last term in (2.B.17), after an integration by parts and use of (2.A.10) and (2.A.11), is seen to equal

$$\begin{aligned}\kappa\int_{\mathbf{x}}\epsilon^{ij}\,\mathrm{tr}\,(g_\tau^{-1}\partial_i g_\tau\partial_j\lambda) &= -\kappa\int_{\mathbf{x}}\mathrm{tr}\,(g_\tau^{-1}\partial_0 g_\tau g_\tau^{-1}\partial_i g_\tau g_\tau^{-1}\partial_j g_\tau)\\ &= -\frac{\kappa}{3}\int_{\mathbf{x}}\epsilon^{\alpha\beta\gamma}\,\mathrm{tr}\,(g_\tau^{-1}\partial_\alpha g_\tau g_\tau^{-1}\partial_\beta g_\tau g_\tau^{-1}\partial_\gamma g_\tau)\\ &= -8\pi^2\kappa\int_{\mathbf{x}} w(g_\tau)\\ &= -8\pi^2\kappa\frac{d}{d\tau}\int_{\mathbf{x}} w^0(g_\tau)\ .\end{aligned} \tag{2.B.18}$$

Thus $e^{iG}e^{-iG_\varphi} = e^{-2\pi i\alpha_1(\varphi,g)}$ where

$$\alpha_1(\varphi,g) = -\frac{\kappa}{2\pi}\int_{\mathbf{x}}\left\{\mathrm{tr}\,(2\varphi\partial_2 g\,g^{-1} + g^{-1}\partial_1 g\,g^{-1}\partial_2 g)\right\} + 4\pi\kappa\int_{\mathbf{x}} w^0(g) \tag{2.B.19}$$

and (2.B.15) becomes

$$U(g)\Psi(\varphi) = e^{-2\pi i\alpha_1(\varphi;g)}\Psi(\varphi^g)\ . \tag{2.B.20}$$

The last term in (2.B.19), which appears also in the 1-cocycle of topologically massive gauge theories [see (2.A.20b)], is multivalued for the same reason.

The Gauss law constraint (2.B.10) requires that physical states $\Psi(\varphi)$ be left unchanged by the action of $U(g)$, since the generator annihilates them.

$$U(g)\Psi(\varphi) = \Psi(\varphi) \tag{2.B.21}$$

Therefore in this theory, as in topologically massive gauge theories, functionals describing physical states are not gauge invariant; rather, according to (2.B.20), they satisfy

$$\Psi(\varphi^g) = e^{2\pi i\alpha_1(\varphi;g)}\Psi(\varphi)\ . \tag{2.B.22}$$

Only when $4\pi\kappa$ is an integer can this condition be met with single-valued functionals.

As indicated earlier, it is generally true that when a symmetric theory is described by a Lagrangian that changes by a total time derivative under a finite symmetry transformation, $L \to L + \frac{d}{dt}2\pi\alpha$, the 1-cocycle is just α, evaluated at fixed time. To verify this for the Chern–Simons theory, we must cast the Lagrangian in phase space form: the kinetic term should involve $p\dot{q}$, *i.e.* $\kappa A_2^a\dot{A}_1^a$ rather than $\frac{\kappa}{2}\epsilon^{ij}\dot{A}_i^a A_j^a$. This is achieved by subtracting $\kappa\frac{d}{dt}\,\mathrm{tr}\,(A_1A_2)$ from (2.B.3).

$$\tilde{L}_{\mathrm{CS}} = L_{\mathrm{CS}} - \kappa\frac{d}{dt}\int_{\mathbf{x}}\mathrm{tr}\,(A_1A_2) \tag{2.B.23}$$

Then from (2.A.8), (2.A.9), (2.A.10) and (2.A.11) it follows that $\tilde{L}_{\mathrm{CS}}$ transforms as

$$\begin{aligned}\tilde{L}_{\mathrm{CS}} &\to \tilde{L}_{\mathrm{CS}}\\ &+ \frac{d}{dt}\kappa\int_{\mathbf{x}}\Big\{\mathrm{tr}\left(\epsilon^{ij}\partial_i g\,g^{-1}A_j - \partial_1 g\,g^{-1}A_2 - A_1\partial_2 g\,g^{-1} - g^{-1}\partial_1 g\,g^{-1}\partial_2 g\right)\\ &+ 8\pi^2\omega^0(g)\Big\}\\ &= \tilde{L}_{\mathrm{CS}} + \frac{d}{dt}\kappa\int_{\mathbf{x}}\left\{-\mathrm{tr}\,(2\varphi\partial_2 g\,g^{-1} + g^{-1}\partial_1 g\,g^{-1}\partial_2 g) + 8\pi^2 w^0(g)\right\}\ ,\end{aligned} \tag{2.B.24}$$

in agreement with the general result and with (2.B.19).

We see that the 1-cocycle for the Chern–Simons theory is essentially the same as the one in(2.A.20) for the topologically massive gauge theory, once differences in polarization are taken into account: in the former there is a single field variable $\varphi^a = A_1^a$, while the latter is described by a pair A_i^a, $i = 1, 2$. In particular, the multivalued contribution to each is the same.

Next we construct explicitly states that obey (2.B.22), thus solving the Gauss law constraint. To this end we write

$$\Psi(\varphi) = e^{2\pi i \alpha_0(\varphi)} \psi(\varphi) \tag{2.B.25}$$

and seek a quantity $\alpha_0(\varphi)$, called a *cochain*, that satisfies

$$\alpha_0(\varphi^g) - \alpha_0(\varphi) = \alpha_1(\varphi; g) \tag{2.B.26}$$

Then (2.B.25) solves (2.B.22) with gauge invariant $\psi(\varphi)$.

If Eq. (2.B.26) holds the 1-cocycle α_1 is *trivial* — it is a *coboundary*. It is known that α_1 is non-trivial in *local* cohomology, but a *spatially non-local* functional that trivializes α_1 can be constructed. It is easy to verify that

$$\alpha_0(\varphi) = 4\pi\kappa \int_x w^0(h) - \frac{\kappa}{2\pi} \int_x \mathrm{tr}\, (\varphi h^{-1} \partial_2 h) \quad , \tag{2.B.27}$$

where h is defined by the non-local relation

$$\varphi \equiv h^{-1} \partial_1 h \quad , \tag{2.B.28}$$

solves (2.B.26) and therefore trivialized the cocycle (2.B.19).

It is to be remembered that the multivalued contribution to the trivializing functional (2.B.27) is related to the effective action of chiral fermions coupled to an external gauge field in two [Euclidean] dimensions. This connection arises because the fermion determinant is not gauge invariant; under a gauge transformation its change is related to the Chern–Simons 1-cocycle.[9]

The gauge invariant functional $\psi(\varphi)$ in (2.B.25) is formed solely from $\varphi = A_1$. It must be constructed from path-ordered exponential integrals of φ along x^1 at fixed x^2; *e.g.* closed Wilson loops $\Phi(C_{x^1}; x^2) = \mathrm{tr}\, P \exp \int_{C_{x^1}} dx^1 \varphi(x^1, x^2)$ where P denotes path ordering. [In two-dimensional, sourceless electrodynamics analogous holonomies around closed loops in the single spatial direction are the only surviving degrees of freedom in the quantized theory; they give rise to the vacuum angle and probe a possible vacuum electric field.[2h]] Whether such one-dimensional closed loops, or other gauge invariant constructions, exist depends on the topology of the two-dimensional space-like manifold on which the fixed-time canonical formalism is defined. Apart from this gauge-invariant functional, physical states of the non-Abelian Chern–Simons theory are given by

$$\Psi(\varphi) = N\, e^{2\pi i \alpha_0(\varphi)} \tag{2.B.29}$$

where N provides normalization and the other factor is related to the two-dimensional chiral fermion determinant — a non-local functional, as is normal for a physical wave functional. The necessary quantization of κ, (2.A.14), is again evident: from (2.B.25) and (2.B.29) we see that $\alpha_0(\varphi)$ contains the multi-valued term $4\pi\kappa \int_x \omega^0(h)$, which with unquantixed κ would render $\Psi(\varphi)$ in (2.B.25) or (2.B.29) multi-valued.

Since the Hamiltonian vanishes, the physical states (2.B.25) or (2.B.29) solve the Chern–Simons theory for all times. It is instructive to demonstrate explicitly that the vector potential A_i acting on the state (2.B.29) is a pure gauge. To exhibit the action of $A_2^a = \delta/i\kappa\delta\varphi^a$ on $\Psi(\varphi)$ we need $\delta\alpha_0(\varphi)/i\kappa\delta\varphi^a$. This can be found from the definition (2.B.28), which implies $\partial_1(\delta h\, h^{-1}) = h\delta\varphi h^{-1}$, and from the formula (2.B.27) for $2\pi\alpha_0$, which has the consequence that $\delta(2\pi\alpha_0) = -2\kappa\int_{\mathbf{x}} \mathrm{tr}\,\{\partial_2 h\, h^{-1}\partial_1(\delta h\, h^{-1})\}$. Thus

$$\begin{aligned} A_2(\mathbf{x})\Psi(\varphi) &= \frac{2\pi}{\kappa}T^a\frac{\delta\alpha_0(\varphi)}{\delta\varphi^a(\mathbf{x})}\Psi(\varphi) \\ &= h^{-1}(\mathbf{x})\partial_2 h(\mathbf{x})\Psi(\varphi)\ . \end{aligned} \qquad (2.B.30a)$$

Together with (2.B.28)

$$A_1(\mathbf{x})\Psi(\varphi) = h^{-1}(\mathbf{x})\partial_1 h(\mathbf{x})\Psi(\varphi) \qquad (2.B.30b)$$

the desired result is obtained.

$$A_i\Psi(\varphi) = h^{-1}\partial_i h\Psi(\varphi) \qquad (2.B.30c)$$

The wave-functional (2.B.29) has φ-independent norm $|N|^2$, hence it cannot be normlized by [funcitonal] integration over φ. This is to be expected of states that satisfy Gauss' law, because the Gauss law operator has a continuous spectrum. The resolution of course is that the integration measure $\mathcal{D}\varphi$ must be gauge fixed — $\delta(\varphi)$ is a natural choice leading to trivial integrals.

Note that it is possible to formulate the theory in terms of a gauge invariant, spatially non-local Lagrangian. From (2.B.24) it follows that the gauge transform of $\tilde{L}_{\mathrm{CS}}(A)$ is

$$\tilde{L}_{\mathrm{CS}}(A^g) = \tilde{L}_{\mathrm{CS}}(A) + \frac{d}{dt}2\pi\alpha_1(\varphi; g) \qquad (2.B.31a)$$

With (2.B.26), this can be presented as

$$\tilde{L}_{\mathrm{CS}}(A^g) - \frac{d}{dt}2\pi\alpha_0(\varphi^g) = \tilde{L}_{\mathrm{CS}}(A) - \frac{d}{dt}2\pi\alpha_0(\varphi) \qquad (2.B.31b)$$

This means that the equivalent Lagrangian

$$\begin{aligned} L_{\mathrm{CS}}^{\mathrm{invariant}}(A) &\equiv \tilde{L}_{\mathrm{CS}}(A) - \frac{d}{dt}2\pi\alpha_0(\varphi) \\ &= \tilde{L}_{\mathrm{CS}}(A) - 2\pi\int_{\mathbf{x}}\dot{\varphi}^a\frac{\delta\alpha_0(\varphi)}{\delta\varphi^a} \\ &= -2\kappa\int_{\mathbf{x}}\mathrm{tr}\,\left(A_2 - h^{-1}\partial_2 h\right)\dot{A}_1 \end{aligned} \qquad (2.B.31c)$$

is invariant against time-independent gauge transformations ($h^g \equiv hg$). In (2.B.31c), $h^{-1}\partial_2 h$ is the non-local functional of $\varphi = A_1$, defined by (2.B.28); $A_2 - h^{-1}\partial_2 h$ does *not* vanish here, only when acting on physical states.

The same results may be presented in the holomorphic representation, wherein states are functionals of a complex function $\mathcal{A}^a$ and $A^{a*} \equiv \frac{1}{\sqrt{2}}(A_1^a - iA_2^a)$ is realized by multiplication by $\mathcal{A}^a$ while $A^a \equiv \frac{1}{\sqrt{2}}(A_1^a + iA_2^a)$ acts by functional differentiation.

$$|\Psi\rangle \longrightarrow \Psi(\mathcal{A}) \tag{2.B.32a}$$

$$A^{a*}(\mathbf{x})|\Psi\rangle \longrightarrow \mathcal{A}^a(\mathbf{x})\Psi(\mathcal{A}) \tag{2.B.32b}$$

$$A^a(\mathbf{x})|\Psi\rangle \longrightarrow \frac{\delta}{\delta\mathcal{A}^a(\mathbf{z})}\Psi(\mathcal{A}) \tag{2.B.32c}$$

This action reproduces the commutator between A and A^*.

$$\left[A^a(\mathbf{x}), A^{*b}(\mathbf{y})\right] = \frac{1}{\kappa}\delta^{ab}\delta(\mathbf{x}-\mathbf{y}) \tag{2.B.33}$$

The adjoint relationship between the two operators is maintained, provided inner products involve a non-trivial measure in the functional integral

$$\langle\Psi_1|\Psi_2\rangle = \int \mathcal{D}A\mathcal{D}A^*\, e^{-\kappa\int_{\mathbf{x}} A^{c*}(\mathbf{x})A^c(\mathbf{x})}\Psi_1^*(\mathcal{A})\Psi_2(\mathcal{A}) \tag{2.B.34a}$$

where

$$\mathcal{D}A\mathcal{D}A^* = \mathcal{D}A_1^a\,\mathcal{D}A_2^a/\det(2\pi i) \tag{2.B.34b}$$

A development paralleling the previous discussion in the Cartesian polarization results in a wave functional in the holomorphic polarization analogous to (2.B.27) and (2.B.28).

$$\Psi(\mathcal{A}) = N\, e^{2\pi i\alpha_0(\mathcal{A})} \tag{2.B.35a}$$

$$\alpha_0(\mathcal{A}) = \frac{i\kappa}{2\pi}\int_{\mathbf{x}} \mathrm{tr}\,\left(\mathcal{A}h^{-1}\partial_+ h\right) + 4\pi\kappa\int_{\mathbf{x}}\omega^0(h) \tag{2.B.35b}$$

Here h is defined through

$$\mathcal{A} \equiv h^{-1}\partial_- h \tag{2.B.36}$$

and $\partial_\pm \equiv \frac{1}{\sqrt{2}}(\partial_1 \pm i\partial_2)$. The multivalued phase is of course encountered once again, while the integration measure $\exp -\kappa\int_{\mathbf{x}}\mathcal{A}^{c*}\mathcal{A}^c$ insures convergence of the functional integrals — no further gauge fixing is needed.

We conclude this presentation of the quantum Chern–Simons theory by noting that the Gauss law constraint is here solved after quantization. One may alternatively solve it classically, and quantize the remaining degrees of freedom. In general the two procedures do not commute, as is seen from the following example.[2d]

Consider the quantum mechanics for planar motion of a point particle described by the two-vector $\mathbf{q} = (q^1, q^2)$. The Lagrangian

$$L = \frac{1}{2}\dot{\mathbf{q}}^2 - V(q) \tag{2.B.37}$$

is invariant under rotations through the angle θ,

$$\delta q^i = -\theta \epsilon^{ij} q^j \tag{2.B.38}$$

when $V(q)$ depends only on the magnitude of $\mathbf{q}$. Here q^i is a function of time t, and the overdot denotes differentiation with respect to t. However, θ is time-independent — the rotation in (2.B.38) is "global." In the usual way, one knows that invariance under (2.B.38) implies conservation of angular momentum $J = \mathbf{q} \times \mathbf{p}$, where $\mathbf{p} \equiv \frac{\partial L}{\partial \dot{\mathbf{q}}} = \dot{\mathbf{q}}$, and the familiar Hamiltonian operator for fixed angular momentum ℓ is

$$H_\ell = -\frac{\hbar^2}{2} \frac{\partial^2}{\partial r^2} + \frac{\hbar^2 \left(\ell^2 - \frac{1}{4}\right)}{2r^2} + V(r) \ , \qquad r \geq 0 \ . \tag{2.B.39}$$

[We temporarily restore Planck's constant.] One may promote the global symmetry under rotations (2.B.38) to a "local" gauge symmetry by introducing into (2.B.37) a "gauge potential" $a(t)$,

$$L_a = \frac{1}{2} \left(\dot{q}^i + a\epsilon^{ij} q^j\right) \left(\dot{q}^i + a\epsilon^{ik} q^k\right) - V(q) \tag{2.B.40}$$

that transforms under time-dependent rotations.

$$\delta a = \dot{\theta} \tag{2.B.41}$$

In the Weyl gauge, $a = 0$, L_a coincides with (2.B.37), but now there is an additional constraint that captures the Lagrangian equation of motion obtained from (2.B.40) by varying a: the rotation generator J must annihilate physical states, which therefore are only s-states. If the constraint is solved classically, the classical Hamiltonian for rotationally invariant motion with vanishing angular momentum is

$$H_{\ell=0}^{\text{classical}} = \frac{p_r^2}{2} + V(r) \ , \tag{2.B.42a}$$

whose quantized form, obtained with the naive replacement $p_r^2 \to -\hbar^2 \partial_r^2$,

$$H_{\ell=0}^{\text{classical}} \longrightarrow -\frac{\hbar^2}{2} \frac{\partial^2}{\partial r^2} + V(r) \tag{2.B.42b}$$

does not reproduce the quantum s-wave Hamiltonian that survives from (2.B.39) if the constraint is imposed *after* quantization.

$$H_{\ell=0} = -\frac{\hbar^2}{2} \frac{\partial^2}{\partial r^2} - \frac{\hbar^2}{8r^2} + V(r) \tag{2.B.43}$$

The $\mathcal{O}(\hbar^2)$ difference between (2.B.42b) and (2.B.43) can be viewed as an ordering ambiguity *i.e.* (2.B.43) follows from (2.B.42a) if p_r^2 is taken to be $-\frac{\hbar^2}{\sqrt{r}} \partial_r r \partial_r \frac{1}{\sqrt{r}}$, but without further information there is no way to justify the "correct" choice. No such ambiguity arises if one imposes the constraint after quantization. [Henceforth we return $\hbar$ to unity.]

This is not to say that there will always be a discrepancy when phase space is reduced before or after quantization. For example, in many-body quantum mechanics, the passage to the center-of-mass rest frame [which may be formulated as a gauge principle for translations[28]] produces the same quantum theory whether it is carried out before or after quantization. Similarly, enforcing the Gauss law in quantum electrodynamics does not involve ordering ambiguities. It is therefore surprising that we find non-commutativity of quantization and phase-space reduction already for the Abelian Chern–Simons theory in flat space, which I shall now describe.

C. Abelian Chern–Simons Gauge Theory with Sources

We begin with the Lagrange density

$$\mathcal{L}_{\rm CS} = \frac{\kappa}{2}\epsilon^{\alpha\beta\gamma}\partial_\alpha A_\beta A_\gamma - A_\mu j^\mu \qquad (2.C.1)$$

where $j^\mu = (\rho, \mathbf{j})$ is the conserved matter current with time-independent charge.

$$Q = \int_{\mathbf{x}} \rho(t, \mathbf{x}) \qquad (2.C.2)$$

Here the fields are real functions, and the coupling constant is absorbed in the definition of the current j^μ. We leave the matter Lagrangian unspecified; indeed we take the current to be an external, conserved, c-number source. The previous canonical development holds in this theory, except that the Hamiltonian does not vanish,

$$H = \int_{\mathbf{x}} A_i j^i \qquad (2.C.3)$$

and the Gauss law constraint acquires an inhomogeneous term.

$$\epsilon^{ij}\partial_i A_j = -B = \frac{1}{\kappa}\rho \qquad (2.C.4)$$

This constraint on the magnetic field B implies that particles with charge Q are also flux-tubes for A-flux, $\Phi = -Q/\kappa$, and leads to exotic statistics and angular momentum of the charge- and flux-carrying particles.[10]

In contrast to the non-Abelian theory, here the gauge field contribution to the constraint (2.C.4) is linear, and may be chosen to be the momentum conjugate to a coordinate θ. This is achieved by decomposing A_i into its longitudinal and transverse components,

$$\begin{aligned} A_i &= \partial_i\theta + \epsilon^{ij}\partial_j^{-1}B \\ \partial_j^{-1} &\equiv \partial_j/\nabla^2 \end{aligned} \qquad (2.C.5)$$

The decomposition (2.C.5) is unique and well-defined provided there are no zero modes of the two-dimensional Laplacian ∇^2. This we assume here; indeed, we consider space-time to be Minkowskian.

The commutation relation (2.B.8) implies that B and θ form a canonical pair.

$$[\theta(\mathbf{x}), B(\mathbf{y})] = \frac{i}{\kappa}\delta^2(\mathbf{x}-\mathbf{y}) \qquad (2.C.6)$$

In the Schrödinger representation we realize B as a functional derivative with respect to the coordinate, θ, $B = \delta/i\kappa\delta\theta$. This is the *rotationally invariant polarization*.

The constraint (2.C.4) reduces to

$$\left[\frac{1}{i}\,\frac{\delta}{\delta\theta(\mathbf{x})} + \rho(\mathbf{x})\right]\Psi = 0 \quad , \qquad (2.C.7)$$

and is solved by

$$\Psi(\theta;t) = N(t)\exp\left[-i\int_{\mathbf{x}}\rho\theta\right] \quad , \tag{2.C.8}$$

where $N(t)$ is a θ-independent, but possibly time-dependent normalization factor. $N(t)$ is then determined by requiring that $\Psi(\theta;t)$ satisfies the time-dependent Schrödinger equation.

$$i\partial_t\Psi(\theta;t) = \left[\int_{\mathbf{x}} j^i A_i\right]\Psi(\theta;t) \tag{2.C.9}$$

Inserting the decomposition (2.C.5) and using the continuity equation for the matter current, we find

$$N(t) = \exp -\frac{i}{\kappa}\int_0^t dt'\int_{\mathbf{x}}\rho(t',\mathbf{x})j(t',\mathbf{x}) \quad , \tag{2.C.10}$$

where we have written j^i in terms of its longitudinal and transverse parts.

$$j^i = -\partial_i^{-1}\dot{\rho} + \epsilon^{ij}\partial_j j \tag{2.C.11}$$

$N(t)$ is normalized to 1 at $t = 0$. Note that for static matter the state Ψ is an eigenstate of the Hamiltonian with energy eigenvalue

$$E = \frac{1}{\kappa}\int_{\mathbf{x}}\rho j \quad . \tag{2.C.12}$$

We see that the theory admits a unique physical state, which in the absence of sources is described by the wave functional $\Psi = 1$.

Of course, the above development can alternatively be presented in the Cartesian polarization $A_1 = \varphi$, $A_2 = \delta/i\kappa\delta\varphi$, which we used for the non-Abelian theory. In the absence of external sources, the unique physical state that satisfies Gauss's law is

$$\Psi(\varphi) = N\exp\left[\frac{i\kappa}{2}\int\varphi\frac{\partial_2}{\partial_1}\varphi\right] \quad , \tag{2.C.13}$$

in agreement with (2.B.25), (2.B.27) and (2.B.29). Ψ responds to a gauge transformation by

$$\begin{aligned}\Psi\left(\varphi^g\right) &= \Psi\left(\varphi+\partial_1\lambda\right) = e^{i2\pi\alpha_1(\varphi;\lambda)}\Psi(\varphi) \quad , \\ \alpha_1\left(\varphi;\lambda\right) &= \frac{\kappa}{2\pi}\int_{\mathbf{x}}\left[\varphi\partial_2\lambda + \frac{1}{2}\partial_1\lambda\partial_2\lambda\right] \quad ,\end{aligned} \tag{2.C.14}$$

in agreement with (2.B.22).

The unitary transformation functional that connects the Cartesian polarization to the rotationally invariant one is

$$\langle\theta|\varphi\rangle = \det{}^{1/2}\left(\frac{\kappa}{2\pi}\,\frac{\Delta}{\partial_2}\right)\exp\left[\frac{i\kappa}{2}\int\left\{\left(\partial_i\theta - \frac{\partial_i}{\partial_1}\varphi\right)\frac{\partial_1}{\partial_2}\left(\partial_i\theta - \frac{\partial_i}{\partial_1}\varphi\right) - \varphi\frac{\partial_2}{\partial_1}\varphi\right\}\right] \tag{2.C.15}$$

$$\Psi(\theta) = \langle\theta|\Psi\rangle = 1 \ , \qquad \Psi(\varphi) = \langle\varphi|\Psi\rangle = N \exp\left[\frac{i\kappa}{2}\int \varphi \frac{\partial_2}{\partial_1}\varphi\right]$$

In the rotationally invariant polarization the physical state, in the absence of external sources, $\Psi = 1$, is obviously gauge invariant. This fact realizes the gauge invariant formulation described previously: in the rotationally invariant polarization the Lagrangian, apart from a total time derivative, is invariant against time-independent gauge transformations.

$$\tilde{L}_{\rm CS} = \kappa \int_{\mathbf{x}} B\dot{\theta} - \frac{\kappa}{2}\frac{d}{dt}\int_{\mathbf{x}} B\theta \qquad (2.C.16a)$$

$$L_{\rm CS}^{\rm invariant} = \kappa \int_{\mathbf{x}} B\dot{\theta} \qquad (2.C.16b)$$

Finally in the holomorphic polarization (2.B.32) the state of the Abelian theory is

$$\Psi(\mathcal{A}) = N \exp \frac{\kappa}{2}\int \mathcal{A}\frac{\partial_+}{\partial_-}\mathcal{A} \qquad (2.C.17)$$

We have already remarked that the Chern–Simons theory may be viewed as the $\kappa \to \infty$ limit of the topologically massive model. It is interesting to examine in detail how one theory passes to the other, and this we can do explicitly for the wave functionals of the Abelian theory. For $\mathcal{L}$ of (2.A.6a) to pass into $\mathcal{L}_{\rm CS}$ of (2.A.6b) it suffices [in the Abelian case] to rescale the potential by $\sqrt{\frac{\kappa'}{\kappa}}$ and set κ to infinity, with κ' remaining as the coefficient of the Chern–Simons Lagrange density. Performing this limit on the ground state wave functional (2.A.22a) leaves

$$\Psi_0(\mathbf{A}) \underset{\kappa\to\infty}{\longrightarrow} \left(\exp i\frac{\kappa'}{2}\int BA_L\right)\left(\exp -\frac{\kappa'}{2}\int \mathbf{A}_T\cdot\mathbf{A}_T\right) \qquad (2.C.18a)$$

In terms of complex variables $\mathcal{A} \equiv \frac{1}{\sqrt{2}}(A_1 - iA_2)$ and $\mathcal{A}^* = \frac{1}{\sqrt{2}}(A_1 + iA_2)$, (2.C.18a) reads

$$\Psi_0(\mathbf{A}) \underset{\kappa\to\infty}{\longrightarrow} \exp\left(\frac{\kappa'}{2}\int \mathcal{A}\frac{\partial_+}{\partial_-}\mathcal{A}\right)\exp -\frac{\kappa'}{2}\int \mathcal{A}^*\mathcal{A} \qquad (2.C.18b)$$

Comparison with (2.C.17) shows that in the limit, the ground state wave functional of the topologically massive theory tends to the unique state of the Chern–Simons theory, times the square root of the measure factor, so that the probability measures correctly pass into each other.[2e] Other details on this limit can be found in the literature.[2e]

D. Quantum Holonomy

In the Abelian Chern–Simons theory defined on the topologically trivial plane there is very little structure. Indeed when the constraints are solved before quantization, there is no structure at all if there are no sources. However, by quantizing first, and computing the quantum holonomy around closed loops at fixed time, we encounter non-trivial results that show an explicit difference between solving the constraints before and after quantization.[2d]

The holonomy operator $\Phi(C)$ is defined by the parallel transport equation around a closed planar loop C parametrized by $\mathbf{x}(\tau)$ for $\tau \in [0,1]$, with $\mathbf{x}(0) = \mathbf{x}(1) \equiv \mathbf{x}_0$, which serves

as a marked point on the loop, where we also specify the initial and final unit tangent vectors, $\hat{\mathbf{v}}_0 = \dot{\mathbf{x}}(0)/|\dot{\mathbf{x}}(0)|$ and $\hat{\mathbf{v}}_1 = \dot{\mathbf{x}}(1)/|\mathbf{x}(1)|$, respectively. The marked point is on a smooth segment of the loop if $\hat{\mathbf{v}}_0 = \hat{\mathbf{v}}_1$; otherwise it is at a cusp with opening angle $\pi \mp \cos^{-1} \hat{\mathbf{v}}_0 \cdot \hat{\mathbf{v}}_1$, where $\mp$ refer to opening angles $< \pi$ and $\geq \pi$, respectively, and $0 \leq \cos^{-1} \hat{\mathbf{v}}_0 \cdot \hat{\mathbf{v}}_1 < 2\pi$.

The equation for the holonomy

$$[i\partial_\tau + V(\tau)]\,\Phi(C) = 0 \quad , \quad V(\tau) \equiv \dot{x}^i(\tau) A_i(\mathbf{x}(\tau)) \tag{2.D.1}$$

is solved at the classical level by the Aharonov–Bohm phase.

$$\Phi^{\text{classical}} = \exp\left[i \int_0^1 d\tau\, V(\tau)\right] = \exp\left[i \int_C dx^i\, A_i\right] \tag{2.D.2}$$

The constraint (2.C.4) in the theory without sources forces A_i to be a pure gauge, and therefore $\Phi^{\text{classical}} = 1$, independent of the loop C.

To solve Eq. (2.D.1) at the quantum level we must recall that $[A_1(\mathbf{x})]$, $A_2(\mathbf{x})] \neq 0$, therefore $[V(\tau), V(\tau')] \neq 0$, and the quantum holonomy operator is given by a path-ordered expression.

$$\Phi(C) = P \exp\left[i \int_0^1 d\tau\, V(\tau)\right] \tag{2.D.3}$$

To determine the action of $\Phi(C)$ on states we first need to undo the path ordering. Since the commutator $[V(\tau), V(\tau')]$ is a c-number, this yields

$$\begin{aligned} \Phi(C) &= \exp\left[-\frac{1}{2} \int_0^1 d\tau \int_0^\tau d\tau' \,[V(\tau), V(\tau')]\right] \exp\left[i \int_0^1 d\tau\, V(\tau)\right] \\ &= \exp\left[-\frac{i}{2\kappa} \int_0^1 d\tau \int_0^\tau d\tau'\, \dot{x}^i(\tau) \epsilon^{ij} \dot{x}^j(\tau') \delta^2\left(\mathbf{x}(\tau) - \mathbf{x}(\tau')\right)\right] \\ &\times \exp\left[i \int_0^1 d\tau\, \dot{x}^i(\tau) A_i\left(\mathbf{x}(\tau)\right)\right] \quad . \end{aligned} \tag{2.D.4}$$

In our chosen polarization, $\Phi(C)$ acts on functionals of the "coordinate" θ, and it is therefore convenient to reorganize (2.D.4) so that the "momentum" B stands on the right. To this end, we split the operator $V(\tau)$ into a self-commuting pair

$$\begin{aligned} V(\tau) &= V_1(\tau) + V_2(\tau) \\ [V_1(\tau), V_1(\tau')] &= [V_2(\tau), V_2(\tau')] = 0 \\ [V_1(\tau), V_2(\tau')] &= c\text{-number} \end{aligned} \tag{2.D.5}$$

Splitting the operator-valued exponential in (2.D.4) gives an additional phase, which combines with the first to yield

$$\begin{aligned} \Phi(C) &= \exp[i\gamma] \exp\left[i \int_0^1 d\tau\, V_1(\tau)\right] \exp\left[i \int_0^1 d\tau\, V_2(\tau)\right] \\ \gamma &= i \int_0^1 d\tau \int_0^\tau d\tau' \,[V_1(\tau), V_2(\tau')] \end{aligned} \tag{2.D.6}$$

With our polarization

$$V_1(\tau) = \dot{x}^i(\tau)\partial_i\theta\left(\mathbf{x}(\tau)\right) = \frac{d}{d\tau}\theta\left(\mathbf{x}(\tau)\right) \quad , \qquad (2.D.7)$$
$$V_2(\tau) = \dot{x}^i(\tau)\epsilon^{ij}\partial_j^{-1}B\left(\mathbf{x}(\tau)\right) \quad ,$$

γ becomes

$$\gamma = \frac{1}{\kappa}\int_0^1 d\tau \int_0^\tau d\tau' \frac{\partial}{\partial\tau}\dot{x}^i(\tau')\epsilon^{ij}\partial_j^{-1}\delta^2\left(\mathbf{x}(\tau)-\mathbf{x}(\tau')\right) \quad , \qquad (2.D.8a)$$

where $\partial_j^{-1}\delta$ is the derivative of the two-dimensional Green's function.

$$\partial_j^{-1}\delta^2(\mathbf{r}) = \frac{1}{2\pi}\,\frac{r^j}{|\mathbf{r}|} = \frac{\epsilon^{jk}}{2\pi}\partial_k \tan^{-1}\frac{r^2}{r^1}$$

The formula relating $\partial_j^{-1}\delta$ to a derivative of $\tan^{-1}$ gives an alternative expression for γ.

$$\gamma = \frac{1}{2\pi\kappa}\int_0^1 d\tau \int_0^\tau d\tau' \frac{\partial}{\partial\tau}\,\frac{\partial}{\partial\tau'}\tan^{-1}\frac{x^2(\tau)-x^2(\tau')}{x^1(\tau)-x^1(\tau')} \qquad (2.D.8b)$$

The derivation may be organized differently. We recognize that the phase arising from splitting $\exp\left[i\int_0^1 d\tau\, V(\tau)\right]$ into $\exp\left[i\int_0^1 d\tau\, V_1(\tau)\right]\exp\left[i\int_0^1 d\tau\, V_2(\tau)\right]$ vanishes because the loop is closed, leaving for the entire contribution to γ just the last quantity in (2.D.4), which arises from undoing the path-ordering, and which is determined by the polarization-independent $[A_i, A_j]$ commutator.

$$\gamma = -\frac{1}{2\kappa}\int_0^1 d\tau \int_0^\tau d\tau' \dot{x}^i(\tau)\epsilon^{ij}\dot{x}^j(\tau')\delta^2\left(\mathbf{x}(\tau)-\mathbf{x}(\tau')\right) \qquad (2.D.8c)$$

Using the identity $2\pi\delta^2(\mathbf{r}) = \epsilon^{ij}\partial_i\partial_j \tan^{-1} r^2/r^1$ we can rewrite (2.D.8c).

$$\gamma = \frac{1}{4\pi\kappa}\int_0^1 d\tau \int_0^\tau d\tau' \left(\frac{\partial}{\partial\tau}\,\frac{\partial}{\partial\tau'} - \frac{\partial}{\partial\tau'}\,\frac{\partial}{\partial\tau}\right)\tan^{-1}\frac{x^2(\tau)-x^2(\tau')}{x^1(\tau)-x^1(\tau')} \qquad (2.D.8d)$$

Equation (2.D.8c) exhibits the elusive nature of γ. Owing to the δ-function enforcing $\mathbf{x}(\tau) = \mathbf{x}(\tau')$, one might conclude that $\dot{x}^i(\tau)\epsilon^{ij}\dot{x}^j(\tau')$, and hence γ, vanishes. However, upon closer examination we recognize that the two-dimensional δ-function is a product of two one-dimensional δ-functions, each enforcing the same constraint on the one-dimensional variables τ, τ'. Thus the integrand in (2.D.8c) involves the ambiguous quantity $\dot{x}^i(\tau)\epsilon^{ij}\dot{x}^j(\tau)/\left|\dot{x}^1(\tau)\dot{x}^2(\tau)\right|\ \delta(\tau-\tau')\delta(\tau-\tau')$, and in the following a careful analysis is performed to obtain an unambiguous result. But it is already clear that γ is non-trivial owing to the continuum properties of space, which give rise to δ-functions. In a discretized world with Kronecker delta's, (2.D.8) does indeed vanish.

Our evaluation of γ is based on the following observation. In spite of the singular nature of (2.D.8c), one can begin with any of the other formulas (2.D.8a), (2.D.8b) and (2.D.8d),

manipulate finite expressions and arrive at an unambiguous answer for γ. By this procedure, we shall derive below the result

$$\gamma = \frac{1}{2\pi\kappa}\Delta\Theta_{\mathbf{r}} - \frac{1}{4\pi\kappa}\Delta\Theta_{\mathbf{v}} \quad . \tag{2.D.9}$$

Here $\Delta\Theta_{\mathbf{r}}$ is the total angle accumulated at the marked point $\mathbf{x}_0$ when the loop is traversed by a vector based at $\mathbf{x}_0$,

$$\begin{aligned} \Theta_{\mathbf{r}}(\tau) &= \tan^{-1}\frac{r^2(\tau)}{r^1(\tau)} \quad , \\ \mathbf{r} &= \mathbf{x} - \mathbf{x}_0 \quad , \\ \Delta\Theta_{\mathbf{r}} &\equiv \Theta_{\mathbf{r}}(1) - \Theta_{\mathbf{r}}(0) \quad , \end{aligned} \tag{2.D.10a}$$

while the quantity $\Delta\Theta_{\mathbf{v}}$ is the accumulated angular change in the tangent to the curve.

$$\begin{aligned} \Theta_{\mathbf{v}}(\tau) &= \tan^{-1}\frac{\dot{r}^2(\tau)}{\dot{r}^1(\tau)} = \tan^{-1}\frac{\hat{v}^2(\tau)}{\hat{v}^1(\tau)} \\ \Delta\Theta_{\mathbf{v}} &\equiv \Theta_{\mathbf{v}}(1) - \Theta_{\mathbf{v}}(0) \end{aligned} \tag{2.D.10b}$$

The evaluation of (2.D.8) that gives (2.D.9) is performed without using regulators. However, to illustrate the subtlety of (2.D.8) we remark here that if regulators *are* introduced, for example by regulating the δ-function, and if the regularization preserves the fact that the Green's function derivative is odd under interchange of argument, $\partial_j^{-1}\delta^2(\mathbf{x}-\mathbf{y}) = -\partial_j^{-1}\delta^2(\mathbf{y}-\mathbf{x})$, so that $\partial_j^{-1}\delta^2(0)$ vanishes, then we again obtain a unique answer that does not depend on the details of the regularization. However, this answer differs from (2.D.9): $\gamma^{\mathrm{reg}} = \frac{1}{2\pi\kappa}\Delta\Theta_{\mathbf{r}}$, the contribution from the change in the tangent is missing.

We now present the derivation of (2.D.9). In (2.D.8a) or (2.D.8b) the τ derivative is interchanged with the τ' integral. Starting from (2.D.8a) we have

$$\begin{aligned} \gamma &= \frac{1}{\kappa}\int_0^1 d\tau \frac{\partial}{\partial\tau}\int_0^\tau d\tau'\, \dot{x}^i(\tau')\epsilon^{ij}\delta^2\left(\mathbf{x}(\tau)-\mathbf{x}(\tau')\right) \\ &\quad - \frac{1}{\kappa}\int_0^1 d\tau \left(\dot{x}^i(\tau')\epsilon^{ij}\partial_j^{-1}\delta^2\left(\mathbf{x}(\tau)-\mathbf{x}(\tau')\right)\right)\Big|_{\tau'=\tau} \\ &= -\frac{1}{\kappa}\int_0^1 d\tau'\, \dot{x}^i(\tau')\epsilon^{ij}\partial_j^{-1}\delta^2\left(\mathbf{x}(\tau')-\mathbf{x}_0\right) \\ &\quad + \frac{1}{\kappa}\int_0^1 d\tau \left(\dot{x}^i(\tau')\epsilon^{ij}\partial_j^{-1}\delta^2\left(\mathbf{x}(\tau')-\mathbf{x}(\tau)\right)\right)\Big|_{\tau'=\tau} \quad . \end{aligned} \tag{2.D.11a}$$

By expressing the derivative of the Green's functions in terms of the inverse tangent, or alternatively by beginning with (2.D.8b), we find

$$\begin{aligned} \gamma &= \frac{1}{2\pi\kappa}\int_0^1 d\tau' \frac{\partial}{\partial\tau'}\tan^{-1}\frac{x^2(\tau')-x_0^2}{x^1(\tau')-x_0^1} \\ &\quad - \frac{1}{2\pi\kappa}\int_0^1 d\tau \left(\frac{\partial}{\partial\tau'}\tan^{-1}\frac{x^2(\tau')-x^2(\tau)}{x^1(\tau')-x^1(\tau)}\right)\Big|_{\tau'=\tau} \end{aligned} \tag{2.D.11b}$$

[It is easy to see that the same formula emerges if one begins with (2.D.8d), treats the $\partial_\tau \partial_{\tau'}$ contribution as in (2.D.11b) and evaluates the $\partial_{\tau'} \partial_\tau$ contribution by first performing the τ' integral.] The integrand of the last term in (2.D.11b) is

$$\left(\frac{\partial}{\partial \tau'} \tan^{-1} \frac{x^2(\tau') - x^2(\tau)}{x^1(\tau') - x^1(\tau)} \right) \Bigg|_{\tau'=\tau} = \frac{1}{2} \frac{\partial}{\partial \tau} \tan^{-1} \frac{\dot{x}^2(\tau)}{\dot{x}^1(\tau)}$$

and so γ becomes

$$\gamma = \frac{1}{2\pi\kappa} \int_0^1 d\tau \frac{\partial}{\partial \tau} \tan^{-1} \frac{x^2(\tau) - x_0^2}{x^1(\tau) - x_0^1} - \frac{1}{4\pi\kappa} \int_0^1 d\tau \frac{\partial}{\partial \tau} \tan^{-1} \frac{\dot{x}^2(\tau)}{\dot{x}^1(\tau)} \qquad (2.D.11c)$$

This establishes (2.D.9). [Note that if a regulator had been used for the δ function in (2.D.11a), then the last term of each equation in (2.D.11) would vanish, with natural regulators such that $\partial_j^{-1} \delta^2(0) = 0$, leaving, as commented above, for $\gamma^{\rm reg}$ just the first term on the right-hand side of (2.D.11c). A possible regularization would be to retain the kinetic Maxwell term, and then decouple it by passing with κ to infinity. It should be no surprise, in view of earlier discussions on this limit [see (2.C.18)], that different answers can be obtained. For more discussion, see the literature.[2e]

The value of γ depends on the curve. Consider first simple curves, traversed in the counterclockwise direction, without self-intersections. For smooth, simple curves, the angle swept out as the marked point is π, and

$$\frac{1}{2\pi\kappa} \Delta\Theta_{\rm r} = \frac{1}{2\kappa} \quad . \qquad (2.D.12a)$$

The angular change in the tangent is 2π, so

$$\frac{1}{4\pi\kappa} \Delta\Theta_{\rm v} = \frac{1}{2\kappa} \qquad (2.D.12b)$$

and therefore

$$\gamma = 0 \qquad (2.D.12c)$$

The same results hold if the curve has cusps, provided the marked point is not a cusp. However, when $\mathbf{x}_0$ does lie at a cusp where $\hat{\mathbf{v}}_0 \neq \hat{\mathbf{v}}_1$, there are shortfalls in the angular traversals and

$$\frac{1}{2\pi\kappa} \Delta\Theta_{\rm r} = \frac{1}{2\pi\kappa} \left(\pi \mp \cos^{-1} \hat{\mathbf{v}}_0 \cdot \hat{\mathbf{v}}_1 \right) \qquad (2.D.13a)$$

$$\frac{1}{3\pi\kappa} \Delta\Theta_{\rm v} = \frac{1}{4\pi\kappa} \left(2\pi \mp \cos^{-1} \hat{\mathbf{v}}_0 \cdot \hat{\mathbf{v}}_1 \right) \qquad (2.D.13b)$$

$$\gamma = \mp \frac{1}{4\pi\kappa} \cos^{-1} \hat{\mathbf{v}}_0 \cdot \hat{\mathbf{v}}_1 \qquad (2.D.13c)$$

where $\mp$ refer to opening angles of the cusp $< \pi$ and $\leq \pi$, respectively.

We now discuss loops with intersections. We set the following conventions: the marked point lies on an outermost smooth segment of the loop, insuring that $|\Delta\Theta_{\rm r}| = \pi$, regardless of the number of intersections and cusps, since $\mathbf{x}_0$ does not lie at any of these exceptional

points; the parametrization takes the contour through $\mathbf{x}_0$ in a counterclockwise direction thus fixing $\Delta\Theta_r = \pi$; an intersection is defined by an actual crossing — touching contours are not intersections; the total number of intersections is ν and we do not consider loops with multiple intersections at the same point.

Only $\Delta\Theta_v$ varies with the intersection number ν. For $\nu = 1$, there are two elementary intersections: the "figure eight" where the two sub-loops are traversed in opposite directions, so that $\Delta\Theta_v = 0$, and the "nested loop" where the two sub-loops are traversed in the same direction, with $\Delta\Theta_v = 4\pi$. Loops with higher ν can be constructed by superposing in various ways these two elementary blocks. For a given ν, the highest possible value for $\Delta\Theta_v$, $\Delta\Theta_v^{\max} = (\nu+1)2\pi$, is achieved by superposing like-oriented nested intersections. The lowest possible value, $\Delta\Theta_v^{\min} = -(\nu-1)2\pi$, is obtained by building out of ν like-oriented intersections a loop whose direction has been reversed by a single figure eight intersection. The possible values of $\Delta\Theta_v$, for fixed ν, interpolate between $\Delta\Theta_v^{\min}$ and $\Delta\Theta_v^{\max}$ in steps of 4π. The different allowed values for $\Delta\Theta_v$, combined with the unique $\Delta\Theta_r = \pi$, give a table of values for γ at fixed ν.

$$\gamma = \frac{m}{\kappa} \quad , \quad m \in \left[-\frac{\nu}{2}, -\frac{\nu}{2}+1, \ldots, \frac{\nu}{2}-1, \frac{\nu}{2}\right] \tag{2.D.14}$$

Finally, we return to the full holonomy operator, and determine its action on the state (2.C.8). In our polarization, we have

$$\Phi(C)\Psi(\theta) = \exp\left[i\gamma(C)\right] \exp\left[i\int_C dx^i \partial_i\theta\right] \exp\left[-\frac{i}{\kappa}\int_C dx^i \epsilon^{ij}\partial_j^{-1}\rho\right]\Psi(\theta) \ . \tag{2.D.15}$$

Since the holonomy of the pure gauge $\partial_i\theta$ is trivial, $\exp\left[i\int_C dx^i\partial_i\theta\right] = 1$. The integral in the exponent of the last factor is evaluated by Stoke's law and gives

$$-\int_C dx^i \epsilon^{ij}\partial_j^{-1}\rho = \int_{S_C} d\mathbf{x}\,\rho = Q(C) \tag{2.D.16}$$

where $Q(C)$ is the total charge contained within the closed curve, with contributions appropriately signed if C is self-intersecting. [For a single point source of charge Q surrounded by a counterclockwise loop, $Q(C) = Q$.] Thus the physical state (2.C.8) is an eigenstate of the holonomy operator with eigenvalue $\phi(C)$.

$$\phi(C) = \exp\left(i\left[\gamma(C) + \frac{1}{\kappa}Q(C)\right]\right) \tag{2.D.17}$$

[Note that the holonomy operator commutes with the Gauss law operator because it is gauge invariant, and with the Hamiltonian (2.C.3) if $Q(C)$ is time-independent.]

Since holonomies around closed loops are the only gauge invariant and generally covariant observables, this shows that the $U(1)$ Chern–Simons theory in Minkowski space is characterized by the strengths of external charges, and by the vacuum holonomy $e^{i\gamma(C)}$, which is missed when the constraints are imposed before quantization. The vacuum holonomy, which can be evaluated without regularization, vanishes for simple loops provided the marked point is not at a cusp; otherwise, it is determined by the opening angle of the cusp. This rich structure is already present for the $U(1)$ Chern–Simons theory in Minkowski space.

The gauge structure in the Chern–Simons theory follows closely that of its topologically massive antecedent: both involve essentially the same cocycle in the action of the gauge group. However, the behavior of the quantum holonomy Φ is quite different. In the latter theory, the vacuum ground state — for the non-interacting Abelian model, the Gaussian (2.A.22) in $\mathbf{A}$ — is not an eigenstate of Φ, and the vacuum expectation value of the holonomy operator is infinite. In the Abelian Chern–Simons theory, the vacuum state — the only state for the source-free model in Minkowski space — is a Φ eigenstate with finite non-trivial eigenvalue, which is not seen when constraints are solved before quantization.

If expectation values of the holonomy operator in a topologically massive theory are to possess physical significance, Φ must be renormalized and its vacuum value is undefined. Neither regularization nor renormalization is needed in the Chern–Simons model; indeed since γ carries a rich loop dependence, a universal renormalization cannot remove the effect.

E. Anomalous Statistics and Spin of Charged Particles

We saw earlier that charged particles interacting through a Chern–Simons [Abelian] gauge field carry flux. This has the consequence that their spin and statistics is modified by the gauge-field interaction[10] — a result which can be established without reference to the detailed nature of the particle dynamics.[2d] Here we first show how the holonomy modifies statistics, and that spin adjusts so that the spin-statistics theorem is preserved. Later, we shall take a point-particle model for the matter and regain these results in an explicit manner.

Consider two identical particles, each with charge Q, and imagine a fixed-time test of statistics by carrying one particle around the other, corresponding to a double interchange of the particles. The wave function of the test particle will acquire, in addition to the conventional statistical factor, the phase

$$P \exp\left[iQ \int_C dx^i \, A_i\right] \quad ,$$

where C is a loop without self-intersections surrounding the particle. The state (2.C.8) is an eigenstate of this operator with eigenvalue $\exp\left[iQ^2/\kappa\right]$, apart from the vacuum contribution, which is absent provided the marked point is not at a cusp. The phase acquired by the wave function under a single interchange of the two particles is half the above, *i.e.* $Q^2/2\kappa$. Thus the statistics phase is an observable, whose value satisfies the spin-statistics theorem because particles in this theory carry anomalous spin S,

$$2\pi S = \frac{Q^2}{2\kappa} \qquad (2.E.1)$$

as we now demonstrate.

Spin fractionization for the charged matter particles is due to the angular momentum of the gauge field associated with the magnetic flux created by the charged particle. The variation of the gauge field under an infinitesimal spatial rotation $\delta x^i = -\epsilon^{ij} x^j$,

$$\delta A_i = -x^j \epsilon^{jk} \partial_k A_i - \epsilon^{ij} A_j \quad , \qquad (2.E.2)$$

leaves the Lagrangian corresponding to (2.C.1) invariant, provided the external current $\mathbf{j}$ is rotationally covariant.

$$x^j \epsilon^{jk} \partial_k j^i + \epsilon^{ij} j^j = 0 \qquad (2.E.3)$$

For the conserved angular momentum operator we take

$$J = -\frac{\kappa}{2}\int_{\mathbf{x}} x^i \epsilon^{ij}\,\{A_j, B\} \quad . \tag{2.E.4}$$

J generates the transformation (2.E.2),

$$\delta A_i = i[J, A_i] \tag{2.E.5}$$

and commutes with the Hamiltonian (2.C.3) when the current is rotationally covariant, *i.e.*, (2.E.3) is satisfied. Also J is gauge invariant, as is seen by replacing A_j in (2.E.4) by $\partial_j\lambda$ and B by $-\rho/\kappa$ according to (2.C.4). When the charge density is spherically symmetric, an integration by parts yields a vanishing response to the gauge transformation. J is not obtained from Noether's energy-momentum tensor, rather from the symmetric tensor, which has no pure gauge field contribution owing to the topological nature of the Chern–Simons term; only the interaction contributes.

$$T^{\mu\nu} = g^{\mu\nu} j^\alpha A_\alpha - j^\mu A^\nu - j^\nu A^\mu \tag{2.E.6}$$

Thus we have

$$\begin{aligned} J = \int_{\mathbf{x}} \epsilon^{ij} x^i T^{0j} &= \int_{\mathbf{x}} \epsilon^{ij} x^i A_j \rho \\ &= -\kappa \int_{\mathbf{x}} \epsilon^{ij} x^i A_j B \quad , \end{aligned} \tag{2.E.7}$$

where (2.C.4) was used. In (2.E.4) the expression is symmetrized to insure Hermiticity; however, it differs from (2.E.7) by a commutator $i\int_{\mathbf{x}} \epsilon^{ij}\partial_j\delta^2(\mathbf{x}-\mathbf{x})$, which although involving $\delta^2(0)$ can be set to zero by an integration by parts. [Alternatively, $\partial_j\delta^2(\mathbf{x})$ is odd in its argument.]

Consider now the action of J on the state (2.C.8)

$$J\Psi(\theta) = \left[\int_{\mathbf{x}} \epsilon^{ij} x^i A_j \rho\right]\Psi(\theta) = \int \rho\epsilon^{ij} x^i \left[\partial_j\theta - \frac{1}{\kappa}\epsilon^{jk}\partial_k^{-1}\rho\right]\Psi(\theta) \tag{2.E.8}$$

The contribution proportional to $\partial_j\theta$ vanishes upon an integration by parts for spherically symmetric ρ. This shows that the physical state (2.C.8) is an angular momentum eigenstate with eigenvalue S.

$$\begin{aligned} S &= \frac{1}{\kappa}\int \rho x^i \partial_i^{-1}\rho \\ &= \frac{1}{2\pi\kappa}\int_{\mathbf{x}}\int_{\mathbf{y}} \rho(t,\mathbf{x})\frac{\mathbf{x}\cdot(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}\rho(t,\mathbf{y}) \\ &= \frac{1}{4\pi\kappa}\int_{\mathbf{x}}\int_{\mathbf{y}} \rho(t,\mathbf{x})\left[1+\frac{\mathbf{x}^2-\mathbf{y}^2}{|\mathbf{x}-\mathbf{y}|^2}\right]\rho(t,\mathbf{y}) \\ &= \frac{Q^2}{4\pi\kappa} \quad , \end{aligned} \tag{2.E.9}$$

Equation (2.E.9) gives the fractional spin carried by the charged matter particles, and agrees with previous results. S is a sharp observable, which satisfies the spin statistics relation (2.E.1).

F. Point-Particles with Abelian Chern–Simons Gauge Fields

A possible model for charged matter consists of point particles.[11] The total Lagrangian is

$$L = L_{\text{matter}} + L_{\text{interaction}} + L_{\text{CS}} \tag{2.F.1}$$

where

$$L_{\text{matter}} = \frac{1}{2}\sum_{p=1}^{N} m_p v_p^2(t) \tag{2.F.2a}$$

$$\begin{aligned} L_{\text{interaction}} &= \sum_{p=1}^{N} e_p \Big(\mathbf{v}_p(t) \cdot \mathbf{A}\left(t, \mathbf{r}_p(t)\right) - A_0\left(t, \mathbf{r}_p(t)\right) \Big) \\ &= -\int_{\mathbf{x}} A_\mu(x) j^\mu(x) \end{aligned}$$

$$j^\mu = \sum_{p=1}^{N} e_p v_p^\mu(t) \delta^2\left(\mathbf{x} - \mathbf{r}_p(t)\right) = (\rho(x), \mathbf{j}(x))$$

$$v_p^\mu = (1, \mathbf{v}_p) \tag{2.F.2b}$$

$$L_{\text{CS}} = \frac{\kappa}{2}\int_{\mathbf{x}} \epsilon^{ij} \dot{A}_i(x) A_j(x) - \kappa \int_{\mathbf{x}} A_0(x) B(x) \tag{2.F.2c}$$

We are considering N point-particles with coordinates $\mathbf{r}_p(t)$, $p = 1, \ldots, N$, which are the particle dynamical variables, and $\mathbf{v}_p(t) = \dot{\mathbf{r}}_p(t)$ are the velocities. The masses and charges are m_p and e_p, respectively. The second expression for the interaction Lagrangian makes use of the point-particle current, which is a δ-function. Thus the integral over all space evaluates $x = (t, \mathbf{x})$, the field point argument of $A_\mu(x)$, at $x = (t, \mathbf{r}_p(t))$. The time component A_0 has not been set to zero.

The [unordered] Euler–Lagrange equations of motion consist of the Lorentz force equation for the matter variables

$$m_p \dot{v}_p^{\mathbf{i}} = e_p \left(E^{\mathbf{i}}(\mathbf{r}_p) + \epsilon^{ij} v_p^j B(\mathbf{r}_p) \right) \tag{2.F.3a}$$

and a field-current identity that relates the electromagnetic fields to the matter currents.

$$E^{\mathbf{i}}(x) = \frac{1}{\kappa}\epsilon^{ij} j^j(x) \tag{2.F.3b}$$

$$B(x) = -\frac{1}{\kappa}\rho(x) \tag{2.F.3c}$$

Point-particle electrodynamics suffers from well-known self-energy problems. Let us observe that these are absent here. Consider the equation of motion for a single particle, $N = 1$ and subscript p suppressed. The force in Eq. (2.F.3a) arises from the electromagnetic fields at the particle position $\mathbf{r}$; by (2.F.3b) and (2.F.3c) they are given by the charge and current densities evaluated at $\mathbf{x} = \mathbf{r}$. However from (2.F.2b) we see that at $\mathbf{x} = \mathbf{r}$ there appears the undefined quantity $\delta^2(\mathbf{r} - \mathbf{r})$ — the density function of a point-particle at its position. Fortunately this

singular object is multiplied by a factor that vanishes, since according to the field-current identity, the quantity

$$E^i(\mathbf{x}) + \epsilon^{ij} v^j B(\mathbf{x}) = \frac{1}{\kappa}\epsilon^{ij}\left(j^j(x) - v^j\rho(x)\right) = \frac{1}{\kappa}\epsilon^{ij}\left(v^j - v^j\right)\delta^2(\mathbf{x}-\mathbf{r})$$

vanishes unambiguously; therefore we shall take it to be zero also at $\mathbf{x} = \mathbf{r}$. In other words the charge and current densities are regulated by non-singular expressions for the evaluation of the self-interaction, which is then shown to vanish.

With this prescription, particles interacting through Chern–Simons gauge fields do not experience self-interactions. Equations (2.F.3) combine to give for the particle coordinates a closed equation of motion, free from undefined quantities.

$$m_p \dot{v}_p^i = \epsilon^{ij}\frac{e_p}{\kappa}\sum_{q\neq p} e_q\left(v_q^j - v_p^j\right)\delta\left(\mathbf{r}_p - \mathbf{r}_q\right) \tag{2.F.4}$$

The Hamiltonian arising from the Lagrangian (2.F.1) and (2.F.2) is

$$H = \frac{1}{2}\sum_{p=1}^{N} m_p v_p^2 + \int_{\mathbf{x}} A_0(x)\left(\kappa B(x) + \rho(x)\right) \tag{2.F.5}$$

It is recognized that the Lagrange multiplier A_0 may be set to zero [by choosing the Weyl gauge] provided Eq. (2.F.3c) is imposed as a constraint. Thus the Hamiltonian is just the free particle one.

$$H = \frac{1}{2}\sum_{p=1}^{N} m_p v_p^2 \tag{2.F.6}$$

Although the gauge field is invisible in (2.F.6), its presence is felt in the commutator algebra.

The commutation relations between canonical variables follow in the usual way, except that vector potentials satisfy the Abelian version of (2.B.8). However, it is useful to present the algebra in terms of the gauge invariant velocity operator, which occurs in the Hamiltonian, rather than in terms of the canonical momentum.

$$\mathbf{p}_p = \frac{\partial L}{\partial \mathbf{v}_p} = m_p\mathbf{v}_p + e_p\mathbf{A}(\mathbf{r}_p) \tag{2.F.7}$$

Thus we have from (2.B.8),

$$\left[A^i(\mathbf{x}), A^j(\mathbf{y})\right] = \frac{i}{\kappa}\epsilon^{ij}\delta^2(\mathbf{x}-\mathbf{y}) \tag{2.F.8}$$

The particle variables satisfy

$$\left[r_p^i, r_q^j\right] = 0 \tag{2.F.9a}$$

$$\left[r_p^i, m_q v_q^j\right] = i\delta^{ij}\delta_{pq} \tag{2.F.9b}$$

$$\left[m_p v_p^i, m_q v_q^j\right] = i\epsilon^{ij}\left(\delta_{pq}e_p B(\mathbf{r}_q) + \frac{1}{\kappa}e_p e_q \delta^2(\mathbf{r}_p - \mathbf{r}_q)\right) \tag{2.F.9c}$$

The velocity commutator does not vanish; rather it contains terms that arise from: (i) the fact that $\mathbf{v}$ differs from $\mathbf{p}$ by $\mathbf{A}(\mathbf{r})$, (2.F.7), and $\mathbf{p}$ does not commute with $\mathbf{A}(\mathbf{r})$ but produces the first term in the parenthesis of (2.F.9c); also (ii) the vector potentials do not commute, (2.F.8), giving rise to the second term in parenthesis of (2.F.9c). There is also the non-vanishing commutator between velocity and field.

$$\left[m_p v_p^i, A^j(\mathbf{x})\right] = -\frac{i}{\kappa}\epsilon^{ij} e_p \delta^2(\mathbf{x} - \mathbf{r}_p) \tag{2.F.10}$$

Finally we remind that (2.F.3c) is imposed as a constraint.

Before proceeding with an analysis of the dynamical problem, it is interesting to record the symmetries of our theory.

First there are the spatial translation and rotation symmetries, under which the coordinates and fields change as

$$\begin{aligned} \delta r_p^i &= a^i \\ \delta A_\mu(t,\mathbf{x}) &= -a^i \partial_i A_\mu(t,\mathbf{x}) \qquad \text{(translations)} \end{aligned} \tag{2.F.11}$$

$$\begin{aligned} \delta r_p^i &= -\epsilon^{ij} r_p^j \\ \delta A_0(t,\mathbf{x}) &= -\epsilon^{jk} x^j \partial_\kappa A_0(t,\mathbf{x}) \\ \delta A_i(t,\mathbf{x}) &= -\epsilon^{jk} x^j \partial_\kappa A_i(t,\mathbf{x}) - \epsilon^{ij} A_j(t,\mathbf{x}) \quad \text{(rotations)} \end{aligned} \tag{2.F.12}$$

The Lagrangian (2.F.1), (2.F.2) is invariant, and the conserved constants of motion are the momentum $\mathbf{P}$ and angular momentum J, respectively.

$$\begin{aligned} \mathbf{P} &= \sum_{p=1}^N (m_p \mathbf{v}_p + e_p \mathbf{A}(\mathbf{r}_p)) + \kappa \int_{\mathbf{x}} \mathbf{A}B \\ &= \sum_{p=1}^N \mathbf{p}_p + \kappa \int_{\mathbf{x}} \mathbf{A}B \end{aligned} \tag{2.F.13}$$

$$\begin{aligned} J &= \sum_{p=1}^N (\mathbf{r}_p \times (m_p \mathbf{v}_p + e_p \mathbf{A}(\mathbf{r}_p))) + \kappa \int_{\mathbf{x}} (\mathbf{x} \times \mathbf{A})B \\ &= \sum_{p=1}^N \mathbf{r}_p \times \mathbf{p}_p + \kappa \int_{\mathbf{x}} (\mathbf{x} \times \mathbf{A})B \ . \end{aligned} \tag{2.F.14}$$

The second formula in both expressions makes use of the canonical momentum, (2.F.7). The vector potential, evaluated at $\mathbf{x} = \mathbf{r}_p$, which is combined with $m_p \mathbf{v}_p$ to form the canonical momentum, arises from the integration Lagrangian (2.F.2b); the last term in (2.F.13) and (2.F.14), proportional to κ, arises from the Chern–Simons kinetic term.

Let us observe that $\sum_{p=1}^N \mathbf{r}_p \times \mathbf{p}_p$ possesses integer eigenvalues when acting on single valued wave functions. Thus point-particles, interacting with gauge fields that are governed by Chern–Simons dynamics, possess in their angular momentum a contribution additional to the

usual integer. The extra term is not quantized but is determined by the gauge field and the Chern–Simons coupling strength κ. This is consistent with the results of the external source analysis, presented earlier, and it will be shown later that in fact there is complete agreement.

Note that for point-particles moving in *prescribed external* gauge fields [rather than *dynamical* ones], the last term in (2.F.13) and (2.F.14) is missing, and $\mathbf{A}(\mathbf{r}_p)$ is a given function of $\mathbf{r}_p$. Therefore, in the external field problem the angular momentum spectrum comprises the conventional integers, even though it differs from the kinematical momentum.[12]

$$\sum_{p=1}^{N} \mathbf{r}_p \times \mathbf{p}_p = \sum_{p=1}^{N} \mathbf{r}_p \times m_p \mathbf{v}_p + \sum_{p=1}^{N} e_p \mathbf{r}_p \times \mathbf{A}(\mathbf{r}_p)$$

[This point is occasionally confused in the literature.]

In addition to the symmetries under the above spatial transformations, the theory is also invariant against transformations of time: obviously time translation $[t \to t + t_0]$ is a symmetry leading to energy [= Hamiltonian] conservation; but there are two further, unexpected time transformations that leave the action invariant: time dilation $[t \to \lambda t]$, and time special conformal transformation $[1/t \to 1/t + 1/t_0]$. Together, the three form a dynamical $SO(2,1)$ symmetry group of conformal transformations.[2f]

Infinitesimally we have for these time reparametrizations

$$\delta t = -f(t) \tag{2.F.15a}$$

$$f(t) = \begin{cases} 1 & \text{translation} \\ t & \text{dilation} \\ t^2 & \text{special conformal transformation} \end{cases} \tag{2.F.15b}$$

The dynamical variables transform as

$$\begin{aligned} \delta_f \mathbf{r}_p(t) &= f(t)\mathbf{v}_p(t) - \frac{1}{2}\dot{f}(t)\mathbf{r}_p(t) \\ \delta_f A_0(t,\mathbf{x}) &= \partial_t\big(f(t)A_0(t,\mathbf{x})\big) + \frac{1}{2}\dot{f}(t)\mathbf{x}\cdot\nabla A_0(t,\mathbf{x}) - \frac{1}{2}\ddot{f}(t)\mathbf{x}\cdot\mathbf{A}(t,\mathbf{x}) \\ \delta_f \mathbf{A}(t,\mathbf{x}) &= \partial_t\big(f(t)\mathbf{A}(t,\mathbf{x})\big) - \frac{1}{2}\dot{f}(t)\mathbf{A}(t,\mathbf{x}) + \dot{f}(t)\mathbf{x}\cdot\nabla\mathbf{A}(t,\mathbf{x}) \end{aligned} \tag{2.F.16}$$

and the Lagrangian changes by a total time derivative. The constants of motion arising from the three transformations are, respectively

$$H = \frac{1}{2}\sum_{p=1}^{N} m_p v_p^2 + \int_{\mathbf{x}} A_0(\kappa B + \rho) \tag{2.F.17a}$$

$$D = tH - \frac{1}{4}\sum_{p=1}^{N} m_p\left(\mathbf{r}_p\cdot\mathbf{v}_p + \mathbf{v}_p\cdot\mathbf{r}_p\right) - \frac{1}{2}\int_{\mathbf{x}} \mathbf{x}\cdot\mathbf{A}(\kappa B + \rho) \tag{2.F.17b}$$

$$K = -t^2 H + 2tD + \frac{1}{2}\sum_{p=1}^{N} m_p r_p^2 \tag{2.F.17c}$$

Of course (2.F.17a) coincides with the Hamiltonian of (2.F.5).

Returning now to the dynamical problem, we observe that when the constraint (2.F.3c) is imposed, all reference to gauge fields disappears from the equation of motion (2.F.4). Hence, we can set $\kappa B + \rho$ to zero throughout, and suppress the gauge degrees of freedom. Therefore, only the dynamical algebra (2.F.9) is relevant, and now it takes the form

$$[r_p^i, r_q^j] = 0 \tag{2.F.18a}$$

$$[r_p^i, m_q v_q^j] = i\delta^{ij}\delta_{pq} \tag{2.F.18b}$$

$$[m_p v_p^i, m_q v_q^j] = i\frac{\epsilon^{ij}}{\kappa}\Big((1-\delta_{pq})e_p e_q \delta^2(\mathbf{r}_q - \mathbf{r}_q) - \delta_{pq}\sum_{n\neq p} e_p e_n \delta^2(\mathbf{r}_p - \mathbf{r}_n)\Big) \tag{2.F.18c}$$

The consistency of these commutation relations is established by realizing the operators through their action on functions of $\mathbf{r}_p$, with $\mathbf{r}_p$ acting by multiplication, while $m_p\mathbf{v}_p$ is redefined as

$$m_p v_p^i = p_p^i - \frac{e_p}{2\pi\kappa}\epsilon^{ij}\sum_{q\neq p} e_q \frac{(r_p^j - r_q^j)}{|\mathbf{r}_p - \mathbf{r}_q|^2} \tag{2.F.19}$$

with $\mathbf{p}_p$ acting as $-i\nabla_{\mathbf{r}_p}$.

The symmetry generators (2.F.13), (2.F.14) and (2.F.17) become

$$\mathbf{P} = \sum_{p=1}^{N} m_p \mathbf{v}_p \tag{2.F.20}$$

$$J = \sum_{p=1}^{N} \mathbf{r}_p \times m_p \mathbf{v}_p \tag{2.F.21}$$

$$H = \frac{1}{2}\sum_{p=1}^{N} m_p v_p^2 \tag{2.F.22a}$$

$$D = tH - \frac{1}{4}\sum_{p=1}^{N} m_p\left(\mathbf{r}_p\cdot\mathbf{v}_p + \mathbf{v}_p\cdot\mathbf{r}_p\right) \tag{2.F.22b}$$

$$K = -t^2 H + 2tD + \frac{1}{2}\sum_{p=1}^{N} m_p r_p^2 \tag{2.F.22c}$$

The presence of the interaction is hidden, but it is in evidence in the commutators (2.F.18) and in the relation (2.F.19) between canonical momentum and velocity, which implies that the effective particle Lagrangian is[11]

$$L_{\text{effective}} = \sum_{p=1}^{N}\left(\frac{1}{2}m_p v_p^2 + e_p \mathbf{v}_p\cdot\mathbf{a}_p\right) \tag{2.F.23a}$$

$$a_p^i(\mathbf{r}_1,\ldots,\mathbf{r}_N) = \frac{1}{2\pi\kappa}\epsilon^{ij}\sum_{q\neq p} e_q \frac{(r_p^j - r_q^j)}{|\mathbf{r}_p - \mathbf{r}_q|^2}$$
$$= \frac{1}{2\pi\kappa}\epsilon^{ij}\frac{\partial}{\partial r_p^j}\sum_{q\neq p} e_q \ln|\mathbf{r}_p - \mathbf{r}_q|$$
$$= -\frac{1}{2\pi\kappa}\frac{\partial}{\partial r_p^i}\sum_{q\neq p} e_q\theta_{pq}$$
$$\tan\theta_{pq} = \frac{y_p - y_q}{x_p - x_q} \qquad (2.F.23b)$$

Explicitly, (2.F.23a) reads

$$L_{\text{effective}} = \frac{1}{2}\sum_{p=1}^{N} m_p v_p^2 + \frac{1}{2\pi\kappa}\sum_{\substack{p,q=1\\ p<q}}^{N} e_p e_q \frac{(\mathbf{v}_p - \mathbf{v}_q)\times(\mathbf{r}_p - \mathbf{r}_q)}{|\mathbf{r}_p - \mathbf{r}_q|^2} \qquad (2.F.24)$$

Note that the angular momentum (2.F.21) is not constructed from the canonical momentum, hence as already remarked, its eigenvalues are non-integral. All the constants of motion $C = \mathbf{P}$, J, H, D and K generate the appropriate transformations (2.F.11), (2.F.12) and (2.F.16) upon commutation.

$$\delta\mathbf{r}_p = i\,[C, .\mathbf{r}_p] \qquad (2.F.25)$$

In particular, the equation of motion (2.F.4), properly symmetrized, emerges upon commuting $\mathbf{v}_p$ with H. Also the $SO(2,1)$ generators satisfy the conformal Lie algebra,

$$[D,H] = -iH \quad , \quad [D,K] = iK \quad , \quad [H,K] = 2iD \qquad (2.F.26)$$

which may be presented in the more familiar Cartan basis by forming linear combinations with the help of a fixed, positive parameter a of time dimensionality

$$\mathcal{R} = \frac{1}{2}\left(\frac{1}{a}K + aH\right) \qquad (2.F.27a)$$
$$\mathcal{S} = \frac{1}{2}\left(\frac{1}{a}K - aH\right) \qquad (2.F.27b)$$
$$L_\pm = (\mathcal{S} \pm iD) \qquad (2.F.27c)$$

$\mathcal{S}$ and D act as non-compact two-dimensional boost generators, while $\mathcal{R}$ generates the rotations that form the compact $SO(2)$ subgroup of $SO(2,1)$.

$$[\mathcal{R}, L_\pm] = \pm L_\pm \qquad (2.F.28a)$$
$$[L_+, L_-] = -2\mathcal{R} \qquad (2.F.28b)$$

J commutes with the conformal generators; it rotates $\mathbf{P}$ in the proper manner.

$$[J, P^i] = i\epsilon^{ij}P^j \qquad (2.F.29)$$

The momentum commutes with H; the commutators with the remaining conformal generators are

$$[D, \mathbf{P}] = -\frac{i}{2}\mathbf{P} \tag{2.F.30}$$

$$[K, \mathbf{P}] = -i\left(t\mathbf{P} - \sum_{p=1}^{N} m_p \mathbf{r}_p\right) \tag{2.F.31}$$

Equation (2.F.30) shows that the scale dimension of the momentum is 1/2, opposite to that of the coordinate $\mathbf{r}_p$. Since the commutator of two constants of motion is again a constant of motion, the right-hand side of (2.F.31) shows that center-of-mass motion is free. This is a consequence of the evident invariance of our theory against Galileo boosts, which are generated by $t\mathbf{P} - \sum_{p=1}^{N} m_p \mathbf{r}_p$.

That the generators are indeed constants of motion may be established by use of the formula

$$\frac{dC}{dt} = \frac{i}{\hbar}[H, C] + \frac{\partial C}{\partial t} \tag{2.F.32}$$

It follows from (2.F.32) that all three generators are time-dependent. Note however, D and K do not commute with the Hamiltonian; their total time derivative vanishes owing to the explicit time-dependence seen in (2.F.22b) and (2.F.22c).

We conclude this discussion of point-particle/Chern–Simons dynamics by recording the Casimir operator $\mathcal{J}^2$ of the SO(2,1) group.

$$\mathcal{J}^2 = \mathcal{R}^2 - \mathcal{S}^2 - D^2 = \frac{1}{2}(KH + HK) - D^2 \tag{2.F.33}$$

G. Quantum Dynamics

The Schrödinger equation governing dynamics of particles interacting with Chern–Simons gauge fields is inferred from (2.F.19), (2.F.22a) and (2.F.23).

$$i\frac{\partial}{\partial t}\Psi(t; \mathbf{r}_1, \ldots, \mathbf{r}_N) = H\Psi(t; \mathbf{r}_1, \ldots, \mathbf{r}_N) \tag{2.G.1a}$$

$$H = \sum_{p=1}^{N} \frac{1}{2m_p}\left(\frac{1}{i}\nabla_{\mathbf{r}_p} - e_p \mathbf{a}_p\right)^2 \tag{2.G.1b}$$

The wave function Ψ is single-valued. We may however make use of the formulas in (2.F.23b) to express $\mathbf{a}_p$ as a gradient, and remove the interaction in (2.G.1) by redefining the phase of the wave function.[13]

$$\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) = e^{i\Theta}\Psi^0(\mathbf{r}_1, \ldots, \mathbf{r}_N) \tag{2.G.2}$$

$$\Theta = \sum_{\substack{p,q=1 \\ p<q}}^{N} \nu_{pq}\theta_{pq} \tag{2.G.3a}$$

$$\nu_{pq} = -\frac{e_p e_q}{2\pi\kappa} \tag{2.G.3b}$$

Then Ψ^0 satisfies the *free* Schrödinger equation.

$$i\frac{\partial}{\partial t}\Psi^0(t;\mathbf{r}_1,\ldots,\mathbf{r}_N) = H^0\Psi^0(t;\mathbf{r}_1,\ldots,\mathbf{r}_n) \tag{2.G.4a}$$

$$H^0 = e^{-i\Theta}H^0e^{i\Theta} = \sum_{p=1}^{N}\left(-\frac{1}{2m_p}\nabla^2_{\mathbf{r}_p}\right) \tag{2.G.4b}$$

Even though H^0 is a sum of one-body Hamiltonians, Ψ^0 cannot be chosen as a simple product of one-body eigenstates [plane waves] because Ψ^0 satisfies complicated aperiodicity conditions, which must hold so that Ψ be single-valued. Of course Ψ^0 is a *superposition* of plane waves, however, determining the precise superposition, which when multiplied by $e^{i\Theta}$ gives a single-valued wave function, is a challenging, non-trivial problem that has been solved only for the two-body case.

The two-body problem is tractable because of the center-of-mass reduction, wherein only the relative coordinate, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, experiences the interaction, while the center-of-mass coordinate $\mathbf{R} = \frac{m_1\mathbf{r}_1+m_2\mathbf{r}_2}{m_1+m_2}$ moves freely. By setting total momentum to zero, the two-body wave function depends only on $\mathbf{r}$ and satisfies

$$i\frac{\partial}{\partial t}\psi(t;\mathbf{r}) = h\psi(t;\mathbf{r}) \tag{2.G.5}$$

where the Hamiltonian for relative motion is

$$h = \frac{1}{2M}\left(\mathbf{p} - \mathbf{a}(\mathbf{r})\right)^2 \tag{2.G.6}$$

M is the reduced mass, $M^{-1} = m_1^{-1} + m_2^{-1}$, and the vector potential a gives rise to a vortex with flux $\Phi = -\frac{e_1e_2}{\kappa}$

$$\begin{aligned} a^i(\mathbf{r}) &= -\frac{\Phi}{2\pi}\epsilon^{ij}\frac{\hat{r}^j}{r} = -\frac{\Phi}{2\pi}\epsilon^{ij}\partial_j \ln r \\ &= \frac{\Phi}{2\pi}\partial_i\theta = \nu\partial_i\theta \end{aligned} \tag{2.G.7}$$

$$\mathbf{r} = (r\cos\theta, r\sin\theta)$$

Hence (2.G.5) requires, after the usual separation of time,

$$\psi(t;\mathbf{r}) = e^{-iEt}\psi_E(\mathbf{r}) \tag{2.G.8}$$

solving the following eigenvalue problem.

$$-\frac{1}{2M}\left(\nabla - i\nu\nabla\theta\right)^2\psi_E(\mathbf{r}) = E\psi_E(\mathbf{r}) \tag{2.G.9}$$

The eigenfunctions necessarily have theform

$$\psi_E(\mathbf{r}) = e^{i\nu\theta}\psi^0_k(\mathbf{r}) \tag{2.G.10}$$

where $\psi_k^0(\mathbf{r})$, though governed by the free Hamiltonian

$$h^0 = e^{-i\nu\theta} h\, e^{i\nu\theta} \tag{2.G.11}$$

$$h^0\psi_k^0 = \frac{k^2}{2M}\psi_k^0 \quad , \qquad E = \frac{k^2}{2M} \tag{2.G.12}$$

is not a conventional plane wave, owing to a non-trivial boundary condition,

$$\psi_k^0(r, \theta = 0) = e^{-i2\pi\nu}\psi_k^0(r, \theta = 2\pi) \tag{2.G.13}$$

which must be met so that $\psi_E(\mathbf{r})$ is single-valued.

Since rotation by 2π corresponds to double exchange of particles, we see that ψ_k^0 acquires a statistics factor $-\pi\nu = e^2/2\kappa$, in agreement with (2.E.1) for $e_1 = e_2$. Moreover, the [relative] angular momentum

$$\begin{aligned} J &= \mathbf{r} \times M\mathbf{v} = \mathbf{r} \times \mathbf{p} - \mathbf{r} \times \mathbf{a} \\ &= \mathbf{r} \times \mathbf{p} + \frac{e^2}{2\pi\kappa} \end{aligned} \tag{2.G.14}$$

indicates that each particle possesses additional spin $e^2/4\pi\kappa$ again in agreement with (2.E.1).

[The angular momentum operator acting on the multi-valued wavefunction ψ_k^0 is

$$\begin{aligned} J^0 = e^{-i\nu\theta} J\, e^{i\nu\theta} &= e^{-i\nu\theta}\frac{1}{i}\,\frac{\partial}{\partial\theta}e^{i\nu\theta} + \frac{e^2}{2\pi\kappa} \\ &= \frac{1}{i}\,\frac{\partial}{\partial\theta} + \nu + \frac{e^2}{2\pi\kappa} = \frac{1}{i}\,\frac{\partial}{\partial\theta} \quad , \end{aligned}$$

i.e. it is just the angular derivative. Nevertheless, its eigenvalues are non-integral, just as those of J in (2.G.14), since $J^0 = \frac{1}{i}\,\frac{\partial}{\partial\theta}$ acts on multi-valued functions which satisfy (2.G.13). It should further be emphasized that, as we have already stated, the reason that J is just $r \times M\mathbf{v}$ and not $\mathbf{r} \times \mathbf{p}$ is because our effective particle theory arises from a *dynamical* model for the gauge potential, where the dynamics is governed by the Chern–Simons term. If the problem (2.G.5) and (2.G.6) is viewed as describing single particle motion in an *externally* prescribed gauge potential $\mathbf{a}(\mathbf{r})$, then the correct angular momentum *is* $\mathbf{r} \times \mathbf{p}$, with integral eigenvalues.[12]]

The Schrödinger equation (2.G.5), (2.G.6) and (2.G.7) leads only to scattering, and the scattering amplitude has been obtained long ago by Aharonov and Bohm, and later by Ruijsenaars.[14] More recently the problem has re-emerged in the context of planar gravity.[3e]. As we shall see, there too one seeks free solutions with unconventional boundary conditions.

I shall now present the solution, using the gravitational techniques, which have the advantage of giving an explicit wave function, in the form of a contour integral.[2f]

$$\psi_k^0(\mathbf{r}) = \oint \frac{dz}{2\pi} e^{i\mathbf{k}(z)\cdot\mathbf{r}}\rho(z) \tag{2.G.15}$$

Here $\mathbf{k}(z) = (k\cos z, k\sin z)$ and it is obvious that $\psi_k^0(\mathbf{r})$ satisfies the free equation. That it also satisfies (2.G.13) requires a special contour C and weight function ρ, which we now derive, by considering the scattering problem, the radial equation and the phase shifts.

The *Ansatz* $u_E^j(r)\frac{e^{ij\theta}}{\sqrt{2\pi}}$ is made for $\psi_E(\mathbf{r})$ in (2.G.9); j is an arbitrary integer to insure single valuedness, $j = 0, \pm 1, \pm 2, \ldots$; $u_E^j(r)$ satisfies the radial equation,

$$\left(-\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} + \frac{(j-\nu)^2}{r^2}\right) u_E^j(r) = k^2 u_E^j(r) \tag{2.G.16}$$

and is given by a Bessel function.

$$u_E^j(r) = \sqrt{M} J_{|j-\nu|}(kr) \tag{2.G.17}$$

The angular momentum of this partial wave is $j - \nu$; see (2.G.14). The normalization is fixed by

$$\int_0^\infty r dr\, u_E^j(r) u_E^j(r) = \delta(E - E') \tag{2.G.18a}$$

and insures

$$\int_0^\infty dE\, u_E^j(r) u^j E(r') = \frac{1}{r}\delta(r - r') \tag{2.G.18b}$$

When the plane wave identity

$$e^{ikr\cos\theta} = \sum_{j=-\infty}^{\infty} e^{ij\left(\theta+\frac{\pi}{2}\right)} J_j(kr) \tag{2.G.19}$$

is recalled, we are led to form the scattering solution by

$$\psi_E(\mathbf{r}) = \sum_{j=-\infty}^{\infty} e^{i\left(\delta_j + \frac{\pi}{2}j\right)} u_E^j(r) \frac{e^{ij\theta}}{\sqrt{2\pi}} \tag{2.G.20}$$

where the phase shift δ_j of $u_E^j(r)$ relative to $J_j(kr)$ is identified from their large r asymptotes.

$$\delta_j = \begin{cases} \nu\frac{\pi}{2} & j > [\nu] \\ -\nu\frac{\pi}{2} - j\pi & j \leq [\nu] \end{cases} \tag{2.G.21}$$

Here the brackets [ ] indicate integer part. The energy independence of the phase shifts is a consequence of scale invariance.[15]

From (2.G.20) and (2.G.21), the wave function is constructed as

$$\begin{aligned} \psi_E(\mathbf{r}) = &\sqrt{\frac{M}{2\pi}} \sum_{j=[\nu]+1}^{\infty} e^{i\frac{\pi}{2}(\nu+j)} J_{j-\nu}(kr) e^{ij\theta} \\ &+ \sqrt{\frac{M}{2\pi}} \sum_{j=-\infty}^{[\nu]} e^{-i\frac{\pi}{2}(\nu+j)} J_{\nu-j}(kr) e^{ij\theta} \end{aligned} \tag{2.G.22}$$

The sums are performed by using the Schläfli contour representation for the Bessel function.

$$J_\alpha(kr) = e^{i\frac{\pi}{2}\alpha} \oint_s \frac{dz}{2\pi} e^{-ikr\cos z} e^{iz\alpha} \tag{2.G.23}$$

The Schläfli contour C_s begins at $z = -3\pi/2 + i\infty$, descends to slightly above the real axis, passes from $z = -3\pi/2 + i(0^+)$ to $z = \pi/2 + i(0^+)$ and ascends to $z = \pi/2 + i\infty$. The sums are now geometric, and give for the two terms in (2.G.22), respectively,

$$\begin{aligned}
\psi_E(\mathbf{r}) &= \left(\frac{M}{2\pi}\right)^{1/2} \oint_s \frac{dz}{2\pi} e^{-ikr\cos s} e^{i([\nu]\pi - \{\nu\}z + [\nu]\theta)} \frac{-1}{1+e^{-i(z+\theta)}} \\
&+ \left(\frac{M}{2\pi}\right)^{1/2} \oint_s \frac{dz}{2\pi} e^{-ikr\cos s} e^{i([\nu]\pi - \{\nu\}z - [\nu]\theta)} \frac{1}{1+e^{i(z-\theta)}} \\
&= \left(\frac{M}{2\pi}\right)^{1/2} \oint_{-s} \frac{dz}{2\pi} e^{-ikr\cos s} e^{i([\nu]\pi + \{\nu\}z + [\nu]\theta)} \frac{1}{1+e^{i(z-\theta)}} \\
&+ \left(\frac{M}{2\pi}\right)^{1/2} \oint_s \frac{dz}{2\pi} e^{ikr\cos s} e^{i([\nu]\pi + \{\nu\}z + [\nu]\theta)} \frac{1}{1+e^{i(z-\theta)}} \\
\{\nu\} &\equiv \nu - [\nu]
\end{aligned} \tag{2.G.24a}$$

In passing from the first to the second equality, the change of variables $z \to -z$ is performed in the first integral. As a consequence, the integration contour for that integral, now called C_{-s}, becomes the mirror image of the Schläfli contour C_s. [C_{-s} starts from $3\pi/2 - i\infty$, ascends to $3\pi/2$ below the real axis, passes to $-\pi/2$ and descends to $-\pi/2 - i\infty$.] As a further consequence, the integrands of the two contour integrals become identical [we use $e^{i[\nu]\pi} = e^{-i[\nu]\pi}$]. To proceed, contours are shifted: C_{-s} is shifted by $\pi/2$ to the left, and C_s by $\pi/2$ to right, so that the vertical portions of both contours are at $z = \pm\pi$. The last step is to redefine the integration variable by $z = -z' + \theta - \pi$. The z' integral now runs over the contour C depicted in Fig. 1a, and $\psi_E(\mathbf{r})$ is represented by [z' is renamed z]

$$\begin{aligned}
\psi_E(\mathbf{r}) &= \left(\frac{M}{2\pi}\right)^{1/2} e^{i\nu\theta} \oint \frac{dz}{2\pi} e^{ikr\cos(z-\theta)} \frac{e^{-i\{\nu\}z}}{1-e^{-iz}} \\
&= \left(\frac{M}{2\pi}\right)^{1/2} e^{i\nu\theta} \oint \frac{dz}{2\pi} e^{i\mathbf{k}(z)\cdot\mathbf{r}} \frac{e^{-i\{\nu\}z}}{1-e^{-iz}}
\end{aligned} \tag{2.G.24b}$$

[A constant phase factor has been suppressed.] Thus we have derived the representation (2.G.15), with contour C as in Fig. 1a and $\rho(z)$ given by

$$\rho(z) = \left(\frac{M}{2\pi}\right)^{1/2} \frac{e^{-i\{\nu\}z}}{1-e^{-iz}} \tag{2.G.25}$$

That $\psi_E(\mathbf{r})$ as given by (2.G.24b) satisfies the free Schrödinger equation is obvious; that it is single-valued — periodic in θ with 2π period — is more easily seen in (2.G.24a).

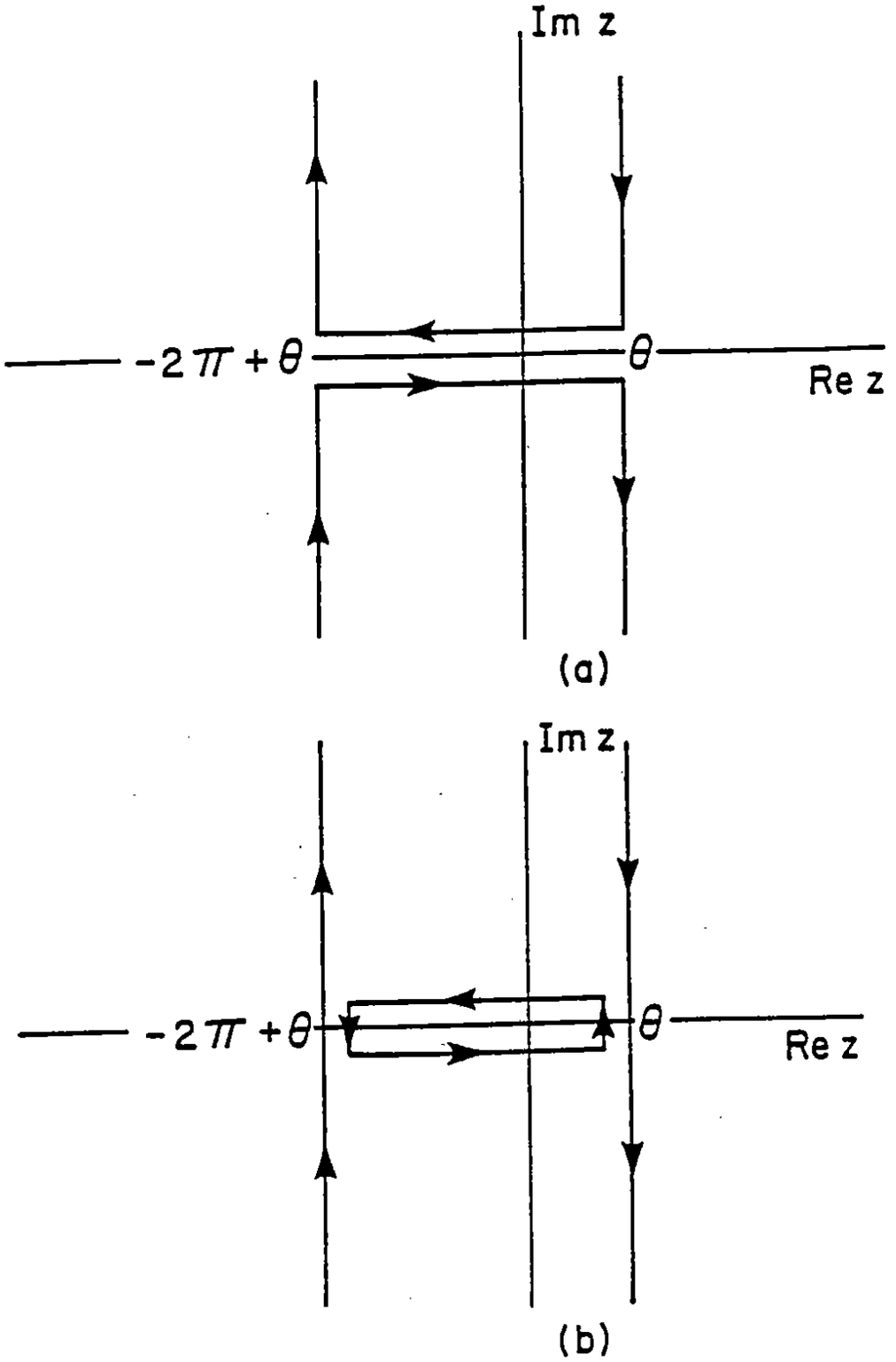


Fig. 1: (a) Integration contour C for the wave function (2.G.24b). The pole at the origin is avoided. (b) Contour $\tilde{C}$ equivalent to contour C. The pole at the origin is enclosed, giving rise to the incoming wave. The vertical contours produce the scattered wave.

The contour C avoids the pole in $\rho(z)$ at $z = 0$. However, we may alternatively enclose the pole and replace the contour C by the three-segmented contour $\tilde{C}$, depicted in Fig. 1b. The portion encircling the pole is evaluated by Cauchy's residue theorem, contributing $(M/2\pi)^{1/2}e^{i(kr\cos\theta+\nu\theta)}$; the portions arising from the vertical axes are presented in terms of real integrals by setting $z = \theta - 2\pi + iy$ and $z = \theta + iy$. Evidently, this separation decomposes the total scattering wave function $\psi_E(\mathbf{r})$ into an incoming wave [pole contribution] and the scattered wave [vertical contour contributions].

$$\psi_E(\mathbf{r}) = \left(\frac{M}{2\pi}\right)^{1/2} (\psi^{\text{in}}(\mathbf{r}) + \psi_E^{\text{sc}}(\mathbf{r}) \tag{2.G.26a}$$

$$\psi_E^{\text{in}}(\mathbf{r}) = e^{i(kr\cos\theta+\nu\theta)} \tag{2.G.26b}$$

$$\psi_E^{\text{sc}}(\mathbf{r}) = e^{i\{\nu\}\theta}e^{i\nu\pi}\sin\nu\pi\int_{-\infty}^{\infty}\frac{dy}{\pi}e^{ikr\cosh y}\frac{e^{\{\nu\}y}}{e^{y-i\theta}-1} \tag{2.G.26c}$$

The large r asymptote of $\psi_E^{\text{sc}}(\mathbf{r})$ defines the scattering amplitude $f(\theta)$ through the formula

$$\psi_E^{\text{sc}}(\mathbf{r}) \underset{r\to\infty}{\longrightarrow} \sqrt{\frac{i}{r}}f(\theta)e^{ikr} \tag{2.G.27}$$

Although the integral (2.G.26c) for $\psi_E^{\text{sc}}(\mathbf{r})$ cannot be performed, its limit at large r can be evaluated. The circular wave formula (2.G.27) is found, with scattering amplitude

$$f(\theta) = \frac{1}{\sqrt{2\pi k}}e^{-i\{\nu\}\theta}e^{i(\nu+1/2)(\theta+\pi)}\frac{\sin\nu\pi}{\sin\theta/2} \tag{2.G.28}$$

Equations (2.G.26b) and (2.G.28) are essentially the results of Aharonov and Bohm and Ruijsenaars;[14] note especially that the incoming wave is not a plane wave, but is modulated by the additional phase $e^{i\nu\theta}$.

Since $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $e^{i\mathbf{k}(z)\cdot\mathbf{r}}$ is a product of two plane waves

$$\begin{aligned} e^{i\mathbf{k}(z)\cdot(\mathbf{r}_1-\mathbf{r}_2)} &= \psi_1^z(\mathbf{r}_1)\psi_2^z(\mathbf{r}_2) \\ \psi_1^z(\mathbf{r}) &= e^{i\mathbf{k}(z)\cdot\mathbf{r}} \\ \psi_2^z(\mathbf{r}) &= e^{-i\mathbf{k}(z)\cdot\mathbf{r}} \end{aligned} \tag{2.G.29}$$

Hence the representation (2.G.24b) shows explicitly how products of one-body plane waves are superposed to form our solution.[2f]

$$\psi_E^0(\mathbf{r}) = \left(\frac{M}{2\pi}\right)^{1/2}\oint\frac{dz}{2\pi}\psi_1^z(\mathbf{r}_1)\psi_2^z(\mathbf{r}_2)\frac{e^{-i\{\nu\}z}}{1-e^{-iz}} \tag{2.G.30}$$

As yet we do not have a similar closed form for the N-body wave function. The problem is reminiscent of the δ-function interaction on a line. There too the many-body wave function is obtained by superposing one-body wave functions in a fashion prescribed by the Bethe *Ansatz*. Perhaps similar ideas will prove useful here.

Finally, we conclude that the two-body relative coordinate problem of course also possesses the $SO(2,1)$ symmetry, with generators given by the relative coordinate parts of the two-body generators (2.F.22)

$$H = (h) = \frac{1}{2}Mv^2 \tag{2.G.31a}$$

$$D = tH - \frac{1}{4}M(\mathbf{r}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{r}) \tag{2.G.31b}$$

$$K = -t^2H + 2tD + \frac{1}{2}Mr^2 \tag{2.G.31c}$$

The algebraic properties of these quantities hold as before; now they are based on the dynamical algebra

$$[r^i, r^j] = 0 \quad , \quad [r^i, Mv^j] = i\delta^{ij} \quad , \quad [Mv^i, Mv^j] = i\epsilon^{ij}2\pi\nu\,\delta(\mathbf{r}) \tag{2.G.32}$$

The Casimir in (2.F.33) can be expressed in terms of the angular momentum (2.G.14)

$$\mathcal{J}^2 = \frac{1}{4}(J^2 - 1) \tag{2.G.33}$$

Since the eigenvalues of J are $j - \nu$, those of $\mathcal{J}^2$ are $\frac{1}{4}\left((j-\nu)^2 - 1\right)$, and the entire motion at fixed angular momentum is described by a single, irreducible, unitary and infinite-dimensional representation of $SO(2,1)$. We have already remarked that the energy independence of the phase shifts is a consequence of the symmetry. Because of the higher symmetry, the time-dependent Schrödinger equation (2.G.5) can be separated in coordinates other than the usual time and space. Indeed group theory may be used to give a complete, alternative analysis of the problem.[2f]

3. PLANAR GRAVITY

A. Introduction

The equations for Einstein's theory of gravity — general relativity — can be presented in any space-time with dimension d equal to or greater than three: The Einstein tensor

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{3.A.1}$$

vanishes in the absence of matter sources,

$$G_{\mu\nu} = 0 \tag{3.A.2a}$$

while in their presence it is proportional to the energy-momentum tensor of matter, $T_{\mu\nu}$.

$$G_{\mu\nu} = 2\pi G\,T_{\mu\nu} \tag{3.A.2b}$$

Here $R_{\mu\nu}$ and R are traces of the four-index Riemann tensor $R_{\alpha\mu\beta\nu}$ in which all local geometrical information about the space-time is encoded. G is the gravitational coupling constant

— the generalization to other dimensions of Newton's constant; in (3.A.2b) G enters with an unconventional normalization that is convenient for the subsequent analysis. The reason that (3.A.2) cannot be posited in two space-time dimensions is because there $G_{\mu\nu}$ vanishes identically. [However other geometrical equations have been proposed at $d = 2$.[16]]

It is obvious from (3.A.2b) that when space-time is flat, *i.e.* when the Riemann tensor vanishes, so also does the Einstein tensor and $T_{\mu\nu}$ must be zero. In general, the converse does not hold: absence of matter implies vanishing Einstein tensor, but the Riemann tensor need not be zero so that empty space-time need not be flat. However, in three dimensions the Riemann tensor is linearly related to the Einstein tensor,

$$R^{\alpha\mu}_{\beta\nu} = \epsilon^{\alpha\mu\gamma}\epsilon_{\beta\nu\delta}G^{\delta}_{\gamma} \tag{3.A.3}$$

so that the vanishing of the latter implies the vanishing of the former: empty space-time is necessarily locally flat.[17]

Several consequences follow immediately: since the vacuum state [empty space-time] is locally flat, there are no gravitational waves in the classical theory, and upon quantization there are no quantum gravitons. Sources produce curvature, but only locally at the location in space-time of the sources. Forces between sources are not mediated by graviton exchange, since there are no gravitons. Rather interactions arise because the locally flat space-time possesses in the large non-trivial geometrical and topological structure that gives rise to non-trivial motions. It also follows that the non-relativistic limit of Einstein's general relativity in three-dimensions is not three-dimensional Newtonian gravity, which involves a conventional force law that decreases with the inverse power of the distance.

It is the purpose of our research program to study in three-dimensional space-time the classical and quantum motions of matter that interacts gravitationally.[3] Since there are no propagating gravitational degrees of freedom, the problem is tractable, and we can learn much about the puzzles that are encountered when a geometrical theory is confronted by quantum mechanics. In four dimensions these puzzles exist as well, and it is my opinion that understanding them is important for understanding quantum gravity; a task quite independent of and perhaps more fundamental than the task of overcoming the unrenormalizable infinities that pollute four-dimensional gravity, but are absent in three dimensions since non-renormalizable graviton exchange does not occur. To conclude these introductory remarks, I note the following points.

(a) The theory can be elaborated by adding a cosmological constant to the field equation. The vacuum is then a space of constant curvature, whose sign depends on the sign of the cosmological constant. While some investigations of such models have been performed, I shall not further discussthem here.[3c]

(b) Another elaboration of the conventional theory involves adding a topological term, analogous to the gauge theoretic modification.[3a] This Chern–Simons addition will be discussed below.

B. Classical Space-Times

We record several interesting space-times that arise from classical sources.[3b] We begin with a single massive but spinless point-particle. Without loss of generality the particle is taken to be at rest at the origin of the coordinate system, *i.e.* it is described by an energy-momentum tensor all whose components, except the energy density, vanish,

$$\begin{aligned} \sqrt{\det g_{\mu\nu}}\, T^{00} &= M\delta(X)\delta(Y) \\ T^{0i} = T^{ij} &= 0 \end{aligned} \qquad (3.B.1)$$

Here M is the particle mass.

The task is to find the metric or equivalently to give a formula for the line element. Clearly it is non-trivial only in its spatial components,

$$(ds)^2 = (dt)^2 - (d\ell)^2 \qquad (3.B.2)$$

and we need to find expressions for $(d\ell)^2$.

We recognize that we seek a space which is everywhere flat [$T^{\mu\nu}$ vanishes] except at the origin where a δ-function singularity concentrates the curvature. It is clear that the desired space is a cone, with the source particle positioned at the apex.[3b, 4d, 17] It remains to give an analytic description of this obvious geometrical fact.

To solve the Einstein equation (3.A.2b) with sources given by (3.B.1), it is necessary to choose a coordinate system, and the conical solution looks different in different coordinates. Of course only the two-dimensional spatial section needs to be considered.

Particularly useful coordinates, which lend themselves to a many-body generalization, are the conformal ones where the metric tensor is a multiple of the flat metric tensor; this can always be locally achieved in two dimensions. The conformally flat spatial metric that solves Einstein's equation then leads to the following spatial interval.

$$(d\ell)^2 = \frac{1}{R^{2GM}}\Big((dR)^2 + R^2(d\Theta)^2\Big) \qquad (3.B.3)$$

Here the variables range over the conventional circles.

$$\begin{aligned} 0 \le R &\le \infty \\ -\pi \le \Theta &\le \pi \end{aligned} \qquad (3.B.4)$$

While (3.B.3) certainly provides the desired solution, it does not seem to produce the cone described earlier. Nor is it manifest that the space is flat except at the origin.

All this can be seen by passing to another coordinate system, attained from (3.B.3) and (3.B.4) by a change of variables.

$$\begin{aligned} r &= \frac{R^{1-GM}}{1-GM} \\ \theta &= (1-GM)\Theta \end{aligned} \qquad (3.B.5)$$

In terms of r and θ the spatial metric is flat, and the line-element is trivial.

$$(d\ell)^2 = (dr)^2 + r^2(d\theta)^2 \tag{3.B.6}$$

However, the range of the new variables is unconventional — an angular region is excised, since according to (3.B.4) the range of (r, θ) is

$$\begin{aligned} 0 \leq r \leq \infty \\ -\pi(1 - GM) \leq \theta \leq \pi(1 - GM) \end{aligned} \tag{3.B.7}$$

This describes a cone, with apex determined by GM. [Henceforth we take $GM \leq 1$. For $GM > 1$, the space changes character and the description becomes more complicated.[3b] At $GM = 1$, it is seen from (3.B.3) that space becomes a cylinder in the variable $r = \ln R$.]

In summary, we say that a point particle of mass M at the origin gives rise to a locally flat space-time, but the global identification of coordinate variables is unconventional and reveals the presence of a massive point-particle: the point (t, r, θ) is identified with

$$(t, r, \theta) \approx (t, r, \theta + 2\pi(1 - GM) \tag{3.B.8a}$$

In terms of a complex variable description of the space, $z = x + iy$, we identify z with

$$z \approx e^{-2\pi i GM} z \tag{3.B.8b}$$

This is the analog in planar gravity of the Schwarzschild solution.

To find the planar analog of the Kerr solution, we endow our point-particle at the origin with spin S, *i.e.* now the energy-momentum tensor possess non-trivial energy density and momentum density, the latter giving rise to no momentum but to angular momentum S.

$$\begin{aligned} \sqrt{\det g_{\mu\nu}} T^{00} &= M\delta(X)\delta(Y) \\ \sqrt{\det g_{\mu\nu}} T^{0i} &= \sqrt{\det g_{\mu\nu}} T^{i0} = S\epsilon^{ij}\partial_j \delta(X)\delta(Y) \\ T^{ij} &= 0 \end{aligned} \tag{3.B.9}$$

In the spatially conformal coordinate system, the metric that solves the field equation leads to a space-time interval, which is non-trivial in time as well as space.

$$(ds)^2 = (dt + GSd\Theta)^2 - \frac{1}{R^{2GM}}\left((dR)^2 + R^2(d\Theta)^2\right) \tag{3.B.10}$$

Once again, by a change of variables one may pass to a locally flat space-time, where the presence of a massive, spinning source is encoded in a non-trivial identification of coordinate variables. Defining new spatial variables as in (3.B.5) and also a new time variable τ by

$$\tau = t + GS\Theta = t + \frac{GS}{1 - GM}\theta \tag{3.B.11}$$

we see that (3.B.10) becomes flat,

$$(ds)^2 = (d\tau)^2 - (dr)^2 - r^2(d\theta)^2 \qquad (3.B.12)$$

but the required identification is

$$(\tau, r, \theta) \approx (\tau + 2\pi GS, r, \theta + 2\pi(1 - GM)) \qquad (3.B.13)$$

Time is helical, space is conical and there are closed time-like contours.

Note that specifying a solution is equivalent to specifying an element of the 2+1-dimensional Poincaré group that effects the identification (3.B.13).

The static one-body solution can be generalized to describe N particles located at $\mathbf{R}_i$, with masses M_i and spins S_i, $i = 1, \ldots, N$.[18] One finds in spatially conformal coordinates

$$(ds)^2 = \left(dt + G \sum_{i=1}^{N} S_i \frac{(\mathbf{R} - \mathbf{R}_i)}{|\mathbf{R} - \mathbf{R}_i|^2} \times d\mathbf{R} \right)^2 - \prod_{i=1}^{N} \frac{1}{|\mathbf{R} - \mathbf{R}_i|^{2GM_i}} (d\mathbf{R})^2 \qquad (3.B.14)$$

The passage to locally flat coordinates is effected by first defining a new time τ.

$$d\tau = dt + G \sum_{i=1}^{N} S_i \frac{(\mathbf{R} - \mathbf{R}_i)}{|\mathbf{R} - \mathbf{R}_i|^2} \times d\mathbf{R} \qquad (3.B.15)$$

This hides the spins in complicated identifications on τ. To flatten the spatial interval, it is useful to express it in complex variables $Z = X + iY$, *etc.*

$$(d\ell)^2 = \left(\prod_{i=1}^{N} \frac{1}{(Z - Z_i)^{GM_i}} \right) dZ \left(\prod_{i=1}^{N} \frac{1}{(\bar{Z} - \bar{Z}_i)^{GM_I}} \right) d\bar{Z} \qquad (3.B.16)$$

Thus the definition

$$dz = \left(\prod_{i=1}^{N} \frac{1}{(Z - Z_i)^{GM_i}} \right) dZ \qquad (3.B.17)$$

gives the flat spatial interval

$$(d\ell)^2 = dz\, d\bar{z} \qquad (3.B.18)$$

but complicated identifications on the complex plane, which generalize (3.B.8b), reveal the presence of N particles with masses M_i. Unlike in the one-body problem, we cannot express z as a closed form function of Z, but for most purposes the integral expression suffices,

$$z = \int^{Z} dZ' \prod_{i=1}^{N} \frac{1}{(Z' - Z_i)^{GM_i}} \qquad (3.B.19)$$

and it can be explicitly evaluated in special cases.

It is easy to show that the above solution also satisfies self-consistently the geodesic equation.[3b] Thus a static N-body configuration exists and is stable in three-dimensional space-time, in contrast to higher dimensions where gravitational attraction would prevent this. This demonstrates vividly the absence of Newtonian attraction in our theory.

With point-particle sources, the two-dimensional space is flat, but curvature is concentrated on a lower-dimensional sub-space: the zero-dimensional collection of points where the particles are located. One may next consider flat space with curvature concentrated on one-dimensional lines; *i.e.* string sources in the plane, which presumably correspond to domain walls in four-dimensional space-time, just as points on the plane correspond to strings in four-dimensional space-time.

When considering strings, it is natural to allow for tension along the string; otherwise the source is an uninteresting pulvarization of the point-particle — a "dust" string.

In the spinless case the results are simple and startling.[3f,19] There are no open strings, only closed ones. A circular source at $r = a$ is described by an energy-momentum tensor whose non-vanishing components are

$$\sqrt{\det g_{\mu\nu}}T^0_0 = \mu\delta(r-a) \qquad (3.B.20a)$$

$$\sqrt{\det g_{\mu\nu}}T^\theta_\theta = \tau\delta(r-a) \qquad (3.B.20b)$$

Here μ and τ are mass and stress density/per unit length; the total mass is $M = 2\pi a\mu$; for a relativistic string $\tau = \mu$. The momentum density and the other stress components vanish. With this source the space-time interval in conformally flat spatial coordinates is

$$(ds)^2 = \begin{cases} \left(1 - 2\pi G a\tau \ln\frac{r}{a}\right)(dt)^2 - \left(\frac{a}{r}\right)^2\left((dr)^2 + r^2(d\theta)^2\right) & r \geq a \\ (dt)^2 - (dr)^2 - r^2(d\theta)^2 & r \leq a \end{cases} \qquad (3.B.21)$$

The exterior spatial interval also reads $\left(d\,a\ln\frac{r}{a}\right)^2 + (ad\theta)^2$, which is a half-cylinder of radius a extending from infinity to $r = a$, where it is capped by the flat disk of the $r \leq a$ region. Moreover, the total mass $M = 2\pi a\mu$ is given by G^{-1}, so that

$$GM = 1 \qquad (3.B.22)$$

We have seen earlier that for point-particles obeying (3.B.22) the space is a cylinder; here (3.B.22) is always obeyed for spinless strings under tension and the space is a capped cylinder.

Although for $\tau > 0$, g_{00} vanishes at a finite distance, this is not a conventional horizon because g_{00} does not change sign, but time does "stand still" there. Clearly there exist solutions with either sign of τ and unrelated to μ. However, for a relativistic string $\tau = \mu > 0$.

For more discussion on these extended objects and inclusion of spin, please consult the research papers.[3f,19]

C. Quantum Dynamics

The simplest non-trivial dynamics arises when we consider the interaction of two point-particles with each other. As in other contexts, it is possible to pass to the center-of-mass frame where the relative coordinate moves in an effective potential that describes the interaction.[3d] The

same problem arises without the center-of-mass reduction, but in the limit when one particle's mass becomes much larger than the other.[3e]

In view of this, it suffices to consider the problem of a test particle [mass m] moving in the field produced by the source particle [mass M] located at the origin.

The classical motion of a spinless test particle is easy to describe: in flat coordinates there is no deviation from straight-line motion. However, when the unconventional identification (3.B.13) is performed, we find a classical scattering angle,

$$\Delta\theta_{\text{classical}} = \pm\pi GM \tag{3.C.1}$$

and a classical time delay,

$$\Delta t_{\text{classical}} = \mp\frac{GS\pi}{1-GM} \tag{3.C.2}$$

where S is the source particle's spin, and the sign depends on which side the source is passed. The classical trajectories are depicted in Fig. 2 [ignore the dotted lines for the moment]. They depend only on the impact parameter, but not on the energy; the scattering angle does not vary with impact parameter, except in its sign.

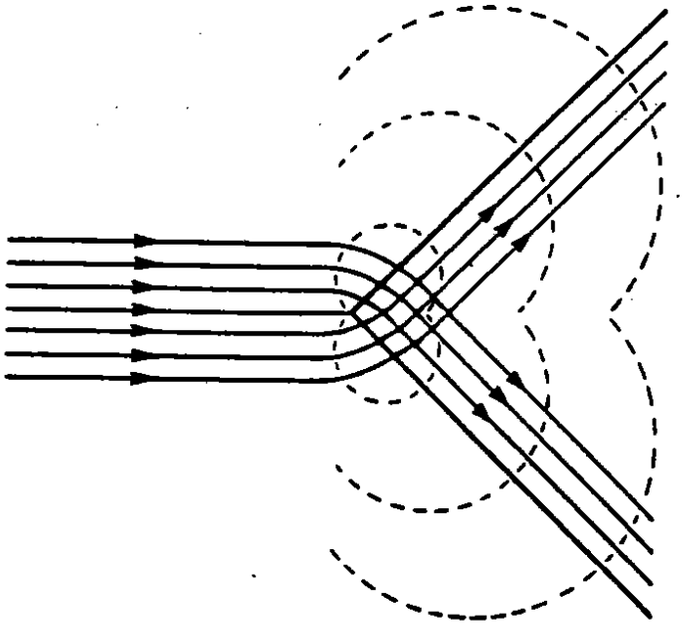


Fig. 2 Qualitatative pictorialization for scattering of waves on an obstacle at the origin. The two sharp lines are classical trajectories with scattering angle $\pm\pi GM$ in (3.C.1), the sign depending on which side the trajectory passes the source. The envelope to the right of the source, formed by heavy diagonal lines, is the sharp geometrical shadow. Broken lines represent diffraction on two sharp edges, even though no edge is actually present — the source [conical defect] produced the "edges."

Next we give a quantum mechanical description and to this end we solve a quantum mechanical equation appropriate to the test particle: Schrödinger or Klein–Gordon for spinless test particles; Dirac for spin 1/2 test particles, *etc.* [We do not second quantize the matter degrees of freedom.] The question that still must be considered is what interaction should we use to describe the influence of the source on the test particle.

The answer that we propose is that no interaction need be considered; rather we solve the free, non-interacting equation but impose on the solution a coordinate condition that reflects the identification (3.B.13).

For example, let us consider the simplest case first — a spinless test particle in a spinless source. The equation we propose to solve is the free [square-root] Klein–Gordon,

$$i\frac{\partial}{\partial t}\psi(t;r,\theta) = \sqrt{-\nabla^2 + m^2}\,\psi(t;r,\theta) \tag{3.C.3}$$

with the requirement that

$$\begin{aligned} \psi(t;r,\theta) &= \psi(t;r,\theta+2\pi\alpha) \\ \alpha &= 1 - GM \end{aligned} \tag{3.C.4}$$

[If non-relativistic motion is of interest, the non-local "square root" operator is replaced by $m - \nabla^2/2m$, which leads to the free Schrödinger equation, with boundary conditions (3.C.4). The mathematical analysis is identical.]

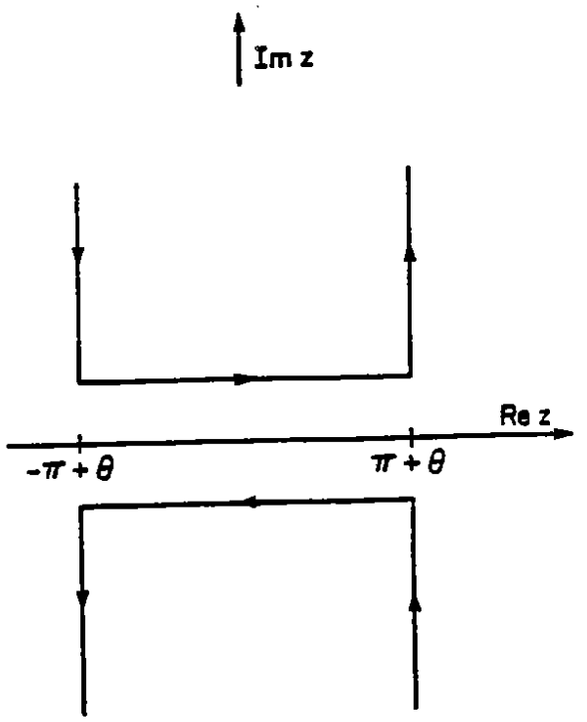
The solution of (3.C.3), which satisfies (3.C.4) is constructed along the same lines as the Aharonov–Bohm scattering solution discussed in detail earlier. I shall not repeat that presentation, beyond remarking that time, radial and angular variables are separated in the usual way, with partial waves carrying angular momentum, ℓ, which is not integer quantized, rather $\alpha\ell$ is an integer. This of course is a consequence of the fact that the angular range is $2\pi\alpha$, not 2π.

The scattering solution is given by a contour integral in which plane waves are superposed, with definite weight[3d,e]

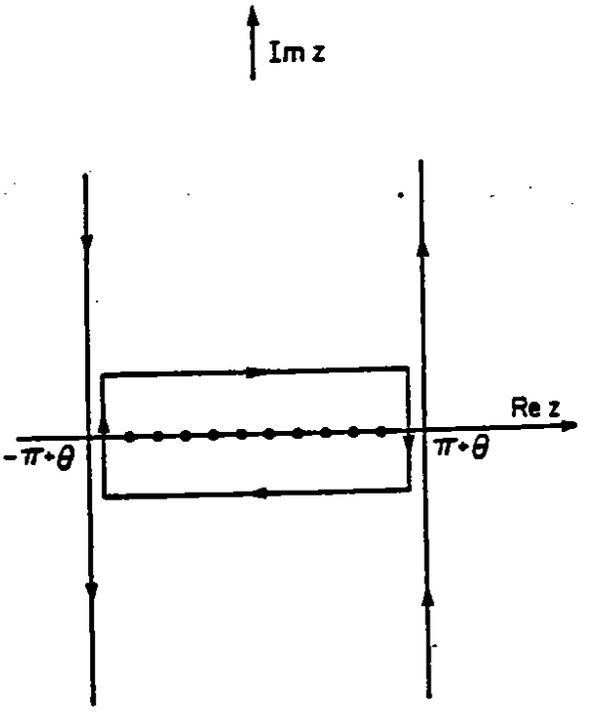
$$\begin{aligned} \psi(t;r,\theta) &= e^{-iEt}\oint \frac{dz}{2\pi} e^{i\mathbf{k}(z)\cdot\mathbf{r}} \frac{1}{1-e^{iz/\alpha}} \\ &\equiv e^{-iEt}\psi(r,\theta) \end{aligned} \tag{3.C.5}$$

Here $E = \sqrt{k^2+m^2}$ and $\mathbf{k}$ is the vector of magnitude k, rotated by the contour integration variable z: $\mathbf{k} = (k\cos z,\ k\sin z)$. That (3.C.5) satisfies (3.C.3) is obvious, that also the boundary condition (3.C.4) is obeyed depends on the specific weight function in (3.C.5) and also on the contour, which is depicted in Fig. 3a.

The weight function has poles on the real axis at $z = z_n = 2\pi n\alpha$ and the contour C avoids them. However, the contour may be deformed as in the discussion of the Aharonov–Bohm problem. We can consider the equivalent, three segment contour $\tilde{C}$, depicted in Fig. 3b, where the poles are encircled and also there are integrals along the vertical lines. The contribution from the encircled poles is evaluated by Cauchy's theorem; it gives the incoming wave. The remaining integrals along the vertical lines give the scattered wave, but the integrations cannot



(a)



(b)

Fig. 3: (a) Integration contour C for the representation of $\psi(r,\theta)$ in (3.C.5). (b) Integration contour $\tilde{C}$ for the representation of $\psi(r,\theta)$ equivalent to that in (a) but giving rise to the decomposition $\psi = \psi^{\text{in}} + \psi^{\text{sc}}$. The incoming wave ψ^{in} is given by the [negative] Cauchy contour around the poles at $z = 2\pi n\alpha$, indicated by heavy dots. The integrals along the left and right verticle contours determine the scattered wave ψ^{sc}, whose large distance asymptote defines the scattering amplitude f.

be evaluated, so no closed form is available. Nevertheless, the large r asymptote is accessible, and the scattering amplitude is determined explicitly.[3d,e]

$$\psi(r,\theta) = \psi^{\text{in}}(r,\theta) + \psi^{\text{sc}}(r,\theta) \tag{3.C.6}$$

$$\psi^{\text{in}}(r,\theta) = \alpha \sum_n{}' e^{i\mathbf{k}(z_n)\cdot\mathbf{r}} \tag{3.C.7a}$$

$$\psi^{\text{sc}}(r,\theta) = i\int_{-\infty}^{\infty} \frac{dy}{2\pi} e^{ikr\cosh y} \left[\frac{1}{1-e^{i\frac{\pi}{\alpha}} e^{-\frac{1}{\alpha}(y+i\theta)}} - \frac{1}{1-e^{-i\frac{\pi}{\alpha}} e^{-\frac{1}{\alpha}(y+i\theta)}} \right]$$

$$\underset{r\to\infty}{\longrightarrow} \sqrt{\frac{i}{r}} f(\theta) e^{ikr} \tag{3.C.7b}$$

$$f(\theta) = \frac{1}{2\sqrt{2\pi k}} \left[\left(\operatorname{ctn}\frac{\theta-\pi}{2\alpha} - i \right) - \left(\operatorname{ctn}\frac{\theta+\pi}{2\alpha} - i \right) \right] \tag{3.C.8}$$

The prime on the sum in (3.C.7a) indicates that z_n must lie in the interval $[-\pi+\theta, \pi+\theta]$. Note that the incoming wave is not a plane wave, rather it is a superposition of variously rotated plane waves. This is analogous to the modulated plane wave found in the Aharonov–Bohm analysis.

We observe that the scattering amplitude $f(\theta)$ in (3.C.8) is real and vanishes when $1/\alpha$ is an integer. Also there are singularities at finite values of θ, where either of the two cotangents blows up. Finally, the optical theorem, which in two dimensions and with our normalization reads

$$\operatorname{Im} f(0) = \sqrt{\frac{k}{4\pi}} \int d\theta\, |f(\theta)|^2 \tag{3.C.9}$$

fails because the left-hand side vanishes and the right-hand side diverges. Nevertheless, there is no loss of unitarity: one can verify from the exact solution (3.C.6) – (3.C.7) that the probability current is conserved. The peculiarities of the scattering amplitude are presumably related to the long-range nature of the "interaction": no matter how far the scattered particle is from the source, it remains on a cone. An interesting problem that here remains is the study of how a wave packet evolves in time.

Going beyond the simplest case, we consider the situation that arises when both the source and the test particle are spinning. The source spin S is arbitrary; for the test particle we consider spins 0 and 1/2, solving the Klein–Gordon and Dirac equations, respectively, but now with the more elaborate identification (3.B.13). One may again give a contour integral representation for the wave function, obtain the incoming wave by performing a Cauchy contour integral, and deduce an explicit formula for the scattering amplitude. The result is an elegant generalization of (3.C.8), which can be presented in universal form, provided the following definitions are made.

$$\begin{aligned}
&S^s = \text{spin of source [can be arbitrary, previously called } S\text{]}.\\
&S^t = \text{spin of test particle [actual calculations done only for } S^t = 0, 1/2\text{]}.\\
&E^s = \text{energy of source [taken to be } M\text{]}.\\
&E^t = \text{energy of test particle } \left(E^t = \sqrt{k^2+m^2}\right) \quad .
\end{aligned} \tag{3.C.10}$$

The scattering amplitude is[3d,e]

$$f(\theta) = \frac{e^{-i[\omega]\theta/\alpha}}{2\sqrt{2\pi k}}\left[e^{-i\{\omega\}\pi/\alpha}\left(\mathrm{ctn}\frac{\theta-\pi}{2\alpha} - i\right) - e^{i\{\omega\}\pi/\alpha}\left(\mathrm{ctn}\frac{\theta+\pi}{2\alpha} - i\right)\right] \qquad (3.C.11)$$

Here ω is the symmetric cross product

$$\omega = E^s S^t + E^t S^w \qquad (3.C.12)$$

while the square and curly brackets denote integer and fractional part, respectively.

$$\omega = [\omega] + \{\omega\} \qquad (3.C.13)$$

For the spinless test particle, $S^t = 0$, one can determine from the phase shift $\delta(E)$ the time-delay by Wigner's formula. Agreement with the classical result (3.C.2) is found.

$$\Delta T = 2\frac{\partial\delta(E)}{\partial E} \qquad (3.C.14)$$

We may understand the scattering amplitude as arising from diffraction effects [like in physical optics] which supplement the classical trajectories [whose analogy is geometrical optics]. These diffraction patterns are indicated by the dotted arcs in Fig. 2 and the two terms in (3.C.8) correspond to the two branches. We observe that scattering consists of a rotation through the angle $\pm\pi GM$, and we recall that in the presence of spin a rotation is accompanied by a phase change in the wave function. This explains the emergence of the additional phases in (3.C.11) as compared to (3.C.8).

The analysis of the Dirac equation is especially interesting owing to the fact that the Dirac Hamiltonian ceases to be self-adjoint on a conical, time-helical space time.[3e] [The same malady afflicts the Dirac equation in the presence of a vortex — the spinning Aharonov–Bohm effect.[20]] Of course the derivatives are formally Hermitian, but consideration of the boundary conditions indicates that a self-adjoint extension, depending on parameters, must be made and different physical results emerge with different values for the parameters. [In deriving Eq. (3.C.11) a definite choice is made to insure universality — but other choices are possible.]

In physical terms what is seen here is the failure of the point-particle description. Extended, smooth objects — described *e.g.* by fields — would lead to a self-adjoint Hamiltonian and in the point-particle limit various parameters, characterizing the extended object, survive as boundary terms on the particle surface and provide the missing information. The situation is similar to what is found for the Dirac equation with a [Dirac] point magnetic monopole. The Hamiltonian needs a one-parameter self-adjoint extension.[21] When a smooth 't Hooft–Polyakov monopole is considered, the parameter is identified as the QCD vacuum angle.[22] For the gravitational [and vortex] problems it remains an open question what model for the extended particle gives a physical origin to the mathematically necessary self-adjoint extension parameters.

The loss of self-adjointness appears to be related to the closed time-like curves that are present in a background metric arising from a spinning source.

We conclude this discussion of quantum motion by remarking that the true two-body problem — in contrast to its test particle source-particle equivalent description — is solved on a space with deficit angle given by the eigenvalues of the two-body Hamiltonian.[3d] This truly "Machian" behavior raises conceptual puzzles — for example it is impossible to superpose or compare energy eigensolutions. Moreover, the three- or more-body problem has thus far not been resolved [apart from a very easy special case[3e]] owing in part to difficulties in describing the multi-conical space on which the physical motion takes place.

D. Topological Elaborations

Up to now the discussion has been based on the three-dimensional version of the Einstein equation (3.A.2). However, in complete analogy to three-dimensional gauge theories, it is possible to modify (3.A.2) by an additional term, because in three dimensions there exists another second rank tensor that is symmetric and covariantly conserved. Sometimes called the *Cotton tensor*, its form is

$$C^{\mu\nu} = \frac{1}{2\sqrt{\det g_{\mu\nu}}}\epsilon^{\mu\alpha\beta}D_\alpha R^\nu_\beta + \mu \leftrightarrow \nu \qquad (3.D.1)$$

Symmetry is manifest, covariant conservation follows from the Bianchi identities. $C^{\mu\nu}$ is traceless as follows from (3.D.1). also with the help of Bianchi identities.

$$C^\mu_\mu = 0 \qquad (3.D.2)$$

Moreover, $C^{\mu\nu}$ may be viewed as the three-dimensional conformal tensor — an odd-parity analog of the Weyl tensor, the latter vanishing identically at $d = 3$. [That is why the Riemann tensor is determined by the Einstein tensor.] $C^{\mu\nu}$ is invariant against conformal redefinition of the metric tensor $g^{\mu\nu}(x) \to \lambda(x)g^{\mu\nu}(x)$ and vanishes if and only if space-time is conformally flat, $g_{\mu\nu}(x) = \lambda(x)\eta_{\mu\nu}$. We may supplement/replace the left-hand of (3.A.2) by the addition of a multiple of $C^{\mu\nu}$.[3a]

$$G^{\mu\nu} + \frac{1}{\kappa}C^{\mu\nu} = 0 \qquad (3.D.3a)$$

$$G^{\mu\nu} + \frac{1}{\kappa}C^{\mu\nu} = 2\pi G T^{\mu\nu} \qquad (3.D.3b)$$

[Also a cosmological constant can of course be added to the equation with or without sources, (3.D.3a) or (3.D.3b) respectively — we shall not do so.]

From its definition (3.D.1), we see that $C^{\mu\nu}$ is of one derivative order higher than $G^{\mu\nu}$, hence the dimension of κ is mass. Analysis of the linearized approximation yields dramatic results. While in the absence of the modification, there are no gravitational excitations, the addition "liberates" a previously "confined" graviton, which now becomes a single propagating mode; moreover, it is massive, while retaining general covariance. The spin is ± 2, the sign being correlated with the sign of κ. [The triple derivative nature of the differential equations (3.D.3) does not give rise to acausality; here, the conformal invariance comes into play, removing possibly dangerous terms from $C^{\mu\nu}$.]

$G^{\mu\nu}$ is obtained variationally from the Einstein–Hilbert action. Similarly, $C^{\mu\nu}$ may be obtained variationally from the Chern–Simons action, for the local Lorentz group in 2+1 dimensions — $SO(2,1)$. Constructing that quantity as in a gauge theory from the connection — either Christoffel or spin — but viewing the connection as a function of the fundamental dynamical variable — either the metric tensor or the *dreibein* respectively — and varying the dynamical variable gives $C^{\mu\nu}$.[3a]

Thus we see that the proposed modification is the complete analog of the situation in the gauge theory, and for that reason the model (3.D.3) is called *topologically massive gravity*.

However, no quantization condition need be imposed on κ.[23] Non-trivial homotopies in a non-compact group like $SO(2,1)$ coincide with those of its maximal compact subgroup, here

$SO(2)$; but $SO(2)$ is trivial in this respect, so the gravitational Chern–Simons action is invariant, just as the field equations are covariant, and κ is unrestricted.

It is not known whether topologically massive gravity is renormalizable.

Of course a theory based solely on the Chern–Simons action/Cotton tensor field equation may also be considered.[3a] Here again, there are no propagating degrees of freedom, and due to the tracelessness of $C^{\mu\nu}$, only massless sources, with trace-free energy-momentum tensor can be coupled. However, owing to its triple derivative structure, the topological term is *not* natural for a low energy description, in contrast to the gauge theoretic Chern–Simons term. On the contrary. The Einstein/Hilbert theory is dominant at low energies, while the Chern–Simons/Cotton term dominates at high energy.

I conclude this discussion of topological elaborations on planar gravity by the following observations.

(a) Just like the gauge theoretic Chern–Simons term, the gravitational $SO(2,1)$ Chern–Simons term is induced by virtual fermions.[24] This raises a puzzle about our treatment of quantum scattering, when the matter degrees of freedom are *second* quantized fermions and the "bare" gravitational action is just the conventional Einstein–Hilbert action. On the one hand the bare gravitational action suggests that there are no propagating gravitational degrees of freedom. On the other, fermion loops induce a Chern–Simons action which when considered together with the bare action indicates the presence of massive, propagating gravitons. So which viewpoint is correct? Is the emergent "graviton" a fermion/anti-fermion bound state? How should perturbative calculations be organized?

(b) The fact that in planar Einstein gravity, the gravitational field is locally determined by matter sources is analogous to the situation in gauge theoretic Chern–Simons theory. Indeed the analogy exposes an identity: the Einstein–Hilbert action is also the Chern–Simons term for $ISO(2,1)$, the inhomogeneous (2+1)-dimensional Lorentz group, *i.e.* the Poincaré group.[25] There are six generators: J^μ rotations and P^μ translations. With these we associate respectively the "gauge" connections ω^μ and e^μ — the spin connection and *dreibein* — and use an off-diagonal "trace," $\langle P^\mu P^\nu\rangle = 0$, $\langle J^\mu J^\nu\rangle = 0$, $\langle J^\mu P^\mu\rangle = \delta^{\mu\nu}$, to construct the Chern–Simons term. The result is the Einstein–Hilbert action in first-order form.

(c) The Lagrangian for topologically massive gravity consists of $\mathcal{L}_{\rm EH} + \frac{1}{\kappa}\mathcal{L}_{\rm CS}$, the Einstein–Hilbert Lagrangian summed with κ^{-1} times the Chern–Simons term. Equivalently we may write it as $\mathcal{L}_{\rm CS} + \kappa\mathcal{L}_{\rm EH}$, and view the higher derivative $\mathcal{L}_{\rm CS}$ as the "kinetic" term and $\kappa\mathcal{L}_{\rm EH}$ as the "mass" term. The former possesses more symmetry than the latter — it is conformally invariant. In some sense that is "too much" symmetry, and no propagation is possible with just the kinetic term. Inclusion of the less symmetric mass [Einstein–Hilbert] term lowers the symmetry and "liberates" the previously confined graviton. One may even promote κ to a scalar field φ with its own [unspecified] dynamics. The combination $\mathcal{L}_{\rm CS} + \varphi\mathcal{L}_{\rm EH} + \mathcal{L}_\varphi$ can be conformally invariant for suitably chosen $\mathcal{L}_\varphi$. Then an expansion about $\langle\varphi\rangle = 0$ contains no propagating gravitons, while the symmetry breaking starting point $\langle\varphi\rangle = \kappa$ liberates the graviton.[26].

(d) Some classical solutions to topologically massive gravity have been found. They are planar analogs of Gödel universes.[27]

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