DEUTSCHES ELEKTRONEN – SYNCHROTRON

DESY 92-191
UTS-DFT-92-33
December 1992

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ISSN 0418-9833

NOTKESTRASSE 85 · D-2000 HAMBURG 52

ISSN 0418-9833

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ABSTRACT

In this paper we extend the standard Moyal formalism to the tangent and cotangent bundle of the phase-space of any hamiltonian mechanical system. In this manner we build the quantum analog of the classical hamiltonian vector-field of time evolution and its associated Lie-derivative. We also use this extended Moyal formalism to develop a quantum analog of the Cartan calculus on symplectic manifolds.

In the last few years there has been a flourishing of studies in the field of quantum groups,[1] [2] and in the associated non-commutative differential structures.[3] [4] [5]

Inspired by these works, we decided to study the differential structures of a typical non-commutative theory, Quantum Mechanics (QM)* . Some old work in this direction already existed[6] and also some more modern ones.[7] [8] Our approach to the study of differential structures in QM is based on the Moyal technique of quantization.[9] [10] This is a technique based, rougly speaking, on c-numbers and not on operators. These c-numbers are known as "symbols" of the associated operators.[11] The commutators of operators are then replaced by what are called the Moyal brackets of the associated symbols which in the classical limit ($\hbar \to 0$) go into the usual Poisson brackets.

The Moyal formalism had been ignored during the fifties and part of the sixties,[12] but it got revived in the seventies[13] by mathematicians who realized that the Moyal brackets, and the associated algebra of observables, belonged to the mathematical topos of *deformation* of algebraic structures. More recently the Moyal formalism has been further studied[14] [15] and applied also to the study of infinite dimensional algebras.[16] [17]

The Moyal formalism is developed in phase-space and it is based on the early intuition of Wigner[18] that it is possible to develop a theory of quantum systems in phase-space which is formally analogous to classical mechanics.

In this approach any Weyl-ordered operator[19] $\hat{\mathcal{O}}(\hat{p},\hat{q})$ is related to a phase-function $\mathcal{O}(p,q)$, referred to as its "symbol"[11], in the following way

$$\hat{\mathcal{O}}(\hat{p},\hat{q}) = \left(\frac{1}{2\pi\hbar}\right)^{2n} \int d^n\alpha\, d^n\beta\, d^np\, d^nq\, \mathcal{O}(p,q)\, e^{-\frac{i}{\hbar}(\alpha(p-\hat{p})+\beta(q-\hat{q}))} \qquad (1)$$

The Map $\mathcal{O} \to \hat{\mathcal{O}}$ is invertible, *i.e.*, there is a one-to-one correspondence between

* Note that "quantum groups" are not necessarily associated with quantum mechanics, but with *deformations* of classical groups whose deformation parameter has nothing to do with Planck's constant $\hbar$ in general.

operators and functions on phase-space. Similarly any state vector $|\psi\rangle$ is represented by the Wigner function $\varrho_{qm}(p,q)$, which is defined as the symbol of the projection operator $|\psi\rangle\langle\psi|$. The function $\varrho_{qm}(p,q)$ is a "pseudo-probability" distribution, *i.e.*, in the limit $\hbar \to 0$ it approaches the classical probability density $\varrho_{cl}(p,q)$, but it is not positive definite in general. Under the isomorphism relating operators to symbols, the operator $\widehat{\mathcal{O}}_3 = \widehat{\mathcal{O}}_1\widehat{\mathcal{O}}_2$ is mapped on the *"star product"* $\mathcal{O}_3(p,q) = \big(\mathcal{O}_1 \star \mathcal{O}_2\big)(p,q)$ of the symbols $\mathcal{O}_1(p,q)$ and $\mathcal{O}_2(p,q)$. The *star-product* is given by

$$\begin{aligned}\big(\mathcal{O}_1 \star \mathcal{O}_2\big)(p,q) &\equiv exp\Big[i\frac{\hbar}{2}\Big(\frac{\partial}{\partial q_1^i}\frac{\partial}{\partial p_2^i} - \frac{\partial}{\partial p_1^i}\frac{\partial}{\partial q_2^i}\Big)\Big]\mathcal{O}_1(p_1,q_1)\mathcal{O}_2(p_2,q_2)|_{\substack{p_1=p_2=p\\ q_1=q_2=q}} \\ &\equiv exp\Big[i\frac{\hbar}{2}\omega^{ab}\,\overset{1}{\partial}_a\overset{2}{\partial}_b\Big]\mathcal{O}_1(\phi_1)\mathcal{O}_2(\phi_2)|_{\phi_1=\phi_2=\phi}\end{aligned} \tag{2}$$

Here $\phi^a = (p^i, q^i)$, $i = 1\cdots n$, $a = 1\cdots 2n$ are the 2n coordinates of phase-space, ω^{ab} is the symplectic matrix[20] and $\overset{1,2}{\partial}_a = \frac{\partial}{\partial \phi^a_{1,2}}$. (We use the same notation as in ref.[21].) The symbol of the commutator $\big[\widehat{\mathcal{O}}_1, \widehat{\mathcal{O}}_2\big] \equiv i\hbar\widehat{\mathcal{O}}_3$ is given by the Moyal bracket[9] (mb) of the symbols $\mathcal{O}_1(\phi)$ and $\mathcal{O}_2(\phi)$:

$$\Big\{\mathcal{O}_1(\phi), \mathcal{O}_2(\phi)\Big\}_{mb} = \frac{1}{i\hbar}\Big(\mathcal{O}_1(\phi)\star\mathcal{O}_2(\phi) - \mathcal{O}_2(\phi)\star\mathcal{O}_1(\phi)\Big) \tag{3}$$

This bracket is a deformation of the Poisson bracket (pb) $\big\{\cdot,\cdot\big\}_{pb}$ to which it reduces in the limit of $\hbar \to 0$. Because the $\star - product$ is associative[13], $\big\{\cdot,\cdot\big\}_{mb}$ obeys the Jacobi identity and defines a derivation with respect to the $\star - product$, *i.e.*,

$$\Big\{\mathcal{O}_1, \mathcal{O}_2\star\mathcal{O}_3\Big\}_{mb} = \Big\{\mathcal{O}_1, \mathcal{O}_2\Big\}_{mb}\star\mathcal{O}_3 + \mathcal{O}_2\star\Big\{\mathcal{O}_1, \mathcal{O}_3\Big\}_{mb} \tag{4}$$

The time evolution of the density matrix $\widehat{\varrho}$, at the level of symbols, is governed by the equation

$$-\dot{\varrho} = \Big\{\varrho, H\Big\}_{mb} = \frac{2}{\hbar}sin\Big[\frac{\hbar}{2}\omega^{ab}\,\overset{1}{\partial}_a\overset{2}{\partial}_b\Big]\varrho(\phi_1)H(\phi_2)|_{\phi_{1,2}=\phi} \tag{5}$$

where $H(\phi)$ represents the Hamiltonian $\widehat{H}$ and $\varrho(\phi)$ the density operator $\widehat{\varrho}$.

In the classical limit eq.(5) reduces to

$$-\dot{\varrho} = \Big\{\varrho, H\Big\}_{pb} = h^a\partial_a\varrho = l_h\varrho$$

which can be interpreted as the action of the Lie-derivative l_h along the hamiltonian vector field, $h^a \equiv \omega^{ab}\partial_b H(\phi)$, acting on the scalar $\varrho(\phi)$. Therefore we may regard the RHS of eq. (5) as the quantum deformation of the Lie-derivative l_h acting on scalars. It is the purpose of the present paper to extend the notion of a quantum deformed Lie-derivative to higher p-forms on phase-space.

After this brief introduction to the formalism, let us now explain which is our strategy for developing a differential calculus in QM. In ref.[21] we developed the standard Cartan calculus for *classical hamiltonian mechanics* (CM) by introducing some new variables besides the phase-space coordinates ϕ^a. These variables were λ_a, c^a, and $\bar{c}_a$. Their meaning was the following: c^a was a basis in the cotangent space to phase-space, $\bar{c}_a$ was a basis in the tangent space to phase-space, and the same for λ_a, with the only difference that λ_a was a basis to express symmetric tensors, while c_a was a basis to express anti-symmetric tensors. All the standard exterior calculus (exterior derivative, exterior co-derivative, relation forms-vector fields via the symplectic 2-form, interior products, hamiltonian vector fields, Lie-derivatives) could then be reproduced via some (BRS-like)charges and graded commutators in the 4x(2n)-dim.-space $(\phi^a, \lambda_a, c^a, \bar{c}_a)$. Another options was to introduce a symplectic structure in this generalized phase-space and have all the standard exterior calculus done not by the commutators but by some extended Poisson brackets (epb). The only nonvanishing extended Poisson brackets were

$$\Big\{\phi^a, \lambda_b\Big\}_{epb} = \delta^a_b \quad , \quad \Big\{\bar{c}_b, c^a\Big\}_{epb} = -i\delta^a_b \tag{6}$$

These brackets are graded Poisson brackets because the ghosts c^a and $\bar{c}_a$ are anticommuting variables. Note that all components of ϕ^a have vanishing extended brackets with one another, $\big\{\phi^a, \phi^b\big\}_{epb} = 0$, even though $\big\{\phi^a, \phi^b\big\}_{pb} = \omega^{ab}$. In a way the epb-formalism considers all ϕ's as "position variables" and the λ's as conjugate "momenta". Because we may think of λ_a as a derivative with respect to ϕ^a, the natural mathematical arena of the epb-formalism is the tangent bundle over phase-space. In ref.[21] we have translated the rules of the classical exterior calculus on phase-space (known as Cartan calculus) to the epb-formalism on the extended phase-space coordinatized by the 4x(2n) variables $(\phi^a, \lambda_a, c^a, \bar{c}_a)$. These rules were embedded in a map, called the hat map $\hat{}$, which associates scalar functions on the extended phase-space to antisymmetric tensor fields on the ordinary one. In particular, the $\hat{}$-map replaces the basis elements $d\phi^{a_1} \wedge d\phi^{a_2} \wedge \cdots \wedge d\phi^{a_p}$ of p-forms by strings of ghosts: $c^{a_1}c^{a_2}\cdots c^{a_p}$. As an example for the epb-rules we only mention the exterior derivative of p-forms $F^{(p)} = \frac{1}{p!}F_{a_1\cdots a_p}d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p}$:

$$\left(dF^{(p)}\right)^{\widehat{}} = i\left\{Q, \widehat{F}^{(p)}\right\}_{epb}, \quad Q \equiv ic^a\lambda_a$$

Here Q can be interpreted as a BRS-operator[21] . Some more rules are listed in the Table at the end of the paper, where we have omitted the $\hat{}$-symbol for simplicity. The Lie-derivative along the hamiltonian vector field h^a is obtained by taking the extended Poisson bracket with $i\widetilde{\mathcal{H}}$ where

$$\widetilde{\mathcal{H}} = \lambda_a h^a + i\bar{c}_a\partial_b h^a c^b \tag{7}$$

is the Hamiltonian on the extended phase-space. It has the property that

$$\left\{H(\phi), \varrho(\phi)\right\}_{pb} = \left\{\widetilde{\mathcal{H}}, \varrho(\phi)\right\}_{epb} \tag{8}$$

i.e., $\widetilde{\mathcal{H}}$ with $\left\{\cdot,\cdot\right\}_{epb}$ gives rise to the same time evolution of $\varrho(t,\phi)$ as the ordinary $H(\phi)$ with $\left\{\cdot,\cdot\right\}_{pb}$. Moreover, defining $\frac{d\mathcal{O}}{dt} = \left\{\mathcal{O}, \widetilde{\mathcal{H}}\right\}_{epb}$ for observables $\mathcal{O} = \mathcal{O}(\phi, \lambda, c, \bar{c})$ which depend also on the new variables $(\lambda, c, \bar{c})$, we can give a dynamics also to that sector of the epb-formalism which is not present at the ordinary pb-level. In particular, setting $\mathcal{O} \equiv \widehat{F}^{(p)} = \frac{1}{p!}F^{(p)}_{a_1\cdots a_p}(\phi)c^{a_1}\cdots c^{a_p}$, we obtain the time evolution of p-forms. Working out the corresponding bracket one finds indeed that the coefficients are time evolved by the standard Lie-derivative:

$$\begin{aligned} -\partial_t F^{(p)}_{a_1\cdots a_p} &= l_h F^{(p)}_{a_1\cdots a_p} \\ &\equiv h^b\partial_b F^{(p)}_{a_1\cdots a_p} + \sum_{j=1}^{p}\partial_{a_j}h^b F_{a_1\cdots a_{j-1}ba_{j+1}\cdots a_p} \end{aligned} \tag{9}$$

We will not dwell on this formalism further but recommend to read ref.[21] as a background for the rest of this paper.

We would like now to do the same in QM by developing an analogous extension of phase-space and of the Moyal brackets . That means we would like to have the analog of forms (whose basis is given in CM by c^a), the analog of vectors and tensor-fields (whose basis is given in CM by $\bar{c}_a, \lambda_a$), the analog of the hamiltonian vector fields (given in CM by h^a) and the analog of the Lie-derivative l_h.

Our *quantum*-Cartan calculus will be developed by extending the ϕ-space equipped with the Moyal brackets, following the same strategy as in CM[21] . Let us go over to (ϕ, λ)-space and let us define and *extended* $\star_e$-product as follows:

$$\begin{aligned} A(\lambda,\phi)\star_e B(\lambda,\phi) &\equiv exp\Big[\frac{i}{2}\Big(\frac{\partial}{\partial\phi_1^a}\frac{\partial}{\partial\lambda_{2a}} - \frac{\partial}{\partial\lambda_{1a}}\frac{\partial}{\partial\phi_2^a}\Big)\Big]A(\lambda_1,\phi_1)B(\lambda_2,\phi_2)\Big|_{\substack{\phi_1=\phi_2=\phi \\ \lambda_1=\lambda_2=\lambda}} \\ &= A(\lambda,\phi)B(\lambda,\phi) + \frac{i}{2}\Big(\frac{\partial A}{\partial\phi^a}\frac{\partial B}{\partial\lambda_a} - \frac{\partial A}{\partial\lambda_a}\frac{\partial B}{\partial\phi^a}\Big) + \cdots \end{aligned} \tag{10}$$

Analogously we define the extended Moyal brackets (emb) as

$$\left\{A(\lambda,\phi), B(\lambda,\phi)\right\}_{emb} \equiv \frac{1}{i}\left(A\star_e B - B\star_e A\right) \tag{11}$$

In exactly the same way as $\left\{\cdot,\cdot\right\}_{mb}$ is a deformation[11] of $\left\{\cdot,\cdot\right\}_{pb}$, the new bracket $\left\{\cdot,\cdot\right\}_{emb}$ is a deformation of $\left\{\cdot,\cdot\right\}_{epb}$. For a reason which will become clear in a moment we have put $\hbar = 1$ for the deformation parameter here. Our basic postulate is that the standard evolution of zero-forms $\varrho(\phi)$ is the same under the normal Moyal brackets (5), and under the extended ones. That means we would like to find a $\widetilde{\mathcal{H}}_b^{\hbar}$ on (ϕ, λ)-space such* that under the extended Moyal brackets it gives the same evolution as the one given by H under the normal Moyal brackets:

$$\left\{H(\phi), \varrho(\phi)\right\}_{mb} = \left\{\widetilde{\mathcal{H}}_b^{\hbar}(\lambda, \phi), \varrho(\phi)\right\}_{emb} \tag{12}$$

This relation is analogous to eq.(8) for CM. It is easy to convince oneself that eq. (12) has no solution in the form it stands[22]. It has a solution only if we impose the weaker requirement

$$\left\{H(\phi), \varrho(\phi)\right\}_{mb} = \left\{\widetilde{\mathcal{H}}_b^{\hbar}(\lambda, \phi), \varrho(\phi)\right\}_{emb|_{\lambda=0}} \tag{13}$$

Its general solution is:

$$\widetilde{\mathcal{H}}_b^{\hbar}(\lambda, \phi) = \frac{1}{\hbar}\sinh\left[\hbar\lambda_a\omega^{ab}\partial_b\right]H(\phi) + \frac{1}{\hbar}G(\lambda, \phi, \omega) \tag{14}$$

where $G(\lambda, \phi, \omega)$ is any arbitrary even function in λ which has to vanish in the classical limit. Among all the possible choices of G, we pick up three which give a particularly simple form of $\widetilde{\mathcal{H}}_b^{\hbar}$. They are the following:

$$\begin{aligned}
\widetilde{\mathcal{H}}_{b-}^{\hbar} &= \frac{1}{\hbar}\left[e^{\hbar\lambda_a\omega^{ab}\partial_b} - 1\right]H(\phi) = \frac{1}{\hbar}\left[H(\phi^a - \hbar\omega^{ab}\lambda_b) - H(\phi)\right] \\
\widetilde{\mathcal{H}}_{b+}^{\hbar} &= \frac{1}{\hbar}\left[1 - e^{-\hbar\lambda_a\omega^{ab}\partial_b}\right]H(\phi) = -\frac{1}{\hbar}\left[H(\phi^a + \hbar\omega^{ab}\lambda_b) - H(\phi)\right] \\
\widetilde{\mathcal{H}}_{b0}^{\hbar} &= \frac{1}{2\hbar}\left[e^{\hbar\lambda_a\omega^{ab}\partial_b} - e^{-\hbar\lambda_a\omega^{ab}\partial_b}\right]H(\phi) = \\
&= \frac{1}{2\hbar}\left[H(\phi^a - \hbar\omega^{ab}\lambda_b) - H(\phi^a + \hbar\omega^{ab}\lambda_b)\right]
\end{aligned} \tag{15}$$

It is possible to show[22] that all three of them satisfy eq. (13). They have the correct classical limit: $\lim_{\hbar\to 0}\widetilde{\mathcal{H}}_b^{\hbar} = \lambda_a h^a \equiv \widetilde{\mathcal{H}}_b$. In the limit $\hbar \to 0$ the bracket

* The index "$\hbar$" is for "*quantum*". Here we are restricting ourselves to zero-forms, so the solutions will not contain the ghosts c and $\bar{c}$, that means in the classical case $\widetilde{\mathcal{H}}$ shall just reduce to $h^a\lambda_a$. In the quantum case we write $\widetilde{\mathcal{H}}_b^{\hbar}$, the "$b$" staying for "bosonic part". Later on we will consider the general solution which gives also the evolution of higher forms.

(11) with (10) remains unaltered; however, as the classical $\widetilde{\mathcal{H}}_b$ is linear in λ_a, the higher derivative terms in (10) become ineffective in the classical limit. (We also could have adopted the opposite point of view: setting $\hbar = 1$ in the Hamiltonian and keeping $\hbar \neq 1$ in the bracket leads to the same equation of motion after setting $\lambda = 0$.)

One might wonder whether it is consistent to relax the condition (12) in the form (8). It is not clear a priori that the "constraint hypersurface" defined by $\lambda_a = 0$ is invariant under time-evolution. It can be checked however, that indeed $\left\{\lambda_a, \widetilde{\mathcal{H}}_b^{\hbar}\right\}_{emb} = 0$ for $\lambda_a = 0$.

The form of the Hamiltonian (15) is very suggestive because it seems to indicate that, in this quantum calculus, there is an intrinsic "lattice structure" with a length scale (or "lattice spacing") set by the value of $\hbar$. In a sense, the quantum Hamiltonians $\widetilde{\mathcal{H}}_b^{\hbar}$ in (15) are obtained from their classical ancestor $H(\phi)$ by applying a kind of backward, forward or symmetric lattice derivative, respectively. This rather intriguing aspect will be further explored in ref.[22] .*

The next step we have to do is to find the form of $\widetilde{\mathcal{H}}^{\hbar}$ which acts also on higher forms. We have dropped the subindex "b" because this $\widetilde{\mathcal{H}}^{\hbar}$ will contain, in analogy with the classical Lie-derivative, a part containing anticommuting variables. Let us indicate this part as $\widetilde{\mathcal{H}}_f^{\hbar}$ where "f" is for "fermionic" or anticommuting. So the over-all $\widetilde{\mathcal{H}}^{\hbar}$ is the sum of these two pieces:

$$\widetilde{\mathcal{H}}^{\hbar} = \widetilde{\mathcal{H}}_b^{\hbar} + \widetilde{\mathcal{H}}_f^{\hbar} \tag{16}$$

To build the fermionic part we have to develop the extended Moyal brackets of the Grassmannian space $\bar{c}_a, c^a$. The extended Poisson brackets were defined[21] as

* That quantum deformations introduce an intrinsic "length" scale is also among the hopes of authors who try to find physical applications of quantum groups, especially in connection with quantum gravity[23] .

$$\{A,B\}_{epb} = A\Big[\frac{\overleftarrow{\partial}}{\partial\phi^a}\frac{\overrightarrow{\partial}}{\partial\lambda_a} - \frac{\overleftarrow{\partial}}{\partial\lambda_a}\frac{\overrightarrow{\partial}}{\partial\phi^a} - i\Big(\frac{\overleftarrow{\partial}}{\partial\bar{c}_a}\frac{\overrightarrow{\partial}}{\partial c^a} + \frac{\overleftarrow{\partial}}{\partial c^a}\frac{\overrightarrow{\partial}}{\partial\bar{c}_a}\Big)\Big]B \tag{17}$$

where A and B are functions of $(\lambda,\phi,c,\bar{c})$. The deformation of (17) proceeds via the star-product for graded phase space, which has been studied by Berezin[11], for instance. It reads

$$A \star_e B = A\ exp\Big[\frac{i}{2}\Big(\frac{\overleftarrow{\partial}}{\partial\phi^a}\frac{\overrightarrow{\partial}}{\partial\lambda_a} - \frac{\overleftarrow{\partial}}{\partial\lambda_a}\frac{\overrightarrow{\partial}}{\partial\phi^a}\Big) + \frac{\overleftarrow{\partial}}{\partial c^a}\frac{\overrightarrow{\partial}}{\partial\bar{c}_a}\Big]B \tag{18}$$

and gives rise to the following graded generalization of the extended Moyal bracket:

$$\{A,B\}_{emb} = \frac{1}{i}\big[A \star_e B - (-)^{[A][B]}B \star_e A\big] \tag{19}$$

Here $[A]$ denotes the grading[11] of A. The bracket (19) has the same algebraic properties as the graded commutator. It satisfies the graded Jacobi identities and identities like

$$\{A \star_e B, C\}_{emb} = A \star_e \{B,C\}_{emb} + (-1)^{[B][C]}\{A,C\}_{emb} \star_e B \tag{20}$$

Let us now turn to the task of finding $\widetilde{\mathcal{H}}_f^\hbar$; it will define a candidate for a quantum Lie-derivative. As there is no uniquely prescribed way of how to proceed, we shall follow an approach which is inspired by our treatment of classical symplectic geometry[21] and which seems particularly natural in our framework. We shall require that:

i) *the fermionic Hamiltonian $\widetilde{\mathcal{H}}_f^\hbar$ is quadratic in the ghosts*

ii) *the complete Hamiltonian $\widetilde{\mathcal{H}}_b^\hbar + \widetilde{\mathcal{H}}_f^\hbar$ is invariant under the same BRS-transformation as its classical counterpart.*

The classical BRS-transformations found in ref.[21] read

$$\delta\phi^a = \epsilon c^a, \quad \delta\bar{c}_a = i\epsilon\lambda_a, \quad \delta c^a = 0 = \delta\lambda_a \tag{21}$$

They are generated by $Q = i\lambda_a c^a$ which satisfies $\{Q,\widetilde{\mathcal{H}}\}_{epb} = 0$, and which was interpreted as the classical exterior derivative. The conditions i) and ii) fix $\widetilde{\mathcal{H}}_f^\hbar$

uniquely once the bosonic part of the Hamiltonian is specified. Starting from a $\widetilde{\mathcal{H}}_b^\hbar$ of the general form $\widetilde{\mathcal{H}}_b^\hbar = \mathcal{F}(\lambda_a\omega^{ab}\partial_b)H(\phi)$, where $\mathcal{F}$ is any of the functions appearing in (15) and requesting $\widetilde{\mathcal{H}}_f^\hbar$ to be of the form $\widetilde{\mathcal{H}}_f^\hbar = i\bar{c}_a W_b^a(\phi,\lambda)c^b$ (with W_b^a a function to be determined), the requirement of BRS invariance of $\widetilde{\mathcal{H}}^\hbar$ gives the following result[22] : $W_b^a = \omega^{ac}\partial_b\partial_c\frac{\mathcal{F}(\lambda\omega\partial)}{(\lambda\omega\partial)}H(\phi)$. So the complete *quantum*-Lie-derivative $\widetilde{\mathcal{H}}^\hbar$ reads

$$\widetilde{\mathcal{H}}^\hbar = \mathcal{F}(\lambda\omega\partial)H(\phi) + i\bar{c}_a\omega^{ac}\partial_b\partial_c\frac{\mathcal{F}(\lambda\omega\partial)}{(\lambda\omega\partial)}H(\phi)c^b \tag{22}$$

This *quantum*-Lie-derivative is invariant also under an anti-BRS[21] charge found in CM which is nothing else than an exterior co-derivative. The full ISp(2) algebra of observables found for CM[21] can also be checked[22] to be an algebra of invariances of (22).

Let us now choose for $\mathcal{F}(\lambda\omega\partial)$ in (22) the one appearing in the third expression of (15), that is $\mathcal{F}(x) = \frac{\sinh(\hbar x)}{\hbar}$. Then the *quantum*-Lie-derivative $\widetilde{\mathcal{H}}^\hbar$ can be written as

$$\widetilde{\mathcal{H}}^\hbar = \lambda_a h_\hbar^a + i\bar{c}_a\partial_b h_\hbar^a c^b \tag{23}$$

where

$$h_\hbar^a(\lambda,\phi) \equiv \frac{\sinh(\hbar\lambda\omega\partial)}{(\hbar\lambda\omega\partial)}h^a(\phi) \tag{24}$$

is the *quantum*-Hamiltonian vector field. Note the formal resemblance between the classical Lie-derivative and the quantum one (23).

Let us now see which is the time-evolution of an antisymmetric (p,q)-tensor, defined as

$$T = T_{a_1\cdots a_p}^{b_1\cdots b_q}(\phi)\bar{c}_{b_1}\bar{c}_{b_2}\cdots\bar{c}_{b_q}c^{a_1}c^{a_2}\cdots c^{a_p}$$

and where, in analogy with the classical case[21] , we define the action of the *quantum*-Lie-derivative, denoted by $L_\hbar$ on T as follows:

$$L_\hbar\, T \equiv -\left\{\widetilde{\mathcal{H}}^\hbar, T\right\}_{emb_{|\lambda=0}}$$

After some lengthy calculations[22] we get, in components,

$$\begin{aligned} L_\hbar T^{b_1\cdots b_q}_{a_1\cdots a_p}(\phi) &= \left\{T^{b_1\cdots b_q}_{a_1\cdots a_p}(\phi), H\right\}_{mb} - \\ &- \sum_{j=1}^{q} \frac{\sinh[\frac{i\hbar}{2}\overset{1}{\partial}\,\omega\,\overset{2}{\partial}]}{[\frac{i\hbar}{2}\overset{1}{\partial}\,\omega\,\overset{2}{\partial}]} T^{b_1\cdots b_{j-1}eb_{j+1}\cdots b_q}_{a_1\cdots a_p}(\phi_1)\,\overset{2}{\partial}_e\, h^{b_j}(\phi_2)|_{\phi_{1,2}=\phi} + \\ &+ \sum_{j=1}^{p} \frac{\sinh[\frac{i\hbar}{2}\overset{1}{\partial}\,\omega\,\overset{2}{\partial}]}{[\frac{i\hbar}{2}\overset{1}{\partial}\,\omega\,\overset{2}{\partial}]} T^{b_1\cdots b_q}_{a_1\cdots a_{j-1}ea_{j+1}\cdots a_p}(\phi_1)\,\overset{2}{\partial}_{a_j}\, h^{e}(\phi_2)|_{\phi_{1,2}=\phi} \end{aligned} \tag{25}$$

Applying the formula above to one forms $\varrho = \varrho_a(\phi)c^a$ we get:

$$L_\hbar \varrho_a = \left\{\varrho_a, H\right\}_{mb} + \frac{\sinh[\frac{i\hbar}{2}\overset{1}{\partial}\,\omega\,\overset{2}{\partial}]}{[\frac{i\hbar}{2}\overset{1}{\partial}\,\omega\,\overset{2}{\partial}]}\varrho_b(\phi_1)\,\overset{2}{\partial}_a\, h^b(\phi_2)|_{\phi_{1,2}=\phi} \tag{26}$$

In the classical tensor calculus the derivative $\partial_a\varrho(\phi)$ of a zero-form $\varrho(\phi)$ transforms as a one form. Let us check if this is true also in the deformed case. We start from the scalar eq. (5) for $-\dot{\varrho} = L_\hbar\varrho$ and act with ∂_a on both sides of this equation. In this manner we obtain an equation for the Lie-derivative of $\varrho_a \equiv \partial_a\varrho$ which coincides exactly with eq. (26). We conclude that for the choice $\mathcal{F}(x) = \frac{\sinh(\hbar x)}{\hbar}$, *i.e.*, for the third Hamiltonian $\widetilde{\mathcal{H}}^\hbar_{b0}$ of eq.(15), $\partial_a\varrho$ transforms as a one-form. We also observe that for all other choices of the function G in eq.(14), including the ones leading to $\widetilde{\mathcal{H}}^\hbar_{b+}$ and $\widetilde{\mathcal{H}}^\hbar_{b-}$, $\partial_a\varrho$ would *not* transform as a one form. This is because the sinh in (26) would be replaced by a different function. We conclude that the conditions i) and ii) above, together with the requirement that $\partial_a\varrho c^a$ should be a one-form, fixes the quantum Lie-derivative (25) uniquely. Using the same strategy as for the Lie-derivative also the other tensor manipulations can be "quantum deformed". To do this, we may

take over the table from CM almost literally: again we replace the basis vectors ∂_a and $d\phi^a$ by the corresponding ghosts, and implement the exterior derivative, the inner product and Lie-derivative by taking appropriate brackets. However, the classical extended Poisson brackets are now replaced by the extended Moyal brackets. Apart from this substitution only one entry of the table at the end of the paper changes, namely the emb-representation of the Hamiltonian vector field:

$$\begin{aligned} h_\hbar(\lambda,\phi) &= \frac{\sinh(\hbar\lambda\omega\partial)}{(\hbar\lambda\omega\partial)} i h^a(\phi)\lambda_a \\ &= i h^a_\hbar(\lambda,\phi)\lambda_a \end{aligned} \tag{27}$$

Due to its dependence on λ, $h^a_\hbar$ is not a vector field on phase-space, but rather a (horizontal) vector field on the cotangent bundle.

Let us note that the extended Moyal brackets $\{\cdot,\cdot\}_{emb}$ is a derivation[22] with respect to the $\star_e$-product. It then follows that all the the operations $\Omega \equiv d, i_h, L_h$ are (graded)-derivations on the space of functions $T(\phi,\lambda,c,\bar{c})$ equipped with the $\star_e$-product because there exists a generalized Leibniz rule for the operations Ω:

$$\left\{\Omega, T_1 \star_e T_2\right\}_{emb} = \left\{\Omega, T_1\right\} \star_e T_2 \pm T_1 \star_e \left\{\Omega, T_2\right\}_{emb}$$

Let us also notice that, if we define as a *quantum*-tensor field an object T which transforms according to $L_h T = -\left\{\widetilde{\mathcal{H}}, T\right\}_{emb}$ under the diffeomorphism induced by some hamiltonian vector field h, then $T_1 \star_e T_2$ is a tensor if $T_{1,2}$ are tensors, but $T_1 \cdot T_2$ is not in general[22] .

To summarize: in this letter we considered quantum phase space as a typical non-commutative manifold. We have presented the first elements of a tensor calculus on this space by starting from the geometrical structures, in particular the extended Poisson bracket found in ref.[21] and by "deforming" these structures using generalized star-products and Moyal brackets. Let us remember that the *standard* Moyal bracket[9] gives rise to a deformation of the inifinite dimensional

Lie-algebra of functions over the ordinary phase-space, while the *extended* Moyal bracket considered here plays an analogous role on its tangent bundle.

Clearly much more work has to be done in order to understand all aspects of this formalism. In ref.[22] we shall present further details on the formal properties of the "*quantum* Cartan calculus", in particular on the question of its symplectic covariance. We will also dwell more on the hidden lattice structure of the extended phase-space and its inherent length scale.

Table

○	Cartan's rules	$\{\cdot,\cdot\}_{epb}$-rules
vector-fields	$v = v^a\partial_a$	$v = v^a\bar{c}_a$
1-forms	$\alpha = \alpha_a d\phi^a$	$\alpha = \alpha_a c^a$
p-forms	$F^{(p)} = \frac{1}{p!}F_{a_1\cdots a_p}d\phi^{a_1}\wedge\cdots\wedge d\phi^{a_p}$	$F^{(p)} = \frac{1}{p!}F_{a_1\cdots a_p}c^{a_1}\cdots c^{a_p}$
p-vectors	$V^{(p)} = \frac{1}{p!}V^{a_1\cdots a_p}\partial_{a_1}\wedge\cdots\wedge\partial_{a_p}$	$V^{(p)} = \frac{1}{p!}V^{a_1\cdots a_p}\bar{c}_{a_1}\cdots\bar{c}_{a_p}$
ext. deriv.	$dF^{(p)}$	$i\left\{Q, F^{(p)}\right\}_{epb}$
int. product	$i_v F^{(p)}$	$i\left\{v, F^{(p)}\right\}_{epb}$
Ham.vec.field	$h = \omega^{ab}\partial_b H\partial_a$	$h = i\omega^{ab}\partial_b H\lambda_a$
Lie-deriv.	$l_h = di_h + i_h d$	$i\widetilde{\mathcal{H}} = i\lambda_a h^a - \bar{c}_a\partial_b h^a c^b$
Lie-deriv. on $T=F^{(p)},V^{(p)}$	$l_h T$	$-\{\widetilde{\mathcal{H}}, T\}_{epb}$

ACKNOWLEDGEMENTS

M.R. acknowledges the hospitality of the Department of Theoretical Physics of the University of Trieste. The authors wish to thank W.D.Thacker for many helpful discussions over the more than two years this research was in progress. This work has been supported in part by grants from INFN and NATO.

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