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SUPERSELECTION SECTORS WITH INFINITE STATISTICAL DIMENSION *

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Abstract

Examples for sectors with infinite statistical dimension are presented

1 Introduction

The occurence of anomalous particle statistics (anyons,plektons) seems to be a relevant feature of low dimensional quantum systems. Two quantities characterize the intrinsic statistics of a particle: the statistics phase which is +1 for bosons and −1 for fermions, may assume any value on S^1 (hence the name "anyons") and is related in $2+1$ dimensions to the spin of the particle, and the statistical dimension which is a number $d \geq 1$ characterizing the number of internal degrees of freedom and showing up in the increase of the number of states with the particle number. This number cannot assume every value, for bosons and fermions it is an integer, and in general it must be the square root of a possible value of the Jones index.

Let me first recall the interplay between statistics and inner degrees of freedom in quantum mechanics as it occurs for instance in atomic physics as long as spin dependent forces are neglected. The single particle space is a space of d-component wave functions,

$$\tilde{\mathcal{H}}_1 = \mathcal{H}_1 \otimes \mathbb{C}^d \quad , \quad \mathcal{H}_1 = \mathcal{L}^2(\mathbb{R}^3)$$

and the symmetric resp. antisymmetric Fock space $\tilde{\mathcal{H}}_\pm$ is the direct sum of symmetrized resp. antisymmetrized n-fold tensor powers of $\tilde{\mathcal{H}}_1$,

$$\tilde{\mathcal{H}}_\pm = \bigoplus_{n=0}^{\infty} (\underbrace{\tilde{\mathcal{H}}_1 \otimes \cdots \otimes \tilde{\mathcal{H}}_1}_{n})_\pm .$$

One may now pass over to a reduced description where the unobservable degrees of freedom are eliminated. The reduced single particle space is

$$\mathcal{H}_n = \underbrace{\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_1}_{n} \otimes \mathbb{C}S_n$$

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where $\mathbb{C}S_n$ is the group algebra of the symmetric group S_n, together with the positive semidefinite scalar product ($e_i, i = 1,..n$ orthonormal basis of $\mathbb{C}^d$)

$$(\Phi_1 \otimes \cdots \otimes \Phi_n \otimes \sigma_1^{-1}, \Psi_1 \otimes \cdots \otimes \Psi_n \otimes \sigma_2^{-1})$$

$$= \sum_{i_1 \ldots i_n} d^{-n} (((\Phi_1 \otimes e_{i_{\sigma_1(1)}}) \otimes \cdots \otimes (\Phi_n \otimes e_{i_{\sigma_1(n)}}))_\pm , ((\Psi_1 \otimes e_{i_{\sigma_2(1)}}) \otimes \cdots \otimes (\Psi_n \otimes e_{i_{\sigma_2(n)}}))_\pm)$$

$$= \sum_{i_1 \ldots i_n} d^{-n} \sum_\sigma (\mathrm{sign}(\sigma)) \prod_j (\Phi_j, \Psi_{\sigma(j)})(e_{i_{\sigma_1(j)}}, e_{i_{\sigma_2\sigma(j)}})$$

$$= \sum_\sigma \prod_j (\Phi_j, \Psi_{\sigma(j)}) d^{-n} \mathrm{tr} U_n(\sigma_1 \sigma^{-1} \sigma_2^{-1})$$

where U_n is the representation

$$U_n(\sigma)\xi_1 \otimes \cdots \otimes \xi_n = (\mathrm{sign}(\sigma))\xi_{\sigma^{-1}(1)} \otimes \cdots \otimes \xi_{\sigma^{-1}(n)}$$

of S_n in $\mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d$. The statistics of the n-particle space is characterized by the functional $\varphi(\sigma) = d^{-n}\mathrm{tr}U_n(\sigma)$ on S_n. φ extends to a Markov trace on S_∞, i.e. it is a positive trace satisfying the relation

$$\varphi(\sigma\tau_n) = \varphi(\sigma)\varphi(\tau_n) \quad , \quad \varphi(\tau_n) = \pm\frac{1}{d}$$

for $\sigma \in S_n$ and τ_n denoting the transposition $(n, n+1)$.

This kind of statistics is called para bose(fermi) statistics of order d. It was a remarkable success of the theory of superselection sectors that this structure of the multiparticle spaces could be derived from general principles of quantum field theory [1]. One starts from a Haag Kastler net of local algebras of observables $\mathcal{A} = (\mathcal{A}(\mathcal{O}))_{\mathcal{O}\in\mathcal{K}}$ with $\mathcal{K}$ denoting the set of open double cones in Minkowski space and looks for all representations π of $\mathcal{A}$ which are related to a distinguished representation π_0 (the vacuum representation) by a localized endomorphism ρ,

$$\pi \simeq \pi_0 \circ \rho.$$

In this theory, one finds to each ρ a unitary operator $\epsilon_\rho \in \mathcal{A}$, the so-called statistics operator. $\epsilon := \epsilon_\rho$ satisfies the relations

$$\epsilon^2 = 1$$

$$\epsilon\rho(\epsilon)\epsilon = \rho(\epsilon)\epsilon\rho(\epsilon)$$

$$\epsilon\rho^2(A) = \rho^2(A)\epsilon \quad , \quad A \in \mathcal{A}$$

hence generates via $\tau_n \mapsto \rho^{n-1}(\epsilon)$ a unitary representation U_ρ of S_∞. ρ has a left inverse ϕ such that $\phi^n \circ U_\rho$ converges to a Markov trace charecerized by the statistics parameter $\phi(\epsilon) = \pm\frac{1}{d}$. (For details see [2].)

2 Statistical dimension and Jones index

From the preceding discussion one might think that the statistical dimension depends on the way how the representation π is related (through ρ) to the vacuum representation π_0. Actually, it turns out, that the statistical dimension is an intrinsic property of the representation π. Due to locality, one always has the inclusion of algebras

$$\pi(\mathcal{A}(\mathcal{O})) \subset \pi(\mathcal{A}(\mathcal{O}'))'.$$

Here $\mathcal{A}(\mathcal{O}')$ is the algebra generated by all algebras $\mathcal{A}(\mathcal{O}_1)$ with $\mathcal{O}_1$ contained in the spacelike complement $\mathcal{O}'$ of $\mathcal{O}$. One says that π satisfies Haag duality if both algebras are equal.

The vacuum representation π_0 is in many cases known to satisfy Haag duality [3,4]. It is this property of the vacuum representation from which the DHR analysis starts. There are also cases where the vacuum representation violates Haag duality, e.g. in the case of spontaneous breakdown of symmetry [5,6].

Let us now consider a theory where the local algebras of observables are the fixpoint algebras of a larger algebra (the "field algebra") under a compact group of automorphisms G, and where the vacuum representation satisfies Haag duality. The vacuum representation of the field algebra decomposes according to the representations of G. One finds that π satisfies Haag duality if and only if it corresponds to a one dimensional representation of G [7].

To get an insight why this might be true we look at the example of a free n-component Dirac field ψ in $d+1 \geq 3$ space time dimensions. The global gauge group $U(n)$ acts in a natural way on ψ, and the observables are defined to be the gauge invariant operators. Let Ω be the vector representing the vacuum state, and let π be the representation of the algebra $\mathcal{A}$ of observables with cyclic vector $\psi(f)\Omega$, f denoting an n-tupel of smooth smearing functions with compact support. The model has conserved currents

$$j_{\mu,a} =: \bar{\psi}\gamma_\mu T_a \psi :$$

where $\{T_a\}$ denotes an orthonormal basis in the Lie algebra of $U(n)$ (with respect to the Killing form). These currents implement the gauge transformations. Now let $\mathcal{O}$ be a double cone with spatial basis $B_{x,r} = \{y, y_0 = x_0, |\mathbf{y}-\mathbf{x}| < r\}$. Let g be a test function with total integral 1 and with support in the double cone $\mathcal{O}_\epsilon = \{x, |x_0| + |\mathbf{x}| < \epsilon\}$. Then the integrals

$$Q_{a,x} := \int_{B_{x,r+\epsilon}} j_{0,a}(g_y) d^d y$$

(g_y denotes the translate by y of g) implement the gauge transformations within $\mathcal{O}$ and therefore commute with all observables $A \in \mathcal{A}(\mathcal{O})$. The local Casimir operator

$$C = \sum_a Q_{a,x}^2$$

is gauge invariant and hence its bounded functions are elements of $\mathcal{A}$. They commute with all elements of $\mathcal{A}(\mathcal{O})$, while their images under π are in general not elements of $\pi(\mathcal{A}(\mathcal{O}'))''$, hence Haag duality is violated.

As was shown in [7], the statistical dimension d_π is greater than 1 if and only if π violates Haag duality. Now there is a quantitative measure for the relative size of a subalgebra A within another algebra B, namely the rank of B as a left or right module over A. In the framework of finite von Neumann algebras this is the Jones index $[B : A]$ [8]. Kosaki [9] generalized this index to properly infinite von Neumann algebras. Longo [10] then established the connection to the theory of superselection sectors by proving the formula

$$[\pi(\mathcal{A}(\mathcal{O}'))' : \pi(\mathcal{A}(\mathcal{O}))] = d_\pi^2 .$$

The formula remains valid in two dimensional Minkowski space. There the relation $\epsilon^2 = 1$ does not necessarily hold, so U_ρ is only a representation of the braid group, and the statistical dimension might be noninteger.

One may use Watatani's [11] generalization of the Jones index to C*-algebras in order to avoid the choice of a distinguished local von Neumann algebra $\mathcal{A}(\mathcal{O})$. One defines

$$\mathcal{A}_\pi = \overline{\bigcup_{\mathcal{O}} \pi(\mathcal{A}(\mathcal{O}'))'}$$

and uses a conditional expectation

$$E : \mathcal{A}_\pi \to \pi(\mathcal{A})$$

which respects the local structure,

$$E(\pi(\mathcal{A}(\mathcal{O}'))') = \pi(\mathcal{A}(\mathcal{O}))''.$$

The statistical dimension d_π appears as the best possible constant in the Pimsner-Popa [12] (resp. DHR) inequality

$$E(A) \geq d_\pi^{-2} A \quad , \quad A \geq 0, A \in \mathcal{A}_\pi .$$

For an irreducible representation π with finite statistical dimension E is uniquely fixed. It is given by the formula

$$E = \pi \circ \phi \circ \pi_0^{-1} \circ \mathrm{Ad}V$$

where V is a unitary intertwiner

$$V\pi(A) = \pi_0 \circ \rho(A) V \quad , \quad A \in \mathcal{A}.$$

ρ is a DHR endomorphism and ϕ the unique left inverse of ρ.

If π is not irreducible but has finite statistical dimension, then the image of any projection $e \in \pi(\mathcal{A})'$ under E is a number greater or equal to d_π^{-2}, hence there can be at most d_π^2 mutually orthogonal projections in $\pi(\mathcal{A})'$, so π must be a direct sum of a finite number of irreducible representations. This shows that representations of $\mathcal{A}$ with finite statistical dimension have a structure similar to those of a compact group. Actually, in $d \geq 3$ dimensions Doplicher and Roberts [13] proved that both structures are equivalent.

3 Finiteness of statistics

In the DHR theory, the composition of representations of the form

$$\pi \simeq \pi_0 \circ \rho$$

with localized endomorphisms ρ is defined by

$$\pi_1 \times \pi_2 \simeq \pi_0 \circ \rho_1 \rho_2$$

The statistical dimensions are multiplicative and additive,

$$d_{\pi_1 \times \pi_2} = d_{\pi_1} d_{\pi_2} \quad , \quad d_{\oplus \pi_i} = \sum d_{\pi_i}$$

hence the product of representations with finite statistical dimensions is always completely reducible into components which themselves have finite statistics. Now for irreducible DHR representations it was proven in [1] that $d_\pi < \infty$ if and only if there exists a representation $\bar{\pi}$ such that $\pi \times \bar{\pi}$ contains a subrepresentation which is equivalent to π_0.

In [14,15] it was shown that the statistical dimension is always finite if π is an irreducible translation covariant representation satisfying the spectrum condition such that the energy momentum spectrum contains an isolated mass shell (i.e. there are vectors in the representation space describing states of a single massive particle).

One does not expect that the proof can be generalized to theories with massless particles. In conformal field theory the situation could be better, since the structure of infrared clouds is determined by conformal covariance. We shall see that nevertheless the answer is negative, by presenting explicit examples of sectors with infinite statistics.

4 Sector composition in terms of states

Algebraic field theory provides an easy proof of the general structure of fusion rules and on restrictions on the allowed values of statistical dimensions. It is much harder to work out fusion rules explicitly in models (see [16,17] for the Ising model and [18] for loop groups). In [19] therefore an alternative way of sector composition was proposed.

Starting point is the definition of a set of localized states relative to a distinguished state ω_0 (interpreted as the vacuum). Let $\mathcal{O}$ be a double cone. Then we define $\mathcal{S}(\mathcal{O})$ as the set of positive linear funtionals ω on $\mathcal{A}$ whose restrictions on $\mathcal{A}(\mathcal{O}')$ are dominated by the restriction of ω_0 , i.e. there is some $\lambda > 0$ with

$$\omega(A) \leq \lambda \omega_0(A)$$

for all positive $A \in \mathcal{A}(\mathcal{O}')$.

Examples for such states can be given in terms of partial intertwiners. Let π_0 be the representation induced by ω_0 and let π be another representation. Let $S : \mathcal{H}_{\pi_0} \to \mathcal{H}_\pi$ be a bounded operator satisfying the intertwining relation

$$S\pi_0(A) = \pi(A)S \quad , \quad A \in \mathcal{A}(\mathcal{O}').$$

Then $\omega(\cdot) = (S\Omega, \pi(\cdot)S\Omega)$, Ω being the vector inducing ω_0 in π_0, is localized in $\mathcal{O}$. Indeed, for $A \in \mathcal{A}(\mathcal{O}')$, $A > 0$

$$\omega(A) = (S\Omega, \pi(A)S\Omega) = (S^*S\Omega, \pi_0(A)\Omega) \leq ||S^*S||\omega_0(A)$$

Actually, all elements in $\mathcal{S}(\mathcal{O})$ are of this type.

One might be tempted to define localized states as those which coincide with ω_0 on $\mathcal{A}(\mathcal{O}')$. This definition, however would have the disadvantage that it is not stable under convex decompositions, simply because the restriction of ω_0 to $\mathcal{A}(\mathcal{O}')$ is not pure.

We require as before that π_0 satisfies Haag duality. Then by

$$\pi_0 \circ \chi_\omega(\cdot) = S^*\pi(\cdot)S$$

we associate with ω a positive mapping χ_ω of $\mathcal{A}$ which is uniquely characterized by $\omega_0 \circ \chi_\omega = \omega$ and

$$\chi_\omega(ABC) = A\chi_\omega(B)C$$

for $A, C \in \mathcal{A}(\mathcal{O}'), B \in \mathcal{A}$.

The product of two localized states is now defined by

$$\omega_1 \times \omega_2 = \omega_1 \circ \chi_{\omega_2}.$$

Note that $\omega_1 \times \omega_2(1) \neq 1$ in general, even if $\omega_1(1) = \omega_2(1) = 1$. The product again belongs to $\mathcal{S}(\mathcal{O})$ [19].

How is this composition related to the DHR composition? If ρ is a DHR endomorphism and $S \in \mathcal{A}(\mathcal{O})$, then $\omega(\cdot) = \omega_0(S^*\rho(\cdot)S)$ is an element of $\mathcal{S}(\mathcal{O})$, and the positive mapping associated to ω is

$$\chi_\omega(\cdot) = S^*\rho(\cdot)S.$$

Let ω_i, $i = 1, 2$ be two such states with corresponding endomorphisms ρ_i and operators S_i. Their product is

$$\omega_1 \times \omega_2 = \omega_0(S_1^*\rho_1(S_2)^*\rho_1\rho_2(\cdot)\rho_1(S_2)S_1)$$

hence the product state is a state in the composed sector $\rho_1\rho_2$. Thus the representation induced by $\omega_1 \times \omega_2$ is equivalent to a cyclic subrepresentation of the DHR product $\pi_{\omega_1} \times \pi_{\omega_2}$. At present it is not known for which states the product of states is cyclic for the product of representations.

Now choose a DHR endomorphism ρ_i from each irreducible sector with finite statistics. Then there are intertwiners $T_{ij}^{k,n}$, $n = 1, ..N_{ij}^k$ with nonnegative integers N_{ij}^k such that

$$T_{ij}^{k,n}\rho_k(\cdot) = \rho_i\rho_j(\cdot)T_{ij}^{k,n}$$

with the orthogonality and completeness relations

$$(T_{ij}^{k,n})^*T_{ij}^{k',n'} = \delta_{k,k'}\delta_{n,n'}$$

$$\sum_{k,n} T_{ij}^{k,n}(T_{ij}^{k,n})^* = 1$$

The product of states $\omega_i \times \omega_j$, $\omega_i = \omega_0(S_i^*\rho_i(\cdot)S_i)$ can be decomposed into a finite number of mutually disjoint primary states $\tilde{\omega_k}$,

$$\omega_i \times \omega_j = \sum \tilde{\omega_k}$$

with

$$\tilde{\omega_k} = \sum_n \omega_0(S_i^*\rho_i(S_j)^*T_{ij}^{k,n}\rho_k(\cdot)(T_{ij}^{k,n})^*\rho_i(S_j)S_i).$$

$\tilde{\omega_k}$ is a state in the sector of ρ_k and it is a mixture of at most N_{ij}^k pure states.

Let us look at a simple example. Let φ be the free scalar massless field in two dimensions. Then $j = \partial_0\varphi - \partial_1\varphi$ is a function of $u = t - x$. The commutation relations are

$$[j(u), j(u')] = 2i\delta'(u - u'),$$

and the two-point function is

$$(\Omega, j(u)j(u')\Omega) = \frac{-1}{\pi(u - u' - i\epsilon)^2} = \frac{1}{\pi}\int_0^\infty e^{-ip(u-u')}p\mathrm{d}p.$$

It is convenient to describe the model in terms of Weyl operators

$$W(f) = \exp ij(f) \quad , \quad j(f) = \int j(u)f(u)\mathrm{d}u$$

with real valued test functions f with compact support. The commutation relations assume the form

$$W(f)W(g) = \exp(-i\int fg')W(f + g)$$

and the vacuum is defined by

$$\omega_0(W(f)) = \exp -||f||^2 \quad , \quad ||f||^2 = \int_0^\infty |\tilde{f}(p)|^2 p\mathrm{d}p$$

with the Fourier transform $\tilde{f}$ of f. The algebra generated by the Weyl operators has outer automorphisms labeled by real valued test functions F with compact support,

$$\rho_F(W(f)) = \exp(2i\int Ff)W(f).$$

ρ_F is outer if and only if $\int F \neq 0$. We have $\rho_F\rho_G = \rho_{F+G}$ and the sectors are labeled by the real number $q = \int F$. The statistics operator is

$$\epsilon_{\rho_F} = \exp i(\int F)^2$$

and the monodromy operator

$$\epsilon(\rho_F, \rho_G)\epsilon(\rho_G, \rho_F) = \exp 2i \int F \int G$$

(see [20]). The fusion structure is

$$[q_1] \times [q_2] = [q_1 + q_2].$$

As suggested by Rehren [21], a simple modification of this model has sectors with nontrivial braid group statistics. Namely, the model has the internal symmetry $j \to -j$, resp. $W(f) \to W(-f)$. The algebra $\mathcal{A}_{\text{inv}}$ of fixpoints under this symmetry consists of linear combinations of operators

$$V(f) = \cos j(f) = \frac{1}{2}(W(f) + W(-f))$$

with the relations

$$V(f)V(g) = \frac{1}{2}(\exp(-i\int fg')V(f+g) + \exp(i\int fg')V(f-g))$$

$$V(0) = 1 \quad , \quad V(f)^* = V(f).$$

The vacuum sector of the original model decomposes after restriction to $\mathcal{A}_{\text{inv}}$ into a direct sum

$$\pi_0|_{\mathcal{A}_{\text{inv}}} = \pi_{\text{even}} + \pi_{\text{odd}}$$

according to the eigenspaces of the unitary which implements the symmetry. In the sector π_q, $q \neq 0$ however, this symmetry is spontaneously broken. Therefore $\pi_q|_{\mathcal{A}_{\text{inv}}}$ is irreducible. On the other hand, in restriction to $\mathcal{A}_{\text{inv}}$, π_q and π_{-q} become equivalent.

Localized states and the associated positive maps can be explicitly given,

$$\omega_F = \omega_0 \circ \rho_F|_{\mathcal{A}_{\text{inv}}},$$

$$\chi_F(V(f)) = \cos(2\int Ff)V(f)$$

i.e. χ_F is obtained from ρ_F by a mean over the symmetry group $\mathbb{Z}_2$. The product is

$$\omega_F \times \omega_G = \frac{1}{2}(\omega_{F+G} + \omega_{F-G}).$$

The states on the right hand side might not be pure if $\int(F \pm G) = 0$. So we find the fusion rules ($q > 0$)

$$[q_1] \times [q_2] = [q_1 + q_2] \oplus [|q_1 - q_2|] \quad , \quad q_1 \neq q_2$$
$$[q] \times [q] = [2q] \oplus [\text{even}] \oplus [\text{odd}]$$
$$[q] \times [\text{odd}] = [q]$$
$$[\text{odd}] \times [\text{odd}] = [\text{even}]$$

The statistical dimensions are $d_q = 2$ and $d_{\text{even}} = d_{\text{odd}} = 1$. Strictly speaking, the arguments given above provide only lower bounds on the statistical dimensions. This comes from the unsufficient control on the cyclicity of product states in product representations. By different methods using Longo's canonical endomorphisms the equality sign can be proven . For our construction of sectors with infinite statistical dimension the lower bound is sufficient.

5 Sectors with infinite statistics

In this section we want to present examples of sectors with infinite statistics. We shall see that these sectors are by no means pathological and that they have a decent fusion structure. Actually, Guido and Longo already succeeded in generalizing much of the general theory to the case of infinite statistics [22].

Our first example is similar to the example treated in the preceding section, but now the symmetry group is an infinite compact group. We consider two commuting chiral U(1) currents j_1, j_2 and Weyl operators

$$W(f) = \exp i(j_1(f_1) + j_2(f_2)) \quad , \quad f = (f_1, f_2).$$

The group SO(2) acts on the Weyl operators in a natural way

$$W(f) \mapsto W(R_\phi f).$$

The sectors of the models are labeled by $\vec{q} = (q_1, q_2) \in \mathbb{R}^2$ and the corresponding localized endomorphisms by pairs of testfunctions with compact support , $F = (F_1, F_2)$, $\int F_i = q_i$,

$$\rho_F(W(f)) = \exp(2i\int (F, f))W(f).$$

The fusion rules are

$$[\vec{q}] \times [\vec{q}'] = [\vec{q} + \vec{q}'].$$

We now consider the SO(2)-invariant algebra $\mathcal{A}_{\text{inv}}$. The algebra has generators

$$V(f) = \int \frac{\mathrm{d}\phi}{2\pi} W(R_\phi f)$$

with the relation

$$V(f)V(g) = \int \frac{\mathrm{d}\phi}{2\pi} V(f + R_\phi g).$$

The localized states and the positive maps are

$$\omega_F = \omega \circ \chi_F \quad , \quad \chi_F(V(f)) = V(f)\mathrm{J}_0(2r)$$

where $r^2 = (\int F_1 f_1 + F_2 f_2)^2 + (\int F_1 f_2 - F_2 f_1)^2$ and J_0 is the modified Bessel function. As in the preceding section, χ_F is obtained from ρ_F by an average over the symmetry group.

If $\int F = 0$ the state ω_F can be decomposed according to the representations of SO(2),

$$\omega_F = \sum_{n\in\mathbb{Z}} \omega_{F,n}.$$

If $\int F \neq 0$ the states ω_F are pure. They induce mutually disjoint representations if the numbers $q = \sqrt{(\int F_1)^2 + (\int F_2)^2}$ are different. The product decomposes into a continuous mixture

$$\omega_F \times \omega_G = \int \frac{\mathrm{d}\phi}{2\pi} \omega_{F+R_\phi G}.$$

Thus we find the fusion rule $(q_1, q_2 > 0)$

$$[q_1] \times [q_2] \supset \int_{[0,\pi]}^{\oplus} \frac{\mathrm{d}\phi}{\pi} [\sqrt{q_1^2 + q_2^2 + 2q_1 q_2 \cos\phi}].$$

We expect an equality sign, but were not yet able to prove it. The proven relation already implies that $d_q = \infty$ for all $q > 0$.

We now look at a second example. We consider again the model of Section 4. It was pointed out by Fuchs [23] that this model has a further sector (the "twisted sector") which formally should have infinite statistics. This sector is analogous to the Ramond sector in the chiral Ising model; one could try to treat it in a similar way as Mack and Schomerus treated the Ising model [16]. There is however an easier way which directly leads to a proof of infinite statistical dimension.

We choose a "kink" function $\alpha : \mathbb{R} \to \mathbb{R}$, namely a smooth function with a derivative of compact support and $\alpha(x) \to 0$ for $x \to -\infty$ and $\alpha(x) \to \pi$ for $x \to +\infty$, and look at the homomorphism $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$,

$$\rho_\alpha : W(f) \mapsto W(f\cos\alpha) \otimes W(f \sin\alpha).$$

Let $S : \mathcal{H}_0 \to \mathcal{H}_0 \otimes \mathcal{H}_0$ denote the isometry

$$S\Phi = \Phi \otimes \Omega$$

and let $\omega_\alpha = (\omega_0 \otimes \omega_0) \circ \alpha$, $\chi_\alpha = S^* \rho_\alpha S$ and

$$\chi_\alpha(W(f)) = W(f\cos\alpha)\exp(-||f \sin\alpha||^2).$$

After restriction to $\mathcal{A}_{\text{inv}}$, ω_α and χ_α become localized, and ω_α can be shown to be a state (not necessarily pure) in the twisted sector.

This fact may be seen by using the angular coordinate $\varphi = 2\arctan u$. We consider the real valued test functions f as functions of φ with support in the open interval $(-\pi, \pi)$. Then the norm characterizing the vacuum state is

$$||f||^2 = -\frac{1}{8\pi}\int \mathrm{d}\varphi \mathrm{d}\varphi' f(\varphi) f(\varphi') (\sin(\frac{\varphi - \varphi'}{2}) - i\epsilon)^{-2}$$
$$= \frac{1}{4\pi}\sum_{n\in\mathbb{Z}} |n| \cdot |\int \mathrm{d}\varphi f(\varphi) \exp i\varphi n|^2.$$

The lowest weight state with respect to the generator L_0 of translations in φ in the twisted sector is given by

$$\omega_{tw}(W(f)) = \exp -||f||^2_{tw}$$

where the norm $||\cdot||_{tw}$ is defined as

$$||f||^2 = -\frac{1}{8\pi}\int \mathrm{d}\varphi \mathrm{d}\varphi' f(\varphi) f(\varphi') (\sin(\frac{\varphi - \varphi'}{2}) - i\epsilon)^{-2} \cos(\frac{\varphi - \varphi'}{2})$$
$$= \frac{1}{4\pi}\sum_{n\in\mathbb{Z}+\frac{1}{2}} |n| \cdot |\int \mathrm{d}\varphi f(\varphi) \exp i\varphi n|^2.$$

The state ω_α is obtained from ω_{tw} by replacing the argument of the cosine by $\alpha(\varphi) - \alpha(\varphi')$. Hence the integral kernels of the corresponding quadratic forms differ by a smooth function $K(\varphi, \varphi')$ which satisfies antiperiodic boundary conditions

$$K(\varphi, \pi) = -K(\varphi, -\pi)$$

and hence has an expansion

$$K(\varphi, \varphi') = \sum_{n,m\in\mathbb{Z}+\frac{1}{2}} K_{n,m} \exp(i\varphi n - i\varphi' m)$$

with fast decreasing coefficients K_{nm}.

It then follows from the theory of quasiequivalence of quasifree states [24] that ω_α is a state in the twisted sector.

We now compute the product (with kink functions α, β)

$$\omega_\alpha \times \omega_\beta(V(f)) = \exp -(f, f)_T$$

where $(\cdot,\cdot)_T$ is a real scalar product defined by

$$(f, f)_T = ||f\cos\alpha\cos\beta||^2 + ||f\cos\beta\sin\alpha||^2 + ||f\sin\beta||^2.$$

In the angular variables the kernel of the quadratic form $(\cdot,\cdot)_T$ differs from that of $||\cdot||^2$ by a smooth function L which satisfies periodic boundary conditions. It has an expansion

$$L(\varphi, \varphi') = \sum_{n,m\in\mathbb{Z}} L_{nm} \exp(i\varphi n - i\varphi' m)$$

with fast decreasing coefficients. Nevertheless, both quadratic forms induce inequivalent topologies, since on $l^2(\mathbb{Z})$ the operator of multiplication by $|n|$ does not dominate an operator with nonvanishing matrix elements L_{n0}. Actually, the symplectic form $\int fg'$ becomes degenerate on the completion of $\mathcal{D}_{\mathbb{R}}(-\pi, \pi)$ with respect to $||\cdot||_T$. The subspace of degenerate vectors in this completion is the space of constant functions c. The corresponding operators $V(c)$ are central elements in the representation induced by $\omega_\alpha \times \omega_\beta$.

The product state can be decomposed according to the spectrum of the central elements $V(c)$, with the result

$$\omega_\alpha \times \omega_\beta = \int_0^\infty \mathrm{d}q \frac{\exp -\frac{q^2}{4(1,1)_T}}{\sqrt{\pi(1,1)_T}} \omega_q$$

with

$$\omega_q(V(f)) = \exp(-(f,f)_{T_0})\cos(q\frac{(1,f)_T}{(1,1)_T})$$

where $(f,f)_{T_0} = (f,f)_T - \frac{(f,1)_T(1,f)_T}{(1,1)_T}$. The integral kernel of $(\cdot,\cdot)_{T_0}$ differs from that of $\|\cdot\|^2$ by a smooth function $L^{(0)}$ whose expansion coefficients are obtained from those of L by

$$L^{(0)}_{nm} = L_{nm} - \frac{L_{n0}L_{0m}}{L_{00}}.$$

It is now easy to see that $(f,f)_{T_0}$ is continuous with respect to $\|\cdot\|$ and vice versa, and that ω_{T_0},

$$\omega_{T_0}(V(f)) = \exp(-(f,f)_{T_0})$$

is a state in the vacuum sector, hence ω_q is a state (not necessarily pure) in the sector $[q]$. So we conclude

$$[\text{twisted}] \times [\text{twisted}] \supset \int^{\oplus}_{\mathbb{R}_+} [q]$$

which again implies infinite statistics.

Further candidates of sectors with infinite statistics are the sectors of the $(c=1)$-Virasoro algebra with nonhalfinteger conformal weights [25].

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