the phase of the parallel transport operator π_p depends on the path p. Choose a base point m, and choose, in each homotopy class $\gamma \in \Gamma_g$ of closed paths at m, a representative path $p(\gamma)$. Parallel transport along $p(\gamma)$ gives a projective unitary representation of Γ_g , $\gamma(m) = \pi_{p(\gamma)}$, in the fiber of W_g at m:

$$\gamma_1(m)\gamma_2(m) = \gamma_2(m)\,\gamma_1(m)\,\sigma(\gamma_1,\gamma_2)(m)\,. \tag{48}$$

The operators $\gamma(m)$ are unitary because the covariant constant hermitian metric h is globally invariant under parallel transport. Conversely, a projectively flat hermitian vector bundle can be reconstructed from a projective unitary representation $\mathcal{H}_g$ of Γ_g, by essentially the same technique used to make a flat hermitian bundle from a unitary representation of Γ_g.

Away from orbifold points and the compactification divisor, we can choose a local basis for W_g of holomorphic sections $w_a(m)$ which are covariant constant up to a common phase,

$$Dw_a(m) = \partial f(\bar{m},m)w_a(m) \tag{49}$$

where f is a locally defined function satisfying $\bar{\partial}\partial f = \eta$. The components of the metric, $h_{ab} = h(\overline{w_a}, w_b)$, are not constant, since $\partial(h_{ab}) = (\partial f)h_{ab}$. But we can write, locally,

$$h_{ab} = e^f h^0_{ab} \tag{50}$$

with h^0_{ab} constant.

Now define $V_g = E_c \otimes W_g$. Let $\psi(m)$ be a holomorphic section of V_g. In components, $\psi(m) = \psi^a(m)w_a$ where $\psi^a(m)$ is a local section of E_c. A conformal field theory in the strict sense requires $\psi(m)$ nowhere zero. The partition function of the gauge system is

$$Z = h(\bar{\psi},\psi) = \overline{\psi^a(m)}\, h_{ab}\, \psi^b(m)\,, \tag{51}$$

which is manifestly a section of $\bar{E}\otimes E$ over $\overline{M}_g$. Locally, the partition function has the form

$$Z(\bar{m},m) = e^f\, \overline{\psi^a(m)}\, h^0_{ab}\, \psi^b(m)\,, \tag{52}$$

so it is real analytic up to the factor e^f.

Note that the gauge system introduces an arbitrary element into the construction of the partition function, namely the particular choice of $(1,1)$-form η representing the cohomology class $[\eta] = (-c/24)[\omega_{wp}/2\pi^2]$. We interpret η as the abstract version of the renormalization scheme. We will see below that the correlation functions of the primary fields are independent of η. But the choice of η affects the definition of the stress-energy tensor and thus of the descendent fields. Different choices of η give different, but equivalent, presentations of the underlying conformal field theory. Given a choice of surface geometry g, with its associated hermitian metric h^g in E_c over M_g, we can fix η so that the metric $\tilde{h} = h^g h$ in V_g is flat over M_g, but singular on the compactification divisor. The flatness, as opposed to projective flatness, of $\tilde{h}$ over M_g allows the partition function to be written as a manifestly single-valued and real analytic function on moduli space, which is singular on the compactification divisor, diverging as $q^{\epsilon_0}\bar{q}^{\epsilon_0}$. The real analyticity of correlation functions is then also manifest. This is the situation we envision in the construction of the gauge system from the conformal field theory using generalized characters.

As an aside, we mention that one possible way to make W_g and $\psi(m)$ at the same time is to follow the pattern in the genus 1 Ising model. Start with one or more sections $\tilde{\psi}_i(t)$ of E_c over Teichmuler space. Γ_g acts on the sections $\tilde{\psi}_i$ to generate a vector bundle V_g over $\overline{M}_g$, in which the $\tilde{\psi}_i(t)$ can be interpreted as a single section $\psi(m)$. W_g is then defined as the projectively flat vector bundle $E_{-c}\otimes V_g$. Then a projectively flat metric h_{ab} must be found in W_g. It is not at all obvious under what conditions the $\tilde{\psi}_i(t)$ yield a projectively flat vector bundle W_g which is hermitian.

Now let us examine the behavior of the abstract partition function of the

gauge system near the compactification divisor. Let m_D be a point in D_g. Let t be a point on the boundary of $\mathcal{T}_g$ in the equivalence class of m_D. Let $\gamma^{tw} \in \Gamma$ be the Dehn twist $q \to e^{2\pi i} q$ around a node in t. Off the compactification divisor, near m_D, W_g decomposes into subbundles $W_g(h)$ whose fibers are the eigenspaces $\mathcal{H}_g(h)$ on which γ^{tw} has eigenvalue $e^{-2\pi i h}$. Regard h as a real number, so $\mathcal{H}_g(h)$ depends only on h modulo integers. Clearly h_{ab} is block diagonal with respect to this decomposition. V_g decomposes into the local subbundles $V_g(h) = E_c \otimes W_g(h)$. $\psi(m)$ can be decomposed near m_D in the form

$$\psi(m_D, q) = \sum_k \psi_k(m_D, q) \tag{53}$$

where $\psi_k(m_D, q) \sim q^{h_k}$ is a local section of $V_g(h_k)$, such that ψ cannot be further decomposed into a sum of local sections of $V_g(h_k)$. By convention, $h_0 = 0$. The partition function now has the expansion

$$\begin{aligned} Z(\bar{m}_D, m_D, \bar{q}, q) &= \sum_{j,k} Z_{j,k}(\bar{m}_D, m_D, \bar{q}, q) \\ Z_{j,k}(\bar{m}_D, m_D, \bar{q}, q) &= \overline{\psi_j^a(m_D, q)}\, h_{ab}\, \psi_k^b(m_D, q)\,. \end{aligned} \tag{54}$$

where $Z_{j,k} \sim \bar{q}^{h_j} q^{h_k}$. Each index (j,k) labels a primary conformal field $\phi_{j,k}$ of conformal weights $\bar{h} = h_j$, $h = h_k$. $Z_{j,k}$ is the contribution of $\phi_{j,k}$ and its descendents in the channel passing through the node. If we choose a local frame such that $h_{ab} = e^f h^0_{ab}$ with h^0_{ab} constant, as in eqn. (50), then the normalized contribution of $\phi_{j,k}$,

$$Z_{j,k}/Z = \left(\overline{\psi_j^a(m_D, q)}\, h^0_{ab}\, \psi_k^b(m_D, q)\right) / \left(\overline{\psi^a(m_D, q)}\, h^0_{ab}\, \psi^b(m_D, q)\right) \tag{55}$$

is real analytic. This shows the real analyticity of the correlation functions of the primary fields reconstructed from the partition function of a gauge system.

7. The Factorization Algebra

We must now ensure that the partition function of the gauge system, eqn. (51), factorizes on the compactification divisor. This is a necessary condition for the gauge system to be a conformal field theory, and we have argued in section 4 that this should also be a sufficient condition.

We want conditions on h_{ab} and on $\psi(m)$ which will imply factorization on the compactification divisor. The form of the generalized characters provides guidance. When a node is formed, a generalized character decomposes into a sum of products of generalized characters on the component surface(s) left by removal of the node. The matrix elements $\tilde{h}_{ab}$, for a surface with node, determine the $\tilde{h}_{ab}$ for the component surfaces. That is, both ψ and h_{ab} should individually satisfy factorization identities on the compactification divisor.

We now describe the abstract mathematical structure in which the factorization of ψ and h_{ab} are expressed. We consider factorization in the vector bundles V_g over $\overline{\mathcal{M}}_g$, but the discussion applies equally to a general vector bundle, projective vector bundle, or projective line bundle. We need only consider the generic case of factorization at a surface with a single node, because factorization at a surface with multiple nodes is given by iteration of single node factorization, since there are simultaneous transversal coordinates for all the nodes. However, factorization on multiple nodes must be independent of the sequence in which the single node factorizations are performed, so we will need to impose associativity conditions on the single node factorization. It is enough to consider surfaces with two nodes to determine the associativity conditions.

Repeating diagram (3), $D_{g,0}$ is a fiber bundle over $\overline{\mathcal{M}}_{g-1}$ and $D_{g,k}$ is a fiber bundle over $\overline{\mathcal{M}}_k \times \overline{\mathcal{M}}_{g-k}$, $k > 0$, where the fibers are the locations (x_1, x_2) of the

punctures:

$$\begin{array}{ccc} D_{g,0} & \to & \overline{\mathcal{M}}_g \\ \downarrow & & \\ \overline{\mathcal{M}}_{g-1} & & \end{array} \qquad \begin{array}{ccc} D_{g,k} & \to & \overline{\mathcal{M}}_g \\ \downarrow & & \\ \overline{\mathcal{M}}_k \times \overline{\mathcal{M}}_{g-k} & & \end{array} \tag{56}$$

A simple example of factorization is provided by the line bundle E_c. E_c is trivial along the punctures. Therefore a local section of E_c over $\overline{\mathcal{M}}_{g-1}$ lifts to give a local section of E_c over $D_{g,0}$, which is constant along the fibers. And, for $k > 0$, the product of a pair of local sections of E_c, one over $\overline{\mathcal{M}}_k$ and one over $\overline{\mathcal{M}}_{g-k}$, lifts to give a local section of E_c over $D_{g,k}$, constant along the fibers.

Given vector bundles V_g over $\overline{\mathcal{M}}_g$, in the extended sense of V-bundle described in section 5, write $\mathcal{V}_g$ for the local holomorphic sections of V_g. A factorization structure for the V_g is a collection of maps F_g and $F_{k,g-k}$ acting on local holomorphic sections:

$$\begin{array}{ccc} \mathcal{V}_{g/D_{g,0}} & \leftarrow & \mathcal{V}_g \\ F_g \uparrow & & \\ \mathcal{V}_{g-1} & & \end{array} \qquad \begin{array}{ccc} \mathcal{V}_{g/D_{g,k}} & \leftarrow & \mathcal{V}_g \\ F_{k,g-k} \uparrow & & \\ \mathcal{V}_k \times \mathcal{V}_{g-k} & & \end{array} \tag{57}$$

satisfying consistency conditions which will be described. Recall that a vector bundle V_g restricted to the compactification divisor degenerates to the twist invariant sub-bundle of V_g nearby.

For $k = 0$ the map F_g takes a local section of V_{g-1} over an open set U in $\overline{\mathcal{M}}_{g-1}$ to a local section of V_g over the lift of U in $D_{g,0}$. For $k > 0$, $F_{k,g-k}$ is a bilinear map from local sections, one of V_k over an open set U_1 in $\overline{\mathcal{M}}_k$, the other of V_{g-k} over an open set U_2 in $\overline{\mathcal{M}}_{g-k}$, to a local section of V_g over the lift of $U_1 \times U_2$ in $D_{g,k}$. The F_g, $F_{k,g-k}$ thus *define* what it means for sections of V_g over $D_{g,0}$ or $D_{k,g-k}$, in the images of F_g or $F_{k,g-k}$, to be constant in the fibers. For this to be possible, those subbundles of V_{g/D_g} must be trivial in the puncture variables.

Schematically, in component form, the factorization maps have the form

$$\begin{aligned} (F_g \eta_{(g-1)})^a &= (F_g)^a_b \, \eta^b_{(g-1)} \\ F_{k,g-k}(\eta_{(k)}, \eta_{(g-k)})^a &= (F_{k,g-k})^a_{bc} \, \eta^b_{(k)} \, \eta^c_{(g-k)} \end{aligned} \tag{58}$$

where $\eta_{(g)}$ is a local section of V_g over the appropriate neighborhood, and we use indices a, b, c for all the vector bundles V_g, even though their fibers are not the same. In more detail,

$$\begin{aligned} &(F_g \eta_{(g-1)})^a(m^{(g-1)}, x_1, x_2) = F_g(m^{(g-1)})^a_b \, \eta^b_{(g-1)}(m^{(g-1)}) \\ &F_{k,g-k}(\eta_{(k)}, \eta_{(g-k)})^a(m_1^{(k)}, m_2^{(g-k)}, x_1, x_2) = \\ &\qquad F_{k,g-k}(m_1^{(k)}, m_2^{(g-k)})^a_{bc} \, \eta^b_{(k)}(m_1^{(k)}) \, \eta^c_{(g-k)}(m_2^{(g-k)}) \, . \end{aligned} \tag{59}$$

We omit indicating the dependence of F_g and $F_{k,g-k}$ on the puncture variables, assuming a frame in which they are constant in (x_1, x_2). If $2k = g$, $F_{k,k}$ must be consistent with exchange symmetry:

$$F_{k,k}(m_1^{(k)}, m_2^{(k)})^a_{bc} = F_{k,k}(m_2^{(k)}, m_1^{(k)})^a_{cb} \, . \tag{60}$$

It is convenient to define $F_{k,g-k}$ for $k > g - k$ so that F_{k_1,k_2} is invariant under exchange symmetry:

$$F_{k_1,k_2}(m_1^{(k_1)}, m_2^{(k_2)})^a_{b_1 b_2} = F_{k_2,k_1}(m_2^{(k_2)}, m_1^{(k_1)})^a_{b_2 b_1} \, . \tag{61}$$

The consistency conditions come from factorizing on two nodes in different orders. There are several cases. In the first case, one node separates the surface into genus k_1 and $k_2 + 1$, and the other node lies in a handle in the genus $k_2 + 1$ component:

$$\begin{array}{ccccc} & & m_D^{(k_1+k_2+1)} & & \\ & \swarrow & & \searrow & \\ (m_1^{(k_1)}, m_3^{(k_2+1)}) & & & & m_4^{(k_1+k_2)} \\ & \searrow & & \swarrow & \\ & & (m_1^{(k_1)}, m_2^{(k_2)}) \, . & & \end{array} \tag{62}$$

The consistency condition is

$$F_{k_1+k_2+1}((m_4^{(k_1+k_2)})_c^a\, F_{k_1,k_2}(m_1^{(k_1)}, m_2^{(k_2)})_{b_1b_2}^c = F_{k_1,k_2+1}(m_1^{(k_1)}, m_3^{(k_2+1)})_{b_1c}^a\, F_{k_2+1}(m_2^{(k_2)})_{b_2}^c \,. \tag{63}$$

In the second case, one node separates the surface into genus k_1 and k_2+k_3, and the other node separates the surface into genus k_1+k_2 and k_3:

$$\begin{array}{ccc} & m_D^{(k_1+k_2+k_3)} & \\ \swarrow & & \searrow \\ (m_4^{(k_1+k_2)}, m_3^{(k_3)}) & & (m_1^{(k_1)}, m_5^{(k_2+k_3)}) \\ \searrow & & \swarrow \\ & (m_1^{(k_1)}, m_2^{(k_2)}, m_3^{(k_3)}) \,. & \end{array} \tag{64}$$

The consistency condition is:

$$F_{k_1,k_2}(m_1^{(k_1)}, m_2^{(k_2)})_{b_1b_2}^c\, F_{k_1+k_2,k_3}((m_4^{(k_1+k_2)}, m_3^{(k_3)})_{cb_3}^a = F_{k_1,k_2+k_3}(m_1^{(k_1)}, m_5^{(k_2+k_3)})_{b_1c}^a F_{k_2,k_3}(m_2^{(k_2)}, m_3^{(k_3)})_{b_2b_3}^c \,. \tag{65}$$

The remaining configurations of two nodes give analogous consistency conditions.

The definition of the gauge system is now completed by imposing factorization conditions on the sections $\psi_{(g)}$ of V_g over $\overline{\mathcal{M}}_g$ and on the metric $h^{(g)}$ in W_g over $\overline{\mathcal{M}}_g$. For ψ,

$$\psi_{(g)/D_{g,0}} = F_g\psi_{(g-1)} \qquad \psi_{(g)/D_{g,k}} = F_{k,g-k}(\psi_{(k)}, \psi_{(g-k)}) \,. \tag{66}$$

Schematically, in components,

$$\psi^a_{(g)/D_{g,0}} = (F_g)^a_b\, \psi^b_{(g-1)} \qquad \psi^a_{(g)/D_{g,k}} = (F_{k,g-k})^a_{bc}\, \psi^b_{(k)}\, \psi^c_{(g-k)} \,. \tag{67}$$

For the metric, and arbitrary sections η of the W_g,

$$\begin{aligned} h^{(g)}(\overline{F_g\eta_{(g-1)}}, F_g\eta_{(g-1)})_{/D_{g,0}} &= h^{(g-1)}(\overline{\eta_{(g-1)}}, \eta_{(g-1)}) \\ h^{(g)}(\overline{F_{k,g-k}(\eta_{(k)}, \eta_{(g-k)})}, F_{(k,g-k)}(\eta_{(k)}, \eta_{(g-k)}))_{/D_{g,k}} &= \\ h^{(k)}(\overline{\eta_{(k)}}, \eta_{(k)})\, h^{(g-k)}(\overline{\eta_{(g-k)}}, \eta_{(g-k)}) & \end{aligned} \tag{68}$$

where we have left implicit the factorization maps for E_c. Schematically, in components,

$$\begin{aligned} h^{(g)}_{ab}\, (\overline{F_g})^a_c\, (F_g)^b_d &= h^{(g-1)}_{cd} \\ h^{(g)}_{ab}\, (\overline{F_{k,g-k}})^a_{c_1c_2}\, (F_{k,g-k})^b_{d_1d_2} &= h^{(k)}_{c_1d_1}\, h^{(g-k)}_{c_2d_2} \,. \end{aligned} \tag{69}$$

It follows from factorization of the metric that the metric connection D is compatible with the factorization structure F. This means that that the connection D on the compactification divisor D_g agrees via F with the connection D on the component surface(s), or equivalently that F is covariant constant.

From the factorization conditions (66,68) on ψ and h it follows immediately that

$$\begin{aligned} h^{(g)}(\bar\psi_{(g)}, \psi_{(g)})_{/D_{g,0}} &= h^{(g-1)}(\bar\psi_{(g-1)}, \psi_{(g-1)}) \\ h^{(g)}(\bar\psi_{(g)}, \psi_{(g)})_{/D_{g,k}} &= h^{(k)}(\bar\psi_{(k)}, \psi_{(k)})\; h^{(g-k)}(\bar\psi_{(g-k)}, \psi_{(g-k)}) \end{aligned} \tag{70}$$

which are exactly the required factorization identities (26,28) for the partition function of the gauge system.

8. Universal Moduli Space

The gauge system, in order to be equivalent to a conformal field theory, must obey the factorization conditions described in the previous section. These

factorization conditions are natural from the point of view of conformal field theory. In this section we show that they are also mathematically natural.

We define the *universal moduli space* $\overline{\mathcal{R}}$ of stable Riemann surfaces. We show that a holomorphic vector bundle V on the universal moduli space $\overline{\mathcal{R}}$ is exactly a collection of holomorphic vector bundles V_g on the moduli spaces $\overline{\mathcal{M}}_g$ along with a factorization structure at the compactification divisor. In particular, the projective line bundles $E_c = (\lambda_H)^{c/2}$ on the moduli spaces $\overline{\mathcal{M}}_g$ define a projective line bundle E_c on $\overline{\mathcal{R}}$. The partition function $Z(m,\bar{m})$, satisfying factorization at the compactification divisor, defines a section of $\bar{E}_c \otimes E_c$ over $\overline{\mathcal{R}}$. The holomorphic sections $\psi_{(g)}$ of the vector bundles V_g, satisfying the factorization conditions, define a holomorphic section ψ of V over $\overline{\mathcal{R}}$. The metrics $h^{(g)}$ on $W_g = E_{-c} \otimes V_g$, satisfying the factorization conditions, define a metric h on $W = E_{-c} \otimes V$.

A gauge system, or two dimensional conformal field theory, is simply a holomorphic vector bundle V over universal moduli space $\overline{\mathcal{R}}$, a holomorphic section ψ of V, and a projectively flat hermitian metric h in $W = E_{-c} \otimes V$. The partition function is $Z = h(\bar{\psi}, \psi)$.

The definition of universal moduli space is motivated by the intuitive picture of factorization. From the point of view of two dimensional conformal field theory, when a smooth Riemann surface of genus g develops a node and the node is removed, the surface becomes a possibly disconnected Riemann surface of genus $g-1$. More generally, if k nodes are removed the genus becomes $g-k$. For disconnected Riemann surfaces, the Euler number is $\chi = 2-2g = 2(1-\#_h-\#_c)$, where $\#_h$ is the number of handles and $\#_c$ is the number of components. The genus is therefore $g = \#_h - \#_c + 1$. Removing a node raises the Euler number by two and lowers the genus by one.

The natural setting for the factorization conditions is the space of all compact stable Riemann surfaces, connected and disconnected. The moduli space $\mathcal{R}$ of all compact, smooth Riemann surfaces consists of infinitely many disconnected

components:

$$\mathcal{R} = \prod_{g=0}^{\infty} \left(\bigcup_{k=0}^{\infty} \mathrm{Sym}^k(\mathcal{M}_g) \right) = \bigcup_{\{n_g\}} \prod_{g=0}^{\infty} \mathrm{Sym}^{n_g}(\mathcal{M}_g) \tag{71}$$

where Sym^k is the k-fold symmetric product. The connected components of $\mathcal{R}$ are indexed by the multiplicities $\{n_g\}$, where $n_g \geq 0$ is the number of connected components of genus g in the Riemann surface. On the other hand, the moduli space of all compact *stable* Riemann surfaces,

$$\overline{\mathcal{R}} = \prod_{g=0}^{\infty} \left(\bigcup_{k=0}^{\infty} \mathrm{Sym}^k(\overline{\mathcal{M}}_g) \right) , \tag{72}$$

is a connected space. Given any two surfaces, there is a third surface of higher genus which can be continuously, even analytically, deformed into either of the two original surfaces by forming nodes and forgetting the punctures. The universal compactification divisor is $\mathcal{D} = \overline{\mathcal{R}} - \mathcal{R}$.

We define an analytic structure on $\overline{\mathcal{R}}$ which makes this intuition precise. The naive analytic structure on $\overline{\mathcal{R}}$ is the product structure inherited from the $\overline{\mathcal{M}}_g$. But we impose a stronger analytic structure by requiring that the removal of nodes be analytic. Define, for $n \geq 0$,

$$\begin{aligned} \mathcal{R}_n &= \bigcup_{\sum_g g n_g \leq n} \prod_{g=0}^{\infty} \mathrm{Sym}^{n_g}(\mathcal{M}_g) \\ \overline{\mathcal{R}}_n &= \bigcup_{\sum_g g n_g \leq n} \prod_{g=0}^{\infty} \mathrm{Sym}^{n_g}(\overline{\mathcal{M}}_g) \\ \mathcal{D}_n &= \overline{\mathcal{R}}_n - \mathcal{R}_n . \end{aligned} \tag{73}$$

The universal moduli space $\overline{\mathcal{R}}$ is the limit of the inclusions

$$\overline{\mathcal{R}}_0 \to \overline{\mathcal{R}}_1 \to \overline{\mathcal{R}}_2 \to \cdots \to \overline{\mathcal{R}} . \tag{74}$$

Let $\pi\colon \mathcal{D}_n \to \mathcal{R}_{n-1}$ be the map which takes a surface m_D with nodes to the smooth surface $\pi(m_D)$ which results from removing the nodes and healing the

punctures that are left. Along with the inclusions $D_n \to \overline{\mathcal{R}}_n$ and $\mathcal{R}_n \to \overline{\mathcal{R}}_n$, the maps π form diagrams,

$$\begin{array}{ccccc} & D_n & \to & \overline{\mathcal{R}}_n & \qquad\qquad & D & \to & \overline{\mathcal{R}} \\ \cdots & \pi\downarrow & \nearrow & & \cdots & \pi\downarrow & \nearrow & \\ & \mathcal{R}_{n-1} & & & & \mathcal{R} & & \end{array} \tag{75}$$

which do *not* commute in the naive analytic structure on $\overline{\mathcal{R}}$. We define the analytic structure on $\overline{\mathcal{R}}$ to be the strongest analytic structure, having the fewest local holomorphic functions, which includes the local holomorphic functions on $\mathcal{R}$ and for which the diagrams (75) commute as diagrams of analytic maps. In terms of the local analytic functions $\mathcal{O}(\overline{\mathcal{R}})$ on $\overline{\mathcal{R}}$, $\mathcal{O}(D)$ on D, $\mathcal{O}(\mathcal{R})$ on $\mathcal{R}$, there are the commuting diagrams

$$\begin{array}{ccccc} & \mathcal{O}(D_n) & \leftarrow & \mathcal{O}(\overline{\mathcal{R}}_n) & \qquad\qquad & \mathcal{O}(D) & \leftarrow & \mathcal{O}(\overline{\mathcal{R}}) \\ \cdots & \pi^*\uparrow & \swarrow & & \cdots & \pi^*\uparrow & \swarrow & \\ & \mathcal{O}(\mathcal{R}_{n-1}) & & & & \mathcal{O}(\mathcal{R}) & & \end{array} \tag{76}$$

A local holomorphic function f defined on a neighborhood U of $\overline{\mathcal{R}}$ near the compactification divisor D and also on $\pi(U \cap D)$ must satisfy $f_{/D} = f \circ \pi$. In other words, $f_{/D}$ must be constant in the puncture variables and must be consistent, via removal of nodes, with its value at lower genus. It is easy to verify, by induction on n in the diagrams (75) or (76), that the only globally holomorphic functions on $\overline{\mathcal{R}}$ are the constants. In this sense, $\overline{\mathcal{R}}$ is a connected, compact analytic space.

The diagrams (75) or (76) give a natural factorization structure for analytic functions on $\overline{\mathcal{R}}$. The factorization condition described in section 7, for the special case of holomorphic functions, expresses exactly the requirement of analyticity on $\overline{\mathcal{R}}$.

A holomorphic vector bundle V over $\overline{\mathcal{R}}$ consists of ordinary vector bundles over the components of $\mathcal{R}$, patched together over the compactification divisor by maps π^* analogous to those in diagrams (76). In terms of the local analytic

sections $\mathcal{V}(\overline{\mathcal{R}})$ of V over $\overline{\mathcal{R}}$, $\mathcal{V}(D)$ over D, $\mathcal{V}(\mathcal{R})$ over $\mathcal{R}$, V is defined by the commuting diagrams

$$\begin{array}{ccccc} & \mathcal{V}(D_n) & \leftarrow & \mathcal{V}(\overline{\mathcal{R}}_n) & \qquad\qquad & \mathcal{V}(D) & \leftarrow & \mathcal{V}(\overline{\mathcal{R}}) \\ \cdots & \pi^*\uparrow & \swarrow & & \cdots & \pi^*\uparrow & \swarrow & \\ & \mathcal{V}(\mathcal{R}_{n-1}) & & & & \mathcal{V}(\mathcal{R}) & & \end{array} \tag{77}$$

A local holomorphic section η of V must satisfy $\eta_{/D} = \pi^*\eta$. This is exactly a factorization structure in bundles V_g over $\overline{\mathcal{M}}_g$ as described in section 7. Connections and hermitian metrics in V must respect the maps π^*.

One possible approach to studying the universal moduli space is to describe its universal analytic covering space and covering group, the universal modular group. The covering space cannot be the universal Teichmuller space described in the mathematics literature, because that universal Teichmuller space is the same as its universal modular group; the quotient, the moduli space, is a single point[36].

In the analytic formulation of string theory,[29] nonperturbative string effects are to be described in terms of world surfaces of infinite genus. It might also be useful to study conformal field theory extended to some appropriate class of infinite genus surfaces. Here we only make some formal comments about infinite genus surfaces. There are two universal moduli spaces of interest. The larger space is

$$\overline{\mathcal{R}}_\infty = \mathcal{R} \times \bigcup_{k=0}^{\infty} \mathrm{Sym}^k(\overline{\mathcal{M}}_\infty) \tag{78}$$

where $\overline{\mathcal{M}}_\infty$ is the space of connected, stable Riemann surfaces of infinite genus. We think of these infinite genus surfaces as essentially compact, almost all of the infinitely many handles being very small. Except for the finite genus parts of $\overline{\mathcal{R}}_\infty$, the surfaces with nodes should be dense in $\overline{\mathcal{M}}_\infty$, since adding or subtracting a very small handle, or splitting off a very small component surface, should be a very small disturbance. Except for the finite genus parts, the analytic structure

on $\overline{\mathcal{R}}_\infty$ should be very smooth. The crucial question is whether it is possible to extend a conformal field theory, and a gauge system, to $\overline{\mathcal{R}}_\infty$.

The second infinite genus universal moduli space is a subspace of $\overline{\mathcal{R}}_\infty$:

$$\overline{\mathcal{R}}^0_\infty = \bigcup_{k=0}^{\infty} \mathrm{Sym}^k(\overline{\mathcal{M}}^0_\infty) \tag{79}$$

where $\overline{\mathcal{M}}^0_\infty$ is the space of connected, stable Riemann surfaces of infinite genus, which have the property that removing a finite number of nodes leaves only infinite genus connected components. $\overline{\mathcal{R}}^0_\infty$ should be extremely regular. If the extension to infinite genus is possible, it would be interesting to know the relation among gauge systems on $\overline{\mathcal{R}}$, on $\overline{\mathcal{R}}_\infty$ and on $\overline{\mathcal{R}}^0_\infty$. It might be conjectured that any gauge system on $\overline{\mathcal{R}}$ would extend to $\overline{\mathcal{R}}_\infty$ and thus to $\overline{\mathcal{R}}^0_\infty$, and that every gauge system on $\overline{\mathcal{R}}^0_\infty$ would come this way. But to reduce a surface in $\overline{\mathcal{R}}^0_\infty$ to finite genus, an infinite number of nodes would have to be formed and removed. We might suppose that there could be more than one way of extending a gauge system on $\overline{\mathcal{R}}^0_\infty$ to $\overline{\mathcal{R}}_\infty$ and $\overline{\mathcal{R}}$. The gauge systems on $\overline{\mathcal{R}}^0_\infty$ might serve as universal approximations to the more various collection of ordinary conformal field theories, the gauge systems on $\overline{\mathcal{R}}$. We might even speculate that the factorization algebra, and projectively flat hermitian metric, associated with the projective vector bundle W over $\overline{\mathcal{R}}^0_\infty$ might have an especially simple structure, perhaps equivalent to a Fock space. The holomorphic section ψ would be an analytic field over $\overline{\mathcal{R}}^0_\infty$ of free field ground states in this Fock space.

9. Discussion

We have formulated two dimensional conformal field theories as gauge systems on universal stable moduli space $\overline{\mathcal{R}}$. A gauge system is a vector bundle W over universal stable moduli space, a projectively flat metric h in W, and

a holomorphic section ψ of $V = E_c \otimes W$, where E_c is the line bundle $(\lambda_H)^{c/2}$. The partition function is $Z(\bar{m},m) = h(\overline{\psi(m)},\psi(m))$. The factorization conditions which are equivalent to analyticity of ψ and smoothness of h on $\overline{\mathcal{R}}$ imply factorization of the partition function and allow reconstruction of the correlation functions of the conformal field theory from the gauge system. Projective flatness of h and global analyticity of ψ imply modular invariance and real analyticity of the partition function, which in turn imply crossing symmetry and real analyticity of the reconstructed correlation functions.

The description of two dimensional conformal field theory as geometry on universal moduli space exposes an underlying gauge invariance, which is the projective unitary gauge group of the projective vector bundle W. Note that these gauge transformations do not necessarily leave the metric invariant, but can change the curvature form η in its (1,1)-cohomology class. These extended gauge transformations change the partition function, but only by changing the overall factor e^f in eqn. (52). The correlation functions of the primary fields are unchanged, so these gauge transformations take a conformal field theory to an equivalent one. Presumably this gauge symmetry includes the fundamental redundancy in the field algebra of a conformal field theory, since the redundant operators[47] are perturbations of the system which do not change the partition function.

The main technical task in establishing the foundations of this analytic formulation of conformal field theory is to make rigorous the arguments outlined in sections 4 and 6 that a conformal field theory can be reconstructed from the partition function of every gauge system and that the partition function of every conformal field theory is given by a gauge system.

It would be interesting to have a natural way of constructing projectively flat bundles, for example from natural analytic functions on the universal analytic covering space of the universal moduli space, in analogy to the character $\chi_0(q)$, eqn. (39), for the Ising model in genus 1. In any case, effective means of

constructing gauge systems are certainly needed.

The classification of gauge systems seems a worthwhile mathematical enterprise. Present knowledge of conformal field theory puts interesting constraints on the structure of gauge systems. We list some of these known facts. For $c < 1$, unitarity forces c to be in the discrete series, the highest weights h_a are restricted to a finite list, and the multiplicities are highly constrained. In particular, the rank of each vector bundle W_g is finite, for finite g. In all known conformal field theories, the weights h_a form a discrete set, with multiplicities which can be calculated or estimated under some circumstances, e.g., in the $c < 1$ discrete series or in the infinite volume, $\alpha' \to \infty$, limit of Calabi-Yau spaces. The identity operator is the only field with conformal weights $0, 0$. In general, the multiplicities h_{ab} of the genus 1 partition function are integers. When a symmetry group S acts, these integers h_{ab} become the multiplicities of the representations of S in the Hilbert space of the two dimensional conformal field theory. It is especially important to understand how the constraints which follow from the structure of the Virasoro algebra arise in the gauge systems.

There is a straightforward generalization of gauge systems to super gauge systems, which are the abstract versions of superconformal field theories. Super Riemann surfaces[48] replace ordinary Riemann surfaces; super moduli space replaces ordinary moduli space; and super vector bundles replace ordinary vector bundles. The super gauge systems are of interest because of their application to fermionic string theory and supersymmetric critical phenomena.

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