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Integrability Conditions For Potential Flow Of The Renormalisation Group *

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ABSTRACT

The renormalisaton group (RG) flow for a quantum field theory in flat D-dimensional space is considered. Some conditions on the trace of the stress operator are derived which ensure that the RG flow is derivable from a potential on the space of couplings. This requires introducing a metric on the space of interactions and two possibilities are considered. Assuming positive definiteness of the metric the potential provides a function on the space of couplings which is non-increasing along the RG trajectories. Thus any theory which satisfies the integrability conditions also satisfies a c-theorem in D-dimensions.

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§1 The c-theorem And Renormalisation Group Flow

The question of the nature of the renormalisation group (RG) flow on the space of coupling constants for a quantum field theory is a recurrent one in physics. It has been shown [1] in two dimensional Euclidean field theory, assuming certain positivity conditions on the Hilbert space of the theory, that there exists a function on the space of coupling constants which is non-increasing along the RG trajectories (the c-theorem). This has very important and far reaching implications for the theory because it puts constraints on the way that the RG flow can be realised, for example it can never come back to visit a point where it has already been, thus eliminating the possibility of limit cycles. The non-increasing function, c, can be interpreted as a measure of the number of degrees of freedom of the theory and its decreasing nature as the length scale, l, is increased as being due to "integrating out" the degrees of freedom on scales less than l. At critical points (conformal field theories) it is the central charge of the theory.

The proof of the c-theorem in [1] relies heavily on the special properties of two dimensions and it is crucial that there are only four different bilinears (in terms of index structure) that can be constructed from the stress tensor, $T_{\mu\nu}$, in two dimensions. Attempts to generalise the c-theorem to higher dimensions have met with difficulties because, for $D > 2$, there is a fifth independent bilinear which cannot be eliminated [2]. Some authors have tried to generalise Zamalodchikov's result using spectral representations of stress tensor bilinears, [3] [4] [5], but so far without complete success.

A stronger condition than the c-theorem is that of potential flow. The possibility of potential flow was emphasised by Wallace and Zia in [6] and [7]. In the latter reference the three loop β-functions for massless φ^4 theory, with two φ^4 couplings, were shown to be derivable from a potential, and it has been conjectured that this property should hold to all orders in perturbation theory (at least in two dimensions) [8]. It can also be shown that a single scalar field coupled with Yukawa interactions to a four component fermion in four dimensions exhibits potential flow to sixth order in the couplings [9].

If the space of couplings is equipped with an invertible positive definite metric, G_{ab}, and the β functions are derivable from a potential, $c(g)$, in coupling constant space,

$$\beta^b G_{ba} := \beta_a = -\frac{\partial}{\partial g^a} c(g), \tag{1}$$

then a c-theorem follows easily. (Here $a = 1, \ldots, n$ labels the dimensionless couplings g^a which can be thought of as real co-ordinates on the n-dimensional space of interactions, denoted by $\mathcal{G}$.) The metric is necessary since the β-functions are naturally defined as vectors,

$$\beta^a = l\frac{d}{dl}g^a = \frac{d}{dt}g^a, \tag{2}$$

where $t = \ln l$, and a gradient is necessarily a co-vector. Note that the β-functions in (2) are defined by differentiation with respect to a length rather than a momentum and so have the opposite sign to the usual field theoretic definition.

The c-theorem follows from (1) by differentiation of the potential,

$$\frac{dc}{dt} = \beta^a \partial_a c = -G_{ab}\beta^a\beta^b \leq 0, \tag{3}$$

where the last inequality follows when positive definiteness of the metric is assumed. Hence the potential, $c(g)$, is itself a function on $\mathcal{G}$ which is non-increasing along the RG flow.

Of course the above statement of the c-theorem depends crucially on the choice of metric, [1] [6] [7] [10] [11] [12]. In [7] a metric was constructed by *assumming* potential flow and using this as a criterion for defining a metric. But there are other considerations which might go into the definition of a physically reasonable metric, [10] [11] [12]. In this work both the metrics of Zamalodchikov [12] and that of O'Connor and Stephens [11] will be considered. In both cases conditions that the trace of the stress operator must satisfy, in order for the RG flow to be a potential flow, are obtained.

Before discussing the conditions for potential flow, in §4, some relevant properties of the space of local interactions $\mathcal{G}$ and the renormalised stress operator are discussed in §2 and of the operator product expansion in §3. §5 gives a summary and conclusions.

§2 The Trace Of The Stress Tensor

Consider a field theory in D dimensions which is described by an action,

$$S[g,\varphi] = \int_{\mathbf{R}^D} H(g,\varphi(x),\partial_\mu\varphi(x))d^Dx, \tag{4}$$

which depends on some set of fields, $\varphi(x)$, and their first derivatives, together with a finite set of real couplings, g^a. The couplings are all taken to be dimensionless. If there are any massive couplings in the theory these can always be made massless by multiplying by appropriate powers of the renormalisation length, l. Thus, for example, the dimensionless coupling associated with a mass, m^2, would be l^2m^2 giving rise to a β-function $\beta = 2-\delta$, where δ is the usual beta function associated with a mass. It will be assumed that the theory can be renormalised and that a quantum stress operator $\hat{T}_{\mu\nu}$, satisfying $\partial_\mu \hat{T}^\mu{}_\nu = 0$ and $<\hat{T}^\mu{}_\nu>=0$, can be defined. Of course, quantum mechanically, the trace of the stress operator can be a non-zero operator, $\hat{T}^\mu{}_\mu = -\hat{\Theta} \neq 0$, even when its classical counterpart vanishes. This happens when there is a conformal anomaly.

Following ref. [1], we consider the renormalised operators

$$\hat{\Phi}_a(x) = \frac{\partial\hat{H}}{\partial g^a} - <\frac{\partial\hat{H}}{\partial g^a}>, \tag{5}$$

to be a basis for all relevant or marginal operators of the theory, i.e. any relevant or marginal operator, which is a scalar in $\mathbf{R}^D$, can be written as a linear combination of $\hat{\Phi}_a(x)$ and the identity. To incorporate tensor (or spinor) operators would require a larger basis which we will denote by $\hat{O}_A(x)$, where A includes tensor indices μ as well as couplings a and possibily also spinor indices. A consequence of the definition of $\hat{\Phi}_a(x)$ is

$$\partial_a\hat{\Phi}_b = \partial_b\hat{\Phi}_a. \tag{6}$$

Note that, in terms of the "simple" fields $\varphi(x)$, which one might use in a functional integral to do perturbation theory for example, $\hat{\Phi}_a(x)$ would be composite operators and

therefore must be regularised, e.g. by introducing a cut-off $L << l$. We shall assume that this can be done. The $\hat{\Phi}_a$ are defined so that $<\hat{\Phi}_a(x)>=0$. This does not require that the renormalised $\partial_a\hat{H}(x)$ have zero expectation value, though they should be finite and independent of x from translation invariance. They will be denoted by

$$<\partial_a\hat{H}(x)>=h_a. \tag{7}$$

Since $\hat{\Phi}_a(x)$ are a basis for scalar operators with length dimension $-D$, $\hat{\Theta}(x)$ can be expanded as

$$\hat{\Theta}(x) = \beta^a(g)\hat{\Phi}_a(x) \tag{8}$$

for some set of functions, β^a. It can be shown, e.g. [12], that the β^a in (8) are just the β functions of equation (2). Note that $<\hat{\Theta}>=0$ because it is defined to be, as is usual for renormalisation of the stress tensor in flat space.

Now the operator $\hat{\Theta}$ should not depend on the renormalisation length, l, therefore

$$l\frac{d}{dl}\hat{\Theta} = l\frac{\partial}{\partial l}\Big|_g \hat{\Theta} + \beta^a\partial_a\hat{\Theta} = 0. \tag{9}$$

In terms of differential forms on $T^*_g(\mathcal{G})$ the quantities

$$\hat{\Phi}(x) = \hat{\Phi}_a(x)dg^a \tag{10}$$

can be thought of as operator valued one-forms. These also should be independent of l giving

$$l\frac{d}{dl}\hat{\Phi} = l\frac{\partial}{\partial l}\Big|_g \hat{\Phi} + \mathcal{L}_{\vec{\beta}}\hat{\Phi} = 0, \tag{11}$$

where $\mathcal{L}_{\vec{\beta}}$ is the Lie derivative with respect to the vector field $\vec{\beta}$ on the space of couplings $\mathcal{G}$,

$$\mathcal{L}_{\vec{\beta}}\hat{\Phi} = (di_{\vec{\beta}} + i_{\vec{\beta}}d)\hat{\Phi} = (\partial_a\beta^b)\hat{\Phi}_b dg^a + \beta^b(\partial_b\hat{\Phi}_a)dg^a = d\hat{\Theta} \tag{12}$$

(d and $i_{\vec{\beta}}$ are respectively the exterior derivative on $T^*_g(\mathcal{G})$ and contraction with the vector $\vec{\beta}$).

These equations can then be summarised as

$$d\hat{\Phi} = 0, \qquad \mathcal{L}_{\vec{\beta}}\hat{\Phi} = d\hat{\Theta} = -l\frac{\partial}{\partial l}\Big|_g \hat{\Phi} \quad \text{and} \quad \mathcal{L}_{\vec{\beta}}\hat{\Theta} = -l\frac{\partial}{\partial l}\Big|_g \hat{\Theta}. \tag{13}$$

Note that

$$l\frac{\partial}{\partial l}\Big|_g \hat{\Phi}_a(x) = \partial_a\Big(l\frac{\partial}{\partial l}\Big|_g \hat{H}(x)\Big) - l\frac{\partial}{\partial l}\Big|_g <\partial_a\hat{H}(x)>. \tag{14}$$

Expressions for $l\frac{\partial}{\partial l}|_g\hat{H}(x)$ for scalar field theory are considered in [13] and [14].

§3 The Operator Product Expansion And The Metric

The operator product expansion (OPE) was proposed in[15] and use will be made here of the version in[16] and[17]. Let $\hat{O}_A(x)$ be a basis for all operators, not just the basic ones of the theory, (5). In general this is an infinite set, even for renormalisable theories (e.g. in free scalar field theory this set would include the identitiy plus the operators : $\hat{\varphi}^p(x)$: for all integers $p \geq 1$ plus all possible derivatives, with respect to x, of these operators). Then the OPE for the product of two operators $\hat{O}_A(x)$ and $\hat{O}_B(x)$ can be written as

$$\hat{O}_A(x)\hat{O}_B(y) = C_{AB}{}^0(x-y)\,\mathbf{1} + C_{AB}{}^C(x-y)\hat{O}_C\left(\frac{x+y}{2}\right), \tag{15}$$

where the summation convention is used for the index C, and the identity operator has been written explicitly. The $C_{AB}{}^C(x-y)$ are c-number functions which clearly depend on the couplings, g^a, and which generically diverge as $x \to y$.

In expectation values it does not matter in what order the operators $\hat{O}_A(x)$ and $\hat{O}_B(y)$ occur so $C_{AB}{}^C(x-y)$ are symmetric under interchange of *both* A with B *and* x with y but not under either separately, in general. For example, if $\hat{O}_A(x) = \partial_\mu \hat{\Phi}_a(x)$ and $\hat{O}_B(y) = \hat{\Phi}_b(y)$ then $C_{AB}{}^C(x-y)$ has no particular symmetry under interchange of x and y alone. However if the three indices A, B and C are all scalar indices in $\mathbf{R}^D$ (i.e. do not involve space indices μ) then $C_{AB}{}^C(x-y)$ are functions of $|x-y|$ only and so are symmetric under intechange of A and B alone with the argument held fixed. Thus, for example,

$$C_{ab}{}^c(z) = C_{ba}{}^c(z). \tag{16}$$

More generally it is always true that

$$C_{AB}{}^C(z) = C_{BA}{}^C(-z) \tag{17}$$

for any argument z.

In the next section we will need to know something about the symmetry properties of three point correlators under interchange of the indices A, B and C. Consider therefore the three point correlator $< \hat{O}_A(x)\hat{O}_B(y)\hat{O}_C(z) >$. The OPE is associative and so can be applied first either to $\hat{O}_A(x)\hat{O}_B(y)$ or to $\hat{O}_B(y)\hat{O}_C(z)$. Alternatively $\hat{O}_C(z)$ can be pulled to the front and the OPE applied first to $\hat{O}_C(z)\hat{O}_A(x)$. A second application of (15) then gives (using $< \hat{O}_A(x) >= 0$ except for the identity operator)

$$\begin{aligned} < \hat{O}_A(x)\hat{O}_B(y)\hat{O}_C(z) > &= C_{AB}{}^F(x-y)C_{FC}{}^0\left(\frac{x+y}{2} - z\right) \\ &= C_{BC}{}^F(y-z)C_{AF}{}^0\left(x - \frac{y+z}{2}\right) \\ &= C_{CA}{}^F(z-x)C_{FB}{}^0\left(\frac{z+x}{2} - y\right). \end{aligned} \tag{18}$$

These equations are completely equivalent to the statement of associativity of the OPE and will be required in the derivation of the integrability conditions for potential flow.

Finally this section is ended with two possible definitions for a metric on $\mathcal{G}$. Following Zamalodchikov, [1] [12], a dimensionless metric on $\mathcal{G}$ can be obtained from the two point correlators $< \hat{\Phi}_a(x)\hat{\Phi}_b(y) >$ by defining

$$\begin{aligned} \overline{G}_{ab}(g) &= l^{2D} < \hat{\Phi}_a(x)\hat{\Phi}_b(y) > \big|_{|x-y|=l} \\ &= l^{2D} C_{ab}{}^0(l), \end{aligned} \tag{19}$$

and the assumption of positivity of the theory ensures that $\overline{G}_{ab}(g)$ is a positive definite metric on $\mathcal{G}$. Note that $\overline{G}_{ab}(g)$, being dimensionless, has no explicit l dependence and depends on the renormalisation point only implicitly, through g dependence i.e. $l\frac{\partial}{\partial l}|_g \overline{G}_{ab}(g) = 0$. For small separations

$$C_{ab}{}^0(|x-y|) = \sum_{s=-2D}^{\infty} C_{ab}^{(s)0}(g)|x-y|^s, \tag{20}$$

where $C_{ab}^{(s)0}(g)$ are independent of $|x-y|$. Thus, for small l, $\overline{G}_{ab}$ is essentially $C^{(-2D)0}_{ab}(g)$ i.e. the leading term in the OPE.

An alternative metric, that of O'Connor and Stephens [11], is obtained by integrating the two point functions over all space, rather than evaluating them at a specific separation,

$$\tilde{G}_{ab}(g) = l^D \int d^D y < \hat{\Phi}_a(y)\hat{\Phi}_b(x) > . \tag{21}$$

From translational invariance $\tilde{G}_{ab}(g)$ is independent of the point x but it must be regularised, due to the singularity at $y \sim x$. It will be assumed that this can be done.

§4 Integrability Conditions For Potential Flow

Integrability conditions on the β-functions for the RG flow to be a potential flow will now be derived. Potential flow requires that the one form $\beta = \beta_a dg^a$ on $T^*(\mathcal{G})$ is closed and hence locally exact (as stated before, questions concerning the global structure of $T^*_{\mathcal{G}}$ are not addressed here).

Consider first the β_a obtained from the metric (19). We have

$$\begin{aligned} \partial_a \beta_b &= l^{2D} \partial_a < \hat{\Phi}_b(x)\hat{\Theta}(y) > \big|_{|x-y|=l} \\ &= l^{2D} < \partial_a \hat{\Phi}_b(x)\hat{\Theta}(y) > \big|_{|x-y|=l} + l^{2D} < \hat{\Phi}_b(x)\partial_a\hat{\Theta}(y) > \big|_{|x-y|=l} \\ &\quad - l^{2D}\beta^c \int d^D z < \hat{\Phi}_a(z)\hat{\Phi}_b(x)\hat{\Phi}_c(y) > \big|_{|x-y|=l} . \end{aligned} \tag{22}$$

The last term in equation (22) can be understood by using a path integral formalism for the calculation of amplitudes and taking into consideration the variation of the couplings in the action, [12]. Of course it requires regularisation, due to singularities when $z \sim x$

and $z \sim y$, but it is assumed that this can be done. At this point it is crucial that the three point correlator dies off sufficiently fast at large separations for the integral in (22) to converge. This requires that it falls off faster than $|z-x|^{-D}$.

Now define

$$S_{abc}(|x-y|) := l^{2D} \int d^D z < \hat{\Phi}_a(z)\hat{\Phi}_b(x)\hat{\Phi}_c(y) > . \tag{23}$$

Making use of (6), equation (22) now yields

$$\partial_{[a}\beta_{b]} = -l^{2D} < \hat{\Phi}_{[a}(x)\partial_{b]}\hat{\Theta}(y) > |_{|x-y|=l} - S_{[ab]c}(l)\beta^c. \tag{24}$$

The vanishing of the right hand side of this equation provides integrability conditions for potential flow.

These conditions can be interpreted as follows. Since $\hat{\Phi}_a(x)$ are a basis for scalar operators of dimension $-D$, we can expand

$$\partial_b\hat{\Theta}(y) = \eta_b{}^c\hat{\Phi}_c(y). \tag{25}$$

(In principle there could be a term proportional to the identity operator on the right hand side of equation (25) but this does not affect the ensuing analysis and will be omitted.) This gives

$$\partial_{[a}\beta_{b]} = \overline{\eta}_{[ab]} - S_{[ab]c}\beta^c, \tag{26}$$

where the matrix $\overline{\eta}_{ab}$ is defined as

$$\overline{\eta}_{ab} = \eta_a{}^c\overline{G}_{cb}, \tag{27}$$

and S_{abc} is $S_{abc}(|x-y|)$ evaluated at $|x-y| = l$.
Now from (13) and (25)

$$\partial_a\Theta(x) = -l\frac{\partial}{\partial l}\Big|_g \hat{\Phi}_a(x) = \eta_a{}^c\hat{\Phi}_c(x). \tag{28}$$

It follows that $\eta_a{}^b$ can be thought of as the linear transformation matrix that generates the change in the basis $\hat{\Phi}_a(x)$ under an infinitesimal change in the renormalisation point, with the renormalised couplings kept fixed. Using the metric this linear transformation can be decomposed into a rotation, a shear and a dilation. Thus the condition for potential flow relates the rotational part of $\overline{\eta}_{ab}$ to the antisymmetric part of $S_{abc}\beta^c$,

$$\overline{\eta}_{[ab]} = S_{[ab]c}\beta^c. \tag{29}$$

Let us now examine the two sides of this equation in detail. Consider first the right hand side. As mentioned earlier, S_{abc} requires regularisation in order to be properly defined. It will now be shown that the antisymmetric part, $S_{[ab]c}$, is finite. Consider the three point correlator in (23), clearly it is symmetric in b and c keeping x and y fixed

and, furthermore, is independent of the direction of the vector $x-y$. Using (18) it can be written as

$$\begin{aligned} S_{abc}(|x-y|) &= l^{2D} \int d^D z\, C_{ab}{}^F(z-x) C_{Fc}{}^0\Big(\frac{z+x}{2} - y\Big) \\ &= l^{2D} \int d^D z\, C_{bc}{}^F(x-y) C_{aF}{}^0\Big(z - \frac{x+y}{2}\Big) \\ &= l^{2D} \int d^D z\, C_{ca}{}^F(y-z) C_{Fb}{}^0\Big(\frac{y+z}{2} - x\Big). \end{aligned} \tag{30}$$

Again, since $\hat{\Phi}_a(x)$ have dimension $-D$ by definition (5), simple dimensional analysis shows that $C_{ab}{}^0$ have length dimension $-2D$ and $C_{ab}{}^c$ have dimension $-D$ while $C_{ab}{}^F$ for other values of F have higher length dimensions.

Compare the first and third expressions on the right hand side of equation (30). First shift the integration variable $z \to z + x$ in the first expression. Now interchange b and c in the last expression on the right hand side and shift the integration variable $z \to y - z$. This gives, using (17)

$$\begin{aligned} S_{abc}(|x-y|) &= l^{2D} \int d^D z\, C_{ab}{}^F(z) C_{Fc}{}^0\Big(\frac{z}{2} + x - y\Big) \\ &= l^{2D} \int d^D z\, C_{ba}{}^F(z) C_{cF}{}^0\Big(\frac{z}{2} + x - y\Big). \end{aligned} \tag{31}$$

Since $S_{abc}(|x-y|)$ is independent of the direction of $x-y$ we can integrate $C_{cF}{}^0(\frac{z}{2}+x-y)$ over all directions, keeping $|x-y| = l$ fixed, and normalise by dividing out the volume of the unit S^{D-1} sphere. Denote the resulting integrated functions by $\overline{C}_{cF}{}^0(\frac{z}{2})$. Now from (17) it is clear that

$$\overline{C}_{cF}{}^0\Big(\frac{z}{2}\Big) = \overline{C}_{Fc}{}^0\Big(-\frac{z}{2}\Big). \tag{32}$$

Using this property in (31) yields the result

$$\begin{aligned} S_{[ab]c} &= l^{2D} \int d^D z\, C_{ab}{}^F(z)\Big[\overline{C}_{Fc}{}^0\Big(\frac{z}{2}\Big) - \overline{C}_{Fc}{}^0\Big(-\frac{z}{2}\Big)\Big] \\ &= l^{2D} \int d^D z\, C_{[ab]}{}^F(z)\overline{C}_{Fc}{}^0\Big(\frac{z}{2}\Big). \end{aligned} \tag{33}$$

From their definition $C_{0c}{}^0$ vanish, so only $F \neq 0$ appear in this expression. Since $\hat{\Phi}_a(x)$ are a basis for all scalar operators of length dimension $-D$, dimensional arguments show that the only terms in (33) which diverge as $\sim z^{-D}$ near the origin are those for which F is an Euclidean scalar, with $F = f = 1, \ldots, n$ labelling only couplings. All other possibilities for F diverge less strongly for $z \sim 0$ and so give a finite contribution to the integral at $z \sim 0$. $\overline{C}_{Fc}{}^0(z)$ are regular at the origin, provided $l \neq 0$. Thus the only singular terms in (33) are those with $F = f$. But $C_{ab}{}^f(z)$ is symmetric under $z \to -z$ or, equivalently, under interchange of a and b hence the singular terms in (33) cancel and $S_{[ab]c}$ is finite as claimed. Thus the integrability conditions (29) for potential flow with the metric (19) are well defined.

Now consider the left hand side of (29). We can obtain some information about $\overline{\eta}_{[ab]}$ by considering $l\frac{\partial}{\partial l}|_g\hat{\Theta} = -\beta^a\partial_a\hat{\Theta}$, which is the full dimension (canonical plus anomalous) of the operator $\hat{\Theta}$. The anomalous dimension of the stress operator vanishes because it consists of D conserved currents, one for each generator of translations in D Euclidean dimensions, and conserved currents do not get renormalised. $\hat{\Theta}$ has canonical dimension $-D$, hence

$$-l\frac{\partial}{\partial l}\bigg|_g \hat{\Theta} = \beta^a\partial_a\hat{\Theta} = D\hat{\Theta} = D\beta^a\hat{\Phi}_a. \tag{34}$$

This equation is derived, from general arguments, in [12] and it can be verified, to all orders in perturbation theory, for massive $\lambda\varphi^4$ in $D \leq 4$ using the techniques in [18], provided the improved stress tensor is used. Actually, since the stress operator is only conserved when the equations of motion are used, (34) is only true modulo the equations of motion. (Note that for the metric (19) any linear combination of $\hat{\Phi}_a$ which is proportional to the equations of motion must be removed from our basis set anyway, since otherwise the metric is degenerate.)

If we further assume that the basis for operators $\hat{\Phi}_a$ is linearly independent and that $\partial_a\hat{\Theta}$ are linearly independant operators too, and therefore also constitute a basis since they are the same in number as $\hat{\Phi}_a$, we can conclude from (34) that

$$\partial_a\hat{\Theta} = D\hat{\Phi}_a \qquad \Leftrightarrow \qquad \eta^a{}_b = D\delta^a{}_b. \tag{35}$$

It follows that

$$\overline{\eta}_{ab} = D\overline{G}_{ab} \tag{36}$$

and so

$$\overline{\eta}_{[ab]} = 0, \tag{37}$$

since $\overline{G}_{ab}$ is symmetric by definition. i.e. $\overline{\eta}_{ab}$ is irrotational. The assumption that $\partial_a\hat{\Theta}$ are linearly independent does not seem to be a particularly strong one - for instance equation (35) can be shown to be true for massive $\lambda\varphi^4$ theory in four dimensions. When this assumption holds the integrability conditions for the metric (19) reduce to the statement

$$S_{[ab]c}\beta^c = 0. \tag{38}$$

Now consider a parallel analysis applied to the other metric under consideration, (21). Then we have

$$\begin{aligned}\partial_a\beta_b &= l^D \int d^Dx\partial_a < \hat{\Phi}_b(x)\hat{\Theta}(y) > \\ &= l^D \int d^Dx < \partial_a\hat{\Phi}_b(x)\hat{\Theta}(y) > + l^D \int d^Dx < \hat{\Phi}_b(x)\partial_a\hat{\Theta}(y) > \\ &\quad - l^D\beta^c \int d^Dz \int d^Dx < \hat{\Phi}_a(z)\hat{\Phi}_b(x)\hat{\Phi}_c(y) >_; .\end{aligned} \tag{39}$$

The first and third terms in the last expression on the right hand side of this equation are manifestly symmetric under interchange of a and b. Hence the integrability conditions reduce to

$$\tilde{\eta}_{[ab]} = 0, \tag{40}$$

where

$$\tilde{\eta}_{ab} := \eta_a{}^c\tilde{G}_{cb}. \tag{41}$$

Thus the situation here is somewhat simpler - the condition of integrability of the RG flow is completely equivalent to the statement that $\tilde{\eta}_{ab}$ is irrotational.

Again, if it is assumed that $\partial_a\hat{\Theta}$ are linearly independent, equation (35) implies that (40) is identically true, since $\tilde{G}_{cb}$ is symmetric by definition. Hence, for the metric (21) potential flow is assured if $\partial_a\hat{\Theta}$ are linearly independent.

§5 Conclusions

For a local renormalisable quantum field theory in flat D-dimensional space integrability conditions on the β functions for the RG flow to be a potential flow have been derived. This requires the introduction of a metric on the space of local interactions and two possibilities,

$$\overline{G}_{ab}(g) = l^{2D} < \hat{\Phi}_a(x)\hat{\Phi}_b(y) > \big|_{|x-y|=l} \tag{19}$$

and

$$\tilde{G}_{ab}(g) = l^D \int d^Dy < \hat{\Phi}_a(y)\hat{\Phi}_b(x) >, \tag{21}$$

have been considered. These give rise to integrability conditions

$$\overline{\eta}_{[ab]} = S_{[ab]c}\beta^c \tag{29}$$

and

$$\tilde{\eta}_{[ab]} = 0 \tag{40}$$

respectively, where $\overline{\eta}_{ab} = \eta_a{}^c\overline{G}_{cb}$ and $\tilde{\eta}_{ab} = \eta_a{}^c\tilde{G}_{cb}$ are determined by the linear transformation matrix $\eta_a{}^b$ defined in equation (28). If it is further assumed that $\partial_a\hat{\Theta}$ form a linearly independent set of operators then (40) is identically satisfied and (29) reduce to the statement that the antisymmetric part of the three point function (23) contracted with β^c vanishes,

$$S_{[ab]c}\beta^c = 0. \tag{38}$$

The nature of the potential itself has not been addressed here but this is clearly of prime importance since, if it could be constructed, it would provide a higher dimensional analogue of the two dimensional concept of the central charge of the theory. It has been suggested that if the RG flow is a potential flow then the effective action should provide the required potential [8]. A great deal more work must be done before this hypothesis can be either confirmed or refuted.

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References

[1] A.B. Zamolodchikov, Pis'ma Zh. Eksp. Teor. Fiz.**43**, (1986) 565 (JETP Lett. **43**, (1986), 730)

[2] J.L. Cardy, Phys. Lett. B **215**, (1988), 749

[3] A. Cappelli, D. Friedan, J.I. Latorre, Nuc. Phys. **B352**, (1991) 616

[4] G.M. Shore, Phys. Lett. B **53**, (1991), 380
Phys. Lett. B **256**, (1991), 407

[5] I. Jack, Renormalisation Group '91, 2nd International Conference, Dubna, USSR Ed. D.V. Shirkov and V.B. Priezzhev (1991) World Scientific

[6] D.J. Wallace and R.K.P. Zia, Phys. Lett. **48A**, (1974), 325

[7] D.J. Wallace and R.K.P. Zia, Ann. Phys. **92**, (1975), 142

[8] T. Banks and E. Martinec, Nucl. Phys. **B249**, (1987), 142

[9] B.P. Dolan, *Potential Flow Of The Renormalisation Group In A Simple Model* in preparation

[10] S. Amari, *Differential-Geometric Methods In Statistics* Lecture Notes In Statistics, Vol 28, Springer-Verlag (1985)

[11] D. O'Connor and C.R. Stephens, *Geometry The Renormalisation Group And Gravity* D.I.A.S. Preprint (1992)

[12] A.B Zamalodchikov, Rev. Math. Phys. **1**, (1990), 197

[13] K.G. Wilson and J. Kogut, Phys. Rep. **12C**, (1974), 75

[14] J. Polchinski, Nuc. Phys. **B231** (1984) 269

[15] K.G. Wilson, Phys. Rev. **179** , (1968), 1499

[16] A.M. Polyakov, Zh. Eksp. Teor. Fiz.**66**, (1974), 23 (Sov. Phys. JETP **39**, (1974), 10)

[17] A.A. Belavin, A.M. Polyakov and A.B. Zamalodchikov, Nuc. Phys. **B241**, (1984), 333

[18] L.S. Brown, Ann. Phys. **126**, (1979), 135