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Abstract

Using standard techniques of Quantum Field Theory we determine the fluctuations of the components of the energy-momentum tensor in the presence of plates. For this aim we derive the Wightman functions of the electromagnetic field using canonical quantization. For the two- plate system the correlation functions contain an infinite set of poles corresponding to events connected by n-times reflected light- signals. Taking into account the conditions of the measuring process the fluctuations of the Casimir pressure depend critically on the chosen quantization procedure which itself reflects the model of the plates.

1 Introduction

In the following we continue the investigation of the vacuum state in Quantum Field Theory. In Quantum Mechanics the ground state is equally well investigated compared with other states. The wave function yields the necessary basic information. In Quantum Field Theory the situation is quite different. The vacuum state is usually represented by the formal Fock space vector $|0>$, only the Green functions of the field operators contain further information. Already in free field theory simple expectation values of the stress tensor or energy densities lead to divergent quantities. Because this seems to be unphysical in most applications these infinities are subtracted by the normal ordering procedure.

But this is not the right way. At least in part these infinities are direct consequences of the quantization procedure. For example the infinities of the ground state energy of QED could be understood as the added up zero point energies of the harmonic oscillators describing all the field modes. So possible infinities should be handled carefully and one should look for physically interesting finite parts of it.

A well-known nontrivial example is the Casimir pressure. Another interesting quantity is the fluctuation of the electromagnetic field strength (considered in Quantum Optics).

In the last time G.Barton [1] has raised the question of the fluctuations of the Casimir pressure. Similar investigations have been performed in [2], [3]. Fluctuations of observables are determined appropriately from correlation functions averaged with characteristic functions describing the measuring procedure. In [1] the correlation functions have been evaluated as matrix elements of operator products using complete set of intermediate states.

Here we prefer another method. We write down the complete expression for the product of the two operators considered and apply then standard methods Quantum Field Theory. These methods allow a simultaneous treatment of different interesting cases for which the Green functions are explicitly known.

As a case of physical importance we consider QED with one or two parallel conducting plates [4], [5], [6]. We start with the treatment of ideal conductors characterized by the vanishing of the tangential component of the electric field strength E_t and the normal component of the magnetic field strength B_n on the plates. We study the fluctuations of the components of the energy momentum tensor and also the fluctuations of the Casimir pressure. It turns out that the fluctuations of the components of the energy-momentum tensor depend in an essential manner on the external conditions, i.e. wether there are one or two plates, or no plate at all. Consequently the vacuum fluctuations constitute an important indication of the physical situation. The Wightman functions for the considered case can be constructed by the help of the reflection principle. Accordingly we observe a resonance structure of the correlation functions . Such resonances appear if the distances between the considered events correspond to a classical light signal n-times reflected at the plates.

In connection with the fluctuations in the case of two plates it turns out that in dependence of the chosen quantization procedure different expressions for the fluctuations of the Casimir pressure are obtained. The discrepancy originates from the mode propagating parallel to the plates. For this reason we analyse the canonical quantization of the electromagnetic field in the presence of two plates to clarify this question. We choose a quantization of the electromagnetic field A_μ which takes into account the boundary conditions of ideal conductors and allows the interaction with charged matter fields in a local manner which leads to Dirichlet conditions for the physical modes. The questionable mode appears as a special case of the non-transversal solutions which due to the kinematical decomposition in general do not obey a boundary condition. For the special momentum parallel to the plates this mode becomes transversal and fulfils the boundary conditions. It depends on the model for the plates (infinitely thin or not), whether this mode appears as part of the continuous spectrum or as a discrete solution. Correspondingly different expressions for the fluctuation of the Casimir pressure are obtained.

2 Canoncial Quantization and Wightman Functions in the Presence of Plates

Quantum Electrodynamics with boundary conditions has been used in many calculations. There are many different approaches for the quantization of electrodynamics in the presence of conductors [7], [8], [9], [10], [11], [12], [5] . A straightforward formulation within the covariant gauges has been given by the help of the functional integration. In the perturbation theory it leads to the standard Feynman diagram technique with a modified photon propagator [7]. In the present case we are interested in the Wightman functions of the electromagnetic field. Because it is not a priori clear how to derive, especially in arbitrary gauges, the Wightman functions from the propagators (T-products) we present a straightforward construction based on canonical quantization whereby we discuss different quantization schemes, too. In all cases we consider here ideally conducting plates. The boundary conditions $E_t = B_n = 0$ can be written in terms of the electromagnetic potentials A_μ by

$$\epsilon_{\mu\nu\rho\sigma} n^\rho \partial^\sigma A^\nu|_S = 0 \tag{2.1}$$

where n^ρ denotes the normal vector.

2.1 Quantization in the Case of Infinitely Thin Plates

As physical situation we assume that the plates are infinitely thin and characterized by the boundary conditions (2.1) only. The quantization will be performed on both side of the plate.

As usual, we expand the free photon field A_μ in terms of four polarization vectors e^i_μ

$$A_\mu(x) = \sum_{i=0}^{3} e^i_\mu f_i(x) \tag{2.2}$$

In order to find the boundary conditions to be fulfilled by the wave functions $f_i(x)$ one has to choose a suitable basis e^i_μ. It follows from eq.(2.1) that the boundary conditions act actually in the space perpendicular to the vectors n_ρ and ∂_σ. at the surface S. Because of the triviality of the surface considered here we are able to introduce globally polarization vectors which satisfy the necessary conditions at the surface. As two such vectors we choose for the case that the plates are perpendicular to the x_3-axis

$$e^1_\mu = \frac{1}{\sqrt{\Delta_\perp}} \begin{pmatrix} 0 \\ -\partial_2 \\ \partial_1 \\ 0 \end{pmatrix} \tag{2.3}$$

$$e^2_\mu = \frac{1}{\sqrt{\Delta_\perp \tilde{\Delta}}} \begin{pmatrix} \Delta_\perp \\ \partial_0\partial_1 \\ \partial_0\partial_2 \\ 0 \end{pmatrix} \tag{2.4}$$

with $\tilde{\Delta} = \tilde{\partial}^2 = \partial_0^2 - \partial_1^2 - \partial_2^2$ and $\Delta_\perp = \partial_\perp^2 = \partial_1^2 + \partial_2^2$. The remaining orthogonal polarization vectors are

$$e^3_\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{2.5}$$

$$e^0_\mu = \frac{1}{\sqrt{\tilde{\Delta}}} \begin{pmatrix} \partial_0 \\ \partial_1 \\ \partial_2 \\ 0 \end{pmatrix} \tag{2.6}$$

As it should be the vectors e^i_μ satisfy the relations

$$\begin{aligned} e^i_\mu e^j_\nu g_{ij} &= g_{\mu\nu}, \quad g^{\mu\nu} e^i_\mu e^j_\nu = g^{ij}, \quad a = (\tilde{a}, a_3), \quad \tilde{a} = (a_0, a_1, a_2), \\ \sum_{i=1}^{2} e^i_\mu e^j_\nu g_{ij} &= \tilde{g}_{\mu\nu} - \frac{\tilde{\partial}_\mu \tilde{\partial}_\nu}{\tilde{\Delta}} \quad \sum_{i=0,3} e^i_\mu e^j_\nu g_{ij} = \begin{pmatrix} \frac{\tilde{\partial}_\mu \tilde{\partial}_\nu}{\tilde{\partial}^2} & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{2.7}$$

The boundary condition (2.1) leads to

$$\epsilon_{\mu\nu\rho\sigma} n^\rho \partial^\sigma A^\nu|_S = \frac{1}{\sqrt{\Delta_\perp}} \begin{pmatrix} \Delta_\perp \\ -\partial_0\partial_1 \\ -\partial_0\partial_2 \\ 0 \end{pmatrix} f_1|_S + \frac{1}{\sqrt{\Delta_\perp \tilde{\Delta}}} \begin{pmatrix} 0 \\ \partial_2 \tilde{\partial}^2 \\ -\partial_1 \tilde{\partial}^2 \\ 0 \end{pmatrix} f_2|_S = 0 \tag{2.8}$$

on the plates. All the derivatives act in the (x_0, x_1, x_2)- subspace so that we have the Dirichlet conditions

$$f_i|_S = 0 \quad (i = 1, 2) \tag{2.9}$$

whereas f_0 and f_3 are free of boundary conditions. To obtain the field equation for A_μ one has to start from the Lagrangian of the photon field. As it is well-known quantization of a gauge field theory demands the choice of a gauge fixing condition or a gauge invariance breaking term in the Lagrangian. We choose the Feynman gauge with the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 = \frac{1}{2} A_\mu \partial^2 A^\mu . \tag{2.10}$$

As an interesting example we consider the quantization in the space between two parallel plates located at $x_3 = 0$ and $x_3 = d$. The wave equations $\partial^2 f_i = 0$ together with the boundary conditions (2.9) have the normalized solutions

$$f_i^{\pm} = \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2k_0}}e^{\pm ikx}, \quad k_0 = \sqrt{\vec{k}^2} \quad (i = 3, 0), \tag{2.11}$$

$$f_i^{\pm} = \frac{1}{2\pi}\sqrt{\frac{2}{d}}\frac{1}{\sqrt{2k_0}}e^{\pm i\tilde{k}\tilde{x}}\sin n\frac{\pi}{d}x_3, \tag{2.12}$$

$$k_0 = \sqrt{k_\perp^2 + (\frac{n\pi}{d})^2}, \quad n = 1,\, 2,\, 3,\, ..., \quad (i = 1, 2).$$

According to the standard procedure the quantized field A_μ reads

$$A_\mu(x) = \sum_{i=0,3} e_\mu^i \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2k_0}}[e^{-ikx}a_i(\vec{k}) + e^{ikx}a_i^*(\vec{k})]$$
$$+ \sum_{i=1,2} e_\mu^i \sum_{n=1}^{\infty}\frac{1}{2\pi}\sqrt{\frac{2}{d}}\int\frac{d^2k_\perp}{\sqrt{2k_0}}\sin\frac{n\pi}{d}x_3 \tag{2.13}$$
$$[e^{-i\tilde{k}\tilde{x}}a_i(\vec{k}_\perp, n) + e^{+i\tilde{k}\tilde{x}}a_i^*(\vec{k}_\perp)]$$

with

$$[a_i(\vec{k}), a_j^*(\vec{q})] = -g_{ij}\delta(\vec{q} - \vec{k}), \quad (i = 0,\, 3), \tag{2.14}$$

$$[\, a_i(\vec{k}_\perp, n), a_j^*(\vec{q}_\perp, m)] = -g_{ij}\delta_{nm}\delta(\vec{q}_\perp - \vec{k}_\perp), \quad (i = 1,\, 2).$$

The difficulties of the commutation relation (2.14) with $i = 0$ can be resolved in standard way using the Gupta-Bleuler method of indefinite metric [13]. Defining the Hermitian conjugate operator of a_i as $a_i^\dagger = \eta a_i^* \eta$ (where η is the metric operator), taking into account the special properties of this operator it turns out that for vacuum expectation values a formal calculation using the commutation relations in a straightforward manner leads to the right result. With this procedure we directly define the Wightman function $< 0|A_\mu(x)A_\nu(y)|0 >$. Taking into account eqs.(2.13), (2.14) and (2.7) we obtain:

$$< 0|A_\mu(x)A_\nu(y)|0 >$$
$$= -\sum_{i=1,2} e_\mu^i e_\nu^j g_{ij}\frac{2}{(2\pi)^2 d}\sum_{n=1}^{\infty}\int\frac{d^2k_\perp}{2k_0}e^{-ik(\widetilde{x-y})}\sin\frac{n\pi}{d}x_3\sin\frac{n\pi}{d}y_3$$
$$-\sum_{i=0,3} e_\mu^i e_\nu^j g_{ij}\frac{1}{(2\pi)^3)}\int\frac{d^3k}{2k_0}e^{-ik(x-y)} \tag{2.15}$$
$$= i(g_{\tilde{\mu}\tilde{\nu}} - \frac{\partial_{\tilde{\mu}}\partial_{\tilde{\nu}}}{\tilde{\partial}^2})^s D_2^-(\tilde{x} - \tilde{y}, x_3, y_3) + i\begin{pmatrix} \frac{\tilde{\partial}_\mu\tilde{\partial}_\nu}{\tilde{\partial}^2} & 0 \\ 0 & -1 \end{pmatrix} D^-(x - y) \tag{2.16}$$

with

$$D^-(x - y) = \frac{i}{(2\pi)^3}\int d^4k e^{ik(x-y)}\delta(k^2)\Theta(-k_0) \tag{2.17}$$

and

$${}^sD_2^-(\tilde{x} - \tilde{y}, x_3, y_3) = \frac{2i}{(2\pi)^2 d}\sum_{n=1}^{\infty}\int\frac{d^2k_\perp}{2k_0}e^{-ik(\widetilde{x-y})}\sin\frac{n\pi}{d}x_3\sin\frac{n\pi}{d}y_3 \tag{2.18}$$

Alternative expressions of the foregoing functions are

$$D^-(z) = \frac{-i}{4\pi^2[(z_0 - i\epsilon)^2 - \vec{z}^2]},$$
$$= \int\frac{\tilde{d}k}{(2\pi)^3}e^{i\tilde{k}\tilde{z}}\Theta(-k_0)\Theta(\tilde{k}^2)\frac{i}{2\Gamma}(e^{i\Gamma z_3} + e^{-i\Gamma z_3}) \tag{2.19}$$

and

$${}^sD_2^-(x, y) = +\int\frac{d\tilde{k}}{(2\pi)^3}e^{i\tilde{k}\tilde{z}}\frac{-i}{2\Gamma}\Theta(-k_0)\Theta(\tilde{k}^2)\{\frac{1}{2i\sin(\Gamma d)} - \frac{1}{2i\sin(\Gamma^* d)}\}$$
$$[(2\cos\Gamma(x_3 - y_3)\cos\Gamma d - 2\cos\Gamma(x_3 + y_3 - d)] \tag{2.20}$$
$$= -\frac{1}{8\pi d\zeta}\{\frac{1}{e^{\frac{i\pi}{d}(\zeta - x_3 - y_3)} - 1} + \frac{1}{e^{\frac{i\pi}{d}(\zeta + x_3 + y_3)} - 1}$$
$$-\frac{1}{e^{\frac{i\pi}{d}(\zeta - x_3 + y_3)} - 1} - \frac{1}{e^{[\frac{i\pi}{d}(\zeta + x_3 - y_3)} - 1}\} \tag{2.21}$$

and

$${}^sD_2^-(x, y) = \sum_{l=-\infty}^{+\infty}[D^-(\tilde{z}\,, x_3 - y_3 + 2ld) - D^-(\tilde{z}\,, x_3 + y_3 + 2dl)] \tag{2.22}$$

where we have used the notations with $\tilde{z} = \tilde{x} - \tilde{y}$, $\zeta = \sqrt{(z_0 - i\epsilon)^2 - \vec{z}_\perp^2}$ and $\Gamma = \sqrt{\vec{p}^2}$ (for the derivation compare Appendix A).

In the first representation of ${}^sD_2^-(x, y)$ the mode summation of eq.(2.18) is converted into a Sommerfeld-Watson type integral using

$$\frac{\cos\Gamma d}{\Gamma}\{\frac{1}{2i\sin(\Gamma d)} - \frac{1}{2i\sin(\Gamma^* d)}\} = -\frac{\pi}{dk_0}\delta(k_0 - \sqrt{k_\perp^2 + (\frac{n\pi}{d})^2})$$

The representation (2.22) coincides with the elementary construction based on the reflection principle. It comes as a surprise that ${}^sD_2^-$ can be given by the closed summed up expression (2.21). We remark that the explicitly constructed Wightman functions ${}^sD^-$ are in accordance with the expression obtained from the earlier given propagator [7] using the relation between the propagator ${}^sD^c$ and the Wightman function

$${}^sD^c(x, y) -{}^s D^{c*}(x, y) ={}^s D^-(x, y) -{}^s D^+(x, y). \tag{2.23}$$

The Wightman function D^- is that part in $D^c - D^{c*}$ which allows an analytic continuation $z_0 = x_0 - y_0 \to z_0 - i\eta$, $(\eta > 0)$ and correspondingly carries a $\Theta(-k_0)$ in its Fourier representation.

Looking at other quantizations of the electromagnetic field we underline that in the representation (2.18) the sum over the eigenmodes starts with the term n=1. Obviously the eigenfunctions (2.11) and (2.12) corresponding to the polarization vectors e^1_μ, e^2_μ constitute a complete system. The same applies to the ordinary plane waves

$$f_{3,0} = \frac{1}{(2\pi)^{3/2}\sqrt{2k_0}} e^{\pm ikx} \tag{2.24}$$

belonging to e^2_μ and e^0_μ. In general the latter modes are non-transversal and in this sense unphysical and have not to obey the boundary conditions. However on the special case $k_3 = 0$ the solution

$$A_\mu = e^3_\mu \frac{1}{(2\pi)^{3/2}\sqrt{2k_0}} e^{\pm i\tilde{k}\tilde{x}}$$

becomes transversal and fulfils the boundary conditions. It describes a physical wave propagating parallel to the plates. This wave is not restricted to the space between the plates (as the general solution (2.24)) and has to be distinguished from the discrete solution

$$\frac{1}{2\pi\sqrt{2k_0 d}} e^{i\tilde{k}\tilde{x}}, \quad \tilde{k}^2 = 0$$

normalized on the interval $0 < x_3 < d$.

2.2 Quantization in the Case of Thick Plates

If we consider free electrodynamics only, then the transversal waves cannot penetrate the plates. However the wave propagating parallel to the plates extends over the unrestricted three dimensional space. Like the unphysical waves (which are not transversal and do not obey the boundary conditions) it is normalized over the space R_3. If, however, one restricts the quantization to the space in between the two plates which in some sense corresponds to considering the thickness of the plates as large in comparison to the distance d, then we can adopt quantization methods used in solid state physics. Additional to the boundary conditions (2.9) we impose periodicity conditions on the unphysical amplitudes

$$f_i(\tilde{x}, x_3) = f_i(\tilde{x}, x_3 + \frac{2\pi}{d}) \quad \text{for i=0 and i=3.}$$

Then the solutions (2.24) have to be replaced by

$$f_{3,0} = \frac{1}{2\pi\sqrt{2k_0 d}} e^{\pm ikx} \quad \text{with} \quad k_3 = \frac{2n\pi}{d},\ n = 0, \pm 1, \pm 2, \ldots$$

In fact now all modes are discrete. The mode decomposition of the potential reads now

$$\begin{aligned} A_\mu(x) &= \sum_{i=0,3} e^i_\mu \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)\sqrt{2k_0 d}} [e^{-i\tilde{k}\tilde{x} + x_3(2n\pi)/d} a_i(\vec{k}) + e^{i\tilde{k}\tilde{x} - x_3(2n\pi)/d} a_i^*(\vec{k})] \\ &+ \sum_{i=1,2} e^i_\mu \sum_{n=1}^{\infty} \frac{1}{2\pi} \sqrt{\frac{2}{d}} \int \frac{d^2k_\perp}{\sqrt{2k_0}} \sin\frac{n\pi}{d} x_3 [e^{-i\tilde{k}\tilde{x}} a_i(\vec{k}_\perp, n) + e^{+i\tilde{k}\tilde{x}} a_i^*(\vec{k}_\perp)] \end{aligned}$$

The Wightman function in the form (2.16) remains formally unchanged, only the representation (2.17) for D^- is replaced by

$$D^-(x-y) = \frac{i}{(2\pi)^2 d} \int d\tilde{k} \sum_{n=-\infty}^{+\infty} e^{i(k\widetilde{(x-y)} - (x_3-y_3)(2n\pi)/d)} \delta(k^2)\Theta(-k_0).$$

We underline that the representation for ${}^sD_2^-$ remains unchanged. Again the physical mode propagating parallel to the plates is included into the amplitude f_3, however, as a discrete mode now. The discreteness of the mode changes the standard relation ${}^sD^- = D^- + \bar{D}^-$ into ${}^sD^- = D^-_\cdot + \bar{D}^- + \tilde{D}^-$ where

$$\tilde{D}^-(\tilde{z}) = \frac{i}{(2\pi)^2 d} \int d\tilde{k} e^{ik\widetilde{(x-y)}} \delta(\tilde{k}^2)\Theta(-k_0)$$

This quantization procedure can be extended in a straightforward manner to interacting QED by the inclusion of the electron field. In the last case we have to postulate periodicity conditions for the electron field too.

2.3 Quantization with Hertz Vectors as Polarization Vectors

Because of the importance of this problem we add some further remarks. In the classical treatment of the propagation of waves it is customary to start with electric and the magnetic Hertz vectors (taking into account the $\vec{E}$ - $\vec{B}$ symmetry of free electrodynamics). Historically such vectors have been introduced for solving the Maxwell equations in a simple manner. They are in principle defined globally and adapted for giving acceptable boundary conditions in the case of conducting surfaces [14], [10], [8],[15]. These vectors correspond to an expansion of the physical degrees of freedom of the electromagnetic wave fields according to

$$A_\mu(x) = \sum_{i=1,2} h^i_\mu g_i(x) \tag{2.25}$$

with

$$h^1_\mu = e^1_\mu, \quad h^2_\mu = \frac{1}{\sqrt{\partial_0^2 - \partial_3^2}} \begin{pmatrix} \partial_3 \\ 0 \\ 0 \\ \partial_0 \end{pmatrix}$$

or

$$h^1_\mu = e^1_\mu, \quad h^2_\mu = \frac{1}{\sqrt{\Delta\Delta_\perp}} \begin{pmatrix} 0 \\ \partial_1\partial_3 \\ \partial_2\partial_3 \\ -\Delta_\perp \end{pmatrix} \tag{2.26}$$

Again the functions $g^i(x)$ satisfy the wave equation. If we apply the polarization vectors listed first then the boundary condition (2.1) leads to

$$\epsilon_{\mu\nu\rho\sigma} n^\rho \partial^\sigma A^\nu|_S = \frac{1}{\sqrt{\Delta_\perp}} \begin{pmatrix} \Delta_\perp \\ -\partial_0\partial_1 \\ -\partial_0\partial_2 \\ 0 \end{pmatrix} g_1|_S + \frac{1}{\sqrt{\partial_0^2 - \partial_3^2}} \begin{pmatrix} 0 \\ \partial_2\partial_3 \\ -\partial_1\partial_3 \\ 0 \end{pmatrix} g_2|_S = 0 \tag{2.27}$$

so that the function g_2 satisfies the Neumann boundary condition

$$\partial_3 g_1(x)|_S = 0$$

whereas the first function g_1 satisfies the Dirichlet condition as before. The same conditions are valid for the other choice of the Hertz vectors. The solutions for g_2 satisfying the Neumann condition are:

$$g_2^\pm = \frac{1}{(2\pi)} \frac{1}{\sqrt{k_0 d}} e^{\pm i\tilde{kx}} \cos n\frac{\pi}{d}x_3, \; k_0 = \sqrt{k_\perp^2 + (\frac{n\pi}{d})^2}, \; n = 1,\, 2,\, 3,\, \ldots \tag{2.28}$$

and

$$g_2^\pm = \sqrt{\frac{1}{2dk_0}} \frac{1}{2\pi} e^{\pm i\tilde{kx}}, \quad k_0 = \sqrt{k_\perp^2} \tag{2.29}$$

Here the last solution is necessary for a completion of the set of solutions. So in this case from a mathematical point of view the discrete mode with n=0 is included in a natural way.

It is generally believed that the Quantum Field Theory of the free electromagnetic field without interaction could be formulated with these basic vectors. Starting again from the Lagrangian (2.10), inserting the representation (2.25) of the photon field we obtain

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A^\mu)^2 = \frac{1}{2} g^{ij} g_i \partial^2 g_j \tag{2.30}$$

with $g^{ij} = diag(-1, -1)$, where (i,j) are restricted to (1,2) because we have taken into account only "physical modes". A canonical quantization of the field theory given by the Lagrangian (2.30) and using eq.(2.25) would lead to the following Wightman function

$$< 0|A_\mu(x)A_\nu(y)|0 >= -i(h^1_\mu h^1_\nu \, {}^sD^-_{2D}(x,y) + h^2_\mu h^2_\nu \, {}^sD^-_{2N}(x,y)) \tag{2.31}$$

where ${}^sD^-_{2D}(x,y)$ denotes here the Wightman function (2.18) satisfying the Dirichlet boundary condition and ${}^sD^-_{2N}(x,y)$ the Wightman function satisfying the Neumann boundary condition. The last includes the questionable mode in a natural way.

$${}^sD^-_{2N}((\tilde{x}-\tilde{y}), x_3, y_3) = \frac{i}{(2\pi)^2 d} \sum_{n=-\infty}^{+\infty} \int \frac{d^2k_\perp}{2k_0} e^{-ik\widetilde{(x-y)}} \cos\frac{n\pi}{d}x_3 \cos\frac{n\pi}{d}y_3 \tag{2.32}$$

If we now calculate the Wightman functions for the field strength on the basis of expressions (2.16-18) or (2.31-32) then both expressions are identical up to the questionable mode. This can be seen most easily for the special Wightman function $< 0|F_{03}(x)F_{03}(x')|0 >$. (Compare Appendix B). In case of only one plate where there are no discrete modes the results coincide. The same is true for thick plates with the appropriate quantization of unphysical modes.

For the case of two thin plates the serious problem remains how to understand the discrepancy with respect to correlation functions. This fact is the more embarrassing as in ordinary QED the quantization on the basis of the two vectors h^i (2.26) which corresponds to the Coulomb gauge is fully equivalent to covariant procedures. Obviously the covariant quantization with the polarization vectors e^i (2.3-6) is the appropriate method for thin plates, whereas quantization with the help of the Hertz vectors h^i reflects the situation of thick plates. In any case we consider the existence of the discrete mode (2.29) as a criterion for thick plates, because for thin plates the parallel mode is part of the continuous spectrum.

A covariant quantization of QED demands four independent polarization vectors. One can easily convince oneself that there do not exist four orthogonal and normalized polarization vectors with the properties:

they include the vectors h^1 and h^2 ,

the wave functions of the remaining polarization vectors h^3 and h^0 are free of boundary conditions.

As already stated in the beginning the reason is that the boundary conditions act in the space orthogonal to the normal vector n_μ and four dimensional gradient ∂_μ. Therefore in this case also the unphysical waves satisfy boundary conditions, following from an inappropriate choice of the polarization vectors. The problem is whether these additional boundary conditions have physical consequences [16] or not.

We underline that the quantization in the case of thin plates using all four components of the photon field and the basis vectors e^i is completely equivalent to our former procedure [7] which rests on the functional integral for QED in covariant gauges.

3 Field Theoretic Description of Fluctuations

In general the fluctuation of an observable T in the vacuum state is defined by

$$(\Delta T)^2 =< 0|(T-\bar{T})^2|0 >=< 0|T^2|0 > - < 0|\bar{T}|0 >^2 \tag{3.1}$$

$$\bar{T} = < 0|T|0 >$$

where

$$T = \int f(x)T(x)\,dx. \qquad (3.2)$$

is determined by a local field theoretic observable $T(x)$ and a function $f(x)$ describing the measuring procedure. Therefore the essential information for the fluctuation is contained in the expectation values

$$\begin{aligned} W(x,x') &= < 0|T(x)T(x')|0 > - < 0|T(x)|0 >< 0|T(x')|0 > \\ &= < |T(x)T(x')| >' \end{aligned} \qquad (3.3)$$

In our case we consider the diagonal $T_{\mu\mu}$ components of the energy-momentum tensor. For a discussion of the Casimir pressure we need the 33-component. From this quantity the Casimir pressure on a plate located at $x_3 = a$ can be obtained as the difference of T_{33} across the plates

$$p(x) = T_{33}(x_3 = a + \epsilon) - T_{33}(x_3 = a - \epsilon) \qquad (3.4)$$

For the energy- momentum tensor we use the symmetric tensor

$$T_{\mu\nu} = F^{\rho}_{\mu}F_{\rho\nu} - 1/4 g_{\mu\nu}F_{\sigma\tau}F^{\sigma\tau} \qquad (3.5)$$

with the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

As regularization procedure we use the point splitting technique. So we write for the diagonal elements of the energy-momentum tensor

$$\begin{aligned} T_{\mu\mu} &= \lim_{y\to x} -\frac{1}{2} g_{\mu\mu}[({}^{(\mu)}h^{\rho\lambda\,(\mu)}h^{\sigma\tau} - {}^{(\mu)}h^{\rho\tau\,(\mu)}h^{\sigma\lambda})\partial^x_\rho\partial^y_\lambda]A_\sigma(x)A_\tau(y) \\ &= \lim_{y\to x} -\frac{1}{2} g_{\mu\mu}[\partial^{\overline{xy}(\mu)}h^{\sigma\tau} - \partial^{x\underline{\tau}}\partial^{y\underline{\sigma}}]A_\sigma(x)A_\tau(y) \end{aligned} \qquad (3.6)$$

In the last equation the indexes $\mu\mu$ are suppressed in part, they are included in the following definitions, which are used appropriately.

$$\tilde{a}\tilde{b} = a_0b_0 - a_1b_1 - a_2b_2, \quad \partial^{xy} = g^{\rho\lambda}\partial^x_\rho\partial^y_\lambda, \quad \partial^{\overline{xy}} = {}^{(\mu)}h^{\rho\lambda}\partial^x_\rho\partial^y_\lambda$$
$$\partial^{\widetilde{xy}} = g^{\widetilde{\rho\lambda}}\partial^x_{\tilde\rho}\partial^y_{\tilde\lambda}, \quad \partial^{\widehat{xy}} = {}^{(\mu)}h^{\widetilde{\rho\lambda}}\partial^x_{\tilde\rho}\partial^y_{\tilde\lambda}, \quad \partial^{x\underline{\sigma}} = \partial^x_\rho{}^{(\mu)}h^{\sigma\rho}$$

whereby the matrix ${}^{(\mu)}h_{ij}$ reads

$${}^{(\mu)}h_{\alpha\beta} = \begin{cases} -g_{\alpha\beta}\,g_{\mu\mu} & \alpha \neq \mu \text{ or } \beta \neq \mu \\ +g_{\alpha\beta}\,g_{\mu\mu} & \alpha = \beta = \mu \end{cases}$$

Taking into account (3.6) the product $T_{\mu\mu}(x,y)T_{\mu\mu}(x',y')$ appears as a product of four field operators

$$\begin{aligned} &T_{\mu\mu}(x,y)T_{\mu\mu}(x',y') = \\ &\quad \lim_{y\to x}\lim_{y'\to x'} \frac{1}{2}[\partial^{\overline{xy}(\mu)}h^{\sigma\tau} - \partial^{x\underline{\tau}}\partial^{y\underline{\sigma}}][\partial^{\overline{x'y'}(\mu)}h^{\sigma'\tau'} - \partial^{x'\underline{\tau'}}\partial^{y'\underline{\sigma'}}]A_\sigma(x)A_\tau(y)A_{\sigma'}(x')A_{\tau'}(y') \end{aligned} \qquad (3.7)$$

Because we restrict the consideration to free field theory the Wick theorem can be applied immediately

$$\begin{aligned} < 0|A(x)A(y)A(x')A(y')|0 > = \\ &< 0|A(x)A(y)|0 >< 0|A(x')A(y')|0 > \\ + &< 0|A(x)A(x')|0 >< 0|A(y)A(y')|0 > \\ + &< 0|A(x)A(y')|0 >< 0|A(y)A(x')|0 > . \end{aligned} \qquad (3.8)$$

Therefore the correlation function is reduced to a sum of products of elementary Wightman functions. Due to the subtracted structure of the correlation function $< 0|T_{\mu\mu}(x,y)T_{\mu\mu}(x',y')|0 >'$ the first term of the r.h.s. of eq.(3.8) drops out and the point splitting can be removed in principle. In the following we apply the photon Wightman function in the form

$$< 0|A_\mu(x)A_\nu(y)|0 >= ig_{\mu\nu}D^-(x-y) + i(\tilde{g}_{\mu\nu} - \frac{\tilde\partial^x_\mu\tilde\partial^y_\nu}{\tilde\partial^x\tilde\partial^y})\bar{D}^-(\ddot{x},y) \qquad (3.9)$$

where the decomposition ${}^sD^- = D^- + \bar{D}^-$ has been used (compare (2.16)). The contribution $\bar{D}^-$ takes care of the boundary conditions [7]. Inserting eqs.(3.7),(3.8) and (3.9) into the correlation functions for the stress tensor we obtain

$$\begin{aligned} &< 0|T_{\mu\mu}(x)T_{\mu\mu}(x')|0 >' = \\ &\{-\partial^{\overline{xy}}\partial^{\overline{x'y'}}[D^-(x,x')D^-(y,y') + D^-(x,x')\bar{D}^-(y,y') + \bar{D}^-(x,x')D^-(y,y')] \\ &-\frac{1}{2}[\partial^{yy'}\partial^{\widetilde{xx'}}PD^-(x,x')\bar{D}^-(y,y') + \partial^{xx'}\partial^{\widetilde{yy'}}PD^-(y,y')\bar{D}^-(x,x')] \\ &-\frac{1}{2}[(2-P)\partial^{\overline{xy}}\partial^{\overline{x'y'}} + \partial^{\widetilde{xx'}}\partial^{\widetilde{yy'}}PP + (\partial^{\overline{xy}}\partial^{\widehat{x'y'}} + \partial^{\overline{x'y'}}\partial^{\widehat{xy}})P] \\ &\bar{D}^-(x,x')\bar{D}^-(y,y')\}|_{y\to x, y'\to x'} \end{aligned} \qquad (3.10)$$

where

$$P = (1 - \frac{\partial^{\widehat{xy}}\partial^{\widehat{x'y'}}}{\partial^{\widetilde{xy}}\partial^{\widetilde{x'y'}}}).$$

Taking into account special properties of the Wightman functions considered here, namely the reduction of these functions into a translation invariant part D_- and an "anti"translation invariant part D_+ according to

$$\bar{D}^-(x,y) = \bar{D}^-_-(\tilde{x}-\tilde{y}, x_3 - y_3) + \bar{D}^-_+(\tilde{x}-\tilde{y}, x_3 + y_3) \qquad (3.11)$$

where each part satisfies the field equation

$$\Box \bar{D}_-^-(\tilde{x}-\tilde{y}, x_3-y_3)=0, \quad \Box \bar{D}_+^-(\tilde{x}-\tilde{y}, x_3+y_3)=0.$$

we obtain finally

$$< 0|T_{\mu\mu}(x)T_{\mu\mu}(x')|0 >' = -\partial^{\overline{xy}}\partial^{\overline{x'y'}}[(D^-(x,x')+\bar{D}_-^-(x,x'))(D^-(y,y')+\bar{D}_-^-(y,y')) + \bar{D}_+^-(x,x')\bar{D}_+^-(y,y')]|_{y\to x, y'\to x'}. \quad (3.12)$$

The fluctuation of the Casimir pressure on a plate located at $x_3 = a$ can be reduced to the correlation function (3.10) due to the relation (3.4). One obtains

$$< 0|p(y)p(x')|0 >'|\,|_{x_3=y_3=a} = < 0|T_{33}(x)T_{33}(x')|0 >'|_{x_3=x_3'=a+\epsilon} + < 0|T_{33}(x)T_{33}(x')|0 >'|_{x_3=x_3'=a-\epsilon} \quad (3.13)$$

for ideally conducting plates. The reason for the absence of mixed terms originates from the fact, that physical modes cannot propagate across the plates for ideal conductors.

4 Fluctuation of the Casimir Pressure

In this section we study the correlation functions for the stress tensor and the Casimir pressure for different physical situations. In general we assume the quantization procedure for infinitely thin plates. Modifications for thick plates arising in the case of two plates are given additionally.

4.1 Electromagnetic Field in Free Space

As a first example we consider the correlation function of the stress tensor for the free electromagnetic field in free space. In this case equation (3.12) reads

$$< 0|T_{\mu\mu}(x)T_{\mu\mu}(x')|0 >' = -\partial^{\overline{xy}}\partial^{\overline{x'y'}}D^-(x,x')D^-(y,y')|_{x=y,x'=y'} \quad (4.1)$$

Inserting the explicit expression for the free field Wightman function we get

$$< 0|T_{\mu\mu}(x)T_{\mu\mu}(x')|0 >' = \frac{1}{\pi^4}\frac{[4(^{(\mu)}h^{\sigma\tau}(x-x')_\sigma(x-x')_\tau)^2-((x-x')^2)^2]}{[(x-x')^2]^6}. \quad (4.2)$$

As to be expected the correlation function is translational invariant and has a strong Wightman-type singularity for coinciding points and a power law decreasing behaviour for large distances.

For later use we explicitly write down the momentum space representation at coinciding x_3 coordinates of the fluctuation of the 33-component of the stress tensor.

$$< 0|T_{33}(x)T_{33}(x')|0 >'|_{x_3=x_3'} = \frac{1}{15(2\pi)^2}\int\frac{d\tilde{k}}{(2\pi)^3}e^{-i\tilde{k}\tilde{z}}\Theta(k_0)(\tilde{k}^2)_+^{5/2} \quad (4.3)$$

4.2 Electromagnetic Field in the Presence of one Plate

Here we study the fluctuation of the stress tensor disturbed by one plate at $x_3 = 0$. Again we have to apply the formula (3.12) whereby the Wightman functions D^- and $\bar{D}_1^- = D^-(\tilde{x}-\tilde{y}, x_3+y_3)$ have to be taken into account:

$$< 0|T_{\mu\mu}(x)T_{\mu\mu}(x')|0 >' = -\partial^{\overline{xy}}\partial^{\overline{x'y'}} [D^-(x,x')D^-(y,y') + D^-(\tilde{x}-\tilde{x}'x_3+x'_3)D^-(\tilde{y}-\tilde{y}', y_3+y'_3)]|_{x=y,x'=y'} \quad (4.4)$$

An explicit evaluation leads to

$$< 0|T_{\mu\mu}(x)T_{\mu\mu}(x')|0 >' = \frac{1}{\pi^4}\{[4(^{(\mu)}h^{\sigma\tau}(x-x')_\sigma(x-x')_\tau-((x-x')^2)^2]\frac{1}{[(x-x')^2]^6} + [4(^{(\mu)}h^{\bar{\sigma}\bar{\tau}}(x-x')_{\bar{\sigma}}(x-x')_{\bar{\tau}} + ^{(\mu)}h^{33}(x_3+x_3')^2)^2 - (g^{\bar{\sigma}\bar{\tau}}(x-x')_{\bar{\sigma}}(x-x')_{\bar{\tau}} - (x_3+x_3')^2)^2] [g^{\bar{\sigma}\bar{\tau}}(x-x')_{\bar{\sigma}}(x-x')_{\bar{\tau}} - (x_3+x_3')^2]^{-6}\}. \quad (4.5)$$

In a physical picture we see contributions corresponding to the free propagation from the point x to the point x' and the propagation via a reflection at the plate x_3. However there is no superposition between both types of waves. For fixed times and large values of $(x_3+x_3')^2$ the second term vanishes so that the fluctuations reduce to those of the free field case. Remark that this is not the case if we simultaneously consider large time differences. At last one should notice that not only the fluctuations of the third component of the energy-momentum tensor but of all diagonal elements are changed by the presence of the plate. If we consider the fluctuations near one plate i.e. $x_3 \to 0$ and $x_3' \to 0$ we obtain

$$\begin{aligned} < 0|T_{33}(x)T_{33}(x')|0 >'|_{x_3=x_3'=0} &= \frac{2}{15(2\pi)^2}\int\frac{d\tilde{k}}{(2\pi)^3}e^{-i\tilde{k}\tilde{z}}\Theta(k_0)(\tilde{k}^2)_+^{5/2} \quad (4.6)\\ &= \frac{6}{\pi^4}\frac{1}{[(x-x'_0-i\eta)^2-(\vec{x}-\vec{x}')_\perp^2]^4} \end{aligned}$$

which coincides with the result of G.Barton [1]. It is twice the amount of the fluctuation of the free field at $x_3 = 0$.

4.3 Electromagnetic Field between two Parallel Plates

In the case of two plates the correlation functions for the inner and the exterior regions can be treated separately, because in lowest order of perturbation theory both regions are not correlated. The Wightman functions for one plate at $x_3 = 0$, $\bar{D}_1^-$ and for two plates $\bar{D}_2^-$ at $x_3 = 0$ or $x_3 = d$ are identical for the exterior region $x_3 < 0$, therefore the corresponding correlation functions coincide, too.

The investigation of the inner region is more complicated. Again we start from the general expression (3.12)

$$< 0|T_{\mu\mu}(x)T_{\mu\mu}(x')|0 >'= \tag{4.7}$$
$$-\partial^{\overline{xy}}\partial^{\overline{x'y'}}[(D^-(x,x')+\bar{D}^-_{2-}(x,x'))(D^-(y,y')+\bar{D}^-_{2-}(y,y'))$$
$$+\bar{D}^-_{2+}(x,x')\bar{D}^-_{2+}(y,y')]|_{x=y,x'=y'}.$$

Besides D^- we have to insert here the functions

$$^sD^-_{2+}(x,y)=\bar{D}^-_{2+}=-\frac{1}{8\pi d\zeta}\{\frac{1}{e^{\frac{i\pi}{d}(\zeta-x_3-y_3)}-1}+\frac{1}{e^{\frac{i\pi}{d}(\zeta+x_3+y_3)}-1}\} \tag{4.8}$$

and

$$D^-(x,y)+\bar{D}^-_{2-}(x,y)=\frac{1}{8\pi d\zeta}\{\frac{1}{e^{\frac{i\pi}{d}(\zeta-x_3+y_3)}-1}+\frac{1}{e^{[\frac{i\pi}{d}(\zeta+x_3-y_3)}-1}\} \tag{4.9}$$

An explicit calculation in x-space for general positions x and x' is possible and leads to a very long expression. As it is to be expected it contains infinitely many poles (corresponding to the reflection principle) which are already contained in the Wightman functions. At the position of the poles - corresponding to world distances of definite length - the fluctuations are enhanced.

Let us now discuss the fluctuation of the Casimir force. The fixation of the coordinates $x_3 = x'_3 = 0$ to the position of the right plate and the restriction to the 33-component of the energy-momentum tensor simplifies all calculations considerably. Here we will start with a momentum space representation of the Wightman function (2.20). Taking into account (3.9) (3.11) and (4.7) we obtain after a straightforward calculation

$$< 0|T_{33}(x)T_{33}(x')|0 >'|_{x_3=x'_3=0_+} =$$
$$2\int\frac{d\tilde{p}}{(2\pi)^3}\int\frac{d\tilde{p}'}{(2\pi)^3}e^{i\widetilde{(p+p')}\widetilde{(x-x')}}[\Theta(-p_0)\Theta(\tilde{p}^2)\Theta(-p'_0)\Theta((\tilde{p}')^2)]$$
$$(\tilde{p}^2(\tilde{p}')^2+(\tilde{p}\tilde{p}')^2)\frac{\cos\Gamma d\cos\Gamma' d}{\Gamma\Gamma'}$$
$$\{\frac{1}{2i\sin(\Gamma d)}-\frac{1}{2i\sin(\Gamma^* d)}\}\{\frac{1}{2i\sin(\Gamma' d)}-\frac{1}{2i\sin(\Gamma'^* d)}\}. \tag{4.10}$$

With

$$\frac{\cos\Gamma d}{\Gamma}\{\frac{1}{2i\sin(\Gamma d)}-\frac{1}{2i\sin(\Gamma^* d)}\}=-\frac{\pi}{d}\frac{1}{|p_0|}\delta(p_0-\sqrt{(p_\perp)^2+(\frac{\pi n}{d})^2})$$

we get

$$< 0|T_{33}(x)T_{33}(x')|0 >'|_{x_3=x'_3=0_+} =$$
$$\frac{1}{2d^2}\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\int\frac{d^2p_\perp}{(2\pi)^2}\int\frac{d^2p'_\perp}{(2\pi)^2}e^{-i(p_0+p'_0)(x_0-x'_0)+i(p+p')_\perp(x-x')_\perp}$$
$$[(\frac{\pi n}{d})^2(\frac{\pi n'}{d})^2+(\widetilde{pp'})^2]\frac{1}{p_0p'_0}. \tag{4.11}$$

Further evaluation will be much simplified if we exploit the Lorentz invariance in the (x_0, x_1, x_2)-subspace. This allows to put $z_\perp = 0$ and use in addition rotation invariance in the (p_1, p_2)-plane. This leads to

$$< 0|T_{33}(x)T_{33}(x')|0 >'|_{x_3=x'_3=0_+,z_\perp=0} =$$
$$\frac{1}{(2\pi)^2 2d^2}\{A_1^2(z_0,d)+A_2^2(z_0,d)+2A_3^2(z_0,d)\} \tag{4.12}$$

with

$$A_1 = \sum_{n=1}^{\infty}\int\frac{d^2p_\perp}{2\pi}e^{-ip_0(x_0-x'_0)}\frac{1}{p_0}(\frac{n\pi}{d})^2$$
$$A_2 = \sum_{n=1}^{\infty}\int\frac{d^2p_\perp}{2\pi}e^{-ip_0(x_0-x'_0)}p_0$$
$$A_3 = \sum_{n=1}^{\infty}\int\frac{d^2p_\perp}{2\pi}e^{-ip_0(x_0-x'_0)}\frac{1}{p_0}\frac{p_\perp^2}{2} \tag{4.13}$$

and $p_0 = \sqrt{p_\perp^2+(\frac{\pi n}{d})^2}$.

Taking into account the analytic properties of the Wightman functions for $z_0 \to z_0 - i\eta$, $(\eta > 0)$ the integrations and summations can be carried out without problems. The final result can be written in terms of the variable $\zeta = \sqrt{(z_0-i\eta)^2-(z_\perp^2)}$

$$A_1(\zeta,d) = \frac{1}{i\zeta}(\frac{\pi}{d})^2\frac{e^{i\frac{2\pi\zeta}{d}}+e^{i\frac{\pi\zeta}{d}}}{(e^{i\frac{\pi\zeta}{d}}-1)^3}$$
$$A_2(\zeta,d) = \frac{2}{(i\zeta)^3}\frac{1}{(e^{i\frac{\pi\zeta}{d}}-1)}+\frac{2}{(i\zeta)^2}(\frac{\pi}{d})\frac{e^{i\frac{\pi\zeta}{d}}}{(e^{i\frac{\pi\zeta}{d}}-1)^2}+\frac{1}{i\zeta}(\frac{\pi}{d})^2\frac{e^{i\frac{2\pi\zeta}{d}}+e^{i\frac{\pi\zeta}{d}}}{(e^{i\frac{\pi\zeta}{d}}-1)^3}$$
$$A_3(\zeta,d) = \frac{1}{(i\zeta)^3}\frac{1}{(e^{i\frac{\pi\zeta}{d}}-1)}+\frac{1}{(i\zeta)^2}(\frac{\pi}{d})\frac{e^{i\frac{\pi\zeta}{d}}}{(e^{i\frac{\pi\zeta}{d}}-1)^2} \tag{4.14}$$

so that

$$W_2(\zeta,d) \equiv < 0|T_{33}(x)T_{33}(x')|0 >'|_{x_3=x'_3=0_+} =$$
$$\frac{1}{(2\pi)^2 2d^2}\{A_1^2(\zeta,d)+A_2^2(\zeta,d)+2A_3^2(\zeta,d)\}. \tag{4.15}$$

This result generalizes the corresponding investigation of G.Barton. It is very interesting that the correlation function $W_2(\zeta,d)$ contains infinitely many multiple poles at

$$(\zeta)^2 = (x_0-x'_0)^2-(x_\perp-x'_\perp)^2 = 4n^2d^2.$$

In a physical interpretation these values of ζ^2 correspond to pairs of events $(x_0, x_\perp, x_3 = 0)$ and $(x'_0, x'_\perp, x'_3 = 0)$ connected by n-times reflected light-signals. This implies a resonance behaviour of the fluctuations for such distances ζ^2 For the limiting case $d \to \infty$ the results for one mirror can be recovered.

$$\begin{aligned} A_1(\zeta, d)|_{d\to\infty} &= \frac{2d}{\pi\zeta^4} \\ A_2(\zeta, d)|_{d\to\infty} &= \frac{6d}{\pi\zeta^4} \\ A_3(\zeta, d)|_{d\to\infty} &= \frac{2d}{\pi\zeta^4} \end{aligned} \tag{4.16}$$

The correlation function W_2 has the following scaling and limiting properties (W_1 denotes the correlation function $< 0|T_{33}(x)T_{33}(x')|0 >$ corresponding to one plate, see eq.(4.6))

$$\begin{aligned} W_2(\lambda\zeta, \lambda d) &= \frac{1}{\lambda^8} W_2(\zeta, d) \\ \lim_{d\to\infty} W_2(\zeta, d) &= W_1(\zeta) \\ \lim_{\zeta\to 0} \frac{W_2(\zeta, d)}{W_1(\zeta)} &= 1 \\ W_2(\zeta, d)_{\frac{|\zeta|}{d}\ll 1} &= \frac{1}{d^8}(\frac{d}{\zeta})^2 f(\frac{\zeta}{d}). \end{aligned}$$

The function $f(\frac{\zeta}{d})$ is an analytic and integrable function with poles at $\frac{\zeta}{d} = 2n$.

Besides the already given physical interpretations we see that the correlations at $\zeta^2 \approx 0$ approximately coincide with those of the one mirror problem. It can be understood in a simple physical picture: there is not enough time to receive the reflected signals.

As a simple consequence of the Wightman structure of the correlation functions (the poles are located in the upper z_0-plane) we conclude

$$\int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} dx'_0 W_i(\zeta, d) = 0$$

i.e. the fluctuations of observables measured over an infinite time interval tend to zero. According to eq.(3.13) the fluctuations of the Casimir pressure are the sum of the fluctuations on both sides of the plates.

Let us finally consider thick plates where we have to take into account the discrete mode propagating parallel to the plates. Its contribution has to be added to eq.(4.9) leading to

$$\begin{aligned} D^-(x,y) + \bar{D}^-_{2-}(x,y) &= -\frac{1}{8\pi d\zeta}\{-\frac{1}{e^{\frac{i\pi}{d}(\zeta - x_3 + y_3)} - 1} - \frac{1}{e^{[\frac{i\pi}{d}(\zeta + x_3 - y_3)} - 1}\} \\ &\quad + \tilde{D}^-(x,y). \end{aligned}$$

We will not give the modifications in all the correlation with the exception of the momentum space representation (4.11) where now the summation includes the modes with n=0 :

$$\begin{aligned} &< 0|T_{33}(x)T_{33}(x')|0 >' |_{x_3 = x'_3 = 0_+} = \\ &\frac{1}{4d^2} \sum_{n=-\infty}^{+\infty} \sum_{n'=-\infty}^{+\infty} \int \frac{d^2p_\perp}{(2\pi)^2} \int \frac{d^2p'_\perp}{(2\pi)^2} e^{-i(p_0+p'_0)(x_0-x'_0)+i(p+p')_\perp(x-x')_\perp} \\ &[(\frac{\pi n}{d})^2(\frac{\pi n'}{d})^2 + (\widetilde{pp'})^2]\frac{1}{p_0 p'_0}. \end{aligned}$$

4.4 Fluctuations of the Casimir Force and the Measuring Process

Let us in conclusion of this section combine our results on correlation functions with measuring processes which according to (3.2) make recourse to specific functions which characterize the measuring procedure. We factorize the characteristic function $f(\tilde{x})$ according to $f(\tilde{x}) = g(x_0)h(x_\perp)$. As an example we choose

$$g(x_0) = \frac{\tau}{\pi}\frac{1}{x_0^2 + \tau^2}, \quad \int dx_0 e^{-ip_0 x_0} g(x_0) = e^{-|p_0|\tau}$$

and $h(x_\perp)$ is implicitly defined by

$$\int dx_\perp e^{i\vec{p}_\perp \vec{x}_\perp} h(x_\perp) = e^{\frac{a\pi}{d}} e^{-a\sqrt{p_\perp^2 + (\pi/d)^2}}$$

Both functions $g(x_0)$ and $h(x_\perp)$ are normalized to 1 and its Fourier transforms are dimensionless. By the help of this functions the fluctuation $(\Delta T)^2$ is expressed by means of the correlation function $W(\tilde{x}, \tilde{x}') =< 0|T_{33}(\tilde{x})T_{33}(\tilde{x}')| >'$ as

$$(\Delta T)^2 = \int d\tilde{x} d\tilde{x}' f(\tilde{x}) f(\tilde{x}') W(\tilde{x}, \tilde{x}'). \tag{4.17}$$

At first we consider the case of two thin plates. Here we are interested in the fluctuations of T_{33} on the inner side of the plate at $x_3 = 0$. Combining the foregoing equations with eq.(4.11) we obtain

$$\begin{aligned} (\Delta T)^2 &= \frac{1}{2d^2} e^{2a\pi/d} \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \int \frac{d^2p_\perp}{(2\pi)^2 p_0} \int \frac{d^2p'_\perp}{(2\pi)^2 p'_0} [(\tilde{p}\tilde{p}')^2 + (\frac{n\pi}{d})^2(\frac{n'\pi}{d})^2] \\ &\quad \exp(-p_0\tau - p'_0\tau)\exp(-a(\sqrt{p_\perp^2 + (\frac{\pi}{d})^2} + \sqrt{p'^2_\perp + (\frac{\pi}{d})^2})) \end{aligned} \tag{4.18}$$

with $p_0 = \sqrt{p_\perp^2 + (\frac{n\pi}{d})^2}$ and $p'_0 = \sqrt{p'^2_\perp + (\frac{n'\pi}{d})^2}$. According to realistic possibilities the characteristic time τ of a measuring process is large in comparison with the time interval necessary for a light signal to traverse the plate distance d

$$d \ll \tau. \tag{4.19}$$

Accordingly $(\Delta T)^2$ is dominated by the term with $n = n' = 1$ with the corresponding modifications of p_0 and p'_0 This yields taking into account rotation invariance

$$(\Delta T)^2 = \frac{1}{2d^2} e^{2a\pi/d} \int \frac{d^2 p_\perp}{(2\pi)^2 p_0} \int \frac{d^2 p'_\perp}{(2\pi)^2 p'_0} \{p_0^2 p'^2_0 + p_1^2 p'^2_1 + p_2^2 p'^2_2 + (\frac{\pi}{d})^4\} e^{-(\tau+a)(p_0+p'_0)}. \qquad (4.20)$$

In the limit (4.19) considered here the contributions from $p_1^2 p'^2_1 + p_2^2 p'^2_2$ in the bracket are nonleading whereas the remaining contributions are equal. The final result reads

$$(\Delta T)^2 = \frac{\pi^2}{4d^6(\tau+a)^2} e^{-\frac{2\pi\tau}{d}} \qquad (4.21)$$

We underline that this result is based essentially on the absence of the modes with n=0. The presence of such modes would change (4.21) to a power-like behaviour, which coincides with the result obtained in reference [1]. These results are consequences of different models for the plates transformed into different quantization procedures.

This should be compared with the fluctuations (specialized to $\mu = 3$ in the case of the one-plate system. Performing the same integrations invoking eq.(4.6) instead of eq.(4.11) we obtain (for simplicity taking a=0)

$$(\Delta T)^2 = \frac{2}{15(2\pi)^5} e^{-i\tilde{q}(\tilde{x}-\tilde{x}')} \int d\tilde{q}\theta(q_0)(\tilde{q}_+^2)^5/2 \int d\tilde{x}d\tilde{x}' f(\tilde{x}) f(\tilde{x}')$$
$$= \frac{6}{(4\pi)^4}(\frac{1}{\tau})^8. \qquad (4.22)$$

Of course this expression describes also the fluctuations of T_{33} on the outer side of the two-plate system. Because of the inequality (4.19) the fluctuations of the inner side of the plates are exponentially suppressed in comparison with the fluctuation of T_{33} on the outer sides of the plates. As a consequence the fluctuations of the Casimir pressure (defined by the same characteristic functions) for the one-plate system are twice as large as the fluctuations for the two-plate system. Note that this does not lead to a contradiction if the second plate is removed to infinity because the inequality (4.19) cannot be maintained in this limit.

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A The Photon Wightman Function between Plates

Our aim is to derive alternative representations of the expression (2.18)

$$
\begin{aligned}
{}^{s}D_2^-(\tilde{x}-\tilde{y},x_3,y_3) &= \frac{2i}{(2\pi)^2 d}\sum_{n=1}^{\infty}\int\frac{d^2k_\perp}{2k_0}e^{-ik\widetilde{(x-y)}}\sin\frac{n\pi}{d}x_3\sin\frac{n\pi}{d}y_3 \\
&= \frac{i}{(2\pi)^2 d}\sum_{n=-\infty}^{+\infty}\int d^4k e^{ik\widetilde{(x-y)}}\Theta(-k_0)\delta(k^2)\delta(k_3-\frac{n\pi}{d})\sin k_3x_3\sin k_3y_3
\end{aligned}
$$

with $k_0=\sqrt{k_\perp^2+(\frac{n\pi}{d})^2}$.

By the help of

$$
\sum_{n=-\infty}^{+\infty}\delta(k_3-\frac{n\pi}{d}) = 2d\sum_{n=-\infty}^{+\infty}\delta(2k_3d-2n\pi) = \frac{d}{\pi}\sum_{l=-\infty}^{+\infty}e^{i2k_3dl}
$$

this can be rewritten as

$$
\begin{aligned}
{}^{s}D_2^-(\tilde{x}-\tilde{y},x_3,y_3) &= \frac{2i}{(2\pi)^3}\sum_{l=-\infty}^{\infty}\int d^4k_e^{-ik\widetilde{(x-y)}}\Theta(-k_0)\delta(k^2) \\
&\quad (\frac{1}{2i})^2[e^{ik_3(x_3+y_3)}+e^{-ik_3(x_3+y_3)}-e^{ik_3(x_3-y_3)}-e^{ik_3(-x_3+y_3)}]e^{i2k_3l} \\
&= \sum_{l=-\infty}^{+\infty}[D^-(\tilde{z},x_3-y_3+2ld)-D^-(\tilde{z},x_3+y_3+2dl)].
\end{aligned}
$$

In order to derive the representation (2.21) we exploit the restricted Lorentz invariance in the $\tilde{x}$ subspace which allows us to represent $\tilde{z}=\tilde{x}-\tilde{y}$ by the vector $(z_0,0,0)$. Now the $p_\perp$ integration can be carried out

$$
\int\frac{d^2p_\perp}{p_0}e^{-ip_0z_0}=\frac{2\pi}{iz_0}e^{-i\frac{z_0n\pi}{d}}.
$$

Convergence is guarantied by the Wightman prescription $z_0\to z_0-i\eta,\ \eta>0$ (analyticity in the forward tube). This property also assures the convergence of the following infinite sum

$$
\begin{aligned}
&\sum_{n=1}^{\infty}e^{-i\frac{z_0\pi}{d}n}\sin\frac{n\pi}{d}x_3\sin\frac{n\pi}{d}y_3 \\
&= -\frac{1}{4}\{\frac{1}{e^{\frac{i\pi}{d}(z_0-x_3-y_3)}-1}+\frac{1}{e^{\frac{i\pi}{d}(z_0+x_3+y_3)}-1} \\
&\quad -\frac{1}{e^{\frac{i\pi}{d}(z_0-x_3+y_3)}-1}-\frac{1}{e^{[\frac{i\pi}{d}(z_0+x_3-y_3)}-1}\}.
\end{aligned}
$$

Taking into account this formula and using once more the restricted Lorentz invariance by substituting $z_0-i\eta\to\zeta=\sqrt{(z_0-i\eta)^2-z_\perp^2}$ we obtain directly the representation (2.21).

B A Special Wightman Function in different Quantization Schemes

As an instructive example concerning the result of different quantization procedures in the case of two plates we consider the correlation function of the field strength F_{03}. It is given by the photon Wightman functions by

$$
\begin{aligned}
&<0|F_{03}(x)F_{03}(y)|0>= \qquad (B.1)\\
&\quad \partial_0^x\partial_0^y<0|A_3(x)A_3(y)|0>+\partial_3^x\partial_3^y<0|A_0(x)A_0(y)|0> \\
&- \quad \partial_3^x\partial_0^y<0|A_0(x)A_3(y)|0>-\partial_0^x\partial_3^y<0|A_3(x)A_0(y)|0>
\end{aligned}
$$

Using at first the expression (2.16) we obtain (for $0<x_3,\ y_3<d$)

$$
\begin{aligned}
&<0|F_{03}(x)F_{03}(y)|0>= \qquad (B.2)\\
&\quad i(\partial_0^z)^2D^-(z)+i\partial_3^x\partial_3^y(1-\frac{(\partial_0^z)^2}{(\tilde{\partial}^z)^2})^sD_{2D}^- - i\frac{(\partial_0^z)^2}{(\tilde{\partial}^z)^2})(\partial_3^z)^2D^-(z)
\end{aligned}
$$

and finally with (2.18)

$$
<0|F_{03}(x)F_{03}(y)|0>=\frac{2}{(2\pi)^2d}\sum_{n=1}^{\infty}\int\frac{d^2k_\perp}{2k_0}k_\perp^2e^{-ik\widetilde{(x-y)}}\cos\frac{n\pi}{d}x_3\cos\frac{n\pi}{d}y_3. \qquad (B.3)
$$

At next we also apply the polarization vectors e_μ^i but choose as physical space the interval $-L<x_3<L$ ($x_\perp$ unrestricted , $L>d$). This amounts in imposing periodicity conditions on the solutions. For points between the plates we have the old solution (2.12). Instead of (2.12) the solutions f_0,f_3 now read

$$
f_{0,3}^{\pm}=\frac{1}{2\pi}\sqrt{\frac{1}{2dk_0}}e^{\pm kx},\quad k_3=\frac{n\pi}{L},\quad n=0,\pm1,\pm2,\ldots
$$

Correspondingly D^- is replaced by

$$
D^-(x-y)=\frac{i}{(2\pi)^22L}\int d\tilde{k}\sum_{n=-\infty}^{+\infty}e^{i[k\widetilde{(x-y)}-(x_3-y_3)(n\pi)/L]}\delta(k^2)\Theta(-k_0). \qquad (B.4)
$$

Also in this case the correlation function is given by (B.2) with D^- replaced by the expression (B.4). Whereas in the foregoing case the first and the third term in (B.2) compensate each other completely now the term with n=0 survives, so that

$$
\begin{aligned}
&<0|F_{03}(x)F_{03}(y)|0>= \\
&\quad \frac{1}{(2\pi)^2d}\int\frac{d^2k_\perp}{2k_0}k_\perp^2e^{-i\tilde{k}\tilde{z}}\{\frac{d}{2L}+2\sum_{n=1}^{+\infty}\cos\frac{n\pi}{d}x_3\cos\frac{n\pi}{d}y_3\}. \qquad (B.5)
\end{aligned}
$$

This formula interpolate between the first case ($L\to\infty$) and the case $2L=d$ which we interpret as the case of thick plates.

The calculation of the same correlation function using Hertz vectors (2.26) has to take into account the structure of the tensor $h^i_\mu h^i_\nu$ (compare (2.31)). Obviously we have $< 0|A_0(x)A_0(y)|0 >=< 0|A_0(x)A_3(y)|0 >=< 0|A_3(x)A_0(y)|0 >= 0$ and

$$< 0|A_3(x)A_3(y)|0 >= -i\frac{\Delta_\perp^2}{\Delta_\perp \partial_0^2}\, {}^s D_{2N}^-$$

where ${}^s D_{2N}^-$ is given by (2.32). Insertion into (B.1) leads to

$$< 0|F_{03}(x)F_{03}(y)|0 >= \frac{1}{(2\pi)^2 d} \sum_{n=-\infty}^{+\infty} \int \frac{d^2k_\perp}{2k_0} k_\perp^2 e^{-i\tilde{k}\tilde{z}} \cos\frac{n\pi}{d}x_3 \cos\frac{n\pi}{d}y_3. \qquad \text{(B.6)}$$

Whereas this result differs from (B.3) (infinitely thin plates, four polarization vectors e^i_μ), it coincides with the expression (B.5) for the special choice d = 2L (thick plates). The result (B.6) is in accordance with that given in [15] if an obvious error in their eqn.(2.17) is corrected.