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A New Look at Goldstone's Theorem

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Abstract. The appearance of spontaneously broken symmetries and its bearing on the physical mass spectrum are analyzed in the algebraic setting of local quantum field theory. Within this setting, a generalization of Goldstone's Theorem is established which does not rely on the existence of conserved currents. Continuous symmetries not satisfying the premises of the theorem can be spontaneously broken even in the presence of a mass gap.

1. Introduction

The present understanding of the phenomenon of spontaneous symmetry breaking and its relation to the physical mass spectrum is limited to symmetries which are generated by conserved currents and act on a given set of unobservable fields (cf. the review articles [1,2] and references quoted there). This setting covers the symmetries of Noether type, familiar from Lagrangian field theory. The possible manifestations of symmetries are, however, not yet

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fully clarified and the conventional setting for their study may well be too restrictive. We recall in this context the advent of supersymmetries or, more recently, of quantum groups. It seems therefore worthwhile reconsidering the phenomenon starting from more general physical principles.

In the present article we take a fresh look at the problem of spontaneously broken (global) gauge symmetries in a framework based solely on local observables. Since observables are by definition invariant under gauge transformations we face the problem that in this investigation the relevant structural features, fields and gauge symmetries, are not visible a priori nor is it even obvious that they exist. We therefore first have to discuss how these structures can be recovered from the observables.

As far as the unbroken part of the gauge symmetries is concerned this problem is well understood for theories in $s > 1$ spatial dimensions. Using the analysis of superselection structure [3], it has proved possible to show [4] that in any theory, there is a canonical field algebra $\mathfrak{F}$ with normal Bose-Fermi commutation relations associated with the given algebra $\mathfrak{A}$ of observables. The field algebra $\mathfrak{F}$ is acted on by a compact gauge group G of the first kind. This group G describes those gauge symmetries which are not spontaneously broken and it determines the superselection structure of the theory in the sense that the superselection sectors describing localizable charges are in one-to-one correspondence with the irreducible representations of G.

We are here concerned with the case where G is possibly only part of some larger gauge group Γ involving spontaneously broken symmetries. Local algebras of observables which are not maximal in the vacuum sector so that Haag duality fails are evidence in favour of spontaneously broken symmetries [5]. The local operators which are then missing are, roughly speaking, fields which transform non-trivially under the gauge group Γ but which do not lead out of the vacuum sector. As will be discussed below, these fields generate an intermediate algebra $\mathfrak{A} \subset \mathfrak{A}^d \subset \mathfrak{F}$.

These general facts suggest defining Γ to be the group of all automorphisms of $\mathfrak{F}$ which leave the observable algebra $\mathfrak{A}$ pointwise invariant [4: Sect. 7]. We show in Sect. 2 that the elements of Γ are automatically local automorphisms of $\mathfrak{F}$, i.e. they do not change the localization properties of fields. This result justifies the interpretation of Γ as a gauge group. Under an additional condition, expressing within our setting the assumption that the energy-momentum density is an observable, we can also show that the elements of Γ commute with translations, i.e. they are internal symmetries.

The gauge symmetry $\gamma \in \Gamma$ is said to be spontaneously broken if there is no unitary operator implementing it in the (irreducible) vacuum representation of $\mathfrak{F}$ or, what amounts to the same thing, if the vacuum state is not stable under the action of γ. However, although the gauge group Γ is not unitarily implemented in this case, there always exist "local" unitary representations of Γ which implement the gauge transformations on any given *local* field algebra. This result is to be expected if one tacitly assumes that Γ arises from some algebra of local currents. But it must be stressed that one-parameter symmetry groups are not necessarily associated with conserved currents, as is made clear by an example in Sect. 4. The existence of local implementations of Γ, however, provides an adequate substitute when discussing Goldstone's Theorem, cf. [6].

Goldstone's Theorem is our main objective and will be treated in Sect. 3. As is clear from the preceding remarks we can not from the outset assume the existence of a conserved current affiliated with a one-parameter group of symmetries. Instead we consider the corresponding generating derivation δ. It is well known that the symmetry is unbroken if and only if $\omega_0 \circ \delta = 0$, where ω_0 is the vacuum state on $\mathfrak{F}$ [7]. Now the fact that such symmetries can always be locally implemented implies certain a priori bounds on the expectation values $\omega_0\delta(F)$, where F is taken from the local field algebra of a given double cone $\mathcal{O}$ with radius R, cf. (3.8). The crucial input in our analysis is the way the a priori bounds increase with R. To give an example: if there is a conserved current generating the symmetry one gets a priori bounds growing as $R^{(s-1)/2}$, where s is the number of space dimensions.

In the general case where there may be no such current, we show that

these bounds must grow at least exponentially with R if our symmetry is to be spontaneously broken in a theory with a mass gap. An example of such a theory is presented in Sect. 4.The broken symmetry in question does not fit into the familiar setting of gauge theory and the phenomenon we are discussing is distinct from the Higgs mechanism.

If the bounds grow like R^k with $k > (s-1)/2$, then the symmetry can be spontaneously broken only in theories without a mass gap. This is even possible when there are no particles of mass zero as we illustrate by another example. When the bounds grow like $R^{(s-1)/2}$ spontaneous symmetry breaking is only possible in the presence of particles of mass zero. Hence it is only in this particular case that the existence of Goldstone Bosons can be established. Finally, when the bounds grow like R^k with $k < (s-1)/2$, or $k = 0$ if $s = 1$, the symmetry cannot be spontaneously broken, whatever the mass spectrum. The latter result may be regarded as a generalization of a theorem by Coleman [8].

After this outline of our results, we discuss the assumptions that will be used in the course of this paper and this will at the same time serve to establish our notation and conventions. We suppose that the net $\mathfrak{A}$ of local observables is given as an inclusion preserving map

$$\mathcal{O} \in \mathcal{K} \rightarrow \mathfrak{A}(\mathcal{O}) \tag{1.1}$$

from the set $\mathcal{K}$ of open double cones in Minkowski space to von Neumann algebras acting on the separable vacuum Hilbert space $\mathcal{H}_0$. The C^*-algebra generated by the range of (1.1) will also be denoted by $\mathfrak{A}$ and will be supposed to be irreducible. The net $\mathfrak{A}$ is supposed to be local i.e.

$$\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)' \,, \mathcal{O}_1 \subset \mathcal{O}_2' \,, \tag{1.2}$$

where $\mathcal{O}'$ denotes the spacelike complement of $\mathcal{O}$ and $\mathcal{R}'$ denotes the commutant of a set $\mathcal{R}$ of bounded operators on Hilbert space. Denoting the C^*-subalgebra of $\mathfrak{A}$ generated by all $\mathfrak{A}(\mathcal{O}_1)$ with $\mathcal{O}_1 \subset \mathcal{O}'$ by $\mathfrak{A}(\mathcal{O}')$ as usual, we assume that the dual net $\mathfrak{A}^d$, defined by

$$\mathcal{O} \in \mathcal{K} \mapsto \mathfrak{A}^d(\mathcal{O}) := \mathfrak{A}(\mathcal{O}')' \tag{1.3}$$

is again a local net. This is well known to be equivalent to essential duality

$$\mathfrak{A}^d = \mathfrak{A}^{dd} \tag{1.4}$$

and to imply that

$$\mathfrak{A}^d(\mathcal{O}')^- = \mathfrak{A}(\mathcal{O}')^- , \quad \mathcal{O} \in \mathcal{K} . \tag{1.5}$$

On the evidence of the Bisognano and Wichmann Theorem essential duality is expected to hold quite generally. On the other hand, the stronger property of duality, $\mathfrak{A} = \mathfrak{A}^d$, will not be assumed here since it implies that there are no spontaneously broken gauge symmetries [5].

We assume translation covariance and the spectrum condition in the vacuum sector, that is the continuous unitary representation U_0 of the translation group which induces the action α of the spacetime translations on the net $\mathfrak{A}$ leaves the vacuum vector Ω invariant and has spectrum in the forward light cone V_+. We also assume that Ω has the Reeh-Schlieder property, i.e. Ω is cyclic for each $\mathfrak{A}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$.

Given this setting, the results of [4] can be applied: there is a field net $\mathfrak{F}$ of von Neumann algebras $\mathfrak{F}(\mathcal{O})$ over $\mathcal{K}$ generated by operators with normal Bose and Fermi commutation relations and acting on a Hilbert space $\mathcal{H}$ containing $\mathcal{H}_0$ as a subspace. The representation U_0 can be extended to a continuous unitary representation U of translations on $\mathcal{H}$ also satisfying the spectrum condition and inducing the spacetime translations α on the net $\mathfrak{F}$. Finally, there is a compact group G of unitaries on $\mathcal{H}$, commuting with the translations U and inducing local automorphisms of $\mathfrak{F}$. The vacuum Hilbert space $\mathcal{H}_0$ is precisely the subspace of G-invariant vectors in $\mathcal{H}$, and the fixed-point net of $\mathfrak{F}$ under the action of G is uniquely determined by its restriction to $\mathcal{H}_0$ where it coincides with the net $\mathfrak{A}^d$. Thus $\mathfrak{A}$ and $\mathfrak{A}^d$ can and will be regarded as subnets of $\mathfrak{F}$, i.e. $\mathfrak{A}(\mathcal{O}) \subseteq \mathfrak{A}^d(\mathcal{O}) \subseteq \mathfrak{F}(\mathcal{O})$ for $\mathcal{O} \in \mathcal{K}$. This structure, described in more detail in [4], is the starting point for this paper.

Before we enter into the analysis we should like to point out the limitations of our framework. We deal here with a field net $\mathfrak{F}$ over double cones and

thus treat only localized charges. Even in theories with a mass gap, there might in principle be topological charges present, too, and these can only be localized in spacelike cones [9]. In this case one can define a field net over these cones [4: Sect. 5] and spontaneously broken symmetries should be defined relative to this net. Such a framework might be appropriate in a general discussion of the Higgs mechanism [10] but it lies beyond the scope of this paper. In the case of theories without a mass gap the general structure of a field net describing all the relevant superselection sectors is not known[1]. Thus at present one actually lacks a suitable starting point for a completely general investigation of spontaneously broken symmetries.

2. Gauge Groups and their Local Implementations

We have defined the full gauge group Γ to be the group of all automorphisms of $\mathfrak{F}$ leaving the subnet $\mathfrak{A}$ pointwise fixed and we will think of the group G of unbroken gauge symmetries as being the subgroup of automorphisms of $\mathfrak{F}$ which leave $\mathfrak{A}^d$ pointwise fixed. The main result of this section says that any element $\gamma \in \Gamma$ acts locally on the field net, i.e. $\gamma(\mathfrak{F}(\mathcal{O})) = \mathfrak{F}(\mathcal{O})$ for each double cone $\mathcal{O}$. This fact will enable us to show that local unitary implementations of Γ exist.

The proof of these assertions is based on the following crucial lemma. For the convenience of the reader we recall in the statement of the lemma the relevant properties of the net $\mathfrak{F}$ established in [4].

LEMMA 2.1. *Let $\mathfrak{F}$ be a field net acted on by a compact group G of local automorphisms whose fixed-point net $\mathfrak{A}^d$ has a faithful vacuum representation, satisfies duality in the vacuum sector and is such that*

$$\mathfrak{A}^d(\mathcal{O}_1) \subset \mathfrak{F}(\mathcal{O}_2)', \quad \mathcal{O}_1 \subset \mathcal{O}_2' . \tag{2.1}$$

[1] The field net constructed in [4] would not necessarily carry all possible charges: states in a sector with non-zero electric charge, for instance, would never be localizable in double cones and will be localizable in some but not all spacelike cones [11].

Suppose further that, the action of G on each $\mathfrak{F}(\mathcal{O})$ has full monoidal spectrum, i.e. each equivalence class of irreducible continuous unitary representations of G is realized on some G-invariant Hilbert space of support I in $\mathfrak{F}(\mathcal{O})$[2] *then*

$$\mathfrak{A}(\mathcal{O}')' \cap \mathfrak{F} = \mathfrak{F}(\mathcal{O}) .$$

Proof. By (2.1) we have

$$\mathfrak{F}(\mathcal{O}) \subset \mathfrak{A}^d(\mathcal{O}')' \cap \mathfrak{F} . \tag{2.2}$$

The right hand side of (2.2) is stable under the action of G and is hence generated by its irreducible tensors. If $\chi_1, \chi_2, \ldots, \chi_n$ is such a tensor transforming like an orthonormal basis $\psi_1, \psi_2, \ldots, \psi_n$ of a G-invariant Hilbert space $H \subset \mathfrak{F}(\mathcal{O})$, then we have $\chi_i = B\psi_i$, $i = 1, 2, \ldots, n$, where $B := \sum_i \chi_i \psi_i^*$. Now B is an element of

$$\mathfrak{A}^d(\mathcal{O}')' \cap \mathfrak{F} \cap G' = \mathfrak{A}^d(\mathcal{O}')' \cap \mathfrak{A}^d = \mathfrak{A}^d(\mathcal{O}) ,$$

where the inclusion (2.2) and duality have been used. Thus $\chi_i \in \mathfrak{F}(\mathcal{O})$ and since all irreducible representations are realized on Hilbert spaces in $\mathfrak{F}(\mathcal{O})$, the opposite inclusion holds too. But in view of relation (1.5) and the Reeh-Schlieder property of the vacuum Ω for $\mathfrak{F}(\mathcal{O})$ we have

$$\mathfrak{A}^d(\mathcal{O}')^- = \mathfrak{A}(\mathcal{O}')^- , \quad \mathcal{O} \in \mathcal{K} \tag{2.3}$$

on $\mathcal{H}$, the vacuum Hilbert space of $\mathfrak{F}$, and consequently $\mathfrak{A}(\mathcal{O}')' \cap \mathfrak{F} = \mathfrak{A}^d(\mathcal{O}')' \cap \mathfrak{F} = \mathfrak{F}(\mathcal{O})$, as claimed. □

We will establish a somewhat stronger result at the end of this section by assuming that the energy-momentum density is an observable. We now collect together the main properties of our gauge groups in the form of a proposition.

[2]For an account of the properties of Hilbert spaces inside operator algebras the reader may consult [12].

Proposition 2.2.
i) *Each $\gamma \in \Gamma$ leaves $\mathfrak{F}(\mathcal{O})$ globally stable, $\mathcal{O} \in \mathcal{K}$, and is locally normal.*
ii) *G is the stabilizer in Γ of the vacuum state of $\mathfrak{F}$.*
iii) *The normalizer of G in Γ is the subgroup of automorphisms in Γ leaving $\mathfrak{A}^d$ globally stable.*

Proof. Since $\gamma \in \Gamma$ leaves $\mathfrak{A}$ and a fortiori $\mathfrak{A}(\mathcal{O}')$ pointwise fixed, it induces an automorphism of the von Neumann algebra $\mathfrak{F}(\mathcal{O})$ by Lemma 2.1 and is hence locally normal [13]. If $\gamma \in \Gamma$ and ω is a vector state of $\mathfrak{F}$ with $\omega = \omega \circ \gamma$, then γ is normal and, being trivial on $\mathfrak{A}$, it is also trivial on $\mathfrak{A}^- \cap \mathfrak{F}$. But by (2.3) we have $\mathfrak{A}^- = \mathfrak{A}^{d-}$ and $\mathfrak{A}^{d-} = G'$ (cf. [4: Thm. 3.6c]). Now $\mathfrak{F} \cap G' = \mathfrak{A}^d$ (cf. [4: Sect. 3]) so

$$\mathfrak{A}^- \cap \mathfrak{F} = \mathfrak{A}^d .$$

Thus γ is trivial on $\mathfrak{A}^d$ and hence belongs to G [4: Thm. 3.6b]. Conversely, if $\gamma \in G$ and ω is induced by a vector belonging to a 1-dimensional subrepresentation of G then $\omega = \omega \circ \gamma$ and ii) is proved. Finally iii) follows trivially from the fact that G is the subgroup of all automorphisms of $\mathfrak{F}$ leaving $\mathfrak{A}^d$ pointwise fixed. □

G is a compact group in the strong operator topology and it is natural to enquire what properties Γ has when we topologize it appropriately. There seems to be no easy general answer to this question. We know from examples that Γ may be compact, or merely locally compact, but it may also fail to be locally compact. However, this question plays only a rather minor role in this paper so that we confine ourselves here to defining the topology on Γ and adding a few elementary remarks.

The appropriate topology to put on a group of local automorphisms of a net of von Neumann algebras like Γ would seem to be the topology of pointwise norm convergence on the preduals of the local von Neumann algebras. Thus, since the preduals are the linear span of the normal states, a base of neighbourhoods of $\gamma \in \Gamma$ in this topology is given by sets of the following form: pick locally normal states $\omega_1, \omega_2, \ldots, \omega_n$, pick $\mathcal{O} \in \mathcal{K}$ and $\varepsilon > 0$ and set

(2.4)
$$N_\gamma(\omega_1,\omega_2,\ldots,\omega_n;\varepsilon;\mathcal{O}) := \{\gamma' \in \Gamma : \|\omega_i \circ \gamma' - \omega_i \circ \gamma\|_{\mathcal{O}} < \varepsilon, \quad i = 1,2,\ldots,n\},$$

where the subscript $\mathcal{O}$ denotes the norm of the restriction of the linear functional to $\mathfrak{F}(\mathcal{O})$. It is easily checked that with this topology Γ becomes a topological group. It is in fact not necessary to use all locally normal states in the definition of the topology. Thus, since the local field algebras are in standard form in the vacuum representation, it suffices to consider just the vector states of this representation. Furthermore, since the elements of Γ act trivially on $\mathfrak{A}$, it suffices to consider the states defined by a set of vectors which is cyclic for some $\mathfrak{A}(\mathcal{O})$. When $\mathcal{H}$ is separable, it will therefore suffice to consider a single, suitably chosen vector state. Note that the map $g \mapsto \mathrm{Ad}\, g$ is a continuous homomorphism from G to Γ so that G may be regarded as a compact subgroup of Γ.

With this information we can now turn to the construction of local implementations of Γ. To this end, we fix some double cone $\mathcal{O}$ and note that Ω is cyclic and separating for the von Neumann algebra $\mathfrak{F}(\mathcal{O})$. Hence there is a natural cone $\mathcal{P}^\natural_{\mathfrak{F}(\mathcal{O}),\Omega} \subset \mathcal{H}$ obtained from the modular operator Δ associated with the pair $(\mathfrak{F}(\mathcal{O}),\Omega)$[3]. It is the closure of $\{\Delta^{1/2}F\Omega : F \in \mathfrak{F}(\mathcal{O}),\ F \geq 0\}$. The natural cone has the property that every normal state of $\mathfrak{F}(\mathcal{O})$ is induced by a *unique* vector in the cone. This vector is cyclic for $\mathfrak{F}(\mathcal{O})$ if and only if it is separating for $\mathfrak{F}(\mathcal{O})$, i.e. if and only if the state is a faithful state of $\mathfrak{F}(\mathcal{O})$. Now putting $\gamma \in \Gamma$ the state $\omega_0 \circ \gamma^{-1}$ is normal on $\mathfrak{F}(\mathcal{O})$. We let Ω_γ denote the unique vector in the cone inducing $\omega_0 \circ \gamma^{-1}$ on $\mathfrak{F}(\mathcal{O})$, i.e.

(2.5)
$$(\Omega, \gamma^{-1}(F)\Omega) = (\Omega_\gamma, F\Omega_\gamma), \quad F \in \mathfrak{F}(\mathcal{O}).$$

We thus have a unitary operator $V_{\mathcal{O}}(\gamma)$ inducing γ on $\mathfrak{F}(\mathcal{O})$ and defined by (cf. [14: Thm. 7.1])

(2.6)
$$V_{\mathcal{O}}(\gamma)F\Omega = \gamma(F)\Omega_\gamma, \quad F \in \mathfrak{F}(\mathcal{O}).$$

[3]For the theory of natural cones the reader may consult [14] or the short summary in the Appendix of [15].

The important point is that $V_{\mathcal{O}}(\gamma)$ maps the natural cone onto itself. In fact if Δ_γ denotes the modular operator associated with the cyclic and separating vector Ω_γ then obviously $V_{\mathcal{O}}(\gamma)\Delta = \Delta_\gamma V_{\mathcal{O}}(\gamma)$ so that $V_{\mathcal{O}}(\gamma)$ maps $\mathcal{P}^\natural_{\mathfrak{F}(\mathcal{O}),\Omega}$ onto $\mathcal{P}^\natural_{\mathfrak{F}(\mathcal{O}),\Omega_\gamma}$. However, it can be shown that these two cones coincide because $\Omega_\gamma \in \mathcal{P}^\natural_{\mathfrak{F}(\mathcal{O}),\Omega}$. From the uniqueness of the representing vector, it follows that

(2.7)
$$V(\gamma)\Omega_{\gamma'} = \Omega_{\gamma\gamma'}.$$

This in turn implies that $\gamma \to V_{\mathcal{O}}(\gamma)$ is a unitary representation implementing the action of Γ on $\mathfrak{F}(\mathcal{O})$. The continuity of the representation for the topology on Γ introduced above is a consequence of the inequality [14]

(2.8)
$$\|\Omega_\gamma - \Omega_{\gamma'}\|^2 \leq \|\omega \circ \gamma^{-1} - \omega \circ \gamma'^{-1}\|_{\mathfrak{F}(\mathcal{O})}.$$

We say that the representation $V_{\mathcal{O}}$ is a local implementation of Γ. It must be stressed that since $\mathfrak{F}(\mathcal{O})$ has a large commutant these local implementations are far from being unique. Since the mere existence of such local implementations suffices for our investigation we refrain from discussing this ambiguity further.

At this point we comment on two examples which are sufficiently simple to allow the topological group Γ to be explicitly calculated. To the best of our knowledge this is the first time that the group Γ of all gauge symmetries of a theory has been computed in a case where spontaneously broken symmetries are involved. For the first example we take the free massless scalar field for $s > 1$ and define the observable net $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ by saying that $\mathfrak{A}(\mathcal{O})$ is the von Neumann algebra generated by the Weyl operators $W(f) := e^{i\phi(f)}$ in the vacuum representation, where f is a real smooth function with $\mathrm{supp} f \subset \mathcal{O}$ and satisfying the subsidiary condition $\int f\, d^{s+1}x = 0$. The canonical field net $\mathcal{O} \mapsto \mathfrak{F}(\mathcal{O})$ can be computed and is obtained by omitting the subsidiary condition in the definition of $\mathfrak{A}$. The compact gauge group G is trivial here since, as one would expect, there are no non-trivial sectors corresponding to localizable charges. The full gauge group Γ can be computed and is isomorphic as a topological group to the additive group $\mathbb{R}$. If $\lambda \in \mathbb{R}$, the corresponding automorphism γ_λ of $\mathfrak{F}$ is uniquely defined by

$$\gamma_\lambda(W(f)) = e^{i\lambda \int f \, d^{s+1}x} W(f) \tag{2.9}$$

for each test function f. This 1-parameter group of gauge symmetries is generated by the conserved current $\partial_\mu \phi$. The fixed-point net of $\mathfrak{F}$ under Γ coincides with $\mathfrak{A}$. Details of the arguments leading to these conclusions will be found in Appendix A.

The second example concerns the free electromagnetic field which also admits an internal symmetry group that can be regarded as a group of gauge symmetries. Here we get the appropriate observable net $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ by taking $\mathfrak{A}(\mathcal{O})$ to be the von Neumann algebra generated by the Weyl operators $W(f) := e^{iF_{\mu\nu}(f^{\mu\nu})}$, where $f^{\mu\nu}$ are real smooth functions, antisymmetric in μ and ν with $\mathrm{supp} f^{\mu\nu} \subset \mathcal{O}$ and $\int f^{\mu\nu} d^{s+1}x = 0$. The canonical field net $\mathcal{O} \mapsto \mathfrak{F}(\mathcal{O})$ is again obtained by omitting the subsidiary condition. The compact gauge group G is trivial since, as before, there are no non-trivial sectors. The gauge group Γ is now isomorphic to $\mathbb{R}^{\frac{1}{2}s(s+1)}$ and the gauge automorphisms take the form

$$\gamma_\lambda(W(f)) = e^{i \int \lambda_{\mu\nu} f^{\mu\nu} d^{s+1}x} W(f) \ , \tag{2.10}$$

where $\lambda_{\mu\nu} = -\lambda_{\nu\mu} \in \mathbb{R}$, on the generating Weyl operators. These gauge symmetries are generated by conserved currents of the form

$$j_\mu(x) := c_{\mu\nu\rho\sigma} x^\nu F^{\rho\sigma}(x) \ . \tag{2.11}$$

Note that these currents are not translationally covariant. The fixed-point net of $\mathfrak{F}$ under Γ again coincides with $\mathfrak{A}$ and further details of the necessary arguments can be found in Appendix A.

In the remainder of this section we turn to the important question of how big the algebra $\mathfrak{A}$ of observables has to be in a physically reasonable theory. This question arises naturally in the present setting since, as already mentioned, the local algebras of observables cannot be maximal in the presence of

spontaneous symmetry breaking. One therefore needs some condition which guarantees that they have a certain minimal size.

As a partial answer to this problem we should like to propose as a principle that the energy-momentum density should be an observable. This would in particular imply that local implementations for translations exist. Within an algebraic setting this requirement can be formulated as follows: given any pair $\Lambda = (\mathcal{O}_1, \mathcal{O}_2)$ of double cones with $\mathcal{O}_1^- \subset \mathcal{O}_2$, there is a continuous unitary representation U_Λ of spacetime translations such that

a) $U_\Lambda(a) \in \mathfrak{A}(\mathcal{O}_2)$

b) $U_\Lambda(a) F U_\Lambda(a)^* = \alpha_a(F)$ for $F, \ \alpha_a(F) \in \mathfrak{F}(\mathcal{O}_1)$.

If the net $\mathfrak{F}$ has the split property [15] (related to the growth of the energy level density of localized states [16] and the existence of thermodynamical equilibrium states [17]) the existence of such local translations can be established, provided there are no spontaneously broken symmetries in the theory [18]. In fact there is a canonical choice for these local translations whereby they satisfy the spectrum condition [18] and converge to the global translations U as $\mathcal{O}_1$, $\mathcal{O}_2$ tend to the whole Minkowski space in an appropriate manner [19]. If the gauge symmetry is spontaneously broken local translations still exist, but they need not be observable: there are simple models with the split property where there is no representation U_Λ satisfying a) and b). It seems that in these cases the algebra of observables is unreasonably small and one would like to exclude such theories on physical grounds.

The net $\mathfrak{A}_0$ generated by all local translations must, in our view, be regarded as a minimal net of local observables. It turns out that for this net the analogue of Lemma 2.1 still holds. To see this we pick $F \in \mathfrak{A}_0(\mathcal{O}')' \cap \mathfrak{F}$, $F' \in \mathfrak{F}(\mathcal{O}_1)$, where $\mathcal{O}_1$ is any double cone whose closure is contained in the the interior of $\mathcal{O}'$, and any spacelike translation a such that the closures of $\mathcal{O}_1$, $\mathcal{O}_1 + a$ are contained in some double cone $\mathcal{O}_2 \subset \mathcal{O}'$. According to a) and b) we then have $\alpha_a(F') = U_\Lambda(a) F' U_\Lambda(a)^{-1}$ for some unitary operator $U_\Lambda(a) \in \mathfrak{A}_0(\mathcal{O}_2) \subset \mathfrak{A}_0(\mathcal{O}')$, and consequently

$$\|[F, \alpha_a(F')]\| = \|U_\Lambda(a)[F, F']U_\Lambda(a)^{-1}\| = \|[F, F']\| \ . \tag{2.12}$$

If $F' \in \mathfrak{F}(\mathcal{O}_1)$ is a Bose operator[4] it follows from this equation and locality, by letting a tend spacelike to infinity, that $[F, F'] = 0$. In the same way we see that the Fermi part of F anticommutes with all Fermi operators $F' \in \mathfrak{F}(\mathcal{O}_1)$. Since $\mathcal{O}_1 \subset \mathcal{O}'$ was arbitrary and the field net satisfies twisted duality [4] we conclude that $F \in \mathfrak{F}(\mathcal{O})$, as claimed.

This result suggests introducing the group Γ_0 of all automorphisms of $\mathfrak{F}$ which leave the subnet $\mathfrak{A}_0$ pointwise fixed. Clearly $\Gamma_0 \supseteq \Gamma$, but we see that the elements of Γ_0 still act locally on the field net. This entails that local implementations of Γ_0 exists. It then follows, using conditions a) and b), that Γ_0 commutes with spacetime translations[5], i.e. it is a group of internal symmetries.

In fact, given $\mathcal{O}_1 \in \mathcal{K}$ and any translation a, we have, by assumption, a unitary operator $U_\Lambda(a) \in \mathfrak{A}_0$ such that $\alpha_a(F) = U_\Lambda(a) F U_\Lambda(a)^{-1}$ for all $F \in \mathfrak{F}(\mathcal{O}_1)$. Hence if $\gamma_0 \in \Gamma_0$ we get

(2.13)
$$\gamma_0(\alpha_a(F)) = \gamma_0(U_\Lambda(a) F U_\Lambda(a)^{-1}) = U_\Lambda(a)\gamma_0(F)U_\Lambda(a)^{-1} = \alpha_a(\gamma_0(F)) ,$$

where in the last step we made use of the fact that $\gamma_0(\mathfrak{F}(\mathcal{O}_1)) = \mathfrak{F}(\mathcal{O}_1)$. Since $\mathcal{O}_1$ and a were arbitrary, we conclude that γ_0 commutes with spacetime translations.

These results in no way depend on the fact that $\mathfrak{F}$ is the canonical field net associated with our algebra of observables. They hold for any extension of a net $\mathfrak{A}_0$ generated by local translations to some field net satisfying twisted duality and conditions a) and b): we always find some group Γ_0 of automorphisms which may be regarded as gauge symmetries relative to $\mathfrak{A}_0$ and which may still appear as internal symmetries if looked at from a possibly richer net of observables $\mathfrak{A}$.

[4]Every operator $F \in \mathfrak{F}$ can be decomposed into a Bose and Fermi part $F_\pm = \frac{1}{2}(F \pm k(F))$, respectively, where $k = k^3 \in \mathrm{Ad}\, G$. These Bose and Fermi operators have normal commutation relations at spacelike distances [4].

[5]This result may also be interpreted as a no go theorem: in a conventional field theory where the energy-momentum tensor is an observable there are no gauge automorphisms which do not commute with translations. Gauge transformations of the second kind therefore call for a different mathematical setting.

It has already been pointed out that Γ_0 is in general larger than Γ, in fact Γ_0 may not even leave $\mathfrak{A}$ globally stable. Such a situation, though mathematically possible, seems rather awkward from a physical point of view. Guided by physics one may very well argue that all internal symmetries have to be (global) gauge symmetries[6], i.e. one may postulate that $\Gamma = \Gamma_0$. The net $\mathfrak{A}_0$ would then determine Γ, and hence, appealing to the principle of gauge invariance, all other observables. In this sense $\mathfrak{A}_0$ may be regarded as the core of the observables.

After this digression let us now return to our original goal of discussing the appearance and consequences of spontaneously broken symmetries. In the present section we have seen how the observables determine the global gauge group and how its action on the local field algebras can be implemented. This is the information needed for the general framework of the subsequent section.

3. Goldstone's Theorem

In this section we discuss Goldstone's Theorem in Algebraic Quantum Field Theory. There are two aspects to this. We first establish conditions under which 1-parameter groups of internal symmetries cannot be spontaneously broken in massive theories. Then, using different if related methods, we give conditions under which such a symmetry can be spontaneously broken only if there are massless *particles* in the theory.

The results of this section hold more generally than in the setting adopted in the preceding section. We therefore begin by listing the assumptions which are actually used here. We consider a net $\mathfrak{F}$ of von Neumann algebras over double cones $\mathcal{O}$ which is covariant with respect to spacetime translations α_a and suppose that we are given a pure vacuum state ω_0 of $\mathfrak{F}$. Without loss of generality we can and will suppose that the vacuum representation is faithful and we then regard $\mathfrak{F}$ as being defined concretely through its vacuum

[6]This is not true if one disregards certain forces; think of charge conjugation in the absence of weak interactions.

representation. In the first part of our analysis we make no explicit use of the commutation properties of the net $\mathfrak{F}$ at spacelike separations. However these properties play an important role in our discussion of the particle aspects of Goldstone's Theorem.

We study 1-parameter groups $\lambda \in \mathbb{R} \mapsto \gamma_\lambda$ of automorphisms of this net having the following properties:

a) $\gamma_\lambda(\mathfrak{F}(\mathcal{O})) = \mathfrak{F}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$, $\lambda \in \mathbb{R}$,

b) $\lambda \mapsto \gamma_\lambda(F)$, $F \in \mathfrak{F}$, is continuous in the weak operator topology in the vacuum representation,

c) $\gamma_\lambda \alpha_a = \alpha_a \gamma_\lambda$, $a \in \mathbb{R}^{s+1}$, $\lambda \in \mathbb{R}$.

Note that b) is equivalent to demanding that $\lambda \mapsto \gamma_\lambda(F)$ be σ-continuous as a map from $\mathbb{R}$ to $\mathfrak{F}(\mathcal{O})$ for each $\mathcal{O} \in \mathcal{K}$. The condition is therefore automatically satisfied if we take a continuous homomorphism γ from $\mathbb{R}$ to Γ in the setting of the previous section.

Throughout the subsequent analysis we will deal with the generator δ of the group $\lambda \to \gamma_\lambda$, which is defined on the domain

(3.1) $$\mathcal{D} = \{F \in \mathfrak{F} : \lambda^{-1}(\gamma_\lambda(F) - F) \text{ converges in norm as } \lambda \to 0\}$$

by setting

(3.2) $$\delta(F) = \lim_{\lambda \to 0} \lambda^{-1}(\gamma_\lambda(F) - F) \, .$$

As is well known $\mathcal{D}$ is a $*$-algebra and δ is a $*$-derivation on $\mathcal{D}$ [20]. Moreover, because of the continuity properties of γ_λ and the fact that $\gamma_\lambda(\mathfrak{F}(\mathcal{O})) = \mathfrak{F}(\mathcal{O})$, it can be shown that $\mathfrak{F}(\mathcal{O}) \cap \mathcal{D}$ is weak operator dense in $\mathfrak{F}(\mathcal{O})$. It is also not difficult to see that $\mathcal{D}_0 = \bigcup_{\mathcal{O} \in \mathcal{K}} \mathfrak{F}(\mathcal{O}) \cap \mathcal{D}$ is a core for δ.

Setting $\alpha_f(F) := \int dt f(t) \alpha_t(F)$, where $f \in L^1(\mathbb{R})$ and α_t are time-translations, one deduces from condition c) and the local normality of γ_λ that $F \in \mathcal{D}$ implies $\alpha_f(F) \in \mathcal{D}$ and $\delta(\alpha_f(F)) = \alpha_f(\delta(F))$. A similar remark applies to space translations or translations in spacetime.

The given symmetry is said to be unbroken in the vacuum representation if $\omega_0 \circ \gamma_\lambda = \omega_0$, $\lambda \in \mathbb{R}$, or, equivalently, if $\omega_0 \circ \delta = 0$ on $\mathcal{D}$.

Either condition implies that there exists a continuous unitary representation $\lambda \to V(\lambda)$ on $\mathcal{H}$ which implements the action of γ_λ on the whole of $\mathfrak{F}$.

We want to derive here conditions under which spontaneous symmetry breaking, $\omega_0 \circ \delta \neq 0$, is possible. What matters in this context is, as we shall see, the behaviour of the functional $\omega_0 \circ \delta$ on the algebras $\mathfrak{F}(\mathcal{O}) \cap \mathcal{D}$ for very large $\mathcal{O}$. To fix this behaviour in a quantitative manner we make use of the fact, established as in the previous section, that there exist local implementations of our symmetry group. By Stone's Theorem this implies that for each $\mathcal{O} \in \mathcal{K}$ there exists some skew-adjoint (and, in general, unbounded) operator $J_\mathcal{O}$ such that

(3.3) $$\delta(F) = [J_\mathcal{O}, F] \, , \quad F \in \mathfrak{F}(\mathcal{O}) \cap \mathcal{D} \, ,$$

in the sense of sesquilinear forms on the domain of $J_\mathcal{O}$. Let us assume for a moment that the vacuum vector Ω is in the domain of $J_\mathcal{O}$. Then we obtain from (3.3) the estimate

(3.4) $$|\omega_0 \circ \delta(F)| = |(\Omega, [J_\mathcal{O}, F]\Omega)| \leq \|J_\mathcal{O}\Omega\| \cdot (\|F\Omega\| + \|F^*\Omega\|) \, .$$

The left hand side of this inequality does not depend on the particular choice of the generator $J_\mathcal{O}$. Hence we get

(3.5) $$|\omega_0 \circ \delta(F)| \leq c_\mathcal{O}(\|F\Omega\| + \|F^*\Omega\|) \, , \quad F \in \mathfrak{F}(\mathcal{O}) \cap \mathcal{D} \, ,$$

where we may take $c_\mathcal{O} = \inf \|J_\mathcal{O}\Omega\|$, the infimum being with respect to all generators $J_\mathcal{O}$ satisfying (3.3).

We have thus found an a priori estimate for our functional $\omega_0 \circ \delta$ and what concerns us in our investigation is the behaviour of the constant $c_\mathcal{O}$ in this bound as $\mathcal{O} \nearrow \mathbb{R}^{s+1}$.

Since we cannot infer from the general results in the preceding section that there always exists a generator $J_\mathcal{O}$ with Ω in its domain we next discuss how the estimate (3.5) has to be modified in the general case. We put $R_\eta = (I + \eta \cdot J_\mathcal{O})^{-1}$, $\eta > 0$ and start from the elementary equality

(3.6) $$\delta(F) = \frac{1}{\eta}[F, I - R_\eta] + (I - R_\eta)\delta(F) + R_\eta \delta(F)(I - R_\eta) \, , \quad F \in \mathfrak{F}(\mathcal{O}) \cap D$$

which follows from relation (3.3). Since R_η is a normal operator and $\|R_\eta\| \leq 1$, our equality yields the estimate

$$|\omega_0 \circ \delta(F)| \leq \|(I - R_\eta)\Omega\| \left(\frac{1}{\eta}\|F\Omega\| + \frac{1}{\eta}\|F^*\Omega\| + 2\|\delta(F)\|\right) . \tag{3.7}$$

But $\|(I - R_\eta)\Omega\| \to 0$ as $\eta \to 0$, hence we can fix the parameter η for each $\mathcal{O}$ in such a way that $2 \cdot \|(I - R_\eta)\Omega\| = \varepsilon$, where $\varepsilon > 0$ is arbitrarily small. Having fixed η in this way, the preceding estimate leads to the bound

$$|\omega_0 \circ \delta(F)| \leq c_{\mathcal{O},\varepsilon}(\|F\Omega\| + \|F^*\Omega\|) + \varepsilon \cdot \|\delta(F)\| \tag{3.8}$$

which is the natural generalization of (3.5). Again it is only the behaviour of the constant $c_{\mathcal{O},\varepsilon}$ for large $\mathcal{O}$ that is relevant. In order to get an idea how this constant behaves we examine the behaviour of $c_{R,\varepsilon} := c_{\mathcal{O}_R,\varepsilon}$, where $\mathcal{O}_R$ denotes the double cone of radius R about the origin, in a familiar case.

Examples. Let $j_\mu(x)$ be a conserved current satisfying the Wightman axioms. Let us further assume that the skew-symmetric operators

$$J_R := i \int \mathrm{d}^{s+1}x f(x^0) g\left(\frac{\mathbf{x}}{R}\right) j_0(x) \tag{3.9}$$

induce δ on $\mathfrak{F}(\mathcal{O}_R)$ as in (3.3). Here f and g are test functions with $\int \mathrm{d}x^0 f(x^0) = 1$ and $g(\mathbf{x}) = 1$ for $|\mathbf{x}| < 1$. The Källen-Lehmann representation of the 2-point function of the current takes the form

$$(\Omega, j_\mu(x) j_\nu(y)\Omega) = \int \mathrm{d}\mu(m) \int \frac{d^s p}{p_0}(p_\mu p_\nu - m^2 g_{\mu\nu}) e^{ip\cdot(x-y)} , \tag{3.10}$$

where $d\mu$ is a positive polynomially bounded measure on $\mathbb{R}_+$ and $p = (p_0, \mathbf{p})$, where $p_0 = (\mathbf{p}^2 + m^2)^{1/2}$. With this input one finds by a standard computation that $\|J_R\Omega\| \leq \text{const} \cdot R^{(s-1)/2}$, and consequently $c_{R,\varepsilon} \leq \text{const} \cdot R^{(s-1)/2}$, where const does not depend on ε. A similar bound holds for tensorial currents $j_{\mu\nu\ldots\rho}(x)$.

As a second example let us consider a dilation covariant theory where the dilations act continuously on the field net $\mathfrak{F}$ as a group $\mu \to \beta_\mu$ of automorphisms, satisfying $\beta_\mu(\mathfrak{F}(\mathcal{O})) = \mathfrak{F}(\mu\mathcal{O})$, $\mu \in \mathbb{R}_+$. We suppose, furthermore, that the vacuum state is dilation invariant, $\omega_0 \circ \beta_\mu = \omega_0$, $\mu \in \mathbb{R}_+$.

We now consider a 1-parameter group $\lambda \mapsto \gamma_\lambda$ of internal symmetries which is normalized by β_μ, $\mu \in \mathbb{R}_+$, so that $\beta_\mu \gamma_{\mathbb{R}} \beta_\mu^{-1} = \gamma_{\mathbb{R}}$. Then there is a $k \in \mathbf{R}$ such that $\gamma_\lambda \beta_\mu = \beta_\mu \gamma_{\lambda \cdot \mu^k}, \mu \in \mathbb{R}_+$, $\lambda \in \mathbf{R}$, and a computation shows that $\beta_\mu(\mathcal{D}) \subset \mathcal{D}$, and

$$\delta(\beta_\mu(F)) = \mu^k \beta_\mu(\delta(F)) , \quad F \in \mathcal{D} . \tag{3.11}$$

This makes it clear that the constant $c_{R,\varepsilon}$ in our a priori bounds can, in the situation envisaged above, be chosen to be of the form $c_\varepsilon \cdot R^k$ for $R \geq R_0$, where c_ε does not depend on R. Note that in deriving this bound we did not assume that δ is induced by some current.

This discussion suggests that in order to cover the standard cases the constant $c_{R,\varepsilon}$ must be allowed to grow at a certain rate as R increases. That $c_{R,\varepsilon}$ indeed blows up if the symmetry is spontaneously broken is shown in the following lemma.

LEMMA 3.1. *Let δ be such that for any $\mathcal{O} \in \mathcal{K}$*

$$|\omega_0 \circ \delta(F)| \leq c_\varepsilon \cdot (\|F\Omega\| + \|F^*\Omega\|) + \varepsilon \cdot \|\delta(F)\| \text{ if } F \in \mathfrak{F}(\mathcal{O}) \cap \mathcal{D} ,$$

where c_ε does not depend on $\mathcal{O}$. Then $\omega_0 \circ \delta = 0$.

Proof. It suffices to show that $\omega_0 \circ \delta(F) = 0$ for $F \in \mathcal{D}_0$ as $\mathcal{D}_0$ is a core for δ. Moreover, we may assume that $F = F^*$ (since $\mathcal{D}_0^* = \mathcal{D}_0$) and $\omega_0(F) = 0$ (since $\delta(I) = 0$, i.e. we can subtract from F a multiple of I, if necessary). Given such an operator F we pick any real function $f \in L^1(\mathbf{R})$ such that f has compact support and $\int f(t)\,\mathrm{d}t = 1$. The operator $\alpha_f(F)$ is still a selfadjoint element of $\mathcal{D}_0$ and, taking the invariance of ω_0 under time translations into account, we have

$$\omega_0 \circ \delta(F) = \omega_0(\alpha_f(\delta(F))) = \omega_0 \circ \delta(\alpha_f(F)) .$$

According to our premises this implies

$$|\omega_0 \circ \delta(F)| \leq \inf(2c_\varepsilon \|\alpha_f(F)\Omega\| + \varepsilon \cdot \|\delta(\alpha_f(F))\|)$$

where the infimum is to be taken over all functions f complying with the above conditions. To evaluate this infimum we note that for fixed f the family of functions $f_R(t) = R^{-1} \cdot f(R^{-1} \cdot t)$, $R > 0$ belongs to the admissible class. Moreover[7], $\|f_R\|_1 = \|f\|_1$ so that $\|\delta\alpha_{f_R}(F)\| \leq \|f\|_1 \cdot \|\delta(F)\|$ uniformly in R. Now by the mean ergodic theorem

$$\lim_{R\to\infty} \alpha_{f_R}(F)\Omega = E_0 F\Omega \ ,$$

where E_0 is the projection onto the subspace of $\mathcal{H}$ which is invariant under time translations, i.e. onto $\mathbb{C}\Omega$. Since $\omega_0(F) = 0$ we deduce from the above estimate that $|\omega_0 \circ \delta(F)| \leq \varepsilon \cdot \|f\|_1 \cdot \|\delta(F)\|$, and since $\varepsilon > 0$ was arbitrary the statement follows. □

We note that the only property of the energy-momentum spectrum used in deriving this result is that there is, up to a phase, a unique time-invariant vector Ω. Lemma 3.1 can, for example, be applied to internal symmetries generated by conserved currents in $s = 1$ dimensions (cf. the preceding discussion). It shows that such a symmetry can never be spontaneously broken thereby reproducing Coleman's well known theorem [8]. Lemma 3.1 may thus be regarded as a generalization of Coleman's result.

Let us now turn to the question of how the energy spectrum can inhibit spontaneous symmetry breaking. Our first result in this direction shows that the a priori bound given in the preceding proposition can be relaxed substantially, without changing the conclusions, if there is a gap $m > 0$ between 0 and the rest of the energy spectrum.

PROPOSITION 3.2. *Consider any theory with a gap $m > 0$ in the energy spectrum. There is a model independent constant $c > 0$ such that for any derivation δ satisfying for $F \in \mathfrak{F}(\mathcal{O}_R)$*

$$|\omega_0 \circ \delta(F)| \leq c_\varepsilon \cdot e^{\mu R} \cdot (\|F\Omega\| + \|F^*\Omega\|) + \varepsilon \cdot \|\delta(F)\|$$

for some $0 \leq \mu < c \cdot m$ one has $\omega_0 \circ \delta = 0$.

[7] The L^p-norms of functions are denoted by $\|\cdot\|_p$, $p = 1, 2$.

Proof. We proceed as in the proof of the preceding lemma and pick any $F = F^* \in \mathcal{D}_0$ such that $\omega_0(F) = 0$. Let $\mathcal{O}_{R_0}$ be a double cone in which F is localized. Then, if $f \in L^1(\mathbb{R})$ is any function with support in the interval $[-R, R]$, we have $\alpha_f(F) \in \mathfrak{F}(\mathcal{O}_{R_0+R}) \cap \mathcal{D}$, and, if $\int dt f(t) = 1$, too, we obtain for sufficiently large R the bound

$$|\omega_0 \circ \delta(F)| \leq 2c_\varepsilon \cdot e^{\mu(R_0+R)} \|\alpha_f(F)\Omega\| + \varepsilon \cdot \|f\|_1 \|\delta(F)\| \ .$$

Hence if we further restrict f by requiring $\|f\|_1 \leq 2$, say, we get

$$|\omega_0 \circ \delta(F)| \leq 2c_\varepsilon \cdot e^{\mu(R_0+R)} \inf \|\alpha_f(F)\Omega\| + 2\varepsilon \cdot \|\delta(F)\| \ ,$$

where the infimum is to be taken with respect to all functions f, as specified above. We will show below that

$$\inf \|\alpha_f(F)\Omega\| \leq c'' \cdot e^{-c' m \cdot R} \ ,$$

where $c' > 0$ is a model-independent constant and c'' does not depend on R. Combining this bound with the preceding estimate we find, by first letting R tend to infinity and then ε tend to 0 that $\omega_0 \circ \delta(F) = 0$. The statement then follows with $c = c'$.

It remains to establish the bound for the infimum given above. Since we are not aiming at an optimal result we can proceed as in [21: Lemma 3.3]. We consider the family of functions

$$\tilde{f}_n(\omega) = \frac{1}{\sqrt{2\pi}} \frac{\sin(R\omega/2)}{(R\omega/2)} \cdot s_n(R\omega/2\pi) \ , \quad n \in \mathbb{N}$$

where

$$s_n(w) = \frac{\sin \pi w}{\pi w} \cdot \prod_{j=1}^{n} \left(1 - \frac{w^2}{j^2}\right)^{-1} \ .$$

The Fourier transforms $f_n(t)$ of $\tilde{f}_n(\omega)$ are real, satisfy $\int dt f_n(t) = 1$ and have support in the interval $[-R, R]$ by the Paley-Wiener Theorem. Moreover,

$$\left(\int dt |f_n(t)|\right)^2 \leq 2R \cdot \int dt |f_n(t)|^2 =$$
$$= 2R \cdot \int d\omega |\tilde{f}_n(\omega)|^2 \leq \frac{4}{2\pi} \cdot \int du \frac{\sin^2 u}{u^2} \cdot \sup_w |s_n(w)|^2 = 2 \ ,$$

where in the last step we made use of the fact that $|s_n(w)| \leq 1$ [21: loc. cit.]. Thus $\|f_n\|_1 \leq \sqrt{2}$, and the functions f_n belong to the family considered above. Now since $\omega_0(F) = 0$ and there is a gap $m > 0$ in the energy spectrum we have

$$\|\alpha_{f_n}(F)\Omega\| \leq \sqrt{2\pi} \sup_{|\omega|\geq m} |\tilde{f}_n(\omega)| \cdot \|F\Omega\| .$$

Taking the infimum on both sides with respect to n and making use of another bound for the functions s_n given in [21: loc. cit.], we arrive at the desired estimate with $c' = ln2/\pi$. □

This result shows that in massive theories the constant $c_{\mathcal{O},\varepsilon}$ in (3.8) has to increase exponentially with the radius of $\mathcal{O}$ if the underlying symmetry is to be spontaneously broken. Hence in view of the polynomial bounds which one obtains in the case of symmetries generated by conserved currents one finds that such symmetries can never be spontaneously broken in massive theories. This is of course a well known result [7].

Proposition 3.2 is an improvement on previous results [6] even though we do not know the best possible value of the constant c. For the class of tame derivations δ, where one can put $\varepsilon = 0$ in the a priori bound, one can show by complex variable theory as in [22] that $c = 1$. In the next section we will see that this result is optimal in the sense that there exists a model exhibiting a spontaneously broken continuous symmetry which satisfies the premises of Proposition 3.2 for all $\mu > m$.

There are several variants of Proposition 3.2. On the one hand one may study different regularity conditions for δ, cf. [23]. Although, in view of the general a priori bound (3.8), our approach seems to be the most natural one. On the other hand, instead of assuming an energy gap, one may assume clustering properties of the state ω_0 in spacelike directions. Since this might be of interest in statistical mechanics we present a specific result in this direction.

LEMMA 3.3. *Let ω be a (factorial) state on $\mathfrak{F}$ which is invariant under space translations $\alpha_{\mathbf{a}}$ and for which the correlation functions*

$$\mathbf{a} \rightarrow \omega(F^*\alpha_{\mathbf{a}}(F)) - |\omega(F)|^2, \; F \in \bigcup_{\mathcal{O}} \mathfrak{F}(\mathcal{O})$$

are absolutely integrable. If δ commutes with $\alpha_{\mathbf{a}}$ and satisfies for $F \in \mathfrak{F}(\mathcal{O}_R) \cap \mathcal{D}$

$$|\omega \circ \delta(F)| \leq c_{R,\varepsilon}(\|F\Omega\| + \|F^*\Omega\|) + \varepsilon \cdot \|\delta(F)\| ,$$

where $c_{R,\varepsilon} \cdot R^{-s/2} \rightarrow 0$ as $R \rightarrow \infty$, then $\omega \circ \delta = 0$.

Remarks.

i) If ω_0 is a pure vacuum state in a local theory with a mass gap then the hypothesis about the correlation functions is automatically satisfied.

ii) The condition on the growth of $c_{R,\varepsilon}$ cannot be relaxed. For consider a local theory with a mass gap. Pick any local operator P with $P^* = P$ and $\omega_0(P) = 0$ and set $\delta_P(F) := i \int d^s x [F, \alpha_{\mathbf{x}}(P)]$. Then δ_P is a $*$-derivation which commutes with space translations and, by the cluster theorem, one has $c_{R,\varepsilon} \leq \text{const} \cdot R^{s/2}$. However, almost always $\omega_0 \circ \delta_P \neq 0$, cf. [24].

Proof of Lemma 3.3. Let $F = F^* \in \mathcal{D}_0$ with $\omega(F) = 0$ and pick a real function $f \in L^2(\mathbb{R}^s)$ with $f(\mathbf{a}) = 0$ for $|\mathbf{a}| \geq 1/2$ and $\int d^s a f(\mathbf{a}) = 1$. Setting $f_R(\mathbf{a}) = R^{-s} f(R^{-1} \cdot \mathbf{a})$, $R > 0$ we obtain for sufficiently large R

$$|\omega \circ \delta(F)| \leq 2c_{R,\varepsilon}\|\alpha_{f_R}(F)\Omega\| + \varepsilon \cdot \|f\|_1 \cdot \|\delta(F)\| .$$

Recalling that $\omega(F) = 0$ and using the information about the correlations in the state ω we get

$$\|\alpha_{f_R}(F)\Omega\|^2 \leq \int d^s a \int d^s b |f_R(\mathbf{a}) f_R(\mathbf{b}) \omega(F^*\alpha_{\mathbf{b}-\mathbf{a}}(F))| \leq$$
$$\leq \text{const} \cdot \int d^s a |f_R(\mathbf{a})|^2 = \text{const} \cdot \|f\|_2^2 \cdot R^{-s} .$$

The statement thus follows. □

Let us now turn to the other aspect of Goldstone's Theorem and discuss when spontaneous symmetry breaking is associated with the presence of particles of zero mass. The state of affairs is fully described in the following proposition, which uses the Bose-Fermi commutation relations of fields at spacelike separations.

PROPOSITION 3.4. *Let δ be a derivation in a local theory in $s > 1$ spatial dimensions such that for $F \in \mathfrak{F}(\mathcal{O}_R) \cap \mathcal{D}$*

$$|\omega_0 \circ \delta(F)| \leq c_{R,\varepsilon}(\|F\Omega\| + \|F^*\Omega\|) + \varepsilon \cdot \|\delta(F)\| .$$

i) *If $\liminf_{R\to\infty} c_{R,\varepsilon} \cdot R^{-(s-1)/2} = 0$, then $\omega_0 \circ \delta = 0$.*
ii) *If $\liminf_{R\to\infty} c_{R,\varepsilon} \cdot R^{-(s-1)/2} < \infty$, then $\omega_0 \circ \delta \neq 0$ is only possible if the spectrum of the translations U coincides with V_+ and the boundary $\partial V_+\backslash\{0\}$ has non-trivial spectral measure (i.e. there are massless particles in the theory).*
iii) *If $c_{R,\varepsilon}$ is polynomially bounded in R, then $\omega_0 \circ \delta \neq 0$ is only possible if the spectrum of U coincides with V_+ (but there are not necessarily any massless particles).*

Remark. If $s = 1$ then i) is trivially satisfied and ii) becomes an empty statement because of Lemma 3.1. As to iii) we note that if $\omega_0 \circ \delta \neq 0$ there can be no gap in the energy spectrum according to Proposition 3.2. But if $s = 1$ this does not necessarily imply that the spectrum of U coincides with V_+.

Proof of Proposition 3.4. In the first part of the argument we proceed exactly as in the proof of Lemma 3.3, the only difference being that we here have to pick elements $F \in \mathcal{D}_0$ which are smooth with respect to spacetime translations α_a. This is not a serious restriction as we can smooth out $a \to \alpha_a(F)$ for any $F \in \mathcal{D}_0$ by integrating with a test function. Thus if $\omega_0 \circ \delta(F) = 0$ for all smooth elements $F \in \mathcal{D}_0$ it follows that $\omega_0 \circ \delta = 0$.

With the same notation as in the proof of Lemma 3.3, we obtain for large R when $F = F^*$ and $\omega_0(F) = 0$ the estimate

$$|\omega_0 \circ \delta(F)| \leq 2 \cdot c_{R,\varepsilon}\|\alpha_{f_R}(F)\Omega\| + \varepsilon \cdot \|f\|_1 \cdot \|\delta(F)\| .$$

Thus in order to proceed we have to control the behaviour of $\|\alpha_{f_R}(F)\Omega\|$ for large R. This we can do by a standard application of the Jost-Lehmann-Dyson representation as expounded in [25]. For the convenience of the reader

we gather the relevant results in Appendix B and show that

$$\lim_{R\to\infty} R^{s-1} \cdot \|\alpha_{f_R}(F)\Omega\|^2 = \frac{1}{(2\pi)^s} \int \frac{d^s p}{|\mathbf{p}|} |\tilde{f}(\mathbf{p})|^2 \cdot \int d^s a (\Omega, \alpha_a(F) E_1 P_0 F\Omega)$$

where E_1 is the spectral projection of U corresponding to the Borel set $\partial V_+\backslash\{0\}$ and P_0 is the generator of time translations.

Let us now discuss the consequences of the different behaviour of the constants $c_{R,\varepsilon}$ in our estimates. Since $\|\alpha_{f_R}(F)\Omega\| \leq \text{const} \cdot R^{-(s-1)/2}$ we find in case i) that $\omega_0 \circ \delta(F) = 0$ whatever the spectrum of U may actually be. In case ii) we infer from the specific form of the above limit that $E_1 F\Omega \neq 0$ if $\omega_0 \circ \delta(F) \neq 0$. Hence $E_1 \neq 0$ and the set $\partial V_+\backslash\{0\}$ has non-trivial spectral measure. Since the spectrum of U is additive [26] and has a Lorentz invariant boundary [27] it has to coincide with V_+. Finally, in case iii) we already know from Proposition 3.2 that there can be no gap in the energy spectrum. By the same reasoning as before we conclude that the spectrum of U coincides with V_+. But the boundary $\partial V_+\backslash\{0\}$ may have trivial measure, as is illustrated by an example in the next section. □

The proof that massless Goldstone particles exist in theories with spontaneously broken continuous symmetries obviously depends on the specific a priori bound $c_{R,\varepsilon} \leq c_\varepsilon \cdot R^{(s-1)/2}$. For the case of internal symmetries which are generated by conserved currents, where these bounds can be established, we thus recover a well-known result of Ezawa and Swieca [28].

Let us next discuss what can be said about the nature of the Goldstone particle, should it exist. We first turn to the problem of statistics. Here we use (cf. footnote 4) the fact that each operator $F \in \mathfrak{F}$ can be decomposed into a Bose and Fermi part $F_\pm$. As a consequence our symmetry γ_λ does not change the Bose-Fermi character of the fields. For let $F_+ \in \mathfrak{F}(\mathcal{O})$ and $G = \gamma_\lambda(F_+)$, then $[G^*, \alpha_a(G)] = 0$ for sufficiently large spacelike a since F_+ is local and γ_λ is an automorphism commuting with translations. Now $G = G_+ + G_-$, where $G_\pm \in \mathfrak{F}(\mathcal{O})$ since γ_λ does not change the localization, and plugging this expression into the above commutator we see that $[G_-^*, \alpha_a(G_-)] = 0$. This is compatible with the spacelike anticommutation relations of Fermi fields only if $G_-^* \cdot \alpha_a(G_-) = 0$, which in turn implies $G_- = 0$ by the Reeh-Schlieder Theorem. Hence γ_λ maps Bose operators onto Bose operators and by a similar argument one sees that the same is true of Fermi operators. Consequently we can split the domain $\mathcal{D}_0$ of the derivation δ into a Bose and Fermi part which are separately invariant under the action of δ.

Since the vacuum expectation value of any Fermi operator is zero we find that $\omega_0 \circ \delta$ vanishes on the Fermi part of $\mathcal{D}_0$. Hence if the symmetry is broken there must be a Bose operator $F_+ \in \mathcal{D}_0$ such that $\omega_0 \circ \delta(F_+) \neq 0$. As we have seen in the proof of the preceding proposition, this is only possible in case ii) if $E_1 F_+ \Omega \neq 0$. Thus the massless particles which accompany the spontaneous breakdown of our symmetry are Bosons.

The fact that $E_1 F_+ \Omega$ has to be different from 0 can also be used in certain cases to determine the helicity of the Goldstone particle. Generic examples are symmetries γ_λ which commute with the action α_R of rotations. Setting $\overline{F}_+ = \int d\mu(R) \alpha_R(F_+)$, where μ is the normalized Haar measure on the group of rotations, and using the fact that ω_0 is invariant under rotations we then have $\omega_0 \circ \delta(\overline{F}_+) = \omega_0 \circ \delta(F_+)$. Hence, if $\omega_0 \circ \delta(F_+) \neq 0$ for some F_+ we find that $E_1 \overline{F}_+ \Omega \neq 0$. Since the vector $E_1 \overline{F}_+ \Omega$ is invariant under rotations we conclude that the subspace of massless particles must contain states of helicity 0, i.e. the Goldstone Boson has helicity 0. (These arguments can also be applied if $\alpha_R \gamma_\lambda \alpha_R^{-1} \gamma_\lambda^{-1}$ is an unbroken symmetry for all λ and R.)

Although these conditions on γ_λ might appear to be very general (all symmetries generated by local, conserved, tensorial currents are of this type [29]), it would be premature to conclude that they include all possible spontaneously broken symmetries and that any associated massless particles have to be scalars. A simple counter example is the free electromagnetic field. As we have discussed in Section 2, we have a group of spontaneously broken symmetries shifting the zero point of the electromagnetic field. But the only massless particle in the theory, the photon, has helicity 1.

We conclude this section with a remark pertaining to what are called geometrical symmetries and supersymmetries. As to the former our conditions

fixing the properties of the symmetry group γ_λ, $\lambda \in \mathbb{R}$ have to be relaxed as follows: let $\lambda \to D_\lambda$ be some continuous representation of $\mathbb{R}$ by diffeomorphisms of spacetime (translations, Lorentz transformations etc.). Then condition a) has to be replaced by $\gamma_\lambda(\mathfrak{F}(\mathcal{O})) = \mathfrak{F}(D_\lambda \cdot \mathcal{O})$ for $\mathcal{O} \in \mathcal{K}$, condition b) remains unchanged, and condition c) now reads $\gamma_\lambda \cdot \alpha_a = \alpha_{D_\lambda \cdot a} \cdot \gamma_\lambda$ for $\lambda \in \mathbb{R}$, $a \in \mathbb{R}^{s+1}$. The generating derivation δ thus still has the property that $\omega_0 \circ \delta \alpha_a = \omega_0 \circ \delta$ for $a \in \mathbb{R}^{s+1}$.

Supersymmetries do not act on the field net $\mathfrak{F}$ by automorphisms. Instead, making use of the Bose-Fermi grading of $\mathfrak{F}$, one describes them by densely defined linear maps $\overline{\delta}$ on $\mathfrak{F}$ which satisfy the *graded* Leibnitz rule $\overline{\delta}(F_\sigma \cdot G_\tau) = \overline{\delta}(F_\sigma) \cdot G_\tau + \sigma \cdot F_\sigma \cdot \overline{\delta}(G_\tau)$, where σ, τ stands for $\pm$, respectively. Moreover, the maps $\overline{\delta}$ preserve the local structure of $\mathfrak{F}$ and commute with translations. Hence the relation $\omega_0 \circ \overline{\delta} \alpha_a = \omega_0 \circ \overline{\delta}$ holds once again for $a \in \mathbb{R}^{s+1}$.

As all that was ever used in the various arguments is that $\omega_0 \circ \delta$ is a translationally invariant functional on $\mathfrak{F}$ vanishing on the unit operator, the results can easily be extended to the above symmetries.

We note that in the case of graded derivations $\overline{\delta}$ the power $R^{-(s-1)/2}$ in the asymptotic bounds of Proposition 3.4 may be replaced by $R^{-s/2}$. This follows from the fact that $\overline{\delta}$ changes Bose operators into Fermi operators whose two-point functions decay more rapidly, in general, in spacelike directions than those of Bose operators (cf. Appendix B). Hence we find, as expected [30, Sect. 4] that the Goldstone particles in theories with spontaneously broken supersymmetries are Fermions with half-integer spin.

4. Beyond the Range of Goldstone's Theorem

In this section, we give a brief discussion of a simple class of examples exhibiting spontaneously broken symmetry.

It should, however, be stressed at the outset that we do not attempt the difficult task of giving (rigorous) examples of physically realistic models with spontaneously broken symmetries. Instead we merely want to complement the results in the previous section with simple mathematical models showing that our main results are essentially optimal. Thus there are examples of spontaneous symmetry breaking which lie beyond the range of Goldstone's Theorem, thereby answering a question raised in [6] and [23]. The hypotheses of the relevant results, Prop. 3.2 and 3.4, are violated only slightly and the conclusions do not hold. Hence substantial improvements on our results are only conceivable if the class of theories considered is limited by further hypotheses.

As our class of examples we take the generalized free scalar field ϕ in s space dimensions in the vacuum representation [31]. The two-point function is given by

$$(4.1)\qquad (\Omega, \phi(x)\phi(y)\Omega) = \int \mathrm{d}\mu(m) \frac{1}{(2\pi)^s} \int \theta(p^0)\delta(p^2 - m^2) e^{ip(x-y)}\, \mathrm{d}p$$

and the corresponding norm on the 1-particle space is given by

$$(4.2)\qquad \|f\|^2 = \int \mathrm{d}\mu(m) \int \frac{d^s p}{2\sqrt{\mathbf{p}^2 + m^2}} |\tilde{f}(\sqrt{\mathbf{p}^2 + m^2}, \mathbf{p})|^2 .$$

We will only consider measures μ with $\mathrm{d}\mu(m) = \rho(m)\,\mathrm{d}m$, i.e. where μ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^+$. For this class, the mapping $W(f) \to W(f)e^{i\lambda \tilde{f}(0)}$ defines an automorphism of the Weyl algebra since $\|f\| = 0$ implies $f = 0$ (because $\tilde{f}$ is entire analytic) and hence $\tilde{f}(0) = 0$. However, we are rather interested in whether this automorphism extends to an automorphism of the local von Neumann algebras. We decide this question by applying the criteria of Lemma A.1 but we first need some preliminary estimates.

Let f be any real test function with support in $\mathcal{O}_R$ and even under time reflection and let $\tilde{f}$ be its Fourier transform. Then, for any $m \geq 0$, the function

$$(4.3)\qquad \mathbf{p} \to \tilde{f}(\sqrt{\mathbf{p}^2 + m^2},\ \mathbf{p})$$

has a Fourier transform with support in $|\mathbf{x}| \leq R$. Hence if χ is any test function which is equal to 1 for $|\mathbf{x}| \leq 1$ we have

$$(4.4)\qquad \tilde{f}(m, \mathbf{0}) = (2\pi)^{-s/2} \int \mathrm{d}^s p\ R^s \tilde{\chi}(R\mathbf{p}) \tilde{f}(\sqrt{\mathbf{p}^2 + m^2},\ \mathbf{p})$$

so that

$$(4.5)\qquad |\tilde{f}(m, \mathbf{0})|^2 \leq (2\pi)^{-s} \int \mathrm{d}^s p \sqrt{\mathbf{p}^2 + m^2} R^{2s} |\tilde{\chi}(R\mathbf{p})|^2 \int \frac{\mathrm{d}^s p}{\sqrt{\mathbf{p}^2 + m^2}} |\tilde{f}(\sqrt{\mathbf{p}^2 + m^2}, \mathbf{p})|^2 .$$

Recalling the support properties of f, we next note that $z \to \tilde{f}(z, \mathbf{0})$, is an entire function of exponential type with

$$(4.6)\qquad |\tilde{f}(z, \mathbf{0})| \leq \frac{c_N}{|z|^N} e^{R|\mathrm{Im}\, z|} , \quad N \in \mathbb{N} .$$

To compensate the growth of $\tilde{f}(z, \mathbf{0})$ in imaginary directions, we consider, for given $M > 0$, the auxiliary function

$$(4.7)\qquad z \to e^{-R\sqrt{M^2 - z^2}} , \quad z \in \mathbb{C} \backslash \{z : \mathrm{Im} z = 0 , \quad |\mathrm{Re}\, z| \geq M\} .$$

This function is analytic and continuous at the boundary and

$$(4.8)\qquad |e^{-R\sqrt{M^2 - z^2}}| \leq e^{-R|\mathrm{Im}\, z|} .$$

Hence the function

$$(4.9)\qquad z \to e^{-R\sqrt{M^2 - z^2}} \tilde{f}(z, \mathbf{0})$$

is analytic in the cut plane, continuous at the boundary and decays rapidly at infinity. Applying Cauchy's formula to express the value of the function at 0 by an integral along the cuts leads to the estimate

$$(4.10)\qquad e^{-RM} |\tilde{f}(0, \mathbf{0})| \leq \frac{2}{\pi} \int_M^\infty \mathrm{d}m \frac{|\tilde{f}(m, \mathbf{0})|}{m} .$$

Now if $\rho(m) > 0$ for $m > M$ we obtain with the help of the preceding estimates

$$(4.11)\qquad |\tilde{f}(0, \mathbf{0})| \leq C(R, M)\ \|f\| ,$$

where

$$C(R,M)^2 = \frac{8}{\pi^2} R^{2s} e^{2MR} \int_M^\infty \mathrm{d}m \frac{1}{m\rho(m)} \int \mathrm{d}^s p \sqrt{1+\frac{\mathbf{p}^2}{m^2}} |\tilde{\chi}(R\mathbf{p})|^2 \tag{4.12}$$
$$\leq \text{ const. } R^s \left(1+\frac{1}{RM}\right) e^{2MR} \int_M^\infty \mathrm{d}m \frac{1}{m\rho(m)} \ .$$

Since, for real f, $\|f\|^2 = \|f_+\|^2 + \|f_-\|^2$, where $f_\pm$ are respectively the even and odd parts of f under time reflections, we get the following result.

LEMMA 4.1. *Suppose that for some $M > 0$*

$$k(M)^2 := \int_M^\infty dm \frac{1}{m\rho(m)} < +\infty$$

then, for any real test function f with support in $\mathcal{O}_R$

$$|\tilde{f}(0,0)| \leq \text{ const. } k(M) R^{s/2} \left(1+\frac{1}{MR}\right)^{1/2} e^{MR} \|f\| \ .$$

If, furthermore $k(0) < +\infty$, then

$$|\tilde{f}(0,0)| \leq \text{ const. } R^{s/2} \|f\| \ .$$

We need only remark that the second statement follows from the first by putting $M = \frac{1}{R}$.

We conclude from Lemmas 4.1 and A.1 that if $k(M) < +\infty$ for some $M \geq 0$ then there is a unique 1-parameter group $\lambda \mapsto \gamma_\lambda$ of local automorphisms of the net generated by our generalized free field such that

$$\gamma_\lambda(W(f)) = e^{i\lambda \tilde{f}(0)} W(f) \ , \quad f \in \mathcal{D}(\mathbb{R}^{s+1}) \ , \tag{4.13}$$

and commuting with spacetime translations. If we furthermore suppose that $\rho(m) = 0$ for $m < M$ and $k(M) < +\infty$ then we have a theory with a mass gap M and a spontaneously broken symmetry for which, from the remarks following Lemma A.1, we can take

$$c_R = \text{ const. } R^{s/2} e^{MR} \ . \tag{4.14}$$

The bounds in Proposition 3.2 are only weakly violated and, from the form of these bounds, it is clear that the symmetry cannot be derived from tempered conserved currents.

If we suppose that our generalized free field has no mass gap but has $k(0) < +\infty$, then we get an example of a spontaneously broken symmetry where we can take

$$c_R = \text{ const. } R^{s/2}$$

but where there is no Goldstone particle. The absence of a mass gap in this case is in agreement with Proposition 3.4 iii).

From a physical point of view our model is, to be sure, pathological. However, we see no mechanism ruling out such symmetries from the outset. It should be noted that the Coleman-Mandula Theorem [32] and the Weinberg-Witten Theorem [33] do not apply to such symmetries since they only consider unbroken symmetries. So there is room for surprises when treating spontaneously broken symmetries.

5. Concluding Remarks

In the present investigation we have seen that whether a symmetry is spontaneously broken or not depends on the balance between the action of the generating derivation on the local algebras of large domains and the nature of correlations in the vacuum state. To outrun the decay of correlations at large distances and lead to spontaneous symmetry breaking, the effects of the derivation have to increase polynomially or exponentially, depending on the nature of the mass spectrum.

For symmetries generated by conserved currents we have strong a priori bounds on the derivations and this is all that is needed to establish the familiar Goldstone Theorem. On the other hand we have exhibited simple examples of symmetries for which Goldstone's Theorem does not hold in spite of the fact that the relevant bounds are only barely violated. These examples served to reveal the limitations of Goldstone's Theorem within our general setting.

It seems conceivable that in theories of physical interest the generating derivations automatically satisfy substantially stronger a priori bounds than the one given in relation (3.8). To support this conjecture let us consider a theory satisfying the time-slice axiom [34] and the split property [15]. Both properties are believed to be characteristic features of physically reasonable theories. Now the split property implies that the action of the derivation δ on any local algebra $\mathfrak{F}(\mathcal{O})$ can be implemented by a skew-adjoint operator $J_{\mathcal{O}}$ which is itself localized in some slightly larger region $\tilde{\mathcal{O}}$[8]. The time-slice axiom, on the other hand, should imply that $J_{\mathcal{O}}$ can be written as a sum of $J_{\mathcal{O}_i}$, where the $\mathcal{O}_i$ cover the base of $\mathcal{O}$. (It is this additivity of the local generators which is missing in our examples violating Goldstone's Theorem.) Covering the base of $\mathcal{O}_R$ with double cones of fixed size and disregarding domain problems, we are led to conclude that for large R

$$\|J_{\mathcal{O}_R}\Omega\|^2 \le \sum_{i,k} |(J_{\mathcal{O}_i}\Omega, J_{\mathcal{O}_k}\Omega)| \le \text{const} \cdot R^{s+1} , \tag{5.1}$$

where we made use of the cluster theorem [25]. Although this bound is still not as good as the one holding for symmetries generated by conserved currents, it would suffice to rule out spontaneous symmetry breaking in massive theories.

Appendix A

In this appendix we first discuss in the abstract results relating to certain automorphisms of Weyl algebras. These results are needed in the second part when discussing examples illustrating various phenomena connected with spontaneously broken symmetries and Goldstone's Theorem.

We consider here Weyl operators $W(k)$, $k \in K$, in a fixed Fock representation such as the vacuum representation of a quasifree Boson system.

[8]In fact this construction leads to a local implementation of Γ by a continuous unitary representation which takes values in $\mathfrak{F}(\tilde{\mathcal{O}})$ and is covariant for the action of the unbroken part G of the gauge group. The problem of the existence of local implementations with more specific properties will be dealt with elsewhere.

Since the Fock representation remains fixed throughout the discussion, we may without loss of generality take K to be a complex Hilbert space. The commutation relations are then

$$W(k)W(k') = W(k+k')e^{-i\text{Im }(k,k')}$$

and the Fock state in question corresponds to the functional

$$(\Omega, W(k)\Omega) = e^{-\frac{1}{2}\|k\|^2}$$

on K. If M is a *real* subspace of K then we let $R(M)$ denote the von Neumann algebra generated by the $W(k)$, $k \in M$. Since $k \to W(k)$ is continuous in the strong operator topology, $R(M) = R(M^-)$, where M^- denotes the closure of M in K.

Now let t be a real linear functional on M then, setting

$$\gamma_\lambda(W(k)) := e^{i\lambda t(k)}W(k) , \quad k \in M , \quad \lambda \in \mathbb{R} ,$$

$k \to \gamma_\lambda(W(k))$ still satisfies the Weyl relations over M. However, γ does not necessarily extend to an automorphism of $R(M)$. In fact, as is easily checked [35], we have

LEMMA A.1. *The following statements are equivalent*

a) *γ_λ extends to an automorphism of $R(M)$, $\lambda \in \mathbb{R}$.*
b) *t is a continuous linear functional on M.*
c) *There is a $k_0 \in iM^-$ such that $t(k) = -2\text{Im }(k_0, k)$.*
d) *There is a $k_0 \in K$ such that $\gamma_\lambda(W(k)) = W(\lambda k_0)W(k)W(\lambda k_0)^*$, $k \in M$.*

In what follows, we will suppose that t is a continuous linear functional so that $\lambda \to \gamma_\lambda$ is a continuous 1-parameter group of automorphisms of the von Neumann algebra $R(M)$. We denote the derivation which is its infinitesimal generator by δ and let $\mathcal{D}$ denote the domain of δ (cf. Eq. (3.1) and (3.2)). We also write $W(\lambda k_0) = e^{i\lambda\phi(k_0)}$ so that $i\phi(k_0)$ implements δ (cf. Eq. (3.3)) and we have (cf. Eq. (3.4))

$$|(\Omega, \delta(F)\Omega)| \le \|k_0\|(\|F\Omega\| + \|F^*\Omega\|) , \quad F \in \mathcal{D} ,$$

since $\|k_0\| = \|i\phi(k_0)\Omega\|$. On the other hand, it follows from c) above that we may take $\|k_0\| = \frac{1}{2}\|t\|$ so that we deduce

$$|(\Omega, \delta(F)\Omega)| \leq \frac{1}{2}\|t\|(\|F\Omega\| + \|F^*\Omega\|) .$$

We next compute the fixed-point algebra $R(M)^\gamma$ of $R(M)$ under the action of the 1-parameter group:

$$R(M)^\gamma = \{F \in R(M) : \gamma_\lambda(F) = F , \quad \lambda \in \mathbb{R}\} .$$

LEMMA A.2. *$R(M)^\gamma = R(M^t)$, where*

$$M^t = \{k \in M : t(k) = 0\} .$$

Proof. Trivially, $R(M^t) \subset R(M)^\gamma$. Picking k_0 as above,

$$\gamma_\lambda(F) = W(\lambda k_0)FW(\lambda k_0)^* , \quad F \in R(M) .$$

Hence $R(M)^\gamma = R(M) \cap R(\mathbb{R}k_0)' = R(M^- \cap (\mathbb{R}k_0)')$, where we have used [36; Thm. 1] and N' denotes the symplectic complement of $N \subset K$,

$$N' = \{k \in K : \mathrm{Im}(k, k') = 0 , \quad k' \in N\} .$$

Thus $R(M)^\gamma = R(M^{-t})$. Now given $k_n \in M$, $k_n \to k \in M^{-t}$, we have $t(k_n) \to t(k) = 0$. If $M = M^t$, the lemma is trivially true and otherwise there is a $\tilde{k} \in M$ with $t(\tilde{k}) = 1$. But then $k_n - t(k_n)\tilde{k} \in M^t$ and $k_n - t(k_n)\tilde{k} \to k$. In this case $W(k_n - t(k_n)\tilde{k}) \to W(k)$ in the strong operator topology and $R(M^t) = R(M^{-t})$ completing the proof. □

We now consider $\mathrm{Aut}R(M)$, the group of automorphisms of $R(M)$, as a topological group giving it the topology of pointwise norm convergence on the predual of $R(M)$. Then in this topology the convergence of automorphisms, $\alpha_n \to \alpha$, is equivalent to $\|f \circ \alpha_n - f \circ \alpha\| \to 0$ for all $f \in R(M)_*$, the predual of $R(M)$. The elements of the predual are the linear functionals f on $R(M)$ of the form

$$f(F) = TrTF , \quad F \in R(M) ,$$

where T is a trace class operator on Fock space.

We consider the set of continuous linear functionals on M as a topological group under addition by using the norm topology.

LEMMA A.3. *There is a unique homomorphism $t \to \gamma_t$ from the group of continuous linear functionals on M into the group of automorphisms of $R(M)$ such that*

$$\gamma_t(W(k)) = e^{it(k)}W(k) , \quad k \in M .$$

This homomorphism is a topological isomorphism onto its image in $\mathrm{Aut}R(M)$.

Proof. The existence of γ_t follows from Lemma A.1 and its uniqueness shows that $t \to \gamma_t$ is a homomorphism which is obviously injective. Now, given t, there is a unique $k_t \in iM^-$ with

$$t(k) = -2\mathrm{Im}\,(k_t, k) , \quad k \in M .$$

The mapping $t \to k_t$ is linear and $\|t\| = 2\|k_t\|$. Furthermore $W(k_t)W(k)W(k_t)^* = e^{it(k)}W(k)$. Since $k \to W(k)$ is continuous in the strong operator topology, $k \to W(k)^*TW(k)$ is continuous with respect to the trace norm for each operator T of trace class. But this implies that $t \to f \circ \gamma_t$ is norm continuous for each $f \in R(M)_*$ so that our homomorphism $t \to \gamma_t$ is continuous. Now let ω_0 be the state on $R(M)$ induced by Ω, let $k \in M$, $\|k\| = 1$ and $t(k) \neq 0$ then

$$\|\omega_0 \circ \gamma_t - \omega_0\| \geq \sup_{\mu \in \mathbb{R}} |\omega_0(W(\mu k))|\, |e^{it(k)\mu} - 1|$$
$$= \sup_{\mu \in \mathbb{R}} e^{-\frac{1}{2}\mu^2}(2 - 2\cos t(k)\mu)^{1/2} .$$

If we set $\mu = \frac{\pi}{t(k)}$, we get

$$\|\omega_0 \circ \gamma_t - \omega_0\| \geq 2e^{-\frac{\pi^2}{2t(k)^2}}$$

and varying k with $\|k\| = 1$ yields

$$\|\omega_0 \circ \gamma_t - \omega_0\| \geq 2e^{-\frac{\pi^2}{2\|t\|^2}} .$$

We conclude that if γ_t tends to the identity automorphism in $\mathrm{Aut}R(M)$, then $\|t\| \to 0$ and our homomorphism is indeed a topological isomorphism onto its image. □

We next consider a real subspace $N \subset M$ and show that it is possible, under certain circumstances, to compute all automorphisms of $R(M)$ which leave $R(N)$ pointwise invariant.

LEMMA A.4. *Let $N \subset M$ be real linear subspaces of K and suppose that $M^- \cap N' = \{0\}$. Then the automorphisms of $R(M)$ which leave $R(N)$ pointwise invariant are precisely the automorphisms γ_t with*

$$\gamma_t(W(k)) = e^{it(k)}W(k) , \quad k \in M ,$$

where t is a continuous linear functional on M with $t(k) = 0$ for $k \in N$.

Proof. The specified automorphisms of $R(M)$ obviously leave $R(N)$ pointwise invariant. Now let β be an automorphism of $R(M)$ leaving $R(N)$ pointwise fixed and pick $k \in M$, $k' \in N$. Then

$$\begin{aligned}\beta(W(k))W(k')\beta(W(k))^* &= \beta(W(k)W(k')W(k)^*) = \beta(e^{-2i\mathrm{Im}(k,k')}W(k')) \\ &= e^{-2i\mathrm{Im}(k,k')}W(k') = W(k)W(k')W(k)^* .\end{aligned}$$

Thus

$$W(k)^*\beta(W(k)) \in R(M) \cap R(N)' = R(M^- \cap N') = R(\{0\}) = \mathbb{C}I ,$$

where we have again used [36; loc. cit.]. Hence $W(k)^*\beta(W(k))$ is just a phase $u(k)$, say, and it follows from the Weyl relations that $u(k+k') = u(k)u(k')$. Since $\lambda \to W(\lambda k)^*\beta(W(\lambda k))$ is continuous in the strong operator topology, $\lambda \to u(\lambda k)$ is a continuous homomorphism from $\mathbb{R}$ into $\mathbb{T}$. Thus $u(\lambda k) = e^{i\lambda t(k)}$ and, since $t(k)$ is uniquely determined, $k \to t(k)$ is a linear functional on M which must be continuous by Lemma A.1. Thus $\beta = \gamma_t$, proving the lemma. □

When $M^- \cap N' = \{0\}$, we say that N is irreducible in M^- since, as we have seen in the proof of Lemma A.4, it implies that $R(M) \cap R(N)' = \mathbb{C}I$. The question of irreducibility is an interesting one and we shall shortly present conditions allowing one to conclude that N is irreducible in M^-. However, we shall be dealing with two nets of von Neumann algebras and asking for the automorphisms of the larger net which leave the elements of the smaller net pointwise invariant and here the question of irreducibility arises in modified form. To model this situation we have

LEMMA A.5. *Consider two increasing sequences $\{M_n\}_{n\in\mathbb{N}}$ and $\{N_n\}_{n\in\mathbb{N}}$ of subspaces with $N_n \subset M_n$. Set $M_\infty = \cup_n M_n$ and $N_\infty = \cup_n N_n$ then, if $M_\infty^- \cap N_\infty' = \{0\}$, the automorphisms of the net $n \to R(M_n)$ which leave the subnet $n \to R(N_n)$ pointwise invariant are precisely the automorphisms γ_t with*

$$\gamma_t(W(k)) = e^{it(k)}W(k) , \quad k \in M_\infty ,$$

where t runs over the linear functionals on M which are continuous in restriction to each M_n and satisfy $t(k) = 0$ for $k \in N_\infty$.

Proof. The proof of Lemma A.4 applies with trivial modifications. □

There is no difficulty in applying Lemma A.5 to our examples because it is easy to check that $N_\infty' = \{0\}$ in these cases. It is more difficult to show that Lemma A.4 can be applied to the local algebras in our examples. To this end, we must first prove some useful results which exploit the fact that N will then be the kernel of some finite set of continuous linear functionals on M.

LEMMA A.6. *Let $t_1, t_2, \ldots, t_n$ be linearly independent real linear functionals on the real linear subspace M and set $N = \bigcap_{i=1}^n \ker t_i$, then $M \cap N' = \cup_t N_t \cap N_t'$, where t runs over all linear combinations of the t_i, i.e. $t = \sum_{i=1}^n c_i t_i$ and $N_t = \ker t$.*

Proof. Pick $k \in M^- \cap N'$ then the linear functional t with $t(f) = \mathrm{Im}\,(k,f)$ has N in its kernel and is hence a linear combination of the t_i. Furthermore $k \in N_t \cap N_t'$ so $M \cap N' \subset \cup_t N_t \cap N_t'$. Now $N \subset N_t \subset M$ so $N_t \cap N_t' \subset M \cap N'$ completing the proof. □

Applying Lemma A.6 to the closure M^- of M in K we get

COROLLARY A.7. *Under the assumptions of Lemma A.6 suppose that $t_1, t_2, \ldots, t_n$ are continuous, then $M^- \cap N' = \cup_t N_t^- \cap N_t'$. Furthermore, the following conditions are equivalent*

a) *$R(N)$ is irreducible in $R(M)$*
b) *Each $R(N_t)$ is a factor*

c) *Each automorphism γ_t on $R(M)$, $t \neq 0$, is outer and $R(M)$ is a factor.*

Proof. In view of Lemma A.6 it suffices to demonstrate the equivalence of a) and c). If some γ_t is not outer the implementing unitary lies in $R(M) \cap R(N)'$. Conversely if $k \in M^- \cap N'$ and $t(f) = \text{Im}\,(k, f)$ then $W\left(-\frac{k}{2}\right)$ implements γ_t and lies in $R(M)$. If $t = 0$, $W\left(-\frac{k}{2}\right)$ lies in the center of $R(M)$. □

Note that in particular if N is defined by a single continuous linear functional, $R(N)$ is irreducible in $R(M)$ if and only if $R(M)$ and $R(N)$ are factors.

Actually to show that $R(N)$ is irreducible in $R(M)$, it is not necessary to check that each $R(N_t)$ is a factor. If $N^- \cap N' = \{0\}$, it suffices to check it on a basis $t_1, t_2, \dots, t_n$. In fact

$$M^- \cap N' = \left(\sum_{i=1}^{n} N_{t_i}^- \cap N_{t_i}'\right)^- .$$

To see this we apply the first part of Corollary A.7 and compute

$$\left(\sum_i N_{t_i}^- \cap N_{t_i}'\right)^- = \left(\sum_i M^- \cap N_{t_i}'\right)^- = M^- \cap \left(\sum_i N_{t_i}'\right)^- .$$

But $\left(\sum_i N_{t_i}'\right)^- = \left(\cap_i N_{t_i}^-\right)' = N'$, since the corresponding identity holds for orthogonal complements in a real Hilbert space and $N' = (iN)^\perp$.

If we pursue the above strategy for showing that N is irreducible in M^-, we shall need to know that certain local algebras are factors. One route for proving that local algebras of free Bose fields are factors was opened up by Araki [37]. The examples we shall be treating are the free massless scalar field and the functional $t(f) = \tilde{f}(0)$ and the free electromagnetic field and functionals of the form $t(f) = \lambda_{\mu\nu}\tilde{f}^{\mu\nu}(0)$. The kernels of these functionals are invariant under translation and *dilations* (but not, in the case of the electromagnetic field, under Lorentz transformations). This allows us to avoid explicit computations by using the following abstract argument.

LEMMA A.8. *Let $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ be a translation covariant local net over double cones in an irreducible vacuum representation. If the net is covariant for dilations and dilations are the modular group for $\mathfrak{A}(V_+)^-$ and the vacuum vector Ω, then $\mathfrak{A}(\mathcal{O})$ is a factor.*

Proof. Without loss of generality, we may suppose that the double cone $\mathcal{O}$ has one vertex at the origin and the other within V_+ so that $x \in \mathcal{O}$ implies $e^{\lambda\pi}x \in \mathcal{O}$ for $\lambda \leq 0$. Let β_λ denote the automorphism corresponding to a dilation with parameter $\lambda \in \mathbb{R}$. Pick $Z \doteq Z^* \in \mathfrak{A}(\mathcal{O}) \cap \mathfrak{A}(\mathcal{O})'$ then if $\lambda \leq 0$ we have $[Z, \beta_\lambda(Z)] = 0$ since $\beta_\lambda(Z) \in \mathfrak{A}(\mathcal{O})$. However, if $\lambda \geq 0$, we have $[Z, \beta_\lambda(Z)] = \beta_\lambda([\beta_{-\lambda}(Z), Z]) = 0$ since $\beta_{-\lambda}(Z) \in \mathfrak{A}(\mathcal{O})$. Hence, in particular, $\lambda \to (\Omega, [Z, \beta_\lambda(Z)]\Omega)$ vanishes identically. Now since $Z \in \mathfrak{A}(V_+)^-$ and β_λ is the modular group of $(\mathfrak{A}(V_+)^-, \Omega)$, $\lambda \to (\Omega, Z\beta_\lambda(Z)\Omega)$ can be continued to a bounded analytic function on the strip $0 < \text{Im}\, z < 1$ and, with an obvious notation

$$(\Omega, Z\beta_{\lambda+i}(Z)\Omega) = (\Omega, \beta_\lambda(Z)Z\Omega) = (\Omega, Z\beta_\lambda(Z)\Omega) .$$

This periodicity in the imaginary direction implies that $\lambda \to (\Omega, Z\beta_\lambda(Z)\Omega)$ can be continued to an entire analytic bounded function. Any such function is a constant so that

$$(\Omega, ZZ\Omega) = \lim_{\lambda \to +\infty} (\Omega, Z\beta_{-\lambda}(Z)\Omega) = (\Omega, Z\Omega)^2 .$$

Since Ω is separating for $\mathfrak{A}(\mathcal{O})$, Z is a multiple of the identity. □

By the results of [38], when we have a free dilation covariant theory in even spacetime dimensions where the local nets are derived from underlying Wightman fields, the dilations are the modular group for $(\mathfrak{A}(V_+)^-, \Omega)$ and the above lemma can be applied.

We now turn to the concrete examples: we start with the free massless scalar field in $s+1$-dimensions, $s > 1$, which we describe by introducing Weyl operators in a vacuum representation,

$$W(f) = e^{i\phi(f)} ,$$

where f is a real smooth function with compact support. The vacuum functional is given by

$$\omega_0(W(f)) = (\Omega, W(f)\Omega) = e^{-\frac{1}{2}\|f\|^2} ,$$

$$\|f\|^2 = \int \frac{d^s p}{2p_0} |\tilde{f}(p_0, \mathbf{p})|^2 \ , \quad p_0 = |\mathbf{p}| \ .$$

We let $\mathfrak{F}(\mathcal{O})$ denote the von Neumann algebra generated by the Weyl operators $W(f)$ with $\mathrm{supp} f \subset \mathcal{O}$ and get in this way a net $\mathcal{O} \mapsto \mathfrak{F}(\mathcal{O})$ of von Neumann algebras over double cones. Thus if we take our K to be the Hilbert space derived from the above scalar product on the space of test functions and $M(\mathcal{O})$ to be the subspace corresponding to test functions with support in $\mathcal{O}$ then

$$\mathfrak{F}(\mathcal{O}) = R(M(\mathcal{O}))$$

in the notation used above.

Now if t is a linear functional on $M = \cup_{\mathcal{O}} M(\mathcal{O})$ which is continuous on each $M(\mathcal{O})$ then we have a corresponding automorphism γ_t of the net $\mathfrak{F}$ by Lemma A.1. γ_t will commute with spacetime translations if

$$t(f_a) = t(f) \ , \quad f_a(x) := f(x-a) \ ,$$

i.e. if t is translation invariant. Since t must be a distribution, it can only have the form

$$t(f) = c \cdot \int f(x)\, \mathrm{d}^{s+1}x = c\, (2\pi)^{\frac{1}{2}(s+1)} \tilde{f}(0)$$

for some $c \in \mathbb{R}$. Of course, it is well known that such t do determine automorphisms of $\mathfrak{F}$. Perhaps the quickest way to see this is to note that the 1-parameter group of symmetries is generated by the conserved current $\partial_\mu \phi$ so that we have explicit expressions for Weyl operators $W(-\partial_0 f g_R)$ (cf. the example preceding Lemma 3.1) implementing the automorphisms locally and can appeal to Lemma A.1. The net $\mathfrak{F}$ thus has a 1-parameter group $\lambda \mapsto \gamma_{\lambda t}$ of internal symmetries. To arrive at our model illustrating the standpoint of Section 2, we treat these as gauge symmetries and stipulate that the observable net $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ is the subnet of von Neumann algebras generated by the Weyl operators $W(f)$ with $\int f(x)\, \mathrm{d}^{s+1}x = 0$, i.e. which are invariant under $\gamma_{\lambda t}, \lambda \in \mathbb{R}$.

The observable net $\mathfrak{A}$ is irreducibly represented in the vacuum representation of $\mathfrak{F}$. To see this, note that for any test function f we have

$$W(f)\alpha_a(W(f))^* \in \mathfrak{A}$$

and let a tend spacelike to infinity. Thus we take the vacuum representation of $\mathfrak{A}$ to be defined by the restriction to $\mathfrak{A}$ of the vacuum representation of $\mathfrak{F}$. The same observation shows that

$$\mathfrak{A}(\mathcal{O}')^- = \mathfrak{F}(\mathcal{O}')^- \ , \quad \mathcal{O} \in \mathcal{K} \ ,$$

so that, as $\mathfrak{F}$ satisfies duality in the vacuum representation, $\mathfrak{F}$ is the dual net of $\mathfrak{A}$. Hence $\mathfrak{A}$ satisfies essential duality and $\mathfrak{A}$ and $\mathfrak{F}$ have the same superselection structure [39].

We next show that $\mathfrak{F}$ has trivial superselection structure by appealing to Theorem 3.5 of [39]. To verify the hypotheses of that result, we note that if $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$ and $\mathcal{O}_1^- \subset \mathcal{O}_2$ then $\mathfrak{F}(\mathcal{O}_1) \vee \mathfrak{F}(\mathcal{O}_2')^-$ is, as required, canonically isomorphic to $\mathfrak{F}(\mathcal{O}_1) \otimes \mathfrak{F}(\mathcal{O}_2')^-$ as a consequence of the split property [40], [41] and [42]. We should also check that if $\mathcal{O} \in \mathcal{K}$ and b is a path then

$$\bigcap_{\partial b' = \partial b} \mathfrak{F}(\mathcal{O} + b') = \mathfrak{F}(\mathcal{O} + \partial_0 b) \vee \mathfrak{F}(\mathcal{O} + \partial_1 b) \ .$$

However results on the net structure of free fields such as [37] invariably refer to double cones based on some subset of a fixed Cauchy hyperplane. For this reason, it is convenient to restrict ourselves to paths at $t = 0$ and double cones based on balls at $t = 0$. Since the unitary equivalence class of representations of $\mathfrak{F}$ associated with a 1-cocycle of $\mathfrak{F}$ (cf. Eq. (4.3) of [39]) is determined by the restriction of the 1-cocycle to paths at time $t = 0$, the argument of the theorem allows us to conclude that $\mathfrak{F}$ has no non-trivial superselection sectors whenever the above equality is valid for paths at time $t = 0$ and double cones based on $t = 0$. In other words, if $\mathfrak{F}_0$ denotes the net at time $t = 0$ then we must show that if B is an open ball and b is a path at $t = 0$ then

$$\bigcap_{\partial b' = \partial b} \mathfrak{F}_0(B + b') = \mathfrak{F}_0(B + \partial_0 b) \vee \mathfrak{F}_0(B + \partial_1 b) \ . \tag{A.1}$$

For the free massive scalar field such a result follows from additivity and the duality results established in [37]. In fact, it is actually only necessary to consider two paths from $\partial_1 b$ to $\partial_0 b$, the straight-line path b and a smooth path b' chosen so that

$$(B+b')\cap(B+b) = B+\partial b\ .$$

Since this result involves just the field and its derivative at $t=0$, it extends to the case of a free massless scalar field by local normality [42].

Now that we know that $\mathfrak{A}$ has no non-trivial superselection sectors, we know that $\mathfrak{F}$ is not only the dual net but also the canonical field net of Section 2 and furthermore that the compact group G is trivial in this case. By Lemma A.5, the group Γ of gauge automorphisms is precisely the group $\lambda \mapsto \gamma_{\lambda t}$ and bearing Lemma A.3 in mind and the definition of the topology of Γ, we can assert that $\lambda \mapsto \gamma_{\lambda t}$ is a topological isomorphism of $\mathbb{R}$ and Γ. Furthermore, from Lemma A.2, we know that in our model the fixed-point net of $\mathfrak{F}$ under Γ is just $\mathfrak{A}$ so that the principle of gauge invariance is valid here. This is a new result whose interest is enhanced by the fact that $\mathfrak{A}$ from its very definition is just the net generated by the current $\partial_\mu\phi$.

One question remains to be answered when applying the abstract theory of this Appendix to the concrete example of the free massless scalar field and that is to decide whether $\mathfrak{A}(\mathcal{O})$ is irreducible in $\mathfrak{F}(\mathcal{O})$. This means that the gauge group can then be determined locally in the sense that

$$\Gamma(\mathcal{O}) := \{\gamma \in \mathrm{Aut}\mathfrak{F}(\mathcal{O}) : \gamma(A)=A,\ A\in\mathfrak{A}(\mathcal{O})\}$$

coincides with the restriction of Γ to $\mathfrak{F}(\mathcal{O})$. According to our general theory, $\mathfrak{A}(\mathcal{O})$ is irreducible in $\mathfrak{F}(\mathcal{O})$ if and only if both algebras are factors. Although this is presumably true for any $s>1$, we content ourselves with supposing that s is odd when we may appeal to Lemma A.8 and the remark following it.

Finally, our 1-parameter group is spontaneously broken and is normalized by the automorphisms representing the dilations on $\mathfrak{F}$. Hence the discussion

of dilation invariant theories in Section 3 applies here and a computation shows that $k=\frac{1}{2}(s-1)$ and consequently c_R behaves like $R^{\frac{1}{2}(s-1)}$ in this case. This is not surprising in view of the fact that the symmetry is generated by the conserved covariant current $j_\mu = \partial_\mu\phi$.

We now turn to our second example, the free electromagnetic field in $s+1$-dimensions, $s>1$. Although technically more complicated, the discussion follows the pattern established above for the free massless scalar field. We again use Weyl operators but the test functions f are now real antisymmetric second rank tensors,

$$W(f) = e^{iF_{\mu\nu}(f^{\mu\nu})}\ ,$$

and the vacuum functional is given by

$$\omega_0(W(f)) = (\Omega, W(f)\Omega) = e^{-\frac{1}{2}\|f\|^2}\ ,$$

where

$$\|f\|^2 = \int \overline{\tilde{f}^{\mu\nu}(p)} p_\mu g_{\nu\rho} p_\sigma \tilde{f}^{\rho\sigma}(p)\frac{d^s\mathbf{p}}{2|\mathbf{p}|}\ ,\quad p_0 = |\mathbf{p}|\ .$$

As for the case of the free massless scalar field, $\mathfrak{F}$ denotes the net of von Neumann algebras generated by the Weyl operators, and we otherwise retain our previous notation.

We look for linear functionals on $M := \bigcup_{\mathcal{O}} M(\mathcal{O})$ which are continuous on each $M(\mathcal{O})$ and translation invariant. Any such functional t must have the form

$$t(f) = \lambda_{\mu\nu}\int f^{\mu\nu}\, d^{s+1}x\ ,\quad \lambda_{\mu\nu} = -\lambda_{\nu\mu}\in\mathbb{R}\ .$$

The proof that these linear functionals do give rise to automorphisms of the net $\mathfrak{F}$ is now a little bit more complicated. The relevant currents are of the form

$$j_\mu(x) = c_{\mu\nu\rho\sigma}x^\nu F^{\rho\sigma}(x)$$

and the connection between the coefficients $c_{\mu\nu\rho\sigma}$ and the antisymmetric $\lambda_{\mu\nu}$, although relatively involved, can be computed by the interested reader[9]. As

[9] It is worth remarking that, for suitable choices of the coefficients $c_{\mu\nu\rho\sigma}$, the currents j_μ would be conserved quantities even in quantum electrodynamics.

already noted these currents are not translation covariant. With their aid, we may write down explicitly Weyl operators which provide local implementations of the desired symmetries. Hence, by Lemma A.1, we have local automorphisms γ_t of the net $\mathfrak{F}$. We again treat these automorphisms as gauge automorphisms and define the observable net to be the subnet of $\mathfrak{F}$ generated by the Weyl operators $W(f)$ for which

$$\lambda_{\mu\nu} \int f^{\mu\nu} \, d^{s+1}x = 0$$

for all $\lambda_{\mu\nu} = -\lambda_{\nu\mu} \in \mathbb{R}$. These are just the Weyl operators invariant under the γ_t.

The same arguments as in the scalar case show that $\mathfrak{A}$ acts irreducibly on the Hilbert space of the vacuum representation of $\mathfrak{F}$. Furthermore, since $\mathfrak{F}$ satisfies duality [43] we conclude as before that $\mathfrak{F}$ is the dual net of $\mathfrak{A}$ so that $\mathfrak{A}$ satisfies essential duality. Thus $\mathfrak{A}$ and $\mathfrak{F}$ have the same superselection sectors.

It is again the case that $\mathfrak{F}$ and hence $\mathfrak{A}$ have no non-trivial superselection sectors and we again appeal to a modified form of Theorem 3.5 of [39] using paths at $t = 0$ and double cones based on $t = 0$. $\mathfrak{F}$ has the split property as a consequence of [44]. However, there is still one minor obstacle: we again want to verify (A.1) using additivity and duality and it will again suffice to consider two paths as before. However, Theorem II.2.7 of [43] establishes duality for the free electromagnetic field only for bounded sets B of $\mathbb{R}^3$ which are, in strong sense, contractible. The technical reason for this is that dilation invariance was used to give a simple proof of outer regularity for such regions. But the outer regularity is quite irrelevant for our purposes and, if we replace $\mathfrak{F}_0$ by its outer regularization $\mathfrak{F}_1$,

$$\mathfrak{F}_1(B) := \bigcap_{\varepsilon>0} \mathfrak{F}_0(B + B_\varepsilon) \, ,$$

where B_ε is a ball of radius ε centred on the origin then the argument of Theorem II.2.7 of [43] shows that $\mathfrak{F}_1$ satisfies duality whenever the interior

of the complement of $B + B_\varepsilon$ is simply connected for all sufficiently small ε. This then implies that (A.1) holds for the net $\mathfrak{F}_1$. The proof of Theorem 3.5 of [39] shows that (A.1) for $\mathfrak{F}_1$ implies that $\mathfrak{F}$ has no non-trivial superselection sectors. Lemma 3.8 of [39] shows that it is anyway irrelevant whether we use $\mathfrak{F}_0$ or $\mathfrak{F}_1$.

We now know that $\mathfrak{F}$ is the canonical field net associated with $\mathfrak{A}$ so that the compact gauge group G is again trivial. By Lemma A.5, the group Γ of gauge automorphisms is the group $t \to \gamma_t$ and is isomorphic to $\mathbb{R}^{\frac{s(s+1)}{2}}$. Furthermore, from Lemma A.2, we know that in our simple model the fixed-point net of $\mathfrak{F}$ under Γ is just $\mathfrak{A}$ so that the principle of gauge-invariance is valid here, too.

Again there is the question of deciding whether $\mathfrak{A}(\mathcal{O})$ is irreducible in $\mathfrak{F}(\mathcal{O})$ and here we have to appeal to the equivalence of a) and b) in Corollary A.7 and show that for each γ_t the algebra $R(N_t)$ is a factor. However, the linear functionals t here are all translation invariant and have a dilation-invariant kernel so that, if we restrict ourselves to the case that s is odd, $R(N_t)$ will be a factor by Lemma A.8. Hence $\mathfrak{A}(\mathcal{O})$ is irreducible in $\mathfrak{F}(\mathcal{O})$ and the gauge group can be determined locally, i.e.

$$\Gamma(\mathcal{O}) := \{\gamma \in \mathrm{Aut}\mathfrak{F}(\mathcal{O}) : \gamma(A) = A, \ A \in \mathfrak{A}(\mathcal{O})\}$$

coincides with the restriction of Γ to $\mathfrak{F}(\mathcal{O})$.

If $t \neq 0$, the 1-parameter group $\lambda \to \gamma_{\lambda t}$ is spontaneously broken and is normalized by the automorphisms representing the dilations on $\mathfrak{F}$. Hence the discussion of dilation invariant theories in Section 3 again applies. We now have $k = \frac{1}{2}(s+1)$ so that c_R behaves like $R^{\frac{1}{2}(s+1)}$. Thus these symmetries cannot be generated by conserved translation covariant currents. However, as we have already pointed out, there are non-covariant conserved currents generating these symmetries.

Appendix B

We establish in this appendix the relation

$$\lim_{R\to\infty} R^{s-1}\|\alpha_{f_R}(F)\Omega\|^2 = \frac{1}{(2\pi)^s}\int \frac{d^s p}{|\mathbf{p}|}|\tilde f(\mathbf{p})|^2 \cdot \int d^s a(\Omega, \alpha_{\mathbf{a}}(F)E_1 P_0 F\Omega)$$

used in the proof of Proposition 3.4. Here F is a smooth, local and selfadjoint field operator with vanishing vacuum expectation value and $f_R(\mathbf{a}) = R^{-s}\cdot f(R^{-1}\mathbf{a})$.

We begin by noting that F can be decomposed into a Bose and Fermi part (cf. footnote 4) and it suffices to verify the above relation for each of these parts separately, so let us first consider the case of a Bose operator F_+. Making use of the Jost-Lehmann-Dyson representation as in [25] we can represent the two-point function of F_+ in the form ($s > 1$)

$$(\Omega, \alpha_a(F_+)F_+\Omega) = \\ = -2\cdot\int_{\Delta B} d^s\mathbf{b}\int d\mu(p)e^{i\mathbf{pb}}\left(\frac{\partial}{\partial a_0}\Delta^{(+)}(a-\mathbf{b};p^2) + ip_0\Delta^{(+)}(a-\mathbf{b};p^2)\right).$$

Here $\Delta B = \{\mathbf{a}-\mathbf{b} : \mathbf{a},\mathbf{b}\in B\}$, where B is the base of a double cone in which F_+ is localized, $d\mu(p)$ is the signed measure whose Fourier transform is the imaginary part of the two-point function, and $\Delta^{(+)}(x,m^2)$ is the positive frequency part of the Pauli-Jordan distribution. The above relation is to be understood in the sense of distributions with respect to $\mathbf{a}$. Restricting it to the $a_0 = 0$ hyperplane yields

$$(\Omega, \alpha_{\mathbf{a}}(F_+)F_+\Omega) = \\ = \frac{1}{2}(\Omega, [\alpha_{\mathbf{a}}(F_+), F_+]\Omega) - 2i\int_{\Delta B} d^s b\int d\mu(p)p_0 e^{i\mathbf{pb}}\Delta^{(+)}(\mathbf{a}-\mathbf{b};p^2).$$

Bearing in mind that f_R was real, we find upon integration that

$$\|\alpha_{f_R}(F_+)\Omega\|^2 = \frac{1}{(2\pi)^s}\int_{\Delta B} d^s b\int d\mu(p)p_0\int\frac{d^s q}{\sqrt{p^2+\mathbf{q}^2}}|\tilde f(R\cdot\mathbf{q})|^2\cdot e^{i(\mathbf{p}-\mathbf{q})\mathbf{b}}$$

and, applying the dominated convergence theorem, we get

$$\lim_{R\to\infty} R^{s-1}\cdot\|\alpha_{f_R}(F_+)\Omega\|^2 = \frac{1}{(2\pi)^s}\int d^s b\int_{\{p^2=0\}} d\mu(p)p_0 e^{i\mathbf{pb}}\int\frac{d^s q}{|\mathbf{q}|}|\tilde f(\mathbf{q})|^2.$$

Appealing to the Jost-Lehmann-Dyson representation [25] a second time, we find that $a \to \int_{\{p^2=0\}} d\mu(p)e^{-ipa}$ is a solution of the wave equation with Cauchy data in ΔB. Hence

$$d\mu(p)|_{\{p^2=0\}} = (g(\mathbf{p}) + \frac{1}{p_0}\cdot h(\mathbf{p}))\delta(p^2)\, d^{s+1}p,$$

where g, h are smooth. We may thus replace $d\mu(p)$ in the former equation by twice its positive frequency part $d\mu^{(+)}(p)$. The ensuing equality is the desired result for Bose operators. In the case of Fermi operators this method leads to a similar result, the only difference being that commutators have to be replaced by anti-commutators.

After a moment's reflection one finds, however, that the above limit is always zero for Fermi operators. In fact a stronger result holds. Since this is of interest in the analysis of Fermionic charges (supersymmetries) we briefly indicate these improvements: let F_- be a local, selfadjoint Fermi operator. We write

$$2\cdot\|\alpha_{f_R}(F_-)\Omega\|^2 = \\ = \int d^s a\int d^s b f_R(\mathbf{a})f_R(\mathbf{b})\cdot(\Omega, (\alpha_{\mathbf{a}}(F_-)\alpha_{\mathbf{b}}(F_-) + \alpha_{\mathbf{b}}(F_-)\alpha_{\mathbf{a}}(F_-))\Omega)$$

and note that the vacuum expectation value on the right hand side vanishes for large $|\mathbf{a}-\mathbf{b}|$ because of the anti-commutativity of Fermi fields. It is then straightforward to show that

$$\lim_{R\to\infty} R^s\cdot\|\alpha_{f_R}(F_-)\Omega\|^2 = \\ = \frac{1}{2}\int d^s p|\tilde f(\mathbf{p})|^2\cdot\int d^s a(\Omega, (\alpha_{\mathbf{a}}(F_-)F_- + F_-\alpha_{\mathbf{a}}(F_-))\Omega).$$

Note that the spectral projection E_1 does not appear on the right hand side. Nevertheless, considering suitable time averages of F_-, we see that this result suffices to establish the analogue of Proposition 3.4 for graded derivations of $\mathfrak{F}$ (cf. the remarks at the end of Section 3). We refrain from presenting the tedious details.

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