complete divergent piece of the determinant including the contribution of the fiducial metric $\hat{g}$. Rewrite equation (3.50) as

$$\ell n \det' D = - \int_{\varepsilon}^{\infty} \frac{dt}{t} \mathrm{Tr}[\exp(-tD) - K(D)]. \tag{3.60}$$

The only divergence in the above arises from the small t behavior. Inserting asymptotic expansion (3.59) into the above leads to the result:

$$\begin{aligned} \ell n \det' D = & -\varepsilon^{-1} A_{-1}(1,D) \\ & -2\varepsilon^{-\frac{1}{2}} A_{-\frac{1}{2}}(1,D) \\ & +[A_0(1,D) - \dim \ker D] \ell n \varepsilon \\ & +(\text{finite as } \varepsilon \to 0). \end{aligned} \tag{3.61}$$

Section 4

The Determinants

In the previous section we saw that the differential equation for the determinant required solving for the short time behavior of

$$\begin{aligned} & \mathrm{Tr}[f \exp(-tD)] \\ & = \int_M d^2z \sqrt{g(z)}\, f(z) \langle z|\exp(-tD)|z\rangle . \end{aligned} \tag{4.1}$$

One only has to know the diagonal matrix elements of the diffusion evolution operator for very short times. This leads to a problem that only involves the local properties of the manifold because the diffusing particle cannot travel very far. Therefore, the problem is amenable to local coordinate methods and does not require global geometrical methods. If the starting point for diffusion is away from the boundary then for very short times the particle believes it is moving on the flat Euclidean plane. If the particle is near the boundary then for very short times it perceives the boundary as being a straight line and it believes it is diffusing in the upper half plane. The object is to do a calculation perturbing about the flat space limit. We will use the method of Balian and Bloch[20] to isolate the boundary contributions from the bulk contributions.

The diffusion evolution operator

$$G(z,z';t) = \langle z|\exp(-tD)|z'\rangle \tag{4.2}$$

satisfies the initial condition

$$\lim_{t \to 0} G(z,z';t) = \delta(z,z') \tag{4.3}$$

where δ is the Dirac delta function on the curved manifold. It also satisfies an appropriate boundary condition. G is the solution of the differential equation

$$\left(\frac{\partial}{\partial t} - D\right) G = 0 \tag{4.4}$$

with initial condition (4.3) and the appropriate boundary conditions. All the differential operators we have considered may be written as

$$D = -\partial^2 - V \tag{4.5}$$

where ∂^2 is the flat Laplacian in the local coordinate system and V is a perturbation which measures the local Riemannian geometry. If G_0 is the flat space diffusion operator then equation (4.4) may be written as an integral equation

$$G = G_0 + G_0 V G \ . \tag{4.6}$$

If the point of interest is away from ∂M then one can neglect the boundary conditions and use the diffusion operator for the entire plane as G_0. If the point of interest is near the boundary then one chooses G_0 to be the diffusion operator for the upper half plane with the appropriate boundary conditions.

To solve for the asymptotic behavior of G one simply iterates (4.6). The calculation can be simplified by a judicious choice of coordinates. Balian and Bloch do the calculation for the ordinary Laplacian in arbitrary number of dimensions. Since we are only interested in two dimensions we can use very special coordinates. If a point is in the interior then one can find local conformal coordinates such that at

the point of interest, ζ, the following is true:

$$\begin{aligned} \sigma(\zeta) &= 0 \ , \\ \partial_a \sigma(\zeta) &= 0 \ . \end{aligned} \tag{4.7}$$

If ζ is near ∂M, then choose local conformal coordinates such that locally the boundary is mapped into the real axis, and such that

$$\begin{aligned} \lim_{\mathrm{Im}\ \zeta \to 0} \sigma &= 0 \\ \lim_{\mathrm{Im}\ \zeta \to 0} \partial_1 \sigma &= 0 \end{aligned} \tag{4.8}$$

A technical clarification is in order. If M has the topology of a disc then one can map M into the upper half plane, and the discussion given above is clear. If M requires more than one coordinate patch then there is no problem but an explanation is required. Assume the coordinate patch includes a piece of ∂M. Map the coordinate patch into the upper half plane. The section of ∂M gets mapped into a segment of the real axis. The only place where boundary conditions are imposed is on the segment which corresponds to the section of ∂M. If the point of interest is near ∂M then as far as the briefly diffusing particle is concerned the boundary condtions extend all the way out along the real axis. Since we are doing an asymptotic expansion it is safe to assume the boundary conditions are implemented along the entire real axis.

With the choice of coordinates described above an explicit calculation shows that

$$G \to G_0 + G_0 V G_0 + O(\sqrt{t}) \tag{4.9}$$

as $t \to 0$. First order perturbation theory is all that is necessary to

obtain an answer to the required accuracy.

In Section 2 it was proposed that the metrical deformations and the generators of diffeomorphism must respectively satisfy the boundary conditions given by equations (2.45a), (2.47) and (2.46a) and (2.46b)

$$\begin{cases} t^a n^b \, h_{ab} = 0 \, , & (4.10a) \\ n^a \nabla^b \, h_{ab} = 0 \, ; & (4.10b) \end{cases}$$

and

$$\begin{cases} n^a V_a = 0 \, , & (4.11a) \\ t^a n^b (P_1 V)_{ab} = 0 \, . & (4.11b) \end{cases}$$

The equations involving $d\sigma$ and $d\tau$ may be viewed as constraints on $d\sigma$ and $d\tau$. For notational simplicity we have dropped the infinitessimal "d" notation which was used to describe the deformations of the metric. One should think of $\underset{\sim}{h}$ as a tangent vector at $\underset{\sim}{g}$, where $\underset{\sim}{g}$ is viewed as point in the infinite dimensional manifold of admissible metrics on M. $\underset{\sim}{V}$ should be viewed as the vector field which generates the diffeomorphism.

In accordance with a previous discussion, we can treat the boundary as being mapped into the upper half plane. If we write

$$\phi = V^1 + i V^2 \qquad (4.12)$$

$$\psi = h^{11} + i h^{12} \, , \qquad (4.13)$$

then boundary conditions (4.10) and (4.11) may be written as

$$\begin{cases} \text{Im } \psi = 0 & (4.14a) \\ \text{Im } \nabla_z \psi = 0 & (4.14b) \end{cases}$$

$$\begin{cases} \text{Im } \phi = 0 & (4.15a) \\ \text{Im } \nabla^z \phi = 0 & (4.15b) \end{cases}$$

These boundary conditions lead to self-adjoint positive semi-definite operators. First we note that $\phi \in \mathcal{H}^1$ and $\psi \in \mathcal{H}^2$, and P_1 is an operator that maps the infinitesimal generators of diffeomorphism into the symmetric traceless deformations of the metric. P_1^+ goes in the opposite direction, see figure 2 . The choice $\text{Im } \psi = 0 = h^{12} = 0$ forces the tangential deformation satisfying $\text{Im } \phi = 0$ if P_1 and P_1^+ are required to be well defined operators satisfying a local boundary condition consistent with (3.29). One can see from (3.33) and (3.32) that locality, positive semi-definiteness, and self-adjointness require (4.14b) and (4.15b) to hold. One can also check condition (3.44) for the relationship between the boundary conditions in the different spaces. As a consequence of (3.44) and (3.29) one finds the following associations:

$$\begin{array}{ccc} \text{Im } \phi = 0 & \overset{(3.44)}{\longleftrightarrow} & \text{Im } \nabla_z \psi = 0 \\ & \searrow^{(3.29)} & \\ \text{Im } \nabla^z \phi = 0 & \underset{(3.44)}{\longleftrightarrow} & \text{Im } \psi = 0 \end{array} \qquad (4.16)$$

In some sense only one boundary condition is needed since mathematical requirements impose all the others. In particular we note that the most natural starting point is either $\text{Im } \psi = 0$ or $\text{Im } \phi = 0$ as discussed in Section 2.

The boundary conditions (4.14) and (4.15) are in fact a mixed boundary condition: Dirichlet for one component, Neumann for the other.

One can extend boundary conditions (4.14) and (4.15) to an arbitrary

$\mathcal{D}^{n+1}$ and $\mathcal{D}^{n}$. The short time behaviors of the diffusion operators are given by

$$\mathrm{Tr}[f\ \exp(-t\ D_n^{(+)})] = \frac{1}{2\pi t}\int_M d^2z\ \sqrt{g}\ f + \frac{3n+1}{6\pi}\left[\int_{\partial M} ds\ k f + \frac{1}{2}\int_M d^2z\ \sqrt{g}\ R f\right] + O(\sqrt{t}\ ), \tag{4.17}$$

and

$$\mathrm{Tr}[f\ \exp(-tD_{n+1}^{(-)})] = \frac{1}{2\pi t}\int_M d^2z\ \sqrt{g}\ f - \frac{3n+2}{6\pi}\left[\int_{\partial M} ds\ k f + \frac{1}{2}\int_M d^2z\ \sqrt{g}\ R f\right] + O(\sqrt{t}\ ). \tag{4.18}$$

A sample calculation is presented in Appendix C.

One can insert (4.17) and (4.18) into (3.58) to obtain the differential equation for the determinant:

$$\delta\ \ell n\ \det'\ D_n^{(+)} = -\frac{1}{\pi\varepsilon}\int_M d^2z\ \sqrt{g}\ \delta\sigma - \frac{1+6n(n+1)}{3\pi}\left[\int_{\partial M} ds\ k\ \delta\sigma + \frac{1}{2}\int_M d^2z\ \sqrt{g}\ R\ \delta\sigma\right] + 2(n+1)\ \mathrm{Tr}[\delta\sigma\ K(P_n)] - 2n\ \mathrm{Tr}[\delta\sigma\ K(P_n^{\dagger})] \tag{4.19}$$

We are now in a position to integrate the above equations. Let us assume as in Section 2 that the metric $\underset{\sim}{g}$ is conformally related to a metric $\hat{\underset{\sim}{g}}$ by

$$\underset{\sim}{g} = \hat{\underset{\sim}{g}}\ \exp(2\sigma) \tag{4.20}$$

Then one can show that the respective curvatures and geodesic curvatures

are related by

$$R = e^{-2\sigma}(\hat{R} + 2\hat{\Delta}\sigma) \tag{4.21a}$$

$$k = e^{-\sigma}\ (\hat{k} - \hat{n}^a\partial_a\sigma) \tag{4.21b}$$

where $\hat{}$ refers to quantities calculated with respect to $\hat{\underset{\sim}{g}}$. Equations (4.21) can be inserted into (4.19). The first two terms of (4.19) may be written as:

$$\delta\{-\frac{1}{2\pi\varepsilon}\int_M d^2z\ \sqrt{\hat{g}}\ e^{2\sigma} - \frac{1+6n(n+1)}{3\pi}\left[\int_{\partial M} d\hat{s}\ \hat{k}\sigma + \frac{1}{2}\int_M d^2z\ \sqrt{\hat{g}}\ \hat{R}\ \sigma + \frac{1}{2}\int_M d^2z\ \sqrt{\hat{g}}\ \hat{g}^{ab}\partial_a\sigma\ \partial_b\sigma\right]\} \tag{4.22}$$

Next we have to analyze the kernel terms. Let $\phi_r^{(o)}$ be a basis for ker P_n, then according to (3.47), $\phi_r^{(o)}$ is an analytic function of z. Define the metric on ker P_n by

$$H_{rs}(P_n) = \langle\phi_r^{(o)},\ \phi_s^{(o)}\rangle\ . \tag{4.23}$$

It is important that the functions $\phi_r^{(o)}$ be defined to be independent of the conformal factor σ. The analytic functions on a compact space are defined independent of a metric. With this convention it is easy to show that

$$\delta\ \ell n\ \det H(P_n) = 2(n+1)\ \mathrm{Tr}[\delta\sigma\ K(P_n)]\ . \tag{4.24}$$

Let $\phi_\omega^{(o)}$ be a basic for ker $P_n^{\dagger}$, then according to (3.49), $f_\omega = (g_{z\bar{z}})^{n+1}\ \psi_\omega^{(o)}$ is an analytic function of $\bar{z}$. Define the metric on ker $P_n^{\dagger}$ by

$$H_{\omega\upsilon}(P_n^{\dagger}) = \langle\psi_\omega^{(o)},\ \psi_\upsilon^{(o)}\rangle\ . \tag{4.25}$$

Likewise one can show that

$$\delta \ln \det H(P_n^\dagger) = -2n \, \mathrm{Tr}[\delta\sigma \, K(P_n^\dagger)] \, . \tag{4.26}$$

The f_ω should be defined to be independent of σ. Putting together (4.22) (4.24) and (4.25) one reaches the conclusion that

$$\begin{aligned} \ln \det' D_n^{(+)} &= -\frac{1}{2\pi\varepsilon}\int_M d^2z \, \sqrt{\hat{g}} \, \exp(2\sigma) \\ &- \frac{1+6n(n+1)}{3\pi}\left[\frac{1}{2}\int_M d^2z \, \sqrt{\hat{g}} \, \hat{g}^{ab}\partial_a\sigma\,\partial_b\sigma \right. \\ &\left. + \int_{\partial M} ds \, \hat{k}\,\sigma + \frac{1}{2}\int_M d^2z \, \sqrt{\hat{g}} \, \hat{R}\,\sigma\right] \\ &+ \ln \det H(P_n) + \ln \det H(P_n^\dagger) \\ &+ (\text{independent of } \sigma). \end{aligned} \tag{4.27}$$

The above result for a constant curvature metric $\hat{g}$ on a manifold without boundary was first written down by Friedan.

There is an additional piece of information provided by (3.61). This equation tells us that we can extract the divergent parts of the determinant from (4.17):

$$\begin{aligned} \ln \det' D_n^{(+)} &= -\frac{1}{2\pi\varepsilon}\int_M d^2z \, \sqrt{\hat{g}} \, \exp(2\sigma) \\ &+ \left[\frac{3n+1}{3}\chi(M) - \dim \ker P_n\right]\ln \varepsilon \\ &+ (\text{finite as } \varepsilon \to 0) \, , \end{aligned} \tag{4.28}$$

where the Gauss-Bonnet theorem (2.19) was used to simplify the logarithmically divergent piece. One can put (4.27) and (4.28) together to reach

our final result:

$$\begin{aligned} \ln \det' D_n^{(+)} &= -\frac{1}{2\pi\varepsilon}\int_M d^2z \, \sqrt{\hat{g}} \, \exp(2\sigma) \\ &+ \left[\frac{3n+1}{3}\chi(M) - \dim \ker P_n\right]\ln \varepsilon \\ &- \frac{1+6n(n+1)}{3\pi}\left[\frac{1}{2}\int_M d^2z \, \sqrt{\hat{g}} \, \hat{g}^{ab}\partial_a\sigma\partial_b\sigma \right. \\ &\left. + \int_{\partial M} ds \, \hat{k}\,\sigma + \frac{1}{2}\int_M d^2z \, \sqrt{\hat{g}} \, \hat{R}\,\sigma\right] \\ &+ \ln \det H(P_n) + \ln \det H(P_n^\dagger) \\ &+ F_{\text{finite}}[\hat{g}] \, , \end{aligned} \tag{4.28}$$

where $F_{\text{finite}}[\hat{g}]$ is a finite functional of the fiducial metric $\hat{g}$ and is independent of the conformal factor σ.

There are several observations to be made about (4.28). It is a functional which depends on a globally well defined function σ. If one wrote (4.28) in terms of local conformal coordinates then the different patches would have to be pieced together with partitions of unity and one would have a different σ for each patch. There is no linear divergence in (4.28). The logarithmically divergent term is a conformal invariant independent of the conformal factor. It only depends on $\hat{g}$. There are pieces which depend on the kernels of P_n and $P_n^\dagger$. There is a finite piece which depends only on $\hat{g}$. It is important to realize that the divergent piece of the determinant is explicitly known.

We can prove a theorem about the kernels of P_n and $P_n^\dagger$. The index of P_n defined by (3.45),

$$\mathcal{I}(P_n) = \dim \ker P_n - \dim \ker P_n^\dagger \, , \tag{4.29}$$

may be written as

$$\mathcal{A}(P_n) = \mathrm{Tr}\ \exp(-tD_n^{(+)}) - \mathrm{Tr}\ \exp(-tD_{n+1}^{(-)}) \tag{4.30}$$

since the eigenvectors belonging to positive eigenvalues are in one to one correspondence. The right hand side is independent of t and can be computed in the limit $t \to 0$. Inserting asymptotic expansions (4.17) and (4.18) into (4.30) and using the Gauss-Bonnet theorem leads to the result:

$$\mathcal{A}(P_n) = (2n+1)\ \chi(M)\ . \tag{4.31}$$

The above is the Riemann-Roch theorem, a special case of the Atiyah-Singer index theorem.[21] There are three immediate consequences of the above:

$$1)\quad \chi > 0 \Longrightarrow \dim\ker P_n > \dim\ker P_n^\dagger \geq 0 \tag{4.32a}$$

$$2)\quad \chi = 0 \Longrightarrow \dim\ker P_n = \dim\ker P_n^\dagger \geq 0 \tag{4.32b}$$

$$3)\quad \chi < 0 \Longrightarrow \dim\ker P_n^\dagger > \dim\ker P_n \geq 0 \tag{4.32c}$$

If the manifold M has a boundary diffeomorphic to a single circle then $\chi \neq 0$. This means that either $\ker P_n$ or $\ker P_n^\dagger$ must be non-empty. Therefore, at least one of the two kernel terms in (4.28) will be present.

The quantization procedure of Section 2 requires that one sum over all possible metrics on M. It was mentioned that there could be deformations of the metric which were not associated with either conformal deformations or with diffeomorphisms of the manifold. The relationship between the traceless deformations of the metric and the infinitesimal diffeomorphisms is depicted in Figure 3 . The analysis of this section

and of Section 3 shows that only those metrical deformations in $(\ker P_1^\dagger)^\perp$ can be associated with a diffeomorphism in $(\ker P_1^\dagger)^\perp$. It is apparent from Figure that if $\ker P_1^\dagger \neq 0$ then there are metrical deformations which are not associated with diffeomorphisms nor with conformal deformations. We shall refer to the deformations belonging to $\ker P_1^\dagger$ as Teichmüller deformations. What we call a Teichmüller deformation is actually a tangent vector in the Teichmüller space of the manifold M. Teichmüller space has been studied extensively by mathematicians, see reference 13. If the Euler characteristic is negative then (4.32c) tells us that the space of Teichmüller deformations is non-empty. In Appendix B we calculate the dimension of the Teichmüller space.

The naive approach of Section 2 led us to assume that once one "eliminated" the action of the gauge group, the relevant metrics were of the form

$$\underset{\sim}{g} = \hat{\underset{\sim}{g}}\ \exp(2\sigma)\ . \tag{4.33}$$

In other words, the conformal factor σ was the only coordinate in the space of metrics which remains after gauge fixing. The results presented in this section demonstrate the incorrectness of the statement just presented. There are deformations of the metric described by

$$dh_{ab}^{T} = \sum (dt_\omega)(\psi_\omega^{(o)})_{ab}\ , \tag{4.34}$$

where $\omega = 1, \ldots, \dim\ker P_1^\dagger$, which must be taken into account in the path integral. This means that one needs a finite number of coordinates in

addition to the conformal factor $\sigma(z)$ to describe the metrics which enter the gauge fixed path integral. The additional coordinates could be the Riemann normal coordinates defined by the tangent vectors (4.34) at $\hat{\underset{\sim}{g}}$. Let $\hat{\underset{\sim}{g}}_t$ be the metric obtained by the Riemann normal coordinate (t_ω) about $\hat{\underset{\sim}{g}}$. The exact nature of the coordinate system is not essential to the subsequent development. The important fact is that the metrics which enter the gauge fixed path integral may be written as

$$\underset{\sim}{g} = \hat{\underset{\sim}{g}}_t \exp(2\sigma) \tag{4.35}$$

where $\hat{\underset{\sim}{g}}_t$ is obtained by moving in the Teichmüller direction defined by (4.34) and σ is the conformal factor. From (4.34) it is apparent that the contributions of the Teichmüller deformations to the path integral are given by

$$(\Pi dt_\omega)\ [\det H(P_1^\dagger)_t]^{\frac{1}{2}}, \tag{4.36}$$

where we have used the notation (4.25). The subscript was included to emphasize that the determinant not only depends on the conformal factor but also on the Teichmüller coordinate.

The only term of (2.50) which has not been calculated is the determinant of the ordinary Laplacian. The ordinary Laplacian is essentially $D_0^{(+)}$ excpet that we cannot use (4.17) since the boundary conditions are wrong. The short time behavior for the diffusions operator is given by

$$\begin{aligned}\mathrm{Tr}[f\exp(-t\Delta)] \to\ & \frac{1}{4\pi t}\int_M d^2z\ \sqrt{g}\ f \\ & + \frac{\beta}{8\sqrt{\pi t}}\int_{\partial M} ds\ f + \frac{\beta}{8\pi}\int_{\partial M} ds\ n^a\partial_a f \\ & + \frac{1}{12\pi}\left[\int_{\partial M} ds\ k\ f + \frac{1}{2}\int_M d^2z\ \sqrt{g}\ Rf\right] \\ & + O(\sqrt{t}),\end{aligned} \tag{4.37}$$

where

$$\beta = \begin{cases} -1 & \text{Dirichlet} \\ +1 & \text{Neumann} \end{cases} \tag{4.38}$$

We are interested in the Dirichlet case for the problem at hand. The determinant satisfies the equation (3.58) with $n = 0$. In the Dirichlet case there are no zero modes and one can neglect the last term. The equation is easily integrated yielding

$$\begin{aligned}\ln\det\Delta = & -\frac{1}{4\pi\varepsilon}\int_M d^2z\ \sqrt{g}\ \exp(2\sigma) \\ & + \frac{1}{4\sqrt{\pi\varepsilon}}\int_{\partial M} d\hat{s}\ \exp(\sigma) + \frac{1}{4\pi}\int d\hat{s}\ \hat{n}^a\partial_a\sigma \\ & - \frac{1}{6\pi}\left[\frac{1}{2}\int_M d^2z\ \sqrt{\hat{g}}\ \hat{g}^{ab}\partial_a\sigma\partial_b\sigma\right. \\ & \left. + \int_{\partial M} d\hat{s}\ \hat{k}\ \sigma + \frac{1}{2}\int_M d^2z\ \sqrt{\hat{g}}\ \hat{R}\ \sigma\right] \\ & + (\sigma\ \text{independent})\end{aligned} \tag{4.39}$$

The full divergent piece can be calculated using (4.37) and (3.61) with the result

$$\begin{aligned}\ln\det\Delta = & -\frac{1}{4\pi\varepsilon}\int d^2z\ \sqrt{g}\ \exp(2\sigma) \\ & + \frac{1}{4\sqrt{\pi\varepsilon}}\int d\hat{s}\ \exp(\sigma) \\ & + \frac{1}{6}\chi(M)\ \ln\varepsilon \\ & + (\text{finite as } \varepsilon\to 0)\end{aligned} \tag{4.40}$$

Putting (4.39) and (4.40) together we find that the determinant for Dirichlet boundary conditions is given by

$$\begin{aligned}\ell n \det \Delta = & -\frac{1}{4\pi\epsilon}\int d^2z\,\sqrt{\hat{g}}\,\exp(2\sigma) \\ & + \frac{1}{4\sqrt{\pi\epsilon}}\int d\hat{s}\,\exp(\sigma) + \frac{1}{6}\chi(M)\,\ell n\epsilon \\ & - \frac{1}{6\pi}\left[\frac{1}{2}\int_M d^2z\,\sqrt{\hat{g}}\,\hat{g}^{ab}\partial_a\sigma\partial_b\sigma\right. \\ & \left. + \int_{\partial M} d\hat{s}\,\hat{k}\,\sigma + \frac{1}{2}\int_M d^2z\,\sqrt{\hat{g}}\,\hat{R}\,\sigma\right] \\ & + \frac{1}{4\pi}\left\{\int_{\partial M} d\hat{s}\,\hat{n}^a\partial_a\sigma - \int_{\partial M} d\hat{s}\,\hat{k}\right\} \\ & + G_{finite}[\hat{g}] \end{aligned} \tag{4.41}$$

This equation has two distinguishing features with respect to the corresponding one for $D_n^{(+)}$. It has linear divergence proportional to the perimeter. There is a finite term involving the normal derivation of σ. For convenience we grouped a σ independent piece with it in such a way that the quantity in braces is given by

$$\{\qquad\} = -\int_{\partial M} k\,ds\ . \tag{4.42}$$

There is a remark in order. In calculating the determinants we have not worried about the overall scale of the operators. If β is a positive constant then it is easy to show from definition (3.50) that

$$\det_\epsilon(\beta D) = \det_{\beta\epsilon}(D)\ , \tag{4.43}$$

where the subscript denotes the cutoff.

The overall scale of the operator only affects the cutoff dependent terms. The ambiguity in the scale of the operator reflects itself as an arbitrariness in the bare parameters of the theory. The coefficient of the logarithmically divergent term is well defined.

We are now in a position to collect all the results and insert them into (2.50) to obtain the partition function for the Polyakov model:

$$\begin{aligned} Z[C] = & \int\left(\Pi\,\frac{dt_\omega}{\sqrt{2\pi\epsilon}}\right)[d\sigma]\,\Omega^{\blacktriangle}_{Diff;t}[\det H(P_1^\dagger)_t][\det H(P_1)_t]^{\frac{1}{2}} \\ & \times \exp(-S_o[X_t,\hat{g}_t] - S_{conf}[\sigma,\hat{g}_t])\ , \end{aligned} \tag{4.44}$$

where

$$\begin{aligned} S_o[X_t,\hat{g}_t] = & A\,J[X_t,\hat{g}_t] + \frac{1}{2}(\dim\ker P_1)\,\ell n\,\epsilon \\ & + [B + ((d-8)/12)(\ell n\,\epsilon)]\chi(M) \\ & + S_o[\hat{g}_t]_{finite} \end{aligned} \tag{4.45}$$

and

$$\begin{aligned} S_{conf}[\sigma,\hat{g}_t] = & (C - \tfrac{d}{8\pi})\int_{\partial M} ds_t k_t \\ & + \mu_o^2\int_M d^2z\,\sqrt{\hat{g}_t}\,e^{2\sigma} + \lambda_o\int_{\partial M} d\hat{s}_t\,e^{\sigma} \\ & + \frac{26-d}{12\pi}\left[\frac{1}{2}\int_M d^2z\,\sqrt{\hat{g}_t}\,\hat{g}_t^{ab}\partial_a\sigma\partial_b\sigma + \int_{\partial M} d\hat{s}_t\hat{k}_t\sigma\right. \\ & \left. + \frac{1}{2}\int_M d^2z\,\sqrt{\hat{g}_t}\,\hat{R}_t\sigma\right]\ . \end{aligned} \tag{4.46}$$

The subscript t was introduced to emphasize the dependence on the

Teichmüller directions. The coefficients μ_0^2 and λ_0 are respectively quadratically and linearly divergent in the cutoff $\Lambda = \varepsilon^{-\frac{1}{2}}$. The term $S_0[X_t, \hat{g}_t]$ is independent of the conformal factor σ, but it depends on the metric $\hat{g}_t$. Notice that the classical solution X_t depends only on $\hat{g}_t$. $S_{conf}[\sigma, \hat{g}_t]$ is the piece of the action which depends on the conformal factor σ. Equation (4.46) on a manifold without boundary and with $\hat{g}_t$ = identity was first derived by Polyakov. Friedan was the first to suggest choosing $\hat{g}_t$ to be a constant curvature metric. Equation (4.46) without the boundary terms was written down by Friedan.[12] He also stressed the importance of the zero mode contributions to (4.44). The boundary corrections to S_{conf} for $\hat{g}_t$ = identity were calculated independently by Durhuus, Petersen and Olesen using pseudodifferential operator methods,[11] and by the author using the heat evolution operator method. Our results agree.

We still have to work on the above equations before they are put into a final form. Firstly, we return to a problem involving the C coefficient which was alluded to in the discussion following (2.17). Afterwards, we carefully examine additional contributions to the logarithmically divergent terms arising from the Teichmüller deformations and the conformal Killing vectors.

There is a contribution to S_{conf} from the extrinsic curvature given by

$$\left(C - \frac{d}{8\pi}\right) \int_{\partial M} ds_t \, k_t \quad . \tag{4.47}$$

We will argue that $C = d/8\pi$. The first order of business is to discuss the boundary conditions on σ. Equation (2.42) determines σ on the boundary

in terms of the induced metric restricted to the boundary. Equation (2.46c) tells us that $d\sigma$ on ∂M is determined by the action of the diffeomorphisms. In summary, σ is determined on the boundary by (2.42), and (2.46c) tells us that there is no $d\sigma$ integration that has to be performed on ∂M. Equation (4.21b) tells us that k_t depends on the normal derivative of σ. Therefore, S_{conf} depends on the normal derivative of σ on ∂M. This is troublesome since it means that one has to specify both σ and its normal derivative on the boundary. It was already mentioned that this is a disaster for an elliptic equation. In addition, our boundary conditions give no information about the normal derivative of σ.

A similar problem happens in general relativity. The Einstein-Hilbert action $\int d^4x \sqrt{g} R$ depends on the normal derivatives of the metric on the boundary. Gibbons and Hawking[22] have argued that it is necessary to add a piece involving the extrinsic curvature of the boundary to the Einstein-Hilbert action to eliminate the normal derivatives of the metric. Their argument is based on the principle that the transition amplitude to go from an initial state to a final state is obtained by summing over all states at an intermediate stage. Their arguments may be applied to (4.46).

The only place a normal derivative of σ appears in (4.46) is through (4.47). To eliminate the normal derivative of σ on ∂M we must choose $C = d/8\pi$. This is a legitimate choice since the contribution from $(\det \Delta)^{-d/2}$ was a finite number $d/8\pi$. This requires a bit more clarification. By itself, the strictly renormalizable interaction $\int ds\, k$ cannot be induced with a logarithmically divergent term. It can appear with a logarithmically divergent term only through the Gauss-Bonnet

theorem

$$2\pi\ \chi(M) = \frac{1}{2}\int_M d^2z\ R\ \sqrt{g} + \int_{\partial M} ds\ k\ .$$

The reason is that $\int ds\ k$ is not conformally invariant by itself. Therefore, it cannot be induced by itself with a logarithmically divergent coefficient. Dimensional analysis requires that the induced coefficient be a pure number. The same argument applies to the (26-d) term in (4.46). All the interactions within the brackets are strictly renormalizable but not conformally invariant. The coefficient C may be chosen to be equal to $d/8\pi$ since it cannot undergo an infinite renormalization.

We return to the question of what is $d\sigma$ on ∂M. The diffeomorphism group can be used to specify σ on the boundary. In other words, one can do some gauge fixing on the boundary. Equation (2.46c) tells us that under a diffeomorphism one can change $\delta\sigma$ on the boundary in such a way that the measured lengths are the same. Since (2.46c) only requires knowledge of the tangential derivative of the generator of diffeomorphisms one can use the fact that $dV^a = (df)t^a$ on ∂M. It follows from (2.46a) that

$$0 = d\sigma + t^a\partial_a(df)\ .$$

We can choose df in such a way that in the new gauge σ is a constant

on the boundary. The constant may be taken to be zero. This discussion shows that gauge fixing determines σ on the boundary. There is no integral to be performed on the boundary.

There is a second simplification to (4.46). Since the metric on the boundary is required to be the physical metric it follows that

$$\lambda_0 \int_{\partial M} d\hat{s}_t\ e^{\sigma} = \lambda_0\ L \qquad (4.48)$$

where L is the physical length of the physical boundary C.

We now turn to a discussion of the logarithmically divergent terms. We will see that factors of $\ell n\ \varepsilon$ arise in a more careful study of the measure in (4.44). These combine with the dim ker P_1 term in (4.45) via the Riemann-Roch theorem to generate a $(\ell n\ \varepsilon)\ \chi(M)$ term. We choose a convention where the partition function is dimensionless. The first step is to normalize the measure for a Gaussian integral. We are interested in Laplacians with eigenvalues of dimensionality $(\text{length})^2$. The definition provided by (3.50) requires the determinant of a Laplacian to be dimensionless. If a_n is the expansion coefficient for the n-th eigenmode with eigenvalue λ_n then the contribution of this mode to a Gaussian path integral is defined

by

$$\int \frac{da_n}{\sqrt{2\pi\varepsilon}}\, e^{-\frac{1}{2}\lambda_n a_n^2} = (\varepsilon\lambda_n)^{-\frac{1}{2}} \quad . \tag{4.49}$$

This definition is chosen to lead to a dimensionless path integral and a dimensionless determinant. The reason for such care is that the term involving dim ker P_1 in (4.45) will combine with residual powers of ε in the measure to generate a term involving the Euler characteristic. If we write dh_{ab} as

$$dh_{ab} = \sum{}' (da_\alpha)(\Psi_\alpha)_{ab} + \sum (dt_\omega)(\Psi_\omega)_{ab} \tag{4.50}$$

then the measure on the traceless metrical deformations is given by

$$[dh] = \left(\prod \frac{dt_\omega}{\sqrt{2\pi\varepsilon}}\right)\left(\prod{}' \frac{da_\alpha}{\sqrt{2\pi\varepsilon}}\right) \quad . \tag{4.51}$$

We can write an infinitesmimal diffeomorphism as

$$dV_a = \sum{}' (db_\alpha)(\Phi_\alpha)_a + \sum (dc_r)(\Phi^0_r)_a \ , \tag{4.52}$$

where dc_r are the expansion coefficients for the conformal Killing vectors. The relationship (2.33) between dh and dV allows us to re-express (4.51) as

$$\left(\prod \frac{dt_\omega}{\sqrt{2\pi\varepsilon}}\right)\left(\prod{}' \frac{db_\alpha}{\sqrt{2\pi\varepsilon}}\right)\left(\prod{}' \lambda_n\right)^{\frac{1}{2}} \tag{4.53a}$$

$$= \left(\prod{}' \frac{dt_\omega}{\sqrt{2\pi\varepsilon}}\right)\left(\prod{}' \frac{db_\alpha}{\varepsilon\sqrt{2\pi}}\right)(\det{}' P_1^\dagger P_1)^{\frac{1}{2}}. \tag{4.53b}$$

We had to introduce an ε with each λ_n in such a way that the determinant would be dimensionless. As a consequence the group measure

$$\Omega^{\perp}_{Diff} = \prod{}' \frac{db_\alpha}{\varepsilon\sqrt{2\pi}} \tag{4.54}$$

involves terms of the form db_α/ε and <u>not</u> $db_\alpha/\sqrt{\varepsilon}$. One can verify that the combination db_α/ε is dimensionless. The volume of conformal Killing vectors is given by

$$\Omega_{CKV} = \int \prod \frac{dc_r}{\varepsilon\sqrt{2\pi}} [\det H(P_1)]^{\frac{1}{2}} \tag{4.55}$$

It follows that the volume of the full diffeomorphism group is given by

$$\Omega_{Diff} = \Omega^{\perp}_{Diff} \times \Omega_{CKV} \quad . \tag{4.56}$$

Contrary to the impression given by (2.36) we will argue that Ω_{Diff} and Ω_{Kill} are independent of the metric on M. There are several ways of seeing this. The naivest argument is the following. The diffeomorphisms and the analytic diffeomorphisms are defined on a compact Riemann surface independently of the metrical structure on M. Therefore, Ω_{Diff} and Ω_{Kill} should be independent of the metric. For example, in the case of a disc the conformal Killing vectors are the generators of $SL(2,\underset{\sim}{R})$ Ω_{Kill} is the volume of $SL(2,\underset{\sim}{R})$, a divergent computable using the right invariant measure on $SL(2,\underset{\sim}{R})$. Likewise, Ω_{Kill} may be computed by formally defining the right invariant measure of the diffeomorphism group.

Since the Ω's are independent of the metrics we may schematically write (4.44) as

$$Z[C] = \Omega_{Diff} \times \left[\Omega^{-1}_{CKV} \times \int \ldots\right] . \tag{4.57}$$

This representation of the partition function is of special interest to us. Since Z and Ω_{Diff} are gauge invariant quantities, it follows that the quantity in brackets is gauge invariant. In a different gauge, Ω_{CKV} would be replaced by the volume of the residual gauge group after gauge fixing,

and the integral would be the corresponding gauge fixed path integral. The corresponding product in the brackets would yield the same result as in the conformal gauge we have used. No matter which gauge is used, one always gets the same factor of Ω_{Diff} up front. This is an important fact since the physical partition function is given by

$$Z[C]_{phys} = Z[C]/\Omega_{Diff} \cdot \tag{4.58}$$

This is an observation which we will use in Section 5 when we discuss the sum over topologies. There are several arguments that motivate why Z_{phys} is the gauge invariant quantity that contains the correct physics. The original idea of the Feynman path integral is that each physical path should contribute to the path integral. Summing over gauge equivalent paths means that one is counting the same physical configuration many times. A different argument can be borrowed from super-symmetry. Assume one has a super-symmetric Yang-Mills theory then

$$Z \sim (\text{vol } G)^V \times \int_{\text{gauge fixed}} \ldots$$

when G is the gauge group and V is the volume of space time. One gets an additive shift in the ground state energy of the system. We know that within perturbation the gauge fixed path integral gives zero for the ground state energy. This suggests that the correct physical partition function is obtained by dividing Z by the volume of the gauge group. Additional arguments will be given in Section 5 when we discuss the sum over topologies.

We are now in a position to extract the residual powers of ε in the measure. There are two places where residual powers of ε enter. There

are some ε's in the integration over Teichmüller directions in (4.44), and there are others in Ω_{CKV} as shown in (4.55). We would like Z_{phys} to be well defined as $\varepsilon \to 0$. We will show that ε dependence of the measure can be absorbed into a renormalization of the B coefficient in (4.45). The $\ell n\, \varepsilon$ terms in (4.45) are actually $\ell n(\varepsilon M^2)$ where M^2 is some arbitrary renormalization scale. The strategy is to trade the cutoff ε for M^2 in the Teichmüller integrations and in Ω_{CKV}. It is easy to see that one gets the following factors of εM^2:

$$(\varepsilon M^2)^{(\dim \ker P_1)} (\varepsilon M^2)^{-\frac{1}{2}\dim \ker P_1^\dagger} (\varepsilon M^2)^{-\frac{1}{2}\dim \ker P_1} . \tag{4.60}$$

The first factor is the contribution from Ω_{CKV}, the second factor is the contribution from the Teichmüller directions, the third factor is from the second term in (4.45). The Riemann-Roch theorem (4.31) allows one to combine these three terms into a single term involving the Euler characteristic:

$$\exp[(\tfrac{3}{2})\, \chi(M)\, \ell n(\varepsilon M^2)] . \tag{4.61}$$

One can see that this term can be absorbed into the B coefficient in (4.45).

We can now combine all the results into the final expression for the physical partition function:

$$Z[C]_{phys} = \exp(-\lambda_o L) \times \Omega^{-1}_{CKV;ren} \times \exp\{-\chi[B + (\ell n\, \varepsilon M^2)\cdot(d-26)/12]\}$$

$$\times \int \prod \left(\frac{M^2}{2\pi}\right)^{\frac{1}{2}} dt_\omega \, \{\exp(-\tilde{S}[X_t, \hat{g}_t]) \times Z[C, \hat{g}_t]_{conf}\}, \tag{4.62}$$

where

$$\Omega_{CKV;ren} = \int\prod\frac{M^2}{\sqrt{2\pi}}\, dc_r [\det H(P_1)]^{\frac{1}{2}}, \tag{4.63}$$

$$\tilde{S}[X_t,\hat{g}_t] = A\,J[X_t,\hat{g}_t] + S_o[\hat{g}_t]_{finite} \tag{4.64}$$

and

$$Z[C,\hat{g}_t]_{conf} = \int [d\sigma]_t \{[\det H(P_1^\dagger)]_{\sigma,t}\, [\det H(P_1)]^{\frac{1}{2}}_{\sigma,t} \times \exp(-\tilde{S}[\sigma,\hat{g}_t]_{conf})\}. \tag{4.65}$$

The conformal factor action is given by

$$\begin{aligned}\tilde{S}[\sigma,\hat{g}_t]_{conf} &= \mu_o^2 \int_M d^2z\, \sqrt{\hat{g}_t}\, e^{2\sigma} \\ &+ \frac{26-d}{12\pi}\left[\frac{1}{2}\int_M d^2z\, \sqrt{\hat{g}_t}\, \hat{g}_t^{ab}\, \partial_a\sigma\partial_b\sigma\right. \\ &\left. + \int_{\partial M} ds_t\, \hat{k}_t\sigma + \frac{1}{2}\int_M d^2z\, \sqrt{\hat{g}_t}\, \hat{R}_t\sigma\right].\end{aligned} \tag{4.66}$$

There are several important remarks that have to be made concerning the series of equations beginning with (4.62). The first term in (4.65) is the linear divergence associated with the perimeter, see (4.48). We have extracted the cutoff dependence from the Teichmüller integrations and from Ω_{CKV}, see (4.62) and (4.63). The Teichmüller integral is over a finite dimensional space. The factors of M^2 in dt_ω and dc_r in (4.62) and (4.63) respectively, are required by dimensional analysis. Note that in the case of a disc, Ω_{CKV} is divergent since $SL(2,\underset{\sim}{R})$ is non-compact. As was previously mentioned, the cutoff dependence in question renormalizes the B coefficient in (4.64). There can be additional renormalizations to B

from Z_{conf}. These will be discussed in Section 5.

In the calculation of the piece of the partition function which depends on the conformal factor, see (4.65), one encounters some finite dimensional determinants which depend on σ. The conformal factor action $\tilde{S}_{conf}$ is the Liouville theory in a space with fiducial metric $\hat{g}_t$. The quantization of S_{conf} uses a measure $[d\sigma]_t$ determined by (2.26):

$$\|d\sigma\|^2 = \int d^2z\, \sqrt{\hat{g}_t}\, e^{2\sigma}(d\sigma)^2 . \tag{4.67}$$

The measure depends on the Teichmüller parameters.

Section 5

Discussion and Speculation

Sections 3 and 4 developed the mathematics and derived the mathematical results necessary to discuss the Polyakov string model in an arbitrary topology. In this section we discuss the physics of the Polyakov model. Some parts of the discussion will be of a speculative nature. Many of the speculations arise because the quantum Liouville theory is not well understood.[23] Others arise in discussing the sum over the topologies. The discussion will proceed by working from the inside towards the outside in the physical partition function (4.62-66).

The boundary condition on the σ field plays a very important role relating physical lengths to parameter lengths. The coordinate field x^μ enters (2.17) only through the A term. Since the action is quadratic in x^μ, the purpose of A is to set a scale for x^μ. If one chooses x^μ to be dimensionless just as a scalar field in two dimensions then one can choose $A = 1$. The string tension $T_o = (2\pi\alpha')^{-1}$, where α' is the Regge slope, enters through the boundary condition (2.42) on σ. With a dimensionless x^μ, equation (2.42) should read

$$(\hat{g}_{ab})\, e^{2\sigma}\, \dot{z}^a \dot{z}^b = T_o^{-1}\, \frac{\partial x^\mu}{\partial z^a}\, \frac{\partial x^\mu}{\partial z^b}\, \dot{z}^a \dot{z}^b \text{ on } \partial M\ . \tag{5.1}$$

If one chooses x^μ to have the physical dimension of length then $A = T_o$ and the boundary condition reads

$$(g_{ab})_t\, e^{2\sigma}\, \dot{z}^a \dot{z}^b = \frac{\partial x^\mu}{\partial z^a}\, \frac{\partial x^\mu}{\partial z^b}\, \dot{z}^a \dot{z}^b \text{ on } \partial M\ . \tag{5.2}$$

The role of the string tension is to set a scale for the intrinsic metric $g = \hat{g}\exp(2\sigma)$. This is how the physical boundary couples to the intrinsic boundary which in turn is needed to define the Liouville theory. It was shown in Section 4 that σ can be chosen to be zero on ∂M. It follows that the physical length scale can be absorbed into the fiducial metric $\hat{g}$.

We will briefly discuss the functional integral quantization of the Liouville theory. We are interested in some of the general features of the renormalization of the Liouville theory. Since we are only interested in the overall structure of the renormalization counterterms, a discussion of the loop expansion to one loop will suffice.

The Liouville action may be written as

$$I = \frac{1}{\beta^2}\int d^2z\, \sqrt{\hat{g}}\, e^{2\sigma}\left[\frac{1}{2}\, e^{-2\sigma}\, \hat{g}^{ab}\partial_a\sigma\partial_b\sigma + m_o^2 + \frac{1}{2}\, e^{-2\sigma}\, \hat{R}\, \sigma\right] \tag{5.3}$$

The coupling constant β^2 is given by

$$\beta^{-2} = (26-d)/12\pi\ . \tag{5.4}$$

The loop expansion requires β^2 to be small which corresponds to $d \to -\infty$. In the discussion that follows we will not worry about boundary correction nor about zero modes. The quantization measure for the path integral is given by (4.67). A realistic calculation would have to include the determinant factors in (4.65). We will neglect these in the discussion.

The one loop effective action is given by

$$\Gamma_1 = I[\hat{g},\sigma] + \frac{1}{2}\, \mathrm{Tr}\, \ell n(\Delta + 4\, m_o^2) \tag{5.5}$$

where Δ is the Laplacian with respect to the metric $g = \hat{g}\exp(2\sigma)$. The second term in the above may be written as

$$[\tfrac{1}{2}\, \mathrm{Tr}\, \ell n\, \Delta] + [2\, m_o^2\, \mathrm{Tr}\, \Delta^{-1}] + \tfrac{1}{2}[\mathrm{Tr}\, \ell n(1+4m_o^2\, \Delta^{-1}) - 4m_o^2\, \mathrm{Tr}\, \Delta^{-1}]\ . \tag{5.6}$$

Power counting shows that the contribution from the last bracket is finite. The first bracket was evaluated in (4.41). The divergent contribution from the second bracket is easily evaluated by using the identity

$$y^{-1} = \int_0^\infty dt\, e^{-ty} ,$$

and the asymptotic expansion (4.37). The results are given by

$$\begin{aligned}(5.5) = &\left[-\frac{1}{8\pi\varepsilon}\int d^2z\,\sqrt{g}\,e^{2\sigma} + \frac{1}{12}\chi\,\ell n(\varepsilon M^2)\right.\\ &\left.-\frac{1}{12\pi}\int d^2z\,\sqrt{g}\,(\frac{1}{2}\,\hat{g}^{ab}\partial_a\sigma\partial_b\sigma + \frac{1}{2}\,\hat{R}\,\sigma) + \text{finite}\right]\\ &+\left[-\frac{m_o^2}{2\pi}\,\ell n(\varepsilon M^2)\int d^2z\,\sqrt{g}\,e^{2\sigma} + \text{finite}\right]\\ &+ [\text{finite}] \qquad (5.7)\end{aligned}$$

Each bracket corresponds to the contribution from the respective bracket in (5.5). Collecting all the terms leads to the result

$$\begin{aligned}\Gamma_1 = &[B - \frac{\pi}{\beta^2}(1 - \frac{\beta^2}{12\pi})\,\ell n(\varepsilon M^2)]\,\chi\\ &+\frac{1}{\beta^2}(1 - \frac{\beta^2}{12\pi} + \ldots)\int d^2z\,\sqrt{g}(\frac{1}{2}\,\hat{g}^{ab}\partial_a\sigma\partial_b\sigma + \frac{1}{2}\,\hat{R}\,\sigma)\\ &+\frac{1}{\beta^2}\,(m_o^2 - \frac{\beta^2}{8\pi\varepsilon} - \frac{\beta^2}{2\pi}\,m_o^2\,\ell n(\varepsilon M^2) + \ldots)\int d^2z\,\sqrt{g}\,e^{2\sigma}\\ &+ \ldots , \qquad (5.8)\end{aligned}$$

where ... represents the finite contributions we did not calculate. We have also included the Euler characteristic term of (4.62).

A first glance at (5.8) seems to indicate that there is a logarithmic renormalization of the topological chemical potential B, there is also a finite renormalization of β^2, and there are logarithmic and quadratic ($\varepsilon^{-1}=\Lambda^2$) renormalizations of m_o^2. The physics of the Liouville theory requires a different interpretation. We claim that the logarithmically divergent Euler characteristic term should not be viewed as a renormalization of B, but as a piece of the rest of the action. The reason for this viewpoint is the interpretation of σ as a conformal factor. All the analysis of Sections 2, 3, and 4 shows that adding a constant to σ is equivalent to a change in the cutoff ε. To one loop one can argue that there is no ... term in the kinetic energy. Adding a constant to σ generates an Euler characteristic term from the $\hat{R}$ piece. This may be absorbed into a redefinition of ε. Also note that one has to simultaneously renormalize m_o^2 under such a change. To one loop it is consistent to view the chemical potential as fixed.

Following Wilson[24], we define a dimensionless bare mass r_o by $m_o^2 = r_o\Lambda^2$. Changing the cutoff from ε to ε' leads to the following renormalizations:

$$2\sigma = 2\sigma' + \ell n(\varepsilon/\varepsilon') , \qquad (5.9a)$$

$$\beta^2 = \beta'^2 , \qquad (5.9b)$$

$$\left(1 - \frac{\beta^2}{2\pi}\,\ell n\,\varepsilon M^2\right)r_o - \frac{\beta^2}{8\pi} = \left(1 - \frac{\beta'^2}{2\pi}\,\ell n\,\varepsilon' M^2\right) - \frac{\beta'^2}{8\pi} . \qquad (5.9c)$$

We have to assume that ε' is small and very close to ε. This is required since the divergent terms in (5.8) were extracted by using the asymptotic short time behavior of the heat evolution operators. From (5.9c) one derives the following renormalization group equation for the bare

mass r_0:

$$\Lambda \frac{dr_0}{d\Lambda} = -(\beta^2/\pi)r_0 \ . \qquad (5.10)$$

The above discussion shows that this equation is only valid for large Λ. Equation (5.10) is very different from most of the one loop renormalization group equations for a bare mass. In most situations one would have, for example,

$$\Lambda \frac{dr_0}{d\Lambda} = -[2 + O(\beta^2)]r_0 \ . \qquad (5.11)$$

In the above, 2 is the classical contribution and the order β^2 term is the one loop correction. Equation (5.10) automatically begins at one loop. The reason is that in the classical theory once Λ^2 is chosen it is always possible to choose $r_0 = 1$ after an additive redefinition of σ as long as $m_0^2 \neq 0$. In comparison the massive free field depends on one dimensionless number r_0.

Equation (5.10) has an infrared unstable (ultraviolet stable) fixed point at $r_0 = 0$. Near the fixed point, r_0 scales with dimensionality β^2/π. Finally we mention that naive power counting suggests that there might be no higher order corrections to (5.10). If this is true then one can make some interesting conjectures based on scaling and known properties of other two dimensional field theories. First we consider the classical limit $\beta^2 = 0$. Note that r_0 does not scale. This is what happens in a massless free field. It is well known that the classical Liouville theory can be mapped via a canonical transformation into a massless free field[23]. We conjecture that at $\beta^2 = 2\pi$, the Liouville theory describes a massive free boson. The reason for this conjecture is that r_0 scales with dimension 2, i.e. like a mass term of a free scalar field. At

$\beta^2 = \pi$, r_0 scales with dimension one, i.e. like a mass term for a massive free fermion. A wilder speculation is that the Liouville theory at $\beta^2 = \pi$ describes a free massive fermion.[fl] These scaling arguments are actually true in the Thirring – sine-Gordon system[26]. The respective analogies correspond to $\beta^2_{Coleman} = 8\pi$, $\beta^2_{Coleman} = 0$, and $\beta^2_{Coleman} = 4\pi$. We emphasize the only input into these speculations is the renormalization group equation. There are no reasons for believing these conjectures to be true.

If the manifold M has the topology of a torus or an annulus, i.e. $\chi(M) = 0$, then one can give a slightly different renormalization prescription. In this restricted case, it might not be necessary to do an additive field renormalization. The renormalization group equation is given by

$$\Lambda \frac{dr_0}{d\Lambda} = -2\left(1 + \frac{\beta^2}{2\pi}\right)r_0 - \frac{\beta^2}{4\pi} \ . \qquad (5.12)$$

There is an infrared unstable fixed point at $r_0 = \beta^2/8\pi$. Near the fixed point r_0 scales with dimensionality $2(1 + \beta^2/2\pi)$. The term in parenthesis is precisely the modification factor that Curtright and Thorn[22] found for the improved energy momentum tensor. Note that these authors chose M to be a torus.

The above discussion shows that all the terms in (5.7) should be grouped together except the B term. It is an open question whether this pattern persists to higher loops, we will assume it does.

We are now ready to speculate on what happens when one sums over topologies. We assume that M has the topology of a disc with h attached handles. The Euler characteristic is given by $\chi = 1-2h$, and the dimension of Teichmüller space is $-3\chi = 3(2h-1)$ if $h \geq 1$. One can guess

that the grand partition function for surfaces is given by

$$Z = \sum_{h=0}^{\infty} e^{-B\chi} Z_\chi , \qquad (5.13)$$

where Z_χ is the Z_{phys} of (4.62) except we explicitly have taken out the chemical potential term. In Z_χ we have left the divergent renormalization involving χ. Previous discussions show that Z_χ can be made cutoff independent. In Section 4 we gave reasons why one should not include a factor of Ω_{Diff} in Z_χ. We note that if such a factor was present in (5.13) then we would not know how to proceed. The volumes of diffeomorphisms for different topologies are not easily related. As a practical matter the factor must be eliminated. There is an equivalent but slightly different viewpoint which parallels the discussion of Section 4. There we argued that physically equivalent configurations should only be counted once. Our viewpoint follows the discussion of how the quantum mechanical indistinguishability of identical particles requires the classical partition function for N particles to be divided by N! . One can take the viewpoint that one sums over all configurations, but statistics require one to divide by the number of indistinguishable ones.

Let us qualitatively study Z_χ. There is an exponentially damped contribution which depends on the classical action for the coordinate fields x^μ, see (4.64). It is a reasonable expectation that J should increase as the number of handles increases. There is a contribution from the $S_o[\hat{g}_t]_{finite}$ term in (4.64). We are unable to make any statement about this term. There can also be an additional exponential suppression factor from classical solutions to the Liouville theory. As $d \to -\infty$,

the string tension A in (4.64) should grow like $|d|$. We can define $\tilde{A} = \beta^2 A$, and study the theory as $\beta^2 \to 0$ with $\tilde{A}$ fixed. The Teichmüller integrations should be amenable to steepest descent methods. Since Teichmüller space is compact, the integrand for the Teichmüller integrations (after doing the σ integral) has a maximum somewhere. If the maximum is unique, the Teichmüller integrations are dominated by a single $\hat{g}_t$. In other words, each topological sector could be dominated by a specific Teichmüller class.

The radius of convergence of the series (5.13) is determined by the behavior of Z_χ as $h \to \infty$. There are three distinct possibilities. If the radius of convergence is zero, then (5.13) is at best an asymptotic expansion about $\tilde{B} = 0$ where $\tilde{B} = \exp(B)$. The first few terms of (5.13) could possibly provide a reasonable approximation for small $\tilde{B}$. A finite radius of convergence leads to a very interesting possibility. For $\tilde{B}$ small, Z would be dominated by the simplest topologies. As $\tilde{B}$ approaches the radius of convergence, the series could diverge. In such a case all topologies become important at the radius of convergence. If the singularity that determines the radius of convergence is on the real $\tilde{B}$ axis then one would have a genuine phase transition. If the singularity is off the axis then one would remain in the same phase except that the physics could be qualitatively different. In both these cases one winds up with a topological soup near the radius of convergence. If the radius of convergence is infinite then nothing interesting happens. By adjusting $\tilde{B}$ to be small one can suppress the higher topologies.

The object of this study of the Polyakov model is to do a static potential calculation. The major obstacle is our inability to calculate

in the Liouville theory. In ref. 6, the static potential at $d = \infty$ was calculated. The transition is probably an artifact of the large d approximation. It would be interesting to know the behavior of the static potential in the Polyakov model. The model satisfies an area law for large loops. What happens as the width R goes to zero? Since the Polyakov model is renormalizable, the static potential could be well defined as $R \to 0$. Does it behave in the scale invariant fashion $1/R$ as $R \to 0$. In the loop expansion we found an infrared unstable fixed point at $r_0 = 0$. How does this fixed point affect the dynamics of the Liouville theory? These are questions that can only be answered by better understanding the quantum Liouville theory.

In his original paper on the 1/N expansion in QCD, 't Hooft pointed out the possible connection of QCD to dual model.[25] A Green's function in the 1/N expansion is given by

$$G = \sum N^{\chi} G_{\chi} ,$$

where G_{χ} are the graphs with Euler characteristic χ. The number of colors plays the role of a topological coupling constant. It is a fixed parameter in the model.

One of the results of this paper is that the topological coupling constant seems to be a fixed number. At first hand it appears to have a logarithmic renormalization, but our interpretation shows that this is not the case. Therefore, the parameter B is at our disposal. This observation does not contradict the analogy between string models and QCD. Within the context of this paper, the conjecture that B and log N are related seems to hold. If B was renormalized, then one would have

to rethink the analogy since B would become a running coupling constant.

Assume the string model is obtained by an effective coarse graining of the QCD variables when one calculates the Wilson loop. It is difficult to imagine how one would obtain a factor of the volume of the diffeomorphism group from the coarse graining. After all, the Wilson loop is finite after renormalization. This remark was put in to complement the discussion of Section 4.

Throughout this paper we concentrated on compact manifolds with boundary. There are two dimensional manifolds which are compact without boundary, and others non-compact. We did not discuss these since they are not related to the Wilson loop. The compact manifolds without boundary are related to the closed string, and the non-compact ones are related to the open string. There is a very mysterious phenomena related to these two cases. The Liouville theory in both those cases seems to be independent of the physical world. There is no boundary which brings in the physical scale. One has to introduce sources which couple to the physical coordinates x^{μ}. These sources will couple to the conformal factor and introduce the external physical scale into the Liouville theory. The mass parameter of the Liouville theory appears not to be related to the string tension. There is a scale that always has to be introduced into the Liouville theory. It is the scale of the fiducial metric $\hat{g}$. For example, in the case of a sphere $\hat{g}_{ab} = \delta_{ab}(1 + c|z|^2)^{-2}$ is the standard constant curvature metric. The dimensional constant c sets the scale. In the case of a manifold with boundary, the boundary conditions set the scale. Unless one introduces external sources, we do not know how to set the scale for the fiducial metrics $\hat{g}_t$.

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Appendix A

A Classification of Metrics

In this appendix we discuss the classification of metrics on two dimensional manifolds.[13] We will argue that every metric is conformal to a constant curvature metric. A proof of this statement lies way beyond the scope of this paper. We will motivate this theorem at two levels. In Part 1 of the appendix we will attack at the level of a non-linear partial differential equation. We will actually prove the theorem if the Euler characteristic is zero. In Part 2 of the appendix we will use the notion of the universal covering space of the manifold to reach the same conclusions. The presentation in Part 2 is more mathematically sophisticated than in Part 1, but it is essential for the calculation of the dimension of the space of Teichmüller deformations given in Part 2 of Appendix B.

Part 1:

One of the triumphs of algebraic topology is the topological classification of surfaces. If M is a compact orientable two dimensional manifold then M is topologically equivalent to a sphere with a certain number of holes cut out. The Euler characteristic of M is given by

$$\chi(M) = 2 - 2h - b \ , \qquad (A.1)$$

where h is the number of handles, and b is the number of holes (boundaries). The genus of M is h.

It is also possible to classify the allowed metrics on a manifold M. In Part 1 we shall motivate the classification by qualitative

arguments and examples. Proving the classification requires deep mathematical results on the solutions of a certain non-linear partial differential equation.

Let M be a compact orientable two manifold without boundary and let $\underset{\sim}{g}$ be a metric on M. We shall argue that $\underset{\sim}{g}$ is conformal to a constant curvature metric $\hat{\underset{\sim}{g}}$. Assume there exists such a conformal factor σ:

$$\underset{\sim}{g} = \hat{\underset{\sim}{g}} \exp(2\sigma) \tag{A.1}$$

then the respective curvatures are related by

$$\hat{R} = e^{2\sigma}[R - 2\Delta\sigma] . \tag{A.2}$$

The existence of a constant curvature metric boils down to solving the equation

$$2\Delta\sigma = R - \hat{R}\, e^{-2\sigma} \tag{A.3}$$

for σ where $\hat{R}$ is a constant. If $R = 0$ then the above is the Liouville equation which has solutions.

We shall now try to motivate why (4.3) has solutions. Normalize σ by requiring that

$$\int_M d^2z \sqrt{g} \exp(-2\sigma) = 4\pi . \tag{4.4}$$

Normalization (4.4) will impose a normalization condition on $\hat{R}$. The Guass-Bonnet theorem states that

$$\chi(M) = (4\pi)^{-1} \int d^2z \sqrt{\hat{g}}\, \hat{R} . \tag{A.5}$$

Since R is a constant it follows that

$$\hat{R} = \chi .$$

The sign of the curvature in the constant curvature metric is determined by the Euler characteristic.

The constant curvature metric $\hat{\underset{\sim}{g}}$ is determined by a conformal factor which is the solution to

$$2\Delta\sigma = R - \chi e^{-2\sigma} , \tag{A.7}$$

where

$$\int d^2z \sqrt{g}\, e^{-2\sigma} = 4\pi. \tag{A.8}$$

An equation of the type $\Delta\sigma = f$ has a solution only if f is orthogonal to the harmonic functions. Since the harmonic functions are the constant functions, it is easy to show by using the Gauss-Bonnet theorem that the right hand side of (4.7) is orthogonal to the harmonic functions. Therefore, there exists a possibility for a solution to exist. It is a deep theorem that equation (4.7) has solutions.

If the Euler characteristic is zero then (4.7) reduces to the Poisson equation

$$2\Delta\sigma = R . \tag{A.9}$$

This equation is always soluble. Therefore, if the manifold has the topology of a torus then every metric is conformal to a zero curvature metric.

If the manifold has a boundary then every metric is conformal to a constant curvature metric such that the boundary is geodesic. To motivate this theorem we note that the geodesic curvatures are related by

$$\hat{k} = e^{\sigma}(k + n^a \partial_a \sigma) . \tag{A.10}$$

The boundary is a geodesic with respect to the $\hat{g}$ metric if $\hat{k} = 0$. The aforementioned theorem boils down to proving that the equation

$$2\Delta\sigma = R - \chi e^{-2\sigma} \text{ on } M , \tag{A.11}$$

satisfying the Neumann boundary condition

$$k + n^a \partial_a \sigma = 0 \text{ on } \partial M , \tag{A.12}$$

and the normalization condition (A.8) has solutions. If $\chi = 0$ then the above reduces to the Neumann problem for the Laplacian. If $\chi \neq 0$ then we have to appeal to a deep theorem for the existence of a solution.

Part 2:

For the remainder of Appendix A we assume that the reader is familiar with the notion of the universal covering space of a manifold. The presentation will be more mathematical than in Part I, but the viewpoint is very different and offers a new insight to the classification of the metrics.

The universal covering space of a manifold M is a simply connected manifold $\tilde{M}$ along with a projection p: $\tilde{M} \to M$. If W is a neighborhood in M then $p^{-1}(W)$ is a discrete union of disjoint open sets, each of which is diffeomorphic to W. The fundamental group of M acts on $\tilde{M}$ as the group of "deck" transformations. The classic example is the circle, S^1, with universal covering space $\underset{\sim}{R}$ and projection $p(x) = \exp(2\pi i x)$. Note that a non-trivial closed loop in S^1 lifts to an open curve in $\underset{\sim}{R}$ which takes one from one deck to another, see Figure 4.

Assume M is a compact Riemann surface without boundary. A classic

theorem states that M is analytically equivalent to either

1) sphere (if genus $h = 0$);
2) torus = plane/lattice (if $h = 1$);
3) U/G where $U = \{z \mid \text{Im } z > 0\}$ and G is a crystallographic subgroup of $SL(2,\underset{\sim}{R})$, (if $h > 1$).

A consequence of the same theorem is that there are essentially three distinct simply connected Riemann surfaces up to holomorphic equivalence: the sphere, the plane, and the upper half plane. These are the universal covering spaces for the compact Riemann surfaces.

We shall discuss the case of genus $h > 1$. The group of analytic diffeomorphisms of the upper half plane consists of the linear fractional transformations with real coefficients. It is well known that this group is isomorphic to $SL(2,\underset{\sim}{R})$. Let G be a crystallographic subgroup of $SL(2,\underset{\sim}{R})$. The set U/G consists of the equivalence classes of points in U which are identified under the action of G, i.e., $z' \sim z$ if there exists a $\gamma \in G$ such that $z' = \gamma(z)$. The reader might wish to consider the simpler case of genus 1, i.e. the torus, to familiarize himself with the statements made above. A metric on $M = U/G$ automatically is a metric on U but the converse is not true.

The first step in the classification of metrics on M is to study the metrics on the upper half plane U. We can use to our advantage the fact that a single coordinate patch covers U. Denote the standard complex coordinates on U by z. The most general metric on U is of the form

$$ds^2 = \lambda |dz + \mu\, d\bar{z}|^2 , \tag{A.13}$$

where $\lambda = \lambda(z,\bar{z}) > 0$ and $|\mu(z,\bar{z})| < 1$. These two requirements reflect the

positive definiteness of the metric. It is important to notice that if $|\mu_0(z,\bar{z})| < 1$ and $|\mu_1(z,\bar{z})| < 1$ then $\mu_t = (1-t)\mu_0 + t\mu_1$ also satisfies $|\mu_t(z,\bar{z})| < 1$ for $0 \leq t \leq 1$. Therefore the set of μ's forms a convex set.

There is a "natural" metric on the upper half plane. It is the constant negative curvature metric of Poincaré:

$$ds_p^2 = \frac{dzd\bar{z}}{(Imz)^2} \quad . \tag{A.14}$$

This metric is natural in the sense that it is invariant with respect to the action of $SL(2,\underset{\sim}{R})$ in the same way that the Euclidean metric is invariant with respect to the Euclidean transformations of the plane. This observation leads to our first result about metrics on M: there exists a metric of constant negative curvature on M. Since $M = U/G$ and since ds_p^2 is invariant under the subgroup G of $SL(2,\underset{\sim}{R})$ it immediately follows that ds_p^2 projects onto a smooth metric on M.

The next step is to argue that every metric on U is conformal under a diffeomorphism to the Poincaré metric. Let $f:(U,ds^2) \to (U,ds_p^2)$ be a diffeomorphism. Let $f^*ds_p^2$ denote the pullback of the Poincaré metric then

$$f^*ds_p^2 = \frac{|f_z|^2}{|Imf|^2}\,|dz + (f_{\bar{z}}/f_z)d\bar{z}|^2 \quad . \tag{A.15}$$

(A.15) is conformal to (A.13) if and only if

$$f_{\bar{z}} = \mu\, f_z \quad . \tag{A.16}$$

Equation (A.16) is known as the Beltrami equation. The existence theorem for solutions to the Beltrami equation essentially goes back to Gauss.

We may take this theorem to state that given a μ there exists a diffeomorphism f of the upper half plane which satisfies (A.16).

The viewpoint in Part 2 is different than the viewpoint presented in Part 1. In Part 1 the strategy was to construct a conformal factor σ such that $e^{-2\sigma}ds^2$ was a constant curvature metric. In Part 2 we are exploiting the Poincaré metric by constructing a diffeomorphism such that the constant curvature metric $f^*ds_p^2$ was conformal to the desired metric ds^2 . Let us now apply this approach to the metrics on M.

We already know that a metric on $M = U/G$ is automatically a metric on U. We would like to use our results on metrics on U to prove a theorem about metrics on $M = U/G$. The requirement that metric (A.13) on U projects to a smooth metric on $M = U/G$ is that ds^2 be invariant under the action of all $\gamma \in G$. This imposes the following restrictions on the functions λ and μ:

$$\lambda(\gamma(z))\,|\gamma'(z)|^2 = \lambda(z) \quad , \tag{A.17a}$$

$$\mu(\gamma(z))\,\overline{\gamma'(z)}/\gamma'(z) = \mu(z) \; ; \tag{A.17b}$$

for all $\gamma \in G$. It is easy to show that the set of μ's which satisfy (A.17b) forms a convex set. Therefore, the set of all metrics on a manifold $M = U/G$ is a "linearly" connected set.

We would now like to show that every metric on $M = U/G$ is conformal under a diffeomorphism of M to a constant curvature metric. Assume ds^2 in (A.13) is an admissible metric on $M = U/G$, i.e. λ and μ satisfy (A.17). By the existence theorem there exists a diffeomorphism f of U

which satisfies the Beltrami equation (A.16). Is f a smooth diffeomorphism of $M = U/G$? The answer may be found by inspecting the following diagram:

$$\begin{array}{ccccc} (U,ds^2) & \xrightarrow{\text{conf}} & (U,f^*ds_p^2) & \xrightarrow{f} & (U,ds_p^2) \\ \gamma\uparrow & & \uparrow & & \uparrow\gamma \\ (U,ds^2) & \xrightarrow{\text{conf}} & (U,f^*ds_p^2) & \xrightarrow{f} & (U,ds_p^2) \end{array} \qquad (A.18)$$

Both ds^2 and ds_p^2 are invariant under G and the conformal factor is the same in the upper and lower levels, therefore the action of γ is well defined along the dotted line and f projects to a diffeomorphism of $M = U/G$. This completes our discussion of the space of metrics on compact boundaryless Riemann surfaces.

If M has a boundary then one can apply the machinery of the previous section by using a clever construction. Let $\overline{M}$ be the mirror image of M. The reader should think of $\overline{M}$ as being obtained from M by a generalization of the Schwarz reflection principle. The analytic structure on M becomes an anti-analytic structure on $\overline{M}$. Construct the double of M, denoted M^d, by glueing M and $\overline{M}$ about the corresponding points on the respective boundary, see Figure 5 . Note that M^d is a compact orientable manifold without boundary.

Our previous arguments show that one can choose a constant curvature metric on M^d. The restriction of the metric to M is a constant curvature metric on M. With more work one can use the double construction to show that the constant curvature metric can be chosen such that the boundary is a geodesic.

Appendix B

The Dimension of Teichmüller Space

In this appendix we calculate the dimension of Teichmüller space for compact manifolds. We study the space of conformal Killing vectors on the manifold, and then we use the Riemann-Roch theorem,

$$\begin{aligned} \mathcal{I}(P_1) &= \dim \ker P_1 - \dim \ker P_1^\dagger \\ &= 3\,\chi(M)\ , \end{aligned} \qquad (B.1)$$

to calculate the dimensionality of the Teichmüller space. The calculations in the case $\chi < 0$ is more mathematically sophisticated than the other cases.

If one views the sphere as the extended complex plane then the analytic diffeomorphisms of the sphere are given by the linear fractional transformations with complex coefficients. The infinitesimal analytic diffeomorphisms, i.e. conformal Killing vectors, are described by six real parameters. Since $\chi(S^2) = 2$ it follows from (B.1) that dim ker $P_1^\dagger = 0$. There are no Teichmüller deformations on the sphere.

If M has the topology of a disc then one can map M via a linear fractional transformation onto the upper half plane. In Appendix A we used the fact that the analytic diffeomorphisms of the upper half plane were given by linear fractional transformations with real coefficient and determinant 1. The space of conformal Killing vectors is described by three real parameters. Since $\chi(\text{disc}) = 1$, it follows from (B.1) that dim ker $P_1^\dagger = 0$. There are no Teichmüller deformations on the disc.

There are many ways of calculating the dimension of the Teichmüller space of the torus T^2. We shall discuss two methods. The universal

covering space of the torus is the complex plane $\underset{\sim}{C}$. According to Appendix A, a torus is given by $T^2 = \underset{\sim}{C}/\Gamma$ where Γ is a lattice:

$$\Gamma = \{n_1\omega_1 + n_2\omega_2\} \ . \qquad (B.2)$$

In the above equation n_1, n_2 are integers, and ω_1,ω_2 are complex numbers which are linearly independent if viewed as elements of $\underset{\sim}{R}^2$, i.e. $\mathrm{Im}(\omega_1/\omega_2) \neq 0$. Note that Γ is an Abelian group under addition. A conformal Killing vector V^z is an analytic function of z on the torus. Since T^2 is compact, V^z is bounded. V^z defines a bounded periodic entire function on $\underset{\sim}{C}$, the universal covering space of the torus. Liouville's theorem tells us that V^z must be a constant. Therefore, the space of conformal Killing vectors depends on two real parameters: dim ker $P_1 = 2$. Since $\chi(\text{torus}) = 0$ it follows that the space of Teichmüller deformations of a torus is two dimensional. This is easily verified by noting that h_{zz} is an analytic function of z, and repeating the argument just given. Incidentally, we have just proven the well known theorem that an everywhere analytic elliptic function is a constant.

A second proof can be given using properties of the universal covering space. We give a sketchy discussion since the method is described in detail for the case $\chi(M) < 0$ in the next paragraph. The analytic diffeomorphisms of $\underset{\sim}{C}$ are given by $z \to az + b$ where a and b are complex numbers. The analytic diffeomorphisms of $\underset{\sim}{C}$ which project smoothly to $T^2 = \underset{\sim}{C}/\Gamma$ are $z \to z + b$.

Assume that $\chi(M) < 0$ and $\partial M = 0$, then the universal covering space is the upper half plane U as described in Part 2 of Appendix A. An analytic diffeomorphism of $M = U/G$ is an analytic diffeomorphism of U. Therefore,

the analytic diffeomorphisms of $M = U/G$ correspond to elements of $SL(2,\underset{\sim}{R})$ acting on U that project smoothly to $M = U/G$. An immediate consequence is that dim ker $P_1 \leq 3$ since $SL(2,\underset{\sim}{R})$ is a three dimensional Lie group. We shall need two facts about the crystallographic group G: a) G is discrete, b) there are no limit points in G. If $f \in SL(2,\underset{\sim}{R})$ then f projects to an analytic diffeomorphism of $M = U/G$ if and only if for all $\gamma \in G$ there exists a $\gamma_1 \in G$ such that $\gamma \circ f = f \circ \gamma_1$. Define N(G) to be the set of $f \in SL(2,\underset{\sim}{R})$ that satisfy the previously mentioned condition, i.e.

$$N(G) = \{f \in SL(2,\underset{\sim}{R}) \mid Gf = fG\} \ .$$

N(G) is a group called the normalizer of G. The group of analytic diffeomorphisms of M is given by N(G)/G. The reason for dividing by G is that the identity transformation on $M = U/G$ may be represented by an arbitrary element of G in U. We shall prove that N(G) is discrete and has no limit points. This means that N(G)/G is discrete and has no limit points. Therefore, there are no infinitesimal analytic diffeomorphisms. The proof of the discreteness of N(G) is by contradiction. Assume N(G) was not discrete then there exists a sequence $\{\nu_j\}_{j=1}^{\infty}$ where $\nu_j \in N(G)$ and $\nu_j \to$ identity as $j \to \infty$. One can use the properties of a group to choose the limit point of the sequence to be the identity. Since $\nu_j \in N(G)$ this means that for every $\gamma \in G$, there exists a γ_j defined by $\gamma_j = \nu_j \gamma \nu_j^{-1}$ which is in G. The sequence $\{\gamma_j\}$ in G converges to γ. This contradicts the fact that G has no limit points. With a lot more work one can show that N(G)/G is a finite group.

We have just proven that there are no conformal Killing vectors if $\chi(M) < 0$ and if $\partial M = 0$. By the Riemann-Roch theorem we conclude that

the dimension of Teichmüller space is given by

$$\begin{aligned} \dim \ker P_1^{\dagger} &= -\,3\,\chi(M) \\ &= 6(h-1) \end{aligned} \tag{B.3}$$

if $\chi(M) < 0$ and $\partial M = 0$. Equation (B.3) was stated by Riemann without proof. One can show that the "finite" Teichmüller deformations (we have only discussed the infinitesimal deformation of h_{zz}) form a bounded convex domain in $\underset{\sim}{R}^{6(h-1)}$.

If M has a boundary and $\chi(M) < 0$ then one can use the previous arguments on the double of M. One concludes that there are no conformal Killing vectors on the double of M and thus none on M either. Since dim ker $P_1 = 0$ it follows that the dimension of Teichmüller space is given by

$$\begin{aligned} \dim \ker P_1^{\dagger} &= -\,3\,\chi(M) \\ &= 3(b+2h-2) \end{aligned} \tag{B.4}$$

where $\chi(M) < 0$.

Appendix C

A Sample Calculation

In this appendix we present a sample calculation which demonstrates the straight forward nature of the method explained in Section 4 for calculating the short time behavior of the diffusion operator. For simplicity we shall study the ordinary Laplacian Δ and we shall assume there is no boundary.

The first step is to choose local conformal coordinates with the origin being the point of interest. According to equation (4.7) one can choose σ to satisfy

$$\begin{aligned} \sigma(0) &= 0 \,, \\ \partial_a \sigma(0) &= 0 \,. \end{aligned} \tag{C.1}$$

In conformal coordinates the Laplacian is given by

$$\Delta = -\,e^{-2\sigma}\,\partial^2 \,. \tag{C.3}$$

The potential V defined in (4.5) is given by

$$V = (e^{-2\sigma} - 1)\partial^2 \,. \tag{C.4}$$

We are required to solve equation (4.6)

$$G = G_0 + G_0 V G \tag{C.5}$$

to order t^0, where the flat space diffusion operator operator is given by

$$G_0(z,0;t) = (4\pi t)^{-1} \exp(-|z|^2/4t) \,. \tag{C.6}$$

Since the diffusion length is of order $t^{\frac{1}{2}}$, and since

$$G_0(0,0;t) = (4\pi t)^{-1} \tag{C.7}$$

is of order t^{-1} we note that all one requires is two powers of the diffusion length to obtain the solution of (C.5) to order t^0. Near the origin V is given by

$$V = -z^a z^b [\partial^2_{ab}\sigma(0)]\partial^2 + O(z^3) \tag{C.8}$$

where we have used (C.1). Notice that the potential is quadratic in the diffusion length. Therefore to order t^0, the first order perturbation suffices:

$$\begin{aligned} G(0,0;t) &= G_0(0,0;t) \\ &+ \int d^2z \int_0^t dt' \; G_0(0;z;t-t')[-z^a z^b \partial^2_{ab}\sigma(0)]\partial^2 \; G_0(z;0;t') \\ &+ O(t^{\frac{1}{2}}) \end{aligned} \tag{C.9}$$

It is straightforward to evaluate the above

$$G(0,0;t) = (4\pi t)^{-1} - (12\pi)^{-1}\,\partial^2\sigma(0) + O(t^{\frac{1}{2}}) \tag{C.10}$$

Finally one uses the fact that $R(0) = -2\partial^2\sigma(0)$ to obtain the bulk contribution to (4.37).

The inclusion of a boundary requires that one choose G_0 to be the upper half plane diffusion operator. The calculation greatly simplifies if one uses the coordinates described in (4.8). At the point of interest the geodesic curvature is given by $k = -\partial_2\sigma(\zeta)$. The method of Balian and Bloch is very useful in sorting out the boundary contributions. The actual calculation is messy but straight forward. The key point is that one only has to do first order perturbation theory if one makes a clever choice of coordinates.

Appendix D. Pathologies with Dirichlet Boundary Conditions

We discuss a major problem that occurs if one tries to impose Dirichlet boundary conditions in the calculation of $\det D_1^{(+)}$. As in Section 4 we choose the domain to be the upper half plane, but we require the generators of diffeomorphisms to satisfy $V^a = 0$ on ∂M. If we use the notation of Section 4 then $\phi = V^1 + iV^2 = 0$ on ∂M. It is convenient to work in the space $\mathcal{T}^1$ instead of $\mathcal{S}^1$.

We are interested in computing $\det \Delta_n^{(+)}$ with $\phi \in \mathcal{T}^n$ and ϕ satisfying $\phi = 0$ on ∂M. The results of Section 3 show that $\psi \in \mathcal{T}^{n+1}$ satisfies $\nabla_z \psi = 0$ on ∂M. This boundary condition on $\mathcal{T}^{n+1}$ leads to major problems. Notice that $\ker \Delta_{n+1}^{(-)}$ consists of those ψ that satisfy $\nabla_z \psi = 0$ on M with the boundary condition $\nabla_z \psi = 0$ on ∂M. The kernel of $\Delta_{n+1}^{(-)}$ consists of any $(g_{z\bar{z}})^{n+1}\psi$ which is an anti-analytic function on the upper half plane. This is an infinite dimensional vector space. Notice that $\ker \Delta_n^{(+)}$ consists only of the zero function.

The zeroth order heat evolution operators are given by

$$G_0^{(+)}(z,w;t) = \frac{1}{4\pi t}\left(e^{-\frac{|z-w|^2}{4t}} - e^{-\frac{|z-\bar{w}|^2}{4t}}\right) \tag{D.1}$$

$$\begin{aligned} G_0^{(-)}(z,w;t) &= \frac{1}{4\pi t}\, e^{-\frac{|z-w|^2}{4t}} \\ &- \frac{1}{4\pi t}\,\frac{z-w}{(\bar{z}-w)^2}\, e^{-\frac{|z-\bar{w}|^2}{4t}} \\ &+ \frac{1}{\pi}\,\frac{1}{(\bar{z}-w)^2}\left(1 - e^{-\frac{|z-\bar{w}|^2}{4t}}\right) \end{aligned} \tag{D.2}$$

Note that $G_o^{(-)}$ has a term which is an analytic function of $\bar{z}$ which is a reflection of the kernel kernel of the Laplacian. It is the $G_o^{(-)}$ which leads to all the troubles. A long calculation shows that

$$\begin{aligned}
&\mathrm{Tr}[f\ \exp(-t\Delta_{n+1}^{(-)})] \xrightarrow{t\to 0} \\
&\frac{1}{4\pi t}\int_M d^2z\ \sqrt{g}\ f - \frac{1}{8\sqrt{\pi}}\ \frac{1}{\sqrt{t}}\int ds\ f \\
&+ \frac{1}{8\pi}\int_{\partial M} ds\ n^a\partial_a f[1 - \text{"}\ell n\ L^2/t\text{"}] \\
&- \frac{n+1}{4\pi}\int_{\partial M} ds\ k\ f \\
&- \frac{3n+2}{12\pi}\left[\int_{\partial M} ds\ k\ f + \frac{1}{2}\int_M d^2z\ \sqrt{g}\ R f\right] + O(\sqrt{t}) \qquad (D.3)
\end{aligned}$$

In the above L is an infrared cutoff with dimensions of length. The infrared divergence in the above arose because of the non-diffusive term in (D.2). The determinant is given by

$$\begin{aligned}
\mathrm{Tr}\ \ell n\ \Delta_n^{(+)} = &- \frac{1}{4\pi\epsilon}\int_M d^2z\ e^{2\sigma} \\
&+ \frac{1}{4\sqrt{\pi\epsilon}}\int_{\partial M} ds \\
&- \frac{1+6n(n+1)}{6}\left[\int_M d^2z\ \frac{1}{2}(2\sigma)^2 + \int_{\partial M} ds\ \hat{k}\,\sigma\right] \\
&-(2n+1 - n\cdot\ell n\ L^2/\epsilon)\ \frac{1}{4\pi}\int_{\partial M} ds\ k \\
&+ (\text{indep of } \sigma) \qquad (D.4)
\end{aligned}$$

In the above we have chosen $\hat{g}_{ab} = \delta_{ab}$ for simplicity. We note that the infrared divergence in (D.3) manifests itself in the determinant.

Strict Dirichlet boundary conditions are troublesome in the calculation of the Faddeev-Popov determinant. A corollary of (D.4) is that the gauge fixing scheme where one chooses a fixed parametrization of the boundary and afterwards one integrates over all parametrizations is troubled. This follows from the fact that fixing a parametrization of the boundary requires $V = 0$ on ∂M. The determinant of Faddeev and Popov will be infrared divergent.

Appendix E: Manifolds without Boundary

We discuss the modifications to equation (4.62) when $\partial M = 0$. The only change comes from the zero mode of the Laplacian when one performs the x^μ integration. Choose units such that A in (2.17) is equal to one. The x^μ integration gives

$$\int \prod \frac{d^{d} f_0^{\mu}}{\sqrt{2\pi\epsilon}} (\det{}'\Delta)^{-d/2} \tag{E.1}$$

where f_0^μ are the components of the normalized zero mode

$$\left(\int_M d^2z \sqrt{\hat{g}_t}\, e^{2\sigma} \right)^{-\frac{1}{2}} \tag{E.2}$$

One trades the zero-mode coordinates f_0^μ for a collective coordinate ξ^μ which corresponds to the center of mass of the surface. Since dx^μ may be written as either $dx^\mu = d\xi^\mu$ or $dx^\mu = \text{eq.}(E.2) \times df_0^\mu$ it follows that (E.1) is equivalent to

$$\int \prod \frac{d^{d}\xi^{\mu}}{\sqrt{2\pi\epsilon}} \left[\int d^2z \sqrt{\hat{g}_t}\, e^{2\sigma\, d/2} \right] (\det{}'\Delta)^{-d/2} . \tag{E.3}$$

The Laplacian satisfies (3.58) with $n = 0$. The contribution from the $\mathrm{Tr}[\delta\sigma\, K(\Delta)]$ term in the evaluation of $(\det{}'\Delta)^{-d/2}$ cancels the zero mode term in (E.3). The logarithmically divergent term in (3.61) corresponding to dim ker Δ cancels the ϵ dependence in (E.3). The upshot is that the physical partition function is given by (4.62) with all boundary terms set equal to zero but with an additional integration over the location of the surface

$$Z_{\text{phys}} = \int \prod \left(\frac{M^2}{2\pi} \right)^{\frac{1}{2}} d^{d}\xi^{\mu} \times \text{eq.}(4.62) . \tag{E.4}$$

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Figure Captions

Figure 1: A manifold that has the topology of a disc with an attached handle.

Figure 2: The tensor spaces $\mathcal{S}^n$ and $\mathcal{S}^{n+1}$ are mapped into each other by the operators P_n and $P_n^\dagger$. The relationship between the boundary conditions, and the special roles of the kernels are illustrated in the figure.

Figure 3: This figure explicitly demonstrates that there can be deformations of the metric which are not related to infinitesimal diffeomorphisms. The special role of conformal Killing vectors is also depicted.

Figure 4: The universal covering space of the circle is the real line with projection $p(x) = \exp(2\pi i x)$. Notice that a non-trivial closed loop in S^1 takes one from one "deck" to the next in the universal covering space. The thickened section between 0 and 1 illustrates the "lift" of loop in S^1.

Figure 5: The manifold M and its mirror image $\bar{M}$ are joined to make the double of M. Note that ∂M and $\partial \bar{M}$ have opposite orientations such that M and $\bar{M}$ have the same relative orientation.

Footnote

f1. We do not have a candidate for a fermion.

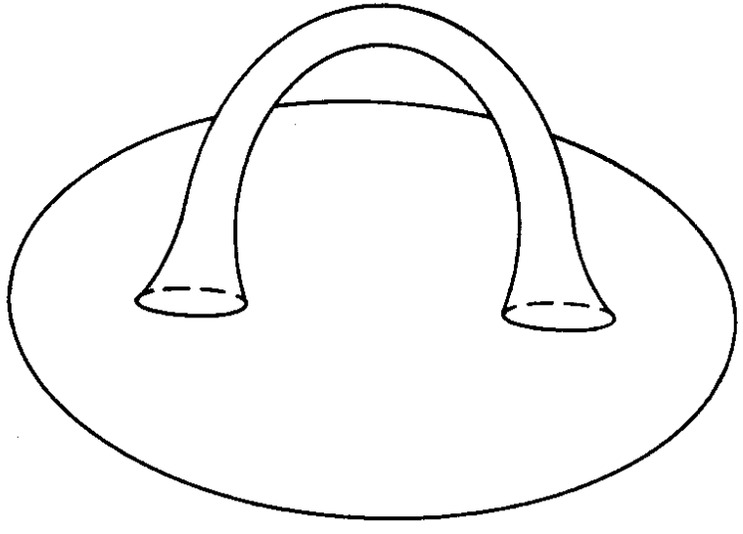

Figure 1

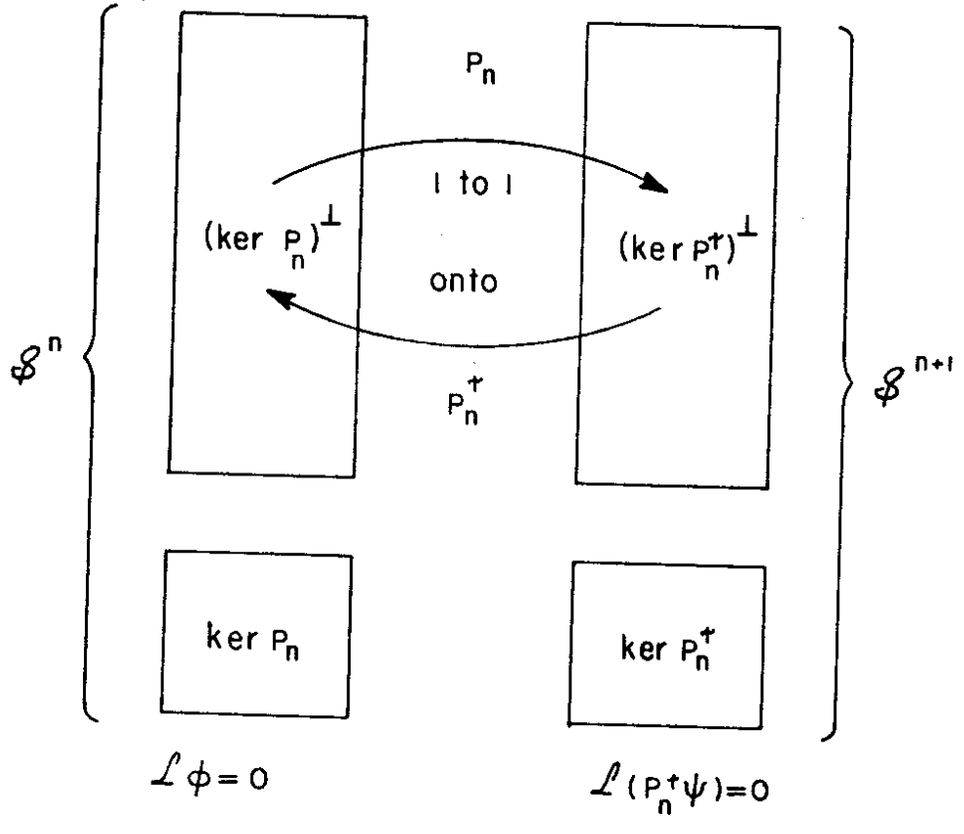

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Figure 2

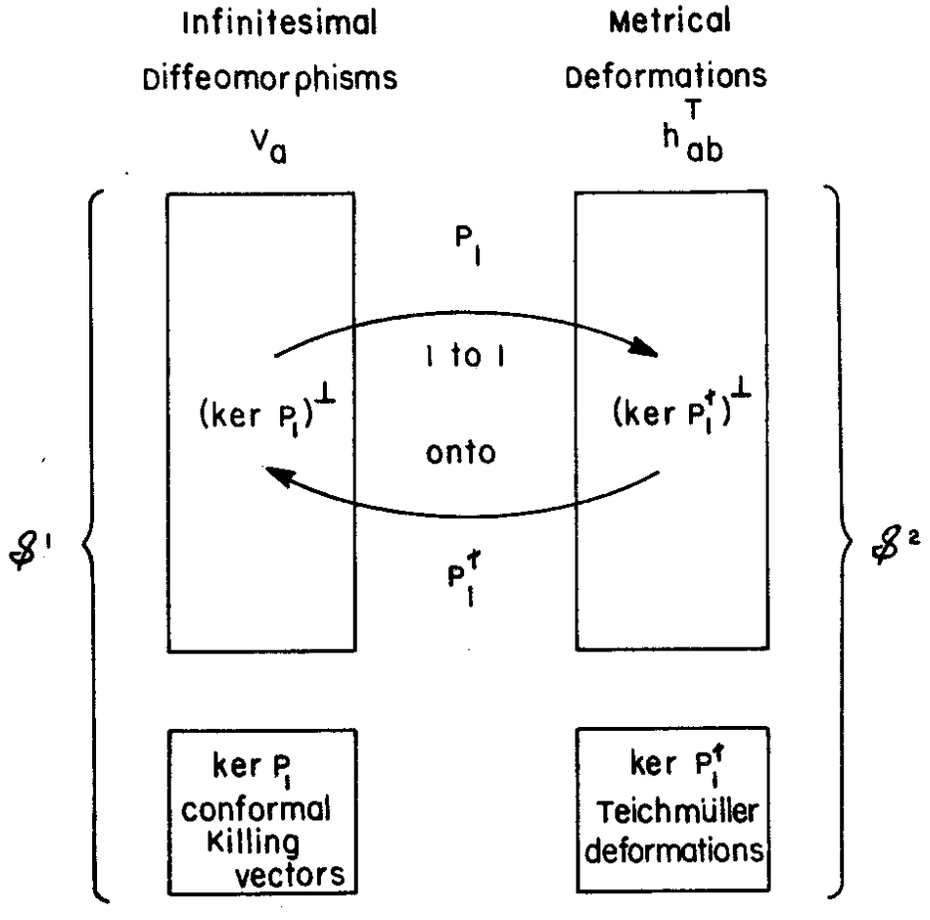

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Figure 3

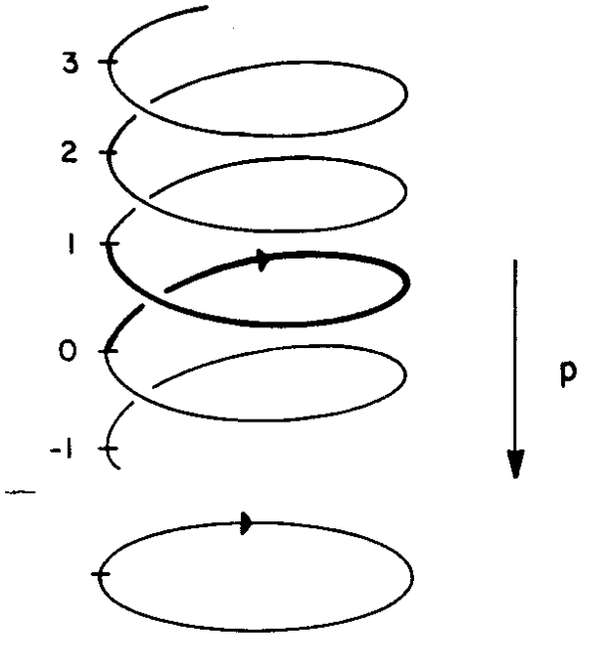

Figure 4

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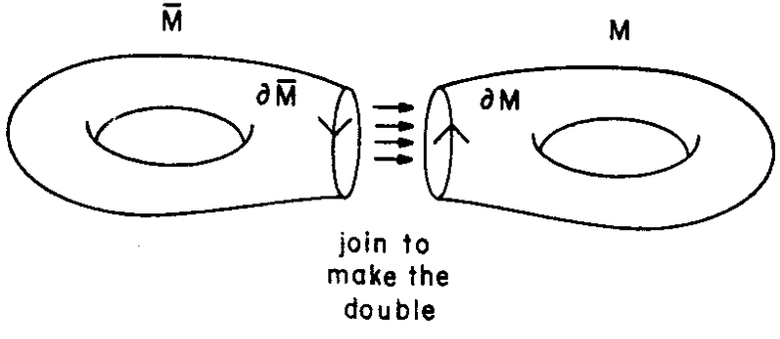
$\overline{M}$
$\partial\overline{M}$
M
∂M
join to
make the
double

Figure 5