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Abstract: On the set of asymptotically vacuumlike states there is a product which induces the composition of superselection sectors. Important concepts of the DHR theory of superselection sectors (statistical dimension, conjugation, fusion rules) are expressed directly in terms of states.

In quantum physics, observables can be multiplied. The physical meaning of this operation is by no means evident, but the fact that observables can be considered as selfadjoint elements of an operator algebra on Hilbert space is the constituting mathematical property of quantum theory.

States can be characterized as positive linear functionals on the algebra of observables. (The normalization condition which normally is imposed on states will be ignored within this note.) In the lecture of Professor Woronowicz we learned that an associative algebra with a coproduct is a natural generalization of the concept of a group. One may ask whether the algebra of observables in quantum theory also has a coproduct, so that observables may be interpreted as generalized symmetries.

Now a coproduct on an algebra immediately leads to a product on the space of linear functionals on the algebra, and it restricts to a product on the state space if the coproduct respects the *-structure. Actually, in quantum field theory there is a multiplicative structure on a certain subset of the state space. This product does not induce a coproduct on the whole algebra but it is indeed connected with the internal symmetries of the theory and their generalized group structure. It arises in the context of the theory of superselection sectors, and I will try to use it for a reformulation of this theory.

Let me first sketch the theory of superselection sectors following [1] (see [2] for the present stage of this field and [3] for a treatment in a textbook). Starting point is the description of a quantum field theory in terms of a Haag-Kastler net of local observable algebras

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$$\mathcal{A} = (\mathcal{A}(\mathcal{O}))_{\mathcal{O}\in\mathcal{K}} \tag{1}$$

where $\mathcal{K}$ is the set of open double cones in Minkowski space. We consider the algebras $\mathcal{A}(\mathcal{O})$ as von Neumann algebras in a defining representation π_0 (the "vacuum representation") on a Hilbert space $\mathcal{H}_0$ which satisfies Haag duality

$$\pi_0(\mathcal{A}(\mathcal{O}'))' = \pi_0(\mathcal{A}(\mathcal{O})) \quad ,\mathcal{O}\in\mathcal{K} \tag{2}$$

(here $\mathcal{O}'$ is the interior of the spacelike complement of $\mathcal{O}$, $\mathcal{A}(\mathcal{O}')$ is the C*-algebra generated by the algebras $\mathcal{A}(\mathcal{O}_1)$, $\mathcal{O}_1 \in \mathcal{K}$, $\mathcal{O}_1 \subset \mathcal{O}'$ and $\mathcal{M}'$ denotes for a subset $\mathcal{M} \subset \mathcal{B}(\mathcal{H}_0)$ the commutant) and contains a vector Ω which is cyclic and separating for each algebra $\mathcal{A}(\mathcal{O})$, $\mathcal{O}\in\mathcal{K}$ (Reeh-Schlieder property), i.e. $\mathcal{A}(\mathcal{O})\Omega$ is dense in $\mathcal{H}_0$ and $A\Omega = 0$, $A \in \mathcal{A}(\mathcal{O})$ implies $A = 0$. The state ω_0 induced by this vector

$$\omega_0(A) = (\Omega, \pi_0(A)\Omega), \quad A\in\mathcal{A} \tag{3}$$

is called the vacuum. We will heavily rely on the well known fact that by the GNS construction every state ω gives rise to a representation π_ω with a cyclic vector Φ_ω inducing the state in the above sense.

The DHR theory of superselection sectors treats representations of the algebra of observables which satisfy the selection criterion

$$\pi|_{\mathcal{A}(\mathcal{O}')} \simeq \pi_0|_{\mathcal{A}(\mathcal{O}')}, \quad \mathcal{O}\in\mathcal{K}. \tag{4}$$

Due to Haag duality for each $\mathcal{O}\in\mathcal{K}$ there is an endomorphism ρ of $\mathcal{A}$ acting trivially on $\mathcal{A}(\mathcal{O}')$ such that

$$\pi \simeq \pi_0 \circ \rho. \tag{5}$$

Moreover, two such endomorphisms are related by an inner automorphism of $\mathcal{A}$. The encoding of representations in terms of endomorphisms induces a product of representations

$$\pi_0\circ\rho_1 \times \pi_0\circ\rho_2 = \pi_0\circ\rho_1\rho_2 \tag{6}$$

which extends to a composition law for the equivalence classes. Due to locality of the underlying net the composition law is commutative on the level of classes. On the level of endomorphisms unitary intertwiners occur which induce a representation of the braid group (in $D = 2$ dimensions) or the symmetric group (in $D > 2$ dimensions) and which provide an intrinsic definition of particle statistics. The composition of irreducible representations may be reducible. This effect can be controlled in terms of a number $d(\pi)\in[1,\infty]$ (the "statistical dimension" of π). Products of representations with finite statistical dimension are completely reducible

$$\rho_i\rho_j \simeq \bigoplus N_{ij}^k \rho_k \tag{7}$$

with nonnegative integers N_{ij}^k, and the statistical dimensions satisfy

$$d(\rho_i)d(\rho_j) = \sum N_{ij}^k d(\rho_k), \tag{8}$$

hence they behave under product and decomposition like dimensions of group representations. Furthermore each irreducible representation with finite statistical dimension has a conjugate in the sense that the product contains a subrepresentation equivalent to the vacuum representation. The conjugate may be chosen to be irreducible, and it then is unique up to equivalence.

The sketched formalism is a beautiful description of the general structure of superselection sectors, and it is that structure which has been observed more recently in conformal field theory [4,5,6]. It has, however, a drawback, since it is often very difficult to apply the general formalism to specific examples. (See however [7,8, 9,10] for successful attempts.) It is the aim of the present note to start a reformulation of the theory in terms of states replacing the often nonaccessible endomorphisms. My hope is that this formulation will be useful for the analysis of models, and I will apply it to a simple model suggested by Rehren. Similar ideas have been presented by Buchholz [11] who aimed at a formulation in terms of Wightman fields.

We consider the set of positive linear functionals

$$\mathcal{S} = \{\omega \in \mathcal{A}_+^* \,|\, \pi_\omega \text{ satisfies the DHR criterion}\} \quad . \tag{9}$$

A norm dense subset $\mathcal{S}_0$ of $\mathcal{S}$ is formed by those functionals which are dominated at spacelike infinity by ω_0, $\mathcal{S}_0 = \bigcup_{\mathcal{O}\in\mathcal{K}} \mathcal{S}(\mathcal{O})$, where

$$\mathcal{S}(\mathcal{O}) = \{\omega \in \mathcal{S} \,|\, \exists \lambda > 0 \text{ such that } \omega(A') \leq \lambda\omega_0(A') \,\forall A' \in \mathcal{A}(\mathcal{O}'), A' \geq 0\} \quad . \tag{10}$$

As an example for a state in $\mathcal{S}(\mathcal{O})$ consider a partial intertwiner (a "vertex operator") for a DHR representation π which is localized in $\mathcal{O} \in \mathcal{K}$, i.e. an operator $S : \mathcal{H}_0 \to \mathcal{H}_\pi$ with

$$SA' = \pi(A')S \quad , A' \in \mathcal{A}(\mathcal{O}') \tag{11}$$

We have $S^*SA' = A'S^*S$, $A' \in \mathcal{A}(\mathcal{O}')$ and hence, by Haag duality, $S^*S \in \mathcal{A}(\mathcal{O})$, thus the state ω induced by $S\Omega$,

$$\omega(A) = (S\Omega, \pi(A)S\Omega) \quad , A \in \mathcal{A} \tag{12}$$

satisfies

$$\omega(A') = \omega_0(S^*SA') \leq ||S^*S||\omega_0(A') \tag{13}$$

for $A' \in \mathcal{A}(\mathcal{O}')$, $A' \geq 0$, i.e. $\omega \in \mathcal{S}(\mathcal{O})$. Actually, all states in $\mathcal{S}_0$ are obtained in this way. Namely, let $\omega \in \mathcal{S}_0$ and let $(\Phi_\omega, \mathcal{H}_\omega, \Phi_\omega)$ denote the GNS triple associated to ω. Let $\mathcal{O} \in \mathcal{K}$ such that $\omega \in \mathcal{S}(\mathcal{O})$. We then define a partial intertwiner S_ω localized in $\mathcal{O}$ by

$$A'\Omega \mapsto \pi_\omega(A')\Phi_\omega \quad , A' \in \mathcal{A}(\mathcal{O}') \tag{14}$$

Due to the Reeh-Schlieder property, S_ω is well defined on a dense domain. Moreover, we have

$$||S_\omega A'\Omega||^2 = ||\pi_\omega(A')\Phi_\omega||^2 = \omega(A'^*A') \leq \lambda\omega_0(A'^*A') = \lambda||A'\Omega||^2 \tag{15}$$

hence S_ω is bounded and may be extended to all of $\mathcal{H}_0$. In particular, S_ω is independent of the choice of $\mathcal{O} \in \mathcal{K}$ with $\omega \in \mathcal{S}(\mathcal{O})$. Moreover, S_ω has the intertwining property (11). We now use this operator for the definition of a completely positive mapping on $\mathcal{A}$,

$$\chi_\omega(A) = S_\omega^* \pi_\omega(A) S_\omega \quad , A \in \mathcal{A} \quad . \tag{16}$$

χ_ω satisfies $\omega_0 \circ \chi_\omega = \omega$ and

$$\chi_\omega(A'BC') = A'\chi_\omega(B)C' \quad , A', C' \in \mathcal{A}(\mathcal{O}'), B \in \mathcal{A} \quad , \tag{17}$$

χ_ω is uniquely determined by these properties; namely, on the dense domain $\mathcal{A}(\mathcal{O}')\Omega$ the matrix elements of $\chi_\omega(A)$ are given by

$$(B'\Omega, \chi_\omega(A)C'\Omega) = (\Omega, \chi_\omega(B'^*AC')\Omega) = \omega(B'^*AC'). \tag{18}$$

As an easy further consequence we find that the association $\omega \mapsto \chi_\omega$ respects positive linear combinations. Actually, by Haag duality, χ_ω maps $\mathcal{A}(\mathcal{O}_1)$ into $\mathcal{A}(\mathcal{O}_1)$ for $\mathcal{O}_1 \supset \mathcal{O}$, hence $\mathcal{A}$ into $\mathcal{A}$. The argument is analogous to that which leads to endomorphisms in the DHR theory. Namely, let $A \in \mathcal{A}(\mathcal{O}_1)$, $\mathcal{O}_1 \in \mathcal{K}$ and $\mathcal{O}_1 \supset \mathcal{O}$ and let $B' \in \mathcal{A}(\mathcal{O}_1')$. Then by (16) and by locality

$$[\chi_\omega(A), B'] = \chi_\omega([A, B']) = 0, \tag{19}$$

hence by Haag duality $\chi_\omega(A) \in \mathcal{A}(\mathcal{O}_1)$. We conclude that the positive mappings associated to states in $\mathcal{S}_0$ can be composed.

This fact will be used for a definition of a product of states. Let $\omega_1, \omega_2 \in \mathcal{S}_0$. Then we set

$$\omega_1 \times \omega_2 := \omega_0 \circ \chi_{\omega_1}\chi_{\omega_2} \quad . \tag{20}$$

$\omega_1 \times \omega_2$ clearly is a positive linear functional. It is again an element of $\mathcal{S}_0$. For, if $\omega_1, \omega_2 \in \mathcal{S}(\mathcal{O})$ and $A' \in \mathcal{A}(\mathcal{O}')$ we have by (17)

$$\omega_1 \times \omega_2(A') = \omega_0(\chi_{\omega_1}\chi_{\omega_2}(1)A'). \tag{21}$$

But (17) also implies, in connection with Haag duality, that $\chi_{\omega_1}\chi_{\omega_2}(1) \in \mathcal{A}(\mathcal{O})$, hence for $A' \geq 0$

$$\omega_1 \times \omega_2(A') \leq ||\chi_{\omega_1}\chi_{\omega_2}(1)||\omega_0(A'), \tag{22}$$

thus $\omega_1 \times \omega_2 \in \mathcal{S}(\mathcal{O})$. We also also observe the homomorphism property

$$\chi_{\omega_1 \times \omega_2} = \chi_{\omega_1}\chi_{\omega_2}. \tag{23}$$

We now want to compare this law of composition with the composition within the DHR theory. Let $\mathcal{O} \in \mathcal{K}$, $\omega \in \mathcal{S}(\mathcal{O})$ and $\rho \in \mathrm{End}(\mathcal{A})$ such that ρ acts trivially on $\mathcal{A}(\mathcal{O}')$ and

$$\pi_\omega \simeq \pi_0 \circ \rho. \tag{24}$$

Let V be a unitary mapping from $\mathcal{H}_0$ to $\mathcal{H}_\omega$ which realizes this equivalence. Then

$$\Psi = V^* \Phi_\omega \tag{25}$$

is a vector in $\mathcal{H}_0$ which is cyclic for $\rho(\mathcal{A})$ and satisfies

$$(\Psi, \rho(A)\Psi) = \omega(A), \quad A \in \mathcal{A}. \tag{26}$$

Let $S = V^* S_\omega$. Then $S \in \mathcal{A}(\mathcal{O})$, $\chi_\omega(A) = S^*\rho(A)S$, $A \in \mathcal{A}$ and

$$\omega = \omega_{0,S} \circ \rho \tag{27}$$

where we use the notation

$$\phi_A(B) = \phi(A^*BA) \quad , \phi \in \mathcal{A}^*_+, A, B \in \mathcal{A}. \tag{28}$$

S and ρ are, in contrast to χ_ω, not uniquely determined by ω. If we require that $S\Omega$ is cyclic for $\rho(\mathcal{A})$ the freedom consists in replacing S by US and ρ by $\mathrm{Ad}U \circ \rho$ for unitaries $U \in \mathcal{A}(\mathcal{O})$. ($\mathrm{Ad}U$ means the adjoint action $\mathrm{Ad}U(A) = UAU^{-1}$ on $\mathcal{A}$.)

Now let us compute the product of states $\omega_1, \omega_2 \in \mathcal{S}(\mathcal{O})$. We have

$$\omega_i = \omega_{0,S_i} \circ \rho_i \quad , i = 1, 2 \tag{29}$$

and thus

$$\omega_1 \times \omega_2(A) = \omega_{0,S_1} \circ \rho_1(S_2^*\rho_2(A)S_2) = \omega_{0,\rho_1(S_2)S_1} \circ \rho_1\rho_2(A). \tag{30}$$

So $\omega_1 \times \omega_2$ is a state in the representation $\pi_0 \circ \rho_1\rho_2$, thus the law of composition for states (21) leads to the DHR law of composition of sectors (6). It may happen, however, that the vector $\rho_1(S_2)S_1\Omega$ is not cyclic for the algebra $\rho_1\rho_2(\mathcal{A})$, so $\pi_{\omega_1 \times \omega_2}$ is, in general, only equivalent to a subrepresentation of the DHR product $\pi_{\omega_1} \times \pi_{\omega_2}$.

In the next step we want to analyze the fusion rules (7) on the level of states. Let ρ_i be irreducible, localized in $\mathcal{O} \in \mathcal{K}$ and mutually inequivalent where i varies over the set of sectors. There are isometries $T_{ij}^{k,n} \in \mathcal{A}(\mathcal{O})$, $n = 1, \ldots, N_{ij}^k$ such that

$$\rho_i\rho_j(A)T_{ij}^{k,n} = T_{ij}^{k,n}\rho_k(A) \quad , A \in \mathcal{A} \tag{31}$$

satisfying the orthonormality and completeness relations [6]

$$(T_{ij}^{k,n})^* T_{ij}^{k',n'} = \delta_{kk'}\delta_{nn'} \tag{32}$$

and

$$\sum_{k,n} T_{ij}^{k,n}(T_{ij}^{k,n})^* = 1 \tag{33}$$

Hence we obtain the decomposition

$$\rho_i\rho_j(A) = \sum_{k,n} T_{ij}^{k,n}\rho_k(A)(T_{ij}^{k,n})^* \quad , A \in \mathcal{A}. \tag{34}$$

Now let $\omega_i = \omega_{0,S_i} \circ \rho_i$. Then

$$\omega_i \times \omega_j = \omega_{0,\rho_i(S_j)S_i} \circ \rho_i\rho_j = \sum_{k,n} \omega_{0,(T_{ij}^{k,n})^*\rho_i(S_j)S_i} \circ \rho_k =: \sum_k \omega_k \tag{35}$$

where ω_k is a state in the representation ρ_k which is a sum of at most N_{ij}^k pure states. The decomposition of the product of states into the components ω_k is unique, hence the fusion rules may be obtained directly in terms of the states.

We now turn to the discussion of conjugates. Let $\rho, \bar\rho$ be irreducible, localized in $\mathcal{O}$ and conjugate to each other, i.e. there are isometries $R, \bar R \in \mathcal{A}(\mathcal{O})$ such that

$$\bar\rho\rho(A)R = RA, \quad \rho\bar\rho(A)\bar R = \bar R A. \tag{36}$$

According to the DHR theory, the phases of these isometries can be chosen such that

$$R^*\bar\rho(\bar R)d(\rho) = 1 = \bar R^*\rho(R)d(\rho). \tag{37}$$

Now let $\omega = \omega_{0,S} \circ \rho$. We define a conjugate state $\bar\omega$ by

$$\bar\omega = d(\omega)\omega_{0,\bar\rho(S^*)R} \circ \bar\rho. \tag{38}$$

The conjugate state is uniquely determined. This follows from the fact that the freedom in the choice of S, ρ, $\bar\rho$ and R consists in replacing S by US, ρ by $\mathrm{Ad}U \circ \rho$, $\bar\rho$ by $\mathrm{Ad}V \circ \bar\rho$ and R by $V\bar\rho(U)R$ for unitaries $U, V \in \mathcal{A}(\mathcal{O})$. Moreover, the double conjugate of ω coincides with ω. In fact, by (38)

$$\bar{\bar\omega} = d(\omega)^2\omega_{0,\rho(R^*\bar\rho(S))\bar R} \circ \rho, \tag{39}$$

but by (36) and (37)

$$\rho(R^*\bar\rho(S))\bar R = \rho(R^*)\rho\bar\rho(S)\bar R = \rho(R^*)\bar R S = d(\omega)^{-1}S, \tag{40}$$

hence $\bar{\bar\omega} = \omega$.

We now look for a characterization of the conjugate which does not refer to the endomorphisms. Let us consider the products $\omega \times \bar\omega$ and $\bar\omega \times \omega$. For performing the straightforward calculations it is convenient to use the positive mappings ψ_A, $A \in \mathcal{A}$,

$$\psi_A(B) = ABA^* \quad , B \in \mathcal{A}. \tag{41}$$

We note the rules

$$\psi_A\psi_B = \psi_{AB},\ \psi_{\lambda A} = |\lambda|^2\psi_A,\ \rho\psi_A = \psi_{\rho(A)}\rho,\ \psi_A = \mathrm{Ad}A \text{ for } A \text{ unitary.} \tag{42}$$

Now $\chi_\omega = \psi_{S^*}\rho$ and $\chi_{\bar\omega} = d(\omega)\psi_{R^*\bar\rho(S)}\bar\rho$, hence

$$\chi_\omega\chi_{\bar\omega} = d(\omega)\psi_{S^*}\rho\psi_{R^*\bar\rho(S)}\bar\rho = d(\omega)\psi_{S^*\rho(R^*)\rho\bar\rho(S)}\rho\bar\rho. \tag{43}$$

By (36) $\rho\bar\rho \geq \psi_{\bar R}$, hence

$$\chi_\omega\chi_{\bar\omega} \geq d(\omega)\psi_{S^*\rho(R^*)\rho\bar\rho(S)\bar R} \tag{44}.$$

By the use of (40) and (42) we finally get

$$\omega \times \bar\omega \geq d(\omega)^{-1}\omega_{0,\chi_\omega(1)}. \tag{45}$$

In a similar way

$$\bar\omega \times \omega \geq d(\omega)^{-1}\omega_{0,\chi_{\bar\omega}(1)}. \tag{46}$$

The inequalities (45) and (46) are the desired relations which contain only the statistical dimension and the conjugate state. It is conceivable that they characterize the conjugate state as well as the statistical dimension completely in the following sense: let $\mathcal{S}_\omega$ denote the set of all $\omega' \in \mathcal{S}_0$ for which there exists some $\lambda > 0$ such that

$$\omega \times \omega' \geq \lambda\omega_{0,\chi_\omega(1)}, \quad \omega' \times \omega \geq \lambda\omega_{0,\chi_{\omega'}(1)}. \tag{47}$$

Let $\lambda(\omega')$ be the supremum of all numbers λ satisfying (47) for a given $\omega' \in \mathcal{S}_\omega$. Then the conjecture is

(i) $d(\omega) = \inf_{\omega' \in \mathcal{S}(\omega)} \lambda(\omega')^{-1}$.
(ii) $\bar\omega$ is the only state in $\mathcal{S}_\omega$ with $\lambda(\bar\omega) = d(\omega)^{-1}$.

A proof of this conjecture is not available at the moment. I will instead illustrate the concepts discussed in this note on a simple model which was suggested by Rehren [12].

Let φ be the massless scalar free field in two dimensions, and let $j = \partial_0\varphi - \partial_1\varphi$. Then j depends only on the light cone variable $u = t - x$ and has the commutation relation

$$[j(u), j(u')] = 2i\delta'(u - u'). \tag{48}$$

The 2-point function of j is

$$(\Omega, j(u)j(u')\Omega) = \frac{1}{\pi}\int_0^\infty dp\, p e^{-ip(u-u')} = \frac{-1}{\pi(u - u' - i\epsilon)^2}. \tag{49}$$

We introduce smeared fields

$$j(f) = \int du\, j(u) f(u) \tag{50}$$

with real valued test functions $f \in \mathcal{D}(\mathbb{R})$ and find

$$[j(f), j(g)] = -2i\int du\, f'(u)g(u) = 2i\int du\, f(u)g'(u) =: 2i\sigma(f,g) \tag{51}$$

and

$$||j(f)\Omega||^2 = 2\int_0^\infty dp\, p|\tilde f(p)|^2 =: 2||f||^2. \tag{52}$$

For an analysis of this model in the framework of algebraic field theory (see e.g. [7]) it is convenient to introduce the Weyl operators

$$W(f) = e^{ij(f)} \tag{52}$$

which satisfy the Weyl relations

$$W(f)W(g) = e^{-i\sigma(f,g)}W(f+g) \tag{53}$$

as a mathematical precise version of the canonical commutation relations. Let $\mathcal{I}$ denote the set of bounded open intervalls on $\mathbb{R}$ and let $\mathcal{A}_0(I)$ denote the algebra generated by all Weyl operators $W(f)$ with $f \in \mathcal{D}(I)$. The vacuum is characterized by the expectation functional

$$(\Omega, W(f)\Omega) = \omega_0(W(f)) = e^{-||f||^2}, \tag{54}$$

and the local algebras $\mathcal{A}(I)$, $I \in \mathcal{I}$ are defined as the weak closures of $\mathcal{A}_0(I)$ in the vacuum Hilbert space.

The Haag-Kastler net $(\mathcal{A}(I))_{I\in\mathcal{I}}$ has nontrivial outer automorphisms which act trivially on the complement I' of some $I \in \mathcal{I}$. Namely, let F be a smooth real function such that $F' \in \mathcal{D}(I)$. Then $\sigma(F,g)$ is well defined for all $g \in \mathcal{D}(I)$ and vanishes when $g \in \mathcal{D}(I')$. The automorphism is then defined on $\mathcal{A}_0$ by

$$\alpha_F(W(g)) = e^{2i\sigma(F,g)}W(g). \tag{55}$$

For each intervall $J \in \mathcal{I}$ there is some test function $f_J \in \mathcal{D}$ such that $F' = f_J'$ on J, hence

$$\alpha_F|_{\mathcal{A}_0(J)} = \mathrm{Ad}W(f_J)|_{\mathcal{A}_0(J)}. \tag{56}$$

This shows in particular that α_F can be uniquely extended to an automorphism of the von Neumann algebra $\mathcal{A}(J)$.

These automorphisms generate superselection sectors which are classified by the real number

$$q(F) = F(\infty) - F(-\infty), \tag{57}$$

and because of the product rule $\alpha_F\alpha_G = \alpha_{F+G}$ they obey the fusion rules

$$q_1 \times q_2 = q_1 + q_2. \tag{58}$$

As a model with nontrivial fusion rules Rehren proposed the restriction of this model to the even subalgebras $\mathcal{A}_e(I)$. These are the algebras which are invariant under the symmetry $\gamma : j(u) \mapsto -j(u)$; they can be generated by the hermitian operators

$$V(f) = \frac{1}{2}(W(f) + W(-f)) \tag{59}$$

which satisfy the relations

$$V(f)V(g) = \frac{1}{2}e^{-i\sigma(f,g)}V(f+g) + \frac{1}{2}e^{i\sigma(f,g)}V(f-g). \tag{60}$$

The sectors of $\mathcal{A}$ may split after restriction to $\mathcal{A}_e$. This is true for the vacuum, because there the automorphism γ is unitarily implemented by an operator Γ with $\Gamma^2 = 1$; the eigenspaces of Γ reduce the vacuum representation which splits into an even and an odd sector. It is not true for the charged sectors, since there the symmetry γ is spontaneously broken. Therefore there exists a sequence of odd elements of $\mathcal{A}$ which converges weakly to 1, thus each odd element can be obtained as a weak limit point of elements of $\mathcal{A}_e$, and $\mathcal{A}_e$ acts irreducibly on these sectors. As an example for such a sequence take

$$A_n = \frac{\lambda}{2i}(W(f_n) - W(-f_n)) \tag{61}$$

where $\lambda \in \mathbb{R}$ and $f_n(u) = f(\frac{u}{n})$. Then $[A_n, W(g)] \to 0$ for all $g \in \mathcal{D}$ and

$$\omega_0 \circ \alpha_F(A_n) = \lambda \sin 2\sigma(F, f_n)e^{-||f_n||^2} \to -\lambda \sin 2q(F)f(0)e^{-||f||^2}, \tag{62}$$

hence for a suitable choice of λ and f $A_n \to 1$ weakly.

We now want to study the fusion rules of this model. It seems to be difficult to exhibit the DHR endomorphisms, but it is very easy to give explicit formulae for the positive mappings. We find

$$\omega_F := \omega_0 \circ \alpha_F|_{\mathcal{A}_e} = \omega_0 \circ \chi_F \tag{63}$$

where

$$\chi_F(V(f)) = \cos\sigma(F,f)V(f) \quad , f \in \mathcal{D}. \tag{64}$$

We obtain the composition law

$$\chi_F\chi_G = \frac{1}{2}\chi_{F+G} + \frac{1}{2}\chi_{F-G}. \tag{65}$$

If $q(F+G)$ and $q(F-G)$ are nonzero, the corresponding decomposition of states,

$$\omega_F \times \omega_G = \frac{1}{2}\omega_{F+G} + \frac{1}{2}\omega_{F-G} \tag{66}$$

is a decomposition into pure states. If, however, $q(F-G) = 0$, say, there is some $f \in \mathcal{D}$ with $f = F-G$, so we have to investigate the state $\omega_f = \omega_0 \circ \chi_f$. Now χ_f has the decomposition

$$\chi_f = \chi_f^{(e)} + \chi_f^{(o)} \tag{67}$$

where $\chi_f^{(e)} = \psi_{V(f)}$ and

$$\chi_f^{(o)}(V(g)) = \frac{1}{2}\cos 2\sigma(f,g)V(g) - \frac{1}{4}V(g+2f) - \frac{1}{4}V(g-2f). \tag{68}$$

$\chi_f^{(o)}$ is not unit preserving, namely

$$\chi_f^{(o)}(1) = \frac{1}{2}(1 - V(2f)). \tag{69}$$

The statistical dimension may now be infered from the conjugation symmetry. We have

$$\omega_F \times \omega_F \geq \frac{1}{2}\omega_0 \tag{70}$$

which is consistent with $d(\omega_F) = 2$, and with $\omega_f^{(o)} = \omega_0 \circ \chi_f^{(o)}$

$$\omega_f^{(o)} \times \omega_f^{(o)} = \omega_{0,\frac{1}{2}(1-V(2f))}, \tag{71}$$

hence $d(\omega_f^{(o)}) = 1$. Actually, as Rehren showed, all statistical dimensions as well as the statistical phases of this model can be computed, up to one phase, once the fusion rules are known [12].

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