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Q–CREATION AND ANNIHILATION TENSORS FOR THE TWO PARAMETERS DEFORMATION OF U(SU(2))

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ABSTRACT. The Jordan–Schwinger construction for the Hopf algebra $U_{qp}(su(2))$ is realized. The creation and annihilation tensor operators together with their tensor products including the Casimir operators are calculated.

1. Introduction

Symmetries that are considered in classical mechanics always build a group of transformations. Several authors showed that a more general mathematical structure than a group can be used for the symmetry in quantum mechanics. From the requirement of compatibility with the statistics (Braid statistics) they are lead to consider quasitriangular Hopf algebras, i.e., quantum groups (see [1] and the references therein) or even more general quantum symmetries as discussed in [2].

Since the quantum symmetries include a wider class of operations, it is natural to investigate its implications on well established symmetry methods which considered only groups of transformations. We are interested here in the theory of tensor operators. In the last three years several papers [3-5] treating tensors for Hopf algebras have appeared; recently, Rittenberg and Scheunert [6] have developed a complete theory for tensor operators and shown as an application the construction of the modified q–creation and annihilation tensors for $U_q(su(2))$. The explicit construction of particular cases following [6] however, constitutes an additional effort.

We study here the Jordan–Schwinger construction for the Hopf algebra $U_{qp}(su(2))$. The two parameters deformed algebra appears to be better adapted for this construction than the one parameter deformed case $U_q(su(2))$. The two types of oscillators required emerge in a natural way in the former case: one oscillator is of qp–type, the other of pq–type. For this two parameters deformation of the $su(2)$ symmetry we then introduce the creation and annihilation q–tensor operators. Further, using standard properties of the tensor product, the modified tensor generators for $U_{qp}(su(2))$, $T_\pm$ and T_0 together with its Casimir operators are calculated.

The paper is organized in 5 sections. After the introduction, in section 2 we briefly describe the quantum symmetry considered here and introduce the quantum groups $U_q(su(2))$ and $U_{qp}(su(2))$. In section 3, following McFarlane and Biedenharn [7], the Jordan–Schwinger construction for $U_{qp}(su(2))$ is presented. In section 4, $q-$tensor operators are introduced and their general properties recalled. In section 5, the creation and annihilation tensor operators are constructed, further, tensor products of these operators are studied. An Appendix containing the explicit calculation of the tensor properties of these operators is included. Finally a short conclusion in section 6 summarizes the results.

2. Quantum symmetry

The quantum symmetry considered here is defined in the following manner [2]: Consider a quantum mechanical system with Hamiltonian H whose Hilbert space of states $\mathcal{H}$ is generated from the ground state $|0>$ by the field operators b_i. A Hopf algebra A with unit element e, co–product Δ, co–unit ϵ and antipode S is called a symmetry of the system if

- $\mathcal{H}$ carries a unitary representation U of A,

- the ground state $|0>$ is invariant: $U(g)|0>=|0>\epsilon(g)$,

- all representation operators U(g) commute with the Hamiltonian,

- the field operators transform covariantly, i.e.,

$$U(g)b_i = \sum_m \sum_p b_m D_{mi}(g_p^1)U(g_p^2),$$

here D_{mi} are the matrix elements of a finite dimensional representation of A and

$$\Delta(g) = \sum_p g_p^1 \otimes g_p^2$$

where Δ is the co–product in A. The rank of the tensor operator b is equal to the rank of the representation matrix D.

As mentioned in the introduction we are interested here in the quantum symmetries given by the Hopf algebras $U_q(su(2))$ and $U_{qp}(su(2))$.

The Hopf algebra $U_q(su(2))$ is an associative algebra generated by the operators:

$$J_+, \qquad J_-, \qquad q^{\frac{H}{2}}, \qquad q^{-\frac{H}{2}}, \qquad 1$$

which obey the following relations:

$$q^{\frac{H}{2}}q^{-\frac{H}{2}} = q^{-\frac{H}{2}}q^{\frac{H}{2}} = 1,$$

$$q^{\frac{H}{2}}J_\pm q^{-\frac{H}{2}} = q^{\pm}J_\pm,$$

$$[J_+, J_-] = [H] = \frac{q^H - q^{-H}}{q - q^{-1}}.$$

Its co–product is

$$\Delta(J_\pm) = J_\pm \otimes q^{\frac{H}{2}} + q^{-\frac{H}{2}} \otimes J_\pm$$

and

$$\Delta(q^{\pm\frac{H}{2}}) = q^{\pm\frac{H}{2}} \otimes q^{\pm\frac{H}{2}}.$$

Its co–unit is

$$\epsilon(J_\pm) = 0 \qquad \epsilon(q^{\pm\frac{H}{2}}) = 1$$

and its antipode is given by

$$S(J_\pm) = -q^{\pm} J_\pm \,, \qquad S(q^{\pm\frac{H}{2}}) = q^{\mp\frac{H}{2}} \quad .$$

The two parameters deformation of $su(2)$, the Hopf algebra $U_{qp}(su(2))$ is also generated by the same generators as above, together with a new "scalar" parameter p which behaves similar to q:

$$p^{\frac{H}{2}} p^{-\frac{H}{2}} = p^{-\frac{H}{2}} p^{\frac{H}{2}} = 1 \qquad \text{and} \qquad p^{\frac{H}{2}} J_\pm p^{-\frac{H}{2}} = p^{\pm} J_\pm.$$

Also the commutator is slightly different:

$$[J_+, J_-]_{qp} := J_+ J_- - pq^{-1} J_- J_+ = [H]_{qp},$$

where the qp-brackets are:

$$[n]_{qp} = \frac{q^n - p^{-n}}{q - p^{-1}}.$$

The co–product takes the form:

$$\Delta(J_\pm) = J_\pm \otimes q^{\frac{H}{2}} + p^{-\frac{H}{2}} \otimes J_\pm$$

and the antipode is given by:

$$S(q^{\pm\frac{H}{2}}) = q^{\mp\frac{H}{2}} \qquad S(p^{\pm\frac{H}{2}}) = p^{\mp\frac{H}{2}}$$

$$S(J_+) = -q \left(\frac{q}{p}\right)^{-\frac{H}{2}} J_+ \qquad S(J_-) = -q^{-1} \left(\frac{q}{p}\right)^{-\frac{H}{2}} J_- .$$

Parallel to the qp deformation, there exists the pq deformation which results by interchanging the roles of q and p. The pq deformation is quite different from the qp version. In fact, we have another bracket, also the commutators are different. The total overlay of the two algebras only occurrs when $q = p$. The representation and the Clebsch-Gordan coefficients of $U_{qp}(su(2))$ can be found in [8] and in the references therein.

3. q– and qp–oscillators

$U_q(su(2))$ has being constructed in the Jordan–Schwinger representation in [7]. We recall here the main features of this representation to proceed later with the similar construction in the two parameters deformed case.

In $U_q(su(2))$ the creation and annihilation operators have the following properties:

$$b|0>= 0, \qquad b|n>= \sqrt{[n]}|n-1>, \qquad b^+|n>= \sqrt{[n+1]}|n+1>,$$

where [ ] are the q–brackets. The number of particles operator N is defined by:

$$N|n>= n|n> .$$

It follows that

$$bb^+ = [N+1], \qquad b^+b = [N], \qquad [N, b^+] = b^+ \qquad \text{and} \qquad [N, b] = -b.$$

These operators obey the q–deformed boson commutation relations:

$$bb^+ - qb^+b = q^{-N} \qquad \text{or} \qquad bb^+ - q^{-1}b^+b = q^N.$$

It is an exercise to see that for two independent q–oscillators, b_1 and b_2, the operators :

$$J_+ = b_1^+ b_2, \quad J_- = b_2^+ b_1, \quad H = N_1 - N_2$$

generate the Hopf algebra $U_q(su(2))$.

We now introduce the two parameters deformed oscillators to construct $U_{qp}(su(2))$. The qp–oscillator is a two parametric issue of the deformation of the classical oscillator or Weyl-Heisenberg algebra. To define the qp–oscillator we consider the operator b and its hermitian conjugated b^+ together with the number of particles operator N obeying the qp–bosons commutation relation:

$$bb^+ - qb^+b = p^{-N}.$$

As usual, we require that b destroys the ground state i.e.,

$$b|0>= 0.$$

For the number of particles operator we have again $N|n >= n|n >$. The action of b^+ is given by $b^+|n >= \mathcal{N}_{n+1}|n+1>$ where $\mathcal{N}_n$ is a constant determined in the following manner: using the commutation relation, by induction it follows,

$$b(b^+)^{n+1} - q^{n+1}(b^+)^{n+1}b = \sum_{i=0}^{n}(qb^+)^i p^{-N}(b^+)^{n-i}.$$

Then, the expectation value of this equation on the state with n particles allows us to write down the recurrence relation for $\mathcal{N}_n$. Choosing a suitable phase factor we find :

$$\mathcal{N}_n = \sqrt{[n]_{qp}}.$$

We also have again

$$bb^+ = [N+1]_{qp}, \qquad b^+b = [N]_{qp}, \qquad [N, b^+] = b^+ \qquad \text{and} \qquad [N, b] = -b.$$

The dual pq–oscillator having the commutation relation

$$aa^+ - pa^+a = q^{-N}$$

which is related with the qp–oscillator through the replacement $q \leftrightarrow p$ can be constructed in a total analogous way.

Now, in order to construct $U_{qp}(su(2))$ in the Jordan–Schwinger representation, we consider the two independent deformed oscillators: one of qp–type, b_1, and the other of pq–type, b_2. The following combinations generate then $U_{qp}(su(2))$:

$$J_+ = b_1^+ b_2, \qquad J_- = b_2^+ b_1, \qquad H = N_1 - N_2.$$

Note that the presence of the two independent bosonic operators required for the construction of the Jordan–Schwinger representation appear in a natural way in $U_{qp}(su(2))$. Further, also note that the order of the bosons in the above construction is essential. We take first a qp-boson and second a pq-boson in order to obtain the qp-deformation of

$su(2)$. To obtain its pq-deformation the order has to be reversed. The non–commutative character of the background geometry thus emerges in this construction.

Concluding this section, we state the following relations valid for both types of bosons:

$$c^+ f(N) = f(N-1)c^+ \qquad \text{and} \qquad cf(N) = f(N+1)c$$

where f is an arbitrary function of the operator N and c is a qp– or a qp–boson operator. These relations which immediately follow from the properties of the boson operators are very useful when performing any explicit calculation involving q–tensors.

4. General statements about tensor operators

In this section we present some general aspects about tensor operators for the quantum symmetries [3-6]. Tensor operators are operators which transform covariantly under the representations of the considered Hopf algebra. The properties of deformed tensors are similar to those of the classical tensors, however, for certain values of the deforming parameters of $su(2)$ the tensor product of irreducible representations are not fully reducible like in classical semisimple algebras. These special values of the parameters are excluded here, i.e., neither q nor $\sqrt{pq}$ are roots of the unity.

We now draw our attention to the representations of $U_q(su(2))$ and $U_{qp}(su(2))$. For both examples the finite representation are completely reducible and are characterized by integer or by half-integer numbers l. If the representation D^l corresponds to the integer

l then the carrier space V_l of D^l has an orthogonal basis e_m , $m = -l, -l+1, \ldots, l$, such that for $U_q(su(2))$:

$$U(J_\pm)e_m = \sqrt{[l \mp m][l \pm m + 1]}e_{m\pm 1} \qquad He_m = me_m$$

and for $U_{qp}(su(2))$:

$$U(J_\pm)e_m = \sqrt{[l \mp m]_{pq}\,[l \pm m + 1]_{qp}}\, e_{m\pm 1}.$$

As in the classical case we have the expansion :

$$|ln> = \sum_{m+m'=n} <jm\ km' \mid ln> |jm> \otimes |km'>$$

which defines the deformed Clebsch–Gordan coefficients of the tensor product $D^j \otimes D^k$. For the one parameter deformation a rigorous calculus is given [9] and for $U_{qp}(su(2))$ in [8][1].

Also, we note that as in the classical case, the following equation is valid:

$$\sum_p \sum_{m+m'=n} <jm\ km' \mid ln> D^j_{rm}(g^1_p) D^k_{sm'}(g^2_p) = \sum_o D^l_{on}(g) <jr\ ks \mid lo>. \qquad (*)$$

For the convenience of the reader, we now state two simple propositions about the deformed tensor product which in the sequel is simply refered as tensor product. These propositions are used later in the paper to construct further tensor operators.

[1] In ref. 8 a slightly different co–product is used, $(q \leftrightarrow p^{-1})$. This is to be taken into account when using the Clebsch-Gordan coefficients defined there.

PROPOSITION 1. *The tensor product of tensor operators is a tensor operator.*

PROOF: First, let as recall the definition of the deformed tensor product. For this purpose let t_m be a covariant tensor operator of rank j and t'_m, a covariant tensor operator of rank k. The tensor product of these operators is:

$$T_n = (t \otimes t')^l_n = \sum_{m+m'=n} < jm\ km' \mid ln > t_m t'_{m'},$$

for the rank l we have the same range as in the classical case.

The transformation law for T_n is then

$$U(g)T_n = \sum_{m+m'=n} < jm\ km' \mid ln > \sum_{p,q,r,s} D^j_{rm}(g^1_p) D^k_{sm'}(g^{21}_{pq}) t_r t'_s U(g^{22}_{pq}).$$

Using the co–associativity property, $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$ we obtain

$$U(g)T_n = \sum_{m+m'=n} < jm\ km' \mid ln > \sum_{p,q,r,s} D^j_{rm}(g^{11}_{pq}) D^k_{sm'}(g^{12}_{pq}) t_r t'_s U(g^2_p)$$

and with (*)

$$U(g)T_n = \sum_{o,r,s,p} D^l_{on}(g^1_p) < jr\ ks \mid lo > t_r t'_s U(g^2_p).$$

Further, from the definition of T_n, it follows

$$U(g)T_n = \sum_{o,p} D^l_{on}(g^1_p) T_o U(g^2_p),$$

showing that T_n is indeed a tensor operator of rank l.

PROPOSITION 2. *The scalar tensor operator is an invariant operator.*

PROOF: For C a scalar operator, i.e., C a 0–rank tensor operator, the transformation law gives:

$$U(g)C = \sum_p D^0(g^1_p) C U(g^2_p),$$

Now, the 0–rank representation is given by the co–unit ϵ. Then

$$\sum_p D^0(g^1_p) C U(g^2_p) = \sum_p \epsilon(g^1_p) C U(g^2_p) = CU\Big(\sum_p \epsilon(g^1_p) g^2_p\Big)$$

and using the co–unit property,i.e.:$(\epsilon \otimes id)\Delta = id$,we obtain:

$$U(g)C = CU(g).$$

5. Creation and annihilation tensor operators

In the standard IBM method **[10]** the 2l+1 quantum oscillator operators are endowed with spherical tensor structure in the following manner: the annihilation operators are changed to

$$\tilde{b}_{lm} = (-1)^{l-m} b_{l,-m} \quad (m = -l, -l+1, \ldots, l),$$

the creation operator $\tilde{b}^+_{lm} = b^+_{lm}$ are kept because they already are tensors.

In this section we show how to endow the pair of quantum oscillators generating $U_{qp}(su(2))$ with tensor structure.

First we recall that l–rank tensor operators obey the following transformation law:

$$U(J_\pm)t_m = \sum_n D^l_{nm}(J_\pm)t_n U(q^{\frac{H}{2}}) + D^l_{nm}(q^{-\frac{H}{2}})t_n U(J_\pm),$$
$$U(q^{\frac{H}{2}})t_m = \sum_n D^l_{nm}(q^{\frac{H}{2}})t_n U(q^{\frac{H}{2}}).$$

for $U_q(su(2))$, and for $U_{qp}(su(2))$,

$$U(J_\pm)t_m = \sum_n D^l_{nm}(J_\pm)t_n U(q^{\frac{H}{2}}) + D^l_{nm}(p^{-\frac{H}{2}})t_n U(J_\pm),$$
$$U(s^{\frac{H}{2}})t_m = \sum_n D^l_{nm}(s^{\frac{H}{2}})t_n U(s^{\frac{H}{2}})$$

where s is p or q.

Let us first discuss the quantum symmetry $U_q(su(2))$ (see [3,6,11]). Here it is easily verified that neither b^+ nor b are tensor operators. Thus we see that for quantum symmetries both the creation and annihilation operators must be redefined in order to obtain tensor operators. The one parameter deformed spherical tensors have been found:

$$\begin{cases} \tilde b_1^+ = q^{-\frac{N_2}{2}} b_1^+ \\ \tilde b_2^+ = q^{\frac{N_1}{2}} b_2^+ \end{cases} \qquad \begin{cases} \tilde b_1 = q^{\frac{N_1+1}{2}} b_2 \\ \tilde b_2 = -q^{-\frac{N_2+1}{2}} b_1. \end{cases}$$

Note that actually the most general form for the annihilation tensor operators is:

$$\begin{cases} \tilde b_1 = q^{\alpha} q^{\frac{N_1+1}{2}} b_2 \\ \tilde b_2 = -q^{\beta} q^{-\frac{N_2+1}{2}} b_1 \end{cases}$$

with $\alpha - \beta = 1$. The choice above, $\alpha = 1/2$, $\beta = -1/2$, is the most symmetrical one.

Using the propositions of the preceding section we now construct the tensor operators of rank 0 and one as bilinear combinations of the elementary tensors $\tilde b$.

We define the invariant operator $\mathcal{N}$ as:

$$\mathcal{N} = (\tilde b^+ \otimes \tilde b)^0 = \sum_{m,m'} < \tfrac12 m\ \tfrac12 m' \mid 00 > \tilde b^+_m \tilde b_{m'}$$

where $b_{1/2} := b_1$ and $b_{-1/2} := b_2$.

After some calculation, it follows,

$$\mathcal{N} = \frac{[N]}{\sqrt{[2]}} \qquad \text{where } N = N_1 + N_2.$$

The rank one tensor operator T is defined by

$$T_l = (\tilde b^+ \otimes \tilde b)^1_l = \sum_{m,m'} < \tfrac12 m\ \tfrac12 m' \mid 1l > \tilde b^+_m \tilde b_{m'}.$$

We then have:

$$T_+ = T_1 = q^{\frac{H}{2}} J_+$$
$$T_- = T_2 = -q^{\frac{H}{2}} J_-$$
$$T_0 = \frac{1}{\sqrt{[2]}}(q^{N_1+1}[N_2] - q^{-N_2-1}[N_1]).$$

This operator allows us to obtain the invariant Casimir operator:

$$\mathcal{C} = (T \otimes T)^0 = \sum_{m,m'} < 1m\ 1m' \mid 00 > T_m T_{m'}.$$

Explicitly we have,

$$\mathcal{C} = \frac{1}{\sqrt{[3]}}(qT_+T_- + q^{-1}T_-T_+ - T_0^2).$$

To find eigenvalues of this q–Casimir, we write $\mathcal{C}$ in the form form:

$$\mathcal{C} = -\frac{1}{\sqrt{[3]}}\left(2q^H J_- J_+ + q^H[H] + \frac{1}{[2]}\left(\frac{q^{N+1}+q^{-N-1}}{q-q^{-1}} - q^H\frac{q+q^{-1}}{q-q^{-1}}\right)^2\right)$$

Acting with C on the maximal weight state $|jj>$ the eigenvalues, $c(n,j)$, depending on j and n follow:

$$c(n,j) = q^2 \sqrt{\frac{q^2-1}{q^6-1}} \left(\frac{q^{4j}-1}{q^2-1} + \frac{1}{q^2+1} \left(\frac{q^{n+2}+q^{-n}-q^{2j+2}-q^{2j}}{q^2-1} \right)^2 \right).$$

Note that $c(n,j)$ has a non-linear dependence on the quantum numbers j and n. In the limit $q \longrightarrow 1$ the eigenvalues are proportional to the classical values $j(j+1)$.

In figure 1, these eigenvalues have been plotted for several values of the deformation parameter.

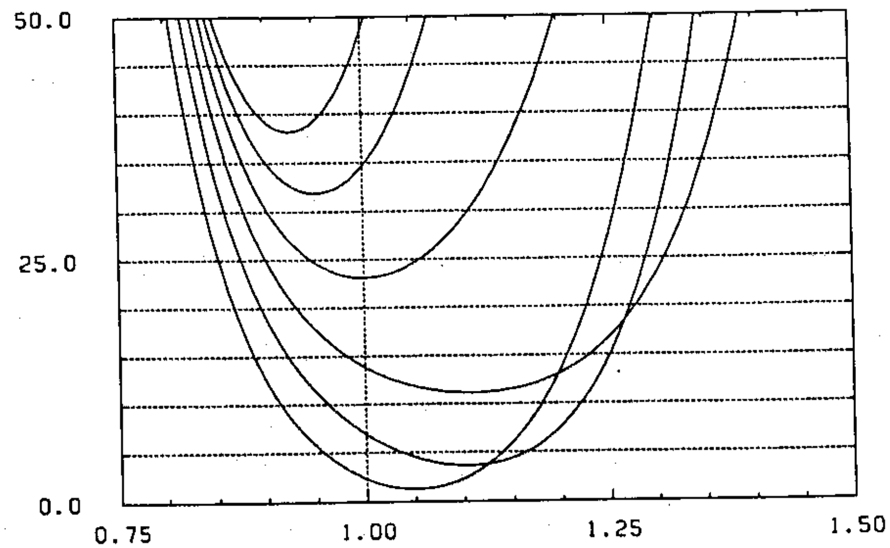

Figure 1a

The eigenvalues $c(n,j)$ of the Casimir operator for the Hopf algebra $U_q(su(2))$ for $q \in [0.75, 1.5]$, $n = 4$ and $j = 1,2,3,4,5$ and 6. The dotted line are drawn to facilitate the comparison of the quantum case with the classical one (q=1), $c_{q=1}(4,j) = \frac{2}{\sqrt{3}} j(j+1)$.

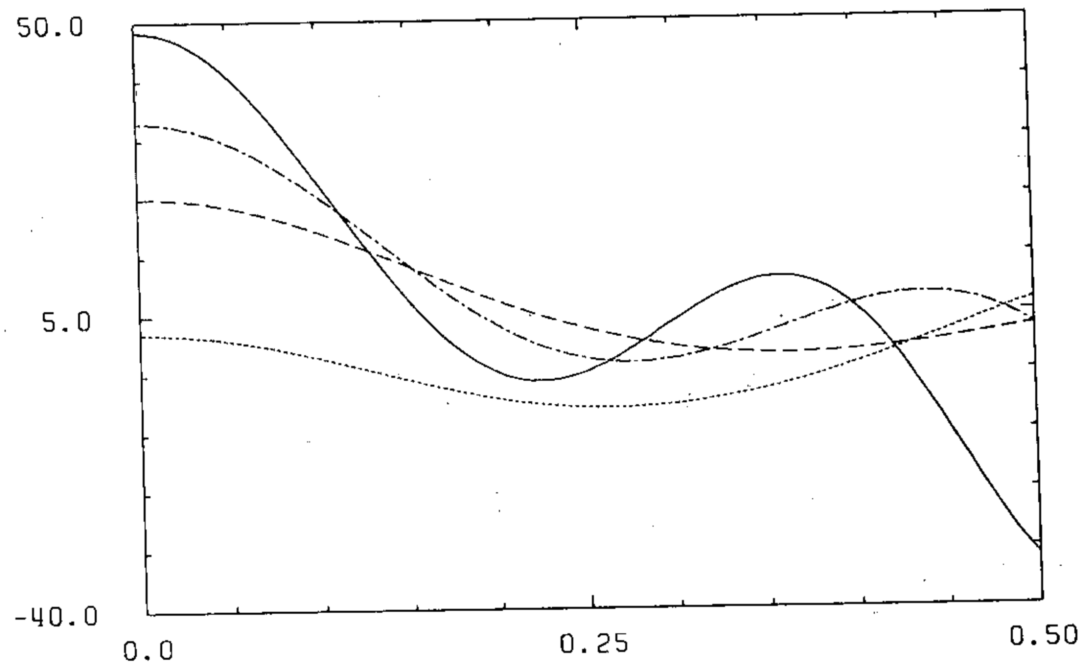

Figure 1b

Real part of the eigenvalues $c(n,j)$ of the Casimir operator for the Hopf algebra $U_q(su(2))$for $q = 1+it$, with $t \in [0., 0.5]$, $n = 4$, $j = 1$ (dotted line), $j = 4$ (dashed line), $j = 5$ (dashed–dotted line) and $j = 6$ (full line).

Solving the set of equations defining the covariance of a tensor operator, we find for $U_{qp}(su(2))$ that the deformed spherical tensors are given by:

$$\begin{cases} \tilde{b}_1^+ = p^{-\frac{N_2}{2}} b_1^+ \\ \tilde{b}_2^+ = q^{\frac{N_1}{2}} b_2^+ (qp^{-1})^{\frac{N_2}{2}} \end{cases} \qquad \begin{cases} \tilde{b}_1 = p^{\frac{N_1+1}{2}} b_2 \\ \tilde{b}_2 = -q^{-\frac{N_2+1}{2}} b_1 (pq^{-1})^{\frac{N_1-1}{2}} . \end{cases}$$

The proof of the tensor properties of these operators and of the following results involving their tensor products is given in the Appendix.

As for $U_q(su(2))$, we construct now from these basic tensor operators tensor operators of rank zero and one. To build tensor products of these operators the following Clebsch–Gordan coefficients are necessary[8][1]:

$$\begin{aligned}
<\tfrac12\tfrac12\ \tfrac12-\tfrac12 \mid 10> &= <\tfrac12-\tfrac12\ \tfrac12\tfrac12 \mid 00> = \frac{p^{-\frac12}}{\sqrt{[2]_{qp}}} \\
<\tfrac12-\tfrac12\ \tfrac12\tfrac12 \mid 10> &= -<\tfrac12\tfrac12\ \tfrac12-\tfrac12 \mid 00> = \frac{q^{\frac12}}{\sqrt{[2]_{qp}}} \\
<\tfrac12\tfrac12\ \tfrac12\tfrac12 \mid 11> &= <\tfrac12-\tfrac12\ \tfrac12-\tfrac12 \mid -1-1> = 1 \\
<11\ 1-1 \mid 00> &= (qp)^{-1} <1-1\ 11 \mid 00> = q \\
& \quad <10\ 10 \mid 00> = -(qp^{-1})^{\frac12} .
\end{aligned}$$

The invariant total number of particles operator becomes with $N := N_1 + N_2$ (see Appendix),

$$\mathcal{N} = \frac{[N]_{qp}}{\sqrt{[2]_{qp}}} (pq^{-1})^{\frac{N-1}{2}}$$

or in a more suggestive form :

$$\mathcal{N} = -\frac{1}{\sqrt{[2]_{qp}}} \ \frac{(pq)^{\frac{N}{2}} - (pq)^{-\frac{N}{2}}}{(pq)^{\frac12} - (qp)^{-\frac12}} .$$

[1] see the footnote before

In this later form the equivalence of the parameters p and q becomes evident. Replacing $\sqrt{pq} \longrightarrow Q$ the one parameter deformed total number of particles operator follows. This equivalence however, is no longer valid for the rank one tensor:

$$T_+ = T_1 = p^{\frac{H}{2}} J_+ ,$$

$$T_- = T_2 = -(pq^{-1})^{\frac12} p^{\frac{H}{2}} J_- ,$$

$$T_0 = \frac{1}{\sqrt{[2]_{qp}}} p^{\frac{H}{2}} (pq^{\frac{N}{2}} [N_2]_{pq} - p^{-1} q^{-\frac{N}{2}} [N_1]_{qp}).$$

T_0 can also be written in the following form which allows checking, for q=p, the correspondence with the one parameter deformed case,

$$T_0 = \frac{1}{\sqrt{[2]_{qp}}} ((qp^{-1})^{\frac{N}{2}} p^{N_1+1} [N_2]_{pq} - (pq^{-1})^{\frac{H}{2}} q^{-N_2} p^{-1} [N_1]_{qp}).$$

In fact at any stage the correspondence

$$U_{qp}(su(2)) \xrightarrow{q=p} U_q(su(2)) \xrightarrow{q=1} U(su(2))$$

is valid.

To conclude this section we construct the Casimir operator:

$$\mathcal{C} = (T \otimes T)^0 .$$

Using the explicit values for the Clebsch–Gordan coefficients, we find

$$\mathcal{C} = \frac{1}{\sqrt{[3]_{qp}}} (qT_+T_- + p^{-1}T_-T_+ - (qp^{-1})^{\frac12} T_0^2).$$

The eigenvalues of this Casimir depend again on the quantum numbers n and j:

$$c(n, j) =$$

$$\sqrt{qp\frac{qp-1}{(qp)^3-1}\left(q\frac{(qp)^{2j}-1}{qp-1}+\frac{p}{qp+1}\left(\frac{(pq)^{1+n/2}-(pq)^{j+1}+(pq)^{-n/2}-(pq)^{j}}{pq-1}\right)^2\right)}.$$

In figure 2, these eigenvalues have been plotted for several values of the deformation parameters.

6. Conclusion

We have provided here the Jordan–Schwinger construction for $U_{qp}(su(2))$. The two types of oscillators required are chosen such that one is of qp–type, the other of pq–type. The q–creation and annihilation tensor operators have been introduced. Also the tensor products of these bosonic operators including the Casimir operators have been studied and the resulting eigenvalues plotted. The richness in the spectra obtained suggests several applications to account for dynamical symmetries close to the spherical one like those developed in [**12-15**].

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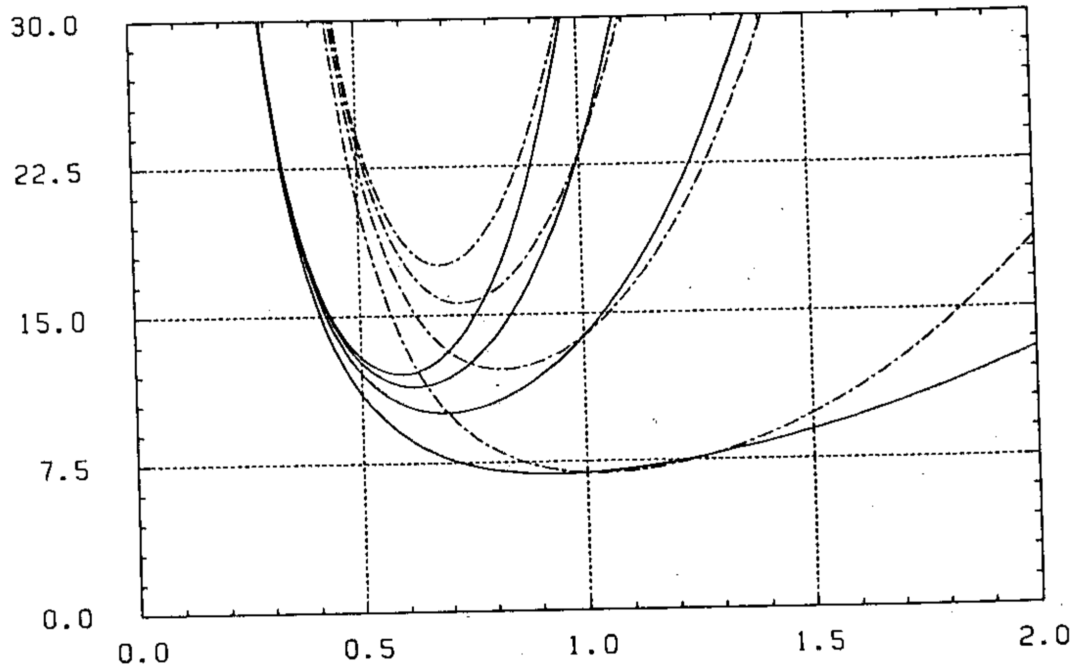

Figure 2

The eigenvalues $c(n,j)$ of the Casimir operator for the Hopf algebra $U_{qp}(su(2))$for $n=4$ and $j=2,3,4$ and 5. The full line corresponds to $p=1$ and $q\in[0.,2.]$. The dotted line corresponds to $q=1$ and $p\in[0.,2.]$.

Appendix

A.- Tensor properties of the $U_{qp}(su(2))$ boson operators $\tilde{b}$ and $\tilde{b}^+$

Tensor operators of rank l in $U_{qp}(su(2))$ obey the following transformation law:

$$U(J_\pm)t_m = \sum_n D^l_{nm}(J_\pm)t_n U(q^{\frac{H}{2}}) + D^l_{nm}(p^{-\frac{H}{2}})t_n U(J_\pm),$$

$$U(s^{\frac{H}{2}})t_m = \sum_n D^l_{nm}(s^{\frac{H}{2}})t_n U(s^{\frac{H}{2}}).$$

where s is p or q.

Here, we prove that the operators, $\tilde{b}_1^+$, $\tilde{b}_2^+$, $\tilde{b}_1$ and $\tilde{b}_2$ generating $U_{qp}(su(2))$ where

$$\tilde{b}_1^+ = p^{-\frac{N_2}{2}} b_1^+ \qquad \tilde{b}_1 = p^{\frac{N_2+1}{2}} b_2$$

and

$$\tilde{b}_2^+ = q^{\frac{N_1}{2}} b_2^+ (qp^{-1})^{\frac{N_2}{2}} \qquad \tilde{b}_2 = -q^{-\frac{N_2+1}{2}} b_1 (pq^{-1})^{\frac{N_1-1}{2}},$$

have tensorial structure.

For this purpose we note that in the Jordan–Schwinger realization $U(J_+) = b_1^+ b_2$, $U(J_-) = b_2^+ b_1$ and $U(H) = N_1 - N_2$. Further, the representation matrices $D^{\frac{1}{2}} \equiv D$ are given by:

$$D(J_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad D(J_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad D(H) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We beging showing that $\tilde{b}_1^+$ is a tensor operator.

The transformation law for $U(J_+)$ reads,

$$\begin{aligned} b_1^+ b_2 \tilde{b}_1^+ =& D_{11}(J_+)\tilde{b}_1^+ q^{\frac{N_1-N_2}{2}} + D_{11}(p^{-\frac{H}{2}})\tilde{b}_1^+ b_1^+ b_2 \\ &+ D_{21}(J_+)\tilde{b}_2^+ q^{\frac{N_1-N_2}{2}} + D_{21}(p^{-\frac{H}{2}})\tilde{b}_2^+ b_1^+ b_2 \\ =& p^{-\frac{1}{2}}\tilde{b}_1^+ b_1^+ b_2, \end{aligned}$$

and using the explicit expression for $\tilde{b}_1^+$, we obtain,

$$b_1^+ b_2 p^{-\frac{N_2}{2}} b_1^+ = p^{-\frac{1}{2}} p^{-\frac{N_2}{2}} b_1^+ b_1^+ b_2.$$

The validity of this equation follows now from the equation given in section 3,

$$b_2 f(N_2) = f(N_2+1) b_2.$$

The transformation law for $U(J_-)$ reads,

$$\begin{aligned} b_2^+ b_1 \tilde{b}_1^+ =& D_{11}(J_-)\tilde{b}_1^+ q^{\frac{N_1-N_2}{2}} + D_{11}(p^{-\frac{H}{2}})\tilde{b}_1^+ b_2^+ b_1 \\ &+ D_{21}(J_-)\tilde{b}_2^+ q^{\frac{N_1-N_2}{2}} + D_{21}(p^{-\frac{H}{2}})\tilde{b}_2^+ b_2^+ b_1 \\ =& p^{-\frac{1}{2}}\tilde{b}_1^+ b_2^+ b_1 + \tilde{b}_2^+ q^{\frac{N_1-N_2}{2}}, \end{aligned}$$

and using the explicit expressions for $\tilde{b}_1^+$ and for $\tilde{b}_2^+$, it follows,

$$b_2^+ b_1 p^{-\frac{N_2}{2}} b_1^+ = p^{-\frac{1}{2}} p^{-\frac{N_2}{2}} b_1^+ b_2^+ b_1 + q^{\frac{N_1}{2}} b_2^+ (qp^{-1})^{\frac{N_2}{2}} q^{\frac{N_1-N_2}{2}}.$$

With $b_2^+ f(N_2) = f(N_2-1) b_2^+$ it becomes,

$$p^{-\frac{N_2-1}{2}} b_1 b_1^+ b_2^+ = p^{-\frac{N_2-1}{2}} p^{-1} b_1^+ b_1 b_2^+ + p^{-\frac{N_2-1}{2}} q^{N_1} b_2^+.$$

The validity of this equation follows now from the qp–bosons commutation relation,

$$b_1 b_1^+ - q b_1^+ b_1 = p^{-N_1},$$

or better from its equivalent relation $(q \leftrightarrow p^{-1})$,

$$b_1 b_1^+ = p^{-1} b_1^+ b_1 + q^{N_1}.$$

The transformation law for $U(s^{\frac{H}{2}})$

$$s^{\frac{N_1-N_2}{2}}\bar{b}_1^+ = D_{11}(s^{\frac{H}{2}})\bar{b}_1^+ s^{\frac{N_1-N_2}{2}} + D_{21}(s^{\frac{H}{2}})\bar{b}_2^+ s^{\frac{N_1-N_2}{2}} = s^{\frac{1}{2}}\bar{b}_1^+ s^{\frac{N_1-N_2}{2}}$$

using the explicit expression for $\bar{b}_1^+$, becomes,

$$s^{\frac{N_1-N_2}{2}}p^{-\frac{N_2}{2}}b_1^+ = s^{\frac{1}{2}}p^{-\frac{N_2}{2}}b_1^+ s^{\frac{N_1-N_2}{2}}.$$

The validity of this equation follows again from $b_1^+ f(N_1) = f(N_1-1)b_1^+$.

We now show that $\bar{b}_1$ is a tensor operator.

The transformation law for $U(J_+)$ reads,

$$\begin{aligned} b_1^+ b_2 \bar{b}_1 =& D_{11}(J_+)\bar{b}_1 q^{\frac{N_1-N_2}{2}} + D_{11}(p^{-\frac{H}{2}})\bar{b}_1 b_1^+ b_2 \\ &+ D_{21}(J_+)\bar{b}_2 q^{\frac{N_1-N_2}{2}} + D_{21}(p^{-\frac{H}{2}})\bar{b}_2 b_1^+ b_2 \\ =& p^{-\frac{1}{2}}\bar{b}_1 b_1^+ b_2, \end{aligned}$$

and using the explicit expression for $\bar{b}_1$, we obtain,

$$b_1^+ b_2 p^{\frac{N_1+1}{2}} b_2 = p^{-\frac{1}{2}}p^{\frac{N_1+1}{2}} b_2 b_1^+ b_2.$$

The validity of this equation follows now from

$$b_1^+ f(N_1) = f(N_1-1)b_1^+.$$

The transformation law for $U(J_-)$ reads,

$$\begin{aligned} b_2^+ b_1 \bar{b}_1 =& D_{11}(J_-)\bar{b}_1 q^{\frac{N_1-N_2}{2}} + D_{11}(p^{-\frac{H}{2}})\bar{b}_1 b_2^+ b_1 \\ &+ D_{21}(J_-)\bar{b}_2 q^{\frac{N_1-N_2}{2}} + D_{21}(p^{-\frac{H}{2}})\bar{b}_2 b_2^+ b_1 \\ =& p^{-\frac{1}{2}}\bar{b}_1 b_2^+ b_1 + \bar{b}_2 q^{\frac{N_1-N_2}{2}}, \end{aligned}$$

and using the explicit expressions for $\bar{b}_1$ and for $\bar{b}_2$, it follows,

$$b_2^+ b_1 p^{\frac{N_1+1}{2}} b_2 = p^{-\frac{1}{2}}p^{\frac{N_1+1}{2}} b_2 b_2^+ b_1 - q^{-\frac{N_2+1}{2}} b_1 (pq^{-1})^{\frac{N_1-1}{2}} q^{\frac{N_1-N_2}{2}}.$$

With $b_1 f(N_1) = f(N_1+1)b_1$ it becomes,

$$p^{\frac{N_1}{2}} p b_2^+ b_2 b_1 = p^{\frac{N_1}{2}} b_2 b_2^+ b_1 - q^{-N_2} p^{\frac{N_1}{2}} b_1.$$

The validity of this equation follows now from the pq–bosons commutation relation,

$$p b_2^+ b_2 = b_2 b_2^+ - q^{-N_2}.$$

The transformation law for $U(s^{\frac{H}{2}})$

$$s^{\frac{N_1-N_2}{2}}\bar{b}_1 = D_{11}(s^{\frac{H}{2}})\bar{b}_1 s^{\frac{N_1-N_2}{2}} + D_{21}(s^{\frac{H}{2}})\bar{b}_2 s^{\frac{N_1-N_2}{2}} = s^{\frac{1}{2}}\bar{b}_1 s^{\frac{N_1-N_2}{2}}$$

using the explicit expression for $\bar{b}_1$, becomes,

$$s^{\frac{N_1-N_2}{2}} p^{\frac{N_1+1}{2}} b_2 = s^{\frac{1}{2}} p^{\frac{N_1+1}{2}} b_2 s^{\frac{N_1-N_2}{2}}.$$

The validity of this equation follows again from $b_2 f(N_2) = f(N_2+1)b_2$.

We now show that $\bar{b}_2^+$ is a tensor operator.

The transformation law for $U(J_+)$ reads,

$$\begin{aligned} b_1^+ b_2 \bar{b}_2^+ =& D_{12}(J_+)\bar{b}_1^+ q^{\frac{N_1-N_2}{2}} + D_{12}(p^{-\frac{H}{2}})\bar{b}_1^+ b_1^+ b_2 \\ &+ D_{22}(J_+)\bar{b}_2^+ q^{\frac{N_1-N_2}{2}} + D_{22}(p^{-\frac{H}{2}})\bar{b}_2^+ b_1^+ b_2 \\ =& \bar{b}_1^+ q^{\frac{N_1-N_2}{2}} + p^{\frac{1}{2}}\bar{b}_2^+ b_1^+ b_2, \end{aligned}$$

and using the explicit expressions for $\tilde{b}_1^+$ and for $\tilde{b}_2^+$, we obtain

$$b_1^+ b_2 q^{\frac{N_1}{2}} b_2^+ (qp^{-1})^{\frac{N_2}{2}} = p^{-\frac{N_2}{2}} b_1^+ q^{\frac{N_1-N_2}{2}} + p^{\frac{1}{2}} q^{\frac{N_1}{2}} b_2^+ (qp^{-1})^{\frac{N_2}{2}} b_1^+ b_2.$$

With $b_1^+ f(N_1) = f(N_1-1) b_1^+$ it becomes,

$$q^{\frac{N_1+N_2-1}{2}} p^{-\frac{N_2}{2}} b_1^+ b_2 b_2^+ = q^{\frac{N_1+N_2-1}{2}} p^{-\frac{N_2}{2}} q^{-N_2} b_1^+ + q^{\frac{N_1+N_2-1}{2}} p^{-\frac{N_2}{2}} p b_1^+ b_2^+ b_2.$$

The validity of this equation follows from the pq–bosons commutation relation,

$$b_2 b_2^+ = q^{-N_2} + p b_2^+ b_2.$$

The transformation law for $U(J_-)$ reads,

$$\begin{aligned} b_1^+ b_2 \tilde{b}_2^+ =& D_{12}(J_-) \tilde{b}_1^+ q^{\frac{N_1-N_2}{2}} + D_{12}(p^{-\frac{H}{2}}) \tilde{b}_1^+ b_1^+ b_2 \\ &+ D_{22}(J_-) \tilde{b}_2^+ q^{\frac{N_1-N_2}{2}} + D_{22}(p^{-\frac{H}{2}}) \tilde{b}_2^+ b_1^+ b_2 \\ =& p^{\frac{1}{2}} \tilde{b}_2^+ b_1^+ b_2, \end{aligned}$$

and using the explicit expression for $\tilde{b}_2^+$, we obtain,

$$b_2^+ b_1 q^{\frac{N_1}{2}} b_2^+ (pq^{-1})^{\frac{N_2}{2}} = p^{\frac{1}{2}} q^{\frac{N_1}{2}} b_2^+ (pq^{-1})^{\frac{N_2}{2}} b_2^+ b_1.$$

Using the relations $b_i^+ f(N_i) = f(N_i - 1) b_i^+$, with $i = 1, 2$, the equality follows. The transformation law for $U(s^{\frac{H}{2}})$

$$s^{\frac{N_1-N_2}{2}} \tilde{b}_2^+ = D_{12}(s^{\frac{H}{2}}) \tilde{b}_1^+ s^{\frac{N_1-N_2}{2}} + D_{22}(s^{\frac{H}{2}}) \tilde{b}_2^+ s^{\frac{N_1-N_2}{2}} = s^{-\frac{1}{2}} \tilde{b}_2^+ s^{\frac{N_1-N_2}{2}}$$

using the explicit expression for $\tilde{b}_2^+$, becomes,

$$s^{\frac{N_1-N_2}{2}} q^{\frac{N_1}{2}} b_2^+ (qp^{-1})^{\frac{N_2}{2}} = s^{-\frac{1}{2}} q^{\frac{N_1}{2}} b_2^+ (qp^{-1})^{\frac{N_2}{2}} s^{\frac{N_1-N_2}{2}}.$$

The validity of this equation follows from $b_2^+ f(N_2) = f(N_2 - 1) b_2^+$.

Finally we show that $\tilde{b}_2$ is a tensor operator.

The transformation law for $U(J_+)$ reads,

$$\begin{aligned} b_1^+ b_2 \tilde{b}_2 =& D_{12}(J_+) \tilde{b}_1 q^{\frac{N_1-N_2}{2}} + D_{12}(p^{-\frac{H}{2}}) \tilde{b}_1 b_1^+ b_2 \\ &+ D_{22}(J_+) \tilde{b}_2 q^{\frac{N_1-N_2}{2}} + D_{22}(p^{-\frac{H}{2}}) \tilde{b}_2 b_1^+ b_2 \\ =& \tilde{b}_1 q^{\frac{N_1-N_2}{2}} + p^{\frac{1}{2}} \tilde{b}_2 b_1^+ b_2, \end{aligned}$$

and using the explicit expressions for $\tilde{b}_1$ and for $\tilde{b}_2$, we obtain

$$-b_1^+ b_2 q^{-\frac{N_2+1}{2}} b_1 (pq^{-1})^{\frac{N_1-1}{2}} = p^{\frac{N_1+1}{2}} b_2 q^{\frac{N_1-N_2}{2}} - p^{\frac{1}{2}} q^{-\frac{N_2+1}{2}} b_1 (pq^{-1})^{\frac{N_1-1}{2}} b_1^+ b_2.$$

With $b_i f(N_i) = f(N_i + 1) b_i$, $i = 1, 2$, it becomes,

$$q^{-\frac{N_1+N_2+1}{2}} p^{\frac{N_1+1}{2}} p^{-1} b_1^+ b_1 b_2 = -q^{-\frac{N_1+N_2+1}{2}} p^{\frac{N_1+1}{2}} q^{N_1} b_2 + q^{-\frac{N_1+N_2+1}{2}} p^{\frac{N_1+1}{2}} b_1 b_1^+ b_2.$$

The validity of this equation follows from the qp–bosons commutation relation,

$$p^{-1} b_1^+ b_1 = -q^{N_1} + b_1 b_1^+.$$

The transformation law for $U(J_-)$ reads,

$$\begin{aligned} b_1^+ b_2 \tilde{b}_2 =& D_{12}(J_-) \tilde{b}_1 q^{\frac{N_1-N_2}{2}} + D_{12}(p^{-\frac{H}{2}}) \tilde{b}_1 b_1^+ b_2 \\ &+ D_{22}(J_-) \tilde{b}_2 q^{\frac{N_1-N_2}{2}} + D_{22}(p^{-\frac{H}{2}}) \tilde{b}_2 b_1^+ b_2 \\ =& p^{\frac{1}{2}} \tilde{b}_2 b_1^+ b_2, \end{aligned}$$

and using the explicit expression for $\tilde{b}_2$, we obtain,

$$b_2^+ b_1 q^{-\frac{N_2+1}{2}} b_1 (pq^{-1})^{\frac{N_1-1}{2}} = p^{\frac{1}{2}} q^{-\frac{N_2+1}{2}} b_1 (pq^{-1})^{\frac{N_1-1}{2}} b_2^+ b_1.$$

Using the relation $b_i^+ f(N_i) = f(N_i - 1)b_i^+$, with $i = 1, 2$, the equality follows. The transformation law for $U(s^{\frac{H}{2}})$

$$s^{\frac{N_1-N_2}{2}}\tilde{b}_2 = D_{12}(s^{\frac{H}{2}})\tilde{b}_1 s^{\frac{N_1-N_2}{2}} + D_{22}(s^{\frac{H}{2}})\tilde{b}_2 s^{\frac{N_1-N_2}{2}} = s^{-\frac{1}{2}}\tilde{b}_2 s^{\frac{N_1-N_2}{2}}$$

using the explicit expression for $\tilde{b}_2$, becomes,

$$s^{\frac{N_1-N_2}{2}} q^{-\frac{N_2+1}{2}} b_1 (pq^{-1})^{\frac{N_1-1}{2}} = s^{-\frac{1}{2}} q^{-\frac{N_2+1}{2}} b_1 (pq^{-1})^{\frac{N_1-1}{2}} s^{\frac{N_1-N_2}{2}}.$$

The validity of this equation follows from $b_1 f(N_1) = f(N_1 + 1)b_1$.

For the rest of the Appendix we require the following Clebsch–Gordan coefficients:

$$\begin{aligned}
< \tfrac{1}{2}\tfrac{1}{2}\ \tfrac{1}{2}\text{-}\tfrac{1}{2} \mid 10 > &= < \tfrac{1}{2}\text{-}\tfrac{1}{2}\ \tfrac{1}{2}\tfrac{1}{2} \mid 00 > = \frac{p^{-\frac{1}{2}}}{\sqrt{[2]_{qp}}} \\
< \tfrac{1}{2}\text{-}\tfrac{1}{2}\ \tfrac{1}{2}\tfrac{1}{2} \mid 10 > &= -< \tfrac{1}{2}\tfrac{1}{2}\ \tfrac{1}{2}\text{-}\tfrac{1}{2} \mid 00 > = \frac{q^{\frac{1}{2}}}{\sqrt{[2]_{qp}}} \\
< \tfrac{1}{2}\tfrac{1}{2}\ \tfrac{1}{2}\tfrac{1}{2} \mid 11 > &= < \tfrac{1}{2}\text{-}\tfrac{1}{2}\ \tfrac{1}{2}\text{-}\tfrac{1}{2} \mid -1-1 > = 1 \\
< 11\ 1-1 \mid 00 > &= (qp)^{-1} < 1-1\ 11 \mid 00 > = q \\
&\quad\ < 10\ 10 \mid 00 > = -(qp^{-1})^{\frac{1}{2}}.
\end{aligned}$$

B.- The total number of particles operator for $U_{qp}(su(2))$.

The total number of particles operator $\mathcal{N}$ is given by

$$\mathcal{N} = (\tilde{b}^+ \otimes \tilde{b})^0 = \sum_{m,m'} < \tfrac{1}{2}m\ \tfrac{1}{2}m' \mid 00 > \tilde{b}_m^+ \tilde{b}_{m'}$$

where $b_{1/2} := b_1$ and $b_{-1/2} := b_2$.

Hence

$$\mathcal{N} = < \tfrac{1}{2}\text{-}\tfrac{1}{2}\ \tfrac{1}{2}\tfrac{1}{2} \mid 00 > \tilde{b}_2^+\tilde{b}_1 + < \tfrac{1}{2}\tfrac{1}{2}\ \tfrac{1}{2}\text{-}\tfrac{1}{2} \mid 00 > \tilde{b}_1^+\tilde{b}_2.$$

Inserting the explicit values for the Clebsch-Gordan coefficients, it follows

$$\mathcal{N} = \frac{p^{-\frac{1}{2}}}{[2]_{qp}}\tilde{b}_2^+\tilde{b}_1 - \frac{q^{\frac{1}{2}}}{[2]_{qp}}\tilde{b}_1^+\tilde{b}_2.$$

and using the explicit expressions for $\tilde{b}_i$, $i = 1, 2$,

$$\mathcal{N} = \frac{p^{-\frac{1}{2}}}{[2]_{qp}} q^{\frac{N_1}{2}} b_2^+ (qp^{-1})^{\frac{N_2}{2}} p^{\frac{N_1+1}{2}} b_2 + \frac{q^{\frac{1}{2}}}{[2]_{qp}} p^{-\frac{N_2}{2}} b_1^+ q^{-\frac{N_2+1}{2}} b_1 (pq^{-1})^{\frac{N_1-1}{2}}.$$

Using for $i = 1, 2$,

$$b_i^+ f(N_i) = f(N_i - 1)b_i^+ \qquad \text{and} \qquad b_i f(N_i) = f(N_i + 1)b_i,$$

it follows,

$$\mathcal{N} = \frac{1}{[2]_{qp}} p^{\frac{N_1-N_2+1}{2}} q^{\frac{N_1+N_2-1}{2}} b_2^+ b_2 + p^{\frac{N_1-N_2-1}{2}} q^{-\frac{N_1+N_2-1}{2}} b_1^+ b_1.$$

Now, we have

$$b_1^+ b_1 = [N_1]_{qp} = \frac{q^{N_1} - p^{-N_1}}{q^1 - p^{-1}} \qquad \text{and} \qquad b_2^+ b_2 = [N_2]_{pq} = \frac{p^{N_2} - q^{-N_2}}{p^1 - q^{-1}}$$

which yields

$$\mathcal{N} = \frac{1}{[2]_{qp}} \frac{p^{\frac{N_1-N_2-1}{2}} q^{\frac{1}{2}}}{q - p^{-1}} \left(q^{\frac{N_1+N_2}{2}} (p^{N_2} - q^{-N_2} + (q^{-\frac{N_1+N_2}{2}} (q^{N_1} - p^{-N_1}) \right) =$$

$$\frac{1}{[2]_{qp}}\frac{1}{q-p^{-1}}\left(p^{\frac{N_1+N_2-1}{2}}q^{\frac{N_1+N_2+1}{2}}-p^{\frac{-N_1-N_2-1}{2}}q^{\frac{-N_1-N_2+1}{2}}\right).$$

Using

$$[N]_{qp}=\frac{q^N-p^{-N}}{q^1-p^{-1}}$$

we have

$$\mathcal{N}=\frac{[N]_{qp}}{\sqrt{[2]}_{qp}}(pq^{-1})^{\frac{N-1}{2}}.$$

C.- The rank one tensor operator T for $U_{qp}(su(2))$.

The rank one tensor operator T is defined by,

$$T_l=(\bar{b}^+\otimes\bar{b})^1_l=\sum_{m,m'}<\tfrac{1}{2}m\ \ \tfrac{1}{2}m'\mid 1l>\bar{b}^+_m\bar{b}_{m'}.$$

Let $T_+=T_1$, $T_-=T_2$, $b_+=b_1$,$b_-=b_2$, $N_+=N_1$, and $N_-=N_2$, we then have:

$$T_\pm=<\tfrac{1}{2}\pm\tfrac{1}{2}\ \ \tfrac{1}{2}\pm\tfrac{1}{2}\mid 1\pm1>\bar{b}^+_\pm\bar{b}_\pm.$$

The resulting Clebsch-Gordan coefficients are equal to one, hence inserting the explicit expressions for $\bar{b}_\pm$, it follows

$$T_+=p^{-\frac{N_2}{2}}b_1^+p^{\frac{N_1+1}{2}}b_2=p^{\frac{N_1-N_2}{2}}J_+$$

and

$$T_-=-q^{\frac{N_1}{2}}b_2^+(qp^{-1})^{\frac{N_2}{2}}q^{-\frac{N_2+1}{2}}b_1(pq^{-1})^{\frac{N_1-1}{2}}=-(pq^{-1})^{\frac{1}{2}}p^{\frac{N_1-N_2}{2}}J_-.$$

For T_0 we have

$$T_0=<\tfrac{1}{2}-\tfrac{1}{2}\ \ \tfrac{1}{2}\tfrac{1}{2}\mid 10>\bar{b}^+_2\bar{b}_1+<\tfrac{1}{2}\tfrac{1}{2}\ \ \tfrac{1}{2}-\tfrac{1}{2}\mid 10>\bar{b}^+_1\bar{b}_2.$$

Using the explicit expressions for the Clebsch-Gordan coefficients and for $\bar{b}_i$, $i=1,2$, we obtain

$$T_0=\frac{q^{\frac{1}{2}}}{[2]_{qp}}q^{\frac{N_1}{2}}b_2^+(qp^{-1})^{\frac{N_2}{2}}p^{\frac{N_1+1}{2}}b_2-\frac{p^{-\frac{1}{2}}}{[2]_{qp}}p^{-\frac{N_2}{2}}b_1^+q^{-\frac{N_2+1}{2}}b_1(pq^{-1})^{\frac{N_1-1}{2}}=$$

$$\frac{1}{[2]_{qp}}\left(p^{\frac{N_1-N_2+2}{2}}q^{\frac{N_1+N_2}{2}}[N_2]_{pq}-p^{\frac{N_1-N_2-2}{2}}q^{-\frac{N_1+N_2}{2}}[N_1]_{qp}\right)$$

which leads to

$$T_0=\frac{1}{\sqrt{[2]}_{qp}}p^{\frac{N_1-N_2}{2}}\left(pq^{\frac{N_1+N_2}{2}}[N_2]_{pq}-p^{-1}q^{-\frac{N_1+N_2}{2}}[N_1]_{qp}\right).$$

The Casimir tensor operator $\mathcal{C}$ for $U_{qp}(su(2))$.

The Casimir tensor operator $\mathcal{C}$ is defined by

$$\mathcal{C}=(T\otimes T)^0=\sum_{m,m'}<1m\ \ 1m'\mid 00>T_mT_{m'}.$$

Hence we have,

$$\mathcal{C}=<11\ \ 1-1\mid 00>T_+T_-+<1-1\ \ 11\mid 00>T_-T_+-<10\ \ 10\mid 00>T_0^2.$$

Using the explicit values for the Clebsch–Gordan coefficients, we find

$$\mathcal{C}=\frac{1}{\sqrt{[3]}_{qp}}\left(qT_+T_-+p^{-1}T_-T_+-(qp^{-1})^{\frac{1}{2}}T_0^2\right).$$

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