DEUTSCHES ELEKTRONEN – SYNCHROTRON

DESY 93-009
January 1993

Global Observables in Local Quantum Physics

K. Fredenhagen

II. Institut für Theoretische Physik, Universität Hamburg

ISSN 0418-9833

NOTKESTRASSE 85 · D-2000 HAMBURG 52

DESY 93-009
January 1993

ISSN 0418-9833

Global Observables in Local Quantum Physics*

dedicated to Huzihiro Araki on the occasion of his 60th birthday

Klaus Fredenhagen
II. Institut für Theoretische Physik
Universität Hamburg

Abstract

An algebraic construction of global observables which characterize superselection sectors of chiral conformal field theories is described.

Keywords: Quantum field theory, operator algebras, superselection sectors, conformal field theory

Local quantum physics [1], better known under its traditional name algebraic quantum field theory, was developed by Araki, Haag, Kastler and others [2-7] in the beginning sixties as a conceptual framework in which the principles of special relativity are incorporated into quantum physics, thereby formalizing the essential physical content of quantum field theory. The basic feature of this approach is the emphasis on local observables. Only local observables are considered to be fundamental, and the physical interpretation is based solely on the knowledge of the system of von Neumann algebras $\mathcal{A} = (\mathcal{A}(\mathcal{O}))_{\mathcal{O}\in\mathcal{K}}$ where $\mathcal{K}$ is the set of open double cones in Minkowski space $\mathcal{M}$ and $\mathcal{A}(\mathcal{O})$ is the algebra of observables measurable within $\mathcal{O}$. $\mathcal{A}$ is supposed to satisfy the requirements of locality and isotony. The global algebra $\mathcal{A}(\mathcal{M})$ is defined as the C*-inductive limit

$$\mathcal{A}(\mathcal{M}) = \overline{\bigcup_{\mathcal{O}\in\mathcal{K}} \mathcal{A}(\mathcal{O})}, \tag{1}$$

and algebras of arbitrary regions $\mathcal{S}$ in Minkowski space are defined as the smallest C*-algebras $\mathcal{A}(\mathcal{S}) \subset \mathcal{A}(\mathcal{M})$ such that $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mathcal{S})$ for all $\mathcal{O} \in \mathcal{K}$ with $\mathcal{O} \subset \mathcal{S}$.

Global observables, such as energy, charge, univalence etc. are not elements of $\mathcal{A}(\mathcal{M})$. They only arise as weak limit points of local observables in certain representations of $\mathcal{A}(\mathcal{M})$. It was therefore very important to develop a representation theory for $\mathcal{A}(\mathcal{M})$, a project which was carried through in several steps in [7-17], see [1] and [18] for further references and more details. A typical feature is that $\mathcal{A}(\mathcal{M})$ is simple, so all representations are faithful, and the state space of one single representation is weakly dense in the whole state space [19] (all sectors are "physically equivalent" in the sense of [7]).

In the DHR theory [10] one starts from a distinguished representation π_0 (interpreted as the vacuum representation) which satisfies Haag duality

$$\pi_0(\mathcal{A}(\mathcal{O})) = \pi_0(\mathcal{A}(\mathcal{O}'))' \quad , \mathcal{O} \in \mathcal{K}. \tag{2}$$

*to appear in the proceedings of the Oji conference on quantum analysis, Kyoto 1992

That this property is satisfied for the vacuum representation of the free scalar field was verified by Araki [4,6]. It is now known to hold in more general cases, but there are also exceptions, typically, but not always associated with spontaneous symmetry breakdown [20,22,23]. One then selects representations π which are local excitations of π_0 in the sense that π and π_0 are unitarily equivalent at infinity, i.e.

$$\pi|_{\mathcal{A}(\mathcal{O}')} \simeq \pi_0|_{\mathcal{A}(\mathcal{O}')} \quad , \mathcal{O} \in \mathcal{K} \tag{3}$$

(DHR criterion). These representations violate Haag duality in general, and the degree of violation can be measured in terms of a number $d(\pi) \geq 1$, called the statistical dimension, whose square was shown by [14] to coincide with the Jones index [21] of the inclusion

$$\pi(\mathcal{A}(\mathcal{O})) \subset \pi(\mathcal{A}(\mathcal{O}'))' \tag{4}$$

which is actually independent of $\mathcal{O}$.

In $d \geq 3$ space time dimensions the structure of all DHR representations with finite statistical dimension was completely clarified by Doplicher and Roberts [12]. There is a uniquely determined compact group G, a $\gamma \in G$ with $\gamma^2 = 1$ which changes the sign of Fermi fields, a net of von Neumann algebras $(\mathcal{F}(\mathcal{O}))_{\mathcal{O}\in\mathcal{K}}$, an action $g \mapsto \alpha_g$ of G by automorphisms of $\mathcal{F}(\mathcal{M})$ such that each local algebra $\mathcal{F}(\mathcal{O})$ is stable under the action of G and that $\mathcal{A}(\mathcal{O})$ is the set of G-invariant elements of $\mathcal{F}(\mathcal{O})$, and $\mathcal{F}(\mathcal{O})$ satisfies graded locality with respect to the grading induced by γ. Moreover, there is a faithful irreducible representation $\hat{\pi}_0$ of $\mathcal{F}(\mathcal{O})$ such that

$$\hat{\pi}_0|_{\mathcal{A}(\mathcal{M})} \simeq \bigoplus_{\xi\in\hat{G}} d(\xi)\pi_\xi \tag{5}$$

with a bijective mapping $\xi \mapsto [\pi_\xi]$ of equivalence classes ξ of irreducible representations of G (with dimension $d(\xi)$) to equivalence classes of irreducible representations of $\mathcal{A}(\mathcal{M})$ satisfying the DHR criterion and having finite statistical dimension $d(\pi_\xi) = d(\xi)$. Furthermore, there is a strongly continuous unitary representation U of G in $\mathcal{H}_{\hat{\pi}_0}$ which implements the automorphisms α_g and whose restriction to $\mathcal{H}_{\pi_\xi}$ is equivalent to a multiple of ξ.

At the very end, one finds global charge operators in the weak closure $\hat{\pi}_0(\mathcal{A})''$ as conjugacy class means,

$$Q([g]) = \int_G \mathrm{d}h U(hgh^{-1}) \quad , g \in G, \tag{6}$$

and the character table of G arises from

$$\chi_\xi(g) 1_{\mathcal{H}_{\pi_\xi}} = Q([g])|_{\mathcal{H}_{\pi_\xi}} \tag{7}$$

In $d = 2$ space time dimensions (as well as for nonlocalizable charges [11] in $d = 3$) the statistical dimensions can be nonintegers, so a simple correspondence to group representations is no longer possible. But the smallest possible values have the form of "quantum dimensions" $[d]_q = (q^d - q^{-d})/(q - q^{-1})$ for $q = e^{i\pi n}$, $n \in \mathbb{Z}$ where d is the dimension of a representation of the quantum group $\mathrm{sl}(2)_q$, so quantum groups are the natural candidates for a generalized internal symmetry. Actually, the situation is more complicated as may be seen from the analysis of the chiral Ising model [24].

A detailed analysis of DHR sectors in 2 dimensions [13,14] led to the following results. Associated with each sector there is a unitary representation of the braid group characterizing the statistics of particles, and there are Markov traces given in terms of the left inverses of the DHR theory inducing link and ribbon invariants in the usual way and also invariants of 3-manifolds, both of the Reshetikin-Turaev [25] and of the Turaev-Viro [26] type [13, II].

Examples for such a structure are chiral conformal field theories in 2 dimensions where the structure of sectors ("primary fields") fits perfectly into the DHR scheme. There is, however, one important gap in the identification of sectors of conformal field theory with DHR sectors. Namely, as Buchholz and Schulz-Mirbach [23] showed, Haag duality is violated in typical cases (Virasoro algebras with central charge $c > 1$).

Let me briefly describe what a chiral theory is from the algebraic point of view. Let $(\mathcal{A}(\mathcal{O}))_{\mathcal{O}\in\mathcal{K}}$ be a translation covariant Haag-Kastler net on 2-dimensional Minkowski space. 2-dimensional Minkowski space may be written in terms of light cone coordinates $u = t - x$ and $v = t + x$. A chiral theory is characterized by the triviality of translations in one lightlike (say v-) direction. A double cone is a product of two open bounded intervals, and in a chiral theory the algebra of a double cone $I \times J$, where I and J are open bounded intervals on the real axis, does not change under shifts of J on the v-axis. Therefore one may introduce a net of algebras indexed by open bounded intervals on $\mathbb{R}$ by

$$\mathcal{A}(I) = \bigvee_J \mathcal{A}(I \times J) \quad . \tag{8}$$

Locality of the original net implies that algebras of disjoint intervals commute.

We now consider the case that not only the translations but the whole real Moebius group $\mathrm{M} = \mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2$ acts covariantly on this net,

$$\alpha_g(\mathcal{A}(I)) = \mathcal{A}(gI) \tag{9}$$

provided I and gI are bounded intervals. Let (π_0, U) be a vacuum representation of $(\mathcal{A}(\mathbb{R}), \alpha)$, i.e. a representation π_0 of $\mathcal{A}(\mathbb{R})$ in a Hilbert space $\mathcal{H}_0$ together with a positive energy representation U of M in $\mathcal{H}_0$ which implements α,

$$\mathrm{Ad}U(g) \circ \pi_0 = \pi_0 \circ \alpha_g \quad , g \in \mathrm{M} \tag{10}$$

and has a one dimensional subspace $\mathbb{C}\Omega$ of invariant vectors. It is known that under quite general assumptions the vacuum representation satisfies essential Haag duality,

$$\pi_0(\mathcal{A}(\mathbb{R}_+))' = \pi_0(\mathcal{A}(\mathbb{R}_-))'' \tag{11}$$

(see [27,23,28-31]). One may then extend the net of local algebras to $\mathrm{S}^1 = \mathbb{R} \cup \{\infty\}$ by defining

$$\pi_0(\mathcal{A}(I)) = U(g)\pi_0(\mathcal{A}(g^{-1}I))U(g)^{-1} \quad , g \in \mathrm{M} \tag{12}$$

where I is an open interval on S^1 with noncoinciding boundary points and $g^{-1}I$ is a bounded interval on $\mathbb{R} = \mathrm{S}^1 \setminus \{\infty\}$. By Moebius covariance, Haag duality on S^1 follows from essential duality on $\mathbb{R}$,

$$\pi_0(\mathcal{A}(I))' = \pi_0(\mathcal{A}(I')) \tag{13}$$

where I' is the interior of the complement of I in S^1. Haag duality on $\mathbb{R}$, however, might not hold since the complement of a bounded interval in $\mathbb{R}$ is a union of two disjoint intervals $I_>$ and $I_<$ having ∞ as a joint boundary point, and the algebra $\pi_0(\mathcal{A}(I_>)) \vee \pi_0(\mathcal{A}(I_<))$ might be smaller than $\pi_0(\mathcal{A}(I_> \cup \{\infty\} \cup I_<))$. Actually, this is the case for the algebra generated by

the energy momentum tensor with central charge $c > 1$ [23], and a similar phenomenon was observed much earlier in a different context by Hislop and Longo (failure of timelike duality for a massless scalar field in 4 dimensions [32]).

A positive energy representation may be defined as a family (π^I) of representations of $\mathcal{A}(I)$ in a Hilbert space $\mathcal{H}_\pi$, $I \in \mathcal{K}$ where $\mathcal{K}$ is now the set of open intervals on S^1 with noncoinciding boundary points, with consistent restrictions

$$\pi^I|_{\mathcal{A}(J)} = \pi^J \quad \text{for} \quad I \supset J \tag{14}$$

together with a positive energy representation U of the covering group of the Moebius group such that

$$\mathrm{Ad}U(g) \circ \pi^I = \pi^{gI} \circ \alpha_g. \tag{15}$$

By a result of Buchholz, Mack and Todorov [33] which relies on a result of Takesaki and Winnink [34] the representations π^I are automatically normal.

Instead of dealing with families of representations of local algebras, it turns out to be useful to deal with representations of a global algebra associated to S^1. $\mathcal{A}(\mathrm{S}^1)$ may be defined as the free C*-algebra generated by all local algebras $\mathcal{A}(I)$ modulo all relations within local algebras. $\mathcal{A}(\mathrm{S}^1)$ can be characterized by the following universality condition: for each family of normal representations (π^I) in some Hilbert space $\mathcal{H}$ which satisfies the consistency condition (14) there is a unique representation π of $\mathcal{A}(\mathrm{S}^1)$ in $\mathcal{H}$ such that $\pi|_{\mathcal{A}(I)} = \pi^I$. (See [35,36] and [13, II].)

These global algebras are not as abstract as one might think. Actually, they have already been used in the algebraic treatment of models of conformal field theory. A simple example I want to mention is the theory of the U(1) current algebra. This model was analyzed in [33]. One considers the *-algebra generated by unitaries $W(f)$ with $f \in C^\infty(\mathrm{S}^1, \mathbb{R})$ which satisfy the Weyl relation

$$W(f)W(g) = \mathrm{e}^{-i\int f'g}W(f+g) \tag{16}$$

where $\int f'g = \int_{\mathrm{S}^1} \mathrm{d}z \frac{\mathrm{d}f}{\mathrm{d}z} g$. The local algebras $\mathcal{A}_0(I)$ are defined as the subalgebras generated by $W(f)$ with $\mathrm{supp} f \subset I$. Using the Weyl relation one easily sees that each consistent family of local representations characterizes a unique representation of $\mathcal{A}_0(\mathrm{S}^1)$. One now may look at the vacuum representation π_0 induced by the vacuum state

$$\omega_0(W(f)) = \mathrm{e}^{-||f||^2} \tag{17}$$

where $||f||^2 = \sum_{n\in\mathbb{N}} n|f_n|^2$, f_n being the nth Fourier coefficient of f. The local algebra is defined as the completion of $\mathcal{A}_0(I)$ with respect to the family of seminorms $||\cdot||_T = |\mathrm{Tr}\pi_0(\cdot)T|$, with trace class operators T in $\mathcal{H}_{\pi_0}$. One then considers the set of all states ω on $\mathcal{A}_0(\mathrm{S}^1)$ such that the induced representation π_ω has restrictions to the local algebras $\mathcal{A}_0(I)$ which extend to normal representations of $\mathcal{A}(I)$ and introduces the "universal locally normal" representation $\hat{\pi} = \bigoplus \pi_\omega$. The map $A \mapsto \hat{\pi}(A)$, $A \in \mathcal{A}_0(I)$, extends to an isomorphism of $\mathcal{A}(I)$ and $\hat{\pi}(\mathcal{A}_0(I))''$. $\mathcal{A}(\mathrm{S}^1)$ can now be defined as the C*-algebra in $\mathcal{B}(\mathcal{H}_{\hat{\pi}})$ generated by $\hat{\pi}(\mathcal{A}_0(I))''$, $I \in \mathcal{K}$.

We finally have to check that $\mathcal{A}(\mathrm{S}^1)$ satisfies the universality condition. So let (π^I) denote a consistent family of local representations of $\mathcal{A}(I)$. The family $(\pi^I|_{\mathcal{A}_0(I)})$ defines a unique representation $\pi^{(0)}$ of $\mathcal{A}_0(\mathrm{S}^1)$. $\pi^{(0)}$ is a direct sum of cyclic representations π_i each of which is equivalent to a subrepresentation of $\hat{\pi}$. Let V_i be an isometric intertwiner from $\mathcal{H}_{\pi_i} \subset \mathcal{H}_{\pi^{(0)}}$ into $\mathcal{H}_{\hat{\pi}}$. Then the representation π of $\mathcal{A}(\mathrm{S}^1)$ is given by

$$\pi(A) = \sum_i V_i^* A V_i \quad , A \in \mathcal{A}(\mathrm{S}^1) \subset \mathcal{B}(\mathcal{H}_{\hat{\pi}}). \tag{18}$$

Let us return to the general discussion. We are interested in representations of the universal algebra $\mathcal{A}(S^1)$ satisfying the DHR criterion which assumes here the form

$$\pi \in (\mathrm{DHR}) \Longleftrightarrow \pi|_{\mathcal{A}(I)} \simeq \pi_0|_{\mathcal{A}(I)} \quad \forall I \in \mathcal{K} \quad . \tag{19}$$

According to [33] positive energy representations with a cyclic vector automatically satisfy this criterion. In analogy to the standard DHR theory, we look for endomorphisms ρ of $\mathcal{A}(S^1)$ with the property $\pi \simeq \pi_0 \circ \rho$. But since π_0 is not necessarily faithful, the construction of ρ is somewhat delicate. It has been described in [35,36] and, in more detail, in [13, II]. One obtains endomorphisms which are localizable in the following sense:

Definition: *An endomorphism ρ of $\mathcal{A}(S^1)$ is called localizable within $I \in \mathcal{K}$ if for all $I_0 \subset I$, $I_0 \in \mathcal{K}$ there is a unitary $U_0 \in \mathcal{A}(I)$ such that*

$$\rho(A) = \mathrm{Ad}U_0(A) \quad , A \in \mathcal{A}(I_0'), \tag{20}$$

$$\mathrm{Ad}U_0^* \circ \rho(\mathcal{A}(I_1)) \subset \mathcal{A}(I_1) \quad for \quad I_1 \supset I_0 \quad , I_1 \in \mathcal{K}. \tag{21}$$

We remark that an endomorphism which is localizable within I in the above sense acts trivially on $\mathcal{A}(I')$ (so it is localized in I in the usual sense) and restricts to an endomorphism of $\mathcal{A}(I_1)$ for $I_1 \supset I$. It has the further property that the endomorphism $\rho_0 = \mathrm{Ad}U_0^* \circ \rho$ used in the definition is localizable within I_0. Namely, let $I_{00} \subset I_0$, $I_{00} \in \mathcal{K}$. Since ρ is localizable within I and $I_{00} \subset I$ there is some unitary $U_{00} \in \mathcal{A}(I)$ satisfying (20) and (21) with I_0 being replaced by I_{00}. Then the unitary $U_0^* U_{00}$ satisfies (20) and (21) with ρ being replaced by ρ_0 and I_0 by I_{00}. It remains to show that $U_0^* U_{00} \in \mathcal{A}(I_0)$. From (20) and the fact that ρ_0 acts trivially on $\mathcal{A}(I_0')$ we conclude that $\pi_0(U_0^* U_{00}) \in \pi_0(\mathcal{A}(I_0'))'$ which coincides with $\pi_0(\mathcal{A}(I_0))$ by Haag duality. But $U_0^* U_{00} \in \mathcal{A}(I)$ and π_0 is faithful on $\mathcal{A}(I) \supset \mathcal{A}(I_0)$, hence $U_0^* U_{00} \in \mathcal{A}(I_0)$.

It is crucial for the whole construction that intertwiners between representations can be lifted to intertwiners between localizable endomorphisms:

Proposition: *Let ρ, ρ' be endomorphisms of $\mathcal{A}(S^1)$ which are localizable within $I \in \mathcal{K}$ (in the sense of the above definition), and let $S \in \mathcal{B}(\mathcal{H}_{\pi_0})$ be an intertwiner from $\pi_0 \circ \rho$ to $\pi_0 \circ \rho'$, i.e.*

$$S\pi_0 \circ \rho(A) = \pi_0 \circ \rho'(A) S \quad , A \in \mathcal{A}(S^1). \tag{22}$$

Then there exists a unique $S_0 \in \mathcal{A}(I)$ such that $\pi_0(S_0) = S$, and S_0 intertwines ρ with ρ', i.e.

$$S_0 \rho(A) = \rho'(A) S_0 \quad , A \in \mathcal{A}(S^1). \tag{23}$$

Proof: ρ and ρ' act trivially on $\mathcal{A}(I')$, hence $S \in \pi(\mathcal{A}(I'))' = \pi_0(\mathcal{A}(I))$, and there is a unique $S_0 \in \mathcal{A}(I)$ with $\pi_0(S_0) = S$. We have to show that S_0 satisfies the intertwining relation (23).

Since $\mathcal{A}(S^1)$ is generated, as a norm closed algebra, by local operators, it is sufficient to check (23) for local algebras $\mathcal{A}(I_1)$. For a given $I_1 \in \mathcal{K}$ there are $I_2, I_0 \in \mathcal{K}$ such that $I_1 \subset I_2$ and $I_0 \subset I_2 \cap I$. We choose unitaries $U_0, U_0' \in \mathcal{A}(I)$ such that (20) and (21) hold for ρ and I_0, and ρ' and I_0, respectively. Let $\rho_0 = \mathrm{Ad}U_0^* \circ \rho$, $\rho_0' = \mathrm{Ad}U_0'^* \circ \rho'$ and $S_1 = U_0'^* S_0 U_0$. Then by (22)

$$\pi_0(S_1 \rho_0(A)) = \pi_0(\rho_0'(A) S_1) \quad , A \in \mathcal{A}(S^1). \tag{24}$$

ρ_0 and ρ_0' act trivially on $\mathcal{A}(I_0')$, so $\pi_0(S_1) \in \pi_0(\mathcal{A}(I_0'))' = \pi_0(\mathcal{A}(I_0))$, and by the faithfulness of π_0 on $\mathcal{A}(I) \supset \mathcal{A}(I_0)$ we get $S_1 \in \mathcal{A}(I_0)$. Now let $A \in \mathcal{A}(I_1)$. ρ_0 and ρ_0' are localizable within I_0 and therefore induce endomorphisms of $\mathcal{A}(I_2)$. Hence (24) is a relation in the vacuum representation of the local algebra $\mathcal{A}(I_2)$ and thus remains valid in the universal algebra $\mathcal{A}(S^1)$, i.e.

$$S_1 \rho_0(A) = \rho_0'(A) S_1 \quad , A \in \mathcal{A}(I_1). \tag{25}$$

But this implies the desired relation (23) on $\mathcal{A}(I_1)$. Since $I_1 \in \mathcal{K}$ was arbitrary the proof is complete. □

A representation $\pi \in$ (DHR) leads to localizable endomorphisms which are in addition transportable in the following sense:

Definition: *An endomorphism ρ of $\mathcal{A}(S^1)$ is called transportable if for each $I \in \mathcal{K}$ there exists some endomorphism ρ' of $\mathcal{A}(S^1)$ which is localizable within I and which is inner equivalent to ρ, i.e. there is some unitary $U \in \mathcal{A}(S^1)$ such that $\rho' = \mathrm{Ad}U \circ \rho$.*

We have a 1-1 correspondence between equivalence classes of DHR representations of $\mathcal{A}(S^1)$ and inner equivalence classes of transportable endomorphisms of $\mathcal{A}(S^1)$. We denote by Δ the set of all transportable endomorphisms and by $\Delta(I)$ the subset of endomorphisms which are localizable within $I \in \mathcal{K}$. Note that transportable endomorphisms which are localizable within some interval I are automatically also localizable in every larger interval. This property might not hold without the transportability assumption.

One now can proceed as in the DHR analysis, using as crucial input the correspondence between intertwiners of representations and local intertwiners of localizable endomorphisms established in the Proposition. There are, however, new features which are connected with the existence of nonlocal intertwiners. The following structure was found in [13, II].

Let $I, I_1 \in \mathcal{K}$ have disjoint closures, and let $J_\pm \in \mathcal{K}$ with $J_\pm \supset I \cup I_1$ and $J_+ \cup J_- = S^1$, where, within J_+, I_1 lies on the right hand side of I. Let $\rho \in \Delta(I)$, $\rho_1 \in \Delta(I_1)$ and $V \in \mathcal{B}(\mathcal{H}_{\pi_0})$ unitary such that $\mathrm{Ad}V \circ \pi_0 \circ \rho = \pi_0 \circ \rho_1$. According to the Proposition there exist unique intertwiners $V_\pm \in \mathcal{A}(J_\pm)$ with $\pi_0(V_\pm) = V$ and $\mathrm{Ad}V_\pm \circ \rho = \rho_1$. The unitary

$$V_I(\rho) = V_+^{-1} V_- \tag{26}$$

is an intertwiner from ρ to ρ with $\pi_0(V_I(\rho)) = 1$. It depends only on ρ and, possibly, also on the choice of $I \in \mathcal{K}$ with $\rho \in \Delta(I)$. The latter dependence is compatible with inclusions, $V_I(\rho) = V_J(\rho)$ if $I \subset J$. $V_I(\rho)$ transforms covariantly under local intertwiners; if $\rho, \sigma \in \Delta(I)$ and $S \in \mathcal{A}(I)$ intertwines ρ with σ we have

$$S V_I(\rho) = V_I(\sigma) S \quad . \tag{27}$$

The global selfintertwiners $V_I(\rho)$ have an interesting relation to the statistics operators of the DHR theory. Let $\rho, \sigma \in \Delta(I)$. Then, with the above notations,

$$\sigma(V_I(\rho)) = (\sigma(V_+^{-1})V_+) \cdot (V_+^{-1}V_-) \cdot (V_-^{-1}\sigma(V_-)) \tag{28}$$

$$= \varepsilon_I(\rho,\sigma) \cdot V_I(\rho) \cdot \varepsilon_I(\sigma,\rho) \tag{29}$$

where $\varepsilon_I(\cdot,\cdot)$ are the statistics operators of the DHR theory. They satisfy $\varepsilon_I = \varepsilon_J$ for $I \subset J$, but they depend, in general, on the choice of the interval I with $\rho, \sigma \in \Delta(I)$. Relation (29) together with the usual braid relations of the statistics operators means that the operators $\rho^{i-1}(V_I(\rho))$, $\rho^{i-1}(\varepsilon_I(\rho,\rho))$, $i \in \mathbb{N}$ generate a unitary representation of the braid group of the cylinder. Here $\rho^{i-1}(V_I(\rho))$ represents a braid where the i-th strand is wrapped once around the cylinder whereas all other strands are straight.

In the vacuum representation, (29) implies

$$\pi_0\sigma(V_I(\rho)) = \pi_0(\varepsilon_I(\rho,\sigma)\varepsilon_I(\sigma,\rho)) \tag{30}$$

where the operator on the right hand side is the so-called monodromy operator which is nontrivial for some pair ρ,σ if and only if the theory has nontrivial braid group statistics. We observe that, in general, π_0 is not faithful and $\mathcal{A}(S^1)$ has nontrivial ideals (cf. [12] for a related observation).

$\mathcal{A}(S^1)$ even has a nontrivial center if it has sectors with finite statistical dimension and braid group statistics. Let $I, I_1, J_\pm$ as before, let $\rho \in \Delta(I)$ and let $\bar{\rho} \in \Delta(I_1)$ be conjugate to ρ in the sense that $\pi_0 \circ \rho\bar{\rho}$ contains a subrepresentation equivalent to π_0. Let R be an isometric intertwiner from π_0 to $\pi_0 \circ \rho\bar{\rho}$, and let $R_\pm$ be its preimages under π_0 in $\mathcal{A}(J_\pm)$, respectively. Then

$$R_+^* R_- \in Z(\mathcal{A}(S^1)) \quad . \tag{31}$$

It turns out that for ρ irreducible (i.e. the representation $\pi_0 \circ \rho$ is irreducible) the central element $C_\rho = d(\rho)R_+^* R_-$ is uniquely determined by the equivalence class of ρ. One finds $C_\rho^* = C_{\bar{\rho}}$, the multiplication rule

$$C_\rho C_\sigma = \sum_{[\tau]} N_{\rho\sigma}^\tau C_\tau \tag{32}$$

where the nonnegative integers $N_{\cdot\cdot}^{\cdot}$ are identical to the fusion coefficients occuring in the reduction of composed representations

$$\pi_0 \circ \rho\sigma = \bigoplus_{[\tau]} N_{\rho\sigma}^\tau \pi_0 \circ \tau \tag{33}$$

and the generalized character table

$$d(\sigma)\pi_0\sigma(C_\rho) = \sum_{[\tau]} N_{\rho\sigma}^\tau d(\tau) \frac{\kappa(\tau)}{\kappa(\rho)\kappa(\sigma)} =: Y_{\rho\sigma} \quad . \tag{34}$$

where $\kappa(\cdot)$ denotes the statistics phase. In case the number of DHR sectors is finite ("rational theories") and the vacuum sector is the only one with trivial monodromies with all other sectors, i.e. for all $\rho \in \Delta(I)$, $\rho \not\simeq \mathrm{id}$ there exists some $\sigma \in \Delta(I)$ such that $\varepsilon_I(\rho,\sigma)\varepsilon_I(\sigma,\rho) \neq 1$, the matrix $(Y_{\rho,\sigma})$ is invertible [37], hence the system of charge operators (C_ρ) distinguishes the sectors. The matrix Y is proportional to Verlinde's [38] matrix S [37,17]. So in these cases the global charge operators are obtained by an algebraic construction from local observables, and also remainders of a generalized group structure occur (32) in analogy to the multiplication structure of conjugacy classes of finite groups. Note that this multiplicative structure is identical to that of the representation classes (33); this selfduality property is in the group case true only for finite abelian groups.

References

[1] Rudolf Haag, *Local Quantum Physics. Fields, Particles, Algebras* Springer-Verlag Berlin Heidelberg 1992

[2] R. Haag: Discussion des "axiomes" et des propriétés asymptotiques d'une théorie des champs locales avec particules composées. In *Les problèmes mathématiques de la théorie quantique des champs*, Lille 1957. Paris: Centre National de la Recherche Scientifique 1959

[3] R. Haag, B. Schroer: Postulates of Quantum Field Theory. J. Math. Phys. **3**, 248 (1962)

[4] H. Araki: A Lattice of von Neumann Algebras Associated with the Quantum Theory of a Free Bose Field. J. Math. Phys. **4**, 1343 (1963)

[5] H.J. Borchers, R. Haag, B. Schroer: The vacuum state in quantum field theory. Nuovo Cim. **24**, 214 (1963)

[6] H. Araki: Von Neumann Algebras of Local Observables for Free Scalar Field. J. Math. Phys. **5**, 1 (1964)

[7] R. Haag, D. Kastler: An Algebraic Approach to Field Theory. J. Math. Phys. **5**, 848 (1964)

[8] H.J. Borchers: Local Rings and the Connection of Spin with Statistics. Comm. Math. Phys. **1**, 281 (1965)

[9] S. Doplicher, R. Haag, J.E. Roberts: Fields, Observables and Gauge Transformations. Comm. Math. Phys. **13**, 1, (1969) and Comm. Math. Phys. **15**, 173 (1969)

[10] S. Doplicher, R. Haag, J.E. Roberts: Local Observables and Particle Statistics. Comm. Math. Phys. **23**, 199 (1971) and **35**, 49 (1974)

[11] D. Buchholz, K. Fredenhagen: Locality and the Structure of Particle States. Comm. Math. Phys. **84**, 1 (1982)

[12] S. Doplicher, J.E. Roberts: Why there is a Field Algebra with a Compact Gauge Group Describing the Superselection Structure in Particle Physics. Comm. Math. Phys. **131**, 51 (1990)

[13] K. Fredenhagen, K.-H. Rehren, B. Schroer: Superselection Sectors with Braid Group Statistics and Exchange Algebras I: General Theory. Comm. Math. Phys. **125**, 201 (1989) and II: Geometric Aspects and Conformal Covariance (to appear in Rev. Math. Phys.)

[14] R. Longo: Index of Subfactors and Statistics of Quantum Fields. Comm. Math. Phys. **126**, 217 (1989) and **130**, 285 (1990)

[15] J. Fröhlich, F. Gabbiani, P.-A. Marchetti: Superselection Structure and Statistics in Three-dimensional Local Quantum Theory. In *Knots, Topology, and Quantum Field Theories*, ed. G. Lusanna, p. 335 (World Scientific, 1989)

[16] J. Fröhlich, P.-A. Marchetti: Spin-Statistics Theorem and Scattering in Planar Quantum Field Theory with Braid Statistics. Nucl. Phys. **B356**, 533 (1991)

[17] J. Fröhlich, F. Gabbiani: Braid Statistics in Local Quantum Theory. Rev. Math. Phys. **2**, 251 (1991).

[18] D. Kastler (ed.): *The Algebraic Theory of Superselection Sectors. Introduction and Recent Results.* World Scientific, 1990

[19] J.M.G. Fell: The dual spaces of C*-algebras. Trans. Amer. Math. Soc. **94**, 365 (1960)

[20] J.E. Roberts: Spontaneously Broken Gauge Symmetries and Superselection Rules. Proceedings of the International School of Mathematical Physics, Camerino 1974, ed. G. Gallavotti, Universita di Camerino 1976

[21] V.F.R. Jones: Index for Subfactors. Invent. Math. **72**, 1 (1983)

[22] D. Buchholz, S. Doplicher, R. Longo, J. E. Roberts: A New Look at Goldstone's Theorem. Rev. Math. Phys. (to appear)

[23] D. Buchholz, H. Schulz-Mirbach: Haag duality in conformal quantum field theory. Rev. Math. Phys. **2**, 105 (1990)

[24] G. Mack, V. Schomerus: Conformal field algebras with quantum symmetry from the theory of superselection sectors. Comm. Math. Phys. **134**, 391 (1990)

[25] N.Yu. Reshetikhin, V.G. Turaev: Invariants of 3-manifolds via link polynomials and quantum groups. Invent. Math. **103**, 547 (1991).

[26] V.G. Turaev, O. Viro: State sum invariants of 3-manifolds and quantum $6j$-symbols. Topology **31**, 865 (1992)
M. Karowski, W. Müller, R. Schrader: State sum invariants of compact 3-manifolds with boundary and $6j$-symbols. Journ. Phys. A **25**, 4847 (1992)

[27] J.J. Bisognano, E.H. Wichmann: On the Duality Condition for a Hermitean Scalar Field. J. Math. Phys. **16**, 985 (1975)

[28] M. Jörß: Lokale Netze auf dem eindimensionalen Lichtkegel. Diploma thesis FU Berlin (1991).
M. Jörß: On the existence of pointlike localized fields in conformally invariant quantum physics. DESY 92-156

[29] H.J. Borchers: The CPT theorem in two-dimensional theories of local observables. Comm. Math. Phys. **143**, 315 (1992)

[30] F. Gabbiani, J. Fröhlich: Operator algebras and conformal field theory. ETH-TH/92-30 (preprint)

[31] R. Brunetti, D. Guido, R. Longo: Modular structure and duality in conformal quantum field theory. (To appear)

[32] P.D. Hislop, R. Longo: Modular structure of the local algebras associated with the free massless scalar field theory. Comm. Math. Phys. **84**, 71 (1982)

[33] D. Buchholz, G. Mack, I. Todorov: The Current Algebra on the Circle as a Germ for Local Field Theories. In *Conformal Field Theories and Related Topics*, eds. P. Binetruy et al., Nucl. Phys. B (Proc. Suppl.) **5B**, 20, (1988)

[34] H. Takesaki, W. Winnink: Local normality in quantum statistical mechanics. Comm. Math. Phys. **30**, 129 (1973)

[35] K. Fredenhagen: Generalizations of the Theory of Superselection Sectors. In [18], p. 379 (1990)

[36] D. Guido, R. Longo: Relativistic invariance and charge conjugation in quantum field theory. Comm. Math. Phys. **148**, 521 (1992)

[37] K.H. Rehren: Braid group statistics and their superselection rules. In [18] p. 333 (1990)

[38] E. Verlinde: Fusion rules and modular transformations in 2D conformal field theory. Nucl. Phys. **B300**, 360 (1988)