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A TREATMENT OF HARD PROCESSES SENSITIVE TO THE INFRA-RED STRUCTURE OF QCD

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ABSTRACT

We propose a modified jet evolution equation which resums large corrections to the usual leading logarithmic approximation when phase space constraints expose the singular infra-red structure of QCD.

The modification, which consists simply of a rescaling of the argument of the running coupling constant, is based on perturbative arguments verified at the fourth order level.

Processes analyzed by this method include the quark (Sudakov) form factor; the large moments of structure and fragmentation functions; the asymptotic behaviour of multiplicities; the clustering of final quanta in colourless systems which occupy finite regions of (momentum and position) phase space.

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1. - INTRODUCTION AND OUTLINE OF THE PAPER.

Hard processes have been computed[1),2)] in perturbative QCD by techniques that allow a simple resummation of all corrections of the type $(\alpha_s(Q^2)\log Q^2/\Lambda^2)^n$ to the Born term. This so-called leading logarithm approximation[3)] (LLA) revealed the validity of a simple semiclassical picture[4)] (essentially that of the parton model) for describing and relating to one another hard inclusive processes. A similar result can be obtained for inclusive multiparticle correlation functions within the jet fragmentation process[5)]. The above quantities turn out to be insensitive to the infra-red (IR) structure of the theory, since the leading logs, originating from collinear singularities, have coefficients which are IR regular if real and virtual contributions are added. By the Kinoshita-Lee-Nauenberg[6)] (KLN) theorem, this cancellation is effective provided the quantity under consideration is totally inclusive in the emission of coloured quanta.

The above results are accurate up to terms of order $\alpha_s(Q^2)$, which decrease to zero for large Q^2 because of asymptotic freedom. Nonetheless, accurate computations of non leading log corrections[7)] have shown that they can be numerically large especially near the boundary of phase space. The importance of being able to take into account these large non leading terms to all orders is then obvious. Besides, one would like to know if the semiclassical parton-like picture described above is invalidated by non leading corrections.

In order to understand the origin of these large corrections, let us consider, as an example, e^+e^- annihilation into hadrons or, more generally, the problem of jet fragmentation. The degradation of invariant mass and energy from the original value $\sqrt{Q^2}$ can be followed perturbatively (in the sense specified above) down to the appearance of "final" quanta of squared mass Q_o^2 as long as

$$\frac{\alpha_s(Q_0^2)}{2\pi} \lesssim 1 \quad . \tag{1.1}$$

The fact that final quanta are out of the mass shell implies an automatic infra-red (IR) regularization in which Q_o^2 acts as the IR cut-off. Quantities which are singular in Q_o^2 are therefore sensitive to the IR structure of the theory. This is the case for several quantities of physical interest which have been recently investigated[8),9)]. Let us mention, for instance, the large multiplicity of final states, the mass spectrum of colour singlet systems of quarks and gluons (whose average mass turns out to be of the order of Q_o) or the strong damping of all processes in which real emission of quanta above Q_o^2 is inhibited : e.g., the quark (Sudakov) form factor and the behaviour of structure functions near the phase space boundary $(x \sim 1)$.

In such cases the real emission of soft coloured quanta is restricted, and is therefore unable to fully compensate the strong (reducing) effects of virtual contributions. The outcome is the appearance of large corrections of the type mentioned above, which, for instance, affects the behaviour of a structure function near $x = 1$, in the form:

$$\frac{d}{d \log Q^2} F(Q^2, x) \simeq P(x)\alpha_s(Q^2)\left(1 + \alpha_s(Q^2)\log(1-x) + \dots\right) \tag{1.2}$$

and must be resummed if

$$\alpha_s(Q^2) \log(1-x) \sim O(1) \quad . \tag{1.3}$$

This indeed happens near the phase space boundary

$$x \sim 1 - Q_0^2/Q^2 \quad . \tag{1.4}$$

A simplified treatment of some of these IR sensitive quantities has been recently given in refs. 8) and 9). When applied to the Sudakov form factors, however, such treatment gives results at variance with those known from more rigorous methods[10),11)]. In this paper we wish to present a unified and more refined approach which is summarized by a simple modification of the Altarelli-Parisi (AP) equation[4)] (and consequently of the jet calculus rules[5)]) and which preserves the probabilistic partonic interpretation mentioned before.

We warn the reader that the proof that the modified AP evolution equation resums correctly all IR singular contributions is actually incomplete, i.e. only checked at the leading IR singularity level. Our conjecture that this is the correct equation is nevertheless comforted by the fact that we have found agreement with other approaches, whenever comparable[10)-12)].

Our improved treatment confirms the physical picture stemming from the more naive approach[8),9)]. It actually enhances the strong damping of the (semi) exclusive quantities described above but, as we shall discuss later, the actual form of the tails of these strongly damped (i.e. faster than power) exclusive distributions have probably little physical content. Indeed, when leading logarithms are subject to such a strong cancellation, it becomes possible to identify other contributions (e.g. some which are down by powers) that will become dominant (see Section 6).

To be more specific, both for the Sudakov form factor F_q and for the colour singlet mass spectrum[(*)] we shall find expansions of the type

$$-\log F_q = \log Q^2/Q_o^2 \sum_{\ell=0}^{\infty} c_\ell \left(\alpha_s \log Q^2/Q_o^2\right)^{\ell+1} \left(1+O(\alpha_s)\right), \tag{1.5}$$

where the above series is resummed by our modified evolution equation.

Similarly, for the moments of structure and fragmentation functions ($F_n(Q,Q_o^2)$, $D_n(Q,Q_o^2)$ at large n (i.e. $x \to 1$) we shall obtain expressions of the type:

$$-\log F_n(Q^2,Q_o^2) \sim -\log D_n(Q^2,Q_o^2) =$$
$$= \sum_{\ell,m=1} c_{\ell,m}\, \alpha_s^{\ell+m-1} (\log n)^\ell \left(\log Q^2/Q_o^2\right)^m \left(1+O(\alpha_s)\right) \tag{1.6}$$

and we shall be able to resum the series to a simple function of Q^2 and n, which approaches the Sudakov form factor (1.5) for $n \simeq Q^2/Q_o^2$.

The outline of the paper is the following.

In Section 2, starting from the Dyson equation, we obtain an expression for the quark propagator in a class of axial gauges.

This result is used in Section 3 for deriving the modified AP equation for ordinary non singlet structure and fragmentation

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(*) A behaviour related to eq.(1.5) is also obtained for the colour-connecting distributions introduced in ref. 9).

functions. From this we obtain the asymptotic form of the Sudakov form factor and of $F_n^{NS}(Q^2,Q_0^2)$, $D_n^{NS}(Q^2,Q_0^2)$. (A few technical points, as well as the proof of gauge independence of the form factor are presented in the Appendices).

In Section 4 the preceding analysis is extended to the singlet case and its implications for multiplicities are worked out.

In Section 5 we turn to the colour structure of jets, by giving a modified equation for the generating function of multiparton spectra. We confirm that final states can be arranged in colourless clusters with a mass spectrum showing a damping as strong as the one of the quark form factor. We also show that quanta belonging to the same colourless cluster have a small relative momentum .

These strongly damped contributions are compared with higher twist effects in Section 6, where some final comments are also made.

In Appendix A we discuss at the two loop level the cancellation of the leading IR singularities needed in order to arrive at the modified AP equation.

In Appendix B we check the gauge independence of the Sudakov form factor.

2.- THE QUARK PROPAGATOR AND ITS DYSON EQUATION IN AXIAL GAUGE.

As stated in the introduction, we shall start by computing, at the leading log level, the fermion propagator $S(k)$ (or rather its logarithm) in a class of axial gauges.

Besides being a necessary ingredient for constructing evolution equations of the Altarelli-Parisi type, this calculation also illustrates the origin of the effects we are looking for as well as the method and the approximations used to find them.

We shall work in a set of axial gauges defined by

$$A \cdot \eta = 0 \tag{2.1}$$

with η a vector in the t-z plane parametrized as $\eta = (a+b,0,0,a-b)$, with $\eta^2 = 4ab \ll 1$.

The Dyson-type equation for $\mathrm{Disc}_{k^2}\left[S^{-1}(k)\right]$ (equivalent to a flavour conservation sum rule) is shown in Fig.1 together with the skeleton expansion of the 2-particle irreducible flavour non singlet kernel K.

As is well known[1),2)], the leading log mass singularities are fully given, in the gauges (2.1), by the first term in Fig.1b which generates dressed rainbow diagrams. The fact that vertices and propagators appearing in Fig.1 are renormalized is what makes the running coupling constant

$$\begin{aligned} &\alpha_s(k^2) \simeq \left(b \log k^2/\Lambda^2\right)^{-1} , \\ &b = (11 N_c - 2 N_f)/12\pi , \quad \Lambda \sim 500 \text{ MeV}, \end{aligned} \tag{2.2}$$

appear for each rung of the rainbow diagrams.

We shall now repeat the usual analysis[1)-3)], with some more care, in order to find which is the variable appearing in the running coupling constant (2.2) when the fraction z of the momentum carried by the cut quark line of Fig.1a approaches one. We show that this variable takes indeed a value given by the kinematical upper limit on the emitted gluon mass which behaves as 1-z as $z \to 1$. Following the reasoning of ref.1),this suggests that the correct variable to appear in α_s is $k^2(1-z)$. In order to prove this statement, one should check that no other 1-z dependence is generated by gauge dependent terms in vertices and propagators. Such terms enter in the form of logarithms of $(k\cdot\eta)^2/k^2\eta^2$ which have an infra-red origin (they diverge for $k^2 \to 0$ at fixed $k_{\parallel} \sim k\cdot\eta$). In order to prove our assertion on the argument of α_s one should then check the complete cancellation of all these IR singularities in the appropriate diagrams. We shall be able to check this cancellation explicitly only at the leading IR level ($\alpha_s \log^2$ terms), but we conjecture that this will continue to be true for less singular infra-red divergences (single logarithms). An actual complete check of this conjecture would necessitate the computation of the non rainbow diagrams generated by the crossed kernel in Fig.1b, as well as some collinear-non leading terms of the ladder diagrams.

Let us restrict ourselves for the moment to the planar piece of the kernel (an elegant way to dispose of the non planar piece would be to take the large N limit). Then, the r.h.s. of the equation, represented in Fig.1a, can be just written in terms of the quark and gluon dressed propagators S and $D_{\mu\nu}$ and of the quark gluon vertex Γ_μ as

$$\mathrm{Disc}_{k''^2}\left[\Gamma_\mu(k,k',k'')\left(\mathrm{Disc}_{k'^2}S(k')\right)D_{\mu\nu}(k'')\,\Gamma_\nu(k,k',k'')\right]. \tag{2.3}$$

We now define

$$S(k) = \left[\not k + \Sigma(k)\right]^{-1} = d_q(k^2, k_\parallel)\,\frac{\not k}{k^2} + \hat{d}_q\,\frac{\not\eta}{\eta k}\ ,$$

$$D_{\mu\nu}(k) \doteq \frac{d_g(k^2, k_\parallel)}{k^2}\left(g_{\mu\nu} - \frac{k_\mu\eta_\nu + k_\nu\eta_\mu}{k\eta} + \eta^2\frac{k_\mu k_\nu}{(k\eta)^2}\right) + \cdots ,$$

$$\Gamma_\mu = \gamma_\mu\,\Gamma_{qqg}(k_i^2, k_{i_\parallel}) + \ldots \quad ; \quad k_\parallel^2 = \frac{(k\cdot\eta)^2}{\eta^2}\ , \tag{2.4}$$

where we have set to zero the mass of the quark since we work off-shell. In computing the r.h.s. of the Dyson equation (Fig.1a), the $\hat{d}_q$ term in S(k) as well as other contributions to $D_{\mu\nu}$ and Γ_μ indicated by dots in eq.(2.4) do not contribute to the LLA since they do not keep the propagator poles needed to generate leading logs.

We thus find the simplified Dyson equation:

$$\frac{\mathrm{Disc}_{k^2} S^{-1}(k)}{2\pi i} = \frac{\mathrm{Disc}_{k^2}\Sigma(k)}{2\pi i} \simeq$$

$$\simeq C_F\, g^2 \int \frac{d^4k'}{(2\pi)^4 i}\ \frac{1}{2\pi i}\left[\mathrm{Disc}_{k'^2}\,\frac{d_q(k')}{k'^2}\right] \cdot$$

$$\cdot\left[\mathrm{Disc}_{k''^2}\left(\Gamma^2_{qqg}(k_i^2, k_{i_\parallel})\,\frac{d_g(k'')}{k''^2}\right)\right] \cdot \tag{2.5}$$

$$\cdot\left[-2\,\frac{(k'\cdot k'')\not\eta + (k'\cdot\eta)\not k''}{k''\eta} + O(\eta^2)\right]\ ,$$

where we have just indicated terms of $O(\eta^2)$ which become irrelevant in the limit $\eta^2 \to 0$. Dropping terms in the bracket proportional to k'^2, k''^2 which do not contribute to leading logs (strong ordering approximation), we can decompose k" as follows:

$$K'' = (1-z)K - \left(\tfrac{1}{2} - z\right) K^2 \frac{\eta}{K\cdot\eta} + O(\eta^2) + K''_T \,,$$
$$z = \frac{K'_0 + K'_3}{K_0 + K_3} \,, \quad K''_T \cdot \eta = K''_T \cdot K = 0 \,, \tag{2.6}$$

so that the square bracket term in (2.5) can be rewritten as

$$\left[2zK + K^2 \eta \frac{1+2z^2-z}{K''\cdot\eta} + \text{transverse terms} + O(\eta^2) \right] . \tag{2.7}$$

We can now write

$$d^4 k' = \frac{\pi}{2} \, dk'^2 \, dk''^2 \, dz \; \Theta\left(k^2 - \frac{k'^2}{z} - \frac{k''^2}{1-z} \right) . \tag{2.8}$$

In order to perform the dk''^2 and the dk'^2 integrations, let us state a general formula we shall exploit also later.

Let us say that we are interested in the leading log q^2 behaviour of

$$\int^{q^2} dq'^2 \, \frac{\mathrm{Disc}}{2\pi i} \left[\frac{f(q'^2)}{q'^2} \right] = \frac{1}{2\pi i} \int_{C(q^2)} \frac{dq'^2}{q'^2} f(q'^2) , \tag{2.9}$$

where $f(q'^2)$ is a function that has $\left[\log(-q'^2)\right]^n$ type singularities. The $C(q^2)$ contour is depicted in Fig.2. The phase dependence of $f(q^2e^{i\phi})$ on $C(q^2)$ is then a non leading $\log q^2$ effect(*) and therefore

$$\int^{q^2} dq'^2 \, \frac{\mathrm{Disc}}{2\pi i}\left[\frac{f(q'^2)}{q'^2}\right] = f(-q^2)\,\frac{1}{2\pi i}\int_{C(q^2)} \frac{dq'^2}{q'^2} = f(-q^2)\,, \tag{2.10}$$

or, equivalently,

$$\frac{d}{dq^2}\left[f(-q^2)\,\theta(q^2)\right] \simeq \frac{\mathrm{Disc}\, f(q^2)}{2\pi i\; q^2} \quad . \tag{2.11}$$

Moreover, if $g(q'^2)$ is a function which is regular inside $C(q^2)$, then analogously, we obtain

$$\int^{q^2} dq'^2 g(q'^2)\, \frac{\mathrm{Disc}}{2\pi i}\left[\frac{f(q'^2)}{q'^2}\right] \simeq f(-q^2)\int_{C(q^2)} \frac{dq'^2}{q'^2}\,\frac{g(q'^2)}{2\pi i} = f(-q^2)\,g(0). \tag{2.12}$$

By using (2.12) we start now integrating eq.(2.5) over k''^2. Its maximum value is

$$k''^2_{max} = (1-z)\left(k^2 - \frac{k'^2}{z}\right) \quad . \tag{2.13}$$

All functions of k''^2 have only logarithmic singularities and we use

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(*) Correction terms are down by two powers of log, because $\int_{-\pi}^{\pi} \varphi\, d\varphi = 0.$

therefore eqs.(2.10),(2.11). We obtain, at the leading log level,

$$\frac{\mathrm{Disc}_{k^2}\, S^{-1}(k)}{2\pi i} = \frac{\mathrm{Disc}_{k^2}\, \Sigma(k)}{2\pi i} \simeq$$

$$\simeq -\frac{\pi C_F g^2}{2(2\pi)^3}\int dz \int^{zk^2} dk'^2 \left\{ 2z\not{k} + k^2 \not{\eta}\, \frac{1+2z^2-z}{k^{\parallel}\cdot\eta} \right\}\cdot$$

$$\Gamma^2_{qqg}\left(k^2, k'^2, -(1-z)\left(k^2-\frac{k'^2}{z}\right); k_\parallel, z k_\parallel, (1-z) k_\parallel\right)\cdot \tag{2.14}$$

$$\cdot\, d_g\left(-(1-z)\left(k^2-\frac{k'^2}{z}\right)\right) \qquad \frac{\mathrm{Disc}_{k'^2}}{2\pi i}\left[\frac{d_q(k')}{k'^2}\right] \; .$$

We now notice that the singularity of $d_g(-(1-z)(k^2-\frac{k'^2}{z}))$ lies outside the circle $C(zk^2)$ (cf. Fig.3) and that Γ^2 in eq.(2.14) has no leading singularity in k'^2 in the whole region of interest. This second statement stems from the fact (cf. Appendix A) that the effective vertex function evaluated in a "collinear" configuration (i.e., $k_\perp = 0$ if k^2 is replaced by $-k^2$).

$$G^{coll}(k^2, z, k_\parallel) = \frac{g^2}{4\pi}\, d_q(-k^2, k_\parallel)\, d_q(-zk^2, zk_\parallel)\,\cdot$$

$$\cdot\, d_g(-(1-z)k^2, (1-z)k_\parallel)\, \Gamma^2_{qqg}\left(k^2, -k^2\xi z, -k^2(1-\xi)(1-z); k_\parallel, zk_\parallel, (1-z)k_\parallel\right) \tag{2.15}$$

has no leading singularity in the variable $\xi = k'^2/zk^2$.

Then eq.(2.12) can be applied in performing the k'^2 integral. This implies setting $k'^2=0$ in the arguments of $\Gamma^2_{qqg}\, d_g$ and $k'^2 = k^2 z$ (i.e.: its kinematical limit) in d_q. We then find that vertices and propagators appear in the combination

$$\frac{g^2}{4\pi}\,\Gamma^2_{qqg}\left(k^2, 0, -k^2(1-z)\,;\,k_{i_{/\!/}}\right)\, d_g\left(-k^2(1-z)\right) d_q\left(-k^2 z\right) =$$

$$= d_q^{-1}\left(-k^2\right)\, G^{coll}\Big|_{\xi=0} \,. \tag{2.16}$$

We will now argue that it is legitimate to replace $G^{coll}\Big|_{\xi=0}$ in eq.(2.16) by $\alpha_s(k^2(1-z))$. Let us first notice that for fixed z the kernel depicted in Fig.1b is the same as for the gauge invariant non singlet fragmentation function (cf.Section 3). Then, its complete expression cannot depend on the gauge vector η, and therefore on the parameters $(K_i\eta)^2/K_i^2\eta^2 = K_{i_{/\!/}}^2/K_i^2$. This means that the $k_{/\!/}$ dependent singularities of the various contributions should eventually cancel . The remaining singular mass dependence has then an ultraviolet origin, and therefore can be computed in the kinematical region $k_i^2 \gg k_{/\!/\,i}^{\;2}$. In this regime neglecting the crossed diagrams is justified in LLA, and G^{coll} in eq.(2.15), thanks to the Ward identities[1),2)] is just $\alpha_s(k^2(1-z))$.

Let us now discuss the actual cancellation of IR divergences. The leading ones (of $\alpha_s \log^2\left[k^2\eta^2/(k\eta)^2\right]$ type) do not appear in the (neglected) crossed diagrams of Fig.1b and therefore should cancel in the expression (2.15) of G^{coll}. In the Appendix A we show that this is the case at the one loop level. Non leading IR singularities ($\alpha_s \log k^2/k_{/\!/}^2$) must cancel in a much more complicated way that we have not checked explicitly. Indeed, besides G^{coll}, the cancellation must involve crossed diagrams we have neglected in the kernel as well as non strong-ordered iterations of the ladder. A check of this cancellation is already the absence of α^2 $(\log n)^2$ terms in the two-loop anomalous dimensions[13]. Moreover, a complete 4^{th} order calculation of the NS structure and fragmentation functions has been recently performed[14] (Fig. 4). All singularities for $z \to 1$ of the effective kernel thus found coincide to this order with the expansion of the simple expression $\alpha_s(k^2(1-z))P(z)$ that we will obtain.

The final equation for the quark propagator is derived from eq.(2.14) by inserting the decomposition

$$\Sigma(k) = A\not{k} + B\frac{k^2}{k\cdot\eta}\not{\eta} \quad , \tag{2.17}$$

$$d_q(k^2) = (1+A+2B)^{-1} \quad , \quad \hat{d}_q \simeq d_q \frac{B}{1+A} \quad , \quad (\eta^2 \ll 1) \tag{2.18}$$

and using the identification of (2.15) with $\alpha_s(k^2(1-z))$. One finds

$$-\left(\not{k}\,\frac{\mathrm{Disc}\,A}{2\pi i} + \frac{k^2\not{\eta}}{k\eta}\,\frac{\mathrm{Disc}\,B}{2\pi i}\right) = \frac{c_F}{2\pi}\, d_q^{-1}(-k^2)\,\cdot$$
$$\cdot\int_0^1 dz\left(2z\not{k} + \frac{k^2\not{\eta}}{k\eta}\,\frac{1+2z^2-z}{1-z+\epsilon(k_{/\!/})}\right)\alpha_s\left(k^2(1-z)\right) \tag{2.19}$$

where $\epsilon(k_{/\!/}) = k^2/4k_{/\!/}^2 = k^2\eta^2/4(k\cdot\eta)^2 \ll 1$ and therefore

$$-d_q(k^2)\,\frac{\mathrm{Disc}\,d_q^{-1}(k^2)}{2\pi i} \simeq \int_0^1 dz\, P_q^{qg}(z,\epsilon)\,\frac{\alpha_s(k^2(1-z))}{2\pi} = -\frac{d}{d\log k^2}\log d_q^{-1}(-k^2) \tag{2.20}$$

where

$$P_q^{qg}(z,\epsilon) = C_F\,\frac{1+z^2}{1-z+\epsilon(k_{/\!/})} \tag{2.21}$$

and use has been made of eq.(2.11).

We obtain thus the result

$$d_q(-k^2, k_{/\!/}^2) \simeq \exp\left[-\int_{k^2}^{\mu^2} \frac{dk'^2}{k'^2} \int_0^1 dz\, P_q^{qq}(z,\epsilon)\, \frac{\alpha_s(k'^2(1-z))}{2\pi} \right], \tag{2.22}$$

where μ^2 is the usual subtraction point, here to be taken of order $k_{/\!/}^2 >> k^2$ in order for eq.(2.11) to be valid.

The relation of this result to the (gauge invariant) Sudakov form factor will be discussed in the next section. We want, however, to stress immediately that, by expanding $\alpha_s(k'^2(1-z))$ of eq.(2.22) in powers of $\alpha_s(k'^2)\log(1-z)$, a series of the type discussed in the introduction (eq.(1.5)) is obtained. The evolution equation (2.20) can be seen as a simple way to resum that series of large logarithms.

3.- NON SINGLET EVOLUTION EQUATION AND SUDAKOV FORM FACTOR.

The method of embodying IR divergences in the invariant charge can be easily extended to the parton fragmentation functions and, with some modifications, to the jet calculus .

Let us first consider the flavour non singlet fragmentation function $D^{NS}(Q^2,x)$. Let us recall that $D^{NS}(Q^2,x)$ is related in a standard way to the spin average of a $q\bar{q}$ forward absorptive part, in which one of the external masses is fixed at Q_o^2 and the other is integrated between Q_o^2 and Q^2. It then follows that D^{NS} satisfies the Bethe-Salpeter equation represented in Fig.5 where the kernel is the one of Fig.1b.

We parallel the treatment of Section 2 by neglecting the crossed diagrams of Fig.1b which do not contribute to the leading logarithms ($\alpha_s \log^2 q^2$). As before their contribution is important in order to restore at the single log level the gauge independence of the effective coupling constant.

The Bethe-Salpeter equation for D^{NS} then reads

$$\kappa^2 \frac{d}{d\kappa^2}\left(d_q D^{NS}(\kappa^2, x)\right) = \kappa^2 d_q(-Q_0^2)\, \delta(\kappa^2 - Q_0^2)\, \delta(x-1) +$$

$$+ d_q^2(-\kappa^2) \frac{g^2}{4\pi} \int d\kappa'^2 d\kappa''^2 \frac{dz}{z} P_q^{qg}(z, \epsilon) \cdot$$

$$\cdot \frac{\mathrm{Disc}_{\kappa''^2}}{2\pi i}\left[\Gamma_{qqg}^2(\kappa^2, \kappa'^2, \kappa''^2) \frac{d_g(\kappa''^2)}{\kappa''^2}\right] \frac{d}{d\kappa'^2}\left[d_q D^{NS}(\kappa'^2, \frac{x}{z})\right]. \tag{3.1}$$

The k'^2, k''^2 integrations can now be performed as in Section 2, and we similarly obtain the invariant charge (2.15) and therefore the running coupling constant $\alpha_s(k^2(1-z))$.

Equation (3.1) can then be rewritten as

$$\frac{1}{d_q(-\kappa^2)} \frac{d}{d\log\kappa^2}\left(d_q(-\kappa^2) D^{NS}(\kappa^2, x)\right) =$$

$$= \int_x^1 \frac{dz}{z} \frac{\alpha_s(\kappa^2(1-z))}{2\pi} P_q^{qg}(z, \epsilon(\kappa_\parallel))\, D^{NS}(z\kappa^2, \frac{x}{z}) \tag{3.2}$$

where P_q^{qg} is given in eq.(2.21). By finally using the virtual contribution (2.20) in the l.h.s. of (3.2) we can define the generalized Altarelli-Parisi densities

$$\Pi_q^q(k^2,z) = \alpha_s\left(k^2(1-z)\right) P_q^{qg}(z,\epsilon(k_{\parallel})) - \delta(z-1)\ \cdot$$

$$\cdot \int_0^1 dx\, \alpha_s\left(k^2(1-x)\right) P_q^{qg}(x,\epsilon(k_{\parallel})) \quad , \tag{3.3}$$

$$\tilde{\Pi}_q^q(n,k^2) = \int_0^1 dz\ P_q^{qg}(z,\epsilon(k_{\parallel}))\, \alpha_s\left(k^2(1-z)\right)\left(z^n-1\right)$$

with the resulting modified AP equation:

$$\frac{d}{d\log k^2} D^{NS}(k^2,x) = \frac{1}{2\pi}\int_x^1 \frac{dz}{z}\, \Pi_q^q(k^2,z)\, D^{NS}\left(zk^2,\frac{x}{z}\right). \tag{3.4}$$

Since for the non singlet case the kernel is non singular at $z = 0$, one can neglect the rescaling $k^2 \to k^2 z$ in eq.(3.4) which would only contribute non leading terms. As a result, eq.(3.4) can be diagonalized by going to the usual moments and gives

$$D_n^{NS}(Q^2,Q_0^2) = \exp\left(\frac{1}{2\pi}\int_{Q_0^2}^{Q^2}\frac{dk^2}{k^2}\,\tilde{\Pi}_q^q(n,k^2)\right) . \tag{3.5}$$

The cancellation of IR divergences $(z \to 1)$ is apparent in (3.3), because -- for n finite -- $\Pi(n)$ is regular for $\varepsilon \to 0$. The usual AP density $\left[P(z)\right]_+$, which is k^2 independent, is replaced however by $\left[\alpha_s\left(k^2(1-z)\right)\ P(z)\right]_+$. The difference is of course non leading for finite n but becomes relevant for $\alpha_s(k^2)\log n \sim O(1)$, the case we are interested in .

For $1 \ll n < Q^2/Q_o^2$ we have, in the light-cone gauge(*) ($\varepsilon(k_{/\!/})=0$)

$$\tilde{\Pi}_q^q(n,Q^2) = -\frac{C_F}{\pi}\int_0^{1-1/n}\frac{dz}{1-z}\,\alpha_s(Q^2(1-z)) + O(\alpha_s(Q^2)) =$$

$$= \frac{C_F}{\pi b}\log\left(1-\frac{\log n}{\log Q^2/\Lambda^2}\right) + O(\alpha_s(Q^2)) \qquad (3.6)$$

and from (3.5)

$$D_n^{NS}(Q^2,Q_o^2) \simeq \exp\left(-\frac{C_F}{\pi b}\log n \log\frac{\alpha_s(Q_o^2)}{\alpha_s(Q^2)}\right) \cdot$$

$$\cdot \exp\left[-\frac{C_F}{\pi b}\left(-\log n + \log\frac{Q^2}{n\Lambda^2}\log\frac{\alpha_s(\frac{Q^2}{n})}{\alpha_s(Q^2)}\right)\right] , \quad \left(1 \ll n < \frac{Q^2}{Q_o^2}\right). \qquad (3.7)$$

We can see that the naive large n anomalous dimension contribution (first exponential in (3.7)) is modified by a coefficient C_n^{NS} which becomes important for n large. In fact, for $\alpha_s(Q^2)\,\ell og\, n \ll 1$, we have

$$C_n^{NS}(\alpha_s(Q^2)) \simeq \exp\left[\frac{C_F}{2\pi}\alpha_s(Q^2)(\log n)^2\right] \qquad (3.8)$$

which is just the exponentiation of the perturbative result[7].

(*) The region $n \sim Q^2/Q_o^2$ is given by (3.6) also in gauges for which $\varepsilon^{-1}=k_{/\!/}^2/Q^2 > Q^2/Q_o^2$. For $\varepsilon^{-1} < Q^2/Q_o^2$ the result becomes gauge dependent . The definition of gauge invariant quantities, which in this case necessarily involve both q and $\bar{q}$ jets, is discussed below and in Appendix B.

In the region $n > Q^2/Q_0^2$, roughly related to $1-z < Q_0^2/Q^2$, eq(3.6) is unwarranted due to the fact that the integrand is sensitive to regions in which α_s is large. The reason for this difficulty is that for $x \to 1$ $D^{NS}(x)$ becomes essentially of exclusive type (i.e. even soft gluons cannot be emitted) while physically significant questions should always be inclusive in soft gluons. In fact, besides the usual energy resolution arguments, we must remember that the identification of a nearly zero mass gluon has no meaning in a confining theory. This proviso must therefore be included also in the definition of "exclusive" quantities as the quark (Sudakov) form factor.

In order to understand this point, let us compute the probability $\Delta_q(k^2, Q_0^2)$ that an initial quark with mass k^2 emits only soft gluons in its evolution down to Q_0^2. Soft quanta are defined as those emitted with a fraction of momentum $(1-z) < Q_0^2/k^2$, where Q_0^2 is the scale for which (1.1) holds. In the strong ordering configuration this includes systems of gluons with masses up to Q_0^2.

Recalling that the emission of an off-shell gluon implies a summation over all final states it can give rise to, we see that the soft gluon emission so defined implies being inclusive in all hadronic states with square momentum smaller than Q_0^2.

The differential probability of soft gluon emission can then be obtained from (3.3) as

$$\frac{d \log \Delta_q}{d \log k^2} = - V_q\left(k^2, Q_0^2, \epsilon(k)\right) = \int_{1-\frac{Q_0^2}{k^2}}^{1} \Pi_q^q\left(k^2, z, \epsilon(k_{\parallel})\right) \frac{dz}{2\pi} =$$

$$= - \int_0^{1-\frac{Q_0^2}{k^2}} \frac{dz}{2\pi} \alpha_s\left(k^2(1-z)\right) P_q^{qq}\left(z, \epsilon(k_{\parallel})\right) = - \int_0^{1-\epsilon(k)} \frac{dz}{2\pi} C_F \frac{1+z^2}{1-z} \alpha_s\left(k^2(1-z)\right) \tag{3.9}$$

where

$$\epsilon(k) = \mathrm{Max}\left(\epsilon(k_{/\!/}),\ \frac{Q_o^2}{k^2}\right),\quad \epsilon(k_{/\!/}) = \frac{k^2}{4k_{/\!/}^2}\ . \tag{3.10}$$

In the light-cone gauge ($k_{/\!/}\to\infty$, $\varepsilon(k) = Q_o^2/k^2$) we then obtain

$$\Delta_q^{LC}(k^2, Q_o^2) = \exp\left[-\int_{Q_o^2}^{k^2}\frac{dk'^2}{k'^2}V_q\left(k'^2, Q_o^2; \frac{Q_o^2}{k'^2}\right)\right]\simeq$$

$$\simeq \exp\left[-\frac{C_F}{\pi b}\left(\log\frac{k^2}{\Lambda^2}\log\frac{\alpha_s(Q_o^2)}{\alpha_s(k^2)} - \log\frac{k^2}{Q_o^2}\right)\right]. \tag{3.11}$$

This quantity is related to the square of the Sudakov form factor provided we define $F_q^2(Q^2,Q_o^2)$ as the probability that a photon of squared momentum Q^2 produces a single energetic $q\bar{q}$ pair ($x_q \sim x_{\bar{q}} \sim 1$) plus an arbitrary number of soft gluons.

In fact, one finds that, in the $k_{/\!/}=\infty$ frame, only the quark or the antiquark (whichever is opposite to the gauge vector η) emits quanta and has, therefore, a non trivial evolution. Hence, we have

$$F_q^2(Q^2, Q_o^2) = \Delta_q^{LC}(Q^2, Q_o^2)\times 1\ . \tag{3.12}$$

In a general gauge, both q and $\bar{q}$ emit quanta. We show in Appendix B how combining both probabilities Δ_q and $\Delta_{\bar{q}}$ the result for F_q^2 becomes gauge (η) independent.

By comparing (3.7) with (3.11) we conclude that the quark fragmentation function in the light-cone gauge reduces to the (gauge invariant) quantity $F_q^2(Q^2,Q_o^2)$ for $n \simeq Q^2/Q_o^2$. Moreover, this discussion shows that the moments (3.3) of the generalized A.P.

densities

$$\widetilde{\Pi}_q^q(n,k^2) = \left(\int_0^{1-\frac{Q_0^2}{k^2}} dz + \int_{1-\frac{Q_0^2}{k^2}}^1 dz \right) \alpha_s\, P_q^{qq}(z,\epsilon(k_{\parallel}))\,(z^n-1) \tag{3.13}$$

consist of two parts, corresponding to the two integration intervals in (3.13). Only the first part is reliably computed in the perturbative approach. The second part is negligible for $n \lesssim Q^2/Q_0^2$ $(1-x > Q_0^2/Q^2)$ but would give a gauge dependent contribution for $n \gg Q^2/Q_0^2$ if the perturbative expression of $\alpha_s P_q^{qq}$ is extrapolated to this region where the gluon concept loses its meaning.

On the other hand, confinement is expected to generate hadronic masses, which automatically regularize the $z \to 1$ singularity of the perturbative $\alpha_s P_q^{qq}$. Therefore, Q_0 representing here a typical hadronic scale, the region $(1-Q_0^2/k^2, 1)$ will give a negligible contribution to all physical quantities, or in other words, virtual and real gluons compensate in this region because of hadronization. The fact already mentioned that the region $1-x < Q_0^2/Q^2$ is rather unphysical makes this assumption irrelevant for the single inclusive distribution. However, it is important for the discussion of the full generating function of Section 4, because it provides the correct normalization condition.

The resulting expression for the moments becomes

$$\widetilde{\Pi}_q^q(n,k^2,\epsilon(k)) \simeq \int_0^{1-Q_0^2/k^2} dz\ \alpha_s(k^2(1-z))\, P_q^{qq}(z,\epsilon(k_{\parallel}))\,(z^n-1) \simeq$$

$$\simeq \int_0^{1-\epsilon(k)} dz\, \alpha_s(k^2(1-z))\, P_q^{qq}(z)\,(z^n-1) \quad , \tag{3.14}$$

where $\varepsilon(k)$ defined in (3.10) summarizes the gauge dependent and phase space cutoffs. The corresponding fragmentation function D_n can be evaluated from (3.14) and (3.5) also for $n > Q^2/Q_o^2$ where it becomes roughly n-independent, and is still given by (3.11). Therefore, we have, in the light-cone gauge

$$D_n(Q^2, Q_o^2) = \begin{cases} F_q^2(Q^2, Q_o^2) \,/\, F_q^2\left(\frac{Q^2}{n}, Q_o^2\right), & n \lesssim \frac{Q^2}{Q_o^2}, \\ F_q^2(Q^2, Q_o^2), & n > \frac{Q^2}{Q_o^2}. \end{cases} \tag{3.15}$$

Note that this expression is of the form $F^2(Q^2, Q_o^2) \cdot f(Q^2/n)$ in agreement with previous arguments[15].

To summarize, the modification of the AP equation we have proposed (eqs. (3.4), (3.6), (3.8)) implies:

i) No change in the moments of structure and fragmentation functions for n finite (non zero) and not too large ($\alpha_s \log n < 1$)

ii) Eq. (3.7) for $1 \ll n < Q^2/Q_o{}^2$, in agreement with other results and conjectures[12].

iii) The Sudakov form factor limit for $n \gtrsim Q^2/Q_o^2$, with a form factor falling faster than any power, in agreement with other analyses[11].

In this section we explicitly dealt with the timelike kinematics characteristic of jet fragmentation. For structure functions the appropriate spacelike kinematics makes the evolution go towards larger $|q^2|$ while the momentum is decreased due to bremsstrahlung. This is the reason why the evolution equation for F^{NS} differs from eq. (3.4) for D^{NS} in that F appears in the r.h.s. with argument k^2/z instead of zk^2. Besides, also the rescaling in the argument of α_s is obtained from that of Section 3 by the replacement $z \to 1/z$. All this is expected from general arguments of analytic continuation[16].

As a consequence, for finite or large n, $F_n(Q^2,Q_0^2)$ and $D_n(Q^2,Q_0^2)$ have the same form at the leading log level discussed here.

For small n, instead, F and D have a very distinct behaviour, as expected from the fact that their physical meaning is also different. Indeed, as we shall see in the next section, $D_{n\to 0}(Q^2,Q_0^2)$ is related to total jet multiplicities and is computable, while $F_{n\to 0}(Q^2,Q_0^2)$ is related to the Regge limit of deep inelastic scattering where the perturbation techniques based on strong ordering become invalid.

Finally, in the case of lepton pair production, the different dependence[12] of k^2_{Max} on the Drell-Yan variable τ has to be taken into account. For large n_{DY} (variable conjugated to τ) we find again Eq. (3.15) with n^2_{DY} replacing n, in agreement with other authors[7),12)].

4. GENERALIZATION TO SINGLET DISTRIBUTIONS AND MULTIPLICITIES

The previous analysis can be extended to the full jet evolution, including flavour singlets and colour structure, on the basis of similar cancellations of IR logs, as in the anomalous dimension case[13].

The most important differences in this case are the occurence of the 3g vertex function, and of gluons of small z's as intermediate (non emitted) quanta. This means that we have also to evaluate the combination

$$\Gamma^2_{qgq}\left(k^2, -\xi z k^2, -(1-\xi)(1-z)k^2\right) d_q(k^2)\, d_g(-zk^2)\, d_q(-(1-z)k^2) \tag{4.1}$$

for small z, and of

$$\Gamma^2_{ggg}\left(k^2, -\xi z k^2, -(1-\xi)(1-z)k^2\right) d_g(k^2)\, d_g(-zk^2)\, d_g\left(-(1-z)k^2\right) \tag{4.2}$$

for either $z \to 0$ or $z \to 1$.

The proof of the absence of $(\log)^2$ terms proceeds as before (Appendix A), and by the same arguments we are led to identify (4.1) and (4.2) with their U.V.behaviour. Thanks to the Ward identities relating Γ_{qgq} to d_q^{-1} and Γ_{ggg} to d_g^{-1} in the UV region, we are entitled to replace (4.1) by $\alpha(zk^2)$ and (4.2) by $\alpha(zk^2)\ \alpha((1-z)k^2)/\alpha(k^2) \simeq \alpha(z(1-z)k^2)$. This means that in any case what matters is the smallest gluon mass, or equivalently $k_\perp^2 \simeq z(1-z)k^2$.

By repeating the analysis of Section 2 for the gluon propagator we then obtain

$$d_g(-k^2, k_\parallel^2) \simeq \exp\left[-\int_{k^2}^{\mu^2} \frac{dk'^2}{k'^2} \int_0^1 dz\ P_g^V(z,\epsilon)\, \alpha_s\left(z(1-z)k'^2\right)\right], \tag{4.3}$$

where the virtual gluon probability density is given in terms of the real emission densities as

$$P_g^V = \frac{1}{2}\, P_g^{gg}(z,\epsilon) + P_g^{q\bar{q}}(z)\ . \tag{4.4}$$

Eqs.(4.3) and (2.22) then represent the solution of the Dyson equations for d_g, d_q in the axial gauge which resum the leading $\log k^2$ singularities. This problem has also been investigated by other authors[17)] in order to determine the long distance behaviour of the theory. In our opinion, however, the singular dependence on $k_\parallel^2/k^2$ induced by the gauge dependence has no physical relevance and disappears when "physical" (i.e. gauge invariant) quantities are computed.

We have shown this explicitly in Section 3 and in Appendix B in computing the quark (Sudakov) form factor.

Let us now consider the singlet fragmentation functions. The generalized A P densities for hard gluon emission -i.e., corresponding to eq. (3.14)- become

$$\Pi_a^b\left(k^2, z, \epsilon(k)\right) = \alpha_s\left(z(1-z)k^2\right) P_a^{bb'}\left(z, \epsilon(k_\nu)\right) \theta\left(1-z-\frac{Q_0^2}{k^2}\right) \cdot \theta\left(z-\frac{Q_0^2}{k^2}\right) - \delta(z-1)\, \delta_a^b \int_{\epsilon(k)}^{1-\epsilon(k)} dx\, \alpha_s\left(x(1-x)k^2\right) P_a^V(x) ,$$

$$(a, b = q, \bar{q}, g) \tag{4.5}$$

and the evolution equations in the light-cone gauge have the customary matrix form

$$\frac{d}{d\log k^2} D_a^c(k^2, x) = \sum_b \int_x^1 \frac{dz}{z} \frac{1}{2\pi} \Pi_a^b\left(k^2, z, \frac{Q_0^2}{k^2}\right) D_b^c\left(zk^2, \frac{x}{z}\right). \tag{4.6}$$

For $1 < n \ll Q^2/Q_0^2$ the moments of (4.6) reduce to the usual ones, apart from corrections of relative order $\alpha_s(Q^2)$. Important differences arise instead for $1 \ll n \lesssim Q^2/Q_0^2$ and for $n \to 0$. In the first case the resummation of corrections $(\alpha_s(Q^2) \log n)^p$ for the gluon evolution yields a fragmentation function analogous to (3.7) with C_F replaced by C_A (only the singularity $2\, C_A/(1-z)$ of P_g^{gg} matters).

In the second case $(n \to 0)$, one has to remember[9] that according to eq. (4.6) the jet squared mass k^2 is rescaled in the evolution, to its kinematical limit zk^2. This rescaling cannot be neglected for the multiplicity whose evolution equation reads ($\eta^2 = 0$)

$$\frac{d}{d\log k^2} N_a^b(k^2) = \sum_c \int \frac{dz}{2\pi} \Pi_a^c(k^2, z) N_c^b(zk^2) ,$$

$$\left(N_a^b(Q_0^2) = \delta_a^b \right) . \tag{4.7}$$

In detail, by taking into account the occurence of $\alpha_s(k^2 z(1-z))$, and defining $zk^2 = k'^2$ whenever $P_a^{bb'}(z)$ is singular, we get

$$\frac{d}{d\log k^2} N_g^b = \frac{C_A}{\pi} \int \frac{dk'^2}{k'^2} \alpha_s(k'^2) N_g^b(k'^2) - (1+2\delta) b\,\alpha_s N_g^b + \delta b \alpha_s (N_q^b + N_{\bar{q}}^b) ,$$

$$\frac{d}{d\log k^2} N_q^b = \frac{C_F}{\pi} \left[\int \frac{dk'^2}{k'^2} \alpha_s(k'^2) N_g^b(k'^2) - \frac{3}{4} \alpha_s(k^2) N_g^b(k^2) \right] , \tag{4.8}$$

where $\delta = N_f/6\pi b$.

The asymptotic multiplicity then becomes

$$N_a^b(Q^2, Q_0^2) = \left(\frac{\log Q^2/\Lambda^2}{\log Q_0^2/\Lambda^2} \right)^{-\frac{p}{2} - \frac{1}{4}} \cdot \begin{vmatrix} \frac{C_F}{C_A} r & \frac{C_F}{C_A} \\ r & 1 \end{vmatrix} \cdot$$

$$\cdot \exp\left[2 \sqrt{\frac{C_A}{\pi b}} \left(\sqrt{\log Q^2/\Lambda^2} - \sqrt{\log Q_0^2/\Lambda^2} \right) \right] , \tag{4.9}$$

where a,b = 1 stands for q, and a,b = 2 for g.

where

$$p = \frac{1}{2} + 2\delta\left(1 - \frac{C_F}{C_A}\right) \quad , \quad \eta = \frac{N_f}{6N_c}\sqrt{\frac{N_c\,\alpha_s(Q_0^2)}{\pi}}$$

(4.10)

and p differs by 1/2 from the one previously given[9]. The reason is that since α_s is evaluated at a smaller mass, the yield of particles must increase.

Finally, one can obtain the evolution equations for the jet generating functions

$$G_a(Q^2, Q_0^2, \{z\}) = 1 + \sum_{\{c_i\}} (n_q!\, n_{\bar q}!\, n_g!)^{-1} (z_{c_1} - 1)\cdots(z_{c_n} - 1)\, D_a^{c_1\cdots c_n}(Q^2, Q_0^2)$$

(4.11)

where $D_a^{c_1 \ldots c_n}$ are the integrated n-particle inclusive distributions. By using the method of Ref.9) with the replacement $\alpha_s(k^2) \to \alpha_s(k^2 z(1-z))$ we obtain in the light-cone gauge

$$\frac{d}{d\log k^2} G_a(k^2, Q_0^2, \{z\}) = - G_a V_a\left(k^2, Q_0^2; \frac{Q_0^2}{k^2}\right) +$$

$$+ \frac{1}{2} \sum_{c_1 c_2} \int \frac{dz}{2\pi} \alpha_s\left(z(1-z)k^2\right) P_a^{c_1 c_2}(z)\, G_{c_1}(zk^2, Q_0^2, \{z\}) \cdot$$

$$\cdot\, G_{c_2}\left((1-z)k^2, Q_0^2, \{z\}\right) \quad ,$$

(4.12)

$$G_a(Q_0^2, Q_0^2, \{z\}) = z_a \quad ,$$

where V_q was defined in (3.9) and

$$V_g(k^2, Q_o^2, \epsilon) = \int_{\epsilon(k)}^{1-\epsilon(k)} dx\, \alpha_s(x(1-x)k^2)\, P_g^V(x) \tag{4.13}$$

are the virtual emission contributions.

It is important to notice at this point that the cut-off $\varepsilon(k) = Q_o^2/k^2$ occurs in the virtual terms (3.9) and (4.13) because of the final expression (3.14) for the Π's in which real and virtual gluons compensate for $Q_o^2/k^2 > 1-z > 0$. It follows that eq.(4.12) satisfies the normalization condition $G_a = 1$ for $z_a=1$ since virtual and real emission phase space correctly match. This confirms the classical picture of the jet evolution as a branching process also when the argument of α_s is modified in order to incorporate correctly soft effects.

5. COLOUR DISTRIBUTIONS.

In Refs. 8),9) it was shown that final quanta in a hard process cluster in colour singlet systems with a strongly damped total mass distribution. This damping was found to be the same as in the Sudakov form factor due to a similar incomplete cancellation of I.R. divergences. We show that this similarity is preserved by the present more refined method so that in analogy with eq.(3.11) the mass distribution of singlet clusters will be damped by a law

$$\frac{m^2}{\sigma}\frac{d\sigma}{dm^2} \underset{m^2 \gg Q_o^2}{\propto} \left(\frac{m^2}{Q_o^2}\right)^{-\frac{N_c}{\pi b}\log\frac{\alpha_s(Q_o^2)}{\alpha_s(m^2)}} \times \left(\text{powers of } \frac{Q_o^2}{m^2}\right). \tag{5.1}$$

A simple way to study the colour distributions of the final state is to introduce the colour connecting distributions[9] Γ_a^q(a=q,g) (Fig.6), which are the inclusive distributions of quarks in a quark or gluon jet under the requirement that a colour line connects a to q through the emission of gluons only.

In order to see the consequences of forbidding $q\bar{q}$ emissions along the colour line, we start by studying the probability[8] $\sigma_g(Q^2,Q_o^2)$ that a gluon jet produces only gluons in its evolution down to Q_o^2 .

This can be obtained from the generating functions $G_a(Q^2,Q_o^2,z)$ of eq. (4.11), which satisfy the equations

$$\frac{d}{d\log k^2} G_q = G_q \frac{C_F}{\pi}\left[-\int_0^{1-\epsilon(k)} \frac{dz}{1-z}\alpha_s(k^2(1-z))\left(1-G_g\left(k^2(1-z)\right)\right) + \right.$$

$$\left. + \frac{3}{4}\alpha_s(k^2)\left(1-G_g\right)\right] \qquad ,$$

$$\frac{d}{d\log k^2} G_g = G_g\left[-\frac{N_c}{\pi}\int_0^{1-\epsilon(k)} \frac{dz}{1-z}\alpha_s(k^2(1-z))\left(1-G_g(k^2(1-z))\right) + \right.$$

$$\left. + \frac{11}{12}\frac{N_c}{\pi}\alpha_s(k^2)\left(1-G_g(k^2)\right)\right] + \frac{N_f}{6\pi}\alpha_s(k^2)\left(G_q G_{\bar{q}} - G_g\right),$$

$$G_a\left(Q_o^2, Q_o^2, \{z\}\right) = z_a \quad . \tag{5.2}$$

In order to forbid quark pairs, one has to be exclusive in the quark variables and inclusive in the gluons, i.e., one has to set $\gamma_q = \gamma_{\bar{q}} = 0$, $\gamma_g = 1$.

Defining $1-z = k'^2/k^2$ we get

$$\frac{d\,\sigma_g(k^2,Q_0^2)}{d\log k^2} = \frac{N_c}{\pi}\,\sigma_g\left[-\int_{Q_0^2}^{k^2}\frac{dk'^2}{k'^2}\,\alpha_s(k'^2)\left(1-\sigma_g(k'^2,Q_0^2)\right) + \right.$$

$$\left. + \frac{11}{12}\,\alpha_s(k^2)\left(1-\sigma_g(k^2,Q_0^2)\right)\right] - \frac{N_f}{6\pi}\,\alpha_s(k^2)\,\sigma_g(k^2,Q_0^2). \tag{5.3}$$

As previously discussed[8] $\sigma_g(k^2,Q_0^2)$ is driven to zero for $k^2 \gg Q_0^2$ by the terms proportional to N_f in eq.(5.3). Therefore, the integral over $\sigma_g(k'^2)$ in the r.h.s. of (5.3) converges to a number. The other integral diverges as $\log\alpha_s(k^2)$ whereas the remaining terms are asymptotically vanishing. In conclusion, the asymptotic solution has the form

$$\sigma_g(Q^2,Q_0^2) \simeq \Delta_g^{LC}(Q^2,Q_0^2)\,\left(Q^2/Q_0^2\right)^{\frac{N_c}{\pi b}B}, \tag{5.4}$$

where, in analogy with Δ_q^{LC} of eq.(3.11),

$$\Delta_g^{LC}(Q^2,Q_0^2) = \exp\left[-\int_{Q_0^2}^{Q^2}\frac{dk^2}{k^2}\int_{\epsilon(k)}^{1-\epsilon(k)} dz\; P_g^V(z)\,\alpha_s\left(k^2 z(1-z)\right)\right]$$

$$\simeq \exp\left[-\frac{N_c}{\pi b}\left(\log\frac{Q^2}{\Lambda^2}\log\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} - \log\frac{Q^2}{Q_0^2}\right)\right] \tag{5.5}$$

and

$$B = \int_{\log Q_o^2/\Lambda^2}^{\infty} \frac{dk^2}{k^2} \alpha_s(k^2)\, \sigma_g(k^2, Q_o^2) \tag{5.6}$$

represents, in a self-consistent way, the real emission contribution. Taking into account that the gluon Sudakov form factor is given by Δ_g^{LC} (as shown for the quark in eq.(3.12)), eq.(5.4) shows the connection between damping due to forbidding the emission of quark pairs $(\sigma_g(Q^2,Q_o^2))$ and to forbidding the emission of every (hard) quantum. Let us also recall that the damping of $\sigma_g(Q^2,Q_o^2)$ was directly related[8] to the damping of the colour singlet cluster mass spectrum which, therefore, is the one anticipated in eq.(5.1).

This correspondence can be generalized to the colour connecting distributions Γ_a^q. Asymptotically, their moments satisfy the equations[9]

$$\left[\frac{d}{d\log k^2} + \frac{N_c}{2\pi}\int_{Q_o^2}^{k^2} \frac{dk'^2}{k'^2}\alpha_s(k'^2)\right] \tilde{\Gamma}_g^q(k^2, Q_o^2; n) =$$

$$= \frac{N_c}{2\pi}\int_{Q_o^2}^{k^2} \frac{dk'^2}{k'^2}\left(\frac{k'^2}{k^2}\right)^n \alpha_s(k'^2)\, \tilde{\Gamma}_g^q(k'^2, Q_o^2; n) \tag{5.7}$$

with

$$\tilde{\Gamma}_q^q(Q^2, Q_o^2; n) \simeq \tilde{\Gamma}_g^q(Q^2, Q_o^2; n) \equiv \tilde{\Gamma}(Q^2, Q_o^2; n) \tag{5.8}$$

in the approximation $C_F \simeq N_c/2$ (large N_c expansion).

The solution of (5.7) can be given in terms of the integrated virtual emission probability $V_q(k^2, Q_o^2; Q_o^2/k^2)$ of eq.(3.9).

In fact, using the variable x, Eq. (5.7) reads:

$$\left[\frac{d}{d\log k^2} + V_q\right]\Gamma(k^2, Q_0^2, x) = \int_{Q_0^2}^{k^2} dk'^2 \left[\frac{d}{dk'^2} V_q(k'^2)\right]\Gamma\left(k'^2, Q_0^2, \frac{xk^2}{k'^2}\right) \tag{5.9}$$

whose solution is

$$\Gamma(Q^2, Q_0^2, x) = \frac{1}{x}\Delta_q(xQ^2, Q_0^2)\left[V_q(xQ^2) + \delta(x-1)\right] \tag{5.10}$$

where x is the fraction of momentum carried by the quark, which is colour-connected to the initial antiquark. This x will be called x_{cc} in the following.

The asymptotic behaviour of Γ is therefore again dictated by the Sudakov form factor Δ_q with the important rescaling $Q^2 \to Q^2 x_{cc}$. This means that $\langle x_{cc} \rangle \simeq Q_0^2/Q^2$ is small for large Q^2. One can verify from eq. (5.10) that this small x_{cc} region saturates the sum rule $\int dx\, \Gamma = 1$.

Going to the rest frame of the antiquark directly coupled to the virtual photon, it is easy to check that the above result implies that the momentum of the first emitted quark (i.e., colour connected to the antiquark) is finite. This property can be extended to any $q\bar{q}$ pair connected by colour (i.e., belonging to the same colour singlet). This is just a restatement of the fact that the singlet average mass is finite[8),9)].

This conclusion has a counterpart in terms of the space-time development of QCD jets[18),19)]. All quanta in the same colour singlet cluster spend, in the cluster rest frame, their life-time of order $1/Q_0$ in the same region of space also of order $1/Q_0$.

6. CONCLUSIONS.

Our goal in this paper was to develop a formalism for hard QCD processes which could also cover regions of phase space (or more general questions) where the emission of quanta is so inhibited as to explore the infra-red singularity of the theory.

We argued that this can be achieved in terms of modified evolution and branching equations (i.e., (3.4), (4.6) and ((5.2)) where the argument of the running coupling constant is scaled down to account for phase space restrictions.

The main consequence of this treatment is the prediction of a very strong (i.e., faster than any power of Q^2) damping in quasi-exclusive processes* such as the large moment ($x\to 1$) limit of structure and fragmentation functions (eq.(3.7)) or the mass spectrum of final state colour singlets in e^+e^- annihilation (eq.(5.1)).

Recalling that all perturbative QCD calculations amount to a method for resumming large logarithms, it is clear that the strong damping we obtained reveals very accurate cancellations.

We are aware of the fact that individually small terms, such as higher twist contributions (down by powers) could be uncovered by the strong cancellation of leading contributions.

An interesting example of a situation of this type is offered by structure functions at large Q^2 and at $x \simeq 1-Q_o^2/Q^2$ (with $Q_o^2 \gtrsim \Lambda^2$). In this case our result, which resums soft gluon emission (see Fig. 7a) and neglects the spectator parton, gives the damping (3.11). This, however, is overcome as $Q^2\to\infty$ by the exclusive hadronic (possibly inelastic) form factor which is known to behave[21] as an inverse power

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*Similar considerations could apply to inclusive $p_\perp$ spectra for $Q_o^2 < p_\perp^2 << Q^2$ [9),20)].

$$F_{\text{hadronic}}(Q^2) \propto \left[\alpha_s(Q^2)\right]^n \left(Q^2/Q_0^2\right)^{-m} , \qquad (6.1)$$

$$n, m = 1, 2 \ldots$$

The origin of (6.1) is not hard to understand if one looks at the relevant diagrams[21)] (Fig. 7b). The two hard propagators (drawn as thick lines in Fig. 7b) lead to an immediate colour screening and cause the loss of a power of Q^2. However, once this price is paid, the remaining ladder-type radiative corrections act within low mass colour singlets and produce no further damping. These kinematical configurations with a few propagators of large q^2 fixed at $O(Q^2)$ were not included in our treatment because strongly non leading (i.e. equivalent to higher twists in the operator product expansion).

In order to have an idea of the value of Q^2, say $\bar{Q}^2$, at which the contribution (6.1) starts to overcome ours we can equate (3.11) with the square of (6.1)(taken for simplicity with n=0, m=1) and solve for $\bar{Q}^2$

$$\exp\left[-\frac{C_F}{\pi b}\left\{\left(\log\frac{\bar{Q}^2}{Q_0^2} + \frac{1}{\alpha_0 b}\right)\log\frac{\alpha_0}{\alpha_s(\bar{Q}^2)} - \log\frac{\bar{Q}^2}{Q_0^2}\right\}\right] = \left(\frac{Q_0^2}{\bar{Q}^2}\right)^2 \qquad (6.2)$$

where $\alpha_0 \equiv \alpha_s(Q_0^2)$. Eq(6.2) becomes easily

$$\frac{\alpha_0}{\alpha_s(\bar{Q}^2)}\log\frac{\alpha_0}{\alpha_s(\bar{Q}^2)} = \left(1+\frac{2\pi b}{C_F}\right)\left(\frac{\alpha_0}{\alpha_s(\bar{Q}^2)} - 1\right) \qquad (6.3)$$

giving approximately

$$\bar{Q}^2/\Lambda^2 \simeq \left(Q_0^2/\Lambda^2\right)^{\exp\left(1+\frac{2\pi b}{C_F}\right)} . \qquad (6.4)$$

The above result suggests that, even if the tail of the distribution (3.11) cannot be trusted, the fact that the distribution itself is strongly damped has physical significance.

An equivalent observation could be made for $e^+e^- \to$ hadrons if one could attach a phenomenological meaning to the concept of a large mass colour singlet system with a single $q\bar{q}$ pairs (plus gluons). Then, if one would look, for instance, at the cross section for $e^+e^- \to 2\ q\bar{q}$ pairs + gluons, our leading log result would predict formation of one large, one small mass cluster with a damping such as in eq.(5.1). On the other hand one could produce two-low mass singlets via a time like hadronic form factor and lose just a power as in eq.(6.1) so that this second process eventually overcomes the other.

In any event the above processes are both negligible fractions of $\sigma_T(e^+e^- \to$ hadrons). What we have learned from previous studies[8),9)] and confirmed in the present paper is that separation of colour is strongly suppressed and that the process $e^+e^- \to$ hadrons is entirely saturated as $Q^2 \to \infty$ by channels with an increasing number of colourless clusters whose average multiplicity grows as in eq.(4.9) and whose invariant mass remains finite.

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We are grateful to G.Curci, K.Konishi and R.Petronzio for communicating to us their results prior to publication and for illuminating discussions.

A P P E N D I X   A

CANCELLATION OF LEADING LOGS IN G^{coll}.

We want to show here that the following combination of vertex and propagator renormalization functions found in the text

$$G^{coll} = \frac{g^2}{4\pi} \Gamma^2_{qqg}\left(k^2, -z\xi k^2, -(1-z)(1-\xi)k^2; k_{\parallel i} \right) \cdot$$

$$\cdot\, d_q\left(-k^2, k_\parallel\right) d_q\left(-zk^2, k_{\parallel 1}\right) d_g\left(-(1-z)k^2, k_{\parallel 2}\right) \tag{A.1}$$

is free of singularities of the form $\log^2 k^2_{\parallel i}/k^2$. At the same accuracy level this also establishes its gauge invariance, and its independence of the parameter ξ.

Let us first note that such $\log^2$ terms come from the combination of collinear and infra-red singularities in the loop integrals. By selecting the relevant region of phase space in the intervals where the gluons in the loop are soft, $q^2 \to 0$, $q_\parallel/k_\parallel \to 0$, we find the following contributions:

a) Quark propagator. From the diagram of Fig.8 we obtain, to leading order

$$S = \frac{\not k}{k^2}\left(1 - 2C_F J(k)\right) \tag{A.2}$$

where

$$J(k) = -i \frac{g^2}{(2\pi)^4} \int \frac{d^4q}{q^2+i\epsilon} \frac{2k\eta}{(k-q)^2+i\epsilon} P\left(\frac{1}{\eta q}\right) \simeq$$

$$\simeq \frac{g^2}{(2\pi)^3} \int_{|q| \leq k_{//}} \frac{d^3q}{|q|} \frac{k\eta}{k^2-2kq+i\epsilon} P\left(\frac{1}{\eta q}\right) . \tag{A.3}$$

b) Gluon propagator. We find, from diagram of Fig.9, that the important vertex contributions are those which conserve helicity along the hard line (cf. Fig.9b). There are two of them according to which gluon in the loop is soft.

Therefore,

$$D_{\mu\nu}(k) \simeq \frac{1}{k^2} d_{\mu\nu} + 2 \frac{1}{k^4} \frac{1}{2} \frac{C_A g^2}{(2\pi)^4} \int \frac{d^4q}{q^2+i\epsilon}$$

$$d_{\mu\lambda}(k)\, d^{\lambda\rho}(k-q)\, d_{\rho\nu}(k)\; 2k_\sigma\, d^{\sigma\tau}(q)\, 2k_\tau \frac{1}{(k-q)^2+i\epsilon} \simeq$$

$$\simeq \frac{1}{k^2}\left(1 - 2C_A J(k)\right) d_{\mu\nu} + \eta_\mu \eta_\nu \text{ terms} . \tag{A.4}$$

c) Vertex function. The Feynman diagrams are given in Fig.10 and the leading terms are

$$\Gamma_\mu(k_i) \simeq \gamma_\mu t^a \Big[1 + \left(C_F - \tfrac{1}{2} C_A\right) I(k,k_1) + \tfrac{1}{2} C_A \Big(I(k_2,-k_1) +$$

$$+ I(k,k_2) \Big) \Big] \tag{A.5}$$

where the last two contributions are from the two diagrams of

Fig.10c corresponding to the two phase space region where the momentum q of the gluon in the loop is soft , and $I(k_1;k_2)$ is defined by

$$I(k_1,k_2) = -ig^2\int \frac{d^4q}{(2\pi)^4}\,\frac{2k_2^{\mu}\,d_{\mu\nu}(q)\,2k_1^{\nu}}{(q^2+i\epsilon)\left[(k_1-q)^2+i\epsilon\right]\left[(k_2-q)^2+i\epsilon\right]} \simeq$$

$$\simeq \frac{g^2}{(2\pi)^3}\int \frac{d^3q}{|q|}\,\frac{2(k_1q)(k_2\eta)+2(k_2q)(k_1\eta)}{(k_1^2-2k_1q+i\epsilon)(k_2^2-2k_2q+i\epsilon)}\,P\left(\frac{1}{q\eta}\right).$$

(A.6)

d) <u>ggg vertex function</u>. The contribution for which the gluon with momentum q in the loop is soft gives rise to three terms (cf. Fig.11b) which reproduce the transverse part of the bare invariants at the opposite vertex (as shown by the helicity saturation diagrams). For instance, the saturation of the first diagram in Fig.11b gives the spin numerator

$$g_{\mu\mu_2}(k+k_2)_\lambda\, d^{\lambda\sigma}(k_1)\, d_{\sigma\mu_1}(k_1)\; 2k_2^{\rho}\, d_{\rho\tau}(q)\, 2k_1^{\tau}$$

which is proportional to the corresponding part of the bare vertex transverse to k_1.

Since there are three possible soft gluons, a straightforward computation gives

$$\Gamma_{\mu\mu_1\mu_2} \simeq \Gamma^0_{\mu\mu_1\mu_2}\left(1+\frac{1}{2}C_A\left[I(k_2,-k_1)+I(k,k_1)+\right.\right.$$
$$\left.\left.+\,I(k,k_2)\right]\right) + \ldots \qquad \text{(A.7)}$$

It is easy to show that both $\mathcal{J}(k)$ and $I(k_1,k_2)$ behave like $\log^2 \frac{(k.\eta)^2}{\eta^2 k^2}$ when $k_{/\!/}^2 = \frac{(k.\eta)^2}{\eta^2} >> |k^2|$. We want to prove that this singularity cancels in the invariant charge (A.1), in which the vertex function occurs with the mass configuration $k_1^2=-k^2 z\xi$, $k_2^2=-k^2(1-z)(1-\xi)$.

Let us note that, because of the colour factor in (A.2) (A.4) and (A.5), this amounts to show that the combination

$$I\left(k^2, k_{/\!/}; -\xi z k^2, z k_{/\!/}\right) - J\left(k^2, k_{/\!/}\right) - J\left(-z k^2, z k_{/\!/}\right) \tag{A.8}$$

and analogous ones for k_1, k_2 etc, do not contain $\log^2$ terms. For instance, the combination (A.8) gives rise by (A.6) and (A.3) to integrals of the type

$$\frac{g^2}{(2\pi)^3} \int_{|q| \leq k_{/\!/}} \frac{d^3 q}{|q|} \frac{\xi k^2}{(k^2 - 2qP)(\xi k^2 - 2qP)} P\left(\frac{k\eta}{q\eta}\right) \simeq \frac{\xi}{1-\xi} \log\xi \log\frac{k_{/\!/}^2}{\sqrt{\xi}\, k^2} \tag{A.9}$$

where $P_\mu = (P, \underset{\sim}{O}, P)$ in the large P limit, and any explicit z dependence drops out automatically.

In the integrand of eq.(A.9) there is a double collinear singularity which is however cancelled by the k^2 factor in the numerator, thus proving our assertion at the double log level.

Similar results hold for the combination k_1, k_2 and k, k_2. The overall ξ dependence is such that leading singularities at $\xi = 0$ ($\xi = 1$) are killed by factors of ξ $(1-\xi)$. This shows that the cut in $k_1{}^2$, $k_2{}^2$ of the collinear vertex function does not contribute to G^{coll} at leading log level.

Finally, by applying exactly the same arguments to the triple gluon vertex (A.7) and to (A.4) we can extend the cancellation of

$(\log)^2$ terms to

$$\Gamma^2_{ggg}\left(\kappa^2, -\xi z\kappa^2, -(1-\xi)(1-z)\kappa^2; \kappa_{\parallel i}\right) d_g(-\kappa^2, \kappa_\parallel) d_g(-z\kappa^2, \kappa_{\parallel 1}) \cdot$$

$$\cdot d_g\left(-(1-z)\kappa^2, \kappa_{\parallel 2}\right) . \qquad \text{(A.10)}$$

A P P E N D I X B

The gauge invariance of the result (3.11) can be checked by generalizing to our case with a moving coupling constant the DDT analysis of the renormalized e.m. vertex .

It is clear from Fig.12 that

$$F_q^2 = \Gamma^2_{\gamma q\bar{q}}\left(Q^2, -Q_0^2, -Q_0^2; \kappa_{\parallel 1}, \kappa_{\parallel 2}\right) d_q(-Q_0^2, \kappa_{\parallel 1}) d_{\bar{q}}(-Q_0^2, \kappa_{\parallel 2}), \qquad \text{(B.1)}$$

where $\Gamma_{\gamma q\bar{q}}$ contains the QCD radiative corrections to the quark e.m. vertex, and d_q is given by (2.22) except that $\epsilon(\kappa_\parallel)$ is replaced by $\varepsilon(k)$ in order to allow soft emission up to Q_0.

Since, at $(\log)^2$ accuracy, $\Gamma_{\gamma q\bar{q}}$ is not infrared singular in the "small" masses, i.e.,

$$\Gamma^2_{\gamma q\bar{q}}\left(Q^2, -Q_0^2, -Q_0^2; \kappa_{\parallel 1}, \kappa_{\parallel 2}\right) \simeq \Gamma^2_{\gamma q\bar{q}}\left(Q^2, Q^2, Q^2; \kappa_{\parallel 1}, \kappa_{\parallel 2}\right), \qquad \text{(B.2)}$$

we can use the invariant charge relationship for the latter to get

$$F_q^2(Q^2, Q_o^2) \simeq \Delta_q(Q^2, Q_o^2; \kappa_{\parallel 1})\, \Delta_{\bar{q}}(Q^2, Q_o^2, \kappa_{\parallel 2}) \tag{B.3}$$

where

$$\Delta_q(Q^2, Q_o^2, \kappa_\parallel) = d_q^{-1}(Q^2, \kappa_\parallel)\, d_q(Q_o^2, \kappa_\parallel) =$$

$$= \exp\left[-\int_{Q_o^2}^{Q^2} \frac{d\kappa^2}{\kappa^2} V_q(\kappa^2, Q_o^2, \epsilon(\kappa))\right] . \tag{B.4}$$

In a planar gauge with $\eta = (a+b, \underline{0}, a-b)$ and in the c.m. system we have

$$\epsilon(\kappa) = \mathrm{Max}\left(\frac{\kappa^2}{Q^2\xi^2}, \frac{Q_o^2}{\kappa^2}\right) \tag{B.5}$$

where $\xi^2 = b/a$ and the longitudinal momentum of k is positive ($k_3 = Q/2$). When k points in the opposite direction ($k_3 = -Q/2$) in eq.(B.5) we have to replace $\xi^2 \to 1/\xi^2$.

A straightforward computation of eq.(B.4) with $\varepsilon(k)$ given in eq.(B.5) then leads to the following results:

a) $\xi > Q/Q_o$. In this range of gauge parameters the result for Δ_q is the same as for the light-cone gauge ($\xi \to \infty$), i.e.,

$$-\log \Delta_q(Q^2, Q_o^2) = \frac{C_F}{\pi b}\left[\log\frac{Q^2}{\Lambda^2}\log\left(\frac{\log Q^2/\Lambda^2}{\log Q_o^2/\Lambda^2}\right) - \log\frac{Q^2}{Q_o^2}\right]. \tag{B.6}$$

On the other hand, it is trivial to see that for $\xi < Q/Q_o$ we get $\Delta_q = 1$, corresponding to the absence of emission from the quark with positive longitudinal momentum nearly parallel to the gauge vector η.

b) $Q_o/Q < \xi < Q/Q_o$. In this range of ξ there is a non-trivial gauge dependence:

$$- \log \Delta_q (Q^2, Q_o^2, \xi) =$$

$$= \begin{cases} \dfrac{C_F}{2\pi b}\left[\log \dfrac{Q^2\xi^2}{\Lambda^2} \log \dfrac{\alpha_s(Q_o^2)}{\alpha_s(Q^2\xi^2)} - \log \dfrac{Q^2\xi^2}{Q_o^2}\right] , & \left(\dfrac{Q_o}{Q} < \xi < 1\right), \quad (a) \\ -\log \Delta_q^{LC} - \dfrac{C_F}{2\pi b}\left[\log \dfrac{Q^2}{\Lambda^2\xi^2} \log \dfrac{\alpha_s(Q_o^2)}{\alpha_s(Q^2/\xi^2)} - \log \dfrac{Q^2}{\xi^2 Q_o^2}\right] , & \left(1<\xi<\dfrac{Q}{Q_o}\right). \quad (b) \end{cases} \tag{B.7}$$

Let us note the difference $\dfrac{C_F}{\pi b} \to \dfrac{C_F}{2\pi b}$ in eq.(B.6) and eq.(B.7). This is important for the gauge invariance of the result. In fact, one has, in the c.m. frame,

$$4K_{\parallel 1}^2 = Q^2\xi^2 \quad , \quad 4K_{\parallel 2}^2 = Q^2/\xi^2 \tag{B.8}$$

and therefore, by (B.3),

$$-2 \log F_q = - \log \Delta_q (Q^2, Q_o^2, \xi) - \log \Delta_{\bar q} (Q^2, Q_o^2, \tfrac{1}{\xi}) =$$

$$= - \log \Delta_q^{LC} (Q^2, Q_o^2) \, ,$$

where the last identity follows trivially from (B.7b) by exchanging $\xi \to 1/\xi$. We conclude that (3.11), (3.12) hold for any gauge of this class.

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F I G U R E   C A P T I O N S

Fig.1: Dyson equation for the inverse propagator and skeleton expansion of the flavour non singlet kernel.

Fig.2: Integration contour $C(q^2)$ for eq.(2.9).

Fig.3: Integration contour for eq.(2.14). The branch point at zk^2 is the one of d_g and possible non leading singularities of Γ_{qqg} are dashed.

Fig.4: Fourth order graphs important for the large n behaviour in the planar gauge.

Fig.5: Bethe-Salpeter equation for D^{NS}.

Fig.6: Graphical representation for the distribution of $q\bar{q}$ within a colour singlet cluster. The black dot represents single parton distribution, half-black dots represent colour connecting distributions with a selection against $q\bar{q}$ emission from the open side. The sum Σ' is restricted to (bc_1c_2) = (qqg), $(\bar{q}g\bar{q})$ and (ggg).

Fig.7: (a) structure function at large Q^2 and $x \simeq 1-Q_o^2/Q^2$ and (b) hadronic form factor.
Thick internal lines in Fig.(b) indicate the two hard propagators.

Fig.8: Second order correction for the quark propagator.

Fig.9: Second order correction for the gluon propagator. Dashed lines indicate the saturation of Lorentz indices at the ggg vertex. Fig.(b) indicates the relevant saturation at the double log level in the phase space of soft q.

Fig.10: Second order corrections for the qqg vertex. Dashed lines indicate the saturation of Lorentz indices of the ggg vertex. Fig.(b) indicates the relevant saturation in the double log phase space region of soft q.

Fig.11: Second order correction for the ggg vertex.Fig.(b) indicates the saturation of Lorentz indices contributing to double log in the phase space region of the soft gluon with momentum q. This saturation reproduces the transverse part of the invariants at the vertex of the gluon of momentum k opposite to the soft gluon with momentum q.

Fig.12: Electromagnetic form factor of the quark.

Fig.1

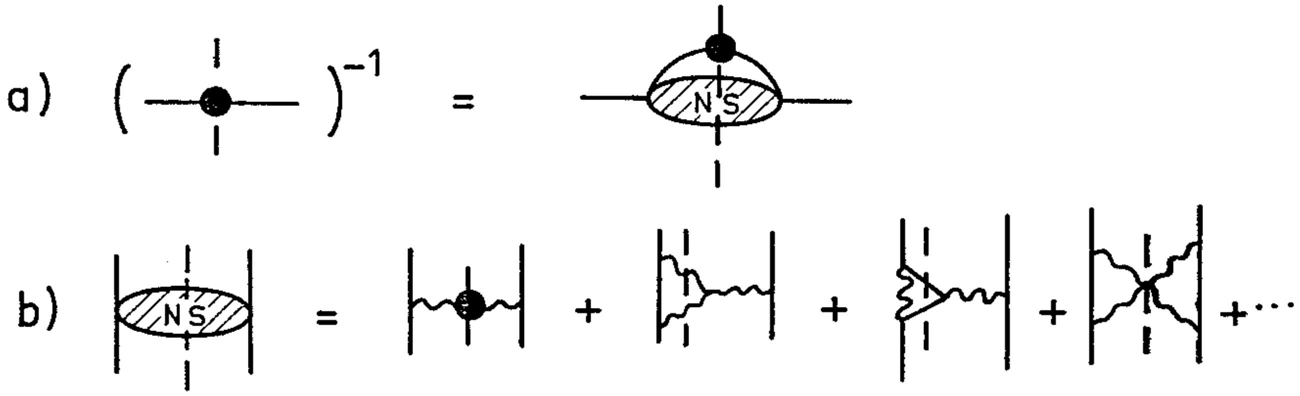

Fig. 2

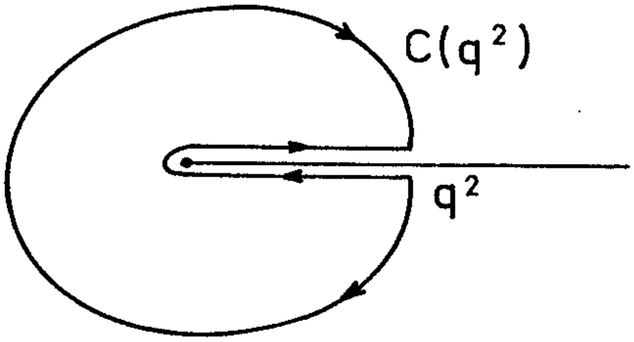

Fig. 3

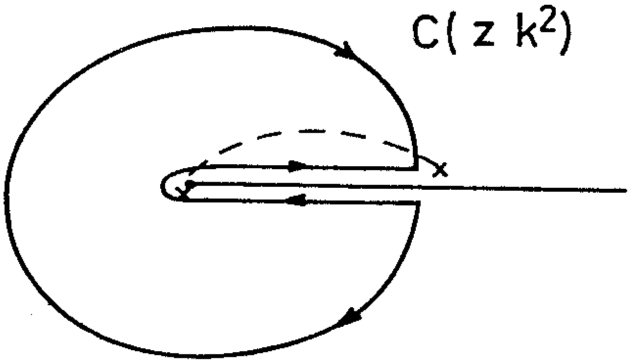

Fig. 4

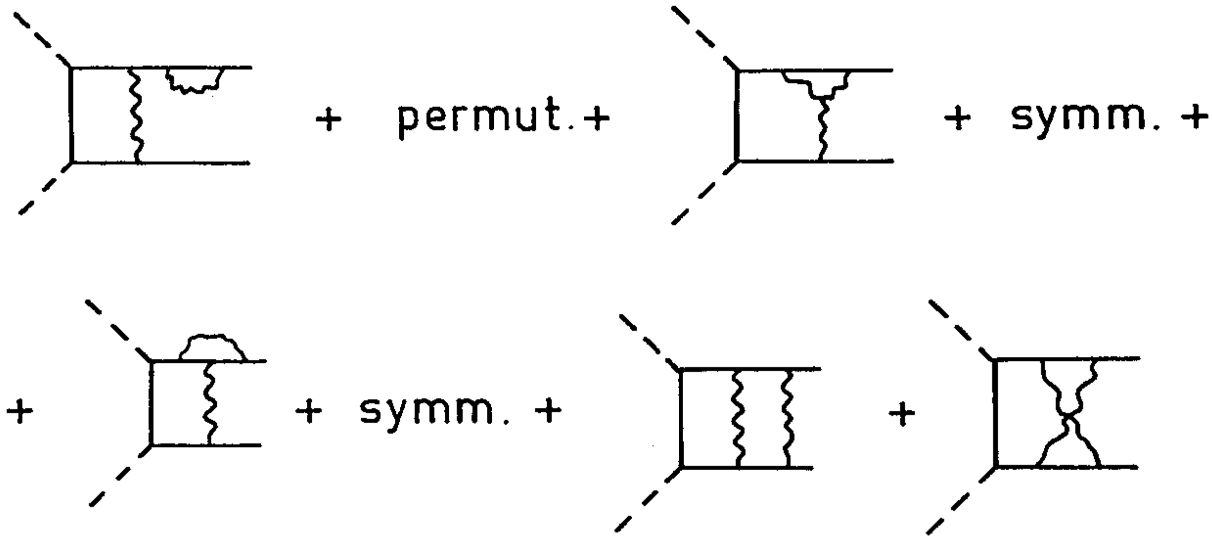

Fig. 5

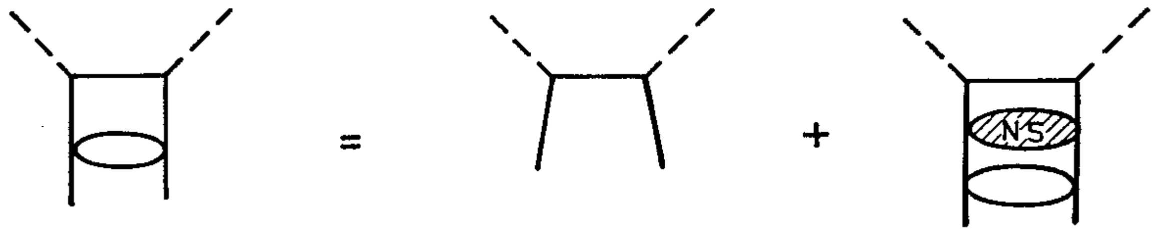

Fig. 6

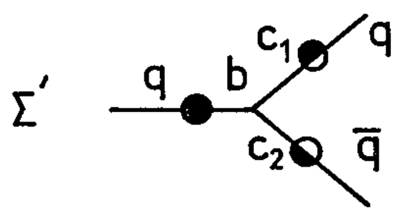

Fig. 7

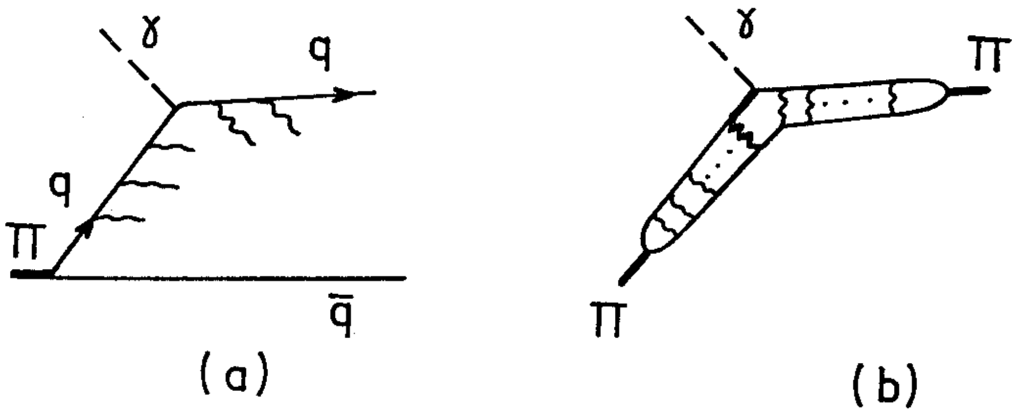

Fig. 8

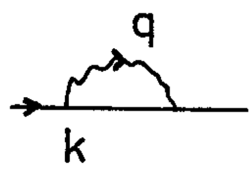

Fig. 9

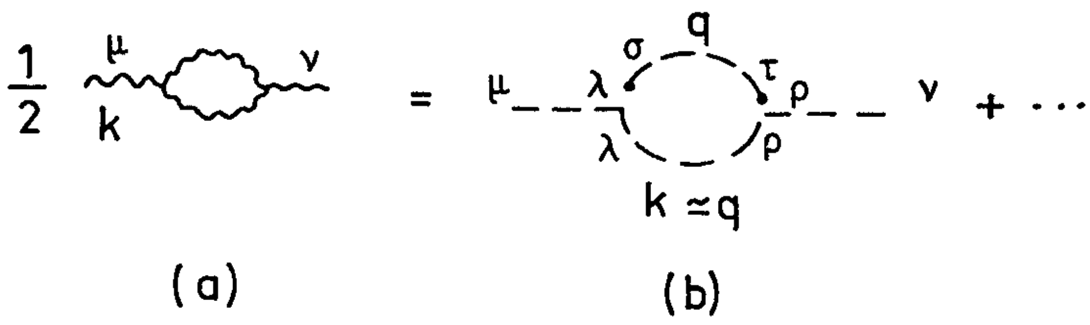

Fig.10

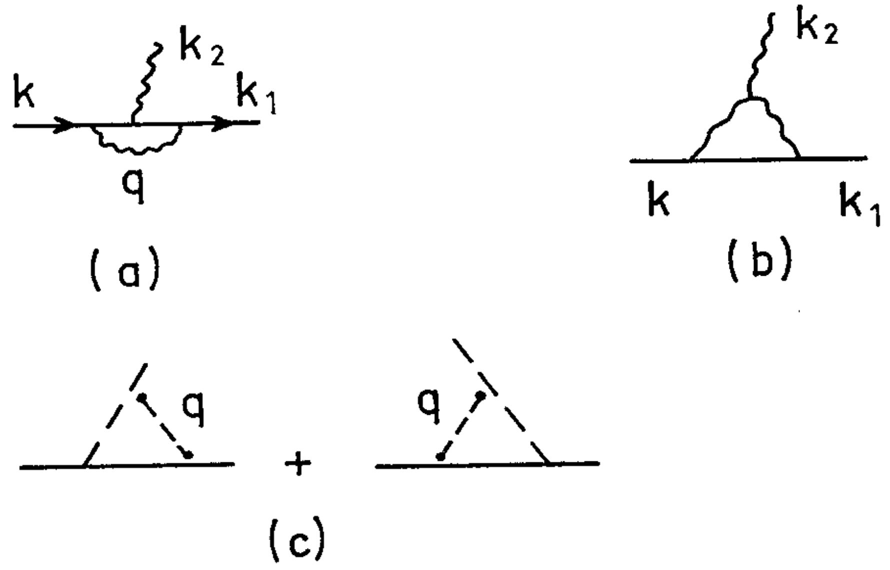

Fig.11

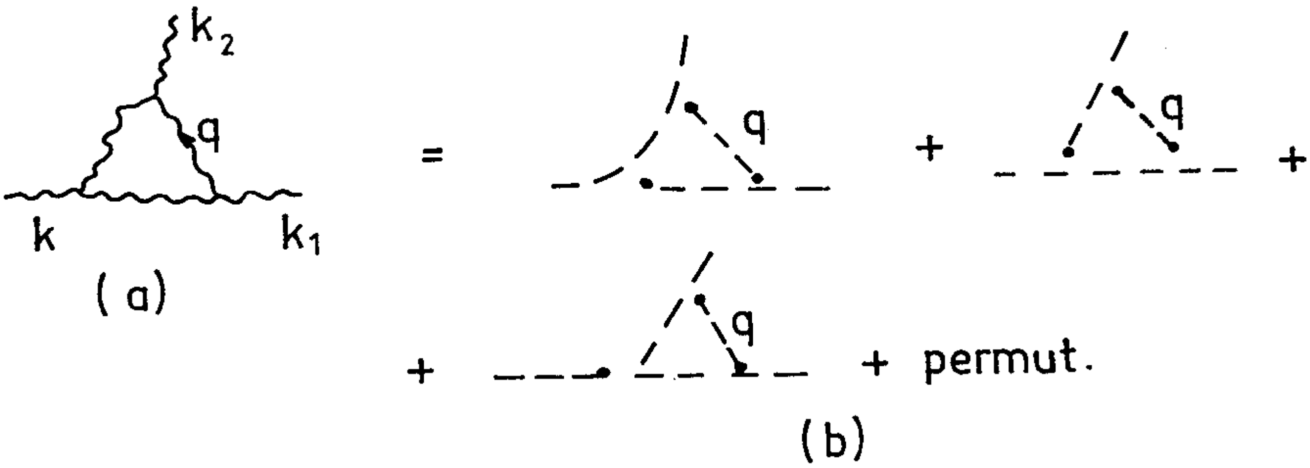

Fig.12