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Dynamical Symmetry as a Tool to Understanding Properties of Supersymmetric Partner Potentials. Example of $so(2,1)$ Symmetry

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Dynamical symmetry as a tool to understanding properties of supersymmetric partner potentials. Example of so(2,1) symmetry

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Analysis of the dynamical symmetry of a system is used to predict properties arising from its supersymmetric quantum mechanical treatment. Two applications of the $so(2,1)$ *algebra, the Coulomb potential and Morse oscillator potential which display different structure with respect to the dynamical symmetry, are studied. This difference is shown to be responsible for the behaviour of the respective supersymmetric partner potentials.*

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Systems displaying dynamical symmetry can be treated by algebraic techniques[1]. They can also be treated by the methods of supersymmetry[2] or factorization[3]. We consider here the possibility of using the group theoretical analysis to deduce, or at least to understand, certain properties of its corresponding supersymmetric approach.

We treat two essentially different applications of the $so(2,1)$ algebra. In one application, the Casimir operator of the subalgebra $so(2)$ is related to the energy of the problem, whereas, in the other, the Casimir operator of the $so(2,1)$ algebra itself is related to the energy. To illustrate these two types of behaviour, we consider the bound states

of the radial Coulomb problem and the Morse oscillator respectively. Note that the conventional application of the algebra $so(2,1)$ to the radial Coulomb problem relates the Casimir of the whole algebra with the energy of the problem. This relation requires however a knowledge of the Greens function[4] or, equivalently, of the inverse of the Casimir operator[5].

The manner in which the $so(2,1)$ algebra enters into the problem has some important consequences, particularly in relation to the behaviour of supersymmetric partner potentials and in the treatment of scattering states. This paper will highlight the differences arising in two different applications of the $so(2,1)$ algebra and show how the group theoretical analysis can be used to predict certain properties of the supersymmetric potentials.

Let us consider first the radial Schrödinger equation for the Coulomb problem

$$\left[-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}-\frac{2Z}{r}-2E_{n_r,l}\right]\psi_{n_r,l}(r)=0,$$

where n_r is the quantum number representing the number of radial nodes and l is the angular momentum quantum number. This equation can be written in the form

$$\left[-r\frac{d^2}{dr^2}+\frac{l(l+1)}{r}+\frac{Z^2}{n^2}r\right]\psi_{n_r,l}(r)=2Z\psi_{n_r,l}(r). \qquad (1)$$

Following a previous $so(2,1)$ approach to the Coulomb problem[6], we consider the operators

$$W_1=\left(\frac{Z}{n}\right)r\ ,$$

$$W_3=\left(\frac{Z}{n}\right)^{-1}\left(-r\frac{d^2}{dr^2}+\frac{l(l+1)}{r}\right)$$

and their commutator

$$[W_1,W_3]=2iW_2=2r\frac{d}{dr}.$$

Introduction of the operators

$$T_1 = (1/2)(W_3 - W_1),$$
$$T_2 = W_2, \qquad (2)$$
$$T_3 = (1/2)(W_3 + W_1),$$

leads to the commutation relations of the $so(2,1)$ algebra,

$$[T_1, T_2] = -iT_3, \quad [T_2, T_3] = iT_1, \quad [T_3, T_1] = iT_2.$$

Equation (1) can be expressed in the form

$$T_3\psi_{n_r,l} = n\psi_{n_r,l} \equiv (n_r + l + 1)\psi_{n_r,l},$$

and the Casimir of $so(2,1)$ becomes

$$C_2 = T_3^2 - T_1^2 - T_2^2 = l(l+1).$$

Hence C_2 determines the angular momentum of the problem and T_3, the Casimir of the subalgebra $so(2)$, determines the energy.
Introducing the ladder operators $T_\pm$ defined by

$$T_\pm = T_1 \pm iT_2 = \pm r\frac{d}{dr} - \frac{Zr}{n} + n$$

we have

$$[T_\pm, T_3] = \mp T_\pm$$

so that

$$T_3T_\pm\psi_{n_r,l} = (n_r + l + 1 \pm 1)T_\pm\psi_{n_r,l} = (n \pm 1)T_\pm\psi_{n_r,l}$$

which implies

$$T_\pm\psi_{n_r,l} \propto \psi_{n_r\pm1,l},$$

showing that $T_\pm$ acts as a ladder operator in the principal quantum number n at fixed l.

We now turn to a related $so(2,1)$ approach to the Morse oscillator[7]. The Morse Hamiltonian is given by

$$H = -\frac{\hbar^2}{2\mu}\frac{d^2}{dR^2} + D_e(1 - e^{-\alpha(R-R_e)})^2,$$

where D_e is the well depth, μ is the reduced mass, R_e is the equilibrium separation and α is a range parameter. Transforming to the dimensionless variable $y = \alpha(R - R_e)$, and defining the well depth parameter λ via the relation

$$\lambda^2 = 2\mu D_e/\alpha^2\hbar^2,$$

we obtain the reduced Hamiltonian (in units of $\alpha^2\hbar^2/2\mu$)

$$H = -\frac{d^2}{dy^2} + \lambda^2(1 - e^{-y})^2.$$

The eigenvalues of this Hamiltonian are well known[8], namely

$$E_v = 2\lambda(v + 1/2) - (v + 1/2)^2,$$

where v, the vibrational quantum number, takes the values $v = 0, 1, 2, .., v_{max}$ with $v_{max} = \mathrm{Int}(\lambda - 1/2)$.
Using the above energy eigenvalue expression, we can write the Schrödinger equation for the Morse oscillator in the form

$$\left[-\frac{d^2}{dy^2} + \lambda^2 e^{-2y} - 2\lambda^2 e^{-y} + (\lambda - v - 1/2)^2\right]\psi_{v,\lambda} = 0 \qquad (3a)$$

which may be rewritten as

$$\left[-e^y\frac{d^2}{dy^2} + (\lambda - v - 1/2)^2 e^y + \lambda^2 e^{-y}\right]\psi_{v,\lambda} = 2\lambda^2\psi_{v,\lambda}. \qquad (3b)$$

We now define the operators

$$W_1 = \lambda e^{-y},$$

$$W_3 = \lambda^{-1}\left(-e^y \frac{d^2}{dy^2} + (\lambda - v - 1/2)^2 e^y\right)$$

with commutator

$$[W_1, W_3] = 2iW_2 = 1 - 2\frac{d}{dy}.$$

Again introducing the operators T_i, $i = 1,2,3$ as in eq. (2) we note that equation (3b) becomes

$$T_3\psi_{v,\lambda} = \lambda\psi_{v,\lambda}.$$

For the Casimir C_2 of the algebra $so(2,1)$ we have, in this case,

$$C_2 = T_3^2 - T_1^2 - T_2^2 = (\lambda - v)(\lambda - v - 1) \equiv (\lambda - v - 1/2)^2 - (1/2)^2.$$

Hence for the Morse potential, the eigenvalues of the operator T_3 are related to the well depth parameter, and the Casimir operator of the algebra $so(2,1)$ itself determines the energy.

Since the $so(2,1)$ representations correspond here to constant energy, it is appropriate to incorporate this fact into the analysis. We define the quantity

$$\Lambda = \lambda - v,$$

where the well depth parameter and vibrational quantum number will vary to as to maintain a constant value of Λ. Then equations for the Casimir operators C_2 and T_3 take the form

$$C_2 = \Lambda(\Lambda - 1) \qquad \text{and} \qquad T_3\psi_{v,\Lambda+v} = (\Lambda + v)\psi_{v,\Lambda+v}\,.$$

For the Morse, the ladder operators $T_\pm$ are given by

$$T_\pm = T_1 \pm iT_2 = \mp\left(\frac{d}{dy} - \frac{1}{2}\right) + \left(\frac{1}{2\lambda}\right)\left(-e^y\frac{d^2}{dy^2} + \left(\lambda - v - \frac{1}{2}\right)^2 e^y - \lambda^2 e^{-y}\right)$$

and obey the relation

$$T_3T_\pm = T_\pm(T_3 \pm 1).$$

Hence

$$T_3T_\pm\psi_{v,\Lambda+v} = (\Lambda + v \pm 1)T_\pm\psi_{v,\Lambda+v}$$

so that

$$T_\pm\psi_{v,\Lambda+v} = \left[\mp\left(\frac{d}{dy} - \frac{1}{2}\right) + \lambda(1 - e^{-y})\right]\psi_{v,\Lambda+v} \propto \psi_{v\pm1,\Lambda+v\pm1}, \qquad (4)$$

where we have used equation (3b) to simplify the effect of $T_\pm$ on the eigenstates $\psi_{v,\Lambda+v}$. These operators are more correctly referred to as shift operators, since they act at constant energy, to distinguish them from ladder operators which act to change the energy. The operators connect different Morse potentials characterized by well depth parameters which differ by one unit, such that the energy remains constant. This effect has been exploited recently in the potential group approach to the Morse oscillator[9].

The two problems treated here can be transformed into each other[7,8] by the change of variable $r \to e^{-y}$. Then equation (3a) for the Morse oscillator becomes

$$\left[-\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) + \frac{(\lambda - v - 1/2)^2}{r^2} - \frac{2\lambda^2}{r} + \lambda^2\right]\psi_{v,\lambda} = 0.$$

The term involving d/dr can be removed by means of the similarity transformation $r^{1/2}Hr^{-1/2}$. It follows that

$$\left[-\frac{d^2}{dr^2} + \frac{(\lambda - v)(\lambda - v - 1)}{r^2} - \frac{2\lambda^2}{r} + \lambda^2\right]\tilde{\psi}_{v,\lambda} = 0,$$

where $\tilde{\psi}_{v,\lambda} = e^{-y/2}\psi_{v,\lambda} = r^{1/2}\psi_{v,\lambda}$. This equation is equivalent to that for the Coulomb problem

$$\left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - \frac{2Z}{r} + \left(\frac{Z}{n}\right)^2\right]\psi_{n_r,l} = 0$$

after the identifications (writing $n_r \equiv v$)

$$l \to \lambda - v - 1, \qquad Z \to \lambda^2, \qquad n \equiv v + l + 1 \to \lambda.$$

Thus the set of bound levels of a single Morse potential are mapped on to the set of degenerate levels of a Coulomb problem with principal quantum number equal to the well depth parameter and with nuclear charge equal to the square of the well depth parameter. Note that this mapping applies when λ is an integer, since n is by definition an integer.

At this point we would like to point out some differences in the two applications of the $so(2,1)$ algebra. We note that the eigenvalues of the Casimir of so(2,1) depend on the particular unitary representation of the algebra. Two such representations are as follows :

(i) The discrete principal series.

$$C_2|j,m\rangle = j(j+1)|j,m\rangle \qquad \text{and} \qquad T_3|j,m\rangle = m|j,m\rangle\,,$$

with $j = -1/2 - n/2$ $(n = 0,1,2,..)$ and $m = -j, -j+1, ...$

(ii) The continuous principal series

$$C_2|j,m\rangle = j(j+1)|j,m\rangle \qquad \text{and} \qquad T_3|j,m\rangle = m|j,m\rangle\,,$$

with $j = -1/2 + ik$ $(0 < k < \infty)$ and $m = 0, \pm1, \pm2, ..$ or $m = \pm 1/2, \pm 3/2, ...$

The bound state problems treated above make use of the discrete series. In the case of the Morse potential, the Casimir operator is directly related to the energy of the system; thus the continuous principal series immediately describes its scattering states at the energy $E = k^2$, with k defined as above. In the case of the Coulomb problem, the continuous series describes a non - physical situation, consisting of a Coulomb potential with a continuous value of angular momentum. The scattering states of the Coulomb problem can, of course, also be treated by the $so(2,1)$ algebraic approach[10], but not in the method described above, which applies specifically to bound state problems.

To study a more subtle consequence of the difference in the application of $so(2,1)$ to the two problems, we consider the application of supersymmetric quantum mechanics. The link between supersymmetric quantum mechanics[2] and the factorization method[3]

has been studied by a number of authors (see for example ref. 11) and we shall adopt a scheme which incorporates aspects of both approaches. We first factorize the Coulomb potential problem

$$\left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - \frac{2Z}{r} - 2E_{n_r,l}\right)\psi_{n_r,l} = 0$$

using the following ansatz :

$$a^\dagger(l)a(l)\psi_{n_r,l} = 2\Delta E_{n_r,l}\psi_{n_r,l},$$

where $\Delta E_{n_r,l} = E_{n_r,l} - E_{0,l}$ and the operators $a(l)$ and $a^\dagger(l)$ are expressed in the form

$$a(l) = \frac{d}{dr} + \frac{\alpha}{r} + \beta$$

$$a^\dagger(l) = -\frac{d}{dr} + \frac{\alpha}{r} + \beta$$

Here, the coefficients α and β and the ground state energy $E_{0,l}$ (for constant value of l) are to be determined by the factorization condition and the relation

$$a(l)\psi_{0,l} \equiv 0$$

which corresponds to annihilation of the ground state for that particular value of l. Hence

$$a(l) = \frac{d}{dr} - \frac{(l+1)}{r} + \frac{Z}{l+1},$$

$$a^\dagger(l) = -\frac{d}{dr} - \frac{(l+1)}{r} + \frac{Z}{l+1},$$

and the ground state energy $E_{0,l}$ becomes

$$E_{0,l} = -\frac{Z^2}{2(l+1)^2}.$$

For the partner Hamiltonian $a(l)a^\dagger(l)$, we find

$$a(l)a^\dagger(l) = -\frac{d^2}{dr^2} + \frac{(l+1)(l+2)}{r^2} - \frac{2Z}{r} + \frac{Z^2}{l+1^2}$$

$$\equiv a^\dagger(l+1)a(l+1) + 2\Delta E_{1,l}. \tag{5}$$

Thus, the partner Hamiltonian corresponds to a Coulomb Hamiltonian with an increase in the l quantum number of one unit, and with the same energy spectrum as the parent Hamiltonian except that there is one less bound level, the lowest level having been shifted from $E_{0,l}$ to $E_{1,l}$. Further, we see that the shift operators act between states of different l values at constant energy i.e.,

$$a(l)\psi_{n_r,l} \propto \psi_{n_r-1,l+1},$$

$$a^\dagger(l)\psi_{n_r-1,l+1} \propto a^\dagger(l)a(l)\psi_{n_r,l} = 2\Delta E_{n_r,l}\psi_{n_r,l},$$

where the constants of proportionality may be determined from the appropriate normalization conditions.

We consider now the supersymmetric approach to the Morse oscillator. We factorize the Hamiltonian

$$H = -\frac{d^2}{dy^2} + \lambda^2(1-e^{-y})^2$$

in the form

$$a^\dagger(\lambda)a(\lambda)\psi_v = \Delta E_{v0}\psi_v,$$

where

$$\Delta E_{v0} = E_v - E_0 = v(2\lambda - v - 1).$$

Then it follows that

$$a(\lambda) = \frac{d}{dy} + \lambda(1-e^{-y}) - 1/2$$

$$a^\dagger(\lambda) = -\frac{d}{dy} + \lambda(1-e^{-y}) - 1/2$$

For the partner Hamiltonian $a(\lambda)a^\dagger(\lambda)$ we have

$$a(\lambda)a^\dagger(\lambda) = -\frac{d^2}{dy^2} + (\lambda-1)^2\left[1 - \left(\frac{\lambda}{\lambda-1}\right)e^{-y}\right] + \lambda - 3/4.$$

This is seen to be a Morse oscillator with its well depth parameter reduced by one unit *and* its equilibrium position shifted from $y_0 = \ln 1 \equiv 0$ to $y_1 = \ln(\lambda/(\lambda-1))$. Note that this constitutes a striking example of shape invariance, as defined by Gendenshtein[12]. Using the commutation relation,

$$[a(\lambda), a^\dagger(\lambda)] = 2\lambda e^{-y} = 2\lambda - 1 - a(\lambda) - a^\dagger(\lambda)$$

and noting that $2\lambda - 1 \equiv \Delta E_{10}$, we have

$$a(\lambda)a^\dagger(\lambda) = (a^\dagger(\lambda)-1)(a(\lambda)-1) + \Delta E_{10}. \tag{6}$$

It then follows that the operators $a(\lambda)$ and $a^\dagger(\lambda)$ act as shift operators between degenerate states of shifted Morse oscillators, such that

$$a(\lambda)\psi_{v,\lambda} \propto \psi_{v-1,\lambda-1},$$

$$a^\dagger(\lambda)\psi_{v-1,\lambda-1} \propto a^\dagger(\lambda)a(\lambda)\psi_{v,\lambda} = \Delta E_{v0}\psi_{v,\lambda}.$$

These equations together with eq. (4) imply that the operators T_+, T_-, and $a^\dagger$, a should be related in the case of the Morse problem. However, the $so(2,1)$ operators $T_\pm$ both act on the *same* state $\psi_{v,\lambda}$, whereas the supersymmetric shift operators $a(\lambda)$ and $a^\dagger(\lambda)$ act on different states, $\psi_{v,\lambda}$ and $\psi_{v-1,\lambda+1}$ respectively. The operators T_- and $a(\lambda)$, both of which act on $\psi_{v,\lambda}$, can be seen to be equivalent. From the properties of the supersymmetric partner potentials, we note that if the shift operator $a^\dagger(\lambda)$ converts $\psi_{v-1,\lambda-1}$ into $\psi_{v,\lambda}$, then the operator $a^\dagger(\lambda)+1$ will convert $\psi_{v,\lambda}$ into $\psi_{v+1,\lambda+1}$. This equivalence of the shift operators T_+ and $a^\dagger(\lambda)+1$ is also apparent from their definitions. This result demonstrates that the shift in equilibrium position associated with the various partner potentials is required to ensure that the shift operators arising from the supersymmetric approach correspond to those arising from the basic $so(2,1)$ dynamical symmetry when operating on the *same* eigenstate.

We now proceed to study how certain properties of the supersymmetric approach follow from an analysis of role played by the algebra $so(2,1)$ in the problem. In particular we are interested in the relations between the parent potential $a^\dagger a$ and the partner potential $aa^\dagger$. For this purpose the following properties of these potential are important.

i.- The energy spectrum of these potentials is the same except that in the $aa^\dagger$ case the ground state of the parent potential is missing.

ii.- Multiplication by the shift operators connects one potential to the other.

Based on these two general properties, a group theoretical analysis will justify and throw light on the relations between the supersymmetric partner potentials i.e., eq. (5) and eq. (6).

Let us consider first the Coulomb problem. Here the shift operators $a(l)$ and $a^\dagger(l)$ change the quantum number l by one unit. This number is connected with the representation of the $so(2,1)$ algebra i.e., with the eigenvalue of the Casimir C_2. Now, looking at the discrete principal series of the algebra, we note that $j = -(l+1)$ so that the eigenvalues of the Casimir T_3 which determine the energy spectrum begin at the value $m = -j = l+1$. Thus, the effect of the shift operators causes the energy spectrum to begin at the shifted stage, $m = l+1+1$. Properties *i* and *ii* can therefore be taken into account by using for the partner potential shift operators with the value of l changed into $l+1$. This is confirmed by equation (5).

$$a(l)a^\dagger(l) = a^\dagger(l+1)a(l+1) + 2\Delta E_{1,l}.$$

In the Morse potential problem, we note that the shift operators $a(\lambda)$ and $a^\dagger(\lambda)$ act to leave the quantity $\Lambda = \lambda - v$ invariant. This quantity characterizes the representation of the $so(2,1)$ algebra, and the eigenvalues of the Casimir C_2 are directly related to the energy spectrum of the problem. Thus, the shift operators relating the parent potential with its partner potential do not change the energy eigenvalues. Hence property *i* can no longer be achieved by the shift operators alone and the operator $a^\dagger a - 1$ together

with some linear combination of the shift operators a and $a^\dagger$ appears in the partner potential expression $aa^\dagger$. In this particular case of the Morse potential, the subtraction of a unit from the shift operators can be used to account for the absence of a state in the otherwise identical energy spectra of the parent and partner potentials. Thus, properties *i* and *ii* together with the group theoretical analysis account for the existence of the simple relation (6),

$$a(\lambda)a^\dagger(\lambda) = (a^\dagger(\lambda) - 1)(a(\lambda) - 1) + \Delta E_{10}.$$

Now, the validity of this relation implies the energy invariance of the Morse potential with respect to a displacement of its equilibrium point, since the the various Morse potentials are shifted by the shift operators not only in well depth parameter, but also in equilibrium position. (see text before eq. (6) and ref. 13). Thus, the group theoretical analysis when applied to the supersymmetric potential implies a non trivial invariance of the system. Another consequence of this result is that the ground states of the various partner potentials correspond to ***integer*** eigenvalues of the annihilation operator corresponding to the parent potential, as has been noted elsewhere[13], in connection with Morse oscillator coherent states.

We can now summarize the results of this paper. We have applied the $so(2,1)$ algebra to the bound states of the Coulomb and Morse oscillator problems. In these applications the role played by the algebra differs. In the Coulomb case the Casimir of the subalgebra $so(2)$ is related to the energy of the problem while the Casimir of the whole algebra describes the angular momentum. In the case of the Morse oscillator, the Casimir of the whole algebra is related to the energy while the Casimir of the subalgebra $so(2)$ is related to the well depth parameter. These two bound state problems are described by the discrete principal series of the algebra. The use of the continuous principal series leads in the Morse case to the treatment of its scattering states. In the Coulomb case, this representation describes a non physical situation in which the Coulomb potential is accompanied by a continuous value of angular momentum. This is a direct consequence of the different usage of the algebra in the two problems.

The way in which the algebra enters into the problems is further reflected in the supersymmetric quantum mechanical approach which generates partner potentials at constant energy. In the supersymmetric treatment of the Coulomb potential, the shift operators a and $a^\dagger$ act to change the angular momentum quantum number by one unit, and hence change the representation of the underlying algebra leading to the connection of the parent and partner potentials in *different* representations, i.e. different angular momentum quantum numbers. In the Morse problem, a and $a^\dagger$ leave the energy invariant and thus also leave invariant the representation of the underlying algebra. As a consequence, the parent and partner potentials are related through shift operators in the *same* representation and the difference in their energy spectrum is taken into account by the subtraction of a unit from the shift operators. This relation is consistent with, or rather, it implies the independence of the energy of the Morse potential with respect to its equilibrium position.

To conclude we note that only the case of $so(2,1)$ symmetry has been treated here; however, we should remark that the precise knowledge of the connection between the Lie algebra and the studied problem can effectively be used to predict the form of the expected relation between the parent potential and the partner potential in the supersymmetric approach for any system displaying dynamical symmetry.

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