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Nuclearity, Split-Property and Duality for the Klein-Gordon Field in Curved Spacetime

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Abstract

Nuclearity, Split-Property and Duality are established for the nets of von Neumann algebras associated with the representations of distinguished states of the massive Klein-Gordon field propagating in particular classes of curved spacetimes.

1 Introduction

In this letter we continue to investigate further the algebraic structure of the Klein-Gordon field propagating in (certain) globally hyperbolic spacetimes and to establish, for a distinguished class of states, most of the properties known for this field theoretical model to hold in Minkowski spacetime. More precisely, we prove the p−nuclearity ("Condition $N_\sharp$" or, equivalently, "Condition $N^\natural$" in [8]) for the net of von Neumann algebras associated with the cyclic representation of the canonical vacuum state on the Weyl-algebra of

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the free, neutral, massive Klein-Gordon field (KG-field, for short) propagating in an ultrastatic spacetime of dimension $n \geq 2$. Using this result in combination with the Reeh-Schlieder theorem for such canonical vacua [26], we obtain the split-property for the afforementioned net of von Neumann algebras. For $n = 4$, we can strengthen this result in that we derive the split-property for the nets of von Neumann algebras associated with quasifree Hadamard states of the KG-field propagating in arbitrary globally hyperbolic spacetimes. Finally, for the ultrastatic situation we gain duality in all irreducible representations of the quasilocal algebra of the KG-field which are locally normal to the canonical vacuum representation.

These results show that the nets of von Neumann algebras associated with the cyclic representations of the states of the said type of the KG-field have "nice" properties which are desirable if the states are to describe reasonable physics. This applies in particular to the nuclearity condition, which can be interpreted as saying that the theory has a phase-space behaviour which entails reasonable thermal properties and a particle interpretation (see the articles [6-9] for further discussion). The nuclearity condition also entails the split property which expresses a strong form of statistical independence. See [9] for further discussion on this point and also [24] for a review.

From our results about the ultrastatic vacuum $\hat{\omega}$ in sections 3 and 4 it follows that for regions $\overline{\mathcal{O}_\flat} \subset \mathcal{O}_\sharp$ with $int\mathcal{O}_\sharp^c \neq \emptyset$ the maps $\Xi^\natural_{\mathcal{O}_\sharp,\mathcal{O}_\flat} : A \mapsto \Delta^{1/4}_{\mathcal{O}_\sharp} A\Omega_{\hat{\omega}}$ $(A \in \mathcal{R}_{\hat{\omega}}(\mathcal{O}_\flat))$, where $\Delta_{\mathcal{O}_\sharp}$ is the modular operator corresponding to $\mathcal{R}_{\hat{\omega}}(\mathcal{O}_\sharp), \Omega_{\hat{\omega}}$, are p-nuclear for each $p > 0$ (see [5]). As remarked in [5,II], the conditions of "modular nuclearity" (in terms of the maps $\Xi^\natural_{\mathcal{O}_\sharp,\mathcal{O}_\flat}$) appear to be the appropriate generalizations of "energy nuclearity" (in terms of the maps $\Xi_{\beta,\mathcal{O}}$ introduced in section 3, respectively the maps θ_β introduced in section 4) to nets of von Neumann algebras over generic spacetimes. It is worth noting that, if the algebras $\mathcal{R}_\omega(\mathcal{O}_\flat)$, $\mathcal{R}_\omega(\mathcal{O}_\sharp)$ are factors for quasifree Hadamard states ω of the Klein-Gordon field in arbitrary globally hyperbolic spacetimes, then their split-inclusion implies that the map $\Xi^\natural_{\mathcal{O}_\sharp,\mathcal{O}_\flat}$ is compact. (The said factoriality is proved for the ultrastatic situation [25]. It is conceivable but not yet proved that it holds generally.) We refer to [5,II] for further discussion on modular nuclearity.

2 The KG-field in globally hyperbolic spacetimes

Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime of dimension $n \geq 2$. If $\mathcal{O}$ is a subset of $\mathcal{M}$, define $J^{\pm}(\mathcal{O})$:= set of all points p in $\mathcal{M}$ which can be reached by future(+)/past(−) -directed causal curves emanating from $\mathcal{O}$, $J(\mathcal{O}) := J^+(\mathcal{O}) \cup J^-(\mathcal{O})$, $D^{\pm}(\mathcal{O})$:= set of all points $p \in J^{\pm}(\mathcal{O})$ such that every past(+)/future(−) -inextendible curve starting at p passes through $\mathcal{O}$, and $D(\mathcal{O}) := D^+(\mathcal{O}) \cup D^-(\mathcal{O})$. As $(\mathcal{M}, g)$ is globally hyperbolic, it possesses a smooth foliation into Cauchy-surfaces. There are situations where such a foliation admits a special form: Let (M, γ) be a d-dimensional, complete Riemannian manifold. If $(\mathcal{M}, g)$ can be realized as $\mathcal{M} = \mathbf{R} \times M$ and $g = dt^2 \oplus (-\gamma)$ (i. e. for a coordinate chart $(t,p) \mapsto (t, x^i(p))$ on a coordinate patch for $\mathcal{M}$, the coordinate expression for g is equal to $dt^2 - \gamma_{ij}dx^i dx^j$, then $(\mathcal{M}, g)$ is called the *n = d + 1 -dimensional ultrastatic spacetime foliated by* (M, γ), and the family $M(t) := \{t\} \times M$, $t \in \mathbf{R}$, constitutes a smooth foliation of $\mathcal{M}$ into Cauchy-surfaces, called the *natural foliation* (see [3,21] for further discussion of spacetime geometry). The Klein-Gordon equation (KG-eqn) on a globally hyperbolic spacetime takes the form

$$(\nabla^a \nabla_a + m^2)\,\varphi = 0 \tag{1}$$

where ∇ is the Levi-Civita connection of the metric g and $m > 0$ is a fixed constant. Notice that in the case where $(\mathcal{M}, g)$ is the ultrastatic spacetime foliated by some complete Riemannian manifold (M, γ), we have

$$\nabla^a \nabla_a = \frac{\partial^2}{\partial t^2} - \Delta_\gamma \tag{2}$$

where Δ_γ is the Laplace-Beltrami operator for (M, γ) and t is the "time-parameter" of the natural foliation.
If $(\mathcal{M}, g)$ is globally hyperbolic, then the Cauchy-problem for (1) is well-posed and the set of solutions of (1) may be identified with the set of its Cauchy-data on an arbitrarily given Cauchy-surface, $\mathcal{C}$; in the simplest case, the set of Cauchy-data $\mathbf{D}_\mathcal{C}$ on $\mathcal{C}$ may be taken as

$$\mathbf{D}_\mathcal{C} := C_0^\infty(\mathcal{C}, \mathbf{R}) \oplus C_0^\infty(\mathcal{C}, \mathbf{R}) \tag{3}$$

and equipped with the symplectic form

$$\delta_\mathcal{C}(u_0 \oplus u_1, v_0 \oplus v_1) := \int_\mathcal{C} (u_0 v_1 - v_0 u_1)\, d\sigma_\mathcal{C} \tag{4}$$

where $d\sigma_\mathcal{C}$ is the metric-induced measure on $\mathcal{C}$. One can form the Weyl-algebra $\mathcal{A}[\mathbf{D}_\mathcal{C}, \delta_\mathcal{C}]$ corresponding to Cauchy-data of solutions of (1), and observing that the propagation according to (1) of Cauchy-data from a Cauchy-surface $\mathcal{C}_1$ to another one, $\mathcal{C}_2$, is a symplectomorphism between the $(\mathbf{D}_{\mathcal{C}_i}, \delta_{\mathcal{C}_i})$, $i = 1, 2$, the Weyl-algebras associated with Cauchy-data on different Cauchy-surfaces can be canonically identified. Therefore $\mathcal{A}[\mathbf{D}_\mathcal{C}, \delta_\mathcal{C}]$ with $\mathcal{C}$ an arbitrary Cauchy-surface, may be referred to as the *Weyl-algebra of the (mass m) Klein-Gordon (KG) -field in* $(\mathcal{M}, g)$, and denoted by $\mathcal{A}_{KG}$.
The most important thing for readers desiring to proceed is to digest the following

Notational Conventions

nota 1. If $\mathcal{C}$ is a Cauchy-surface, then $\mathcal{C}_\flat$, $\mathcal{C}_\sharp$, $\mathcal{C}_{\sharp 1}, \ldots$ *etc.*, will denote arbitrary (nonvoid) open subsets of $\mathcal{C}$ with compact closure, and $\mathcal{C}_\flat^c, \ldots$ *etc.*, will stand for $\mathcal{C} \backslash \overline{\mathcal{C}_\flat}$.

nota 2. In the notation of *nota 1*, we write

$$\mathcal{O}_\flat := int(D(\mathcal{C}_\flat)), \ \ etc.,$$

and

$$\mathcal{O}_\flat^c := int(\mathcal{M} \backslash J(\mathcal{O}_\flat)), \ \ etc.;$$

notice that $\mathcal{O}_\flat^c = int(D(\mathcal{C}_\flat^c))$.

nota 3. If ω is a state on some C^*-algebra A, then $(\mathcal{F}_\omega, \pi_\omega, \Omega_\omega)$ will always denote the GNS-representation of ω. For $\mathsf{A}(\mathcal{O}) \subset \mathsf{A}$, we write

$$\mathcal{R}_\omega(\mathcal{O}) := \pi_\omega(\mathsf{A}(\mathcal{O}))''.$$

If $\mathcal{C}$ is some Cauchy-surface for $(\mathcal{M}, g)$, one obtains the symplectic subspaces $(\mathbf{D}_{\mathcal{C}_\flat}, \delta_{\mathcal{C}_\flat})$ of $(\mathbf{D}_\mathcal{C}, \delta_\mathcal{C})$ (*nota 1*) by defining the objects carrying a $\flat$ in the same manner as those without a $\flat$. Bearing in mind the canonical identification of Weyl-algebras of Cauchy-data on different Cauchy-surfaces, one can show that setting (*nota 1,2*)

$$\mathcal{A}(\mathcal{O}_\flat) := \mathcal{A}[\mathbf{D}_{\mathcal{C}_\flat}, \delta_{\mathcal{C}_\flat}] \tag{5}$$

(and considering $\mathcal{A}[\mathbf{D}_{\mathcal{C}_\flat}, \delta_{\mathcal{C}_\flat}]$ as a sub-Weyl-algebra of $\mathcal{A}[\mathbf{D}_\mathcal{C}, \delta_\mathcal{C}]$) gives rise to an isotonous, causal (or local) and primitively causal net $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \subset \mathcal{A}_{KG}$ of C^*-algebras indexed by the open, relatively compact subsets $\mathcal{O}$ of $\mathcal{M}$ (the just sketched construction is carried out in detail, with all proofs, in [11]).

If $(\mathcal{M}, g)$ is the ultrastatic spacetime foliated by (M, γ), with natural foliation $M(t)$, $t \in \mathbf{R}$, we have the natural time-translations $\tau_{t'} : (t, p) \mapsto (t + t', p)$; they are isometries of the spacetime and hence induce symplectomorphism-groups on the symplectic spaces $(\mathbf{D}_{M(t_0)}, \delta_{M(t_0)})$ for any t_0, and whence give rise to an automorphism group $\alpha_{t'}$ on $\mathcal{A}_{KG}$.

Given a complex Hilbert-space $\mathcal{H}$, we introduce for each $\chi \in \mathcal{H}$ the unitary operator

$$W^F(\chi) := \exp[i\overline{(a^+(\chi) + a(\chi))}] \tag{6}$$

on $F_s(\mathcal{H})$, the symmetric Fock-space over $\mathcal{H}$; a^+ and a are the usual creation and annihilation operators, respectively. For $\mathcal{L}$ a real-linear subspace of $\mathcal{H}$, we shall write

$$\mathcal{W}(\mathcal{L}) := \{W^F(\chi) \,|\chi \in \mathcal{L}\}''. \tag{7}$$

Now let $\mathcal{A}_{KG}$ be represented as $\mathcal{A}[\mathbf{D}_\mathcal{C}, \delta_\mathcal{C}]$ for some Cauchy-surface $\mathcal{C}$ in the globally hyperbolic spacetime $(\mathcal{M}, g)$. A *quasifree state* ω on $\mathcal{A}[\mathbf{D}_\mathcal{C}, \delta_\mathcal{C}]$ is characterized by a one-particle Hilbert-space structure $(\mathbf{k}, \mathcal{H})$, where $\mathcal{H}$ is a complex Hilbert-space and $\mathbf{k} : \mathbf{D}_\mathcal{C} \longrightarrow \mathcal{H}$ is a real-linear injective map satisfying $\mathrm{Im}\langle \mathbf{ku}, \mathbf{kv}\rangle_\mathcal{H} = 2\delta_\mathcal{C}(\mathbf{u}, \mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbf{D}_\mathcal{C}$ and giving rise to an assignment (*nota 1,2*)

$$\mathbf{D}_{\mathcal{C}_\flat} \mapsto \mathbf{k}(\mathbf{D}_{\mathcal{C}_\flat}) =: \mathcal{L}(\mathcal{O}_\flat) \subset \mathcal{H}. \tag{8}$$

In terms of this assignment one finds that one obtains

$$\mathcal{R}_\omega(\mathcal{O}_\flat) = \mathcal{W}(\mathcal{L}(\mathcal{O}_\flat)); \tag{9}$$

in this notation one has that $\mathcal{F}_\omega = F_s(\mathcal{H})$ and $\Omega_\omega = \Omega^F$ = Fock-vacuum (see [1,19] and references cited in [19] for a more detailed discussion of these constructions).

Now consider a d-dimensional complete Riemannian manifold (M, γ) with Laplace-Beltrami operator Δ_γ and metric-induced measure ν_γ, then for fixed $m > 0$ the operator

$$-\Delta_\gamma + m^2 : C_0^\infty(M, \mathbf{C}) \longrightarrow L^2_\mathbf{C}(M, \nu_\gamma) \tag{10}$$

is essentially selfadjoint [10] and we denote its closure by A. Let $(\mathcal{M}, g)$ be the ultrastatic spacetime foliated by (M, γ) and $M(t)$, $t \in \mathbf{R}$ the canonical foliation. For each $t \in \mathbf{R}$, define a quasifree state ω^t on $\mathcal{A}[\mathbf{D}_{M(t)}, \delta_{M(t)}]$ by setting its one-particle Hilbert-space structure $(\mathbf{k}^t, \mathcal{H}^t)$ to be $\mathcal{H}^t := L^2_\mathbf{C}(M, \nu_\gamma)$ and

$$\mathbf{k}^t(u_0 \oplus u_1) := \frac{1}{\sqrt{2}}(A^{1/4}u_0 + iA^{-1/4}u_1) \tag{11}$$

for all $u_0 \oplus u_1 \in \mathbf{D}_{M(t)}$. One can show that ω^t is pure and invariant under the time-translations $\alpha_{t'}$ on $\mathcal{A}_{KG}$; in fact, it is the unique quasifree state on $\mathcal{A}_{KG}$ which is a ground state for the time-translations [17,18], so we may actually drop the $t-$dependence and denote ω^t (t arbitrary) simply by $\hat{\omega}$ and refer to it as the *canonical vacuum state* on the Weyl-algebra of the (mass m) KG-field in the ultrastatic spacetime $(\mathcal{M}, g)$. Notice that $\alpha_{t'}$ is in the GNS-representation of $\hat{\omega}$ implemented by a unitary group $U_{t'} = e^{it'H}$ where the Hamiltonian $H = d\Gamma(\eta)$ is the second quantization of the one-particle Hamiltonian $\eta = A^{1/2}$. There is a distinguished conjugation J on the one-particle Hilbert-space $L^2_\mathbf{C}(M, \nu_\gamma)$ which commutes with η, namely $(Jf)(q) = \overline{f(q)}$.

Having summarized the necessary material about the KG-field in curved spacetimes, we turn to the discussion of the nuclearity condition.

3 Nuclearity

In order to formulate the nuclearity condition ($p-$nuclearity), suppose that we are given a theory in the form of an isotonous, causal net of von Neumann algebras $\mathcal{O} \mapsto \mathsf{R}(\mathcal{O})$ acting on some Hilbert-space $\mathcal{F}$ indexed by open, relatively compact regions $\mathcal{O}$ of some ($n-$dimensional) spacetime $(\mathcal{M}, g)$, and that H is the Hamiltonian of the theory. Further suppose that $\Omega \in \mathcal{F}$ is the unique vector representing the vacuum, and consider for $\beta > 0$ the maps

$$\Xi_{\beta,\mathcal{O}} : \mathsf{R}(\mathcal{O}) \longrightarrow B(\mathcal{F}) \tag{12}$$

defined by

$$\Xi_{\beta,\mathcal{O}}(X) := \mathrm{e}^{-\beta H} X \mathrm{e}^{-\beta H} \quad \forall\, X \in \mathsf{R}(\mathcal{O}) \tag{13}$$

The condition of p−nuclearity in this situation is

Condition $N^\sharp$ (cf. [8]). *The maps $\Xi_{\beta,\mathcal{O}}$ are p−nuclear for all $p > 0$ and any open, relatively compact region $\mathcal{O}$, for sufficiently large $\beta > 0$ (which may depend on $\mathcal{O}$ and p).*

Recall that a linear map $\theta : \mathsf{E} \longrightarrow \mathsf{F}$ between Banach-spaces is called *p-nuclear* if there exist sequences $e_i \in \mathsf{E}^*$, $F_i \in \mathsf{F}$ such that

$$\theta(\,.\,) = \sum_i e_i(\,.\,) F_i \tag{14}$$

in the sense of strong convergence and

$$||\theta||_p := inf \left(\sum_i ||e_i||^p ||F_i||^p \right)^{1/p} < \infty \tag{15}$$

where the infimum is taken over all pairs of e_i, F_i complying with (14). Two things should be realized. (1) If a map is p−nuclear for some $p > 0$, then it is p'−nuclear for every $p' > p$. (2) If $\Xi_{\beta,\mathcal{O}}$ is p−nuclear, then $\Xi_{\beta,\tilde{\mathcal{O}}}$ is p−nuclear for all regions $\tilde{\mathcal{O}} \subset \mathcal{O}$. So when proving Condition $N^\sharp$, it suffices to check the p−nuclearity of the maps $\Xi_{\beta,\mathcal{O}}$ where the $\mathcal{O}$s are taken from a family of spacetime regions exhausting every compact subregion of the spacetime.

Now let $\hat{\omega}$ be the canonical vacuum state on the Weyl-algebra $\mathcal{A}_{KG}$ of the KG-field in the $n = d+1$ −dimensional ultrastatic spacetime $(\mathcal{M}, g)$ foliated by the d−dimensional complete Riemannian manifold (M, γ).

Proposition 1. *The p-nuclearity condition $N^\sharp$ holds for the canonical vacuum $\hat{\omega}$. This means that, if $H = d\Gamma(A^{1/2})$ is the generator of the time-translations in $\mathcal{F}_{\hat{\omega}}$, and $\mathcal{O}$ an open, relatively compact subset of $\mathcal{M}$, there is for each $p > 0$ a $\beta > 0$ such that the map*

$$\Xi_{\beta,\mathcal{O}} : \mathcal{R}_{\hat{\omega}}(\mathcal{O}) \longrightarrow B(\mathcal{F}_{\hat{\omega}})$$

$$\Xi_{\beta,\mathcal{O}}(X) := \mathrm{e}^{-\beta H} X \mathrm{e}^{-\beta H}, \quad \forall\, X \in \mathcal{R}_{\hat{\omega}}(\mathcal{O})$$

is p-nuclear.

Proof. Let $t_0 \in \mathbf{R}$ arbitrary and set $\mathcal{C} := M(t_0)$, then it suffices to prove the p−nuclearity, for suitable $\beta > 0$, for all the maps $\Xi_{\beta,\mathcal{O}_b}$ (*nota 1,2*). Here we have the situation that (cf. (9)) $\mathcal{R}_{\tilde{\omega}}(\mathcal{O}_b) = \mathcal{W}(\mathcal{L}(\mathcal{O}_b))$. Denote by $\mathcal{L}_\varphi$ and $\mathcal{L}_\pi$ the closures of the *complex* linear hulls of $(1+J)\mathcal{L}(\mathcal{O}_b)$ and $(1-J)\mathcal{L}(\mathcal{O}_b)$, respectively, and by E_φ and E_π their corresponding orthogonal projectors in $L^2_{\mathbf{C}}(M, \nu_\gamma)$. Then define the operators

$$S_\varphi := E_\varphi A^{-1/4} \mathrm{e}^{-(\beta/2)A^{1/2}}\,, \quad S_\pi := E_\pi A^{-1/4} \mathrm{e}^{-(\beta/2)A^{1/2}} \tag{16}$$

Now we are in the position to make use of a criterion derived by Buchholz and Porrmann:

Lemma 2. (cf. Lemma 3.4 in [8]) *If S_φ and S_π are p-nuclear for some $0 < p \leq 1$ and some $\beta > 0$, then there exists a $\beta' > 0$ such that $\Xi_{\beta',\mathcal{O}_b}$ is p-nuclear.*

So by the Lemma it suffices to show that for all $\beta > 0$, S_φ and S_π are p−nuclear. We present the arguments showing this for S_φ, those for S_π are strictly analogous. S_φ is p−nuclear ($p > 0$) iff $|S_\varphi|^p$ is trace-class. The aim is therefore to prove that S_φ can be written as a product of arbitrarily many Hilbert-Schmidt operators, and hence as a product of arbitrarily many trace-class operators, entailing p−nuclearity for all $p > 0$. It is a tedious but fairly straightforward task to prove that given $\chi \in C_0^\infty(M(t_0), \mathbf{C})$ and $k \in \mathbf{N}$, we have

$$||A^k(\chi\psi)||_{L^2} \leq C||A^k\psi||_{L^2} \tag{17}$$

for all $\psi \in dom(A^k)$ and some suitably chosen $C > 0$. This entails in particular that $A^k \chi A^{-k}$ and $A^{-k} \chi A^k$ are bounded. Also, for each $\beta > 0$ and $s \in \mathbf{R}$, χA^s maps $C_0^\infty(M(t_0), \mathbf{C})$ into $C_0^\infty(supp(\chi), \mathbf{C})$, and $\chi A^s \mathrm{e}^{-\beta A^{1/2}}$ maps $L^2_{\mathbf{C}}(M(t_0), \nu_\gamma)$ into $C_0^\infty(supp(\chi), \mathbf{C})$ (cf. [27]). Now let $\mu \in \mathbf{N}$ and an arbitrary set of positive integers $r_1, \ldots, r_\mu$ be given. Then we may write

$$S_\varphi = E_\varphi A^{-1/4} \chi A^{-r_1} \tilde{\chi} A^{r_1} \chi A^{-r_2} \tilde{\chi} \cdots A^{r_{\mu-1}} \chi A^{-r_\mu} A^{r_\mu} \mathrm{e}^{-(\beta/2)A^{1/2}} \tag{18}$$

where χ and $\tilde{\chi}$ are arbitrary smooth, positive functions on $M(t_0)$ with compact support and such that $\chi \equiv 1$ on $\mathcal{C}_b$, and $\tilde{\chi} \equiv 1$ on $supp(\chi)$. To see this, observe that from the definition of E_φ one obtains that on $dom(A^{1/4})$

$$E_\varphi = E_\varphi A^{-1/4}\chi^s A^{1/4}$$

for any positive number s. So we have (assuming $s > 1$)

$$\begin{aligned} E_\varphi A^{-1/4}e^{-(\beta/2)A^{1/2}} &= E_\varphi A^{-1/4}\chi^s e^{-(\beta/2)A^{1/2}} \\ &= E_\varphi A^{-1/4}\chi A^{-r_1}A^{r_1}\chi^{s-1}e^{-(\beta/2)A^{1/2}} \\ &= E_\varphi A^{-1/4}\chi A^{-r_1}\tilde{\chi}A^{r_1}\chi^{s-1}e^{-(\beta/2)A^{1/2}} , \end{aligned}$$

since A^{r_1} maps $C_0^\infty(supp(\chi), \mathbf{C})$ into itself. Proceeding in this manner with the insertion of unit operators of the form $A^{-r_j}A^{r_j}$ and adjusting the choice of s to $s = \mu$, we obtain the equation (18). $A^{-\lambda}$ is bounded for all $\lambda > 0$, so in order to show that S_φ can be written as a product of arbitrarily many Hilbert-Schmidt operators, we need only show that given an integer $r_j > 0$, the operator $A^{r_j}\chi A^{-r_{j+1}}\tilde{\chi}$ is Hilbert-Schmidt if the integer r_{j+1} is sufficiently larger than r_j, and that for $\beta > 0$, $r_\mu > 0$ arbitrary, $A^{r_\mu}e^{-(\beta/2)A^{1/2}}$ is bounded. The latter is obvious from the positivity of A. Since $A^{r_j}\chi A^{-r_j}$ is bounded as remarked above, it suffices to show that $A^{-\rho}\tilde{\chi}$ is Hilbert-Schmidt for $\rho > 0$ large enough and $\tilde{\chi} \in C_0^\infty(M(t_0), [0, \infty))$. This in turn amounts to showing that $\tilde{\chi}A^{-2\rho}\tilde{\chi}$ is trace-class. As argued in section IV of [27], it can be seen from the Hadamard-expansion of the heat kernel $e^{-\tau A^{1/2}}$, $\tau > 0$ that $A^{-2\rho}$ is given by an everywhere continuous integral kernel provided that $\rho > 0$ is large enough. (One needs $\rho > d$, the dimension of M.) Then the positive operator $\tilde{\chi}A^{-2\rho}\tilde{\chi}$ is given by a continuous integral kernel with compact support and hence is of trace-class, by a generalization of Mercer's theorem (see section 10.3 in [16]). This completes the proof. □

4 Split Property

Consider for the canonical vacuum state $\hat{\omega}$ and for a fixed region $\mathcal{O}_\flat$ as before the maps

$$\theta_\beta(X) := e^{-\beta H}X\Omega_{\hat{\omega}} \quad (\beta > 0) \tag{19}$$

taking elements of $\mathcal{R}_{\hat{\omega}}(\mathcal{O}_\flat) = \mathcal{W}(\mathcal{L}(\mathcal{O}_\flat))$ to elements of the GNS-Hilbertspace $\mathcal{F}_{\hat{\omega}} = F_s(L^2_{\mathbf{C}}(M, \nu_\gamma))$, where $H = d\Gamma(A^{1/2})$, as above. These maps satisfy a

trace-norm bound, i. e. (with $||\,.\,||_1$ denoting the trace-norm (15))

$$||\theta_\beta||_1 \le e^{(\beta_0/\beta)^k} \quad (\beta > 0) \tag{20}$$

for some positive β_0, k, if the maps

$$T_\varphi(\beta) := E_\varphi e^{-\beta A^{1/2}} , \quad T_\pi(\beta) := E_\pi e^{-\beta A^{1/2}} \tag{21}$$

satisfy some trace-norm bound of the form

$$||T_\varphi(\beta)||_1 \le C_\varphi \beta^{-\kappa_\varphi} \quad (\beta > 0) \tag{22}$$

for some suitable constants $C_\varphi, \kappa_\varphi > 0$, and similarly for $\varphi \to \pi$: This follows from the results of [6]. Using the same methods which in the last section led to the p-nuclearity of the maps S_φ and S_π, it is easily shown that there are a fixed trace-class operator Q_φ on $L^2_{\mathbf{C}}(M(t_0), \nu_\gamma)$ and bounded operators $B_\varphi(\beta)$ with

$$T_\varphi(\beta) = Q_\varphi B_\varphi(\beta) \quad (\beta > 0) \tag{23}$$

such that, with some $\kappa_\varphi > 0$

$$||T_\varphi(\beta)||_1 \le ||\tau_\varphi||_1 ||B_\varphi(\beta)|| \le ||Q_\varphi||_1 ||A^{\kappa_\varphi}e^{-\beta A^{1/2}}|| \tag{24}$$

for all $\beta > 0$. This entails the estimate (22). Replacing φ by π we find a similar estimate. Thus we have the trace-norm bound (20).

Recall the time-translations in our ultrastatic spacetime $\mathcal{M} = \mathbf{R} \times M$,

$$\tau_{t'} : (t_0, p) \mapsto (t_0 + t', p)$$

Given $t_0 \in \mathbf{R}$ and setting $\mathcal{C} = M(t_0)$ as above, one easily finds that for any two $\mathcal{C}_\flat, \mathcal{C}_\sharp$ (*nota 1*) with $\overline{\mathcal{C}_\flat} \subset \mathcal{C}_\sharp$ and nonvoid $\mathcal{C}_\sharp^c$ there is some $\delta > 0$ such that (*nota 2*) $\overline{\tau_{t'}(\mathcal{O}_\flat)} \subset \mathcal{O}_\sharp$ for $|t'| < \delta$. For $\alpha_{t'}$, the action on $\mathcal{A}_{KG}$ induced by $\tau_{t'}$, we then get

$$\alpha_{t'}(\mathcal{A}(\mathcal{O}_\flat)) \subset \mathcal{A}(\mathcal{O}_\sharp) \quad (|t'| < \delta) . \tag{25}$$

If $U_{t'}$ is the unitary group implementing $\alpha_{t'}$ in the GNS-representation of $\hat{\omega}$, we thus have

$$U_{t'}\mathcal{R}_{\hat{\omega}}(\mathcal{O}_\flat)U_{-t'} \subset \mathcal{R}_{\hat{\omega}}(\mathcal{O}_\sharp) \quad (|t'| < \delta) . \tag{26}$$

This means, since we have the trace-norm bound (20), and since the GNS-vacuum $\Omega_{\hat{\omega}}$ is cyclic and separating for algebras of the type $\mathcal{R}_{\hat{\omega}}(\mathcal{O}_\flat)$ (cf. [26])

that we can apply Thm. 17.1.4 in [2] (the original references are [4] and [9]) to obtain that there is a type I -factor $\mathcal{N}$ acting on the GNS-Hilbert-space of $\hat{\omega}$ such that we have split-inclusion,

$$\mathcal{R}_{\hat{\omega}}(\mathcal{O}_\flat) \subset \mathcal{N} \subset \mathcal{R}_{\hat{\omega}}(\mathcal{O}_\sharp). \tag{27}$$

Therefore, we have proved the following

Proposition 3. *Let $M(t)$, $t \in \mathbf{R}$, be the canonical foliation of the $n = d+1$ -dimensional ultrastatic spacetime $(\mathcal{M}, g)$ foliated by the d-dimensional complete Riemannian manifold (M, γ). For $C = M(t_0)$ ($t_0 \in \mathbf{R}$ arbitrary), let $\overline{C_\flat} \subset C_\sharp$ and $C_\sharp^c \neq \emptyset$. Then there is a type I -factor $\mathcal{N}$ acting on $\hat{\omega}$'s GNS-Hilbert-space such that we have the split-inclusion*

$$\mathcal{R}_{\hat{\omega}}(\mathcal{O}_\flat) \subset \mathcal{N} \subset \mathcal{R}_{\hat{\omega}}(\mathcal{O}_\sharp). \tag{28}$$

If the spacetime-dimension is $n = 4$, we have that all quasifree Hadamard states (see [19] for a definition) induce cyclic representations of $\mathcal{A}_{KG}$ which are locally quasiequivalent to $\pi_{\hat{\omega}}$ (cf. [25]). Whence we have the split-property (28) also if we replace $\hat{\omega}$ by any quasifree Hadamard state ω.

Below, we shall generalize this result to quasifree Hadamard states of the (massive) KG-field in arbitrary, not necessarily ultrastatic, globally hyperbolic spacetimes (of dimension $n = 4$). Before doing that, we exploit the results hitherto obtained a little further. If $\mathcal{O}_\sharp$ is "regularly shaped", by which we mean that the boundary of $C_\sharp$ is contained in the union of finitely many smooth, $d-1$ -dimensional submanifolds of $M(t_0)$, then we have in the situation of Proposition 3 that $\mathcal{R}_\omega(\mathcal{O}_\sharp)$ is a factor, and for all $p \in \mathcal{M}$, $\bigcap_{\mathcal{O} \ni p} \mathcal{R}_\omega(\mathcal{O}) = \mathbf{C}1$ by the results of sections 4 and 6 and Appendix E in [25]. (These results do not depend on the dimension d.) If it were true that $\hat{\omega}$ has a scaling limit at every spacetime point in the sense of [28] or section 16.2.4 in [2], then theorem 16.2.18 in [2], or the results of [28], imply that $\mathcal{R}_\omega(\mathcal{O}_\sharp)$ are type III_1 -factors provided that $\mathcal{O}_\sharp$ is small enough. It then follows from the split-property that the algebras $\mathcal{R}_\omega(\mathcal{O}_\sharp)$ are hyperfinite and thus isomorphic to the unique hyperfinite type III_1 -factor.

We are unaware of a proof that $\hat{\omega}$ has a scaling limit at every spacetime point in arbitrary spacetime dimension. But if the spacetime-dimension $n = d+1$ is equal to 4, $\hat{\omega}$ is even an Hadamard state, as proved in [14,19]. We thus have the following result:

Corollary 4. *In the notation of Proposition 3, let $n = d + 1 = 4$. For every quasifree Hadamard state ω (resp., any state which is locally normal to $\hat{\omega}$) and sufficiently small, regularly shaped $\mathcal{O}_\sharp$, $\mathcal{R}_\omega(\mathcal{O}_\sharp)$ is isomorphic to the unique hyperfinite type III_1 -factor.* □

(It should be possible to establish this result along the lines indicated in arbitrary spacetime dimensions.)

Next, we shall generalize the split-property for quasifree Hadamard states to the case of an arbitrary globally hyperbolic spacetime.

Proposition 5. *Let $(\mathcal{M}, g)$ be a four-dimensional globally hyperbolic spacetime, C a Cauchy-surface for $(\mathcal{M}, g)$, and (nota 1) $\overline{C_\flat} \subset C_\sharp$ with $C_\sharp^c \neq \emptyset$. If ω is any quasifree Hadamard state on the Weyl-algebra $\mathcal{A}_{KG}$ of the massive KG-field in $(\mathcal{M}, g)$, then we have the split-inclusion*

$$\mathcal{R}_\omega(\mathcal{O}_\flat) \subset \mathcal{N} \subset \mathcal{R}_\omega(\mathcal{O}_\sharp) \tag{29}$$

with a suitable type I -factor $\mathcal{N}$ acting on the GNS-Hilbert-space of ω.

Proof. Given $(\mathcal{M}, g)$ and C, it will be shown below that there is a globally hyperbolic spactime $(\widetilde{\mathcal{M}}, \tilde{g})$ with the following properties:

(i) A neighbourhood $\mathcal{U}$ of C in $\mathcal{M}$ is isometrically isomorphic to a neighbourhood $\widetilde{\mathcal{U}}$ of a Cauchy-surface $\widetilde{C}$ in $\widetilde{\mathcal{M}}$, and the isometry is also an isometry between C and $\widetilde{C}$.

(ii) $(\widetilde{\mathcal{M}}, \tilde{g})$ coincides to the past of a Cauchy-surface S in $\widetilde{\mathcal{M}}$ with the $(-\infty, \tau) \times \hat{C}$ -part (for some $\tau < 0$) of the ultrastatic spacetime foliated by some complete Riemannian 3-manifold $(\hat{C}, \gamma)$.

(iii) Let $\widetilde{C}_\flat$, $\widetilde{C}_\sharp \subset \widetilde{C}$ be the isometric images of $C_\flat$, $C_\sharp$, and denote by $\hat{C}(t)$, $t \in (-\infty, \tau)$ the natural foliation of the ultrastatic part of $(\widetilde{\mathcal{M}}, \tilde{g})$. Writing

$$\hat{C}_{t\flat} := int(J(\widetilde{C}_\flat)) \cap \hat{C}(t)$$

$$\hat{C}_{\sharp t} := int(D(\widetilde{C}_\sharp)) \cap \hat{C}(t),$$

we find some $t < \tau$ such that

$$\overline{\mathcal{C}_{t\flat}} \subset \hat{\mathcal{C}}_{\sharp t}\,. \tag{30}$$

(Notice that the order of the indices reflects the different definitions.)

Now by (i), ω induces canonically a quasifree Hadamard state $\tilde{\omega}$ on the Weyl-algebra $\tilde{\mathcal{A}}_{KG}$ of the KG-field in $(\widetilde{\mathcal{M}}, \tilde{g})$ which is canonically identified with $\mathcal{A}_{KG}$. So the by (i) the split-inclusion is proved if we can show that we have the split-inclusion

$$\pi_{\tilde{\omega}}(\tilde{\mathcal{A}}(\tilde{\mathcal{O}}_\flat))'' \subset \widetilde{\mathcal{N}} \subset \pi_{\tilde{\omega}}(\tilde{\mathcal{A}}(\tilde{\mathcal{O}}_\sharp))''$$

(where, of course, $\tilde{\mathcal{O}}_\flat = int(D(\tilde{\mathcal{C}}_\flat))$, and same for $\flat \to \sharp$). Writing for t as in (30) $\hat{\mathcal{O}}_{t\flat} := int(D(\hat{\mathcal{C}}_{t\flat}))$, and defining $\hat{\mathcal{O}}_{\sharp t}$ similarly, we have by (ii) and (iii) that

$$\pi_{\tilde{\omega}}(\tilde{\mathcal{A}}(\tilde{\mathcal{O}}_\flat))'' \subset \pi_{\tilde{\omega}}(\tilde{\mathcal{A}}(\hat{\mathcal{O}}_{t\flat}))'' \subset \widetilde{\mathcal{N}} \subset \pi_{\tilde{\omega}}(\tilde{\mathcal{A}}(\hat{\mathcal{O}}_{\sharp t}))'' \subset \pi_{\tilde{\omega}}(\tilde{\mathcal{A}}(\tilde{\mathcal{O}}_\sharp))''$$

with some suitable type I -factor $\widetilde{\mathcal{N}}$ on the GNS-Hilbert-space of $\tilde{\omega}$. The existence of $\widetilde{\mathcal{N}}$ is a consequence of Proposition 3 (and the remarks following it), in view of relation (30).

So we are left with having to construct the spacetime $(\widetilde{\mathcal{M}}, \tilde{g})$ with the properties (i)-(iii). This can be done by using the method of Appendix C in [14], as follows. Choose a neighbourhood $\mathcal{V}$ of $\mathcal{C}$ contained in a normal neighbourhood of $\mathcal{C}$ such that $\mathcal{C}$ is also a Cauchy-surface for $\mathcal{V}$. Then choose some Cauchy-surface Σ in $\mathcal{V}$ lying to the past of $\mathcal{C}$ ($\Sigma \in int\, J^-(\mathcal{C}, \mathcal{V})$) such that $J^-(\mathcal{C}_\flat) \cap J^+(\Sigma) \subset int(D^-(\mathcal{C}_\sharp)) \cap J^+(\Sigma)$. (As a consequence of global hyperbolicity, such a choice of Σ is always possible.) $\mathcal{V}$ is, by the normal exponential map ϕ for $\mathcal{C}$, diffeomorphic to an open neighbourhood $\tilde{\mathcal{V}}$, in $\mathbf{R} \times \mathcal{C}$, of $\tilde{\mathcal{C}} = \{0\} \times \mathcal{C} = \phi(\mathcal{C})$. Using Gaussian normal coordinates (t, x^i) around a point of $\tilde{\mathcal{C}}$, the metric $\phi_* g$ on $\tilde{\mathcal{V}}$ takes the form

$$dt^2 - \gamma_{ij}(t,x)dx^i dx^j\,. \tag{31}$$

Write $\tilde{\mathcal{C}}_\flat := \phi(\mathcal{C}_\flat)$, $\tilde{\Sigma} := \phi(\Sigma)$ etc., and $\hat{\mathcal{C}}(t) = \{t\} \times \mathcal{C}$ for $t \in \mathbf{R}$. Now choose some $t < 0$ such that (in $(\tilde{\mathcal{V}}, \phi_* g)$)

$$\hat{\mathcal{C}}(t) \cap D^-(\mathcal{C}_\sharp) \subset J^+(\Sigma)$$

(which is possible by compactness of $\mathcal{C}_\sharp$). This implies (in $(\tilde{\mathcal{V}}, \phi_* g)$)

$$J^-(\tilde{\mathcal{C}}_\flat) \cap \hat{\mathcal{C}}(t) \subset int(D^-(\tilde{\mathcal{C}}_\sharp)) \cap \hat{\mathcal{C}}(t)\,. \tag{32}$$

Next choose some $\tau_1 \in (t, 0)$, a neighbourhood $\tilde{\mathcal{U}}$ of $\tilde{\mathcal{C}}$ with $\overline{\tilde{\mathcal{U}}} \subset \tilde{\mathcal{V}} \cap int\, J^+(\hat{\mathcal{C}}(\tau_1))$ and a function $f \in C^\infty(\mathbf{R} \times \mathcal{C}, \mathbf{R})$ with $0 \le f \le 1$, $f \equiv 0$ on $\tilde{\mathcal{U}}$, and $f \equiv 1$ outside of the closure of $\tilde{\mathcal{V}}$. Now let $\tilde{\gamma}$ be a complete Riemannian metric for $\tilde{\mathcal{C}}$, and $\beta \in C^\infty(\mathbf{R} \times \mathcal{C}, (0, \infty))$ a function equal to unity on $\tilde{\mathcal{U}}$ and on $(-\infty, \tau_1) \times \mathcal{C}$. Define a Lorentzian metric $\tilde{g}$ on $\mathbf{R} \times \mathcal{C}$ by setting, in coordinates as used for (31), the coordinate expression of $\tilde{g}$ equal to

$$\beta(t,x)dt^2 - \Big(((1 - f(t,x))\gamma_{ij}(t,x) + f(t,x)\tilde{\gamma}_{ij}(x)\Big)\, dx^i dx^j \tag{33}$$

By making β sufficiently small outside of the region where it is demanded to be equal to 1, we can ensure that $(\widetilde{\mathcal{M}} := \mathbf{R} \times \mathcal{C}, \tilde{g})$ is globally hyperbolic, and also that (32) remains valid when J and D are defined with respect to the new metric $\tilde{g}$ (implying (30)). So if we set $\mathcal{U} := \phi^{-1}(\tilde{\mathcal{U}})$, $\hat{\mathcal{C}} := \tilde{\mathcal{C}}$ and $S := \hat{\mathcal{C}}(\tau)$ for some $t < \tau < \tau_1$, we have all the objects with the required properties (i)-(iii). □

5 Haag-Duality

Finally, we show that Haag-duality holds also in certain situations where the spacetime is non-flat. Let $(\mathcal{M}, g)$ be the $n = d + 1$-dimensional ultrastatic spacetime foliated by the complete d-dimensional Riemannian manifold (M, γ), and $\mathcal{C} = M(t_0)$ for an arbitrary $t_0 \in \mathbf{R}$. Then define $\mathsf{A} \subset \mathcal{A}[\mathbf{D}_\mathcal{C}, \delta_\mathcal{C}]$ as the C^*-inductive limit of all the $\mathcal{A}(\mathcal{O}_\flat)$ (*nota 1,2*). A is the algebra of quasilocal observables of the (massive) KG-field in $(\mathcal{M}, g)$; it is independent of the choice of the Cauchy-surface and in general a proper sub-algebra of $\mathcal{A}_{KG}$ unless M is compact. However, if π is a regular representation of $\mathcal{A}_{KG}$, then we have $\pi(\mathsf{A})'' = \pi(\mathcal{A}_{KG})''$. The GNS-representations of quasifree states on $\mathcal{A}_{KG}$ are regular.. Given some $\mathcal{O}_\flat$, we define $\mathsf{A}(\mathcal{O}_\flat^c)$ as the C^*-inductive limit of all the $\mathcal{A}(\mathcal{O}_\sharp)$ with $\overline{\mathcal{C}_\sharp} \subset \mathcal{C}_\flat^c$.

Proposition 6. *Let $\hat{\omega}$ be the canonical vacuum state on $\mathcal{A}_{KG}$, and π an irreducible representation of A, locally normal to $\pi_{\hat{\omega}}$. Then we have duality for π, i. e.*

$$\pi(\mathsf{A}(\mathcal{O}_\flat^c))'' = \pi(\mathcal{A}(\mathcal{O}_\flat))'$$

provided that the boundary of $\mathcal{C}_\flat$ is contained in the union of finitely many smooth, $d-1$-dimensional submanifolds of $\mathcal{C}$.

Remark. For $n = 4$, the GNS-representations of pure quasifree Hadamard states ω fulfil the assumptions of Prop. 6 ([25]), and thus duality holds for $\pi = \pi_\omega$.

Proof. We introduce the following definitions:

$$\mathsf{A}_{\widehat{\omega}}(\mathcal{O}_\flat^c) := \overline{\bigcup_{\overline{\mathcal{O}_\sharp}\subset\mathcal{O}_\flat^c} \mathcal{R}_{\widehat{\omega}}(\mathcal{O}_\sharp)}^{\|\cdot\|}$$

$$\mathsf{A}_{\widehat{\omega}} := \overline{\bigcup_{\mathcal{O}_\sharp\subset\mathcal{M}} \mathcal{R}_{\widehat{\omega}}(\mathcal{O}_\sharp)}^{\|\cdot\|}$$

$$\mathcal{R}_{\widehat{\omega}}(\mathcal{O}_\flat)^c := \mathcal{R}_{\widehat{\omega}}(\mathcal{O}_\flat)' \cap \mathsf{A}_{\widehat{\omega}}$$

If we can show that
(a) $\mathsf{A}_{\widehat{\omega}}(\mathcal{O}_\flat^c)'' = \mathcal{R}_{\widehat{\omega}}(\mathcal{O}_\flat)'$ for all $\mathcal{O}_\flat$ with the boundary of $\mathcal{C}_\flat$ as in the statement of the Proposition, i. e. if Haag-duality holds in the representation $\pi_{\widehat{\omega}}$, and
(b) for all $\mathcal{O}_\flat$, $\mathcal{O}_{\sharp 1}$ with $\overline{\mathcal{O}_\flat} \subset \mathcal{O}_{\sharp 1}$, there is $\mathcal{O}_{\sharp 2}$ with $\overline{\mathcal{O}_{\sharp 1}} \subset \mathcal{O}_{\sharp 2}$ such that

$$\mathcal{R}_{\widehat{\omega}}(\mathcal{O}_\flat)^c \cap \mathcal{R}_{\widehat{\omega}}(\mathcal{O}_{\sharp 1}) \subset (\mathsf{A}_{\widehat{\omega}}(\mathcal{O}_\flat^c) \cap \mathcal{R}_{\widehat{\omega}}(\mathcal{O}_{\sharp 2}))''$$

then the results of [12] and [23] imply, since we have Proposition 3, the assertion of Prop. 6. So let us check that (a) holds. Let $(\mathbf{k},\mathcal{H}) = (\mathbf{k}^{t_0},\mathcal{H}^{t_0})$ be the one-particle Hilbert-space structure of $\widehat{\omega}$. If $\mathcal{L}$ is a real-linear subspace of $\mathcal{H}$, we set $\mathcal{L}^\vee := \{\psi \in \mathcal{H} \mid \mathrm{Im}\langle\psi,\chi\rangle = 0\ \forall\chi\in\mathcal{L}\}$. Then it holds that (cf. (7))

$$\mathcal{W}(\mathcal{L})' = \mathcal{W}(\mathcal{L}^\vee)\,. \tag{34}$$

(This has been proved by several authors [1,13,20] through a variety of methods. The first proof is apparently the one by Araki). Since $\mathsf{A}_{\widehat{\omega}}(\mathcal{O}_\flat^c)'' = \mathcal{W}(\mathcal{L}(\mathcal{O}_\flat^c))$, it is not difficult to deduce from (34) that

$$\mathsf{A}_{\widehat{\omega}}(\mathcal{O}_\flat^c)'' = \mathcal{W}(\mathcal{L}(\mathcal{O}_\flat^c)) = \mathcal{W}(\mathcal{L}(\mathcal{O}_\flat)^\vee) = \mathcal{R}_{\widehat{\omega}}(\mathcal{O}_\flat)' \tag{35}$$

holds exactly if $\mathcal{L}(\mathcal{O}_\flat^c)$ is dense in $\mathcal{L}(\mathcal{O}_\flat)^\vee$, and the latter is the case if and only if $\mathcal{L}(\mathcal{O}_\flat^c) + i\mathcal{L}(\mathcal{O}_\flat)$ is dense in $\mathcal{H}$. (See [15] for further details. One may

also use directly the results of [22] to derive such a criterion for duality.) Equipped with this criterion, we note that under our assumptions on $\partial\mathcal{C}_\flat$, the boundary of $\mathcal{C}_\flat$,

$$\mathcal{L}(\mathcal{O}_\flat^c) + i\mathcal{L}(\mathcal{O}_\flat) = A^{1/4}(C_0^\infty(\mathcal{C}\backslash\partial\mathcal{C}_\flat,\mathbf{R})) + iA^{-1/4}(C_0^\infty(\mathcal{C}\backslash\partial\mathcal{C}_\flat,\mathbf{R}))$$

is dense in $\mathcal{H} = L^2_{\mathbf{C}}(\mathcal{C},\nu_\gamma)$. This follows from section 6 and Appendix E in [25]. (The argument given there is for $n = 4$, but can be seen to be independent of n.) So we have (a).

It remains to be shown that (b) is satisfied. It is obvious that (b) is obtained if we can show that for given $\mathcal{O}_\flat$ and $\mathcal{O}_{\sharp 1}$ with $\overline{\mathcal{O}_\flat} \subset \mathcal{O}_{\sharp 1}$ there is $\mathcal{O}_{\sharp 2}$ with $\overline{\mathcal{O}_{\sharp 1}} \subset \mathcal{O}_{\sharp 2}$ such that

$$\mathcal{R}_{\widehat{\omega}}(\mathcal{O}_\flat^c) \cap \mathcal{R}_{\widehat{\omega}}(\mathcal{O}_{\sharp 1}) \subset \mathcal{R}_{\widehat{\omega}}(\mathcal{O}_\flat^c \cap \mathcal{O}_{\sharp 2})$$

and this amounts to demonstrating that (cf. Thm. 1 (4) in [1])

$$\overline{\mathcal{L}(\mathcal{O}_\flat^c)} \cap \overline{\mathcal{L}(\mathcal{O}_{\sharp 1})} \subset \overline{\mathcal{L}(\mathcal{O}_\flat^c \cap \mathcal{O}_{\sharp 2})}\,. \tag{36}$$

So let $\overline{\mathcal{O}_\flat} \subset \mathcal{O}_{\sharp 1}$ and $\overline{\mathcal{O}_{\sharp 1}} \subset \mathcal{O}_{\sharp 2}$, and let $\psi \in \overline{\mathcal{L}(\mathcal{O}_\flat^c)} \cap \overline{\mathcal{L}(\mathcal{O}_{\sharp 1})}$. Then there are two sequences, $\mathbf{u}^{(j)} \in \mathbf{D}_{\mathcal{C}_\flat^c}$ and $\mathbf{v}^{(j)} \in \mathbf{D}_{\mathcal{C}_{\sharp 1}}$ such that

$$\mathbf{k}(\mathbf{u}^{(j)}) = \frac{1}{\sqrt{2}}\left(A^{1/4}u_0^{(j)} + iA^{-1/4}u_1^{(j)}\right) \to \psi$$

and $\mathbf{k}(\mathbf{v}^{(j)}) \to \psi$. Choose some smooth, real-valued function χ compactly supported in $\mathcal{C}_{\sharp 2}$ and with $\chi \equiv 1$ on $\mathcal{C}_{\sharp 1}$. Set $\chi\mathbf{u}^{(j)} := \chi u_0^{(j)} \oplus \chi u_1^{(j)}$ and define $\chi\mathbf{v}^{(j)}$ analogously. We have $\chi\mathbf{v}^{(j)} = \mathbf{v}^{(j)}$, so $\mathbf{k}(\chi\mathbf{v}^{(j)}) \to \psi$. But we also have $\mathbf{k}(\chi(\mathbf{u}^{(j)} - \mathbf{v}^{(j)})) \to 0$, for we have

$$\|\mathbf{k}(\chi(\mathbf{u}^{(j)} - \mathbf{v}^{(j)}))\|_{L^2} \leq const.\, \|\mathbf{k}(\mathbf{u}^{(j)} - \mathbf{v}^{(j)})\|_{L^2}$$

This follows since

$$\|\mathbf{k}(\mathbf{u})\|_{L^2}^2 = \frac{1}{2}\left(\|A^{1/4}u_0\|_{L^2}^2 + \|A^{-1/4}u_1\|_{L^2}^2\right)$$

and, for $\rho \in (-1,1)$, one has

$$\|A^\rho(\chi f)\|_{L^2} \leq C_\chi \|A^\rho f\|_{L^2} \quad \forall\, f \in C_0^\infty(M) \tag{37}$$

with suitable $C_\chi > 0$. (For the proof of the latter assertion note that it holds for $\rho = 1$. Thus $A\chi A^{-1}$ is continuous in $L^2_{\mathbb{C}}(M, \nu_\gamma)$, and so is $A^{-1}\chi A$. Hence the continuity of $A^\rho \chi A^{-\rho}$ for $-1 < \rho < 1$ can be proved by interpolation.) Therefore ψ is approached by $\mathbf{k}(\chi \mathbf{u}^{(j)}) \in \mathbf{k}(\mathbf{D}_{\mathcal{C}_\flat^c \cap \mathcal{C}_{12}})$ and hence lies in $\overline{\mathcal{L}(\mathcal{O}_\flat^c \cap \mathcal{O}_{\sharp 2})}$. □

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