamloss mechanisms.

In this paper we will regard a simple case of such an instability in which the common motion of the circulating particles can be treated as the oscillation of a rigid body. In this approximation the collective behaviour of all particles is sufficiently characterized by the motion of the bunch center, the dipole oscillation $D(t)$.

As mentioned above, initial dipole oscillations induce electro-magnetic fields (the so called 'wake-fields') which react on the center-of-bunch-motion and so excite a further increase in its oscillation amplitude, the exciting force, F_{exc} , being a linear functional of $D(t)$,

$$F_{exc} = F[D] \quad (1)$$

Simple cavity instabilities are driven by the dipole mode and their structure can be represented by the following feedback-loop



Analyzing the development of such a loop shows, that its self-excitation leads to an exponential growth of the dipole amplitude at the onset of the instability

$$D(t) \sim e^{\alpha t} \sin \Omega t \, e^{-\delta t} \tag{2}$$

where α denotes the growth rate of the instability, Ω the frequency of the collective oscillation and δ represents the effect of damping processes such as radiation damping. α is a increasing function of the beam current because the strength of the wake field which drives the instability is proportional to the total charge of its source. Therefore at small currents, α will be smaller than δ and initial disturbances are damped. But above a certain threshold value for the current $I \geq I_{\text{thresh}}$, the growth rate exceeds the damping rate and the commonly used theory (the linearized Wlassow Theory) predicts a steadily growing dipole amplitude with subsequent beam loss.

This prediction is in strict contrast to the observations made at electron storage rings [5,6,9]:

- Often the growth of the dipole amplitude is limited and stops at a certain value which increases with the beam current;
- After the beam has stabilized one observes strong coherent oscillations but the beam is not lost. Obviously the instability permanently acts on the beam but without further increasing the dipole amplitude;

- The frequency spread of the beam increases with the beam current above I_{thresh};
- Only if the current lies very much above the threshold value, will the beam be lost (example: PETRA electron ring $I_{\text{thresh}} \approx 3$ mA, $I_{max} \approx 24$ mA).

This restabilisation effect has not been explained up to now.

It is of particular importance at injection because it facilitates injection of a much higher current than would otherwise be possible. During ramping the threshold current increases with the particle energy and at operation energy is usually higher than the actual current. Hence the instability disappears and the shaking of the beam stops, permitting efficient operation of the accelerator.

In this paper the self-stabilisation mechanism will be explained by the increasing efficiency of the stabilizing effect of Landau damping during the development of the instability. As a result the maximum storable current I_{max} (instead of the threshold current I_{thresh} as in the linearized Wlassow Theory) can be computed as a function of the available aperture.

This paper gives a presentation of the ideas worked out in [1] in more detail.

2 Landau damping

Oscillating in the focusing guide fields, the circulating charges constitute a set of (mainly harmonic) oscillators in each direction of space. Let $\Phi_i(t)$ be the position of a single oscillator. As an example one may think of Φ_i as the transverse deviation relative to a reference orbit. $\Phi_i(t)$ can be represented as

$$\Phi_i(t) = r_i \cos(\omega_i + \theta_i) \tag{3}$$

with r_i the amplitude, θ_i the phase, and ω_i the frequency of the i'th oscillator.

Because of weak nonlinearities in the guide fields each oscillator in the set has its own frequency ω_i , which is a function of its amplitude r_i

$$\omega_i = Q_0 + \delta Q \, r_i^2 \tag{4}$$

Q_0 is the betatron frequency corresponding to the linear part of the guide field. δQ is the frequency shift due to the nonlinearities. It is small compared to Q_0 because the nonlinearities are weak

$$\frac{\delta Q}{Q_0} \ll 1 \tag{5}$$

A characteristic feature of the motion of a set of particles is the capacity for collective oscillations, the simplest being the dipole oscillation $D(t)$

$$D(t) = \frac{1}{N} \sum_i \Phi_i(t) \tag{6}$$

The coherence of the oscillators in the set is crucial for the development of such a collective motion. Coherence means that all the particles in the bunch oscillate with a definite phase relationship to each other. Because a frequency spread destroys an overall coherence (all relative phases become time dependent), it prevents the system from organizing a collective oscillation. This is the effect of *Landau damping*.

Expressed mathematically (in the framework of the linearized Wlassow Theory), the frequency spread leads to a positive damping rate (a negative growth rate) for small currents

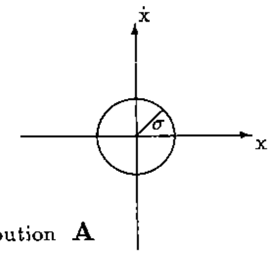
$$\alpha(I) < 0 \qquad \text{for } I < I_{\text{thresh}} \tag{7}$$

where I_{thresh} increases with increasing frequency spread.

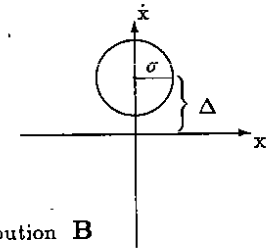
For currents above I_{thresh}, the linearized Wlassow Theory predicts the growth rate at the onset of the instability. But because of its inherent linearisation with respect to the oscillation amplitudes, this theory is incapable of correctly describing the behaviour of the oscillator system after the onset of the instability (for a proof of this statement see [1]). So to describe the stabilized state and to calculate its physical parameters, we will use another, more appropriate method. But first the underlying physical picture of the the stabilisation mechanism must be clarified.

3 The mechanism of the self-stabilisation effect

We consider two different sets of particles. These are characterized by their distributions (called A and B) in a phase space with the coordinates $x = \Phi$ and $\dot{x} = \frac{\dot{\Phi}}{Q_0}$.



distribution **A**



distribution **B**

Distribution A is a stationary one. It is radially symmetric, has no dipole moment ($D(t) \equiv 0$) and, therefore, no forces are acting ($F[D] = 0$). Defining σ as the width of this distribution, the total frequency spread $\delta\omega_A$ is given by

$$\delta\omega_A \sim \delta Q \sigma^2 \tag{8}$$

In simple models, like the one considered in this paper, the threshold current increases linearly with the frequency spread

$$I_{\text{thresh}} \sim \delta Q \sigma^2 \tag{9}$$

Distribution B is similar to A but displaced a distance Δ from the origin. Such a distribution is produced by a kick but also approximates the effect of fast growing collective forces. The total frequency spread in distribution B is approximately given by the squared difference of its maximal and minimal amplitude

$$\delta\omega_B = \delta Q \left\{ (\Delta + \sigma)^2 - (\Delta - \sigma)^2 \right\} \sim \sigma\Delta \tag{10}$$

$\delta\omega_B$, and therefore also the threshold current, increase with Δ

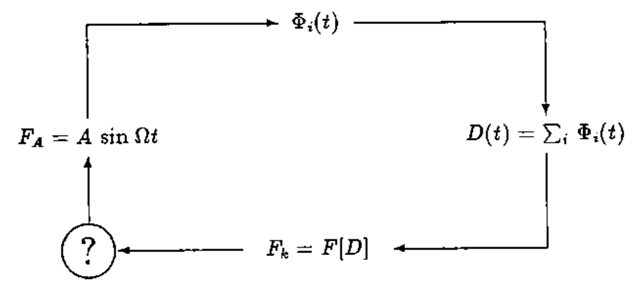
$$I_{\text{thresh}} \sim \sigma\Delta \tag{11}$$

So if Δ is large enough, I_{thresh} can exceed any value of the actual current I.

That means that for any value of the current an unstable stationary distribution A can restabilize, if its subsequent developing dipole amplitude Δ has reached a sufficiently high value. At that amplitude Landau damping has obtained such an efficiency, that it is capable of compensating the instability and the amplitude does not increase any further.

The resulting restabilized state is characterized by a permanent dipole moment generating a continually acting collective force with the right strength to produce just this dipole moment.

It will be shown that the equations of motions for electrons[1] in an storage ring can be solved self-consistently with such a trial solution by analyzing their collective behaviour along the following path



1. We start with a steady harmonic excitation F_A of constant amplitude A and frequency Ω.

$$F_A = A \sin \Omega t \tag{12}$$

acting on a set of electrons. A and Ω are initially free parameters, to be determined from the self-consistency condition.

2. The motion of the i'th electron $\Phi_i(t)$ as a function of its initial values will be derived from the corresponding oscillator equation.

[1] It is clear that a quasistationary state with a permanently exciting force can not be equivalent to the situation in a proton machine, because there exists no damping mechanism for an also inevitable necessary deexcitation.

3. Fixing the initial conditions of all electrons by choosing a concrete initial distribution (gaussian but centered around a finite amplitude Δ like distribution B) and summing over the individual amplitudes $\Phi_i(t)$, we calculate the dipole moment $D(t)$. Later the free parameters will be fixed in such a way that Δ remains asymptotically constant in time. So the chosen initial distribution srarts with the right dipole amplitude (the amplitude of the restabilized state). With this method we avoid solving the initial value problem for the developing instability but focus on the quasistationary distribution finally arising from that process.

4. Then we compute the collective force $F_k[D]$ assuming for it a simple but, in this context, sufficiently accurate functional dependence on the dipole moment D.

5. By adjusting the free parameters A, Ω and Δ, we can arrange the collective force to be equal to the excitation F_A, closing the loop self-consistently. The solution will be a restabilized distribution, rotating at constant amplitude Δ in phase space.

4 Calculation of the Dipolemoment

Assuming a harmonic excitation, the motion of a single electron is determined from the following oscillator equation

$$\ddot{\Phi}_i + 2\lambda\dot{\Phi}_i(t) + \omega_i(r)^2\Phi_i = \xi(t) + A\sin\Omega t \tag{13}$$

where λ is the radiation damping constant, ξ denotes the stochastic forces induced by the quantum nature of the photon emission, and A and Ω are the amplitude and frequency of the collective excitation. $\omega_i(r)$ denotes the eigenfrequency of the oscillation. It depends on the amplitude r_i of the oscillator. Due to the external forces, r_i and also $\omega_i(r)$ are time dependent. But the final solution will be a distribution which is orbiting in phase space at constant amplitude. In that case the individual oscillation amplitudes r_i will not change very much under the combined influence of the damping and excitation mechanisms. Therefore we approximate the eigenfrequency $\omega_i(r)$ by its time independent quasistationary value. This will be justified by the result.

$$\omega(r_i(t)) \approx \omega(r_i(0)) = Q_0 + \delta Q r_i^2(0) \tag{14}$$

Now equation (13) is linear and we can split the solution into a deterministic part, $\Phi_D(t)$ and a stochastic part $\Phi_\xi(t)$

$$\Phi_i(t) = \Phi_D(t) + \Phi_\xi(t) \tag{15}$$

The stochastic part Φ_ξ describes the action of the stochastic forces and is determined by a stochastic equation

$$\ddot{\Phi}_\xi(t) + 2\lambda\dot{\Phi}_\xi + \omega_i^2\Phi_\xi = \xi(t) \tag{16}$$

Because the time average over the stochastic forces vanishes, Φ_ξ gives no contribution to the dipole moment but influences the *shape* of the orbiting distribution. We are only interested in the dipole moment, so we do not consider this stochastic part of $\Phi_i(t)$ leaving out the calculation of the exact form of the self-consistent distribution.

As a starting point for the calculation of the asymptotic amplitude, we use a gaussian of width σ, displaced a distance Δ in phase space. With this choice we fix the initial distribution of the amplitudes r and phases θ

$$\rho(r,\theta,0) = \frac{2}{\sigma^2}\, e^{-\frac{1}{\sigma^2}(r^2+\Delta^2+2\Delta r\sin\theta)} \,. \tag{17}$$

For electrons this is reasonable, due to the stochastic independence of the quantum radiation processes. Moreover this particular choice for the initial conditions has only weak influence on the quasistationary value of the dipole moment which develops with increasing time.

The deterministic part $\Phi_D(t)$ results from a forced oscillator equation

$$\ddot{\Phi}_i + 2\lambda\dot{\Phi}_i(t) + \omega_i(r)\Phi_i = A\sin\Omega t \tag{18}$$

(the subscript 'D' has now been dropped) The frequency spread is small and contains Ω. Therefore we may approximate $\omega \approx \Omega$ and
$\omega^2 - \Omega^2 \approx 2\Omega(\omega - \Omega)$. The solution of (18) for the i'th oscillator as a function of the initial values r_i and θ_i then reads

$$\Phi_i(t) \equiv \Phi(r_i,\theta_i,t) = \left(r_i e^{-\lambda t}\cos\left(\delta\omega(r)t + \theta_i\right) + b\sin\chi\right)\cos\Omega t + \tag{19}$$

$$\left(-r_i e^{-\lambda t}\sin\left(\delta\omega(r)t + \theta_i\right) + b\cos\chi\right)\sin\Omega t$$

where we used the following notation

$$\delta\omega(r) = \frac{\omega - \Omega}{\delta Q} \tag{20}$$

$$\alpha = \frac{A}{2Q_0\delta Q} \approx \frac{A}{2\Omega\delta Q} \tag{21}$$

$$b\sin\chi = -\alpha\,\frac{\frac{\lambda}{\delta Q}}{\delta\omega(r)^2 + (\frac{\lambda}{\delta Q})^2} \tag{22}$$

$$b\cos\chi = \alpha\,\frac{\delta\omega(r)}{\delta\omega(r)^2 + (\frac{\lambda}{\delta Q})^2} \tag{23}$$

Besides the initial values, Φ_i also depends on the (scaled) excitation parameters α and $\delta\omega(r)$. In the next step, the initial amplitude and phase may be approximated by r_i and θ_i based on the same arguments that allowed the choice (17) as initial distribution. Then we obtain the dipole moment by integration of $\Phi_i(t) \equiv \Phi(r_i,\theta_i,t)$ and $\rho(t=0)$ (from now on we measure each length in units of σ and therefore set $\sigma = 1$ in the following)

$$D(t) = \int dr\, r \int \frac{d\theta}{2\pi}\,\Phi(r,\theta,t)\,\rho(r,\theta,0) = \tag{24}$$

$$= \sin\Omega t \int dr\, r \int \frac{d\theta}{2\pi}\, 2e^{-(r^2+\Delta^2+2\Delta r\sin\theta)} \left(r_i\, e^{-\lambda t}\sin\left(\delta\omega(r)t + \theta_i\right) + \alpha\,\frac{\delta\omega(r)}{\delta\omega(r)^2 + (\frac{\lambda}{\delta Q})^2}\right) +$$

$$+ \cos\Omega t \int dr\, r \int \frac{d\theta}{2\pi}\, 2e^{-(r^2+\Delta^2+2\Delta r\sin\theta)} \left(r_i\, e^{-\lambda t}\cos\left(\delta\omega(r)t + \theta_i\right) - \alpha\,\frac{\frac{\lambda}{\delta Q}}{\delta\omega(r)^2 + (\frac{\lambda}{\delta Q})^2}\right)$$

The first part of the integrals vanishes asymptotically and in the second part the θ integration can be performed with

$$\int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{-2\Delta r\sin\theta} = I_0(2\Delta r) \tag{25}$$

where I_0 is the modified Bessel function with index 0.
We introduce the collective amplitude $\hat{D}$ and phase $\hat{\Psi}$ as

$$D(t) = \hat{D}\,\sin\left(\Omega t + \hat{\Psi}\right) \tag{26}$$

$$\frac{\dot{D}(t)}{\Omega} = \hat{D}\,\cos\left(\Omega t + \hat{\Psi}\right)$$

The collective quantities $\hat{D}$ and $\hat{\Psi}$ are obtained from eq (24)

$$\hat{D}\cos\hat{\Psi} = \alpha \int dr\, r\, 2e^{-(r^2+\Delta^2)}\, I_0(2\Delta r)\, \frac{\delta\omega(r)}{\delta\omega(r)^2 + (\frac{\lambda}{\delta Q})^2} \tag{27}$$

$$\hat{D}\sin\hat{\Psi} = \alpha \int dr\, r\, 2e^{-(r^2+\Delta^2)}\, I_0(2\Delta r)\, \frac{\frac{\lambda}{\delta Q}}{\delta\omega(r)^2 + (\frac{\lambda}{\delta Q})^2}$$

These expressions are time independent. So after the transient oscillations (which have no meaning in our considerations), the bunch follows the excitation with constant amplitude and phase.

The evaluation of the integrals can be simplified, if one remembers that in electron storage rings the radiation damping constant, λ, is much smaller than the frequency spread $\frac{\lambda}{\delta Q} \ll 1$. Therefore one can use some special representations leading to the delta function $\delta(x)$ and the principal value distribution $P(\frac{1}{x})$

$$\lim_{\lambda\to 0} \frac{\lambda}{x^2+\lambda^2} = \pi\,\delta(x) \tag{28}$$

$$\lim_{\lambda\to 0} \frac{x}{x^2+\lambda^2} = P(\frac{1}{x}) \tag{29}$$

So we can write[2]

$$\hat{D}\cos\hat{\Psi} = \alpha \fint dr\, r\, 2e^{-(r^2+\Delta^2)}\, \frac{I_0(2\Delta r)}{\omega(r) - \Omega} \tag{30}$$

$$-\hat{D}\sin\hat{\Psi} = \alpha\,\pi \int dr\, r\, 2e^{-(r^2+\Delta^2)}\, I_0(2\Delta r)\, \delta(\omega(r) - \Omega)$$

The use of eq. (28) and (29) is just a numerical simplification and does not mean that radiation damping can be neglected. Indeed radiation damping restricts the single particle amplitude which would otherwise become infinite for resonant particles. It is at this point that inconsistencies arise in trying to apply the present formalism to the case of protons [1].

5 The collective force

The collective force at some time, t, is a linear functional of the dipole moment at all previous times $t' \leq t$.

[2]The expression $\fint\ dx$ denotes a principle value integral

Because we are especially interested in the response of the beam environment to the harmonic excitation of the oscillating bunch we regard the collective force in the frequency domain. In Fourier space the transform of the collective force, $\bar{F}_\omega[D]$, is a product of the Fourier transform of the dipole moment, $\bar{D}(\omega)$, and the machine impedance (or transfer function), $Z(\omega)$, which describes the response of the beam environment [3]

$$F_\omega[D] = i\,D(\omega)\,Z(\omega) \tag{31}$$

The growth rate of the instability is determined only by the *real part* of the transfer function. Indeed $Z(\omega)$ is usually a complicated complex valued function, but, for our purpose (calculating the growth rate of an instability), we only need the strength of its real part at the excitation frequency Ω. Introducing this strenght by a constant real parameter , W, we may use the following simple expression for $Z(\omega)$

$$Z(\omega) = W\,\frac{\omega}{\Omega} \tag{32}$$

Eq. (32) gives a sufficient description of the driving mechanism of the simple cavity instability under investigation, relating its threshold dependence and its growth rate to the feedback strength W. This special form of the impedance simplifies the numerical evaluation of the following conditions but could also be substituted by any other transfer function without changing the argumentation or the qualitative results.

As already mentioned, the feedback strength must be proportional to the beam current $W \sim I$. Hence calculation of a particular current, such as the threshold current I_{thres}, is equivalent to determining the corresponding value, W_{thres}, of W.

In the time domain, eq. (32) reads

$$F[D] = W\,\frac{\dot{D}(t)}{\Omega} \tag{33}$$

Inserting eq. (26), we obtain the expression

$$F_k = W\,\frac{\dot{D}}{\Omega} = W\,\hat{D}\cos(\Omega t + \hat{\Psi}) \tag{34}$$

for the collective force with $\hat{D}$ and $\hat{\Psi}$ given by eq (30).

6 Self-consistency

Self-consistency is achieved if the collective force (33) agrees with the excitation (12) with which we started

$$F_k \stackrel{!}{=} F_A = A \sin \Omega t \tag{35}$$

That means, according to (34)

$$-\, W\, \hat{D} \sin \hat{\Psi} \stackrel{!}{=} A \equiv 2Q_0 \delta Q\, \alpha \tag{36}$$

and

$$\cos \hat{\Psi} \stackrel{!}{=} 0 \tag{37}$$

The trial solution (17) further requires

$$\hat{D} \stackrel{!}{=} \Delta \tag{38}$$

Inserting eq. (30), the self-consistent collective frequency Ω_{krit} results from the condition

$$\int dr\, r\, e^{-(r^2+\Delta^2)} \frac{I_0(2\Delta r)}{\omega(r) - \Omega} \stackrel{!}{=} 0 \quad \Rightarrow \Omega_{\text{krit}}(\Delta)\,. \tag{39}$$

$\Omega_{\text{krit}}(\Delta)$ is the frequency of the continually acting collective force generated by a distribution that has stabilized at the finite amplitude Δ. The value of the feedback strength $W_{\text{krit}}(\Delta)$, for which an initially unstable distribution restabilizes at the amplitude Δ, is determined by condition (36)

$$\frac{-W}{2Q_0\delta Q}\pi \int dr\, r\, e^{-(r^2+\Delta^2)} I_0(2\Delta r)\, \delta(\omega(r) - \Omega) \stackrel{!}{=} 1 \quad \Rightarrow \quad W_{\text{krit}}(\Delta)\,. \tag{40}$$

After scaling the frequency Ω with $f = \frac{\Omega - Q_0}{\delta Q}$, the conditions (39) and (40), for the feedback strength $W_{\text{krit}}(\Delta)$ and collective frequency f_{krit}, read

$$\frac{1}{2} \int ds\ e^{-(s+\Delta^2)} \frac{I_0(2\Delta\sqrt{s})}{s - f} \stackrel{!}{=} 0 \quad \Rightarrow \quad f_{\text{krit}}(\Delta) \tag{41}$$

$$\frac{-W_{\text{krit}}}{2Q_0\delta Q} = \frac{1}{\frac{\pi}{2}\, e^{-(f+\Delta^2)}\, I_0(2\Delta\sqrt{f})}\bigg|_{f=f_{\text{krit}}}$$

7 Predictions of the model

Just as in linear Wlassow theory [2,3], we can use a graphical method for simultaneously solving the two conditions (41). To this end we define two functions $R(f)$ and $I(f)$ of the running parameter f (Δ being fixed) by[3]

$$R(f) = \frac{1}{2} \int ds\ e^{-(s+\Delta^2)} \frac{I_0(2\Delta\sqrt{s})}{s - f} \tag{42}$$

$$I(f) = \frac{\pi}{2}\, e^{-(f+\Delta^2)}\, I_0(2\Delta\sqrt{f}) \tag{43}$$

and define the curve $(x(f), y(f))$, where x and y are cartesian coordinates given by

$$x(f) = \frac{R(f)}{R^2(f) + I^2(f)} \tag{44}$$

$$y(f) = \frac{I(f)}{R^2(f) + I^2(f)}$$

To each value of Δ belongs one special curve. In fig. 2 these curves are drawn for some values of the stabilized amplitude ($\Delta = 0, 0.2, 0.4, \ldots 2.0$). The intersection point of each curve with the abscissa

$$(\,x(f), y(f)\,) = (0, y_0(\Delta)) \qquad \text{for } f = f_{\text{krit}}$$

satisfies eq (41), and the value of $y_0(\Delta)$ determines the critical value $W_{\text{krit}}(\Delta)$ of W which leads to a restabilisation at an amplitude Δ

$$\frac{-W_{\text{krit}}(\Delta)}{2Q_0\delta Q} = y_0\,. \tag{45}$$

$I_{\text{krit}}(\Delta)$ denotes the current for which an initially unstable distribution stabilizes exactly at Δ. In Fig 2, we see that $W_{\text{krit}}(\Delta)$ is a growing function of Δ, indicating that the oscillation amplitude in the stabilized state increases with the beam current.

For small amplitudes, W_{krit} increases quadratically

$$\frac{-W_{\text{krit}}(\Delta)}{2Q_0\delta Q\sigma^2} \approx 0.924\,(1 + \frac{\Delta^2}{\sigma^2}) \quad \text{for} \quad \frac{\Delta}{\sigma} \lesssim 0.5 \tag{46}$$

[3] $R(f)$ and $I(f)$ are the generalizations of the real and imaginary part of the common dispersion integral.

We reintroduced the width σ of the initially unstable distribution by scaling considerations. $W_{\text{krit}}(0)$ is proportional to the upper limit of the stable current, $I_{\text{krit}}(0)$, ($W_{\text{krit}}(0) \sim I_{\text{krit}}(0)$) and is given by

$$\frac{W_{\text{krit}}(0)}{2Q_0\delta Q\sigma^2} = -0.924 \tag{47}$$

The relation of the current $I_{\text{krit}}(\Delta)$ to the maximum stable current $I_{\text{krit}}(0)$

$$\frac{I_{\text{krit}}(\Delta)}{I_{\text{krit}}(0)} = \frac{W_{\text{krit}}(\Delta)}{W_{\text{krit}}(0)} \tag{48}$$

is drawn in fig (3) as function of the collective amplitude (scaled with σ). Except for small Δ/σ , the scaled stabilized current, $I_{\text{krit}}(\Delta)/I_{\text{krit}}(0)$ increases linearly with Δ/σ.

For each value of the parameter W (and thus for each current) there exists a certain frequency width $\delta\omega_{th} = \delta Q\sigma_{th}^2$, for which the collective is just stable. In our model the corresponding width of the distribution, σ_{th}, is given by

$$0.924\,\sigma_{th}^2 = -\frac{W}{2\,\delta Q Q_0} \tag{49}$$

From the linearized theory one would probably conjecture, that a longitudinal unstable bunch will be lengthened up until this value is reached (by the same argument the bunch should become wider in the case of a transversal instability) But this is not true.

According to eq (46), we can see now that at small amplitude σ , σ_{th} , and Δ are connected by the relation

$$\sigma_{th}^2 - \sigma^2 = \Delta^2 \qquad \text{for} \quad \frac{\Delta}{\sigma} \lesssim 0.5 \tag{50}$$

In the limit $\Delta \rightarrow 0$ we get the reasonable result $\sigma = \sigma_{th}$.

For larger amplitudes we obtain from eq (41) or fig (3) the important relation

$$\sigma_{th}^2 = 2.435\,\Delta\,\sigma \qquad \text{for} \ \Delta \gtrsim 0.5 \tag{51}$$

Applied to a longitudinal instabilty, Eq. (51) states that the stabilized bunch length (together with the collective oscillation amplitude) becomes larger as the initial one becomes smaller, their product being proportional to the square of the just-stable bunch length. This qualitative behaviour of the bunch length has been observed for a longer time and is also

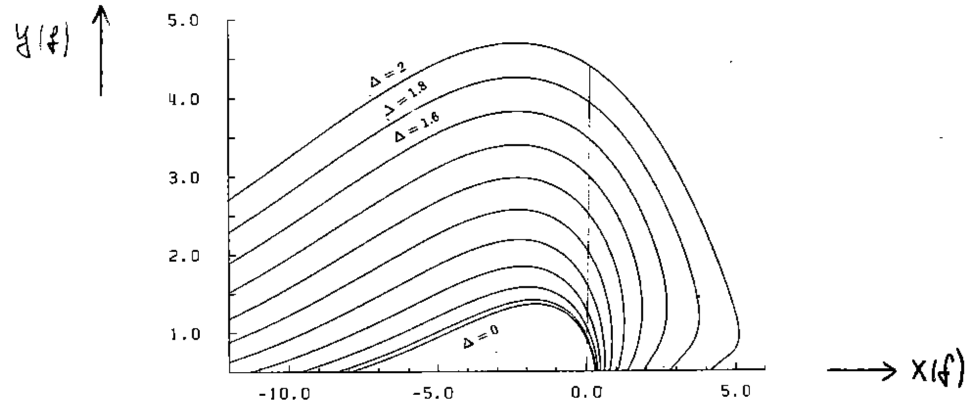


Figure 2: Stability diagrams for different values of the stabilized amplitude Δ ($\Delta = 0, 0.2, 0.4, \ldots, 2.0$). From equ. (44) $x(f)$ and $y(f)$ are drawn as function of the parameter f. For a given value of Δ the condition $x(f) \stackrel{!}{=} 0$ on the corresponding curve defines the self-consistent frequency $f = f_{\text{krit}}$ for that amplitude. The equivalent y-value on the same curve $y_0 = y(f_{\text{krit}})$ gives the corresponding feed-back strength.

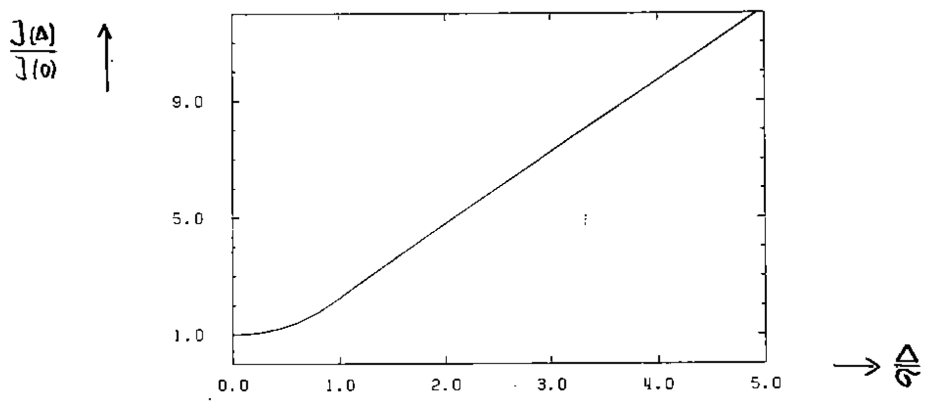


Figure 3: If the actual current, I, exceeds the threshold current $I_{\text{krit}}(0)$ the bunch oscilates with a finite amplitude $\Delta(I)$. The (scaled) function $I(\Delta)$, the current leading to a given amplitude, is drawn here as function of the (scaled) amplitude Δ. Except for small amplitudes the dependence is linear.

confirmed by computer simulations [8,4]. Here it is deduced from the equations of motion and describes the stabilisation effect quantitatively.

If Δ_{max} denotes the largest acceptable collective amplitude, (51) permits the calculation of the maximum storable current I_{max}

$$I_{max} \equiv I_{\text{krit}}\,(\Delta_{max}) = 2.435\,\frac{\Delta_{max}}{\sigma}\,I_{\text{krit}}\,(0) \tag{52}$$

In the stabilized state, besides the oscillation of the whole bunch, one also observes strong excitation of internal modes due to the continuously acting instability. Up to now these modes were not essential for our investigations, but at this point they reduce the value of Δ_{max}. From a 'boiling' distribution, electrons will already be lost if the center of the bunch still remains distant from the limit of the physical aperture. But a bunch lengthening factor of $\Delta_{max} \approx 2\,\sigma$ should be tolerable. For the old PETRA storage ring with a threshold current $I_{thres} = I_{\text{krit}}\,(0) \sim 2$ mA this leads to the observed maximum current of $I_{\text{krit}} \approx 9$ mA. Also new data from the reconstructed PETRA ring ($I_{\text{krit}}\,(0) \approx 3\ldots 5mA$ und $I_{\max} \approx 24mA$) confirm this result.

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Acknowledgements:

The author is indebted to Prof. Peter Schmüser and especially to Prof. Rolf Dieter Kohaupt for continuous encouragement and to Dr. K. Balewski for numerous intense discussions.

The author is grateful to Jan Speth and Susan Wipf for carefully reading the manuscript.