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# Quantum Symmetry for Pedestrians

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**Abstract**

Symmetries more general than groups are possible in quantum theory. Quantum symmetries in the narrow sense are compatible with braid statistics. They are theoretically consistent much as supersymmetry is, and they could lead to degenerate multiplets of excitations with fractional spin in thin films.

## 1 Symmetry in Quantum Theory

Soon after the discovery of quantum mechanics, the symmetry under spacial rotations began to play an important role in atomic physics. Group theoretic methods were developped in order to obtain predictions for atomic spectra. For several decades it was never questioned that symmetries in quantum theory should be groups as they are in classical mechanics. The discovery of supersymmetry as a theoretically consistent form of symmetry showed for the first time that it need not be so [1]. We were able to show that much more general classes of algebras are possible symmetries in quantum theory [2]. Among them are the socalled quantum groups. But they are still not the most general possible symmetries. We emphasize that all of these algebras admit a perfectly conventional interpretation as symmetries. In this article we wish to present an elementary presentation of the main ideas. Technical details can be found in [2].

Supersymmetry can serve to explain how the need for generalization of the group structure comes about. Supersymmetry transformations transform Bosons into Fermions. But Bose fields must obey local commutation relations, whereas Fermi fields must obey anti-commutation relations. The particular structure of supersymmetry algebras arises from the requirement that the transformation law must be consistent with the change in the local commutation relations. An essential aspect of supersymmetry is thus its consistency with the statistics.

It is well known that Bose-Fermi statistics is not the most general possible statistics in two space dimensions. Braid group statistics may occur. This can also happen in one space dimension. Quantum symmetries in the narrow sense are symmetries that are consistent with braid statistics. The existence of mathematical models



shows that such quantum symmetries are consistent possibilities much as supersymmetry is.

It is generally believed that the occurence of braid statistics in nature has been experimentally established by results on the quantum Hall effect. This shows in particular that the necessary embedding of two dimensional systems in three-dimensional space need not destroy the existence of typically two-dimensional physical effects.

There exists a general spin statistics theorem which asserts that Bosons are particles with integer spin and Fermions are particles with half odd integral spin. The possible values of the spin distinguish among the irreducible projective representations of the rotation group. The projective representations of the two dimensional rotation group $SO(2)$ are distinguished by arbitary real spin. It follows that excitations with non integral and non half integral spin have to obey braid group statistics in two space dimensions. If experiments on thin films were able to establish the existence of degenerate multiplets of excitations with non integral and non half integral spin, this would be a strong indication of a quantum symmetry. Such an experimental proof would be easier if one would know how to lift the degeneracy by suitable perturbations. Unfortunately it is not known whether or not and under what conditions quantum symmetries can be broken, either explicitly or spontaneously, and what role the third dimension could play in this.

In order to speak of a symmetry, a group or algebra [1] $\mathcal{G}^*$ must be given whose elements are the symmetry transformations, and the notions of invariance and covariance must be defined. In order to say what is an invariant state, a *"trivial 1-dimensional representation"* $\epsilon$ of $\mathcal{G}^*$ must be specified. The formulation of covariance properties demands that a *tensor product of representations* of $\mathcal{G}^*$ must be defined as a representation. [2] The different stages of generalization of the notion of a symmetry are distinguished by the properties of the the tensor product of representations as follows

| *commutative* | *associative* | *truncated* | *name* |
|---|---|---|---|
| *yes* | *yes* | *no* | group |
| *no* | *yes* | *no* | quantum group |
| *no* | *no* | *no* | quasi quantum group |
| *no* | *no* | *yes* | weak quasi quantumgroup |

The tensor product of representations is called *truncated* if the dimension of the tensor product of two representations of dimension $n_1$ and $n_2$ may be strictly smaller than $n_1 n_2$. This is possible - see section 5. In a supersymmetry the tensor product

[1] In an algebra, addition and multiplication of two elements are defined, and also multiplication of elements with complex numbers. Addition is commutative and associative, and the distributive law holds.

[2] A representation $\tau$ of a group or algebra $\mathcal{G}^*$ is defined when a matrix $\tau(\xi)$ is specified for every element $\xi$ of $\mathcal{G}^*$ such that the representation law (2) below holds and also linearity(3) in case of an algebra. More generally, $\tau(\xi)$ may be linear operators in some linear space $V$.

of representations is associative,but it is not commutative but only graded commutative. We were able to demonstrate the consistency of the further generalizations by construction of a mathematical model with nonabelian braid statistics which posseses a weak quasi quantum group as a symmetry [2]. The model is a conformal field theory which has a socalled "truncated quantum group" with nonassociative tensor product as its symmetry.

Quantum groups were introduced as mathematical objects by Woronowicz [4] and Drinfeld [5]. Signs of quantum group symmetry were found in conformal field theory and in integrable models [6, 7, 8, 9, 10]. Quasi quantum groups were introduced by Drinfeld [11] under another name (quasitriangular quasi Hopf algebras), and related to orbifold models in [12]. The present authors incorporated truncation and proposed the further generalization to weak quasi quantum groups, and they gave an interpretation as symmetries of a conventional kind in quantum theory.

## 2 Symmetry algebras

For the sake of comparison with what is well known, let us consider elementary particles which transform according to some particular representation of an internal symmetry group. As an example we may take the nucleon which exists in two charge states, as a proton and as a neutron. The nucleon is said to have isospin $I = 1/2$. Isospin rotations form an internal symmetry group $SU(2)$, if we disregard electromagnetism and mass differences of the quark constituents for a moment. The isospin rotations can transform the neutron into a proton, and vice versa. Thus, 1-nucleon states transform according to a 2-dimensional representation $\tau^{1/2}$ of the symmetry group. The representation furnishes a $2 \times 2$ representation matrix $(\tau^{1/2}_{\alpha\beta}(g))$. The indices $\alpha$, $\beta$ may assume the values $\pm 1/2$ for instance which label the two charge states of the nucleon. Of course the isospin rotations act not only on 1-nucleon states. Arbitrary N-nucleon states transform in some specific way also under isospin rotations. In particular the 0-nucleon state - the vacuum - is invariant.

We conclude that a linear operator $\mathcal{U}(\xi)$ will be defined for every element $\xi$ of the symmetry group. It acts on physical states, i.e. there is a map in the Hilbert space $\mathcal{H}$ of physical states,

$$\mathcal{U}(\xi) : \mathcal{H} \mapsto \mathcal{H} . \tag{1}$$

When two symmetry transformations are carried out one after the other, then this must have the same effect as the combined transformation $\xi_2\xi_1$. Hence

$$\mathcal{U}(\xi_2)\mathcal{U}(\xi_1) = \mathcal{U}(\xi_2\xi_1) . \tag{2}$$

This is the socalled representation law. The product $\xi_2\xi_1$ is defined through the multiplication law in the group. The special group $SU(2)$ consists of unitary $2 \times 2$ matrices with determinant 1, and their product is defined by matrix multiplication. The representation $\mathcal{U}$ of the group can be extended to a representation of the group



algebra whose elements are formal linear combinations $\sum_i \alpha_i\xi_i$ of group elements $\xi_i$ with complex coefficients $\alpha_i$. This is achieved by the formula

$$\mathcal{U}(\sum_i \alpha_i\xi_i) = \sum_i \alpha_i \mathcal{U}(\xi_i) . \tag{3}$$

Linearity (3) is then automatically true for arbitrary elements $\xi_i$ of the group algebra.

If we use the symbol $e$ to indicate the trivial isospin rotation ("no rotation"), then

$$\mathcal{U}(e) = 1 . \tag{4}$$

$e$ is the unit element of the group (and of the group algebra).

The requirement of unitarity demands that

$$\mathcal{U}(\xi)^* = \mathcal{U}(\xi^*) . \tag{5}$$

This must be true for all symmetry transformations (group elements) $\xi$. If $\xi$ is an isospin rotation then $\xi^*$ is the inverse rotation.

The statement of the invariance of the vacuum $|0>$ says that

$$\mathcal{U}(\xi)|0> = |0> \epsilon(\xi) , \tag{6}$$

where $\epsilon(\xi) = 1$ for all group elements $\xi$. $\epsilon$ can be regarded as a special representation. It is called the "trivial 1-dimensional representation". Invariance of a state means that it transforms according to the trivial 1-dimensional representation.

In case of a symmetry like isospin which does not transform time, the Hamiltonian $H$ must be invariant. $H$ is an operator in the Hilbert space $\mathcal{H}$ of physical states, and its invariance asserts that

$$\mathcal{U}(\xi)H = H\mathcal{U}(\xi) \tag{7}$$

for all symmetry transformations $\xi$.

In elementary particle physics one considers nucleon fields. They are operators $\Psi^{1/2}_m(\mathbf{r}, t)$ in the Hilbert space$\mathcal{H}$ which can create a nucleon in the charge state $m$. The upper index serves to remind us that the nucleon has isospin 1/2. $\mathbf{r}$, $t$ are space and time coordinates. The above mentioned transformation law of N-nucleon states can be deduced from the invariance of the vacuum and the transformation law of nucleon fields. This transformation law tells us how the result compares if one either creates a nucleon first and performs a symmetry transformation afterwards, or the other way round. In order that both operations lead to the same result, one must use the isospin rotated nucleon field in the second case. [3] Therefore

$$\mathcal{U}(\xi)\Psi^{1/2}_m(\mathbf{r}, t) = \sum_n \Psi^{1/2}_n(\mathbf{r}, t)\tau^{1/2}_{nm}(\xi)\mathcal{U}(\xi) .$$

[3]For comparison consider rotations of ordinary space. They are described by orthogonal $3 \times 3$ matrices $\tau = (r_{ab})$. If the operators $(P_1, P_2, P_3)$ are the components of momentum $P$, for instance, then the components of the rotated momentum are $P'_a = \sum_b P_b r_{ba}$. If one considers traceless symmetric tensors of rank $J$ in place of the vector operator $P$, for instance, then the matrix $r$ is replaced by a representation matrix $\tau^J(r)$. By analogy, the isospin rotated field with isospin $I$ has the components $\sum_n \Psi^I_n(\mathbf{r}, t)\tau^I_{nm}(\xi)$. Since isospin rotations are internal symmetries, they do not affect $\mathbf{r}, t$.

This law can be rewritten in a more compact form with the help of the tensor product of the representations $\tau^{1/2}$ and $\mathcal{U}$. Suppose we introduce a basis in the Hilbert space $\mathcal{H}$ of physical states. Then the operator $\mathcal{U}(\xi)$ becomes a matrix $(\mathcal{U}_{\alpha\beta}(\xi))$. If the bases in the representation spaces for two representations $\tau$ and $\mathcal{U}$ are labelled by indices $m$ and $\alpha$, then a basis in the representation space of the tensor product is labelled by index pairs $(m\alpha)$. The rows and columns of the corresponding representation matrices are then also labelled by such index pairs. For groups, the tensor product of representations is defined by the formula

$$(\tau \bigotimes \mathcal{U})_{n\alpha,m\beta}(\xi) = \tau_{nm}(\xi)\mathcal{U}_{\alpha\beta}(\xi) \ .$$

If we omit the indices $\alpha,\beta$ for the basis in the Hilbert space $\mathcal{H}$ again, then we can rewrite the transformation law of the nucleon field in the following equivalent form, with $I = 1/2$.

$$\mathcal{U}(\xi)\Psi^I_m(\mathbf{r},t) = \sum_n \Psi^I_n(\mathbf{r},t)(\tau^I \bigotimes \mathcal{U})_{nm}(\xi) \ . \tag{8}$$

With this, all the fundamental equations which characterize a symmetry have been written in a form which remains true for the most general symmetries.

- Existence and unitarity of a representation $\mathcal{U}$ of the group or symmetry algebra in the Hilbert space $\mathcal{H}$ of physical states, eqs.(1), (2), (4), (5)
- Invariance of the Hamiltonian $H$ and of its ground state, eqs.(6), (7).
- Covariant transformation law of the field operators which create particles or excitations, eq.(8)

Only the properties of the tensor product change. It is better to regard the more general symmetry algebras as generalizations of the group algebra (or of the universal envelopping algebra of the Lie algebra of the group, which is nearly the same), rather than as a generalization of the group itself. The definitions of the trivial 1-dimensional representation, the *-operation, and of the tensor product of representations generalize in a natural way from the group to the group algebra in such a way that all the numbered equations remain valid.

The above mentioned tensor product for group representations has the property that it is commutative and associative:

$$(\tau^I \bigotimes \tau^J)_{ij,lm}(\xi) = (\tau^J \bigotimes \tau^I)_{ji,ml}(\xi) \qquad \text{(commut.)}$$
$$(\tau^I \bigotimes (\tau^J \bigotimes \tau^K))_{ijk,lmn}(\xi) = ((\tau^I \bigotimes \tau^J) \bigotimes \tau^K)_{ijk,lmn}(\xi) \ \text{(assoz.)}$$

For reasons of compatibility with statistics, which will be discussed later on, one does not give up commutativity and associativity completely, but they are only weakened. One demands that

$$(\tau^I \bigotimes \tau^J)(\xi) \ \cong \ (\tau^J \bigotimes \tau^J)'(\xi), \tag{9}$$

$$(\tau^I \bigotimes (\tau^J \bigotimes \tau^K))(\xi) \ \cong \ ((\tau^I \bigotimes \tau^J) \bigotimes \tau^K)(\xi) \ . \tag{10}$$



$\cong$ means that the matrices on both sides are related by a $\xi$-independent similarity transformation, [4] so that the matrices on both sides will still define equivalent representations. $'$ indicates interchange of indices as in eq.(commut.).

The really essential generalization resides not so much in the fact that one does not deal with groups anymore, but in the different definition and the different properties of the tensor product of representations. Giving up associativity is as big a step as giving up commutativity. It is possibly more surprising, because the appearance of noncommutativity in quantum theory in cases where one has commutativity classically, is familiar.

We remark, finally, that the tensor product of representations can be regarded as a genuine product in an algebra $\mathcal{G}$ which is dual to $\mathcal{G}^*$. The algebra $\mathcal{G}$ is a generalization of the (commutative and associative) algebra of "good" functions on a group.

# 3 Compatibility with the statistics

The Bose or Fermi statistics of particles which may be created by field operators may be enforced through the postulate of local commutation relations or anticommutation relations of the field operators.

$$\Psi^I_i(\mathbf{r},t)\Psi^J_j(\mathbf{r}',t) = \pm\Psi^J_j(\mathbf{r}',t)\Psi^I_i(\mathbf{r},t) \text{ when } \mathbf{r} \neq \mathbf{r}' \ , \tag{11}$$

with + for Bose fields and − for Fermi fields. Suppose initially that the field multiplet consists either only of Bose fields or only of Fermi fields, so that the sign $\pm$ does not depend on the indices. (This excludes supersymmetry.)

Let us furthermore assume that the tensor product of representations of the symmetry is associative.

It will then follow from the transformation law (8) for $\Psi^I$ and $\Psi^J$ that the product $\Psi^I_i(\mathbf{r},t)\Psi^J_j(\mathbf{r}',t)$ will also transform covariantly. It will transform according to the representation $(\tau^I \bigotimes \tau^J)$. We see from this that the left hand side of the commutation relations (11) transforms according to the representation $(\tau^I \bigotimes \tau^J)$, whereas the right hand side transforms according to the representation $(\tau^J \bigotimes \tau^I)$. Consistency demands that both sides transform in the same way. This will be the case if the tensor product is commutative (and associative), but not in general otherwise. We conclude that Bose/Fermi statistics is consistent with symmetry groups and -algebras whose representations possess an associative and commutative tensor product.

Let us now give up commutativity, but retain associativity at first. Eq. (9) asserts that there must exist a matrix (or similarity transformation) $\mathcal{R}^{IJ}$ with the

[4] Two matrices $A$ and $B$ are related by a similarity transformation if there exists an invertible matrix $S$ such that $B = SAS^{-1}$.

property that

$$\sum_{km} \mathcal{R}^{IJ}_{ln,km}(\tau^I \bigotimes \tau^J)_{km,ij}(\xi) = \sum_{km} (\tau^J \bigotimes \tau^I)_{nl,mk}(\xi) \mathcal{R}^{IJ}_{km,ij} \ . \tag{12}$$

J. Fröhlich proposed [13] that particles or excitations which obey braid statistics should be described by field operators which obey local braid relations of the following form

$$\Psi^I_i(\mathbf{r},t)\Psi^J_j(\mathbf{r}',t) = \omega^{IJ} \sum_{km} \Psi^J_m(\mathbf{r}',t)\Psi^I_k(\mathbf{r},t)\hat{\mathcal{R}}^{IJ}_{km,ij} \quad \text{when } \mathbf{r} > \mathbf{r}' \ , \tag{13}$$

and similarly if $\mathbf{r} < \mathbf{r}'$. In one space dimensions, $\mathbf{r}, \mathbf{r}'$ are numbers, and the condition $\mathbf{r}' > \mathbf{r}$ has its usual meaning. In two space dimensions particles with braid statistics are described by wave functions with cuts, and the definition of $>$ makes reference to these cuts. We cannot enter into further discussions of this point here. According to Fröhlich, $\hat{\mathcal{R}}^{IJ}$ should be numerical matrices. A phase factor $\omega^{IJ}$ has been extracted from the matrix $\hat{R}^{IJ}$.

It is readily seen from the above mentioned transformation law of the products of two field operators and from eq.(12) that both sides of the local braid relations (13) transform indeed in the same way, provided the matrix $\hat{\mathcal{R}}^{IJ}$ in it equals the matrix $\mathcal{R}^{IJ}$ in eq.(12). The necessary interchange of the factors in $(\tau^J \bigotimes \tau^I)$ is achieved by the commutation of this matrix with $\mathcal{R}^{IJ}$.

The phase factor $\omega^{IJ}$ remains undetermined. It is a generalization of the sign $\pm$ in the case of Bose/Fermi statistics. It need not necessarily assume values $\pm 1$ only. Because of this, one speaks of "fractional statistics" [14]. The matrices $\mathcal{R}^{IJ}$ are supposed to be furnished as part of the statement of the symmetry. One speaks of nonabelian braid statistics if matrices $\mathcal{R}^{IJ}$ appear which are not multiples of the identity. 1-component fields which obey abelian braid statistics are known as anyon fields [14].

When associativity of the tensor product is also given up, then ordinary products of field operators will no longer transform covariantly. In order to find local braid relations nevertheless which are consistent with the symmetry one introduces as an auxiliary construct an additional new product of field operators $(\Psi^I \times \Psi^J)_{ij}$ which transforms covariantly according to the representation $\tau^I \bigotimes \tau^J$. This is possible, assuming the quasiassociativity relation (10). This permits to write down local braid relations again which are manifestly covariant, and therefore consistent with the symmetry. (They have the same form as before, except that $\times$ substitutes for the ordinary product.) Expressing $(\Psi^I \times \Psi^J)_{ij}$ in terms of the ordinary product, these local braid relations can be written in the form (13) again. But the coefficients $\hat{\mathcal{R}}^{IJ}_{ij,lm}$ are no longer equal to the numbers $\mathcal{R}^{IJ}_{ij,lm}$ in eq.(12), but they become operators in the Hilbert space $\mathcal{H}$. In fact they are representation operators $\mathcal{U}(...)$ of certain elements of the symmetry algebra. We shall refrain from reproducing the explicit formula for them here. It is buildt out of the numerical coefficients $\mathcal{R}^{IJ}_{ij,kl}$ from eq.(12) and



operators in Hilbert space which implement the quasi-associativity condition (10), with $\mathcal{U}$ in place of the arbitrary representation $\tau^K$ in it.

No mathematical model is known up to now in which nonabelian local braid relations with *numerical* coefficients $\mathcal{R}^{IJ}$ hold. But a mathematical model with a weak quasi quantum group as its symmetry and with local braid relations with *operator* coefficients $\hat{\mathcal{R}}^{IJ}$ was constructed in [3]. This proves the consistency of local braid relations (13).

The internal consistency of the local braid relations, and also that of eq.(12) demand that the numerical matrices $\mathcal{R}$ in eq.(12) obey the Yang Baxter equations (in the associative case) or quasi Yang Baxter equations (in the non associative case). This ensures (in either case) that a representation of the braid group can be constructed from the matrices $\hat{\mathcal{R}}^{IJ}$.

## 4 Braid group statistics

In the three remaining sections we give some additional details for interested readers.

Braid group statistics may occur in 1 or 2 space dimensions. Let us consider $N$ particles in 2-dimensional space and let us assume that their positions $\mathbf{r}_i$ must all be distinct. Let us now consider continuous shifts of the positions of these particles in space. We require that the condition of non coinciding positions of these particles remains always satisfied, and that a permutation of the original positions of the particles has been achieved at the end of the shift. Each of the curves in space along which one of the particles was moved may be parametrized by real numbers $\tau = 0...1$. There will be classes of shifts which are essentially different. The division into classes is such that the parametrized curves which represent a shift in one class cannot be continuously deformed into the curves associated with another shift if and only if this shift belongs to a different class. These classes are represented by braids on $N$ threads, see figure 1. The braids live in 3 dimensions (2 space dimensions and the parameter $\tau$), but we regard them as projected into a plane, as in the figure 1. The interruption of one of the lines at a crossing indicates which thread crosses behind the other.

The following short description of the braid group follows Artin [15]. The braids are composed of segments $\sigma_i$ and $\sigma_i^{-1}$ which have only one crossing between neighbouring threads $i$ and $i+1$, see figure 2a. A product of braids is defined by juxtaposition as in figure 2b. (As an exercise the reader may check that braid of figure 2b is described in terms of $\sigma_i, \sigma_i^{-1}$ as $\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$). Braids which may be deformed into each other, holding ends fixed, are not distinguished. All possible deformations may be thought to be composed from Artins elementary deformations shown in figure 3.

The equalities between braids on $N$ threads which follow from the elementary deformations read as follows

$$\sigma_i\sigma_k = \sigma_k\sigma_i \quad \text{falls } |k-i| \geq 2 \quad , \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \ , \tag{14}$$

$$\sigma_i \sigma_i^{-1} = \iota = \sigma_i^{-1} \sigma_i \ . \tag{15}$$

$(i, k = 1...N)$. These are the celebrated Artin relations [15].

A braid is represented by a word $b$ on letters $\sigma_i$ and $\sigma_i^{-1}$. Words which can be transformed into each other with the help of the Artin relations represent the same braid. The trivial braid $\iota$ made of 0 segments (parallel threads) acts like a unit element. The relation $\iota$ implies the existence of an inverse $b^{-1}$ to every braid $b$. In particular, $\sigma_i^{-1}$ is the inverse of $\sigma_i$. Figure 2b shows a braid of the form $bb^{-1}$. It is easily verified that this braid can be continuously deformed into the trivial braid.

Since multiplication, a unit element, and an inverse of every element are defined, the braids on $N$ threads form a group $B_N$. The permutation group on $N$ objects is a factor group. A representation $\mu$ of the braid group is at the same time a representation of the permutation group if and only if $\mu(\sigma_i) = \mu(\sigma_i^{-1})$ for all $i$. Conversely, every representation of the braid group defines a representation of the braid group with this property.

If $M < N$ then the group $B_M$ can be embedded into $B_N$. In graphical notation this is done by adding threads $M+1...N$ without crossings. In this way the group $B$ of braids on an arbitrary number of threads is defined. The coefficients $\hat{\mathcal{R}}^{IJ}$ in the local braid relations (13) can be used to construct a representation of this group $B$.

## 5 Tensor products of representations

If $\tau$ is a representation of an algebra $\mathcal{G}^*$, then the matrix $\tau(\xi)$ is defined for every element $\xi$ of $\mathcal{G}^*$. If we want to define the tensor product $(\tau^1 \bigotimes \tau^2)$ of two representations $\tau^1$ and $\tau^2$ then we must define $(\tau^1 \bigotimes \tau^2)(\xi)$ for every $\xi$. In order to do so, we must give arguments to $\tau^1$ and to $\tau^2$. Thus we should associate a pair $(\xi^1, \xi^2)$ of elements of the algebra with every element $\xi$. For groups this association reads

$$\xi \mapsto (\xi, \xi) \ ,$$

and we obtain the tensor product mentioned in section 2,

$$(\tau^1 \bigotimes \tau^2)_{ik,jl}(\xi) = \tau^1_{ij}(\xi)\tau^2_{kl}(\xi) \ .$$

Because of linearity of the representation of an algebra the expression

$$\tau^1_{ij}(\xi^1)\tau^2_{kl}(\xi^2)$$

remains unchanged when the pair

$$(\alpha\xi^1, \alpha^{-1}\xi^2) \tag{16}$$

is substituted for the pair $(\xi^1, \xi^2)$. $\alpha$ can be an arbitrary complex number. If one decides not to distinguish between these pairs then one writes $\xi^1 \otimes \xi^2$ in place of $(\xi^1, \xi^2)$. Formal linear combinations $\sum_a \alpha_a(\xi^1_a \otimes \xi^2_a)$ of such elements with complex coefficients $\alpha_a$ form an algebra $\mathcal{G}^* \otimes \mathcal{G}^*$. They can be added, multiplied with complex numbers, and with each other, using the identities

$$\begin{aligned} \alpha(\xi \otimes \eta) = (\alpha\xi \otimes \eta) &= (\xi \otimes \alpha\eta) \ , \\ (\xi^1 \otimes \xi^2)(\eta^1 \otimes \eta^2) &= (\xi^1\eta^1 \otimes \xi^2\eta^2) \ . \end{aligned} \tag{17}$$

Suppose that a map $\Delta$ from $\mathcal{G}^*$ to $\mathcal{G}^* \otimes \mathcal{G}^*$ is given to us. This maps each element $\xi$ of $\mathcal{G}^*$ into a complex linear combination

$$\Delta(\xi) = \sum_a \alpha_a(\xi^1_a \otimes \xi^2_a) \tag{18}$$

of equivalence classes of pairs as described above. Then we may try to define the tensor product of two arbitrary representations $\tau^1$ and $\tau^2$ by the following formula

$$(\tau^1 \bigotimes \tau^2)_{ik,jl}(\xi) = \sum_a \alpha_a \tau^1_{ij}(\xi^1_a)\tau^2_{kl}(\xi^2_a) \ . \tag{19}$$

The expression of the right hand side is well defined. One can verify that the representation law is fulfilled provided $\Delta$ is an algebra homomorphism, that is

$$\Delta(\xi)\Delta(\eta) = \Delta(\xi\eta)$$

for all $\xi, \eta$ in the algebra $\mathcal{G}^*$. The product on the left hand side is defined by eq.(17). It is not required that $\Delta(e) = e \otimes e$ (cp. below), but the equality $\Delta(e)\Delta(\xi) = \Delta(\xi)$ must hold for all $\xi$.

*Conclusion:* Tensor products of representations of an algebra are determined by a homomorphism

$$\Delta : \mathcal{G}^* \mapsto \mathcal{G}^* \otimes \mathcal{G}^* \ .$$

The tensor product is *commutative* if the expression (19) remains invariant when $\xi^1$ and $\xi^2$ are interchanged.

In order to define a threefold tensor product one must map elements $\xi \in \mathcal{G}^*$ into linear combinations of triples $\xi^1 \otimes \xi^2 \otimes \xi^3$. In order to do so one should first act with $\Delta$ on $\xi$. This yields pairs. Then on can apply $\Delta$ again, either on the first element of the pair or on the second. The tensor product $\bigotimes$ of representations is *associative* if one obtains the same result in both cases.

If $\tau^1(\xi)$ and $\tau^2(\xi)$ are $n_1 \times n_1$ and $n_2 \times n_2$ matrices, then eq.(19) defines $(\tau^1 \bigotimes \tau^2)(\xi)$ as a $n_1 n_2 \times n_1 n_2$ matrix. It acts as a linear transformation in a $n_1 n_2$ dimensional space which is named $V^1 \otimes V^2$. The true representation space of the tensor product representation $\tau^1 \bigotimes \tau^2$ may be a proper subspace of this $n_1 n_2$-dimensional space, however. In this case one speaks of *truncation*. Truncation is introduced by admitting

$$\Delta(e) \neq e \otimes e$$

(e=1-element). In this case, $(\tau^1 \bigotimes \tau^2)(e)$ need not be the unit matrix. Instead it may be a projector which annihilates a nontrivial subspace of $V^1 \otimes V^2$. This subspace will then automatically be annihilated by $(\tau^1 \bigotimes \tau^2)(\xi)$ for all $\xi$. The true representation space of the tensor product representation is the orthogonal complement of the annihilated subspace. Before we come to examples of proper quantum symmetries, we would like to show what the coproduct $\Delta$ looks like in the case of traditional symmetries

1. Group elements $\xi$:

$$\Delta(\xi) = \xi \otimes \xi$$

2. Generators $\xi$ of a Lie algebra:

$$\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi$$

3. Fermionic generators $\xi$ of a SUSY-algebra:

$$\Delta(\xi) = \xi \otimes 1 + (-)^F \otimes \xi$$

Herein $(-)^F$ is the fermionic parity. It anticommutes with all fermionic generators $\xi$: $\mathcal{U}((-)^F) = (-1)^N$ when applied to a state with $N$ fermions. It is seen that commutativity does not hold in this case.

## 6 Example of a quantum group

The best known examples of quantum groups are the algebras $U_q(sl_2)$, where $q$ is a complex number. A suitable $*$-operation exists if either $q$ is real, or $|q| = 1$. If $q$ is a root of unity then the algebra $U_q(sl_2)$ is sick - the tensor products of its representations are not fully reducible in general. But one obtains an acceptable candidate $\mathcal{G}^*$ of a symmetry in this case by a process of truncation. This process of truncation weakens at the same time associativity to quasi associativity (10). We will not enter into a detailed discussion of this here, see ref.[2]. The algebra $U_q(sl_2)$ is generated by generators $q^{\pm H/2}$, $S_+$ and $S_-$. The reader is advised to better read $q^{H/2}$ not as "$q$ to the power..." but as a name like "Bob". A general element of $U_q(sl_2)$ is a sum of products of generators with complex coefficients. Expressions of this kind are not distinguished if they can be transformed one into the other by use of the following relations.

$$\begin{aligned} q^{H/2}q^{-H/2} &= q^{-H/2}q^{H/2} = e , \\ q^{H/2}S_\pm &= q^{\pm 1/2}S_\pm q^{H/2} , \\ [S_+, S_-] &= \frac{q^H - q^{-H}}{q^{1/2} - q^{-1/2}} . \end{aligned} \tag{20}$$

Equations with $\pm$ are understood to be valid when one either takes the upper sign everywhere, or the lower sign.



The tensor product of representations is defined with the help of the following map $\Delta$,

$$\begin{aligned} \Delta(q^{\pm H/2}) &= q^{\pm H/2} \otimes q^{\pm H/2} \\ \Delta(S_\pm) &= S_\pm \otimes q^{H/2} + q^{-H/2} \otimes S_\pm . \end{aligned} \tag{21}$$

One notes that the condition for commutativity of the tensor product of representations is not fulfilled. But the condition for associativity is fulfilled. It is straightforward to verify this.

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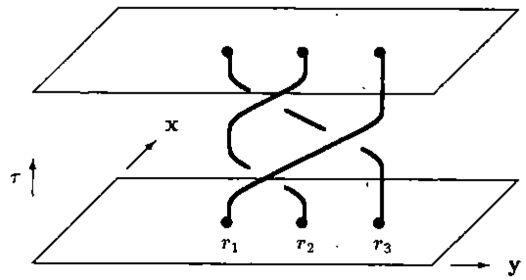



figure 1

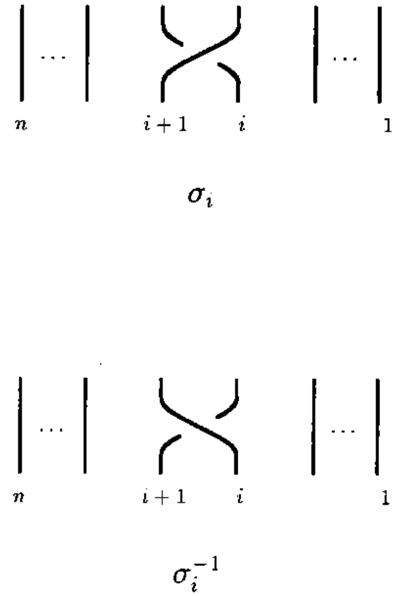



figure 2a

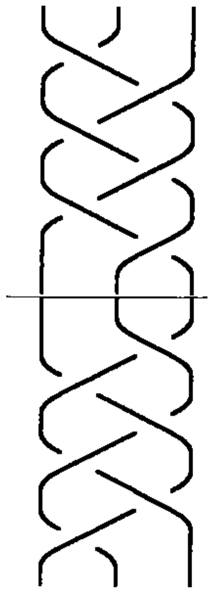
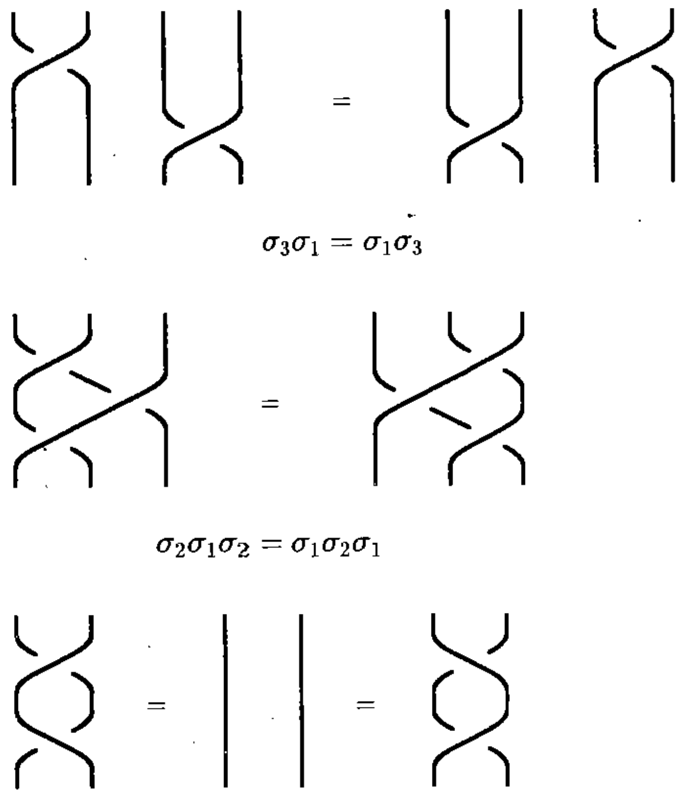


figure 2b



$\sigma_3\sigma_1 = \sigma_1\sigma_3$

$\sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1$

$\sigma_1\sigma_1^{-1} = 1 = \sigma_1^{-1}\sigma_1$

**figure 3**