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The Longitudinal Structure Function $F_L(x, Q^2)$ at Small x [1]

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Abstract

The gluon contributions to the structure function $F_L(x, Q^2)$ is calculated using $k_\perp$–factorization [1]. A generalization of this factorization is given which allows the expression of structure functions and hard cross sections in terms of quantities being well defined within perturbative QCD.

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With the advent of HERA the electromagnetic proton structure functions can be determined with high precision down to values of Bjorken $x \gtrsim 10^{-4}$ at $Q^2 \gtrsim 10\,\mathrm{GeV}^2$ [2]. Among the different structure functions which will be measured, $F_L(x, Q^2)$ is known to be particularly sensitive to the gluon distribution of the proton and may be used as an observable to determine this distribution itself [3, 4].

Usually, the evolution of the proton structure functions is calculated in fixed order perturbation theory assuming the validity of the collinear approach of the parton model and mass–factorization. For not too small values of x, $x \geq 10^{-2}$, this method works well, as demonstrated by various deep–inelastic scattering experiments in the past. However, if x becomes very small, e.g. $x \sim 10^{-3}...10^{-4}$, these assumptions may turn out not to be valid any longer. As discussed in [5] one should thoroughly account for the $k_\perp$ effects of the parton entering the hard scattering process in this range. This leads to a modification of both the parton picture and the factorization scheme used, in comparison with the calculations made for the range of medium values of x, as will be discussed below. Previously, this method has been used in some applications in the high–energy limit, i.e. with further appropriate approximations, in [1], [5]–[7].

In this note we calculate the longitudinal structure function $F_L(x, Q^2)$, as one example, taking the $k_\perp$ effects of the initial state gluon into account without an approximation in x, in order to obtain a coefficient function which is valid in the full x range. This is particularly important due to the fact that the gluon density rises rapidly at small x. Since $F_L(x, Q^2)$ is obtained as a Mellin convolution in x of the gluon density and a coefficient function, the small–x part of the former samples the large–x part of the latter and vice versa. To obtain a consistent perturbative description the $k_\perp$ factorization relation originally used in [1] can not be applied here directly. It has to be transformed into a relation which allows to express $F_L(x, Q^2)$ only in terms of quantities which are fully defined within perturbative QCD.

$k_\perp$ Factorization. The calculation of the deep inelastic scattering cross section presumes factorization of the 'pointlike' hard cross section of the subprocess and the parton distributions. In the case of incoming partons which are collinear with the initial state hadrons the factorization relation is

$$H(x,\mu^2) = \int_0^1 dx_1 dx_2 \delta(x - x_1 x_2) G(x_1,\mu^2) \sigma_H^{pt}(x_2,\mu^2) \qquad (1)$$

for an observable $H(x,\mu^2)$. Here, μ^2 denotes an appropriate factorization scale, $G(x_1,\mu^2)$ the parton distribution and $\sigma_H^{pt}(x_2,\mu^2)$ the cross section of the hard subprocess. The $k_\perp$–dependent factorization structure was derived in [1] for the case that the initial state partons are gluons. One obtains

$$H(x,\mu^2) = \int \frac{d^2 \boldsymbol{k}}{\pi} \int_0^1 dx_1 dx_2 \delta(x - x_1 x_2) \mathcal{F}(x_1, \boldsymbol{k}) \sigma_H^{pt}(x_2, \boldsymbol{k}, \mu^2) \qquad (2)$$

Here, $\mathcal{F}(x, \boldsymbol{k})$ is defined by (cf. [8])

$$G(x,\mu^2) = \int_0^{\mu^2} d\boldsymbol{k}^2 \mathcal{F}(x, \boldsymbol{k}) \qquad (3)$$

Because the gluon momentum k is expressed in the Sudakov representation $k^\mu = \xi p_1^\mu + \eta p_2^\mu + k_\perp^\mu$ it may depend on the choice of the two lightlike vectors p_1 and p_2 in general, which can induce some scheme–dependence in using (2). For the calculation of proton structure functions a natural choice of the lightlike vectors is $q' = q + xP$ and P [1], with q the 4–momentum transferred to the proton, P the proton momentum, and x the Bjorken variable[2].

[1] In ref. [1] the four vectors l_e and P were choosen instead.
[2] Fermion masses were neglected whenever possible.

Since the kinematical range of $|\boldsymbol{k}|$ is $0 \le |\boldsymbol{k}| \le K_{max} = \sqrt{Q^2(1-x)/x}$, eq. (2) is in general not an appropriate definition, because $\mathcal{F}(x, \boldsymbol{k})$ is not defined at values of $\boldsymbol{k}^2$ smaller than $Q_0^2 \sim 1\,\mathrm{GeV}^2$ in perturbative QCD, since non–perturbative terms become significant. Factorization relations are introduced to **separate** non–perturbative parts from perturbative terms. In the case of hadronic structure functions one may, fortunately, rewrite (2) in a way that $H(x,\mu^2)$ can indeed be expressed by quantities defined perturbatively. Since the $K^2 \equiv -k^2$ dependent coefficient function $\sigma_H^{pt}(x, \boldsymbol{k}, \mu^2)$ is calculated in fixed order perturbation theory only, it is not intended to describe the observable $H(x,\mu^2)$ at arbitrary small scales μ^2. One chooses, e.g. in the case of hadronic structure functions, $\mu^2 = Q^2 \equiv -q^2 \gg Q_0^2 \simeq$ few GeV^2. Only for these values of Q^2 a comparison with experimental data shall be done. Therefore, in the range of $K^2 \le Q_0^2$ the coefficient functions $\sigma_H^{pt}(x, K^2/Q^2) \equiv f_i^{qG}(x, K^2/Q^2)$ approach the value $f_i^{qG}(x, K^2/Q^2 \to 0)$. Thus, one may rewrite (2) identically into

$$F_i(x,Q^2) = \sum_q \left\{ \int_x^1 \frac{d\eta}{\eta} f_i^{qG}\left(\frac{x}{\eta}\right) \eta G(\eta, Q_0^2) + \int_x^1 \frac{d\eta}{\eta} \int_{Q_0^2}^{K^2_{max}} dK^2 f_i^{qG}\left(\frac{x}{\eta}, \frac{K^2}{Q^2}\right) \frac{\partial \eta G(\eta, K^2)}{\partial K^2} \right\} \quad (4)$$

if f_i^{qG} contains no collinear singularity for $K^2 \to 0$, which is the case for $F_L(x,Q^2)$ [3]. This is also the reason for the well-known fact that $f_L^{qG}(x, K^2/Q^2 \to 0)$ is scheme–independent in $\mathcal{O}(\alpha_s)$. In (4) $K^2_{max} = Q^2(\eta - x)/x$. Note, that (4) depends on the gluon distribution $xG(x, K^2)$ only at virtualities K^2 which are large enough such that it can be considered as a parton distribution. Eq. (4) will subsequently be used for the calculation of the structure function $F_L(x,Q^2)$.

The Structure Function. The deep inelastic scattering cross section may be written as

$$\frac{d^2\sigma}{dQ^2 dy} = 2\pi\alpha^2 \frac{M s}{(s-M^2)^2} \frac{1}{Q^4} L_{\mu\nu} W^{\mu\nu} \quad (5)$$

For only electromagnetic interactions[4] the leptonic and hadronic tensor $L_{\mu\nu}$ and $W_{\mu\nu}$ are given by

$$\begin{aligned} L_{\mu\nu} &= 2\left[l_\mu l'_\nu + l'_\mu l_\nu - g_{\mu\nu} l.l'\right] \\ W_{\mu\nu} &= \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1(x,Q^2) + \frac{1}{M^2}\left[\left(P_\mu - \frac{P.q}{q^2} q_\mu\right)\left(P_\nu - \frac{P.q}{q^2} q_\nu\right)\right] W_2(x,Q^2) \end{aligned} \quad (6)$$

with l and l' the incoming and outgoing lepton 4–momenta, and M the proton mass. In the Bjorken–limit the longitudinal structure function $F_L(x,Q^2)$ is obtained via the projection

$$\frac{P.q}{M^2} W_2(x,Q^2) - 2x W_1(x,Q^2) \to F_L(x,Q^2) = \frac{8x^3}{Q^2} P_\mu P_\nu W^{\mu\nu} \quad (7)$$

The dominant contribution to $F_L(x,Q^2)$ is obtained from the gluon contribution. The corresponding coefficient function is given by

$$\begin{aligned} f_L^{qG}(K^2, x, Q^2) &= \frac{\alpha_s e_q^2}{4\pi} \left\{ \frac{4Q^4}{K^4 x} G_{1L}(\beta,\zeta) + \frac{xQ^2}{K^2} \frac{1}{\sqrt{1-\zeta}} \log\left|\frac{1+\sqrt{1-\zeta}}{1-\sqrt{1-\zeta}}\right| G_{2L}(\beta,\zeta) \right. \\ &\quad \left. + \frac{2xQ^2}{K^2} G_{3L}(\beta,\zeta) \right\} \end{aligned} \quad (8)$$

[3]The corresponding relation in the case of also collinear singular terms is given in [12].

[4]At $Q^2 \le 500\,\mathrm{GeV}^2$ the contribution due to $\gamma - Z$ interference and $|Z|^2$–terms turns out to be very small in the kinematical range accessible at HERA [9].

where

$$\zeta = \frac{4K^2x^2}{Q^2} \qquad \cos\beta = \frac{1-\zeta/2}{\sqrt{1-x\zeta}} \quad (9)$$

and β denotes the angle between gluon and proton in the virtual photon–virtual gluon cms. The functions $G_{iL}(\beta,\zeta)$ in (8) may be expressed in a polynomial form by

$$G_{iL}(\beta,\zeta) = -\sum_{j=0}^{4} g_{ji}^{L}(\beta) \left(\frac{\zeta}{W(\zeta)}\right)^j \quad (10)$$

where

$$W(\zeta) = 1 - \zeta + \sqrt{1-\zeta} \quad (11)$$

Finally, the coefficients $g_{ji}^{(L)}$ in (10) are:

$$\begin{aligned}
g_{01}^{(L)}(\beta) &= -\frac{1}{8} + \frac{1}{4}\cos\beta - \frac{1}{4}\cos^3\beta + \frac{1}{8}\cos^4\beta \\
g_{02}^{(L)}(\beta) &= -\frac{1}{4} + 2\cos\beta - \cos^2\beta - 3\cos^3\beta + \frac{9}{4}\cos^4\beta \\
g_{03}^{(L)}(\beta) &= -\frac{1}{4} + 6\cos\beta - \frac{9}{2}\cos^2\beta - 10\cos^3\beta + \frac{35}{4}\cos^4\beta \\
g_{11}^{(L)}(\beta) &= \cos\beta - \frac{3}{4}\cos^2\beta - \frac{3}{2}\cos^3\beta + \frac{5}{4}\cos^4\beta \\
g_{12}^{(L)}(\beta) &= \frac{1}{4} + \frac{13}{2}\cos\beta - \frac{15}{2}\cos^2\beta - \frac{21}{2}\cos^3\beta + \frac{45}{4}\cos^4\beta \\
g_{13}^{(L)}(\beta) &= 1 + 18\cos\beta - 24\cos^2\beta - 30\cos^3\beta + 35\cos^4\beta \\
g_{21}^{(L)}(\beta) &= \frac{3}{16} + \frac{9}{8}\cos\beta - \frac{9}{4}\cos^2\beta - \frac{15}{8}\cos^3\beta + \frac{45}{16}\cos^4\beta \\
g_{22}^{(L)}(\beta) &= \frac{5}{4} + \frac{27}{4}\cos\beta - 15\cos^2\beta - \frac{45}{4}\cos^3\beta + \frac{75}{4}\cos^4\beta \\
g_{23}^{(L)}(\beta) &= \frac{7}{2} + 18\cos\beta - 42\cos^2\beta - 30\cos^3\beta + \frac{105}{2}\cos^4\beta \\
g_{31}^{(L)}(\beta) &= \frac{3}{16} + \frac{6}{16}\cos\beta - \frac{15}{8}\cos^2\beta - \frac{5}{2}\cos^3\beta + \frac{35}{4}\cos^4\beta \\
g_{32}^{(L)}(\beta) &= \frac{9}{8} + \frac{9}{4}\cos\beta - \frac{45}{4}\cos^2\beta - \frac{15}{4}\cos^3\beta + \frac{105}{8}\cos^4\beta \\
g_{33}^{(L)}(\beta) &= 3 + 6\cos\beta - 30\cos^2\beta - 10\cos^3\beta + 35\cos^4\beta \\
g_{41}^{(L)}(\beta) &= \frac{3}{64} - \frac{15}{32}\cos^2\beta + \frac{35}{64}\cos^4\beta \\
g_{42}^{(L)}(\beta) &= \frac{9}{32} - \frac{45}{16}\cos^2\beta + \frac{105}{32}\cos^4\beta \\
g_{43}^{(L)}(\beta) &= \frac{3}{4} - \frac{15}{2}\cos^2\beta + \frac{35}{4}\cos^4\beta
\end{aligned} \quad (12)$$

The structure function $F_L(x,Q^2)$ is then given by (4) for $i = L$.

In the limit $K^2 \to 0$ the coefficient function $f_L^{qG}(x,Q^2)$ simplifies considerably and one obtains the well–known result [10]

$$f_L^{qG(0)}(x,Q^2) = \frac{2}{\pi} e_q^2 \alpha_s x^2 (1-x) \quad (13)$$

The gluonic part of $F_L(x,Q^2)$ is represented by

$$F_{L(0)}(x,Q^2) = \sum_q \int_x^1 \frac{dz}{z} f_{2,L}^{qG}\left(\frac{x}{z}\right) zG(z,Q^2) \tag{14}$$

since the integral over K^2 in (4) can be carried out analytically. (4) is the generalization of (14) for the case of $k_\perp$-factorization. Note that in the limit $K^2 \to 0$ the complete coefficient function as derived in the Altarelli–Parisi approach is obtained, which is due to the fact that (4) was derived without further approximations with respect to the x-behaviour. Furthermore, (4) is an expression for $F_L(x,Q^2)$ depending only on quantities accessible in perturbative QCD. We have not been specific in expressing $\partial xG(x,K^2)/\partial K^2$, since at small x various dynamical effects may influence the scaling violations of the the single particle gluon distribution. Among them are effects due to the Lipatov pomeron [8], and gluon–recombination effects [11]. A detailed theoretical investigation of these terms within perturbative QCD requires the completion of further efforts. On the other hand, expressions like (4) allow to extract the gluon distribution from measured structure functions, and thus to compare theoretical predictions on the evolution of $xG(x,Q^2)$ directly.

Results of a numerical comparison between (4) and (14), and on the corresponding behaviour of $F_2(x,Q^2)$ are given elsewhere [12].

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