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Zusammenfassung

Für Theorien, zu deren Beschreibung bei niedrigen Energien zusammengesetzte Felder benötigt werden, wird ein Wilsonscher Renormierungsgruppenzugang untersucht. Dieser liefert eine Reihe von effektiven Wirkungen S_Λ, die von einem UV *cut-off* Λ abhängen. Wir übernehmen Wilsons Forderung, daß diese Wirkungen lokal sein sollten. Dies wird unsere Leitlinie zur Konstruktion von "Blockspins", den Feldern einer effektiven Wirkung, sein. Ausgehend von einer fundamentalen Hochenergietheorie, die keine zusammengesetzten Felder enthält, wird durch schrittweise Ausintegration von hochfrequenten Moden der *cut-off* Λ abgesenkt. Die bei einem bestimmten *cut-off* Λ_c auftretenden Nichtlokalitäten deuten auf die Notwendigkeit hin, zusammengesetzte Freiheitsgrade in die Wirkung einzuführen. Eine auf Symanziks unendlicher Menge von Bethe-Salpeter-Gleichungen für alle n-Punkt-Funktionen basierende Untersuchung zeigt die Möglichkeit, eine lokale effektive Wirkung an der *compositeness* Skala Λ_c zu finden. Eine weitere Ausintegration hochfrequenter Moden erzeugt neue Nichtlokalitäten, die aber von den zusammengesetzten Freiheitsgraden aufgenommen werden können. Es gibt Anzeichen aus einer $\frac{1}{N}$-Entwicklung, daß dies schon ausreicht, um alle hochfrequenten Moden des zusammengesetzten Feldes zu eliminieren, so daß keine weitere Ausintegration hochfrequenter Moden für das zusammengesetzte Feld notwendig ist. Im allgemeinen wird es auch Selbstwechselwirkungen für das zusammengestzte Feld geben.

Abstract

We investigate a Wilson real space renormalization group approach for theories in which composite fields are needed at low energies. It furnishes a sequence of effective actions S_Λ which depend on an UV cut-off Λ. We adopt Wilsons fundamental postulate that these effective actions should be local. This is our basic guiding principle on how to construct "blockspins" , i.e. the fields which appear in the effective actions. Given a fundamental high energy theory which does not contain composite fields we gradually integrate out high frequency modes in order to lower the cut-off Λ. Eventually appearing nonlocalities at some cut-off value Λ_c indicate the necessity to introduce new composite degrees of freedom into the theory. An analysis based on Symanzik's infinite set of Bethe-Salpeter equations for all n-point functions shows that a local low energy effective action containing composite fields can be constructed at the compositeness scale Λ_c. Further integration of high frequency modes generates new nonlocalities which can be absorbed into the composite degrees of freedom. There are indications from an $\frac{1}{N}$ expansion that this suffices already to eliminate high energy degrees of freedom from the composite field so that no separate integration is needed to achieve this. In general the composite field will have self interactions.

Contents

1 Introduction

Non abelian gauge theories with matter fields are the prime candidates for describing elementary particle physics. They show phenomena like asymptotic freedom and confinement. At sufficiently large energy scales the running coupling constant is small enough to justify a perturbative treatment of these theories. For low energies in contrast nonperturbative methods are needed.

To obtain quantitative results with nonperturbative methods lattice regularized quantum field theory is today the most promising framework. Lattice regularization was introduced into quantum field theory by Wilson [28]. Lattice regularized quantum field theories are equivalent to systems of statistical mechanics. This becomes manifest in the euclidean path integral formulation. Therefore methods from statistical mechanics, for example strong coupling expansions, can be used to solve the theory.

Besides analytical work the lattice formulation gives the possibility to do numerical simulations of the theories under consideration. The results of such simulations depend on the regulator, i.e. the lattice spacing. These effects have to be removed. One method is to decrease the lattice spacing, i.e. to increase the number of simulated lattice points in order to come as close as possible to the continuum results. The continuum result is then found by extrapolation. This method is limited by the performance of today computers.

Another possibility is to improve the lattice action in such a way that the simulations reproduce the continuum results. This programme was started by Symanzik [1]. The problem is of course how to change the action in order to improve results. The first idea is to improve the lattice approximation of continuum operators like the lattice laplacian. This will certainly improve results but still cannot remove all effects which come from cutting off the high frequency modes. Symanzik gave a method to improve convergence of lattice results to the continuum limit in a systematical way by including higher dimensional interactions in the action.

Help on the construction of improved actions comes from effective theories. These are cut-off theories which give the same results as the fundamental theory for energy scales below the cut-off. The effects of high energy modes are encoded in the couplings. It is possible to construct effective theories without knowing anything about the underlying high energy dynamics. Choosing the theory to be consistent with symmetries decreases the number of possible interactions. The couplings of the remaining interactions are fitted to experimental data. In this way effective theories with predictive power are constructed. Important examples are chiral lagrangians and linear sigma models [2].

In contrast if the fundamental dynamics is known, it is in principle possible to calculate the coupling constants by integrating out the high frequency modes of the fundamental theory. These couplings depend obviously on the cut-off of the theory. The flow of couplings is given by Wilson's renormalization group. Flow equations for effective lagrangians have been given by Polchinski [19]. These flow equations can be integrated and thereby low energy effective lagrangians can be determined in a nonperturbative way. This approach does not need a lattice formulation. Flow equations can be integrated numerically. This has been done for Wilson's renormalization group equation by Riedl, Golner and Newman [13], for Polchinski's equation by Xylander [10] and for the one particle irreducible effective average action by Tetradis and Wetterich [40]. Important physical information like critical indices can be extracted from the solutions of flow equations.

On the lattice the integration of high frequency modes corresponds to the blockspin method introduced by Kadanoff [29] in the context of statistical mechanics and by Wilson [28] in the context of lattice regularized quantum field theory.

The lattice version of a low energy effective action is called perfect action. Recently it was shown by Hasenfratz and Niedermayer [15] that perfect actions are indeed very useful for numerical simulations. Results which were close to the continuum value were obtained by simulating an approximation of a perfect action on extremely small lattices. Approximations have to be done because an effective action contains an infinite number of interactions. However a finite number is sufficient to reach a wanted precision in the results.

A sophisticated method which incorporates the ideas of renormalization group and effective theories is the multigrid method of Mack and Pordt [4, 5]. Instead of calculating first an effective action which is simulated afterwards it maps the fundamental theory into a special lattice formulation, the multigrid. This is equivalent to considering the theory simultaneously on many different energy scales. Hence a sequence of renormalization group transformations is already included in the construction of the multigrid. Analytical work with this lattice is possible. For instance convergent expansions have been found by Pordt [8]. Numerical simulations of a theory on the multigrid is equivalent to making updates of field configurations on many different energy scales. This multigrid approach is able to defeat critical slowing down for some theories [9]. For a survey of applications of multigrid techniques in connection with effective theories see [5].

In this work we want to address a special problem in calculating effective actions. This problem is connected with composite fields. Wilson's description of the renormalization group was incomplete in the sense that he did not give rules to find a good blockspin. He mentioned only a very general criterion, namely that the resulting action should be local.

Until very recently nearly all applications of the renormalization group dealt with blockspins which were of the same kind as the fundamental field. But phenomenological theories of QCD contain for instance mesons which are bound states of quarks and antiquarks. If the long distance behaviour is described by the exchange of gauge invariant quanta like pions, a corresponding composite field has to be introduced at some scale (cut-off) which is given by the range of this interaction. This cannot be achieved with blockspins which are of the same kind as the fundamental fields.

Although there is a freedom in the choice of the blockspin the choice of the blockspin is not arbitrary. A wrong choice of the blockspin will eventually result in nonlocal terms in the action. An example for this was given by Göpfert and Mack [6] in the context of the discrete gaussian model. For a theory which contains composite particles of mass m, nonlocal terms can appear for cut-offs $\Lambda > m$ if there are no field degrees of freedom in the effective action with the quantum numbers of these particles. An example will be presented which demonstrates this. A low energy effective action of QCD without mesons is nonlocal. This is the reason why composite fields have to be introduced. We will consider the problem when and how composite fields should be introduced and how they develop after introduction.

Recently composite fields were introduced in the framework of flow equations by Ellwanger and Wetterich [42, 41, 43]. In this approach the aspect of locality of the action was not studied. They derive evolution equations which describe in addition to fundamental fields also composite fields as the cut-off drops below a certain compositeness scale. Additional care has to be taken to preserve locality. Without a specific relation between fundamental field flow and composite field flow nonlocal effective actions result in general.

In the present work we study low energy effective actions with composite fields in coordinate space with special emphasis on the concept of locality. A method to calculate effective actions by successively integrating out high energy modes of the fundamental field is suggested. The method is able to keep the action local as the cut-off is lowered. Such a method may serve as a starting point in formulating a multigrid algorithm for theories with composite fields. It might also be possible to translate this method into an algorithm which does analytical blockspin transformations from a fine to a coarse lattice. In this way effective actions can be found which can be used for Monte Carlo simulations.

A motivation for this work is the desire to close the calculational gap between fundamental gauge theories and phenomenological effective theories. Phenomenological low energy effective theories like the Nambu-Jona-Lasinio model [16] are quite successful. They show phenomena like spontaneous breaking of chiral symmetry. When bosonizing the four fermion interaction in the NJL model mesons appear as composite fields. The masses of these field quanta depend on the cut-off of the effective theory. The masses decrease as the cut-off decreases. Eventually the square masses become negative. This signals the spontaneous breakdown of chiral symmetry.

Thus we infer that a four fermion interaction is important for the formation of composite fields

also in general nonabelian gauge theories. In section 3 we show that a four fermion interaction can arise as a result of an integration of high frequency gluon fields. This four fermion interaction can be Fierz transformed into attractive four fermion interactions which can possibly form either mesons or diquarks. At each scale new contributions to this four fermion interactions have to be taken into account. They arise because at each scale high frequency gluon fields are integrated out. This will have an effect on the formation of mesons and diquarks. It is very likely that the formation process will be accelerated by including these new interactions. The method suggested in this work has the quality to deal with these new interactions although this is not done in the present work.

The concept of diquarks is very old. There are experimental phenomena, namely some weak decays, which show evidence that the concept of diquarks is useful [31]. The generation of virtual diquarks states is thought to be a first step in the formation of baryons in B decays. Reliable calculations for the diquark content of the nucleon are however lacking due to the strong nonperturbative effects.

We will demonstrate our method using the Gross-Neveu model as a toy model [18]. It has a four fermion interaction to start with. This will be sufficient to produce composite boson fields which are our interest. In section 4 we introduce a preliminary version of the method. This version is preliminary because we bosonize only the four fermion interaction at the compositeness scale. In this simpler context we can demonstrate the introduction of composite fields and how to treat them in subsequent cut-off lowerings. However the very important concept of locality is violated. This is shown in section 5 where we use a large N expansion to study the locality of the effective action. In section 6 we generalize the provisional method. Using an analysis based on Symanzik's infinite set of Bethe-Salpeter equations for all n-point functions we are able to show that it is possible to construct a local effective action at the compositeness scale Λ_c. All appearing nonlocalities are reformulated as an interaction mediated by a composite field. After this field is introduced into the action the action is again local. A subsequent cut-off lowering is then discussed. It is a generalization of the procedure demonstrated in the provisional outline in section 4.

Afterwards we compare our method with other approaches which use flow equations to calculate the one particle irreducible average action. This shows that locality is a good guiding principle for selecting effective theories. We close with a summary and an outlook onto future work.

2 Essentials of effective field theories

In this section we give a short introduction to the concept of effective theories. Notions like locality, blockspin, renormalization group flow and perturbative relevance are explained. The important connection between the correct choice of blockspin and locality is discussed.

An effective field theory is a low energy theory to some fundamental high energy theory. The fundamental theory can again be an effective theory at a higher energy scale. An effective theory gives the same results as the fundamental theory for low energy processes. In this sense it is exact and its lattice version is called a perfect action [15].

An example for an effective theory is classical electrodynamics. There the electrical properties of a solid are given by the permittivity ϵ and permeability μ. This is sufficient to describe for instance the interaction of a low frequency electromagnetic wave with this solid, which is just the diffraction of light. It is not necessary to apply the fundamental theory of QED to get this result. Classical electrodynamics is able to describe phenomena up to an energy scale of about 1 eV correctly. Then phenomena like photoemission appear which cannot be explained by classical electrodynamics. It is characteristic for effective theories that they are valid only up to a certain energy scale.

The scale up to which an effective theory is valid is given by the cut-off Λ. This cut-off may be a momentum cut-off, but effective theories are not restricted to a special kind of cut-off. A lattice regulator is another important example. For lattice simulations effective theories are wanted because they remove the lattice artefacts concerning the continuum limit. Hasenfratz and Niedermayer [15] have given an example of the practicability of such perfect actions for lattice simulations.

All effects of high energy modes on the low energy physics are encoded in new local interactions. Since by definition we will not try to push the cut-off to infinity, we have no problem with nonrenormalizable interactions. Nonrenormalizable interactions appear naturally in an effective theory. They are needed to account for the effects of high energy states to a certain accuracy. In principle a perfect action needs an infinite number of interactions. To reach a certain precision a finite number of interactions is sufficient. Typically all interactions of energy dimension $(d+n)$, where d is the space time dimension, are needed to reach a precision of order $(\frac{p}{\Lambda})^n$ [21]. p is the momentum scale of the process under consideration. On a lattice the number of possible interactions is constraint by demanding the interactions to be local.

Effective field theories were introduced into physics by Wilson [28]. The blockspin approach [29] is probably the most intuitive way to understand how effective theories can be derived from fundamental theories. There the effective theory is a theory of averages of the fundamental fields over small space-time volumes, for example hypercubes called blocks. These averages are called blockspins. By the averaging process small scale fluctuations are eliminated. Their effect on the large scale phenomena is encoded in a renormalization of couplings and a creation of new couplings. So the couplings of an effective theory are always bare couplings which depend on the cut-off of the theory. Small distance fluctuations in coordinate space correspond to high momentum modes in momentum space. Thus the elimination of small scale fluctuations corresponds to a lowering of the cut-off of the theory. The new cut-off is essentially given by $\frac{1}{a}$, where a determines the size of the averaging volume.

One way to find the effective theory for a given fundamental theory is to integrate out the high frequency modes in a path integral formulation. This requires a split of the fundamental field into a high and a low frequency part. Let Φ denote the low frequency part or blockspin field. Φ could be defined on a lattice which would define the cut-off. It is defined as a functional of the fundamental field.

$$\Phi = C(\phi) \tag{1}$$

where C defines some sort of averaging procedure. For instance it may be the average of the fundamental field over a block. We define an effective action by [4]

$$e^{-S_{eff}(\Phi)} = e^{-S_\Lambda(\Phi)} = \int \mathcal{D}\phi\ \delta(\Phi - C(\phi))e^{-S_{\Lambda'}(\phi)} \tag{2}$$

Λ' is the possibly infinite cut-off of the fundamental theory and Λ is the smaller cut-off of the effective theory.

There are other possibilities to integrate out the high frequency modes. In this work we will use a propagator split. This induces a split of the fundamental field variable ϕ into a blockspin field Φ and a high frequency field ξ. The high frequency field has to be integrated over. We will explain this later in the context of a toy model.

Suppose we lower the cut-off only by an infinitesimal amount, i.e. the increase in the averaging volume is small. The new effective action then differs only little from the previous action and it is possible to derive an evolution equation for this effective action. This was done by Polchinski [19] who used this flow equation to give a new proof of renormalizability. The Polchinski equation determines the renormalization group flow. In principle it can be integrated which is another way to find an effective action from a fundamental one.

An effective field theory has generally less degrees of freedom than the fundamental theory because the small scale structure is no longer present. The fundamental processes are mimicked by new local interactions. Locality means always local on the scale of the effective theory. We can be more precise and define locality as the property of an amplitude to decay in coordinate space faster than $e^{-\Lambda|x|}$ for $|x| \to \infty$. $|x|$ is the distance between two arbitrary arguments of the amplitude. Λ is the scale to which locality refers. We will always have this property in mind if we speak about locality.

Since experiments can probe physics only at finite energy, a theory which can be tested can be at most an effective one. Renormalizability was for a long time a selection criterion for a quantum

field theory however. This can be understood by the enormous success of renormalizable theories. Renormalizability experiences a change of meaning in the context of effective theories. Because nonrenormalizable interactions are infrared irrelevant the coefficients of nonrenormalizable interactions are largely determined by the renormalizable couplings. The effects of nonrenormalizable interactions die out as one goes to lower energy.

The dimensionless couplings flow to values which are solely determined by the renormalizable couplings. Thus in space of dimensionless couplings the renormalization group flow is towards a finite dimensional hyperplane. One could think of inverting this flow and thereby calculating the couplings of a high energy fundamental theory. This extrapolation to larger cut-off is only valid as long as the dimensionless couplings are not to large. For large couplings the scaling behaviour of the fields may change due to large anomalous dimensions. Any deviation from the hypersurface grows large as the renormalization flow goes back to larger cut-off. Therefore an initial deviation from the hypersurface sets already the limit of this extrapolation to larger cut-off. That renormalizable theories are very successful means in this context that the couplings measured in experiments are very close to the hypersurface. Therefore it is possible to invert the renormalization group flow backward to very large cut-off. In case of QED an infinite cut-off is compatible with certain experimental values, for instance the anomalous magnetic moment of the electron.

We will explain this mechanism using an example. Before we do this we recapitulate the notion of renormalizability. It is based on dimensional analysis. The canonical dimension of a field is determined from the kinetic term in the action. For instance, in a scalar field theory in d dimensions, the kinetic term is

$$\frac{1}{2}\int d^dx\ \partial_\mu\phi\partial_\mu\phi. \tag{3}$$

The integration measure contributes dimension of mass $-d$, each derivative contributes $+1$. Hence we determine the dimension of mass of a bosonic field to be $\frac{d-2}{2}$ because the action must be dimensionless. Fermions have only one derivative in their kinetic term and therefore the canonical dimension of mass of fermion fields is $\frac{d-1}{2}$. With this we can determine the dimension δ of an arbitrary interaction.

An interaction is perturbatively renormalizable or infrared relevant if it has a canonical dimension $\delta < d$. It is perturbatively renormalizable and marginal if $\delta = d$. The interaction is called perturbatively nonrenormalizable or infrared irrelevant if $\delta > d$.

We want to investigate the effects of perturbatively renormalizable interactions on the low energy theory. Suppose we start with a cut-off theory with a large cut-off Λ'. We assume for the moment only renormalizable interactions to be present at this scale. In order to be more specific we take a massless ϕ^4 model in four dimensions and ask for the size of the six point function. The lowest order loop contribution is of the form given in figure 1.

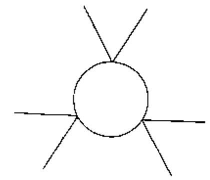


Figure 1: *Correction to the ϕ^6 interaction due to integration of high frequency modes*

There are three propagators in the loop. We integrate the loop momentum from Λ to Λ'. We neglect the external momenta which are assumed to be small compared to the loop momentum. The loop integral is then of the order $\frac{1}{\Lambda^2}$, where Λ is the new cut-off. From this example we learn that nonrenormalizable interactions like ϕ^6 appear naturally in effective theories. They are produced from renormalizable ones via integration of high frequency modes. The exact values of the coefficients of these nonrenormalizable interactions have to be calculated. The order of magnitude, however, can be estimated from the dimension of the interaction. For the above example of a dimension $\delta = 6$ interaction the coefficient is approximately $g = \frac{1}{\Lambda^2}$. As a general rule we find that the coefficient in front of an interaction is approximately $g = \frac{1}{\Lambda^{\delta-d}}$. There may be exceptions if certain contributions are suppressed for symmetry reasons. Also this estimate is only correct up to a logarithmic factor $\propto \ln(\frac{\Lambda'}{\Lambda})$.

We define a dimensionless coupling strength by multiplying the dimensional coupling g with the appropriate power of the actual cut-off.

$$\lambda = g\,\Lambda^{\delta-d} \tag{4}$$

This dimensionless coupling λ for the induced interactions is always of the order one. Since g is a constant, the dimensionless coupling for the original present interaction gets suppressed by $(\frac{\Lambda}{\Lambda'})^{\delta-d}$ if $\delta > d$. In this case the interaction is called infrared irrelevant.

We conclude with the remark that cut-off theories are not in contradiction to renormalizability but include renormalizable theories as special cases.

In this work we regard the fundamental theory as given and try to calculate the low energy effective theory. We mentioned earlier that the coefficients are in principle calculable. This is true but may still be very difficult. The most subtle difficulty lies in the choice of the right low energy field, called blockspin field. To choose the blockspin field to be always of the same kind as the fundamental field may lead to a low energy theory which does not possess the postulated locality properties.

The discrete Gaussian model is one example for this. There one starts with an integer valued field. As one lowers the cut-off, the appropriate blockspin variable should be real valued. It was shown by Göpfert and Mack [6] that the effective action becomes nonlocal if an integer valued blockspin is chosen. Important physical degrees of freedom like spin waves are then completely disguised in the action. A choice of a real blockspin in contrast yields the correct result — a nearly free field theory with a nonvanishing mass.

In this work we want to investigate the case where composite fields become important at a certain scale. If they are not introduced as field variables in their own right into the theory the action will become nonlocal. An effective action should contain only local terms. This guarantees that the essential physics can almost be read off the action. In contrast when nonlocalities appear this indicates that essential physical phenomena are not represented in the choice of variables of the action. The other terms have to compensate this lack. As a consequence they will develop awkward nonlocalities.

A second difficulty is the change in the scaling behaviour of fields for large interactions. Bound states appear typically when some couplings become large. Therefore there may be a transition of terms in the action from irrelevant to marginal or relevant. In nonabelian gauge theories the couplings become large for low energies. This causes bound states to form.

In the next section we study the important case of QCD to show that the resulting effective theory contains among many other interactions a four fermion interaction. Such an interaction is least irrelevant among the nonrenormalizable interactions and hence most likely to become marginal or relevant. This serves as a motivation to investigate a toy model with a four fermion interaction in subsequent sections.

3 Motivation from non-abelian gauge theories

In this section we show that the effective theory associated with QCD contains a four fermion interaction. It results from the integration of high frequency gluon fields. We demonstrate that this four fermion interaction can be Fierz-transformed to new attractive interactions as was pointed out by Cahill et. al. [32, 34]. These new interactions may form bilocal diquark and meson fields which can be expanded into local fields [32]. This serves as motivation to use the Gross-Neveu model [18] as a toy model in later sections.

We start with the QCD action in euclidean space.

$$S = \int d^4x \left(\frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \bar\psi \not{D} \psi \right) + S_{gf} + S_{gh} \tag{5}$$

where

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g C^{abc} A^b_\mu A^c_\nu \tag{6}$$
$$\not{D} = \not{\partial} + g T^a \not{A}^a \tag{7}$$

S_{gf} and S_{gh} are gauge fixing and Faddeev-Popov-ghost actions. The T^a are the antihermitian generators of the non-abelian group. The T^a satisfy the following orthogonality relation $tr(T^aT^b) = -\frac{1}{2}\delta_{ab}$. The connection with the Gell-Mann-matrices is $T^a = -i\frac{\lambda^a}{2}$. The euclidean γ matrices are defined by $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$.

To find the low energy effective theory belonging to QCD one must integrate the high frequency modes out. This is a very difficult task because couplings become large for low energies and perturbation theory is no longer valid. However, as long as the cut-off of the effective energy is large enough, it should be possible to calculate the effective action perturbatively. In gauge theories there is an additional difficulty to implement a frequency split. The difficulty consists in maintaining gauge invariance for the low energy degrees of freedom while integrating the high energy degrees of freedom. The integration of the high frequency modes needs a gauge fixing. Thus gauge fixing has to be done only for the high frequency modes. Bałaban has done such a partial gauge fixing for lattice fields [3].

To demonstrate the appearance of four fermion interactions we choose a much simpler approach. We imagine the gauge being fixed by a yet unspecified term. Hence the gluon propagator exists and we can use a propagator split method to divide the field into a high and a low frequency part.

This uses the convolution theorem for gaussian measures. Given a normalized gaussian measure $d\mu_G(\Phi)$ of the form

$$d\mu_G(\Phi) = (det 2\pi G)^{-\frac{1}{2}} e^{-\frac{1}{2}\Phi G^{-1}\Phi} \prod_x d\Phi(x), \tag{8}$$

the following is true. If $G = G_1 + G_2$ then

$$\int d\mu_G(\Phi) F(\Phi) = \int d\mu_{G_1}(\Phi_1) d\mu_{G_2}(\Phi_2) F(\Phi_1 + \Phi_2) \tag{9}$$

This is immediately clear for the characteristic functional

$$\int d\mu_G(\Phi) e^{iJ\Phi} = e^{-\frac{1}{2}(JGJ)} = e^{-\frac{1}{2}(JG_1J)} e^{-\frac{1}{2}(JG_2J)} = \int d\mu_{G_1}(\Phi_1) e^{iJ\Phi_1} \int d\mu_{G_2}(\Phi_2) e^{iJ\Phi_2} \tag{10}$$

and therefore true for an arbitrary functional.

Thus we take for instance the gluon propagator $D^{ab}_{\mu\nu}(p)$ and use a convenient cut-off function $K_\Lambda(p)$ for the split.

$$D^{ab}_{\mu\nu}(p) = \underbrace{D^{ab}_{\mu\nu}(p) K_\Lambda(p)}_{\equiv D^{s\,ab}_{\mu\nu}(p)} + \underbrace{D^{ab}_{\mu\nu}(p)(1 - K_\Lambda(p))}_{\equiv D^{h\,ab}_{\mu\nu}(p)} \tag{11}$$

$K_\Lambda(p)$ cuts off essentially all momenta above Λ. $K_\Lambda(p)$ has to be chosen such that the high and low frequency propagators are still invertible. The propagator split induces a field split

$$A^a_\mu = A^{a\,s}_\mu + A^{a\,h}_\mu. \tag{12}$$

To calculate a low energy effective action this has to be done as well for the other fields, including the ghost fields which arise because of the gauge fixing. Since we just want to demonstrate the appearance of the four fermion interaction it is sufficient to do only the gaussian part of the high frequency gluon integration. The full treatment would put corrections on the resulting four fermion interaction. This would not change the following Fierz transformations.

To be specific we choose a covariant gauge fixing with the appropriate ghost term.

$$S_{gf} = \frac{1}{2\alpha}(\partial_\mu A_\mu^a)^2 \quad ; \quad S_{gh} = \bar{\eta}\partial_\mu D_\mu \eta \tag{13}$$

The gaussian part of the A^h integration yields

$$\begin{aligned} &\int \mathcal{D}A^h \exp(-\bar{\psi} g T^a A\!\!\!/^{h\,a}\psi - \bar{\eta}\partial_\mu g T^a A_\mu^{h\,a}\eta - \frac{1}{2}A_\mu^{h\,a}(D^{h\,-1})_{\mu\nu}^{ab}A_\nu^{h\,a}) \\ &= \exp(\frac{1}{2}\int\int \left([\bar{\eta}(x)\,\overleftarrow{\partial}_\mu] g T^a \eta(x) - \bar{\psi}(x) g T^a \gamma_\mu \psi(x)\right) \; D_{\mu\nu}^{h\,ab}(x-y) \\ &\qquad \left([\bar{\eta}(y)\,\overleftarrow{\partial}_\nu] g T^b \eta(y) - \bar{\psi}(y) g T^b \gamma_\nu \psi(y)\right) \end{aligned} \tag{14}$$

Among other terms the exponent contains the four fermion interaction we are interested in.

$$\frac{1}{2}\int\int \bar{\psi}(x) g T^a \gamma_\mu \psi(x)\; D_{\mu\nu}^{h\,ab}(x-y)\; \bar{\psi}(y) g T^b \gamma_\nu \psi(y) \tag{15}$$

The high frequency propagator has the form

$$D_{\mu\nu}^{h\,ab}(p) \;=\; \frac{1-K_\Lambda(p)}{p^2}\,(\delta_{\mu\nu} + \frac{p_\mu p_\nu}{p^2}(\alpha - 1))\,\delta_{ab}. \tag{16}$$

In Fierz transforming this four fermion interaction we will mainly follow the treatment of Roberts and Cahill [35, 32]. We start with the following well known relation for the generators of a SU(n) group.

$$\sum_{a=1}^{n^2-1} T_{\alpha\beta}^a T_{\gamma\delta}^a = -\frac{1}{2}\delta_{\alpha\delta}\delta_{\beta\gamma} + \frac{1}{2n}\delta_{\alpha\beta}\delta_{\gamma\delta} \tag{17}$$

where the T^a are normalized such that $tr(T^aT^b) = -\frac{1}{2}\delta_{ab}$. For the special case of SU(3) this reads

$$\sum_{a=1}^{8} T_{\alpha\beta}^a T_{\gamma\delta}^a = -\frac{1}{2}\delta_{\alpha\delta}\delta_{\beta\gamma} + \frac{1}{6}\delta_{\alpha\beta}\delta_{\gamma\delta}. \tag{18}$$

Using the identity

$$\sum_{\rho=1}^{3}\varepsilon_{\rho\alpha\gamma}\varepsilon_{\rho\beta\delta} = \delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha\delta}\delta_{\gamma\beta} \tag{19}$$

this is transformed to

$$\sum_{a=1}^{8} T_{\alpha\beta}^a T_{\gamma\delta}^a = \underbrace{-\frac{1}{3}\delta_{\alpha\delta}\delta_{\beta\gamma}}_{\text{meson}} \underbrace{-\frac{1}{6}\sum_{\rho=1}^{3}\varepsilon_{\rho\alpha\gamma}\varepsilon_{\rho\beta\delta}}_{\text{diquark}}\,. \tag{20}$$

This is the transformation for the colour matrices. This transformation divides the four fermion interaction into two fundamentally different interactions. The first summand on the right hand side describes colour singlet states. The second summand in contrast describes colour triplet states. The indices in the two summands are distributed differently. As a consequence we have to do two different Fierz transformations in Dirac and flavour space to match with this two different distribution of indices. One for the meson part and one for the diquark part.

The flavour space transformation uses again equation (17).

$$\delta_{ij}\delta_{kl} = \frac{1}{n}\delta_{ik}\delta_{lj} - 2\sum_{a=1}^{n^2-1} T_{ik}^a T_{lj}^a = H_{ik}^f H_{lj}^f \tag{21}$$

which for the special case of SU(3) flavour can be arranged into

$$\{H^f, f=1,\ldots,9\} = \{\underbrace{i\sqrt{2}T^2, i\sqrt{2}T^5, i\sqrt{2}T^7}_{\text{antisymmetric}}, \underbrace{\frac{1}{\sqrt{3}}\mathbf{1}, i\sqrt{2}T^1, i\sqrt{2}T^3, i\sqrt{2}T^4, i\sqrt{2}T^6, i\sqrt{2}T^8}_{\text{symmetric}}\}. \tag{22}$$

This flavour space transformation belongs to the diquark part because j and l, i.e. second and fourth index, interchange places. Therefore for the special case of flavour SU(3) diquarks may appear in $\bar{\mathbf{3}}_F$ and $\mathbf{6}_F$ representations.

For the meson part we have to change second third and fourth indices cyclically. This is done again using equation (17).

$$\delta_{ij}\delta_{kl} = \delta_{ij}\delta_{lk} = \frac{1}{n}\delta_{il}\delta_{kj} - 2\sum_{a=1}^{n^2-1} T_{il}^a T_{kj}^a = F_{il}^c F_{kj}^c. \tag{23}$$

This time the special case of SU(3) yields

$$\{F^c, c=1,\ldots,9\} = \{\frac{1}{\sqrt{3}}\mathbf{1}, i\sqrt{2}T^1, \ldots, i\sqrt{2}T^8\}. \tag{24}$$

Hence the meson part appears in $\mathbf{1}_F$ and $\mathbf{8}_F$ representations.

The Fierz-transformation for Dirac-space is taken from Takahashi [30].

$$\begin{aligned} 4(\mu)[\nu] \;=\; & -\delta_{\mu\nu}\{(\,][\,) + (5][5) + (\alpha][\alpha) + (5\alpha][5\alpha)\} \\ & +(\mu][\nu) + (\nu][\mu) + (5\mu][5\nu) + (5\nu][5\mu) \\ & +i\{(\,][\mu\nu) - (\mu\nu][\,)\} + i\{(5][{}^*\mu\nu) - ({}^*\mu\nu][5)\} \\ & +(\mu\alpha][\nu\alpha) + ({}^*\mu\alpha][{}^*\nu\alpha) + \varepsilon_{\mu\nu\alpha\beta}\{(\alpha][5\beta) - (5\beta][\alpha)\} \end{aligned} \tag{25}$$

In this convenient notation the brackets indicate the position of indices. To clarify the notation the beginning of the above formula with all indices written reads

$$4(i\gamma_\mu)_{ab}(i\gamma_\nu)_{cd} = -\delta_{\mu\nu}\{\delta_{ad}\delta_{cb} + (i\gamma_s)_{ad}(i\gamma_s)_{cb} + \ldots \tag{26}$$

The distribution of indices on the right hand side reveals immediately that this transformation is for the meson part. The other abbreviations used are

$$\begin{aligned} (\,) = \mathbf{1}\;;\;(5) = i\gamma_s = -i\gamma_1\gamma_2\gamma_3\gamma_4\;;\;(\mu) = i\gamma_\mu\;;\;(5\mu) = i\gamma_s\gamma_\mu \\ (\mu\nu) = \sigma_{\mu\nu} = \frac{1}{2i}[\gamma_\mu,\gamma_\nu]\;;\;({}^*\mu\nu) = i\gamma_s\sigma_{\mu\nu} = -\frac{i}{2}\varepsilon_{\mu\nu\alpha\beta}\sigma_{\alpha\beta} \end{aligned} \tag{27}$$

The right hand side of the transformation contains the $({}^*\mu\nu)$ matrices which are not basic elements of the γ-algebra, but can be expressed in terms of $\sigma_{\mu\nu}$ matrices.

The Fierz transformation for the diquark part are found from (25) by the following manipulations [32]. We use the properties of the charge conjugation matrix $C = \gamma_2\gamma_4$.

$$C^2 = -\mathbf{1}\;:\; C^{-1}\gamma_\mu C = -\gamma_\mu^T \Rightarrow C^T\gamma_\mu C^T = \gamma_\mu^T \tag{28}$$

The last equation enables us to state the following. Suppose we have given a Fierz transformation of the following form

$$4(\gamma_\mu)_{ab}(\gamma_\nu)_{cd} = \sum_{AB} c_{AB}(\Gamma^A)_{ad}(\Gamma^B)_{cb}, \tag{29}$$

where the matrices Γ^A may form an overcomplete set of matrices in the algebra. Then

$$4(\gamma_\mu)_{ab}(\gamma_\nu)_{cd} = \sum_{AB} c_{AB}(\Gamma^A C^T)_{ac}(C^T\Gamma^B)_{db} \tag{30}$$

is also true because

$$\begin{aligned}\sum_{AB} c_{AB}(\Gamma^A C^T)_{ac}(C^T\Gamma^B)_{db} &= \sum_{AB} c_{AB}\Gamma^A_{ai}C^T_{ic}C^T_{dj}\Gamma^B_{jb} \\ \overset{1.\,\text{Fierz}}{=} C^T_{ic}C^T_{dj}4(\gamma_\mu)_{ab}(\gamma_\nu)_{ji} &= 4(\gamma_\mu)_{ab}(C^T\gamma_\nu C^T)_{dc} = 4(\gamma_\mu)_{ab}(\gamma_\nu)_{cd}.\end{aligned} \tag{31}$$

If we apply this identity to our above Fierz-transformation (25) we will get the transformation for the diquark part. We are now prepared to rearrange the interaction by Fierz transformation.

$$\begin{aligned}
&\frac{1}{2}\iint \bar\psi(x)\gamma_\mu g T^{\hat a}\psi(x)\, D^{h\,\hat a\hat b}_{\mu\nu}(x-y)\, \bar\psi(y)\gamma_\nu g T^{\hat b}\psi(y) \\
=&\frac{g^2}{2}\iint \bar\psi(x)_{a\alpha i}(\gamma_\mu)_{ab}T^{\hat a}_{\alpha\beta}\delta_{ij}\psi(x)_{b\beta j}\, D^{h\,\hat a\hat b}_{\mu\nu}(x-y)\, \bar\psi(y)_{c\gamma k}(\gamma_\nu)_{cd}T^{\hat b}_{\gamma\delta}\delta_{kl}\psi(y)_{d\delta l} \\
=&\frac{g^2}{6}\iint\Big\{ + \bar\psi(x)\, {}^S\mathcal{M}^c\psi(y)\, D(x-y)\, \bar\psi(y)\, {}^S\mathcal{M}^c\psi(x) \\
&+ \bar\psi(x)\, {}^P\mathcal{M}^c\psi(y)\, D(x-y)\, \bar\psi(y)\, {}^P\mathcal{M}^c\psi(x) \\
&+ \bar\psi(x)\, {}^V\mathcal{M}^c_\mu\psi(y)\, \tilde D_{\mu\nu}(x-y)\, \bar\psi(y)\, {}^V\mathcal{M}^c_\nu\psi(x) \\
&+ \bar\psi(x)\, {}^A\mathcal{M}^c_\mu\psi(y)\, \tilde D_{\mu\nu}(x-y)\, \bar\psi(y)\, {}^A\mathcal{M}^c_\nu\psi(x) \\
&+ \bar\psi(x)\, {}^T\mathcal{M}^c_{\mu\alpha}\psi(y)\, D^T_{\mu\nu}(x-y)\, \bar\psi(y)\, {}^T\mathcal{M}^c_{\nu\alpha}\psi(x)\Big\} \\
&+\frac{g^2}{6}\iint\Big\{ + \bar\psi(x)\, {}^S\mathcal{D}^{\rho f}\bar\psi^c(y)\, D(x-y)\, \psi^c(y)\, {}^S\mathcal{D}^{\rho f}\psi(x) \\
&+ \bar\psi(x)\, {}^P\mathcal{D}^{\rho f}\bar\psi^c(y)\, D(x-y)\, \psi^c(y)\, {}^P\mathcal{D}^{\rho f}\psi(x) \\
&+ \bar\psi(x)\, {}^V\mathcal{D}^{\rho f}_\mu\bar\psi^c(y)\, \tilde D_{\mu\nu}(x-y)\, \psi^c(y)\, {}^V\mathcal{D}^{\rho f}_\nu\psi(x) \\
&+ \bar\psi(x)\, {}^A\mathcal{D}^{\rho f}_\mu\bar\psi^c(y)\, \tilde D_{\mu\nu}(x-y)\, \psi^c(y)\, {}^A\mathcal{D}^{\rho f}_\nu\psi(x) \\
&+ \bar\psi(x)\, {}^T\mathcal{D}^{\rho f}_{\mu\alpha}\bar\psi^c(y)\, D^T_{\mu\nu}(x-y)\, \psi^c(y)\, {}^T\mathcal{D}^{\rho f}_{\nu\alpha}\psi(x)\Big\}
\end{aligned} \tag{32}$$

$\mathcal{M}$ denotes a mesonic interaction, $\mathcal{D}$ diquark interactions. Labels S, P, V, A, T denote scalar, pseudoscalar, vector, axial vector and tensor transformation behaviour under Lorentz transformations. Observe that there is a minus sign due to an interchange of Grassmann fields. We calculate the coupling functions $D, \tilde D_{\mu\nu}$ and $D^T_{\mu\nu}$. Since $D_{\mu\nu} = D_{\nu\mu}$ all antisymmetric contributions in (25) vanish. We have to calculate the following

$$\begin{aligned}(\mu)D^h_{\mu\nu}[\nu] &= -\frac{1}{4}\{(\,][\,) + (5][5) + (\alpha][\alpha) + (5\alpha][5\alpha)\}\delta_{\mu\nu}D^h_{\mu\nu} \\ &+\frac{1}{2}\{(\mu][\nu) + (5\mu][5\nu)\}D^h_{\mu\nu} \\ &+\frac{1}{4}\{(\mu\alpha][\nu\alpha) + ({}^*\mu\alpha][{}^*\nu\alpha)\}D^h_{\mu\nu}\end{aligned} \tag{33}$$

We consider first

$$\begin{aligned}&-\frac{1}{4}\{(\alpha][\alpha) + (5\alpha][5\alpha)\}\delta_{\mu\nu}\int\frac{d^4p}{(2\pi)^4}\frac{1-K_\Lambda(p)}{p^2}(\delta_{\mu\nu} + \frac{p_\mu p_\nu}{p^2}(\alpha-1))e^{ipx} \\ &= -\frac{1}{4}\{(\alpha][\beta) + (5\alpha][5\beta)\}\delta_{\alpha\beta}\int\frac{d^4p}{(2\pi)^4}\frac{1-K_\Lambda(p)}{p^2}(3+\alpha)e^{ipx} \\ &= -\{(\mu][\nu) + (5\mu][5\nu)\}\delta_{\mu\nu}\underbrace{\frac{3+\alpha}{4}\int\frac{d^4p}{(2\pi)^4}\frac{1-K_\Lambda(p)}{p^2}e^{ipx}}_{=\frac{1}{4}\delta_{\mu\nu}D^h_{\mu\nu}\equiv D(x)}\end{aligned} \tag{34}$$

If we define $\tilde D_{\mu\nu}$ to be

$$\begin{aligned}\tilde D_{\mu\nu} \equiv -\frac{1}{2}D^h_{\mu\nu} + D &= -\int\frac{d^4p}{(2\pi)^4}\frac{1-K_\Lambda(p)}{p^2}(\delta_{\mu\nu}(\frac{1}{2} - \frac{3+\alpha}{4}) + \frac{p_\mu p_\nu}{p^2}\frac{\alpha-1}{2})e^{ipx} \\ &= \frac{1+\alpha}{4}\int\frac{d^4p}{(2\pi)^4}\frac{1-K_\Lambda(p)}{p^2}(\delta_{\mu\nu} + \frac{p_\mu p_\nu}{p^2}\frac{2-2\alpha}{1+\alpha})e^{ipx},\end{aligned} \tag{35}$$

we find up to this point

$$\begin{aligned}(\mu)D^h_{\mu\nu}[\nu) &= -\{(\,][\,) + (5][5)\}D \\ &-\{(\mu][\nu) + (5\mu][5\nu)\}\tilde D_{\mu\nu} \\ &+\frac{1}{4}\{(\mu\alpha][\nu\alpha) + ({}^*\mu\alpha][{}^*\nu\alpha)\}D^h_{\mu\nu}\end{aligned} \tag{36}$$

To find the function $D^T_{\mu\nu}(x)$ one has to transform $({}^*\mu\alpha][{}^*\nu\alpha)$ [30] according to

$$({}^*\mu\alpha][{}^*\nu\alpha) = -\frac{1}{4}\epsilon_{\mu\alpha\lambda\rho}\epsilon_{\nu\alpha\lambda'\rho'}(\lambda\rho][\lambda'\rho') = (\nu\alpha][\mu\alpha) - \frac{1}{2}\delta_{\mu\nu}(\alpha\beta][\alpha\beta) \tag{37}$$

Therefore we find

$$[(\mu\alpha][\nu\alpha) + ({}^*\mu\alpha][{}^*\nu\alpha)]\,\frac{1}{4}D^h_{\mu\nu}(x) = (\mu\alpha][\nu\alpha)\{\frac{1}{2}D^h_{\mu\nu}(x) - \frac{1}{2}\delta_{\mu\nu}D(x)\} \equiv -(\mu\alpha][\nu\alpha)D^T_{\mu\nu}(x) \tag{38}$$

with

$$\begin{aligned}D^T_{\mu\nu} = -\frac{1}{2}(D^h_{\mu\nu} - \delta_{\mu\nu}D) &= -\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\frac{1-K_\Lambda(p)}{p^2}(\delta_{\mu\nu}(1 - \frac{3+\alpha}{4}) + \frac{p_\mu p_\nu}{p^2}(\alpha-1))e^{ipx} \\ &= \frac{\alpha-1}{8}\int\frac{d^4p}{(2\pi)^4}\frac{1-K_\Lambda(p)}{p^2}(\delta_{\mu\nu} - 4\frac{p_\mu p_\nu}{p^2})e^{ipx}\end{aligned} \tag{39}$$

The Matrices appearing in the interactions are defined by

$$\begin{aligned}&{}^S\mathcal{M}^c = \mathbf{1}\otimes\mathbf{1}\otimes F^c\ ;\ {}^P\mathcal{M}^c = i\gamma_s\otimes\mathbf{1}\otimes F^c\ ;\ {}^V\mathcal{M}^c_\mu = i\gamma_\mu\otimes\mathbf{1}\otimes F^c \\ &{}^A\mathcal{M}^c_\mu = i\gamma_s\gamma_\mu\otimes\mathbf{1}\otimes F^c\ ;\ {}^T\mathcal{M}^c_{\mu\nu} = \sigma_{\mu\nu}\otimes\mathbf{1}\otimes F^c \\ &{}^S\mathcal{D}^{\rho f} = \frac{i}{\sqrt2}\mathbf{1}\otimes\varepsilon_\rho\otimes H^f\ ;\ {}^P\mathcal{D}^{\rho f} = -\frac{1}{\sqrt2}\gamma_s\otimes\varepsilon_\rho\otimes H^f\ ;\ {}^V\mathcal{D}^{\rho f}_\mu = -\frac{1}{\sqrt2}\gamma_\mu\otimes\varepsilon_\rho\otimes H^f \\ &{}^A\mathcal{D}^{\rho f}_\mu = -\frac{1}{\sqrt2}\gamma_s\gamma_\mu\otimes\varepsilon_\rho\otimes H^f\ ;\ {}^T\mathcal{D}^{\rho f}_{\mu\nu} = \frac{i}{\sqrt2}\sigma_{\mu\nu}\otimes\varepsilon_\rho\otimes H^f\end{aligned} \tag{40}$$

We use $\psi^c = C\psi$; $\bar\psi^c = \bar\psi C$. It was shown by Cahill et. al. [34] that these four fermion interactions are attractive. We expect these couplings to grow large as we lower the cut-off until bound states form. We introduce bosonic bilocal fields [37] into the path integral by a generalization of the Hubbard-Stratonovich or auxiliary field method. To bosonize for instance the following four fermion interaction

$$\frac{g^2}{6}\iint\bar\psi(x)\, {}^S\mathcal{D}^{\rho f}\bar\psi^c(y)\, D(x-y)\, \psi^c(y)\, {}^S\mathcal{D}^{\rho f}\psi(x) \tag{41}$$

we introduce the following constant into the path integral.

$$\begin{aligned}const &= \int\mathcal{D}{}^S\Xi^{\rho f*}(x,y)\mathcal{D}{}^S\Xi^{\rho f}(x,y) \\ &\quad e^{\left(-\frac{g^2}{6}\iint({}^S\Xi^{\rho f*}(x,y) - \bar\psi(x)\,{}^S\mathcal{D}^{\rho f}\bar\psi^c(y))\,D(x-y)\delta(x-x')\delta(y-y')\,({}^S\Xi^{\rho f}(x',y') - \psi^c(y')\,{}^S\mathcal{D}^{\rho f}\psi(x'))\right)} \\ &= \int\mathcal{D}{}^S\Xi^{\rho f*}(x,y)\mathcal{D}{}^S\Xi^{\rho f}(x,y) \\ &\quad e^{\left(-\frac{g^2}{6}\iint({}^S\Xi^{\rho f*}(x,y) - \bar\psi(x)\,{}^S\mathcal{D}^{\rho f}\bar\psi^c(y))\,D(x-y)\,({}^S\Xi^{\rho f}(x,y) - \psi^c(y)\,{}^S\mathcal{D}^{\rho f}\psi(x))\right)}\end{aligned} \tag{42}$$

This gaussian integral is a generalization of the following simpler expression

$$\int \frac{d\beta_i^* d\beta_i}{\pi^n} \exp(-\beta_i^* A_{ij}\beta_j + \xi_i^* \beta_i + \beta_i^* \zeta_i) = \frac{1}{det(A)} e^{\xi^* A^{-1}\zeta}. \tag{43}$$

Observe that $\xi \neq \zeta$ in this formula. For interactions like

$$\frac{g^2}{6} \iint \bar\psi(x)\, {}^S\mathcal{M}^c \psi(y)\, D(x-y)\, \bar\psi(y)\, {}^S\mathcal{M}^c \psi(x) \tag{44}$$

we use the following constant

$$const = \int \mathcal{D}^S\Delta^c(x,y) e^{\left(-\frac{g^2}{6}\iint ({}^S\Delta^c(x,y) - \bar\psi(x)\, {}^S\mathcal{M}^c\psi(y))\, D(x-y)\, ({}^S\Delta^c(y,x) - \bar\psi(y)\, {}^S\mathcal{M}^c\psi(x))\right)} \tag{45}$$

where ${}^S\Delta^c(x,y)$ is a hermitian bilocal field, i.e. it satisfies the constraint ${}^S\Delta^{c*}(x,y) = {}^S\Delta^c(y,x)$. Doing this for all four fermion interactions in a similar manner we arrive at the following bosonized interaction

$$\begin{aligned}
&\int \mathcal{D}^S\Delta^c(x,y)\mathcal{D}^P\Delta^c(x,y)\mathcal{D}^V\Delta^c_\mu(x,y)\ldots\mathcal{D}^S\Xi^{\rho f*}(x,y)\mathcal{D}^S\Xi^{\rho f}(x,y)\mathcal{D}^P\Xi^{\rho f*}(x,y)\mathcal{D}^P\Xi^{\rho f}(x,y)\ldots\\
&\exp\Big\{-\iint\Big\{\ {}^S\Delta^c(x,y)D(x-y){}^S\Delta^c(x,y) - 2{}^S\Delta^c(x,y)D(x-y)\bar\psi(y)\,{}^S\mathcal{M}^c\psi(x))\\
&\quad +{}^P\Delta^c(x,y)D(x-y){}^P\Delta^c(x,y) - 2{}^P\Delta^c(x,y)D(x-y)\bar\psi(y)\,{}^P\mathcal{M}^c\psi(x))\\
&\quad +{}^V\Delta^c_\mu(x,y)\tilde D_{\mu\nu}(x-y){}^V\Delta^c_\nu(x,y) - 2{}^V\Delta^c_\mu(x,y)\tilde D_{\mu\nu}(x-y)\bar\psi(y)\,{}^V\mathcal{M}^c_\nu\psi(x))\\
&\quad +{}^A\Delta^c_\mu(x,y)\tilde D_{\mu\nu}(x-y){}^A\Delta^c_\nu(x,y) - 2{}^A\Delta^c_\mu(x,y)\tilde D_{\mu\nu}(x-y)\bar\psi(y)\,{}^A\mathcal{M}^c_\nu\psi(x))\\
&\quad +{}^T\Delta^c_{\mu\alpha}(x,y)D^T_{\mu\nu}(x-y){}^T\Delta^c_{\nu\alpha}(x,y) - 2{}^T\Delta^c_{\mu\alpha}(x,y)D^T_{\mu\nu}(x-y)\bar\psi(y)\,{}^T\mathcal{M}^c_{\nu\alpha}\psi(x))\\
&\quad +{}^S\Xi^{\rho f*}(x,y)D(x-y){}^S\Xi^{\rho f}(x,y) - 2{}^S\Xi^{\rho f*}(x,y)D(x-y)\psi^c(y)\,{}^S\mathcal{D}^{\rho f}\psi(x))\\
&\quad -2\bar\psi(x)\,{}^S\mathcal{D}^{\rho f}\bar\psi^c(x))D(x-y){}^S\Xi^{\rho f}(x,y)\\
&\quad +{}^P\Xi^{\rho f*}(x,y)D(x-y){}^P\Xi^{\rho f}(x,y) - 2{}^P\Xi^{\rho f*}(x,y)D(x-y)\psi^c(y)\,{}^P\mathcal{D}^{\rho f}\psi(x))\\
&\quad -2\bar\psi(x)\,{}^P\mathcal{D}^{\rho f}\bar\psi^c(x))D(x-y){}^P\Xi^{\rho f}(x,y)\\
&\quad +{}^V\Xi^{\rho f*}_\mu(x,y)\tilde D_{\mu\nu}(x-y){}^V\Xi^{\rho f}_\nu(x,y) - 2{}^V\Xi^{\rho f*}_\mu(x,y)\tilde D_{\mu\nu}(x-y)\psi^c(y)\,{}^V\mathcal{D}^{\rho f}_\nu\psi(x))\\
&\quad -2\bar\psi(x)\,{}^V\mathcal{D}^{\rho f}_\mu\bar\psi^c(x))\tilde D_{\mu\nu}(x-y){}^V\Xi^{\rho f}_\nu(x,y)\\
&\quad +{}^A\Xi^{\rho f*}_\mu(x,y)\tilde D_{\mu\nu}(x-y){}^A\Xi^{\rho f}_\nu(x,y) - 2{}^A\Xi^{\rho f*}_\mu(x,y)\tilde D_{\mu\nu}(x-y)\psi^c(y)\,{}^A\mathcal{D}^{\rho f}_\nu\psi(x))\\
&\quad -2\bar\psi(x)\,{}^A\mathcal{D}^{\rho f}_\mu\bar\psi^c(x))\tilde D_{\mu\nu}(x-y){}^A\Xi^{\rho f}_\nu(x,y)\\
&\quad +{}^T\Xi^{\rho f*}_{\mu\alpha}(x,y)D^T_{\mu\nu}(x-y){}^T\Xi^{\rho f}_{\nu\alpha}(x,y) - 2{}^T\Xi^{\rho f*}_{\mu\alpha}(x,y)D^T_{\mu\nu}(x-y)\psi^c(y)\,{}^T\mathcal{D}^{\rho f}_{\nu\alpha}\psi(x))\\
&\quad -2\bar\psi(x)\,{}^T\mathcal{D}^{\rho f}_{\mu\alpha}\bar\psi^c(x))D^T_{\mu\nu}(x-y){}^T\Xi^{\rho f}_{\nu\alpha}(x,y)\Big\}\Big\}
\end{aligned} \tag{46}$$

Now we have introduced the appropriate bosonic degrees of freedom to describe bound states which may be formed by the attractive four fermion interaction. However, bilocal fields are complicated objects which defeat physical intuition. The bilocality reflects the inner structure of the bound states. Since we are interested in a low energy effective theory we do not need information about this inner structure. We may expand a bilocal field into local fields. We introduce a complete orthonormal set of functions $\Gamma_k(x)$. This can be used to expand any of the bilocal fields.

$${}^S\Delta^c(x,y) = \sum_k {}^S\beta^c_k(z)\Gamma_k(w) \tag{47}$$

where $z = \frac{x+y}{2}$; $w = x - y$. ${}^S\beta^c_k(z)$ are the local field variables and $\Gamma_k(w)$ are essentially structure functions of the boson. Observe that our bilocal functions always appeared together with $D(w)$

which is local because the hard gluon propagator has an infrared cut-off. Hence we may truncate the complete orthonormal set such that it also contains only local functions with respect to the cut-off Λ. The kinetic term for this variable is simplified in the following manner.

$$\iint {}^S\Delta^c(x,y)D(x-y)\,{}^S\Delta^c(x,y) = \int_z {}^S\beta^c_k(z)\lambda_{kl}\,{}^S\beta^c_l(z) \tag{48}$$

where $\lambda_{kl} = \int_w \Gamma_k(w)D(w)\Gamma_l(w)$ is the kinetic term of the local boson fields. λ_{kl} is a point propagator, i.e. it has no dynamics at this stage of our calculation. We will see in the next section that such a propagator appears also if we bosonize a point like four fermion interaction.

All this can be done at a cut-off which justifies perturbative calculations. We want to study the behaviour of these terms as we lower the cut-off gradually towards the nonperturbative regime. To do this in the context of the above interactions is clearly too complicated. A local four fermion interaction produces a similar structure when bosonized. Therefore we will study a simple toy model which has a local four fermion interaction to start with. We will study how the bosonic degrees of freedom gain a dynamic and when it really becomes necessary to introduce them into the action.

4 The appearance of composite fields in a toy model

In this and the following sections we use the Gross-Neveu model to demonstrate our ideas about effective theories. An integration of the high frequency part of the fermion field produces nonlocal interactions for cut-offs $\Lambda < \Lambda_c$. It is shown how to eliminate the dominant nonlocalities by use of an effective action which contains boson and fermion fields. The necessary generalizations to obtain a completely local effective action will be considered later.

We want to find a method to calculate effective actions for theories in which composite fields are needed to keep the effective actions local. This requires first of all an interaction which is able to form bound states. The four fermion interaction is the most important candidate for the formation of composite fields. Therefore we choose a toy model which has such a four fermion interaction. We will work with the Gross-Neveu model [18] and its generalization to four dimensions. This has the advantage that we may compare our results with known results for the Gross-Neveu model. When lowering the cut-off gradually we find that we have to generalize the form of the action two times to cope with the new features developing as we lower the cut-off. In the end we find a preliminary version of an effective action containing composite fields. It is preliminary because only the four fermion interaction has the required locality properties. A generalization of this method is given in section 6.2.

Since we are interested in effective theories, we work with a cut-off version of the action. A convenient choice to implement the cut-off is to use a cut-off propagator of the form $u_{\Lambda'} = \frac{i\not p}{p^2} e^{-\frac{p^2}{\Lambda'^2}}$ as in the book of Rivasseau [23]. We make this choice because we will be able to do simple calculations with it afterwards. The action of our model is given by

$$S = \bar\psi u_{\Lambda'}^{-1}\psi - (\bar\psi\psi)\,\lambda\,(\bar\psi\psi) \tag{49}$$

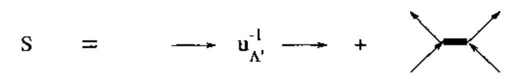


Figure 2

ψ are Dirac spinors. We work in euclidean space. It is understood that the coordinates are integrated and spin and flavour indices are summed. The fermion propagator has Dirac and flavour indices. It is diagonal in flavour space. The coupling λ does carry neither flavour nor spin indices. The $\bar\psi\psi$ in the interaction term are summed over spin and flavour indices, which are suppressed in our notation, i.e. $(\bar\psi\psi) = \sum_{\alpha i}\bar\psi_{\alpha i}\psi_{\alpha i}$. The combination $\bar\psi\psi$ is therefore a scalar.

Since some of the following manipulations create a large number of terms, we introduce a pictorial representation, which allows to follow the line of thought easily. We use this representation as a symbolical writing. It is possible to trace the flavour indices, which is useful in an $\frac{1}{N}$ expansion. An arrow represents a fermion field, the black box represents the interaction λ. Fermion fields which are connected to the same end of the black box must have equal flavour and spin indices. There is always one ingoing arrow and one outgoing arrow on each side of the interaction box. We start with a pointlike four fermion interaction, i.e. $\lambda(x-y)=\lambda\delta(x-y)$. This will change as we lower the cut-off of our model. In general λ is not proportional to a δ-function.

The action has a discrete chiral symmetry

$$\left.\begin{array}{lcl}\psi & \rightarrow & \gamma_s\psi \\ \bar\psi & \rightarrow & -\bar\psi\gamma_s\end{array}\right\}\bar\psi\psi \rightarrow -\bar\psi\psi \tag{50}$$

which forbids a mass term as long as it is not spontaneously broken. $\gamma_s = i^{-\frac{d}{2}}\gamma_1\ldots\gamma_d$ where γ_μ are euclidean γ matrices obeying the relation $\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}$ [22].

The general method we use to obtain an effective action is to integrate out the high frequency modes. This results in an action which has only low frequency modes. The interactions must encode the effects of the high frequency modes. As mentioned above the crucial point is the correct choice of the blockspin. In the first part of our calculation the blockspin field is of the same kind as the fundamental field. But at a certain stage it becomes necessary to introduce in addition composite fields as block spins. These composite fields are boson fields. Without introducing them the effective action would become nonlocal. Hence we have from a certain stage on two different kinds of blockspin fields. The question is when and how to introduce a composite bosonic field.

4.1 Bosonization and integration of high frequencies

In this section we lower the cut-off of the model. Composite boson fields are introduced by bosonizing the four fermion interaction. After checking the decay properties of the composite propagator we find that a composite field is not yet needed and the boson field is integrated out again. This sections serves as an introduction to the calculational methods used.

The first idea to introduce bosonic degrees of freedom is to use the auxiliary field method [17] to eliminate the four fermion interaction and introduce instead a boson field. This is formally done by inserting the following constant into the path integral.

$$const = \int \mathcal{D}\Delta\; e^{-(\Delta-\bar\psi\psi\lambda)\frac{1}{\lambda}(\Delta-\lambda\bar\psi\psi)} \tag{51}$$

The boson field will transform under the discrete chiral symmetry as:

$$\Delta \rightarrow -\Delta. \tag{52}$$

In this way we have formulated the original fermion theory as a boson-fermion theory. The generating functional has the form

$$\begin{aligned} Z &= \int \mathcal{D}\Delta\mathcal{D}\bar\psi\mathcal{D}\psi\; e^{-\bar\psi u_{\Lambda'}^{-1}\psi-\Delta\frac{1}{\lambda}\Delta+2\Delta\bar\psi\psi} \\ &= \int \mathcal{D}\Delta\mathcal{D}\bar\psi\mathcal{D}\psi\; e^{-S_1}. \end{aligned} \tag{53}$$

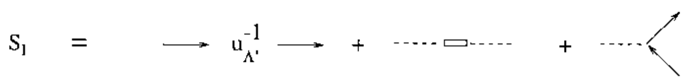


Figure 3

where $\frac{1}{\lambda}$ is depicted as a white box and the auxiliary boson field by a dashed line. The propagator of the boson is $\lambda\delta(x-y)$ which is constant in momentum space. Hence the boson propagator has no momentum cut-off. This is not tolerable in a cut-off theory. Nevertheless the theory is formally equivalent to the previous one and the action is only quadratic in the fermion fields. This allows an easy integration of the high frequency fermion modes. The frequency split is performed using the convolution formula for gaussian measures. The fermion propagator $u_{\Lambda'}$ is split into a high frequency propagator v_0 and a low frequency propagator u_Λ. This induces a split in the field variables. ψ is split into a low frequency blockspin field Ψ and a high frequency fluctuation field ζ, i.e $\psi=\Psi+\zeta$. To be specific let us use the form [23]

$$u_{\Lambda'} = \frac{i\not p}{p^2}e^{-\frac{p^2}{\Lambda'^2}} = \underbrace{\frac{i\not p}{p^2}\,e^{-\frac{p^2}{\Lambda^2}}}_{u_\Lambda} + \underbrace{\frac{i\not p}{p^2}\,(e^{-\frac{p^2}{\Lambda'^2}}-e^{-\frac{p^2}{\Lambda^2}})}_{v_0}. \tag{54}$$

v_0 is small for $\Lambda\approx\Lambda'$ in the following sense. In n dimensions

$$\int\frac{d^nl}{(2\pi)^n}\,tr_\gamma(v_0(l)v_0(l)) = I_n(0,\Lambda',\Lambda)\approx\left\{\begin{array}{ll} -\frac{1}{4\pi^2}(\delta\Lambda)^2 & \text{for}\quad n=4 \\ -\frac{1}{2\pi}\frac{(\delta\Lambda)^2}{\Lambda^2} & \text{for}\quad n=2\end{array}\right.. \tag{55}$$

The calculation is done in the appendix. An infinitesimal cut-off lowering enables us therefore to do a sensible expansion in the small quantity v_0. Upon integrating the high frequency modes ζ belonging to the propagator v_0 we are led to the action

$$e^{-S_2(\bar\Psi;\Psi;\Delta)} = \int d\mu_{v_0}(\zeta)e^{-\bar\Psi u_\Lambda^{-1}\Psi-\Delta\frac{1}{\lambda}\Delta+2\Delta(\bar\Psi\Psi+\bar\Psi\zeta+\bar\zeta\Psi+\bar\zeta\zeta)} \tag{56}$$

Since the action is quadratic in ζ the integration can be done with the result

$$S_2 \;=\; \bar\Psi u_\Lambda^{-1}\Psi - 2\Delta\bar\Psi v(\Delta)\Psi 2\Delta + \Delta\frac{1}{\lambda}\Delta - 2\Delta\bar\Psi\Psi - Tr\ln(1-2\Delta v_0) \tag{57}$$

where we use the notation

$$\begin{aligned} v(\Delta) &= [v_0^{-1}-2\Delta]^{-1} = v_0 + v_0 2\Delta v_0 + \ldots \\ Tr &= tr_\gamma tr_x \quad;\quad tr_x(A) = \int d^nx A_{x,x} \end{aligned} \tag{58}$$

where A is an arbitrary operator. tr_γ is the trace over euclidean γ matrices and includes a sum over fermion flavours.

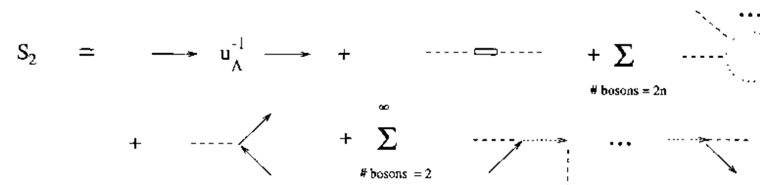


Figure 4

A dotted inner arrow denotes the high frequency propagator v_0. Since $tr_\gamma\gamma_{\mu_1}\ldots\gamma_{\mu_{2n+1}}=0$, only even powers of Δ contribute to the expansion of $Tr\ln(1-2\Delta v_0)$. In equation (57) the action S_2 has been given in closed form. It is invariant under the discrete chiral symmetry. The action S_2 is still not a low energy effective action because the boson propagator has no momentum cut-off. The high frequency parts have to be integrated out to obtain an effective action. Before we do this we investigate the pure bosonic part of the action.

As long as the cut-off is lowered sufficiently little the minimum of the pure bosonic part of the action remains at $\Delta = 0$. To see this we take the pure bosonic part

$$S_{2B} = \Delta\frac{1}{\lambda}\Delta - Tr\ln(1-2\Delta v_0)$$
$$= \Delta\frac{1}{\lambda}\Delta + \sum_{n=1}^{\infty}\frac{1}{2n}Tr(2\Delta v_0 2\Delta v_0)^n \tag{59}$$

and take the functional derivative with respect to Δ.

$$\frac{\delta S_{2B}}{\delta \Delta_x} = \int_y 2\lambda^{-1}_{x,y}\Delta_y + \sum_{n=1}^{\infty} tr_\gamma\left((2v_0 2\Delta v_0)(2\Delta v_0 2\Delta v_0)^{n-1}\right)_{x,x} = 0 \tag{60}$$

x is not integrated over. This is the gap equation which has always the solution $\Delta = 0$. The question is whether there exists another solution with $\Delta \neq 0$. To investigate this we assume that $\Delta = \text{const} \neq 0$ and check whether the equation has a solution. This choice corresponds to the restriction that the vacuum is translationally invariant. It is convenient to work in momentum space.

$$v_{0\,x,y} = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} v_0(p) \quad ; \quad \lambda^{-1}_{x,y} = \frac{1}{\lambda}\int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \tag{61}$$

where we assume $\lambda(x-y) = \lambda\delta(x-y)$. This choice yields the following gap equation:

$$\frac{1}{\lambda} + \sum_{n=1}^{\infty}\int \frac{d^4p}{(2\pi)^4} tr_\gamma\left(2v_0(p)^2((2\Delta)^2 v_0(p)^2)^{n-1}\right) = 0$$
$$\frac{1}{\lambda} + 2\int \frac{d^4p}{(2\pi)^4} tr_\gamma\left(v_0(p)^2 \frac{1}{1-(2\Delta)^2 v_0(p)^2}\right) = 0 \tag{62}$$

The integral

$$-\int \frac{d^4p}{(2\pi)^4} tr_\gamma(v_0(p)^2) \approx \frac{1}{4\pi^2}(\delta\Lambda)^2 \tag{63}$$

has been calculated under the assumption that the propagator has the form

$$v_0(p) = \frac{i\not p}{p^2}\left(e^{-\frac{x^2}{\Lambda'^2}} - e^{-\frac{x^2}{\Lambda^2}}\right) \quad \text{and} \quad \Lambda' \approx \Lambda. \tag{64}$$

This integral can be made as small as necessary. Since $v_0(p)^2 \leq 0$, the integrand in the gap equation is even smaller. As a consequence there exists no nontrivial solution to the gap equation for a sufficiently small v_0. Thus there is no need to shift the boson field in this case.

It will not be possible to perform the following manipulations in closed form. We will therefore do the expansion for small v_0 right away.

$$S_2 = \bar\Psi u_\Lambda^{-1}\Psi + \Delta\frac{1}{\lambda}\Delta - 2\Delta\bar\Psi\Psi - 2\Delta\bar\Psi v_0\Psi 2\Delta + \frac{1}{2}Tr(2\Delta v_0 2\Delta v_0)$$
$$-2\Delta\bar\Psi v_0 2\Delta v_0\Psi 2\Delta + O(v_0^3) \tag{65}$$

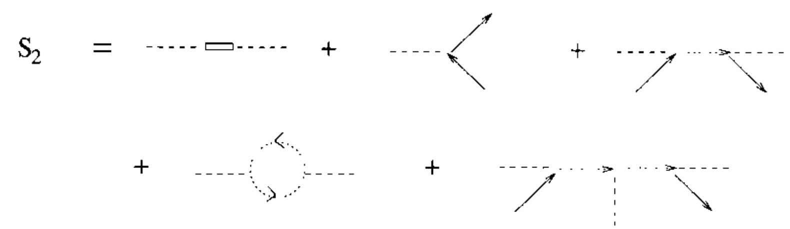


Figure 5

The cut-off for the fermions is determined by the decay properties of the propagator u_Λ. In this first step we have lowered only the cut-off for the fermions. In a proper cut-off theory all physical degrees of freedom should obey the cut-off. Let us consider the behaviour of the boson propagator. The inverse propagator reads

$$G^{-1}_{z,w} = \lambda^{-1}_{z,w} + 2tr_\gamma(v_{0\,z,w}v_{0\,w,z}). \tag{66}$$

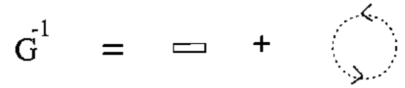


Figure 6

We observe the well known fact that via integrating the high frequency fermions the auxiliary field has gained a dynamics. Before integration its propagator was just λ. The change in the boson propagator is of second order in v_0 and therefore small. The dynamics has not changed much compared to the constant propagator the boson field has had before. Therefore the propagator should still be local, i.e. decay fast.

Remember that we understand locality always as a decay property in coordinate space. This property can be investigated by an explicit calculation using the specific form of v_0 chosen above. We will examine the behaviour of this propagator in the framework of an $\frac{1}{N}$ expansion in a later section. The result is that the boson propagator is completely local for sufficiently small cut-off lowering, i.e. it decays faster than $e^{-\Lambda|x|}$. This means that the auxiliary boson field has to be integrated out again. Since we do an expansion in powers of v_0 it makes no difference whether we regard the term $2\int_{z,w}\Delta(z)tr_\gamma(v_{0\,z,w}v_{0\,w,z})\Delta(w)$ as part of the boson propagator or as an interaction. If we do the latter we will not have to expand the propagator in powers of v_0. The effective action up to third order in v_0 is then given by

$$S_{eff} = \bar\Psi u^{-1}\Psi - \bar\Psi\Psi\lambda\bar\Psi\Psi - 4(\bar\Psi\Psi)\lambda\bar\Psi v_0\Psi\lambda(\bar\Psi\Psi)$$
$$-2\bar\Psi(v_0\bar\lambda)\Psi + 2(\bar\Psi\Psi)\lambda tr_\gamma(v_0\bar v_0)\lambda(\bar\Psi\Psi)$$
$$-8(\bar\Psi\Psi)\lambda\bar\Psi v_0(\bar\Psi\Psi\lambda)v_0\Psi\lambda(\bar\Psi\Psi)$$
$$-4(\bar\Psi\Psi)\lambda\bar\Psi v_0(v_0\bar\lambda)\Psi - 4\bar\Psi(v_0(\lambda\bar\Psi\Psi)v_0\bar\lambda)\Psi$$
$$-4\bar\Psi(v_0\bar\lambda)v_0\Psi\lambda(\bar\Psi\Psi)$$
$$-4\{(\bar\Psi\Psi)\lambda\bar\Psi v_{0\ \ ,x}\Psi + \bar\Psi v_{0\,x,}\ \Psi\lambda(\bar\Psi\Psi)\}[\lambda_x,\ \bar\Psi v_0\Psi\lambda(\bar\Psi\Psi) + (\bar\Psi\Psi)\lambda\bar\Psi v_0\Psi\lambda_{\cdot,x}]$$
$$-2\bar\Psi v_{0\,x,y}\Psi\{\lambda_x,\ \bar\Psi v_0\Psi\lambda_{\cdot,y} + \lambda_y,\ \bar\Psi v_0\Psi\lambda_{\cdot,x}\} + O(v_0^3) \tag{67}$$

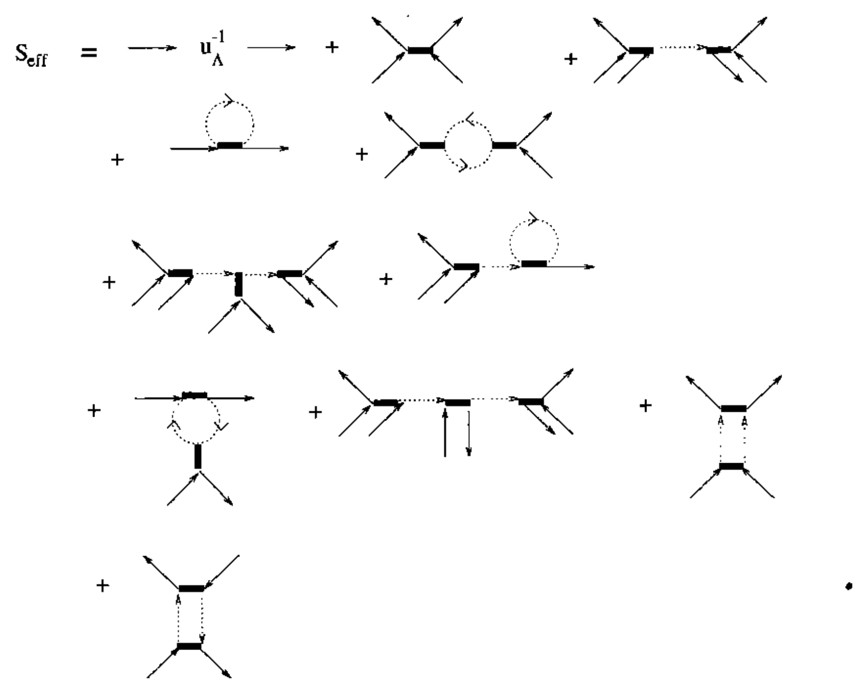


Figure 7

The compact notation is best understood by comparing with the figure 7. v_0 and λ are regarded as integral operators. Coordinate space indices are distributed according to the multiplication rules for such operators. Fields have coordinates according to their position relative to the kernels. The combination $\bar{\Psi}\Psi$ has one common coordinate. An embraced $(\lambda\bar{\Psi}\Psi)$ has to be understood as $\int dy \lambda_{xy} \bar{\Psi}(y)\Psi(y)$ and therefore the whole bracket has one coordinate x. Hence this bracket can be treated as a field and its coordinate is found via its relative position to the surrounding kernels.

A bracket in which a kernel is barred is again a kernel constructed according to

$$(a\bar{b})_{xy} = a_{xy} b_{yx} \tag{68}$$

where a and b may be products of kernels. An example

$$\bar{\Psi}(v_0\bar{\lambda})\Psi = \int_{x,y} \bar{\Psi}(x) v_{0\,xy} \lambda_{yx} \Psi(y) \tag{69}$$

A more complicated example

$$\bar{\Psi}(v_0(\lambda\bar{\Psi}\Psi)v_0\bar{\lambda})\Psi = \int_{xyzw} \bar{\Psi}(x) v_{0\,xy} \lambda_{yz} \bar{\Psi}(z)\Psi(z) v_{0\,yw} \lambda_{wx} \Psi(w) \tag{70}$$

where all coordinates are integrated. At some places the coordinates have to be written explicitly because they appear not at neighbouring places. The dot instead of a coordinate indicates that the multiplication rule for integral operators apply to this coordinate, i.e. neighbouring quantities have the same coordinates which are integrated.

Note that we need a small parameter to perform this integration. This small parameter is v_0. It corresponds to a small cut-off lowering. We make no assumption concerning the size of λ. As a consequence we cannot use λ as a small parameter.

At this point we want to make several remarks.

- As expected new interactions are created in this first renormalization group step. Since we expand to second order in v_0, the number of new interactions is limited in this first step. This will certainly change in future cut-off lowering steps. Therefore we have to generalize our notation in order to accommodate those new interactions. The initial conditions for the renormalization group flow of couplings will not change due to this generalization. We will always start with a pure four fermion interaction.

- Bosonizing was an unnecessary luxury as long as the boson propagator has only a high frequency part. Leaving out the bosonizing step, what we have done corresponds simply to computing

$$e^{-V_{eff}(\Psi)} = e^{-V_\Lambda(\Psi)} = \int d\mu_{v_0}(\zeta)\, e^{-V_{\Lambda'}(\Psi+\zeta)}, \tag{71}$$

where $V_{\Lambda'}$ denotes the interaction part of the action. $V_{eff}(\Psi)$ is up to a volume factor called low energy or Wilsonian effective potential to distinguish it from the textbook effective potential which is the generating function of all one particle irreducible (1PI) Green functions with external momenta set to zero. The fields appearing in the low energy effective potential are low frequency fields. This can be seen from the propagator which has a momentum cut-off Λ.

There exists a very convenient formula to reproduce the perturbation series in v_0 [24]

$$e^{-V_{eff}(\Psi)} = e^{-V_\Lambda(\Psi)} = e^{-\left(\frac{\delta}{\delta\Psi} v_0 \frac{\delta}{\delta\bar{\Psi}}\right)}\, e^{-V_{\Lambda'}(\Psi)} \tag{72}$$

- The Polchinski renormalization group equations [19] are easily derived from this. We just have to differentiate the above formula with respect to Λ. The only Λ dependent term on the right hand side is of course v_0.

$$\begin{aligned} -\frac{d}{d\Lambda} V_\Lambda(\Psi) e^{-V_\Lambda(\Psi)} &= \left(\frac{\delta}{\delta\Psi} \frac{dv_{0\,\Lambda\Lambda'}}{d\Lambda} \frac{\delta}{\delta\bar{\Psi}}\right) e^{-\left(\frac{\delta}{\delta\Psi} v_0 \frac{\delta}{\delta\bar{\Psi}}\right)}\, e^{-V_{\Lambda'}(\Psi)} \\ &= \left(\frac{\delta}{\delta\Psi} \frac{dv_{0\,\Lambda\Lambda'}}{d\Lambda} \frac{\delta}{\delta\bar{\Psi}}\right) e^{-V_\Lambda(\Psi)} \end{aligned} \tag{73}$$

therefore

$$\frac{d}{d\Lambda} V_\Lambda(\Psi) = \left\{ \left(\frac{\delta}{\delta\Psi} \frac{du_\Lambda}{d\Lambda} \frac{\delta}{\delta\bar{\Psi}}\right) V_\Lambda(\psi) - \left(\frac{\delta V_\Lambda}{\delta\Psi} \frac{du_\Lambda}{d\Lambda} \frac{\delta V_\Lambda}{\delta\bar{\Psi}}\right) \right\} \tag{74}$$

where we used

$$\frac{dv_{0\,\Lambda\Lambda'}}{d\Lambda} = -\frac{du_\Lambda}{d\Lambda} \quad \text{which follows from} \quad u_{\Lambda'} = u_\Lambda + v_{0\,\Lambda\Lambda'}. \tag{75}$$

Writing $V_{\Lambda'}$ as a sum of interaction terms it is very easy to derive an infinite system of coupled differential equations for the couplings.

- A remark on the two fermion interaction appearing in S_{eff}. It is truly a two fermion interaction and not a mass term. It has different transformation behaviour from a mass term under the discrete chiral symmetry transformation of the model.

We mentioned that bosonizing is superfluous as long as the boson fields would be integrated out again. Nevertheless we like at least to think of bosonizing first and then lowering the cut-off. The reason for this is that the decay properties of the auxiliary boson propagator give the criterion for introducing a permanent boson field into the theory. A permanent boson field is one which is not integrated out again. This boson field will not be the auxiliary boson field itself because the auxiliary boson propagator has no cut-off. But if the auxiliary boson propagator has components which are nonlocal it will be necessary to introduce a composite field. This composite field should account for the low energy modes of the auxiliary field.

We have done an infinitesimal cut-off lowering and thereby produced a new effective action. Of course we want to repeat such a lowering step in order to get a renormalization group flow for the effective action. There is however a problem concerning the bosonization. As a first disturbing fact we notice that there appear four fermion interactions, which do not have a suitable form for bosonizing. These are the terms belonging to the box diagrams in (67). Since we want a simple toy model to investigate the basic mechanisms, we will neglect this difficulty for the moment. This means that we do not bosonize four fermion interactions which have a form different from

$$\bar{\psi}(x)\psi(x)\lambda_\Lambda(x-y)\bar{\psi}(y)\psi(y). \tag{76}$$

We will see later that in an $\frac{1}{N}$ expansion the terms of the form (76) are in fact the leading order contributions. This will partly justify this procedure a posteriori. As a consequence we will have two four fermion interactions. One has a special form and is used to introduce boson fields into the theory and another one which has a general form. The general form of a four fermion interaction is

$$\int_x g_4(x_{\bar{1}}, x_1, x_{\bar{2}}, x_2)\bar{\psi}(x_{\bar{1}})\psi(x_1)\bar{\psi}(x_{\bar{2}})\psi(x_2). \tag{77}$$

The Dirac and flavour indices of g_4 are suppressed. Here are some examples of interactions which we will not bosonize.

$$\begin{aligned}&-4(\bar{\Psi}\Psi)\lambda\bar{\Psi}v_0(v_0\bar{\lambda})\Psi - 4\bar{\Psi}(v_0(\lambda\bar{\Psi}\Psi)v_0\bar{\lambda})\Psi\\ &-4\bar{\Psi}(v_0\bar{\lambda})v_0\Psi\lambda(\bar{\Psi}\Psi)\\ &-2\bar{\Psi}v_{0\,x,y}\Psi\{\lambda_{x,}\,\bar{\Psi}v_0\Psi\lambda_{,y} + \lambda_{y,}\,\bar{\Psi}v_0\Psi\lambda_{,x}\}\end{aligned} \tag{78}$$

4.2 A generalization of the toy model

A cut-off lowering for generalization of the toy model is done. A preliminary version of an effective action at the compositeness scale Λ_c is given. The local effective action for cut-offs $\Lambda > \Lambda_c$ is implicitly contained in this action.

In the process of lowering the cut-off of the effective theory new interactions are created. Hence we must generalize our notation to incorporate these new interactions conveniently. The original model is retained as a special case. This is done by choosing appropriate initial values for the couplings in the renormalization group flow. At the initial cut-off Λ' only the four fermion coupling is nonzero.

The action is written in the form

$$S = \bar{\psi}u_{\Lambda'}^{-1}\psi - \bar{\psi}\psi\lambda\bar{\psi}\psi + V(\bar{\psi};\psi). \tag{79}$$

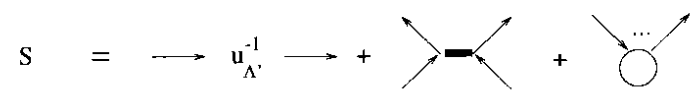


Figure 8

The nonrenormalizable interactions and the part of the four fermion interaction, which does not have the suitable form for bosonization, are contained in

$$V(\bar{\psi};\psi) = \int_{x_i}\sum_n g_n(x_{\bar{1}},\ldots,x_n)\bar{\psi}(x_{\bar{1}})\ldots\psi(x_n). \tag{80}$$

The functions g_n represent couplings. Spinor and flavour indices are suppressed. In a well defined effective theory these couplings should be local. We will return to this point when we investigate the model for large flavour number N.

We proceed exactly as before. In a first step we bosonize the explicit four fermion interaction term in (79) with the auxiliary field method. The resulting action is

$$S_1 = \bar{\psi}u_{\Lambda'}^{-1}\psi + \Delta\frac{1}{\lambda}\Delta - 2\Delta\bar{\psi}\psi + V(\bar{\psi};\psi). \tag{81}$$

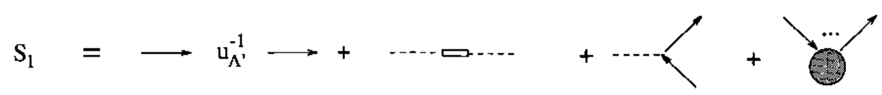


Figure 9

The fermion propagator is split into a high and a low frequency part $u_{\Lambda'} = u_\Lambda + v_0$. Thereby a split in the field variables is induced $\psi = \Psi + \zeta$. The integration of the high frequency field ζ yields a new action, which can be formally written as

$$e^{-S_2(\bar{\Psi};\Psi;\Delta)} = \int d\mu_{v_0}(\zeta)\, e^{-\bar{\Psi}u_\Lambda^{-1}\Psi - \Delta\frac{1}{\lambda}\Delta + 2\Delta(\bar{\Psi}\Psi + \bar{\Psi}\zeta + \bar{\zeta}\Psi + \bar{\zeta}\zeta) - V(\bar{\Psi}+\bar{\zeta};\Psi+\zeta)}. \tag{82}$$

To perform the integration to first order in v_0 it is sufficient to do a Taylor expansion of the interaction part to second order, i.e.

$$V(\bar{\Psi}+\bar{\zeta};\Psi+\zeta) = V(\bar{\Psi};\Psi) + \bar{\zeta}\frac{\delta}{\delta\bar{\Psi}}V - \frac{\delta}{\delta\Psi}V\zeta - \bar{\zeta}\frac{\delta}{\delta\bar{\Psi}}\frac{\delta}{\delta\Psi}V\zeta + \ldots. \tag{83}$$

Hence the ζ integration to first order is still gaussian. After integrating the high frequency fermions we find the action expanded up to first order in v_0

$$\begin{aligned}S_2(\bar{\Psi};\Psi;\Delta) &= \bar{\Psi}u_\Lambda^{-1}\Psi + \Delta\frac{1}{\lambda}\Delta - 2\Delta\bar{\Psi}\Psi + V(\bar{\Psi};\Psi) - (2\Delta\bar{\Psi} - \frac{\delta}{\delta\Psi}V)v_0(\Psi 2\Delta + \frac{\delta}{\delta\bar{\Psi}}V)\\ &\quad + Tr(\frac{\delta}{\delta\bar{\Psi}}\frac{\delta}{\delta\Psi}Vv_0) + O(v_0^2)\end{aligned} \tag{84}$$

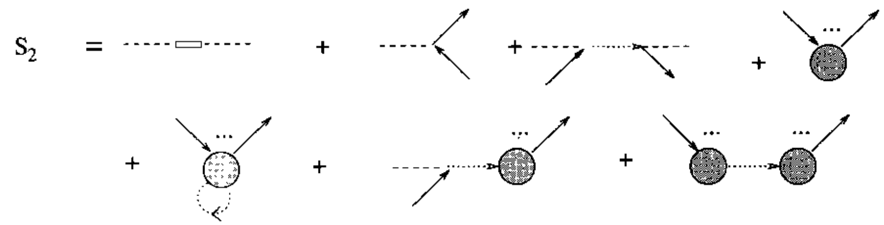


Figure 10

In our graphical representation not all possible permutations of fermion line directions are given. We show only one direction, others can be constructed by symmetry. For example for the last but one term there exists also a diagram with all arrows inverted. As in the previous case the boson field still contains high frequency modes. Its propagator has no momentum cut-off. To get a consistent effective theory we have to integrate out these high frequency modes. Here we have to distinguish between two cases. In the first case the boson propagator is still completely local with respect to the actual cut-off Λ of the effective theory. This is the case as long as the cut-off is larger than a certain value Λ_c. This Λ_c determines the compositeness scale. If the cut-off is below Λ_c then the boson propagator will have a part which is nonlocal. This actually defines Λ_c.

For values of the cut-off larger than Λ_c the propagator is completely local and the integration is done as before. The couplings are renormalized due to the integration of high frequencies. This corresponds to a Polchinski type renormalization group flow. We will not perform the actual calculation since it is contained in the following case.

For values of the cut-off smaller than Λ_c we must do a propagator split for the boson propagator. Otherwise the integration of the boson field would create nonlocal interactions due to the nonlocal parts in the boson propagator. The low frequency part is the propagator of the permanent boson field which has a mass of Λ_c at this stage. Remember that the decay rate of a propagator in coordinate space is essentially the mass of the field. We write the split

$$\lambda_{\Lambda'} = \lambda_\Lambda + \Gamma_{\Lambda'\Lambda} \tag{85}$$

whereby the split in the field variable is induced

$$\Delta = \phi + \xi. \tag{86}$$

The high frequency propagator must be local with respect to the scale set by Λ. We can write the new effective action which results from the integration of the ξ field formally as

$$\begin{aligned} e^{-S_3(\bar\Psi;\Psi;\phi)} &= \int \mathcal{D}\xi e^{-\bar\Psi u_\Lambda^{-1}\Psi - \phi\lambda_\Lambda^{-1}\phi - \xi\Gamma_{\Lambda'\Lambda}^{-1}\xi + 2(\phi+\xi)\bar\Psi\Psi - V(\bar\Psi;\Psi) + (2(\phi+\xi)\bar\Psi - \frac{\delta}{\delta\Psi}V)v_0(\Psi 2(\phi+\xi) + \frac{\delta}{\delta\bar\Psi}V) - Tr(\frac{\delta}{\delta\Psi}\frac{\delta}{\delta\bar\Psi}V v_0)} \\ &= \int \mathcal{D}\xi e^{-\bar\Psi u_\Lambda^{-1}\Psi - \phi\lambda_\Lambda^{-1}\phi - \xi\{\Gamma_{\Lambda'\Lambda}^{-1} - 4\bar\Psi v_0\Psi\}\xi + 2\phi\bar\Psi\Psi - V(\bar\Psi,\Psi) + (2\phi\bar\Psi - \frac{\delta}{\delta\Psi}V)v_0(\Psi 2\phi + \frac{\delta}{\delta\bar\Psi}V) - Tr(\frac{\delta}{\delta\Psi}\frac{\delta}{\delta\bar\Psi}V v_0)} \\ &\quad e^{2\xi\bar\Psi v_0(\Psi 2\phi + \frac{\delta}{\delta\bar\Psi}V) + (2\phi\bar\Psi - \frac{\delta}{\delta\Psi}V)v_0\Psi 2\xi + 2\xi\bar\Psi\Psi} \end{aligned} \tag{87}$$

The ξ integration is gaussian and can be done easily with the result

$$\begin{aligned} S_4(\bar\Psi;\Psi;\phi) &= \bar\Psi u_\Lambda^{-1}\Psi + \phi\lambda_\Lambda^{-1}\phi - 2\phi\bar\Psi\Psi + V(\bar\Psi;\Psi) - (2\phi\bar\Psi - \frac{\delta}{\delta\Psi}V)v_0(\Psi 2\phi + \frac{\delta}{\delta\bar\Psi}V) \\ &+ Tr(\frac{\delta}{\delta\bar\Psi}\frac{\delta}{\delta\Psi}V v_0) - \left(\bar\Psi_x\Psi_x + \bar\Psi_x v_0(\Psi 2\phi + \frac{\delta}{\delta\bar\Psi}V) + (2\phi\bar\Psi - \frac{\delta}{\delta\Psi}V)v_0\Psi_x\right)\{\Gamma^{-1} - 4\bar\Psi v_0\Psi\}^{-1}_{x,y} \\ &\times \left(\bar\Psi_y\Psi_y + \bar\Psi_y v_0(\Psi 2\phi + \frac{\delta}{\delta\bar\Psi}V) + (2\phi\bar\Psi - \frac{\delta}{\delta\Psi}V)v_0\Psi_y\right) + \frac{1}{2}Tr\ln(\Gamma^{-1} - 4\bar\Psi v_0\Psi) \end{aligned} \tag{88}$$

Since S_2 was only given to first order in v_0, we take only linear contributions in v_0 into account for the effective action, i.e.

$$\begin{aligned} S_{eff}(\bar\Psi;\Psi;\phi) &= \bar\Psi u_\Lambda^{-1}\Psi + \phi\lambda_\Lambda^{-1}\phi - 2\phi\bar\Psi\Psi + V(\bar\Psi;\Psi) - (2\phi\bar\Psi - \frac{\delta}{\delta\Psi}V)v_0(\Psi 2\phi + \frac{\delta}{\delta\bar\Psi}V) \\ &+ Tr(\frac{\delta}{\delta\bar\Psi}\frac{\delta}{\delta\Psi}V v_0) - \bar\Psi\Psi\{\Gamma + \Gamma 4\bar\Psi v_0\Psi\Gamma\}\bar\Psi\Psi - \left(\bar\Psi_x v_0(\Psi 2\phi + \frac{\delta}{\delta\bar\Psi}V) + (2\phi\bar\Psi - \frac{\delta}{\delta\Psi}V)v_0\Psi_x\right)\Gamma_{x,\cdot}\bar\Psi\Psi \\ &- \bar\Psi\Psi\Gamma_{\cdot,y}\left(\bar\Psi_y v_0(\Psi 2\phi + \frac{\delta}{\delta\bar\Psi}V) + (2\phi\bar\Psi - \frac{\delta}{\delta\Psi}V)v_0\Psi_y\right) + 2Tr(\Gamma\bar\Psi v_0\Psi), \end{aligned} \tag{89}$$

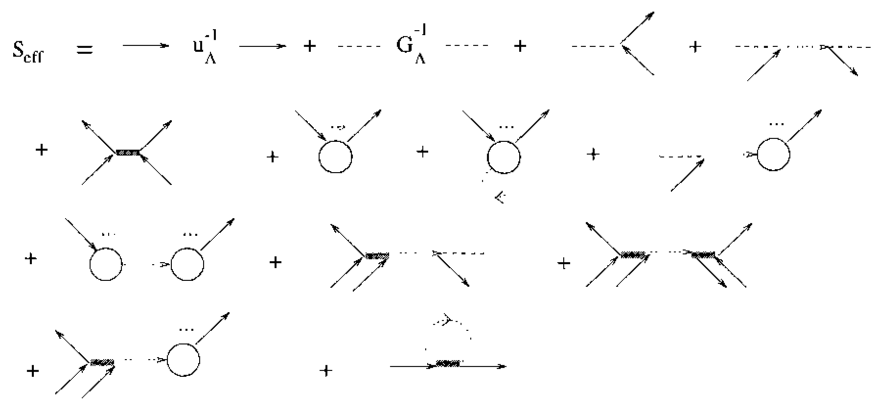


Figure 11

where we dropped a field independent term. In contrast to all preceding actions this effective action contains boson fields.

It is not possible to use Γ as a small parameter in expansions. Γ contains the high frequency part of the auxiliary field. This high frequency part contains at least the inverse of the original four fermion coupling. This starting value for the four fermion coupling needs not to be small.

We mentioned that the case where there is no nonlocal part in the auxiliary field propagator is contained in this calculation. To find the effective action for this case we drop all terms containing a ϕ and write everywhere $\lambda_{\Lambda'}$ instead of Γ. This yields the cut-off lowering for actions with a cut-off above Λ_c.

The effective action at the compositeness scale is the starting point for a further cut-off lowering. This time we must generalize our notation to incorporate boson fields and their interactions.

4.3 An action containing composite boson fields

In this section we will demonstrate how a small renormalization group step is done if a permanent composite boson field is already present in the action. This involves a number of manipulations which have to be performed in a certain order to control the changes in the effective action. As a result we find an effective action which contains a composite field and which four fermion interaction is local.

The first steps are completely analogous to the previous procedure. We bosonize the suitable part of the four fermion interaction. Then we make a frequency split in the fermionic fields and integrate the high frequency part to first order in the small propagator v_0. Only the additional boson field causes some notational inconveniences. We start with the general form of the action

$$S = \bar\psi u_{\Lambda'}^{-1}\psi + \phi G_{\Lambda'}^{-1}\phi - 2\phi h\bar\psi\psi - \bar\psi\psi\lambda\bar\psi\psi + V(\phi;\bar\psi;\psi) \tag{90}$$

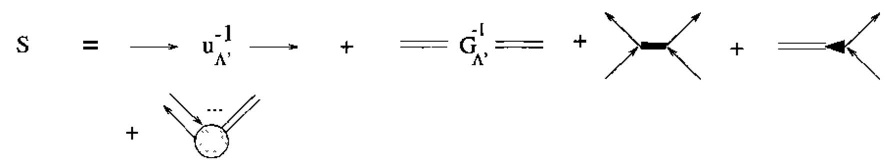


Figure 12

The double line represents the boson field, the triangle represents a possible Yukawa interaction h. The interaction V contains everything which is not explicitly written in the action. It therefore contains all pure bosonic interactions, all pure fermionic interactions including those parts of the four fermion interaction which are not bosonizable and all boson fermion interactions apart from the Yukawa interaction. The action is still invariant under the discrete chiral transformation of equation (50) and (52). We bosonize the λ part of the four fermion interaction and introduce thus a new auxiliary field Δ.

$$\begin{aligned} S_1 &= \bar\psi u_{\Lambda'}^{-1}\psi + \phi G_{\Lambda'}^{-1}\phi - 2\phi h\bar\psi\psi + \Delta\frac{1}{\lambda}\Delta - 2\Delta\bar\psi\psi + V(\phi;\bar\psi;\psi) \\ &= \bar\psi u_{\Lambda'}^{-1}\psi + \phi G_{\Lambda'}^{-1}\phi + \Delta\frac{1}{\lambda}\Delta - \bar\psi 2(\Delta + \phi h)\psi + V(\phi;\bar\psi;\psi) \end{aligned} \tag{91}$$

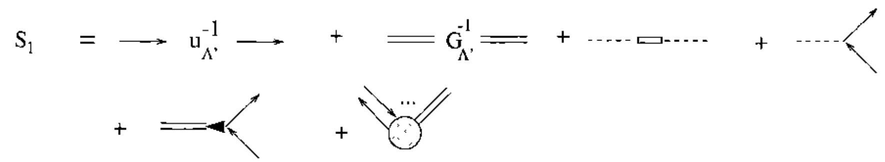


Figure 13

The dashed line denotes the auxiliary boson field. To do the integration of the high frequency fermions to first order in the small fermion propagator v_0 we do again a Taylor expansion of V as in equation (83). The integral is again gaussian and similar in form to the previous case without boson field. Hence we can immediately write the result of the integration by comparison with equation (84).

$$S_2 = \bar\Psi u_\Lambda^{-1}\Psi + \phi G_{\Lambda'}^{-1}\phi + \Delta\frac{1}{\lambda}\Delta - 2(\Delta+\phi h)\bar\Psi\Psi + V(\phi;\bar\Psi;\Psi)$$
$$-(2(\Delta+\phi h)\bar\Psi - \frac{\delta}{\delta\Psi}V)v_0(\Psi 2(\Delta+h\phi) + \frac{\delta}{\delta\bar\Psi}V) + Tr(\frac{\delta}{\delta\bar\Psi}\frac{\delta}{\delta\Psi}Vv_0) + O(v_0^2) \quad (92)$$

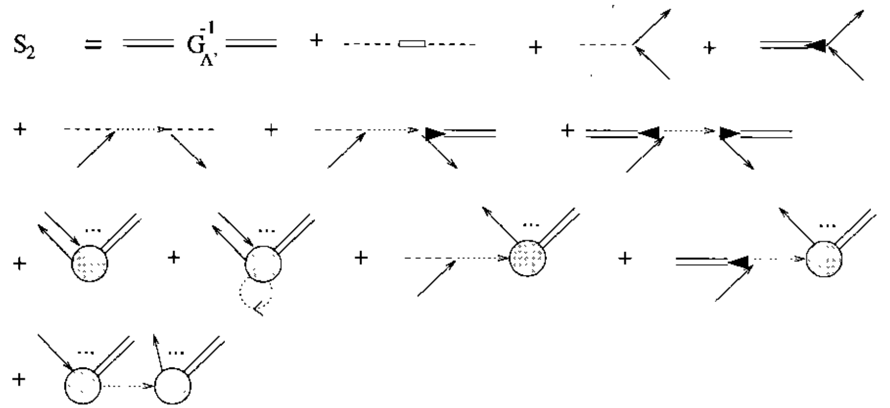


Figure 14

We are now in a situation where we have two boson fields in our theory. The auxiliary field Δ has a high frequency part in its propagator which is not necessarily small because it contains the initial four fermion interaction λ. The auxiliary field propagator might in addition contain a nonlocal part which would generate nonlocal interactions when integrated out again. We do not wish to have two low frequency boson fields in our effective theory. This would mean that for every new step of cut-off lowering a new boson field is created. The obvious thing to do is to merge the low frequency part of the auxiliary field with the already present boson field. This means that the composite field gets at each scale new admixtures from the high frequency fermion fields just integrated. The formation of a composite field may therefore be a continuous process.

The merging of the low frequency part of the auxiliary field with the already present boson field is complicated by the fact that we always want to change the action only by infinitesimal amounts so that the manipulations remain under control. The small parameter is essentially the infintesimal cut-off lowering.

This constraint forbids the following method. First merging the two fields via the convolution theorem for gaussian measures and afterwards integrating the high frequency part of the resulting field. The merging involves two nonsmall propagators and it is therefore not possible to expand this manipulation in a small parameter.

A possible procedure is to do first a frequency split for the auxiliary field. This frequency split is induced by a propagator split as before

$$\lambda = F + \Gamma. \quad (93)$$

Here Γ is the high frequency part. It contains the nonsmall original four fermion coupling λ. F in contrast is the low frequency part which was created by the last high frequency fermion integration. Hence this newly created part is small. Without fixing the splitting procedure we cannot be more

specific. We expect, however, a similar criterion as for the propagator v_0, i.e. integrals containing this propagator F are proportional to some positive power of $\delta\Lambda$.

We now have two small parameters in which we may expand our action. We will expand our actions to first order in either v_0 or F. The precision of the expansion is then given by the larger of the two quantities. By making $\delta\Lambda$ small enough it is always possible to reach a certain precision.

To first order in v_0 the integration of the high frequency auxiliary field yields the new action.

$$S_3 = \bar\Psi u_\Lambda^{-1}\Psi + \phi G_{\Lambda'}^{-1}\phi + \Delta_L F^{-1}\Delta_L - 2(\Delta_L+\phi h)\bar\Psi\Psi + V(\phi;\bar\Psi;\Psi)$$
$$+Tr(\frac{\delta}{\delta\bar\Psi}\frac{\delta}{\delta\Psi}Vv_0) - (2(\Delta_L+\phi h)\bar\Psi - \frac{\delta}{\delta\Psi}V)v_0(\Psi 2(\Delta_L+h\phi) + \frac{\delta}{\delta\bar\Psi}V) - \bar\Psi\Psi\{\Gamma + \Gamma 4\bar\Psi v_0\Psi\Gamma\}\bar\Psi\Psi$$
$$-\left(\bar\Psi_x v_0(\Psi 2(\Delta_L+h\phi) + \frac{\delta}{\delta\bar\Psi}V) + (2(\Delta_L+\phi h)\bar\Psi - \frac{\delta}{\delta\Psi}V)v_0\Psi_x\right)\Gamma_{x,}\bar\Psi\Psi$$
$$\bar\Psi\Psi\Gamma_{,y}\left(\bar\Psi_y v_0(\Psi 2(\Delta_L+h\phi) + \frac{\delta}{\delta\bar\Psi}V) + (2(\Delta_L+\phi h)\bar\Psi - \frac{\delta}{\delta\Psi}V)v_0\Psi_y\right)$$
$$+2Tr(\Gamma\bar\Psi v_0\Psi), \quad (94)$$

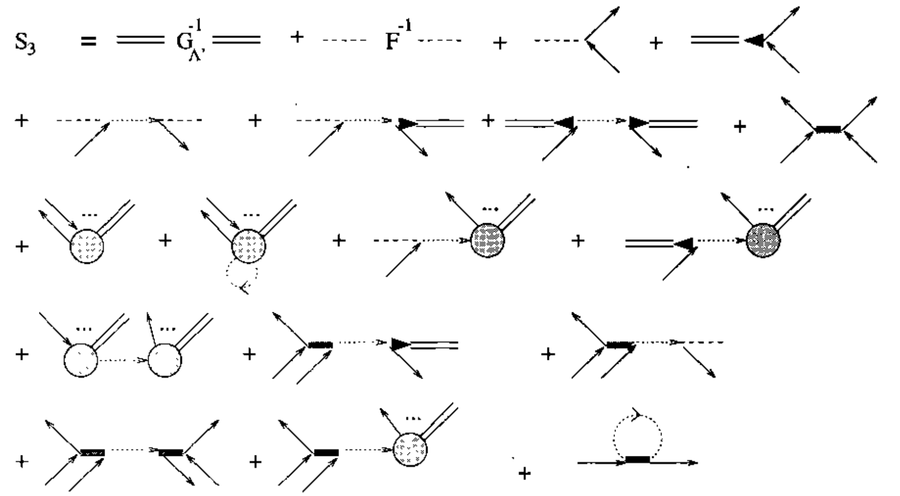


Figure 15

The dashed line depicts now the low frequency auxiliary field Δ_L. Here we are left with the two boson fields which we want to merge. In principle the merging could be done via the convolution theorem for gaussian measures. This would require a rescaling of the Δ_L field. It is technically easier to introduce the field which is the sum of both via a δ-functional.

$$\int \mathcal{D}\chi\, \delta(\chi - \phi - h^{-1}\Delta_L) \quad (95)$$

χ is the new composite field. Integration of Δ_L is done with the help of the δ- functional. This results in the following replacement

$$\Delta_L + \phi h = \chi h \quad (96)$$

wherever this combination appears. This is the case for all appearances of Δ_L except in the kinetic term of the Δ_L field which changes to

$$\Delta_L F^{-1}\Delta_L = (\chi - \phi)hF^{-1}h(\chi - \phi). \quad (97)$$

The resulting action is

$$
\begin{aligned}
S_4 = {} & \bar\Psi u_\Lambda^{-1}\Psi + \phi(G_{\Lambda'}^{-1} + hF^{-1}h)\phi + \chi hF^{-1}h\chi - 2\chi hF^{-1}h\phi - 2\chi h\bar\Psi\Psi + V(\phi;\bar\Psi;\Psi) \\
& + Tr(\frac{\delta}{\delta\bar\Psi}\frac{\delta}{\delta\Psi}V v_0) - (2\chi h\bar\Psi - \frac{\delta}{\delta\Psi}V)v_0(\Psi 2h\chi + \frac{\delta}{\delta\bar\Psi}V) - \bar\Psi\Psi\{\Gamma + \Gamma 4\bar\Psi v_0\Psi\Gamma\}\bar\Psi\Psi \\
& - \left(\bar\Psi_x v_0(\Psi 2h\chi + \frac{\delta}{\delta\bar\Psi}V) + (2\chi h\bar\Psi - \frac{\delta}{\delta\Psi}V)v_0\Psi_x\right)\Gamma_{x,}\bar\Psi\Psi \\
& - \bar\Psi\Psi\Gamma_{,y}\left(\bar\Psi_y v_0(\Psi 2h\chi + \frac{\delta}{\delta\bar\Psi}V) + (2\chi h\bar\Psi - \frac{\delta}{\delta\Psi}V)v_0\Psi_y\right) \\
& + 2Tr(\Gamma\bar\Psi v_0\Psi)
\end{aligned}
\tag{98}
$$

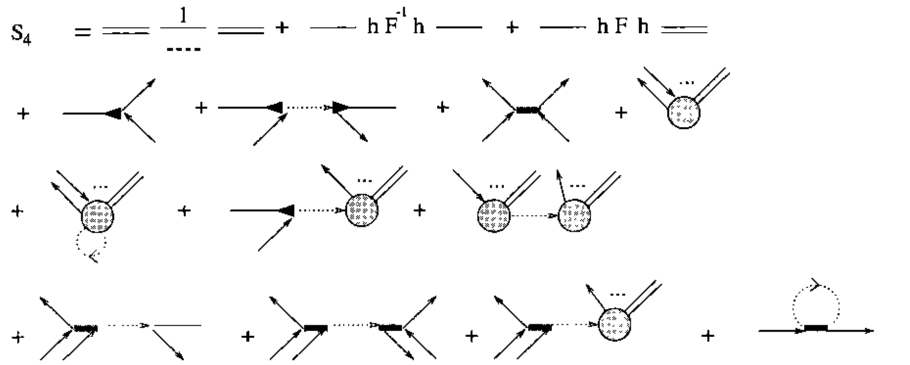


Figure 16

The new composite field χ is denoted by the solid line. Remember that the propagator F is small. Therefore the kinetic term in the ϕ field is dominated by $hF^{-1}h$. This assumes that $G_{\Lambda'}$ is not small by itself. $G_{\Lambda'}$ could be small at the compositeness scale Λ_c. We do not regard this case now. Therefore the ϕ propagator is small. This makes an integration of the ϕ field possible. We regard only contributions of first order in our small quantities v_0 and F. There is of course the ϕ, χ interaction which is large because it contains F^{-1}. In fact it almost compensates the ϕ propagator up to an infinitesimal change. To be specific

$$
[G_{\Lambda'}^{-1} + hF^{-1}h]^{-1}hF^{-1}h\chi \approx [1 + h^{-1}Fh^{-1}G_{\Lambda'}^{-1}]\chi \tag{99}
$$

This has the result that the contributions to the new action arising from terms containing boson fields ϕ and a propagator v_0 are simply this very same terms with ϕ changed to χ. This is because we expand only to first order either in v_0 or F. The other contributions which appear must be expanded to first order in F.

We arrive at the action

$$
\begin{aligned}
S_5 = {} & \bar\Psi u_\Lambda^{-1}\Psi + \chi[G_{\Lambda'} + h^{-1}Fh^{-1}]^{-1}\chi - 2\chi h\bar\Psi\Psi + V(\chi;\bar\Psi;\Psi) \\
& + Tr(\frac{\delta}{\delta\bar\Psi}\frac{\delta}{\delta\Psi}V v_0) + \frac{\delta}{\delta\chi}V\, h^{-1}Fh^{-1}G_{\Lambda'}^{-1}\chi - (2\chi h\bar\Psi - \frac{\delta}{\delta\Psi}V)v_0(\Psi 2h\chi + \frac{\delta}{\delta\bar\Psi}V) \\
& - \bar\Psi\Psi\{\Gamma + \Gamma 4\bar\Psi v_0\Psi\Gamma\}\bar\Psi\Psi - \left(\bar\Psi_x v_0(\Psi 2h\chi + \frac{\delta}{\delta\bar\Psi}V) + (2\chi h\bar\Psi - \frac{\delta}{\delta\Psi}V)v_0\Psi_x\right)\Gamma_{x,}\bar\Psi\Psi \\
& - \bar\Psi\Psi\Gamma_{,y}\left(\bar\Psi_y v_0(\Psi 2h\chi + \frac{\delta}{\delta\bar\Psi}V) + (2\chi h\bar\Psi - \frac{\delta}{\delta\Psi}V)v_0\Psi_y\right) \\
& + 2Tr(\Gamma\bar\Psi v_0\Psi) + \frac{1}{4}tr_x\left((G_{\Lambda'}^{-1} + hF^{-1}h)^{-1}\frac{\delta^2 V}{\delta\chi\delta\chi}\right) - \frac{1}{4}\frac{\delta V}{\delta\chi}(G_{\Lambda'}^{-1} + hF^{-1}h)^{-1}\frac{\delta V}{\delta\chi},
\end{aligned}
\tag{100}
$$

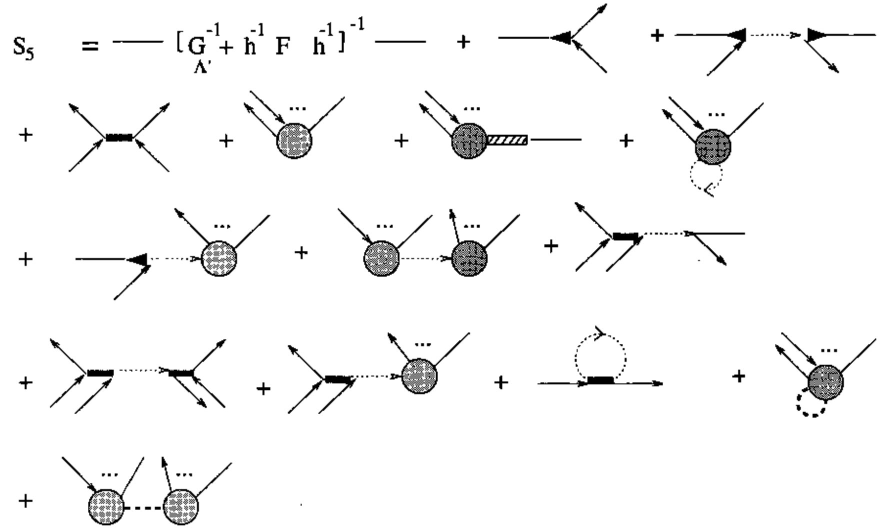


Figure 17

This is not necessarily the final effective action. There might still be a high frequency part in the boson propagator for the χ field. This poses no new problem. We do yet another frequency split and integrate out this high frequency modes. To perform the split in the propagator we have to collect the complete kinetic part first. There is another correction from the integration of high frequency fermion fields.

$$
\chi\left\{G_{\Lambda'} + h^{-1}Fh^{-1} + \frac{1}{2}\frac{\delta^2}{\delta\chi\delta\chi}Tr(\frac{\delta}{\delta\bar\Psi\delta\Psi}V v_0)\right\}\chi \equiv \chi\tilde G\chi. \tag{101}
$$

We do the split for this new propagator $\tilde G$

$$
\tilde G = G_\Lambda + \Omega_{\Lambda'\Lambda} \tag{102}
$$

where $\Omega_{\Lambda'\Lambda}$ is the small high frequency propagator. Here we have introduced a third small quantity. There is, however, some hope from the following $\frac{1}{N}$ expansion that it is not necessary to introduce this quantity at all. There the boson propagator is already of low frequency due to the corrections from the high frequency fermion integration. Whenever this is the case S_5 is already the final effective action.

In any case we can formally do the integration. We write the split in the field variable

$$
\chi = \Phi + \xi \tag{103}
$$

The integration yields four new diagrams in the perturbative expansion of our effective action. Here we assumed again that all small quantities are of the same order.

$$
\begin{aligned}
S_{eff} = {} & \bar\Psi u_\Lambda^{-1}\Psi + \Phi G_\Lambda^{-1}\Phi - 2\Phi h\bar\Psi\Psi + V(\Phi;\bar\Psi;\Psi) \\
& - (\bar\Psi\Psi h - \frac{1}{2}\frac{\delta}{\delta\Phi}V)\Omega(h\bar\Psi\Psi - \frac{1}{2}\frac{\delta}{\delta\Phi}V) + \frac{1}{4}tr_x\left(\Omega\frac{\delta^2 V}{\delta\Phi\delta\Phi}\right) \\
& + Tr(\frac{\delta}{\delta\bar\Psi}\frac{\delta}{\delta\Psi}V v_0) - \frac{1}{2}\frac{\delta^2}{\delta\Phi\delta\Phi}Tr(\frac{\delta}{\delta\bar\Psi}\frac{\delta}{\delta\Psi}V v_0) + \frac{\delta}{\delta\Phi}V\, h^{-1}Fh^{-1}G_\Lambda^{-1}\Phi
\end{aligned}
$$

$$
\begin{aligned}
&-(2\Phi h\bar{\Psi} - \frac{\delta}{\delta\Psi}V)v_0(\Psi 2h\Phi + \frac{\delta}{\delta\bar{\Psi}}V) - \bar{\Psi}\Psi\{\Gamma + \Gamma 4\bar{\Psi}v_0\Psi\Gamma\}\bar{\Psi}\Psi \\
&-\left(\bar{\Psi}_x v_0(\Psi 2h\Phi + \frac{\delta}{\delta\bar{\Psi}}V) + (2\Phi h\bar{\Psi} - \frac{\delta}{\delta\Psi}V)v_0\Psi_x\right)\Gamma_{x,\cdot}\bar{\Psi}\Psi \\
&-\bar{\Psi}\Psi\Gamma_{\cdot,y}\left(\bar{\Psi}_y v_0(\Psi 2h\Phi + \frac{\delta}{\delta\bar{\Psi}}V) + (2\Phi h\bar{\Psi} - \frac{\delta}{\delta\Psi}V)v_0\Psi_y\right) + 2Tr(\Gamma\bar{\Psi}v_0\Psi) \\
&+\frac{1}{4}tr_x\left((G_{\Lambda'}^{-1} + hF^{-1}h)^{-1}\frac{\delta^2 V}{\delta\Phi\delta\Phi}\right) - \frac{1}{4}\frac{\delta V}{\delta\Phi}(G_{\Lambda'}^{-1} + hF^{-1}h)^{-1}\frac{\delta V}{\delta\Phi},
\end{aligned}
\qquad (104)
$$

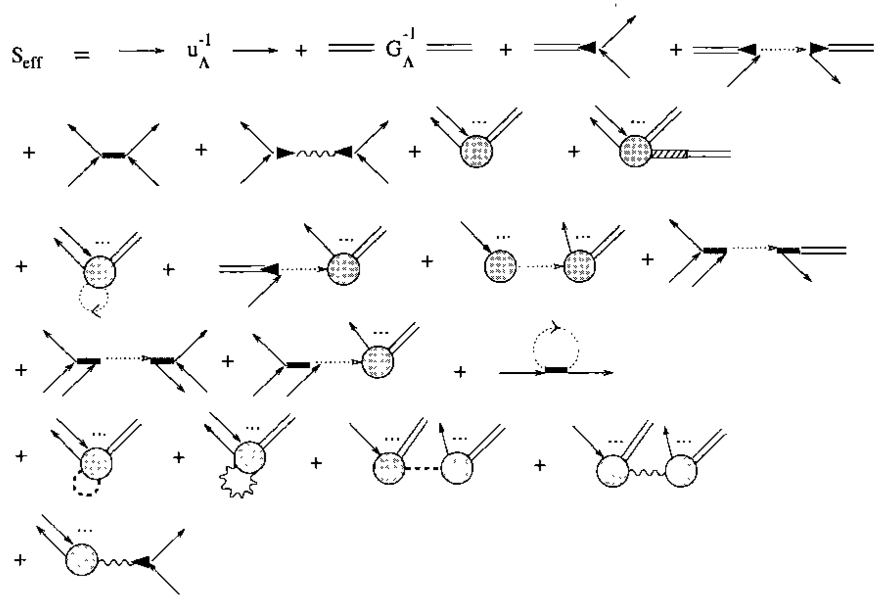


Figure 18

This is the final effective action. Note that this procedure merges at each step of cut-off lowering some nonlocal part of the four fermion coupling with the already existing boson field. This makes a continuous formation process of the composite field possible. This merging is manifest in the $hF^{-1}h$ contribution to the boson propagator.

It is possible to deduce an infinite system of differential equations from this effective action, since the only Λ dependent terms are the propagators v_0,F,Ω. The exact form of those propagators depends of course on the splitting procedure.

We should mention, however, that the effective action found by this procedure has a defect. It may contain terms which do not have the required locality properties. Locality is, however, an essential ingredient for an effective action. The reason for the failure is that we confine our bosonization to the four fermion term. Other interaction like the six fermion interaction may be nonlocal. This is best explained in the framework of an $\frac{1}{N}$ expansion because there we are able to calculate the boson propagator to leading order in this expansion. Thereby we identify the source of nonlocality for the other actions. A generalization of this provisional outline is given in section 6.2 where we bosonize all appearing nonlocal interactions.

5 Investigation of the toy model for large flavour number N

In this section we study the locality properties of the low energy effective actions of the toy model to leading order in a large N expansion. We show that it is possible to find a local effective action using composite fields. This local effective action is rederived using the technique of small cut-off lowering steps. This needs a nonlocal rescaling of the composite field in each step.

In section 5.1 we demonstrate that nonlocal interactions can appear in a pure fermionic formulation of the effective action as we lower the cut-off below a certain threshold Λ_c. This Λ_c depends on the initial dimensionless four fermion coupling. In two dimensions every finite value of this four fermion coupling leads to nonlocal fermion interactions below a certain threshold Λ_c. In four dimensions the action becomes nonlocal if the dimensionless four fermion coupling is bigger than a certain minimal value.

In the rather technical section 5.2 we investigate the properties of the auxiliary field propagator. This shows how nonlocalities appear in the pure fermionic interactions. The mass of the auxiliary field is a measure of the locality of the pure fermionic interactions. For sufficiently large dimensionless couplings this mass decreases as the cut-off of the effective theory is lowered and eventually overtakes the cut-off. At this stage fermion interactions become nonlocal.

In section 5.3 we show to leading order in an $\frac{1}{N}$ expansion that a local formulation of a low energy effective action is possible by introducing composite fields into the action. We give a formal expression for the action.

Since the cut-off dependence of this local boson-fermion interaction is known it is possible to derive a flow equation for this effective action in section 5.4. This is a flow equation for a local boson fermion effective action.

In section 5.5 we also derive the local effective action of section 5.3 but using small cut-off lowering steps this time. This is done because we want to use this method in the general case when no $\frac{1}{N}$ expansion is possible. This derivation exhibits several important features.

- In the process of cut-off lowering the action can become nonlocal although a composite boson has been introduced.
- This nonlocal terms can be absorbed into the interactions which are mediated by the composite boson. This changes the composite boson in a specific way.
- A nonlocal rescaling of the composite boson is necessary during this absorption process.
- It is possible to find the local effective action by performing subsequent small cut-off lowering steps.
- In is not necessary to integrate high frequency composite field modes in this cut-off lowering step. The absorption of the nonlocal interactions changes the composite boson in such a way that it has always the correct cut-off.

Thus a composite boson is not independent fom the fundamental field. Although this statement seems to be trivial it means that flow equations have to reflect this relation. A violation should result in either nonlocal interactions or composite fields which have propagators with an additional mass below the cut-off.

We finish the section with some comments concerning our provisional outline in section 4.

5.1 Nonlocalities in a pure fermionic low energy effective action

In this section we show that a purely fermionic action becomes nonlocal as we lower the cut-off of the theory beyond a certain scale, namely the compositeness scale Λ_c. To leading order in $\frac{1}{N}$ the low energy effective potential can be formally given as trees which vertices are local. The locality depends then only on the links in these trees.

We use in an intermediate step a bosonized formulation of the theory. In this bosonized formulation it is very easy to give a formal expression for the effective action to leading order in $\frac{1}{N}$. It can be seen in this formal expression that the pure fermionic action becomes nonlocal if the auxiliary boson propagator becomes nonlocal. Exactly this will happen if we lower the cut-off below Λ_c.

We start with the action

$$S = \bar\psi u_{\Lambda'}^{-1}\psi - \bar\psi\psi\frac{\lambda}{N}\bar\psi\psi. \tag{105}$$

To have a convenient form for the $\frac{1}{N}$ expansion we rescale the fermion fields with a factor $\sqrt{N}$. The resulting action is proportional to N.

$$S = N\left\{\bar\psi u_{\Lambda'}^{-1}\psi - \bar\psi\psi\lambda\bar\psi\psi\right\} \tag{106}$$

Hence all propagators are proportional to $\frac{1}{N}$ and all interactions are proportional to N in all diagrams which are created by integrating out high frequency modes. A fermion loop may include a summation over flavour numbers which gives an additional factor of N.

We do a propagator split $u_{\Lambda'} = u_\Lambda + v_0$ where v_0 is the high frequency propagator. This induces a field split $\psi = \Psi + \zeta$ where ζ is the high frequency field which is integrated out. We do not assume v_0 to be small. We can write the result of the integration formally using equation (72)

$$e^{-V_\Lambda(\Psi)} = e^{-\left(\frac{\delta}{\delta\Psi}v_0\frac{\delta}{\delta\bar\Psi}\right)}\, e^{-V_{\Lambda'}(\Psi)} \tag{107}$$

The integration produces diagrams which can be expanded in powers of v_0. As mentioned above v_0 is not a good expansion parameter here because it is not small. But since we have done the expansion to second order in the high frequency propagator v_0 already in figure 7, we take the opportunity to redraw this expansion in figure 19 with powers of N explicitly written. This is just an exercise in counting the order in $\frac{1}{N}$ of diagrams.

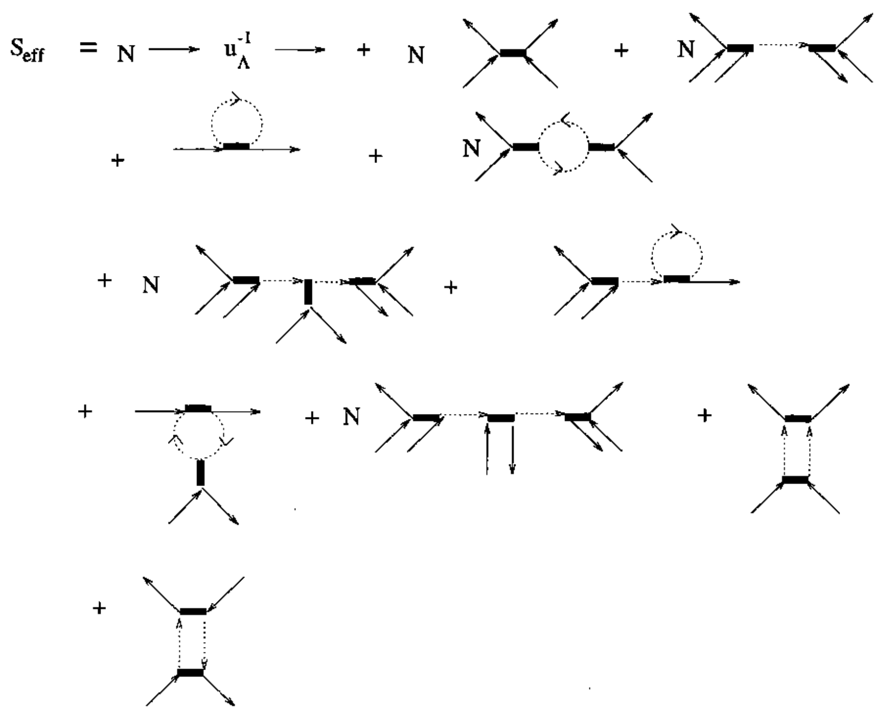


Figure 19

The higher order in v_0 contributions which have not been drawn here contain also order N diagrams. Hence this representation is not well suited for the $(\frac{1}{N})$ expansion. We prefer to bosonize the action and to integrate the high frequency modes of the fermion fields in this bosonized version. We have done this before and so the result is

$$\begin{aligned} S &= N\bar\Psi u_\Lambda^{-1}\Psi + \Delta\frac{N}{\lambda}\Delta - 2N\Delta\bar\Psi\Psi - 2N\Delta\bar\Psi\{v_0^{-1} - 2\Delta\}^{-1}\Psi 2\Delta - Tr\ln(1 - 2\Delta v_0) \\ &= N\bar\Psi u_\Lambda^{-1}\Psi + \Delta\underbrace{\{\frac{N}{\lambda} + 2tr_\gamma(v_0\bar v_0)\}}_{\equiv\lambda_\Lambda^{-1}}\Delta - 2N\Delta\bar\Psi\Psi - 2N\Delta\bar\Psi\{v_0^{-1} - 2\Delta\}^{-1}\Psi 2\Delta \\ &\quad + \sum_{n\geq 2}\frac{1}{2n}Tr\left[(2\Delta v_0 2\Delta v_0)^n\right] \end{aligned} \tag{108}$$

where we introduced the scale dependent auxiliary field propagator $\lambda_\Lambda = \frac{\lambda}{N+2\lambda tr_\gamma(v_0\bar v_0)}$. This auxiliary field propagator has no momentum cut-off. All propagators are of order $\frac{1}{N}$ and all interactions are of order N. If we integrate out the Δ field we have again a pure fermionic action. As before this integration can be written formally as.

$$e^{-V_\Lambda(\Psi)} = e^{\frac{1}{4}\left(\frac{\delta}{\delta\Delta}\lambda_\Lambda\frac{\delta}{\delta\Delta}\right)}\, e^{-\tilde V(\Delta|\bar\Psi,\Psi)}\Big|_{\Delta=0} \tag{109}$$

where

$$\begin{aligned} \tilde V_\Lambda(\Delta|\Psi,\Psi) &= -2N\Delta\bar\Psi\Psi - 2N\Delta\bar\Psi\{v_0^{-1} - 2\Delta\}^{-1}\Psi 2\Delta + \sum_{n\geq 2}\frac{1}{2n}Tr\left[(2\Delta v_0 2\Delta v_0)^n\right] \\ &= -2N\Delta\bar\Psi\Psi - 2N\Delta\bar\Psi\{v_0^{-1} - 2\Delta\}^{-1}\Psi 2\Delta - 2\Delta tr_\gamma(v_0\bar v_0)\Delta - Tr\ln(1 - 2\Delta v_0) \end{aligned} \tag{110}$$

In this formulation it is very simple to extract the leading order diagrams. Since we have integrated out the bosons there can be no external boson line. Hence the boson propagator λ_Λ which is proportional to $\frac{1}{N}$ appears always as an inner line. We have to count only the number of inner lines and vertices in a given diagram. A tree diagram has one more interaction than inner lines and therefore the order of such a tree diagram is N. Any loop made with boson propagators increases the number of inner boson lines against the number of vertices by 1 and suppresses therefore the order of the diagram by a factor $\frac{1}{N}$. There is no factor of N from such a boson loop. Hence we find to leading order in an $\frac{1}{N}$ expansion the formal expression

$$e^{-V_\Lambda(\Psi)} = e^{\frac{1}{4}(\frac{\delta}{\delta\Delta}\lambda_\Lambda\frac{\delta}{\delta\Delta})}\, e^{-\tilde{V}_\Lambda(\Delta|\Psi,\Psi)}\Big|_{\substack{\Delta=0\\ \text{trees only}}}. \tag{111}$$

where only tree diagrams contribute on the right hand side. The inner lines of these trees are λ_Λ, i.e. the auxiliary boson propagator.

The structure of these diagrams is very simple. We show an example in figure 20.

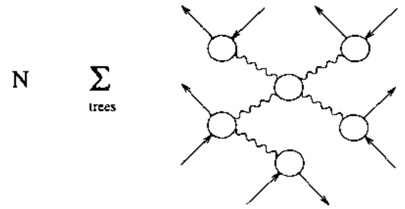


Figure 20

A wiggly line denotes the auxiliary field propagator λ_Λ which is a geometric series in fermion loops.

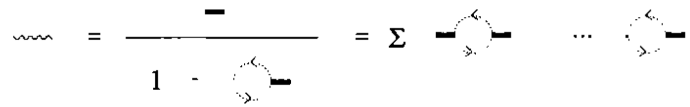


Figure 21

We will see later that this propagator can become nonlocal and is therefore the source of nonlocalities.

The interaction blobs in figure 20 are local interactions because inside the blob the lines are connected by local high frequency propagators v_0. There is maximally one loop inside this blobs. A single loop cannot cause nonlocalities. Observe that a vertex in figure 20 is either a pure boson vertex with an even number of boson fields or a boson fermion vertex with one ingoing and one outgoing fermion line. There are neither pure fermion vertices nor are there boson fermion vertices with more than two fermion lines. We give some examples of these local interactions in figure 22.

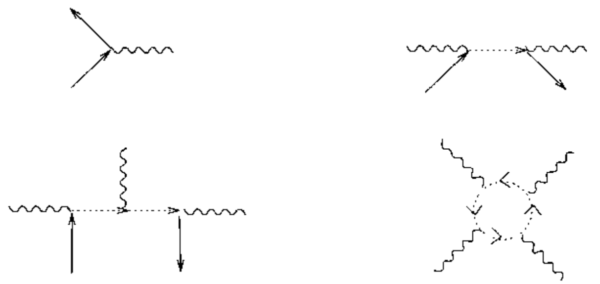


Figure 22

Thus the four and six fermion interactions have to leading order in $\frac{1}{N}$ only the contributions shown in figure 23.

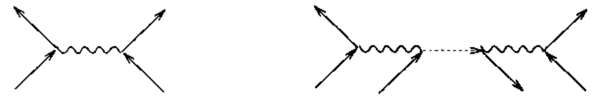


Figure 23

If the auxiliary propagator λ_Λ becomes nonlocal the pure fermionic action will contain nonlocal interactions. We will now study the locality properties of the auxiliary field propagator. This will reveal λ_Λ as the source of nonlocalities to leading order in $\frac{1}{N}$.

5.2 Investigation of the auxiliary field propagator

Here we investigate the properties of the auxiliary propagator λ_Λ. Using an explicit cut-off function this propagator can be calculated analytically. This shows how the exponential tails in the interactions develop and how the decay rate of these tails decreases as the cut-off is lowered until the interaction become eventually nonlocal.

The propagator we want to investigate reads in momentum space

$$\lambda_\Lambda(p) = \frac{\lambda}{N + 2\lambda\int\frac{d^n l}{(2\pi)^n}tr_\gamma(v_0(l+p)v_0(l))}. \tag{112}$$

We are interested in the decay properties of this propagator. Therefore we must investigate the pole structure for imaginary momenta. The smallest pole for imaginary momenta determines the decay rate of the propagator in coordinate space. This can be shown by shifting the integration path in the Fourier integral towards the pole. Hence we are looking for the zeros of

$$N + 2\lambda\int\frac{d^n l}{(2\pi)^n}tr_\gamma(v_0(l+p)v_0(l)) \tag{113}$$

for imaginary momenta. This pole structure does not depend on N and so we set $N = 1$ in this section. To do this calculation we choose the specific form of v_0

$$v_0(p) = \frac{i\not{p}}{p^2}\{e^{-\frac{p^2}{\Lambda'^2}} - e^{-\frac{p^2}{\Lambda^2}}\}. \tag{114}$$

where $\Lambda' > \Lambda$. The calculation is done in the appendix. The result for two dimensions is

$$I_2(p^2,\Lambda',\Lambda) = \int\frac{d^2 l}{(2\pi)^2}\,tr_\gamma(v_0(l+p)v_0(l)) = \frac{1}{2\pi}\left\{2Ei(-\frac{p^2}{\Lambda'^2+\Lambda^2}) - Ei(-\frac{p^2}{2\Lambda'^2}) - Ei(-\frac{p^2}{2\Lambda^2})\right\} \tag{115}$$

where $Ei(x)$ is the exponential integral the definition of which is given as well in the appendix. The limit of this integral for large arguments $|p^2|$ is

$$I_2(p^2,\Lambda',\Lambda) \overset{p^2\to-\infty}{\longrightarrow} \frac{1}{2\pi}\frac{2\Lambda^2}{p^2}e^{-\frac{p^2}{2\Lambda^2}} \quad ; \quad I_2(p^2,\Lambda',\Lambda) \overset{p^2\to\infty}{\longrightarrow} \frac{1}{2\pi}\frac{2\Lambda'^2}{p^2}e^{-\frac{p^2}{2\Lambda'^2}} \tag{116}$$

We plot $-I_2$ in figure 24 for three different values of the integrated momentum range given by $x = \frac{\Lambda}{\Lambda'}$.

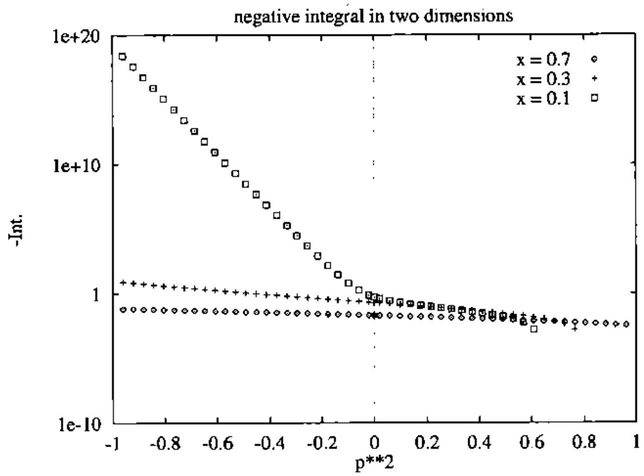


Figure 24

The integral $-I_2$ has no mass dimension in two dimensions. p^2 is given in units of the initial upper cut-off Λ'. Therefore the allowed momentum range of the effective theory is from 0 to x. $-I_2$ will become negative but only for values above this cut-off of the effective theory Λ. This cannot be shown in this plot because of the logarithmic scale.

There is a strong rise for negative p^2 as the integrated range increases, i.e. x decreases. For small values of λ and small cut-off lowering $\Lambda' - \Lambda \ll \Lambda$ the position of the pole will be at values $-p^2 > \Lambda^2$. The mass of the auxiliary field is therefore larger than the cut-off and the field has to be integrated out completely to obtain the low energy effective theory. It is interesting to see the mass of the auxiliary boson as a function of the integrated range of momentum $x = \frac{\Lambda}{\Lambda'}$. In the following plot the mass is given in units of the initial cut-off Λ'. We have plotted the mass as a function of x for three different values of the initial coupling λ in figure 25.

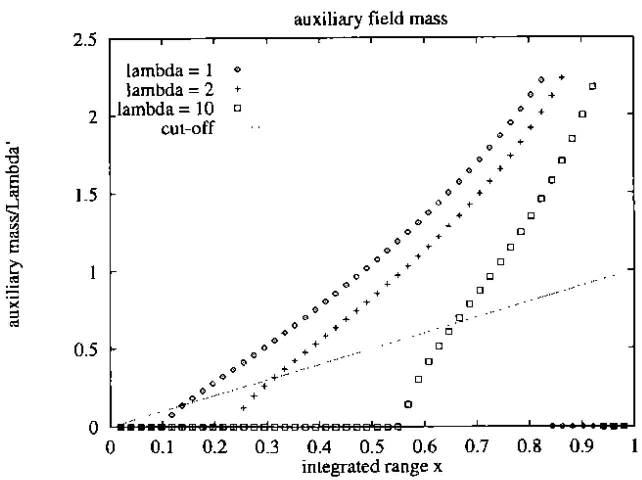


Figure 25

All quantities are given in units of the initial cut-off Λ'. The dotted line gives the position of the cut-off of the actual effective theory. It is simply equal to x in these units. The values on the x-axis have no meaning because negative values and too large values are set to zero by the plot routine.

For values of the auxiliary mass above the dotted line the auxiliary propagator is local and hence the pure fermionic action is local, too. Below the dotted line the auxiliary propagator contains nonlocal parts. Hence the pure fermionic effective theory is nonlocal. This means that we have to introduce a composite field into the effective theory in addition to the low energy fermion field. This can be done by integrating out only the high frequency components of the auxiliary field as mentioned before.

The intersection of the mass function with the dotted line defines the compositeness scale. Let us scrutinize the dependence of the compositeness scale on the initial coupling λ. The defining condition is that the pole value $-p_{pole}^2 = \Lambda_c^2$. The defining equation is then given by

$$1 + 2\lambda I_2(-\Lambda_c^2, \Lambda', \Lambda_c) = 0. \tag{117}$$

This equation depends only on the compositeness range x_c and λ. Expressed in these quantities it reads

$$1 + \frac{\lambda}{\pi}\left\{2Ei(\frac{x_c^2}{1+x_c^2}) - Ei(\frac{x_c^2}{2}) - Ei(\frac{1}{2})\right\} = 0. \tag{118}$$

This determines $x_c(\lambda)$. However it is obviously more convenient to plot the inverse function, i.e. $\lambda(x_c)$. This is done in figure 26.

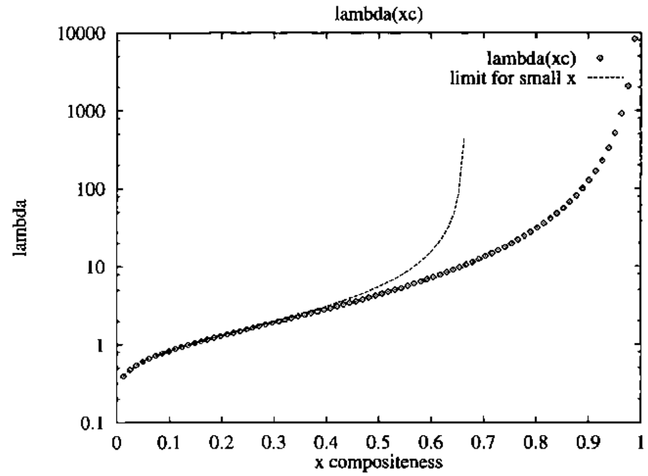


Figure 26

For small x_c the limiting function is given by

$$\lambda(x_c) \stackrel{x_c \to 0}{\longrightarrow} -\frac{\pi}{\ln(2x_c^2) + \gamma - Ei(\frac{1}{2})} \tag{119}$$

Therefore the function goes to zero for $x_c \to 0$. This means that in two dimensions for every initial coupling there exists a scale given by x_c, below which a composite boson is necessary to avoid nonlocal interactions in the effective theory.

We repeat the study for the four dimensional case. The four dimensional integral is

$$I_4(p^2, \Lambda', \Lambda) = -\int \frac{d^4l}{(2\pi)^4} tr_\gamma(v_0(l+p)v_0(l)) = \frac{1}{4\pi^2}\left\{(\frac{\Lambda'^4}{p^2} - \Lambda'^2)e^{-\frac{p^2}{2\Lambda'^2}} + (\frac{\Lambda^4}{p^2} - \Lambda^2)e^{-\frac{p^2}{2\Lambda^2}} - \left(\frac{\Lambda'^4+\Lambda^4}{p^2} - (\Lambda'^2+\Lambda^2)\right)e^{-\frac{p^2}{\Lambda'^2+\Lambda^2}} + \frac{p^2}{2}\left[2Ei(-\frac{p^2}{\Lambda'^2+\Lambda^2}) - Ei(-\frac{p^2}{2\Lambda'^2}) - Ei(-\frac{p^2}{2\Lambda^2})\right]\right\} \tag{120}$$

This time the integral has dimension (mass)^2. We are interested in poles of $\lambda_\Lambda(p)$ (see equation (112)) for imaginary momenta. The integral has the following behaviour for large arguments $|p^2|$.

$$I_4(p^2,\Lambda',\Lambda) \stackrel{p^2\to-\infty}{\longrightarrow} \frac{1}{4\pi^2}\frac{\Lambda^4}{p^2}e^{-\frac{p^2}{2\Lambda^2}} \quad ; \quad I_4(p^2,\Lambda',\Lambda) \stackrel{p^2\to\infty}{\longrightarrow} \frac{1}{4\pi}\frac{\Lambda'^4}{p^2}e^{-\frac{p^2}{2\Lambda'^2}} \tag{121}$$

The following plot in figure 27 shows $-\frac{I_4}{\Lambda'^2}$ for three different ranges $x_c = \frac{\Lambda}{\Lambda'}$ of the integrated high frequency modes.

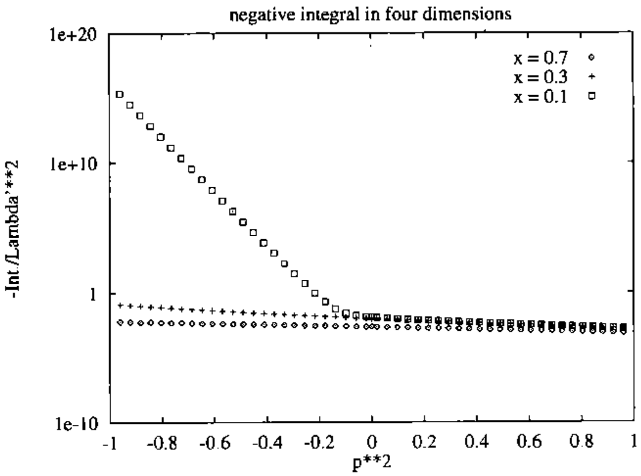


Figure 27

All values are given in units of the initial cut-off Λ'. We observe the same strong rise for negative p^2 as x is decreasing as in the two dimensional case.

The mass function is plotted in figure 28 for three different values of the dimensionless coupling $\lambda_{dl} \equiv \lambda\Lambda'^2$ defined at the initial cut-off.

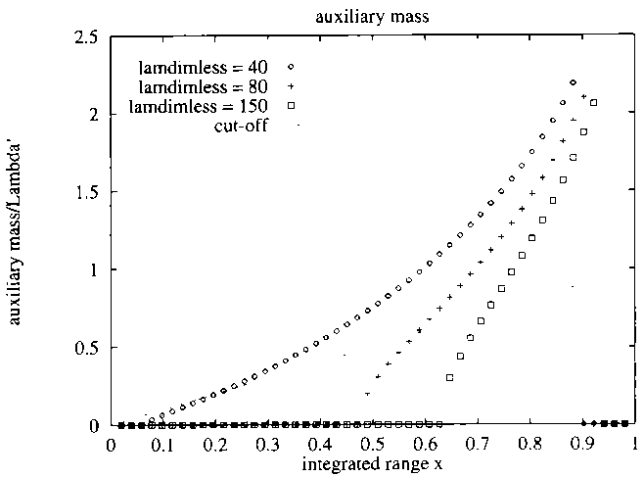


Figure 28

Note the large values of λ_{dl}. For values of $\lambda_{dl} < 4\pi^2$ there is no intersection of the mass function with the dotted line. This means that in four dimensions a minimal dimensionless coupling is necessary to form composite fields in a low energy effective theory. This feature is even more prominent in the plot $\lambda_{dl}(x_c)$ in figure 29. The defining equation for the compositeness scale in four dimensions is

$$1 + \frac{\lambda_{dl}}{2\pi^2}\left\{\frac{1+x_c^2}{x_c^2}e^{\frac{x_c^2}{2}} + 2x_c^2e^{\frac{1}{2}} - \frac{2x_c^4+x_c^2+1}{x_c^2}e^{\frac{x_c^2}{1+x_c^2}} + \frac{x_c^2}{2}\left(2Ei(\frac{x_c^2}{1+x_c^2}) - Ei(\frac{x_c^2}{2}) - Ei(\frac{1}{2})\right)\right\} = 0 \tag{122}$$

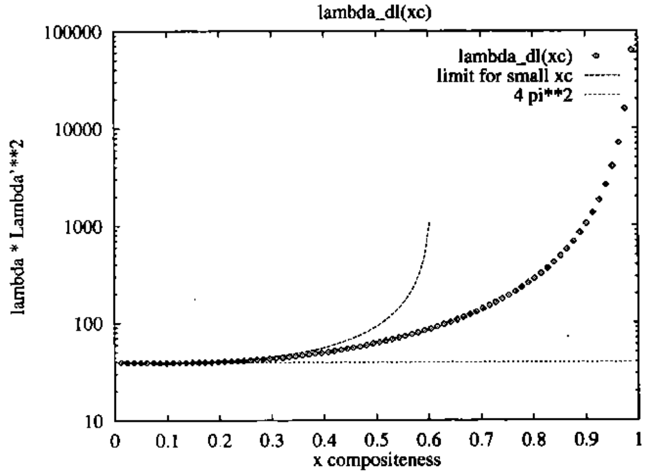


Figure 29

We have plotted again the limiting function for small x_c

$$\lambda_{dl}(x_c) \stackrel{x_c\to 0}{\longrightarrow} \frac{4\pi^2}{1 - 2x_c^2\{2e^{\frac{1}{2}} - \frac{15}{8} + \frac{1}{2}[\gamma - Ei(\frac{1}{2}) + \ln(2x_c^2)]\}} \tag{123}$$

This finishes the study of the auxiliary field propagator to leading order in an $\frac{1}{N}$ expansion. We have seen how the mass lowers and that it can overtake the dropping cut-off of the effective theory. This always happens for two dimensions and in four dimensions for sufficiently large dimensionless coupling. In this case the introduction of a composite field as a permanent field into the action is necessary.

5.3 Existence of a local effective action with a composite boson field

Here we formulate a local effective action with composite boson fields. It is formally given as trees with local vertices and local links.

In fact we do not really have to introduce composite fields into the action because we have done this actually by introducing the auxiliary field. We rather leave a low frequency part of the auxiliary field in the action instead of integrating out the whole auxiliary field. All we have to do is to integrate only the high frequency part of the auxiliary field and this is done by performing a frequency split in the auxiliary field propagator $\lambda_\Lambda = \lambda_\Lambda^L + \lambda_\Lambda^H$. This split has to be done in such a way that the low frequency propagator has a momentum cut-off Λ and the high frequency propagator is local. The high frequency part has no momentum cut-off. The low energy effective action can be given formally to leading order in $\frac{1}{N}$

$$e^{-V_\Lambda(\phi|\bar\Psi,\Psi|\lambda_\Lambda^H)} = e^{\frac{1}{4}(\frac{\delta}{\delta\xi}\lambda_\Lambda^H\frac{\delta}{\delta\xi})}\, e^{-\tilde V_\Lambda((\phi+\xi)|\bar\Psi,\Psi)}\Big|_{\substack{\xi=0\\ \text{trees only}}} = e^{\frac{1}{4}(\frac{\delta}{\delta\phi}\lambda_\Lambda^H\frac{\delta}{\delta\phi})}\, e^{-\tilde V_\Lambda(\phi|\bar\Psi,\Psi)}\Big|_{\text{trees only}} \tag{124}$$

where ϕ is low frequency part of the auxiliary field Δ. The potential $\tilde V_\Lambda$ is given in equation (110). The approximation to leading order in $\frac{1}{N}$ is contained in the restriction to tree diagrams. These trees are generalizations of the trees considered in figure 20.

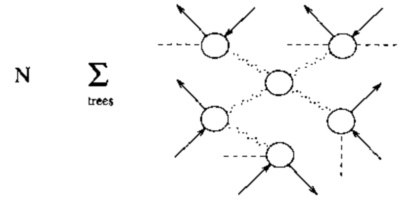


Figure 30

The connecting lines are now high frequency auxiliary propagators λ^H_Λ and the vertex blobs can contain external boson lines. The vertices involved have the same structure as before. Examples are given in figure 22. The effective action is local because local vertices are connected by local propagators and no loops are allowed.

If we go beyond leading order in $\frac{1}{N}$ the locality is no longer guaranteed by the locality of the high frequency part of the auxiliary field propagator. Loops appear which might possibly sum up to produce nonlocalities. In a later section we show how nonlocal fermion amplitudes have to be decomposed in the general case.

The complete action contains the kinetic terms of the low frequency fermion fields and the low frequency boson fields in addition.

$$S_{eff} = \bar\Psi u_\Lambda^{-1}\Psi + \phi G_\Lambda^{-1}\phi + V_\Lambda(\phi|\bar\Psi,\Psi|\lambda^H_\Lambda) \quad (125)$$

where $G_\Lambda = \lambda^L_\Lambda$. This shows that, to leading order in $\frac{1}{N}$, it is at every scale possible to formulate a local low energy effective action. This is at least true as long as the square mass of the boson field is positive. If it drops below zero we face spontaneous symmetry breaking. Below the compositeness scale it is necessary to introduce composite fields to retain locality.

5.4 A flow equation for the local composite boson fermion effective action

Using equation (124) it is possible to derive a flow equation for this effective action to leading order in $\frac{1}{N}$. This is a flow equation which respects locality.

This is simply done by taking the derivative of equation (124) with respect to Λ. This results in

$$\begin{aligned}&-[\partial_\Lambda V_\Lambda(\phi|\bar\Psi,\Psi|\lambda^H_\Lambda)]e^{-V_\Lambda(\phi|\bar\Psi,\Psi|\lambda^H_\Lambda)} = \\ &\frac{1}{4}(\frac{\delta}{\delta\phi}[\partial_\Lambda\lambda^H_\Lambda]\frac{\delta}{\delta\phi})\, e^{-V_\Lambda(\phi|\bar\Psi,\Psi|\lambda^H_\Lambda)} - e^{\frac{1}{4}(\frac{\delta}{\delta\phi}\lambda^H_\Lambda\frac{\delta}{\delta\phi})}\,[\partial_\Lambda\tilde V_\Lambda(\phi|\bar\Psi,\Psi)]e^{-\tilde V_\Lambda(\phi|\bar\Psi,\Psi)}|_{\text{trees only}}\end{aligned} \quad (126)$$

The restriction to tree diagrams makes a further evaluation of the last term on the right hand side possible. Only one derivative of the operator $\frac{1}{4}(\frac{\delta}{\delta\phi}\lambda^H_\Lambda\frac{\delta}{\delta\phi})$ is allowed to act on $\partial_\Lambda\tilde V_\Lambda$. Otherwise we would have a boson loop which suppresses the contribution by a factor $\frac{1}{N}$. Hence the other derivative must act on the exponential to the right of $\partial_\Lambda\tilde V_\Lambda$. The second derivative in $(\frac{1}{4}(\frac{\delta}{\delta\phi}\lambda^H_\Lambda\frac{\delta}{\delta\phi})$ acts like a constant with regard to $\partial_\Lambda\tilde V_\Lambda$. We use the well known formula

$$e^{A\frac{\delta}{\delta\phi}}F[\varphi] = F[\varphi + A]e^{A\frac{\delta}{\delta\phi}} \quad (127)$$

where A is any function independent of ϕ. Applying this formula to the above case we find

$$\begin{aligned}&-[\partial_\Lambda V_\Lambda(\varphi|\bar\Psi,\Psi|\lambda^H_\Lambda)]e^{-V_\Lambda(\phi|\bar\Psi,\Psi|\lambda^H_\Lambda)} = \\ &\frac{1}{4}(\frac{\delta}{\delta\phi}[\partial_\Lambda\lambda^H_\Lambda]\frac{\delta}{\delta\phi})\, e^{-V_\Lambda(\phi|\bar\Psi,\Psi|\lambda^H_\Lambda)} - [\partial_\Lambda\tilde V_\Lambda(\phi + \frac{1}{2}(\lambda^H_\Lambda\frac{\delta}{\delta\varphi})|\bar\Psi,\Psi)]e^{-V_\Lambda(\varphi|\bar\Psi,\Psi|\lambda^H_\Lambda)}|_{\varphi=\phi}\end{aligned} \quad (128)$$

where a factor 2 in the shift of the argument in $\partial_\Lambda\tilde V_\Lambda$ is due to the fact that there are two functional derivatives in the operator $\frac{1}{4}(\frac{\delta}{\delta\phi}\lambda^H_\Lambda\frac{\delta}{\delta\phi})$. We have renamed the functional derivative in $\tilde V_\Lambda$ because it is not allowed to act on the ϕ which is also present in the argument. This can be further simplified using the restriction to trees which forbids any loops.

$$\begin{aligned}&-[\partial_\Lambda V_\Lambda(\phi|\bar\Psi,\Psi|\lambda^H_\Lambda)] = \\ &\frac{1}{4}\left(\frac{\delta V_\Lambda(\phi|\bar\Psi,\Psi|\lambda^H_\Lambda)}{\delta\phi}[\partial_\Lambda\lambda^H_\Lambda]\frac{\delta V_\Lambda(\phi|\bar\Psi,\Psi|\lambda^H_\Lambda)}{\delta\phi}\right) - [\partial_\Lambda\tilde V_\Lambda(\phi - \frac{1}{2}(\lambda^H_\Lambda\frac{\delta V_\Lambda(\phi|\bar\Psi,\Psi|\lambda^H_\Lambda)}{\delta\phi})|\bar\Psi,\Psi)]\end{aligned} \quad (129)$$

This together with

$$\partial_\Lambda\tilde V_\Lambda = 2N\Delta\bar\Psi\{v_0^{-1} - 2\Delta\}^{-1}[\partial_\Lambda v_0^{-1}]\{v_0^{-1} - 2\Delta\}^{-1}\Psi 2\Delta - 4\Delta tr_\gamma([\partial_\Lambda v_0]\bar v_0)\Delta + Tr\left(\frac{2\Delta[\partial_\Lambda v_0]}{1 - 2\Delta v_0}\right) \quad (130)$$

constitutes the flow equation for scales below the compositeness scale Λ_c. This flow equation is different from the flow equations one would expect for a theory of fundamental fermion and boson fields. In a theory with fundamental fields one could expect an equation of the form

$$\begin{aligned}-[\partial_\Lambda V_\Lambda(\phi|\Psi)] &= \frac{1}{4}\left(\frac{\delta V_\Lambda(\phi|\Psi)}{\delta\phi}[\partial_\Lambda G^H]\frac{\delta V_\Lambda(\phi|\Psi)}{\delta\phi}\right) - \frac{1}{4}\left(\frac{\delta}{\delta\phi}[\partial_\Lambda G^H]\frac{\delta}{\delta\phi}\right)V_\Lambda(\phi|\Psi) \\ &+ \left(\frac{\delta V_\Lambda(\phi|\Psi)}{\delta\Psi}\frac{du_\Lambda}{d\Lambda}\frac{\delta V_\Lambda(\phi|\Psi)}{\delta\bar\Psi}\right) - \left(\frac{\delta}{\delta\Psi}\frac{du_\Lambda}{d\Lambda}\frac{\delta}{\delta\bar\Psi}\right)V_\Lambda(\phi|\Psi).\end{aligned} \quad (131)$$

where G^H is the high frequency boson propagator which should not be confounded with λ^H_Λ. λ^H_Λ is just a convenient means to write the interaction in closed form. An equation like (131) does not account for a possibly necessary bosonization of nonlocal interactions which may appear when one lowers the cut-off further. We show in the next subsection that nonlocal interactions will appear if the cut-off of the theory is lowered further and the composite fields are treated naïvely as if they were independent fundamental fields. Hence the Polchinski equations for fundamental boson and fermion fields is not sufficient to deal with composite fields. It needs alterations which take into account the relation between fundamental and composite field.

The form of the flow equations given in (129) might not be the best one. Since the choice of the blockspin, i.e. here the composite field, is ambiguous, the form of a possible flow equation is not unique. Locality is not sufficient to determine the low energy effective action and hence the flow equations. Nevertheless locality should be a necessary criterion for the construction of low energy effective actions. Locality should hence also constrain possible flow equations for low energy effective actions.

We close this part with a remark on the continuation of the above flow equation (129) to scales above the compositeness scale. Above the compositeness scale a composite boson field does not exist. Still we may define the continuation of (129) to scales above Λ_c.

$$\begin{aligned}&-[\partial_\Lambda V_\Lambda(\phi|\bar\Psi,\Psi|\lambda_\Lambda)] = \\ &\frac{1}{4}\left(\frac{\delta V_\Lambda(\phi|\bar\Psi,\Psi|\lambda_\Lambda)}{\delta\phi}[\partial_\Lambda\lambda_\Lambda]\frac{\delta V_\Lambda(\phi|\bar\Psi,\Psi|\lambda_\Lambda)}{\delta\phi}\right) - [\partial_\Lambda\tilde V_\Lambda(\phi - \frac{1}{2}(\lambda_\Lambda\frac{\delta V_\Lambda(\phi|\bar\Psi,\Psi|\lambda_\Lambda)}{\delta\phi})|\bar\Psi,\Psi)]\end{aligned} \quad (132)$$

For scales $\Lambda > \Lambda_c$ the effective potential of the pure fermionic theory to leading order $\frac{1}{N}$ is $V_\Lambda(\psi) \approx V_\Lambda(0|\bar\Psi,\Psi|\lambda_\Lambda)$. The field ϕ has no interpretation. It is only used in defining the flow. Observe that this is the leading order in $\frac{1}{N}$ approximation of the exact Polchinski equation.

$$\frac{d}{d\Lambda}V_\Lambda(\Psi) = \left\{\left(\frac{\delta}{\delta\Psi}\frac{du_\Lambda}{d\Lambda}\frac{\delta}{\delta\bar\Psi}\right)V_\Lambda(\psi) - \left(\frac{\delta V_\Lambda}{\delta\Psi}\frac{du_\Lambda}{d\Lambda}\frac{\delta V_\Lambda}{\delta\bar\Psi}\right)\right\} \quad (133)$$

5.5 Treatment of new nonlocalities which appear when the cut-off is lowered further

In this section we rederive the local effective action of section 5.3 by means of small cut-off lowering steps. It is shown that the integration of high frequency fermion fields produces nonlocalities although a composite boson exists in the action. These nonlocalities can be absorbed into interactions mediated by the composite field. The change of the composite field due to this absorption adjusts already the cut-off of the composite field to the new lower cut-off. Hence no integration of high frequency composite field modes is necessary to obtain the local effective action. However at an intermediate step a nonlocal rescaling of the composite field is necessary.

We work in the framework of a leading order $\frac{1}{N}$ expansion. Suppose that we have integrated out the high frequency fermion fields without introducing a composite field until we reach the cut-off $\Lambda = \Lambda_c - \varepsilon$, i.e. a little below the compositeness scale. At this scale we introduce a composite boson to make the effective action local.

A subsequent integration of high frequency fermion modes to lower the cut-off to $\Lambda - \delta\Lambda$ introduces new nonlocalities as we show below. At the same time the corrections from this integration changes the cut-off of the composite field propagator in such a way that it is below the new cut-off of the theory. That means the highest momentum modes of the theory are lacking in the composite field.

The removal of the nonlocalities in the interactions in turn changes the composite propagator in such a way that the cut-off is increased again to match the new cut-off of the theory. The removal of the nonlocalities in the interactions introduces therefore the lacking momentum modes. To perform the same thing by integrating out certain modes of the composite field is not possible because these modes do not exist anymore due to the corrections from the high frequency fermion integration. One could at most integrate formally field components which have a negative propagator.

The low energy effective potential at the scale $\Lambda = \Lambda_c - \varepsilon$ can be written as in equation (124)

$$e^{-V_\Lambda(\phi|\bar{\Psi},\Psi|\lambda_\Lambda^H)} = e^{\frac{1}{4}(\frac{\delta}{\delta\phi}\lambda_\Lambda^H\frac{\delta}{\delta\phi})}\, e^{-\bar{V}_\Lambda(\phi|\bar{\Psi},\Psi)}\Big|_{\text{trees only}} \tag{134}$$

where λ_Λ^H is the high frequency auxiliary field propagator, i.e a link in the tree of figure 30 and $\bar{V}_\Lambda$ is given by equation (110). The locality of V_Λ depends basically on the locality of λ_Λ^H. We show that a subsequent high frequency fermion field integration makes corrections to V_Λ in such a way that the corrected auxiliary propagator $\hat{\lambda}_\Lambda^H$ becomes nonlocal. Hence the resulting potential $V_{\Lambda-\delta\Lambda}^I$ is also nonlocal. The superscript I indicates that this potential is only an intermediate result because we still have to do certain manipulations to make this potential local.

We do now a further integration of high frequency fermion fields. Let us introduce the notation

$$v_0(p) = \frac{i\not{p}}{p^2}\{e^{-\frac{p^2}{\Lambda'^2}} - e^{-\frac{p^2}{\Lambda^2}}\} \tag{135}$$

and

$$v_1 = \frac{i\not{p}}{p^2}\{e^{-\frac{p^2}{\Lambda^2}} - e^{-\frac{p^2}{(\Lambda-\delta\Lambda)^2}}\} \tag{136}$$

where for the moment we make no assumption about the size of $\delta\Lambda$. Later we choose $\delta\Lambda$ to be small. Then v_1 will be a small propagator compared to v_0 which is finite. We can write the resulting potential as

$$e^{-V_{\Lambda-\delta\Lambda}^I(\phi|\bar{\Psi},\Psi)} = e^{-\left(\frac{\delta}{\delta\Psi}v_1\frac{\delta}{\delta\bar{\Psi}}\right)}\, e^{\frac{1}{4}(\frac{\delta}{\delta\phi}\lambda_\Lambda^H\frac{\delta}{\delta\phi})}\, e^{-\bar{V}_\Lambda(\phi|\bar{\Psi},\Psi)}\Big|_{\text{trees only}} \tag{137}$$

The action of the first operator on $\bar{V}_\Lambda$ is easy to calculate. This calculation is done in the appendix. It uses the already known property of the bosonized action of equation (108) to retain its form in subsequent fermion cut-off lowerings. The result up to field independent constants is:

$$e^{-\left(\frac{\delta}{\delta\Psi}v_1\frac{\delta}{\delta\bar{\Psi}}\right)}\, e^{-\bar{V}_\Lambda(\phi|\bar{\Psi},\Psi)} = e^{-\phi[4tr_\gamma(v_0\bar{v}_1)+2tr_\gamma(v_1\bar{v}_1)]\phi}\, e^{-\bar{V}_{\Lambda-\delta\Lambda}(\phi|\bar{\Psi},\Psi)} \tag{138}$$

Before we investigate the action of the operator $e^{\frac{1}{4}(\frac{\delta}{\delta\phi}\lambda_\Lambda^H\frac{\delta}{\delta\phi})}$ on this potential we do some trivial manipulations. Let us call

$$X = [4tr_\gamma(v_0\bar{v}_1) + 2tr_\gamma(v_1\bar{v}_1)] \tag{139}$$

for the moment.

$$\begin{aligned} e^{-V_{\Lambda-\delta\Lambda}^I(\phi|\bar{\Psi},\Psi)} &= e^{\frac{1}{4}(\frac{\delta}{\delta\phi}\lambda_\Lambda^H\frac{\delta}{\delta\phi})}\, e^{-\phi X\phi} e^{-\bar{V}_{\Lambda-\delta\Lambda}(\phi|\bar{\Psi},\Psi)}\Big|_{\text{trees only}} \\ &= e^{\frac{1}{4}(\frac{\delta}{\delta\xi}\lambda_\Lambda^H\frac{\delta}{\delta\xi})}\, e^{-(\phi+\xi)X(\phi+\xi)} e^{-\bar{V}_{\Lambda-\delta\Lambda}((\phi+\xi)|\bar{\Psi},\Psi)}\Big|_{\substack{\xi=0\\ \text{trees only}}} \end{aligned} \tag{140}$$

At this point it is convenient to change back to the path integral formulation. Remember that the operator $e^{\frac{1}{4}(\frac{\delta}{\delta\xi}\lambda_\Lambda^H\frac{\delta}{\delta\xi})}$ is just a convenient way to write the path integral.

$$\begin{aligned} e^{-V_{\Lambda-\delta\Lambda}^I(\phi|\bar{\Psi},\Psi)} &= \int \mathcal{D}\xi\, e^{-\xi\lambda_\Lambda^{H\,-1}\xi-(\phi+\xi)X(\phi+\xi)} e^{-\bar{V}_{\Lambda-\delta\Lambda}((\phi+\xi)|\bar{\Psi},\Psi)}\Big|_{\substack{\xi=0\\ \text{trees only}}} \\ &= \int \mathcal{D}\xi\, e^{-\xi[\lambda_\Lambda^{H\,-1}+X]\xi-\phi X\phi-2\phi X\xi} e^{-\bar{V}_{\Lambda-\delta\Lambda}((\phi+\xi)|\bar{\Psi},\Psi)}\Big|_{\substack{\xi=0\\ \text{trees only}}} \\ &= e^{\frac{1}{4}\frac{\delta}{\delta\xi}[\lambda_\Lambda^{H\,-1}+X]^{-1}\frac{\delta}{\delta\xi}}\, e^{-\phi X\phi-2\phi X\xi}\, e^{-\bar{V}_{\Lambda-\delta\Lambda}((\phi+\xi)|\bar{\Psi},\Psi)}\Big|_{\substack{\xi=0\\ \text{trees only}}} \end{aligned} \tag{141}$$

Observe that it is possible to do a second fermion cut-off lowering on $V_{\Lambda-\delta\Lambda}^I(\phi|\bar{\Psi},\Psi)$ with the result that the form is reproduced with v_1 changed into $v_1 + v_2$ where v_2 is the propagator corresponding to the second cut-off lowering.

We now stop this derivation for a moment to study the locality of the resulting potential. The locality of the potential $V_{\Lambda-\delta\Lambda}^I(\phi|\bar{\Psi},\Psi)$ depends obviously on the locality of the propagator

$$\hat{\lambda}_\Lambda^H = [\lambda_\Lambda^{H\,-1} + X]^{-1} = \frac{1}{\lambda_\Lambda^{H\,-1} + 4tr_\gamma(v_0\bar{v}_1) + 2tr_\gamma(v_1\bar{v}_1)} \tag{142}$$

We want to show that the potential $V_{\Lambda-\delta\Lambda}^I(\phi|\bar{\Psi},\Psi)$ can become nonlocal because $\hat{\lambda}_\Lambda^H$ can become nonlocal. To this end we specify a certain split of λ_Λ. The momentum space representation of λ_Λ is given by

$$\lambda_\Lambda(p) = \frac{\lambda}{N + 2\lambda\int\frac{d^nl}{(2\pi)^n}tr_\gamma(v_0(l+p)v_0(l))} = \frac{\lambda}{N + 2\lambda I_n(p^2,\Lambda',\Lambda,\Lambda',\Lambda)} \tag{143}$$

If $\Lambda < \Lambda_c$ this function $\lambda_\Lambda(p)$ has a pole for imaginary momenta $p^2 = -\hat{\Lambda}$ with $\hat{\Lambda} < \Lambda$. We expand the denominator around the pole value

$$\frac{\lambda}{2\lambda I_n'(-\hat{\Lambda}^2,\Lambda',\Lambda,\Lambda',\Lambda)(p^2+\hat{\Lambda}^2)} = \frac{1}{2I_n'(-\hat{\Lambda}^2)(p^2+\hat{\Lambda}^2)}. \tag{144}$$

where $I_n' = \frac{dI_n}{dp^2}$. Hence we do a split of the following form

$$\lambda_\Lambda(p) = \underbrace{\frac{1}{2I_n'(-\hat{\Lambda}^2)(p^2+\hat{\Lambda}^2)}}_{\text{low frequency}} + \underbrace{\left\{\frac{\lambda}{N+2\lambda I_n(p^2)} - \frac{1}{2I_n'(-\hat{\Lambda}^2)(p^2+\hat{\Lambda}^2)}\right\}}_{\text{high frequency}} \tag{145}$$

The first term on the right hand side should become the composite field propagator. Obviously it has yet no cut-off. We decide to introduce a Pauli-Villar cut-off in the following form.

$$\begin{aligned} \lambda_\Lambda(p) &= \underbrace{\left[\frac{1}{2I_n'(-\hat{\Lambda}^2)(p^2+\hat{\Lambda}^2)} - \frac{1}{2I_n'(-\hat{\Lambda}^2)(p^2+\Lambda^2)}\right]}_{\lambda_\Lambda^L\equiv G} \\ &+ \underbrace{\left\{\frac{\lambda}{N+2\lambda I_n(p^2)} - \frac{1}{2I_n'(-\hat{\Lambda}^2)(p^2+\hat{\Lambda}^2)} + \frac{1}{2I_n'(-\hat{\Lambda}^2)(p^2+\Lambda^2)}\right\}}_{\lambda_\Lambda^H} \end{aligned} \tag{146}$$

By this we have extracted the nonlocal parts with a momentum cut-off from the interactions. The remaining interactions are just at the border of being still local because the remaining auxiliary propagator λ_Λ^H has its pole at $p^2 = -\Lambda^2$.

With this explicit split we can study the locality of $\hat{\lambda}_\Lambda^H$. We do this in the vicinity of the compositeness scale, i.e. $\Lambda = \Lambda_c - \varepsilon$ where ε is assumed to be small. In addition we assume here our subsequent cut-off lowering $\delta\Lambda$ also to be small. Near the pole of λ_Λ^H we can neglect the regular parts.

$$\lambda_\Lambda^H = \frac{1}{2I_n'(-\hat{\Lambda}^2)(p^2+\Lambda^2)} \tag{147}$$

and therefore we find for the corrected auxiliary propagator $\hat{\lambda}_\Lambda^H$

$$\hat{\lambda}_\Lambda^H = \frac{1}{2I_n'(-\hat{\Lambda}^2)(p^2+\Lambda^2)+X} = \frac{\frac{1}{2I_n'(-\hat{\Lambda}^2)}}{p^2+\Lambda^2+\frac{X}{2I_n'(-\hat{\Lambda}^2)}} \tag{148}$$

The term

$$\frac{X}{2I_n'(-\hat{\Lambda}^2)} = \frac{1}{2I_n'(-\hat{\Lambda}^2)}[4tr_\gamma(v_0\bar{v}_1) + 2tr_\gamma(v_1\bar{v}_1)] \tag{149}$$

is responsible for the shift in the pole. We show in the appendix that $tr_\gamma(v_0\bar{v}_1)$ is proportional to $\delta\Lambda$. For $p^2 \approx -\Lambda^2$ we may neglect the p dependence of this term since this is a second order effect. Hence we can write the shifting term as

$$\frac{1}{2I_n'(-\hat{\Lambda}^2)}[4tr_\gamma(v_0\bar{v}_1) + 2tr_\gamma(v_1\bar{v}_1)] = -C2\Lambda\delta\Lambda \tag{150}$$

where we neglected higher orders in $\delta\Lambda$. This allows to write the corrected auxiliary propagator $\hat{\lambda}_\Lambda^H$ to first order in $\delta\Lambda$ as

$$\hat{\lambda}_\Lambda^H = \frac{\frac{1}{2I_n'(-\hat{\Lambda}^2)}}{p^2+(\Lambda-C\delta\Lambda)^2} \tag{151}$$

If C is larger than 1 the pole is lower than $\Lambda - \delta\Lambda$ and the corrected four fermion interaction becomes nonlocal as we lower the cut-off. We want to estimate the constant C by comparison with the behaviour of the auxiliary field propagator

$$\lambda_\Lambda(p) = \frac{\lambda}{N+2\lambda tr_\gamma(v_0\bar{v}_0)} \tag{152}$$

just below the compositeness scale. As mentioned above it has its pole at $p^2 = -\hat{\Lambda}^2$.

We have shown in the appendix that it is possible to do two subsequent cut-off lowerings in the bosonized version of the action (108). The action retains its form with the exception that v_0 is changed into $v_0 + v_1$. v_1 is the propagator of the second cut-off lowering. As a consequence we know how the auxiliary propagator at Λ will change if we lower the cut-off further by an amount $\delta\Lambda$. It becomes

$$\lambda_{\Lambda-\delta\Lambda}(p) = \frac{\lambda}{N+2\lambda tr_\gamma([v_0+v_1][\overline{v_0+v_1}])} = \frac{1}{\lambda_\Lambda^{-1}+X} \tag{153}$$

Near the pole value of λ_Λ we may write

$$\lambda_{\Lambda-\delta\Lambda}(p) \approx \frac{1}{2I_n'(-\hat{\Lambda}^2)(p^2+\hat{\Lambda}^2)+X} = \frac{\frac{1}{2I_n'(-\hat{\Lambda}^2)}}{(p^2+\hat{\Lambda}^2)+\frac{X}{2I_n'(-\hat{\Lambda}^2)}} \tag{154}$$

This time we write the mass shifting term as

$$\frac{1}{2I_n'(-\hat{\Lambda}^2)}[4tr_\gamma(v_0\bar{v}_1) + 2tr_\gamma(v_1\bar{v}_1)] = -\hat{C}2\hat{\Lambda}\delta\Lambda \tag{155}$$

Λ and $\hat{\Lambda}$ differ only little for values just below the compositeness scale. Remember that we assumed that $\Lambda = \Lambda_c - \varepsilon$. Hence C and $\hat{C}$ differ only a little bit. Since the mass functions in figures 25, 28 intersect the bisector in a finite angle we infer that $\hat{C} > 1$. For sufficiently small ε it follows that $C > 1$. This shows that under certain circumstances, namely a remaining four fermion amplitude which is just at the border of becoming nonlocal, a small cut-off lowering is sufficient to make interactions nonlocal.

One may object that the choice of the split of λ_Λ is not the best one because the interactions were just at the border of being nonlocal. However, if we try a split which leaves the remaining interaction part more local it will take the nonlocalities only longer to reappear.

Now we resume the derivation of the low energy effective action which we interrupted at equation (141) to study the locality properties of the intermediate potential $V_{\Lambda-\delta\Lambda}^I$. For the further calculation $\delta\Lambda$ needs no longer to be small.

We investigate the result of the integration of the ξ in equation (141). The result can formally be given as

$$\begin{aligned} e^{-V_{\Lambda-\delta\Lambda}^I(\phi|\bar{\Psi},\Psi)} &= e^{\frac{1}{4}\frac{\delta}{\delta\xi}[\lambda_\Lambda^{H\,-1}+X]^{-1}\frac{\delta}{\delta\xi}}\, e^{-\phi X\phi - 2\phi X\xi}\, e^{-\tilde{V}_{\Lambda-\delta\Lambda}((\phi+\xi)|\bar{\Psi},\Psi)}\Big|_{\substack{\xi=0\\ \text{trees only}}} \\ &= e^{-\phi X\phi + \phi X[\lambda_\Lambda^{H\,-1}+X]^{-1}X\phi}\, e^{-V_{\Lambda-\delta\Lambda}(\phi-\phi X[\lambda_\Lambda^{H\,-1}+X]^{-1}|\bar{\Psi},\Psi|[\lambda_\Lambda^{H\,-1}+X]^{-1})} \end{aligned} \tag{156}$$

where $X = [4tr_\gamma(v_0\bar{v}_1) + 2tr_\gamma(v_1\bar{v}_1)]$.

$V_{\Lambda-\delta\Lambda}(\phi - \phi X[\lambda_\Lambda^{H\,-1}+X]^{-1}|\bar{\Psi},\Psi|[\lambda_\Lambda^{H\,-1}+X]^{-1})$ consists out of trees which are connected via nonlocal propagators $[\lambda_\Lambda^{H\,-1}+X]^{-1}$. In addition there is a nonlocality hiding in the interaction blobs. The boson fields are connected to the local interaction blobs via

$$\phi - \phi X[\lambda_\Lambda^{H\,-1}+X]^{-1} = \phi\frac{\lambda_\Lambda^{H\,-1}}{\lambda_\Lambda^{H\,-1}+X} = \phi\frac{1}{1+X\lambda_\Lambda^H} \tag{157}$$

which is also nonlocal.

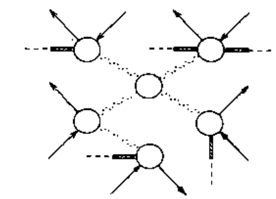
$N \sum_{\text{trees}}$ 

Figure 31

The nonlocality occurring at the connections of the external boson lines to the local vertices can be removed by a rescaling of the ϕ field. Hence the remaining nonlocality is due to the links in the trees. Before we rescale the ϕ field we collect first all kinetic terms belonging to the ϕ field. We find

$$\begin{aligned} -\phi\{G^{-1} + X - X[\lambda_\Lambda^{H\,-1}+X]^{-1}X\}\phi &= -\phi\{G^{-1} + X[\lambda_\Lambda^{H\,-1}+X]^{-1}[X + \lambda_\Lambda^{H\,-1} - X]\}\phi \\ &= -\phi\{G^{-1} + X[1+\lambda_\Lambda^H X]^{-1}\}\phi \\ &= -\phi\{G^{-1}[1+\lambda_\Lambda^H X + GX](1+\lambda_\Lambda^H X)^{-1}\}\phi \end{aligned} \tag{158}$$

Now we do the rescaling of the ϕ field according to equation (157). We multiply ϕ by the factor $(1+\lambda_\Lambda^H X)$. This changes the interaction into $V_{\Lambda-\delta\Lambda}(\phi|\bar{\Psi},\Psi|[\lambda_\Lambda^{H\,-1}+X]^{-1})$ which is still nonlocal due to the nonlocal links $[\lambda_\Lambda^{H\,-1}+X]^{-1}$ in the trees. The kinetic term changes to

$$-\phi\{(1+X\lambda_\Lambda^H)G^{-1}[1+\lambda_\Lambda^H X + GX]\}\phi = -\phi\{(\lambda_\Lambda^{H\,-1}+X)\lambda_\Lambda^H \underbrace{G^{-1}}_{=\lambda_\Lambda^{L\,-1}} \lambda_\Lambda \underbrace{(\lambda_\Lambda^{-1}+X)}_{=\lambda_{\Lambda-\delta\Lambda}^{-1}}\}\phi \tag{159}$$

This new kinetic term has shifted zeros which correspond to new poles in the propagator. Remember that $G^{-1} = \lambda_\Lambda^{L\ -1}$ has two zeros. One at the old cut-off Λ and one at $\hat{\Lambda}$ below this cut-off (compare equation 146). The zero at the old cut-off is cancelled exactly by the pole in λ_Λ^H. Instead a new zero is introduced from the term $(\lambda_\Lambda^{H\ -1} + X)$. Observe that this new pole in the composite field is below the actual cut-off at $\Lambda - \delta\Lambda$ and coincides exactly with the pole in the auxiliary field in the interaction. This is needed afterwards.

The other zero of G^{-1} is compensated by the pole in λ_Λ. Instead a new zero is introduced by $\lambda_{\Lambda-\delta\Lambda}^{-1}$. This means that the lowest pole in the composite boson propagator coincides always with the pole in $\lambda_{\Lambda-\delta\Lambda}^{-1}$ and can therefore be calculated exactly to leading order in $\frac{1}{N}$.

The locations of the zeros is not surprising because we could have come to this result also in a more direct way. Imagine integrating out the high frequency fermion fields down to the cut-off $\Lambda - \delta\Lambda$. We bosonize a part of the interaction using the high frequency auxiliary propagator $(\lambda_\Lambda^{H\ -1} + X)^{-1}$. This yields exactly the same result as above. This must be the case since to leading order in $\frac{1}{N}$ we have made no further approximation.

We still have to make the nonlocal interaction local. This is very simple now. We do a frequency split in the auxiliary propagator $(\lambda_\Lambda^{H\ -1} + X)^{-1}$

$$(\lambda_\Lambda^{H\ -1} + X)^{-1} = \{(\lambda_\Lambda^{H\ -1} + X)^{-1} - \frac{const}{p^2 + (\Lambda - \delta\Lambda)^2}\} + \frac{const}{p^2 + (\Lambda - \delta\Lambda)^2} \qquad (160)$$

where the constant is chosen appropriately to cut off the high frequencies of the propagator in the brackets. The propagator in the brackets is the low frequency part and has two poles. One is at the actual cut-off, i.e. at $\Lambda - \delta\Lambda$. The other is responsible for the nonlocal behaviour of $(\lambda_\Lambda^{H\ -1} + X)^{-1}$ and is below $\Lambda - \delta\Lambda$, say at Λ_1. This pole will cancel the corresponding pole in the composite field propagator. Instead of this pole the new pole at $\Lambda - \delta\Lambda$ is introduced into the composite boson propagator. This sets the cut-off of the composite field to its correct value for the new effective action.

We summarize our knowledge about the locality of the various actions in the following diagram. The diagram shows the absolute values for various quantities. The decay rates of interactions are given as black dots above the momentum scale. The actual cut-off of the considered effective theory is marked by a cross on the momentum scale. The mass and the cut-off of the composite propagator are given by black dots below the momentum scale. Important are only the relative positions of the various quantities. If the conditions for a local effective action are met in a figure a box around this figure is drawn.

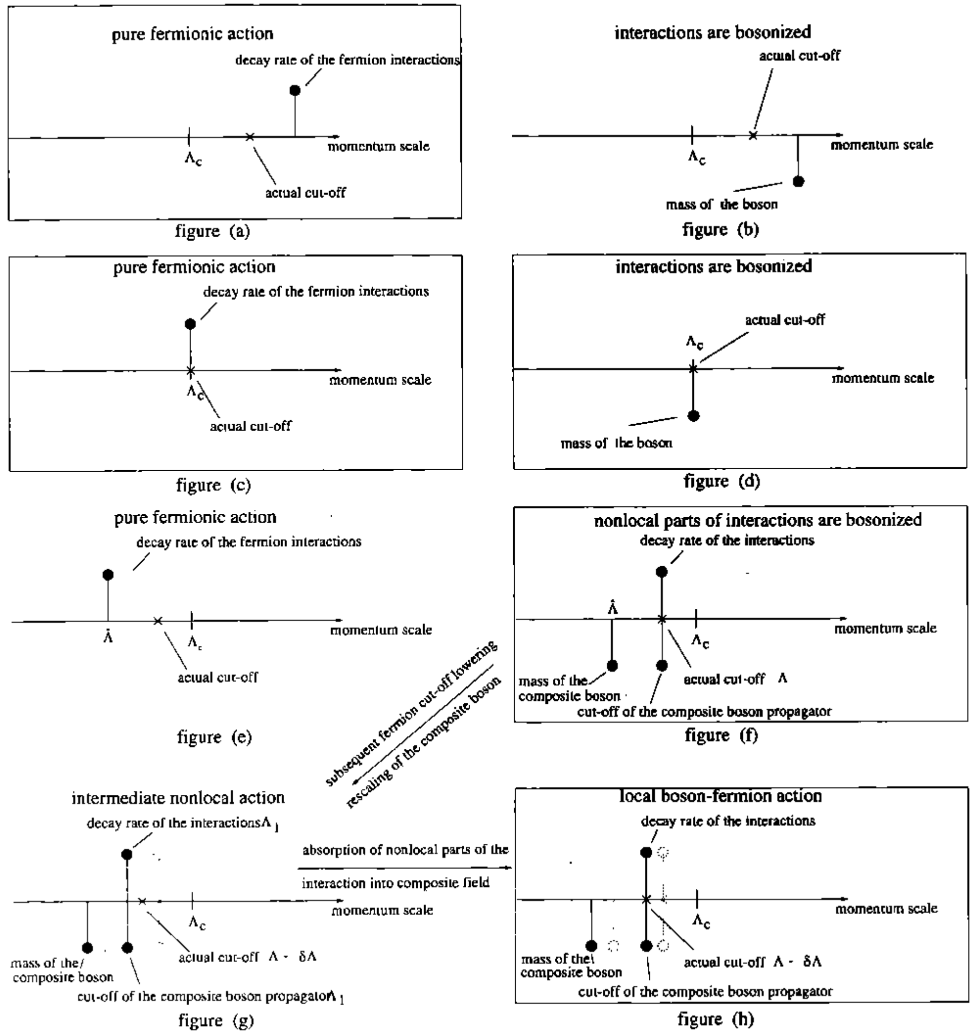


Figure 32

If the cut-off of the pure fermionic action is larger than the compositeness scale Λ_c the pure fermionic interactions will be local because the decay rate of the fermionic interactions in coordinate space is larger than the actual cut-off of the theory. This is depicted in figure (a).

In figure (b) we show the conditions for an effective action with a completely bosonized interaction. Fermion fields are still in the action but no pure fermionic interaction is left. This is the action of equation (108). The cut-off is the same as in figure (a). The composite field has a mass larger than the cut-off which is forbidden in an effective theory.

Figures (c) and (d) show again the pure fermionic formulation and the bosonized version for an actual cut-off which is exactly at the compositeness scale. Here both formulations are valid.

For a cut-off below the compositeness scale the pure fermionic action is nonlocal in figure (e), i.e. the decay rate of the fermionic interactions in the coordinate space is smaller than the actual cut-off.

Bosonizing the nonlocal parts of these interactions yields the action depicted in figure (f). The introduced composite boson has no high momentum modes because they are cut-off at the actual cut-off of the theory. The decay rate of the interactions is also situated at the actual cut-off. This is not surprising because this is just the way we have done the split in the auxiliary field propagator λ_Λ.

We can now see how the poles or decay rates move as we do a subsequent fermion cut-off lowering together with a rescaling of the composite field. It is shown in figure (g) where the former positions are indicated by dotted symbols. This intermediate action has two defects which are of course related. First the decay rate of the interactions is below the cut-off. This means that the interactions are nonlocal. Secondly the composite field propagator has a cut-off which is located below the actual cut-off of the theory. Since both problem are connected they are both removed bosonizing the nonlocal parts of the interactions using an additional composite field.

When merging the two composite fields the new cut-off of the composite field and the decay rate are located at the actual cut-off as they should. This is shown in figure (h). Since figure (h) is qualitatively equal to figure (f) this can be repeated.

Hence we have rederived the local effective action of section 5.3 by integrating out high frequency fermion fields after introducing a composite field. The final result has the same form as the effective action we started with. As a consequence two subsequent steps compose to one large step. This shows the consistency of the method. It is important that a rescaling of the composite field is necessary to keep the action local. This is not necessary in the direct derivation in section 5.3. However for practical problems where no direct derivation is possible one wants to do small consecutive cut-off lowerings. Thus in general a rescaling of the composite field will be necessary in each step.

Another important feature is that we do not integrate high frequency composite boson fields during the procedure. It is not possible to achieve the same result by integrating out certain frequency modes of the composite field. This would require a split in the composite boson propagator which introduces new frequencies. This is not possible without introducing a negative propagator. Therefore it is likely that a naïve treatment of composite fields as fundamental fields fails to produce local effective actions.

5.6 Some remarks on the provisional outline of section 4

We have seen that it is necessary to bosonize the reappearing nonlocalities in the interactions after integrating high frequency fermion fields. In our provisional outline we do this only for the four fermion interaction. Therefore the four fermion interaction is kept local through the whole procedure.

What happens with the other interactions which have more than four fermion fields? At the compositeness scale they certainly have the tendency to become nonlocal. Although we would expect that this trend continues towards lower cut-off we can not exclude that the steady reformulation of the four fermion interaction influences also the other interactions to become more local.

It is very difficult to trace the effect of the manipulations because we bosonize the four fermion term completely and integrate the high frequency modes of the so created auxiliary boson afterwards. This influences all other interaction also. Even in the framework of an $\frac{1}{N}$ expansion the change in the other fermion interactions due to the reformulation of the four fermion interaction will make such a mess that a prediction on the locality of fermion amplitude much below the compositeness scale is too difficult.

To be on the safe side we will give a formulation which is in principle able to reformulate the nonlocalities for these higher fermion interactions as well. This is the decomposition method for nonlocal fermion interaction in the next section.

6 Bosonization of nonlocal fermion interactions

Here we generalize the procedure of section 4 in order to obtain completely local effective actions. Using an approach based on Symanzik's infinite set of Bethe-Salpeter equations [36] we show that it is possible to construct a local effective action using composite fields. A systematic decomposition of nonlocal amplitudes is performed to extract the full interactions of the theory with composite fields. Self interactions of the composite field appear. The bare quantities are extracted from these full quantities using the freedom left in the choice of the **blockspin**.

In the preceding sections we have seen that the interactions of an effective theory might become nonlocal if the blockspin fields are of the same kind as the fundamental field. As mentioned earlier we call an interaction local if it decays **fast enough** in coordinate space. The criterion **fast enough** depends on the cut-off of the effective theory and is defined as follows. If we separate any two subsets of the arguments of the interaction by a distance a, where a becomes large, the interaction must decay faster than $e^{-\Lambda a}$ in order to be local. Λ is the momentum cut-off of the effective theory.

One way nonlocal interactions might develop in an effective theory is the forming of a low mass composite field. By definition the mass of a field is determined by the exponential decay rate of its propagator in coordinate space. Consider for example the propagator of a scalar field $G(p) = \frac{1}{p^2+m^2}$, then for large distances $|x-y|$ the propagator in coordinate space decays exponentially, or more precisely

$$G_{x,y} = \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2+m^2} e^{ip(x-y)} = \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{m}{|x-y|}\right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}(m|x-y|) \qquad (161)$$

where $Re(m) > 0$ and $0 < Re(d) < 5$. K are modified Besselfunctions which have the asymptotic expansion for large arguments and fixed ν [26]

$$K_\nu(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x}\{1 + \frac{4\nu^2-1}{8x} + \frac{(4\nu^2-1)(4\nu^2-9)}{2!(8x)^2} \dots\} \qquad (162)$$

with ν fixed and $|arg\ x| < \frac{3}{2}\pi$.

This example is easily generalized. The decay rate of an arbitrary amplitude is essentially determined by its smallest simple pole in momentum space for imaginary momenta. For simplicity we assume that the amplitude V depends only on p^2. Suppose that the smallest pole for imaginary momenta is at $p^2 = -m^2$. Then $V(p^2)$ can be written as

$$V(p^2) = \frac{V'(-m^2)}{p^2+m^2} + V_R(p^2) \qquad \text{with} \qquad V'(p^2) = \frac{dV(p^2)}{dp^2} \qquad (163)$$

The remaining amplitude $V_R(p^2)$ is regular at $p^2 = -m^2$ and its lowest pole is larger than m, say M. Then the Fourier-transformed of $V_R(p^2)$ decays in coordinate space for large arguments $|x-y|$ faster than $e^{-M|x-y|}$. This can be seen by shifting the path of the Fourier integration in the complex plane towards the pole at iM.

Hence the long range behaviour of the amplitude V is determined by the Fourier transformed of $\frac{V'(-m^2)}{p^2+m^2}$. It is proportional to the long range behaviour of a scalar field propagator with mass m, i.e for large $|x|$

$$V(|x-y|) \overset{|x-y|\to\infty}{\propto} V'(-m^2)e^{-m|x-y|} \qquad (164)$$

Since the exact form of the exponential decay does not matter at this stage of the analysis we approximate the propagator always by $e^{-m|x-y|}$. If the mass m of a composite field is smaller than the cut-off Λ then an interaction which is caused by an exchange of such a composite field will decay also only as e^{-ma}. Here a is the distance between arguments of interactions, which are connected by the exchange of the composite field.

Figure 33

The wiggly line stands for the composite field propagator. We regard nonlocal interactions as exchange processes of light fields in disguise. A general nonlocal interaction may contain all sorts of exchange processes. The aim is to find the suitable decomposition.

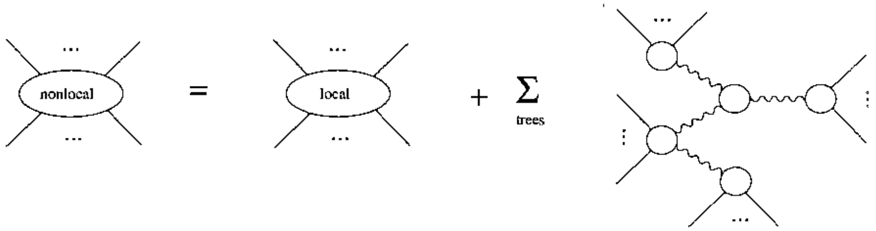


Figure 34

A necessary condition to treat an effective theory in lattice simulations is that all interactions are local. Otherwise an algorithm will not converge in a sufficiently short time. The obvious cure for the appearing nonlocalities in our effective theory is to rewrite it as a local effective theory, which contains composite fields as physical degrees of freedom. The problem is to find a decomposition of a nonlocal amplitude into a local part and a part with tree like structure as on the right hand side of figure 34. All interactions on the right hand side of figure 34, i.e. all blobs, should be local.

In this section we argue first in 6.1 that a low mass bound state leads to a nonlocal part which is factorizable in the above fashion. In section 6.2 we show how the tree like structures of exchange processes are extracted systematically. This procedure is not unique. In section 6.4 we describe how the remaining freedom is used to solve a technical problem in identifying the extracted parts with the new interactions of an effective theory which contains composite fields. This in turn gives a perspective for further lowering of the cut-off of the effective theory.

6.1 Factorization of nonlocal parts in fermion amplitudes

Using an infinite set of Bethe-Salpeter equations as used first by Symanzik [36] we show that in the presence of a low mass bound state the nonlocal parts of the interactions factorize in a specific way. The nonlocal parts can be written as trees with local interactions and connections which have all equal long range behaviour. This long range behaviour is determined by the mass of the light composite field.

This specific factorization of the nonlocal parts is a necessary condition if we want to rewrite these parts as an interaction induced by exchanges of light composite fields.

We assume to be at a scale Λ where nonlocalities appear for the first time in the course of the renormalization group flow. As a consequence the mass of the lightest composite field will be approximately of the order of the cut-off Λ or a little bit less. We consider only those nonlocalities which appear in channels with the same quantum numbers as these low mass bound states. Nonlocalities which have other origin than composite particles are out of the scope of this analysis.

Before we go into detail we want to mention that the interactions of an effective theory have an interpretation as free propagator amputated Green functions of a suitably chosen auxiliary theory. This is quite easy to observe. Take for instance the effective potential of a boson model which is

defined as

$$e^{-V_{eff}(\Phi)} = e^{-V_\Lambda(\Phi)} = \int d\mu_{v_0}(\xi)\, e^{-V_{\Lambda'}(\Phi+\xi)} \tag{165}$$

We do a shift in the ξ field by Φ.

$$\begin{aligned} e^{-V_{eff}(\Phi)} &= \int \mathcal{D}\xi' e^{-\frac{1}{2}(\Phi v_0^{-1}\Phi)-\frac{1}{2}(\xi' v_0^{-1}\xi')-V_{\Lambda'}(\xi')+(\xi' v_0^{-1}\Phi)} \\ &= e^{W_\Lambda[J]-\frac{1}{2}J v_0 J}\big|_{J=v_0^{-1}\Phi} \end{aligned} \tag{166}$$

where

$$e^{W_\Lambda[J]} = \int \mathcal{D}\xi e^{-\frac{1}{2}(\xi v_0^{-1}\xi)-V_{\Lambda'}(\xi)+(\xi J)} \tag{167}$$

By comparing V_{eff} with W_Λ we find that the interactions are exactly the free propagator amputated connected Green functions of the theory belonging to W_Λ. The free propagator of this theory is the high frequency propagator of the fundamental theory. In the following discussion we will work with the connected Green functions. This is done for convenience in the graphical representation. The interactions can be recovered by multiplying each external leg with a factor $\frac{1}{v_0}$.

For the discussion we need the concept of one particle irreducibility in a certain channel. A diagram, which is one particle irreducible (1PI) in a certain channel is one, which cannot be separated into two parts, where one part is on each side of the channel, by cutting a single line across the channel. A connected Green function, which is one particle irreducible in a certain channel, can be defined by [36]

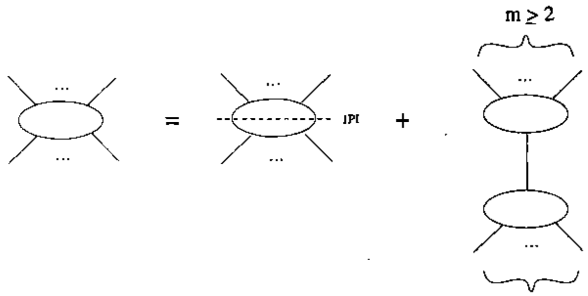


Figure 35

Lines represent the full high frequency propagator v of the fundamental field. The second term on the right hand side appears only for channels which have at least two external legs on each side of the channel. The three point function for instance is already 1PI in all channels and the second term on the right hand side is not necessary.

This concept of one particle irreducibility can be extended to two particle irreducible functions. Let us first consider the special case of the two particle irreducible four point function. This is per definition the Bethe-Salpeter kernel. It is defined implicitly by the equation [36].

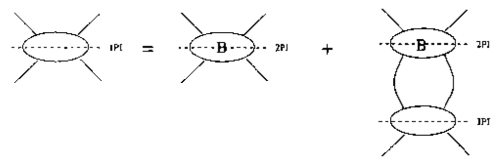


Figure 36

This definition differs from the usual one in the literature because we used connected one particle irreducible amplitudes to define it. Therefore the disconnected piece, which appears usually on the right hand side does not appear here. However the Bethe-Salpeter kernel is the sum of contributions to the connected one particle irreducible four point function, which are not decomposable by a two particle cut across the channel. With this definition of the Bethe-Salpeter kernel we are prepared to decompose a more complicated amplitude. Let the number of coordinates on one side of the channel be $n > 2$. The decomposition into a two particle irreducible part and some explicitly two particle reducible parts reads:

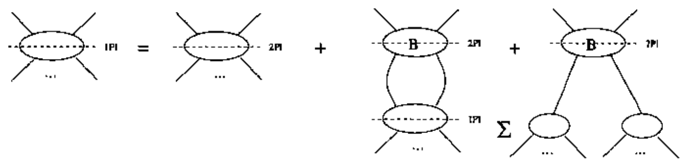


Figure 37

where the number of bottom legs is n. The sum runs over all $(2^{n-1} - 1)$ inequivalent partitions of n coordinates into two undistinguished sets, where no set is empty [36]. This equation can be regarded as a linear integral equation of the second kind. We assume that the above equation in figure 36 for the Bethe-Salpeter kernel can be solved by the Fredholm method. Then this equation can be solved, too. The result is given by

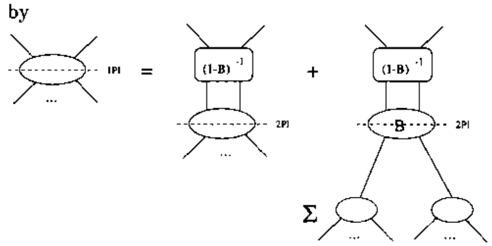


Figure 38

A possible source of the nonlocality is easy to identify in this equation. In momentum space the Bethe-Salpeter kernel may have the eigenvalue 1 for certain values of imaginary momenta running through the channel. The kernel $(1 - B)^{-1}$ has a pole for these momenta. If the pole is a simple one and occurs for momenta $-p^2 = m^2 < \Lambda^2$, then the decay rate of the amplitude will be smaller than the cut-off Λ. See the discussion below equation (161). In this case the fermion amplitude has a nonlocality in that specific channel. This structure holds also for general 1PI amplitudes. A general 1PI amplitude can be decomposed as follows.

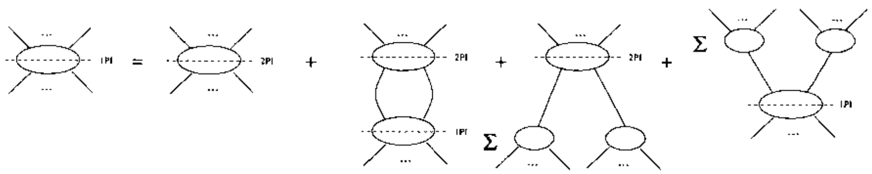


Figure 39

Again this equation can be solved if we substitute the 1PI irreducible amplitudes according to figure 38. The result of this is a somewhat lengthy formula

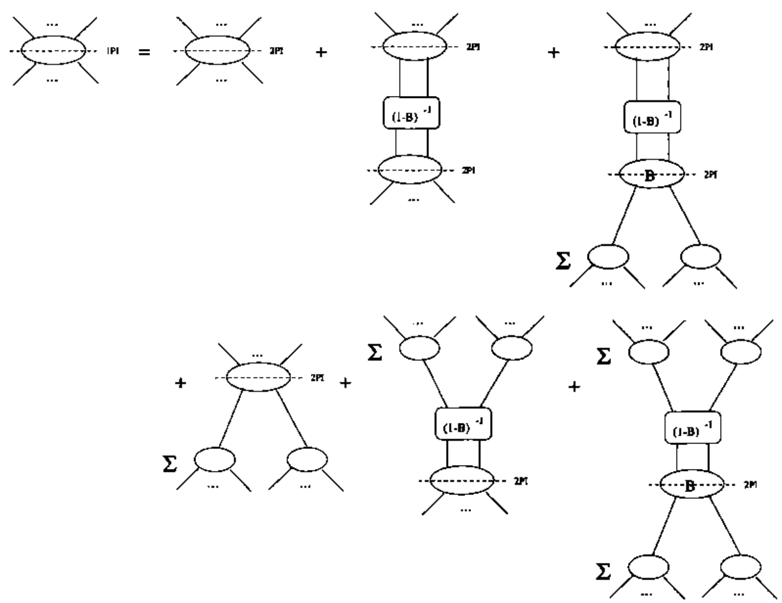


Figure 40

We want to make the separation into a factorizable part and a remaining part more prominent. The factorizable part should contain the nonlocality. Let us define a new amplitude to compress our notation.

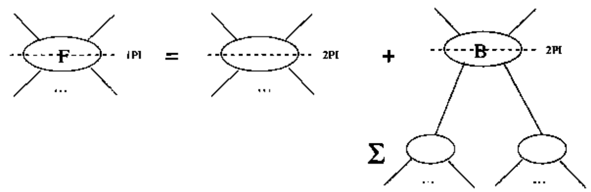


Figure 41

Inserting this into figure 40 we find

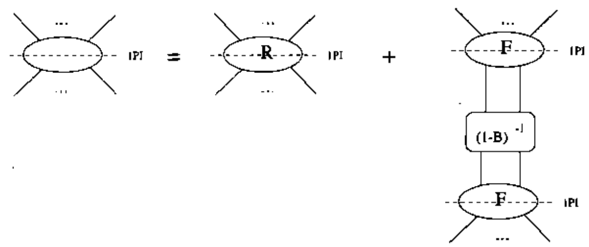


Figure 42

where the remaining amplitude R is given by

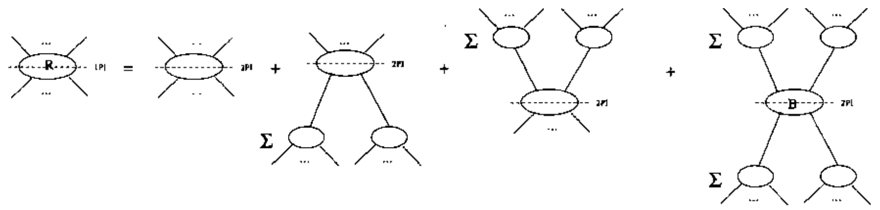


Figure 43

Figure 42 is in principle the wanted decomposition of our nonlocal amplitude. There are two points to clarify.

- We have decomposed only amplitudes which are 1PI in a certain channel. The question arises whether there can be nonlocalities in an one particle reducible channel. A nonlocality in an one particle reducible channel can only occur if the full high frequency propagator of the fundamental field is nonlocal. This can of course never happen since it is local by definition. In fact the cut-off of our effective theory is actually defined by the physical mass of the full high frequency propagator. Therefore it can never happen that a nonlocality occurs in a channel which has the same quantum numbers as the fundamental field.
- The remaining amplitude R should be local. Clearly those parts, which carry the quantum numbers of a fundamental field are local. Still there are some two particle irreducible amplitudes which could be nonlocal. They could in principle contain a three particle bound state with a mass below the cut-off Λ. We will assume that the mass of a three particle boundstate is larger than the mass of a two particle boundstate. In simple words one could say that mesons are lighter than baryons. Since the mass of the two particle bound state is assumed to be of the order of the cut-off Λ, the mass of the three particle bound state would be larger and therefore the 2PI amplitude would be local.

The decomposition in figure 36 uses the knowledge about the Bethe-Salpeter kernel, which is a complicated object. At the compositeness scale neither v_0 nor λ needs to be small. Therefore a perturbative expansion of the Bethe-Salpeter kernel, which for our toy model would look like

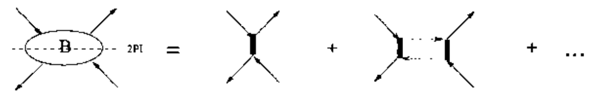


Figure 44

will not converge. At this point we should consider again the purpose of this decomposition. We are looking for an effective theory of fundamental and composite fields which is local in all its interactions and describes all processes up to the scale Λ. These requirements do not determine the effective theory completely. We have for instance the freedom to change the local structure of the composite field propagator if we compensate for this by local changes in the interactions. Local always means a decay faster than $e^{-\Lambda x}$. This is quite intuitive since with processes which are limited in momenta by Λ, it is impossible to resolve any structure below distances of the order $\frac{1}{\Lambda}$. Therefore we have the freedom to shuffle some local parts from the interactions into the composite field propagator and vice versa. We use this freedom to simplify the structure of the composite field propagator.

The Bethe-Salpeter kernel is an integral kernel which acts in an Hilbert-space with a scalar product which involves the full high frequency propagator v. The scalar product of two symmetric wavefunctions ϕ_1 and ϕ_2 is given by

$$< \phi_1|\phi_2 >= \int_{x_1x_2y_1y_2} \phi_1(x_1, y_1)v(x_1 - x_2)v(y_1 - y_2)\phi_1(x_2, y_2). \qquad (168)$$

We define the state $|x_1, x_2>$ by the wavefunction $\phi(y_1, y_2) = \frac{1}{2}(\delta(x_1 - y_1)\delta(x_2 - y_2) + (x_1 \leftrightarrow x_2))$. Thus we can define

$$B(x_1, x_2; x_3, x_4) =< x_1, x_2|B|x_3, x_4 >= \int_{x_1...x_4} v(x_1 - z_1)\dots v(x_4 - z_4)B(\underline{z}_1, \underline{z}_2; \underline{z}_3, \underline{z}_4) \qquad (169)$$

where an underlining of an argument in B denotes the amputation of the full propagator belonging to this coordinate. The action of B on an arbitrary state in this Hilbert-space is given by

$$\begin{aligned} < x_1, x_2|B|\tilde{\varphi} > &= \int \frac{1}{2}\{v(x_1 - z_1)v(x_2 - z_2) + (x_1 \leftrightarrow x_2)\} \underbrace{B(\underline{z}_1, \underline{z}_2; \underline{w}_1, \underline{w}_2)v(w_1 - z_3)v(w_2 - z_4)\tilde{\varphi}(z_3, z_4)}_{\equiv\tilde{\varphi}'(z_1,z_2)} \\ &= < x_1, x_2|\tilde{\varphi}' > \end{aligned} \qquad (170)$$

Therefore we can use the amputated kernel to define the action of B on wavefunctions with two arguments $\tilde{\varphi}(z_1, z_2)$

$$\tilde{\varphi}'(z_1, z_2) = \int_{w_1,w_2,z_3,z_4} \underbrace{B(\underline{z}_1, \underline{z}_2; \underline{w}_1, \underline{w}_2)v(w_1 - z_3)v(w_2 - z_4)}_{=B(\underline{z}_1,\underline{z}_2;z_3,z_4)}\tilde{\varphi}(z_3, z_4) \qquad (171)$$

where (z_1, z_2) and (z_3, z_4) are arguments on different sides of the channel. Because of translational invariance of the Bethe-Salpeter kernel the overall momentum is conserved. We may Fourier transform the partly amputated B with respect to the variables $(z_1 + z_2)$ and $(z_3 + z_4)$. Then this B is a function of the relative coordinates $(z_1 - z_2)$ and $(z_3 - z_4)$ and the momentum p going through the channel. We can do the same with the function $\tilde{\varphi}$, i.e

$$\tilde{\varphi}(z_1, z_2) = \varphi(z_1 + z_2|z_1 - z_2) = \int \frac{d^n p}{(2\pi)^n} \varphi(p|z_1 - z_2)e^{ip(z_1+z_2)}. \qquad (172)$$

Hence the above equation (171) reads

$$\varphi'(p|z_1 - z_2) = \int_{z_3-z_4} B(\underline{z}_1 - \underline{z}_2|z_3 - z_4|p)\, \varphi(p|z_3 - z_4). \qquad (173)$$

Since the Bethe-Salpeter kernel is translational invariant, the operator B commutes with the momentum operator P. Therefore B and P can be simultaneously diagonalized. We assume that $\varphi_\chi(z_1 - z_2|p)$ are eigenfunctions to the Bethe-Salpeter kernel with real eigenvalues $b_\chi(p)$. χ accounts for additional quantum numbers like spin or flavour. These eigenvalues depend on the range of integrated high frequency modes through the propagator v. For the case of degenerate eigenfunctions we may choose an orthonormal set φ_χ^α. The Bethe-Salpeter kernel has a representation in terms of projection operators on the eigenspaces.

$$B(\underline{z}_1, \underline{z}_2; z_3, z_4) \;=\; \sum_{\chi,\alpha} \int dp\, \varphi_\chi^\alpha(z_1 - z_2|p)\, b_\chi(p)\, \varphi_\chi^\alpha(z_3 - z_4|p)^* \qquad (174)$$

Of course this system of eigenfunctions diagonalizes the operator $(1 - B)^{-1}$ as well, which then has the representation

$$(1 - B)^{-1}(\underline{z}_1, \underline{z}_2; z_3, z_4) = \sum_{\chi,\alpha} \int dp\, \varphi_\chi^\alpha(z_1 - z_2|p)\, \frac{1}{1 - b_\chi(p)}\, \varphi_\chi^\alpha(z_3 - z_4|p)^* \qquad (175)$$

The eigenfunction φ can be fused with the F amplitudes to new A amplitudes.

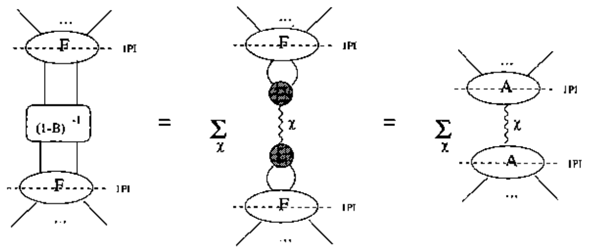


Figure 45

The wiggly lines represent the factor $\frac{1}{1-b_\chi(p)}$, the shaded blobs the eigenfunctions φ_χ. The sum still runs over all eigenvalues. It suffices to determine the lowest mass eigenvalue however, i.e. the χ for which the pole in $\frac{1}{1-b_\chi(p)}$ occurs at momenta for which $m^2 = -p^2$ is minimal. By hypothesis the lowest pole lies just below the cut-off in the situation in which we are interested here. Therefore m is of the order Λ or a little bit smaller.

This decay rate, i.e. the location of the pole in momentum space, is the only knowledge we have to extract from the nonlocal fermion interaction. Everything else is local information, even the form of $\frac{1}{1-b_\chi(p)}$ for momenta away from the pole. We are free to choose the form of the composite field propagator as long as it has a pole at the right mass. We will use the freedom later to choose the propagator to be small. The decomposition reads then

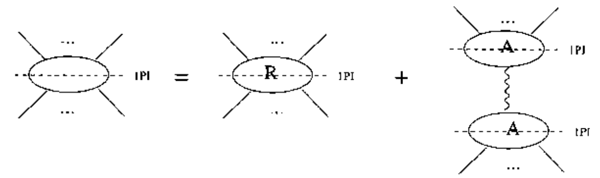


Figure 46

where the wiggly line now denotes a propagator $\frac{1}{p^2+m^2}$ for example. The point we want to make clear with this derivation is the following. No matter which channel we investigate, the decay property of the channel is always determined by its quantum numbers alone. These quantum numbers fix the Bethe-Salpeter kernel and the eigenvalues of the Bethe-Salpeter kernel determine the mass of the composite field. This mass is effectively the decay rate in coordinate space.

6.2 Decomposition of nonlocal amplitudes into local parts plus tree like parts

In this section we show how to extract systematically the tree like structure from the nonlocal interactions. This is done in principle by separating groups of arguments in the amplitudes by large distances to study the long range behaviour. In this way all nonlocal amplitudes can be decomposed.

A nonlocal interaction will be regarded as a sum of exchange processes of composite fields. The problem is to single out certain processes and to identify them with interactions of a new effective theory including self interactions of composite fields. This is done by separating certain groups of arguments in coordinate space by large distances. Alternatively we bring certain sums of external momenta close to pole values.

Before we do this, we want to investigate what kind of structure we expect to find. Suppose we found our effective theory containing composite fields already. The nonlocal theory can be recovered by integrating the composite fields out again.

$$e^{-V_{nl}(\Psi)} = \int d\mu_{v_0}(\zeta)\, e^{-V(\Psi+\zeta)}$$

$$= \int \mathcal{D}\phi\, e^{-\frac{1}{2}(\phi G_0^{-1}\phi)-V_0(\Psi,\phi)}$$
$$= e^{\frac{1}{2}(\frac{\delta}{\delta\phi}G_0\frac{\delta}{\delta\phi})}\, e^{-V_0(\Psi,\phi)}|_{\phi=0} \qquad (176)$$

Although we use the same notation as in our toy model the situation is still general. Ψ denotes either fermion or boson blockspin fields, ζ is the high frequency field and ϕ stands for a general composite field. G_0 and V_0 are the quantities we want to extract from V_{nl}. G_0 is the composite propagator and V_0 is the interaction part. V_0 contains pure composite field interactions as well as pure fundamental field interactions and also interactions between fundamental and composite fields. We demand that all interactions are local in the sense that the amplitudes decay exponentially with the distance between any two arguments with a decay rate bigger than Λ.

The factorization of the nonlocal parts of V_{nl} yields tree like structures. This is in contrast to the above equation which produces all kinds of diagrams containing loops. We have to rewrite it in the following way:

$$e^{-V_{nl}(\Psi)} = e^{\frac{1}{2}(\frac{\delta}{\delta\phi}G\frac{\delta}{\delta\phi})}e^{-V(\Psi,\phi)}\Big|_{\substack{\phi=0\\ \text{trees only}}} \qquad (177)$$

The right hand side generates only terms which have tree structure. G and V are the dressed propagators and interactions respectively. G and V can be given in terms of G_0 and V_0 in a loop expansion as follows

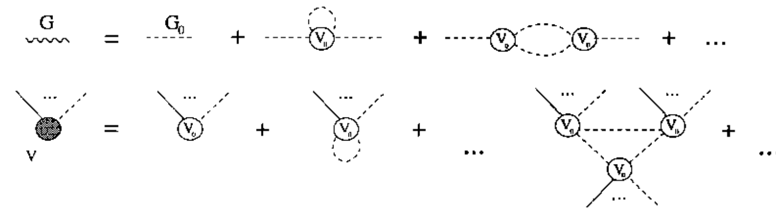


Figure 47

The last summand in the second equation appears only for monomials in the field of sufficient high order on the right hand side, because the outer legs have to match on both sides of the equation. These expansions are manageable for small propagators G or G_0. Before we turn to this problem we want to evaluate equation (177) further.

As mentioned above nonlocalities can occur in principle in all channels which do not carry the quantum numbers of a fundamental field.

The interaction has therefore the form

$$V(\Psi,\phi) = \int_{z\,x} \sum_{n,m} \frac{1}{m!} V^{(n,m)}(z_1,\ldots,z_n|x_1,\ldots,x_m)\Psi(z_1)\ldots\Psi(z_n)\phi(x_1)\ldots\phi(x_m) \qquad (178)$$

where $V^{(1,1)} = V^{(0,2)} = V^{(2,0)} = 0$. Since there can be no nonlocality in the three point function, the first interaction we have to decompose is the four point function. We evaluate equation (177) for the four point function.

$$\int_z V_{nl}^{(4)}(z_1,z_2,z_3,z_4)\,\Psi(z_1)\ldots\Psi(z_4) = \int_z V^{(4,0)}(z_1,z_2,z_3,z_4)\,\Psi(z_1)\ldots\Psi(z_4)$$
$$-\frac{1}{2}\sum_{\text{channels}}\int_{z,x,y} V^{(2,1)}(z_i,z_j|x)\,G_{x,y}V^{(2,1)}(z_k,z_l|y)\Psi(z_1)\ldots\Psi(z_4) \qquad (179)$$

The sum over channels consist of the usual three channels for the four point function (1,2)(3,4), (1,3)(2,4) and (1,4)(2,3). From there we derive immediately

$$V_{nl}^{(4)}(z_1,z_2,z_3,z_4) = V^{(4,0)}(z_1,z_2,z_3,z_4) - \frac{1}{2}\sum_{\text{channels}}\int_{x,y} V^{(2,1)}(z_i,z_j|x)\,G_{x,y}V^{(2,1)}(z_k,z_l|y) \qquad (180)$$

We have to match this with the terms we get from a factorization of the nonlocal parts in the four point function.

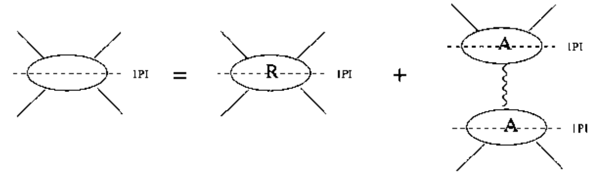


Figure 48

The above figure shows only one channel. In practice the factorization is done as follows. We separate the arguments (z_1, z_2) and (z_3, z_4) for example. The nonlocal amplitude will decay exponentially with the separating distance. The decay rate fixes m. Because we are just at the compositeness scale $m \lesssim \Lambda$. In contrast the amplitudes $V^{(4,0)}$ and $V^{(2,1)}$ are local, i.e. they decay faster than $e^{-\Lambda|x|}$, where $|x|$ is the largest distance between two of the arguments. Suppose $G(p) = \mathcal{M}(p)\frac{1}{p^2+m^2}$ where $\mathcal{M}$ can be a matrix with spin and flavour indices. Then for large distances $|x-y|$ the propagator in coordinate space decays exponentially as in equation (161). We do not need the precise form but approximate $G_{xy} = \mathcal{M}_{xy}\, e^{-m|x-y|}$ for large distances. Inserting this into the above equation we find

$$\begin{aligned} V^{(4)}_{nl}(z_1, z_2, z_3 + a, z_4 + a) &\overset{a\to\infty}{\approx} -\frac{1}{2}\int_{x,y} V^{(2,1)}(z_1, z_2|x)\, \mathcal{M}_{xy}\, e^{-m|x-y|}\, V^{(2,1)}(z_3 + a, z_4 + a|y) \\ &\approx -\frac{1}{2}\int_{x',y'} V^{(2,1)}(z_1, z_2|\frac{z_1+z_2}{2} + x')\, \mathcal{M}(\frac{z_1+z_2}{2} + x' - \frac{z_3+z_4}{2} - a - y') \\ &\quad \times e^{-m|\frac{z_1+z_2}{2} + x' - \frac{z_3+z_4}{2} - a - y'|} V^{(2,1)}(z_3, z_4|\frac{z_3+z_4}{2} + y') \end{aligned} \tag{181}$$

where we used translation invariance of the amplitudes and of the matrix $\mathcal{M}$. Because of the locality of the amplitudes the domains of integration are effectively given by a sphere with radius $\frac{1}{\Lambda}$. We assume that the initial arguments z_1, z_2, z_3, z_4 are also positioned in a sphere of radius $\frac{1}{\Lambda}$. The difference $|\frac{z_1+z_2}{2} - \frac{z_3+z_4}{2}| < \frac{2}{\Lambda}$. Therefore the arguments in the exponent and in the matrix $\mathcal{M}$ are dominated by a. Neglecting the small arguments in the exponent yields

$$\begin{aligned} V^{(4)}_{nl}(z_1, z_2, z_3 + a, z_4 + a) &\overset{a\to\infty}{\approx} -\frac{1}{2}e^{-m|a|}\int_{x'} V^{(2,1)}(z_1, z_2|\frac{z_1+z_2}{2} + x')\, \mathcal{M}_a \int_{y'} V^{(2,1)}(z_3, z_4|\frac{z_3+z_4}{2} + y') \\ &= -\frac{1}{2}\, e^{-m|a|}\, A(z_1, z_2|\frac{z_1+z_2}{2})\, \mathcal{M}_a\, A(z_3, z_4|\frac{z_3+z_4}{2}) \end{aligned} \tag{182}$$

To find the A amplitudes we have to assume a special form of the propagator $G(p)$ because there can be degeneracies, for instance due to a possible spin of the fundamental fields. Let us give an example. Suppose $G(p)$ has the form

$$G(p) = \frac{\delta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}}{p^2 + m^2}, \tag{183}$$

i.e. the propagator of a massive vector boson. For large separations a in coordinate space this gives approximately

$$G(a) = (\delta_{\mu\nu} - a_\mu a_\nu)e^{-m|a|} \tag{184}$$

in coordinate space. We can choose the direction of a to be $(1, 0, 0, 0)$. This selects a certain combination of A amplitudes.

$$V^{(4)}_{nl}(z_1, z_2, z_3 + a, z_4 + a) \overset{a\to\infty}{\approx} -\frac{1}{2}\, e^{-m|a|}(A_\mu A_\mu - A_0 A_0) \tag{185}$$

Here we have written explicitly Lorentz indices but neglected all coordinates. Doing the same for the other directions we can sum the results and extract $A_\mu A_\mu$. This enable us to extract $A_0 A_0$ from the above equation by subtracting the $A_\mu A_\mu$ contribution. The A_0 amplitude can be found by equating $z_1 = z_3$ and $z_2 = z_4$ and taking the square root of the result for $A_0 A_0$ in the limit of large a in $(1, 0, 0, 0)$ direction. This shows that for the actual calculation a special form of the propagator has to be assumed. We assume that this can always be done and that we have found our A amplitudes.

By separating the arguments in the nonlocal $V^{(4)}_{nl}$ amplitude we find at most the A amplitudes. These are approximately integrals of $V^{(2,1)}$ over a sphere of radius $\frac{1}{\Lambda}$. At this stage it is manifest that we have the freedom to choose the local structure of $V^{(2,1)}$, because only the integral is fixed by the long range behaviour. We use the freedom to make the interaction $V^{(2,1)}$ as local as possible. Let us define

$$V^{(2,1)}(z_1, z_2|x) = A(z_1, z_2|\frac{z_1+z_2}{2})\, \delta(x - \frac{z_1+z_2}{2}) \tag{186}$$

We have to do this for the other two channels as well and identify in this way the nonlocal part of the amplitude. $V^{(4,0)}$ is then what is left over when we subtract the identified sum of all three channels from the nonlocal amplitude. Thus we extract $G, V^{(2,1)}$ and $V^{(4,0)}$ from the nonlocal interaction $V^{(4)}_{nl}$.

We proceed to the fivepoint function. Again we evaluate first equation (177) for this case.

$$\begin{aligned} \int_z V^{(5)}_{nl}(z_1, \ldots, z_5)\, \Psi(z_1) \ldots \Psi(z_5) &= \int_z V^{(5,0)}(z_1, \ldots, z_5)\, \Psi(z_1) \ldots \Psi(z_5) \\ &\quad - \int_{z,x} V^{(2,1)}(z_1, z_2|x_1)\, G_{x_1,x_2}\, V^{(3,1)}(z_1, z_2, z_3|x_2)\, \Psi(z_1) \ldots \Psi(z_5) \\ + \frac{1}{2}\int_{z,x,y} & V^{(2,1)}(z_1, z_2|x_1)\, G_{x_1,x_2}\, V^{(2,1)}(z_3, z_4|y_1)\, G_{y_1,y_2}\, V^{(1,2)}(z_5|x_2, y_2)\, \Psi(z_1) \ldots \Psi(z_5) \end{aligned} \tag{187}$$

We compare the integrands where the right hand side is suitably symmetrized.

$$\begin{aligned} V^{(5)}_{nl}(z_1, \ldots, z_5) &= V^{(5,0)}(z_1, \ldots, z_5) - \frac{1}{10}\sum_{\text{permutations}} \int_x V^{(2,1)}(z_i, z_j|x_1)\, G_{x_1,x_2}\, V^{(3,1)}(z_k, z_l, z_m|x_2) \\ &+ \frac{1}{15}\sum_{\text{permutations}} \int_{x,y} V^{(2,1)}(z_i, z_j|x_1)\, G_{x_1,x_2}\, V^{(2,1)}(z_k, z_l|y_1)\, G_{y_1,y_2}\, V^{(1,2)}(z_m|x_2, y_2) \end{aligned} \tag{188}$$

For the following discussion it is more convenient to work in momentum space. Because of translational invariance of the amplitudes in coordinate space we have conservation of overall momentum.

$$\begin{aligned} V^{(5)}_{nl}(p_1, \ldots, p_5) &= V^{(5,0)}(p_1, \ldots, p_5) \\ &\quad - \frac{1}{10}\sum_{\text{permutations}} V^{(2,1)}(p_i, p_j| - (p_i + p_j))\, G(p_i + p_j)\, V^{(3,1)}(p_k, p_l, p_m|(p_i + p_j)) \\ &\quad + \frac{1}{15}\sum_{\text{permutations}} V^{(2,1)}(p_i, p_j| - (p_i + p_j))\, G(p_i + p_j)\, V^{(2,1)}(p_k, p_l| - (p_k + p_l)) \\ &\qquad G(p_k + p_l)\, V^{(1,2)}(p_m|(p_i + p_j), (p_k + p_l)) \end{aligned} \tag{189}$$

This decomposition can be depicted by

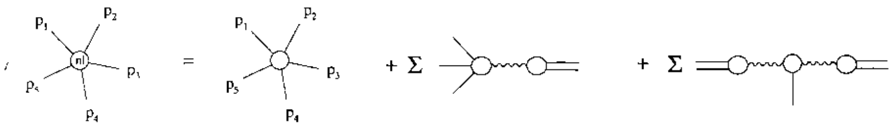


Figure 49

The difficulty consists in identifying the individual terms in the figure by separating the arguments in the nonlocal amplitude. The separation of groups of arguments in coordinate space is equivalent to choose the value for certain sums of momenta $p = (p_i + p_j)$ equal to the pole value $-p^2 = m^2$. In both cases certain channels are singled out because their contributions become much larger than the other contributions.

We first choose two sums, say $p = (p_i + p_j)$ and $q = (p_k + p_l)$, close to the pole value. With this combination we single out the appropriate term in the last sum in figure 49. Remember that $V^{(2,1)}$ is already known from the treatment of the four point function. This allows to fix the amplitude $V^{(1,2)}(p_m|p,q)$ for values of the composite field momenta close to the pole value. The freedom to choose the local structure corresponds to the freedom to choose values of $V^{(1,2)}$ for momenta p, q away from the pole value. $V^{(1,2)}$ must be Lorentz-invariant and analytic. Again we try to make $V^{(1,2)}$ as local as possible. Powers of momenta correspond to derivatives in coordinate space. The degree of nonlocality rises with powers of momenta. Therefore we try to extrapolate to other momentum values without introducing new powers of momenta. The absolute values of p and q are fixed, but there is still the freedom to choose the angle between p and q, which corresponds to the Lorentz-invariant variable $(p\cdot q)$. Remember that the variable $p_m = -(p+q)$ is fixed by momentum conservation. To extrapolate to general p and q values we substitute general values for $(p\cdot q)$. We have to do a Taylor-expansion in $(p\cdot q)$ before in order to know the functional dependence on $(p\cdot q)$. This has to be done for all possible channels.

After fixing $V^{(1,2)}$ for general values of momenta, we free one sum from its constraint. With only one sum, say $p = (p_i + p_j)$, close to the pole value all terms containing a propagator with this sum as argument are large. The only new amplitude in this contributions is $V^{(3,1)}$. Therefore we are able to find $V^{(3,1)}(p_k, p_l, p_m|p)$ for values of the composite field momentum p close to the pole. This we have to extrapolate again. There are two angle variables to extrapolate, $(p\cdot p_k)$ and $(p\cdot p_l)$. The third momentum p_m is fixed by momentum conservation. This renders the amplitude $V^{(3,1)}$.

After we have done this again for all possible channels we define $V^{(5,0)}$ as the remaining part which is left after subtracting all known parts from the nonlocal amplitude.

In this way we decompose successively all appearing nonlocal amplitudes. In practice one has to truncate this hierarchy since this is an infinite process. The sixpoint function yields the first pure composite field interaction. We will not demonstrate this in the general context but rather do the whole decomposition business again for the case of our toy model. Moreover we specialize to the case of large flavour number N.

6.3 Decomposition of the toy model for large N

In this section we exhibit the decomposition for the special case of our toy model for large N. This model is more simple because only one scalar boson appears. This will enable us to do the decomposition of the six fermion amplitude also without excessively complicating notation. We will see that this decomposition leads to pure boson interaction as well.

The procedure should be clear by now. We imagine the high frequency fermions integrated out until for the first time nonlocal pure fermionic interactions appear. The restriction to large N means that only one scalar boson appears as composite field and therefore the interaction of our new effective theory will have the form

$$V(\Psi,\phi) = \int_{z,x}\sum_{n,m}\frac{1}{m!}V^{(2n,m)}(z_{\bar 1}, z_1, \ldots, z_{\bar n}, z_n|x_1,\ldots,x_m)\bar\Psi(z_{\bar 1})\ldots\Psi(z_n)\phi(x_1)\ldots\phi(x_m) \qquad (190)$$

where fermions with arguments $z_{\bar i}$ and z_i have equal flavour indices, which are suppressed in our notation however. We evaluate equation (177) for the four point function using the above interaction V:

$$\int_z V_{nl}^{(4)}(z_{\bar 1}, z_1, z_{\bar 2}, z_2)\,\bar\Psi(z_{\bar 1})\ldots\Psi(z_2) = \int_z V^{(4,0)}(z_{\bar 1}, z_1, z_{\bar 2}, z_2)\,\bar\Psi(z_{\bar 1})\ldots\Psi(z_2)$$
$$-\frac{1}{2}\int_{z,x,y} V^{(2,1)}(z_{\bar 1}, z_1|x)\,G_{x,y}V^{(2,1)}(z_{\bar 2}, z_2|y)\bar\Psi(z_{\bar 1})\ldots\Psi(z_2) \qquad (191)$$

We compare the integrands.

$$V_{nl}^{(4)}(z_{\bar 1}, z_1, z_{\bar 2}, z_2) = V^{(4,0)}(z_{\bar 1}, z_1, z_{\bar 2}, z_2) - \frac{1}{2}\int_{x,y} V^{(2,1)}(z_{\bar 1}, z_1|x)\,G_{x,y}\,V^{(2,1)}(z_{\bar 2}, z_2|y) \qquad (192)$$

This time the number of possible channels is restricted, because the composite field is a scalar boson. We compare with the result from the factorization of nonlocal parts.

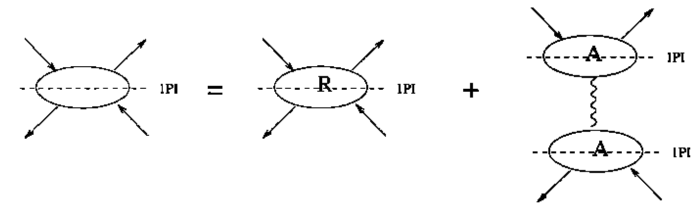


Figure 50

If we assume the form of the composite propagator to be $\frac{1}{p^2+m^2}$ the separation of arguments yields as above

$$V_{nl}^{(4)}(z_{\bar 1}, z_1, z_{\bar 2}+a, z_2+a) \overset{a\to\infty}{\approx} -\frac{const}{2}e^{-m|a|}\int_{x'}V^{(2,1)}(z_{\bar 1}, z_1|\frac{z_{\bar 1}+z_1}{2}+x')\int_{y'}V^{(2,1)}(z_{\bar 2}, z_2|\frac{z_{\bar 2}+z_2}{2}+y')$$
$$= -\frac{const}{2}\,e^{-m|a|}\,A(z_{\bar 1}, z_1|\frac{z_{\bar 1}+z_1}{2})A(z_{\bar 2}, z_2|\frac{z_{\bar 2}+z_2}{2}) \qquad (193)$$

where in this simple case the A amplitudes can be found from

$$A(z_{\bar 1}, z_1|\frac{z_{\bar 1}+z_1}{2}) \overset{a\to\infty}{=} \sqrt{\frac{V_{nl}^{(4)}(z_{\bar 1}, z_1, z_{\bar 1}+a, z_1+a)}{-\frac{const}{2}\,e^{-m|a|}}} \qquad (194)$$

The local structure of $V^{(2,1)}$ is at our choice. We define $V^{(2,1)}$ in the most local way.

$$V^{(2,1)}(z_{\bar 1}, z_1|x) = A(z_{\bar 1}, z_1|\frac{z_{\bar 1}+z_1}{2})\,\delta(x - \frac{z_{\bar 1}+z_1}{2}) \qquad (195)$$

Then $V^{(4,0)}$ is what is left of $V_{nl}^{(4)}$ after subtraction. This time there is no five point amplitude and we proceed immediately to the six point function. Equation (177) for this case is:

$$\int_z V_{nl}^{(6)}(z_{\bar 1},\ldots,z_3)\,\bar\Psi(z_{\bar 1})\ldots\Psi(z_3) = \int_z V^{(6,0)}(z_{\bar 1},\ldots,z_3)\,\bar\Psi(z_{\bar 1})\ldots\Psi(z_3)$$
$$-\int_{z,x} V^{(4,1)}(z_{\bar 1},\ldots,z_2|x_1)\,G_{x_1,x_2}\,V^{(2,1)}(z_{\bar 3}, z_3|x_2)\,\bar\Psi(z_{\bar 1})\ldots\Psi(z_3)$$
$$+\frac{1}{2}\int_{z,x,y} V^{(2,1)}(z_{\bar 1}, z_1|x_1)\,G_{x_1,x_2}\,V^{(2,1)}(z_{\bar 2}, z_2|y_1)\,G_{y_1,y_2}\,V^{(2,2)}(z_{\bar 3}, z_3|x_2,y_2)\,\bar\Psi(z_{\bar 1})\ldots\Psi(z_3)$$
$$-\frac{1}{6}\int_{z,x,y,v} V^{(2,1)}(z_{\bar 1}, z_1|x_1)\,G_{x_1,x_2}\,V^{(2,1)}(z_{\bar 2}, z_2|y_1)\,G_{y_1,y_2}\,V^{(2,1)}(z_{\bar 3}, z_3|v_1)\,G_{v_1,v_2}\,V^{(0,3)}(x_2,y_2,v_2)$$
$$\times\bar\Psi(z_{\bar 1})\ldots\Psi(z_3) \qquad (196)$$

We compare the integrands, where the right hand side is suitably symmetrized.

$$V_{nl}^{(6)}(z_{\bar 1},\ldots,z_3) = V^{(6,0)}(z_{\bar 1},\ldots,z_3) - \frac{1}{3}\sum_{i=1}^{3}\int_x V^{(4,1)}(\ldots|x_1)\,G_{x_1,x_2}\,V^{(2,1)}(z_{\bar i}, z_i|x_2)$$
$$+\frac{1}{6}\sum_{i=1}^{3}\int_{x,y} V^{(2,1)}(\ldots|x_1)\,G_{x_1,x_2}\,V^{(2,1)}(\ldots|y_1)\,G_{y_1,y_2}\,V^{(2,2)}(z_{\bar i}, z_i|x_2,y_2)$$
$$-\frac{1}{6}\int_{x,y,v} V^{(2,1)}(z_{\bar 1}, z_1|x_1)\,G_{x_1,x_2}V^{(2,1)}(z_{\bar 2}, z_2|y_1)\,G_{y_1,y_2}V^{(2,1)}(z_{\bar 3}, z_3|v_1)\,G_{v_1,v_2}V^{(0,3)}(x_2,y_2,v_2) \qquad (197)$$

We switch to momentum space and Fourier transform with respect to the variables $\frac{z_i+z_i}{2}$ suppressing the relative coordinates $z_i - z_i$ for the moment. Because of translational invariance of the amplitude we find conservation of overall momentum at the vertex, i.e. $\sum_{i=1}^3 p_i = 0$. With obvious notation the above equation reads

$$V_{nl}^{(6)}(p_1,p_2,-(p_1+p_2)) = V^{(6,0)}(p_1,p_2,-(p_1+p_2)) - \frac{1}{3}\sum_{i=1}^{3} V^{(4,1)}(\cdot,\cdot|p_i)G(p_i)V^{(2,1)}(p_i)$$
$$+\frac{1}{6}\sum_{i=1}^{3} V^{(2,1)}(\cdot)G(\cdot)V^{(2,1)}(\cdot)G(\cdot)V^{(2,2)}(p_i|\cdot,\cdot)$$
$$-\frac{1}{6}V^{(2,1)}(p_1)G(p_1)V^{(2,1)}(p_2)G(p_2)V^{(2,1)}(-(p_1+p_2))G(p_1+p_2)V^{(0,3)}(p_1,p_2,-(p_1+p_2)) \quad (198)$$

The dots in the arguments stand for the remaining momenta if the i-th momentum is fixed. The decomposition is immediately clear if we draw a picture (see figure 51).

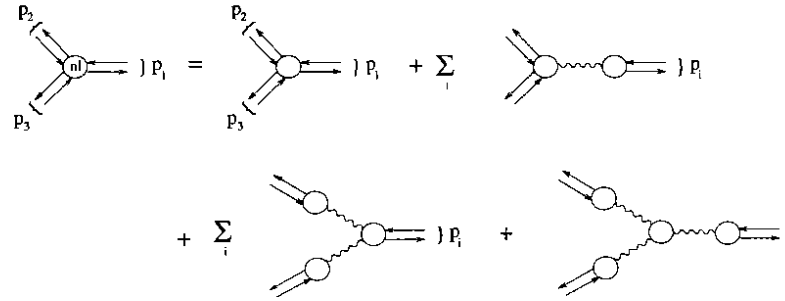


Figure 51

As promised here we have the pure bosonic interaction in the contribution containing three boson propagators. This contribution is singled out by putting all three momenta close to the pole value. Although one of the momenta p_i is fixed by momentum conservation it is possible to choose the directions of momenta such that all three momenta have their values close to the pole. Thereby we find $V^{(0,3)}$ with the already known ambiguities which we fix using the guiding principle to make the interaction as local as possible. Afterwards we release one momentum from its constraint and find $V^{(2,2)}$ and so on.

One may object that the ϕ^3 interaction in our toy model has to be zero because of the discrete chiral symmetry. This is true but nevertheless we have seen how such a pure bosonic interaction might appear. In addition this could be used as a check whether the method works. It should yield a zero ϕ^3 interaction.

By now it should be clear how to find the dressed quantities G, V. We return to the problem of determining the values of G_0, V_0.

6.4 Calculating the non loop corrected quantities G_0, V_0

In this section we study the problem of finding the quantities G_0 and V_0 if G and V are given. This done using the freedom to add any local contribution to the propagator as long as this is compensated by local changes in the interactions. A shift in the pole values of G compared to the pole values of G_0 is calculated.

The relations between G_0, V_0 and G, V are given in figure 47. These series have to be truncated in some way. As mentioned before we may use the freedom to determine the local parts in the interactions to choose G to be small. For values of Λ which lie a little below the compositeness scale Λ_c we may choose the propagator G to be

$$G = \frac{1}{p^2+\Lambda^{*2}-\varepsilon} - \frac{1}{p^2+\Lambda^{*2}+\varepsilon} = \frac{2\varepsilon}{(p^2+\Lambda^{*2}-\varepsilon)(p^2+\Lambda^{*2}+\varepsilon)}. \quad (199)$$

where $\Lambda^{*2} - \varepsilon = m^2$ and $\Lambda^{*2} + \varepsilon = \Lambda^2$. m is the mass of the composite boson field and Λ is the actual cut-off of the theory. G has therefore the suitable nonlocal behaviour and also a momentum cut-off. As long as $\Lambda_c - \Lambda$ is small, ε will also be small. The composite propagator has two poles. We call the pole which is at or above the actual cut-off ghost pole. It was introduced merely to cut off the high frequencies of the composite field. It changes only the local behaviour of the composite propagator.

Since G is of the order ε, we may assume G_0 to be of the order ε, too. Then we can truncate the series in figure 47 and use the approximate equations to find G_0 and V_0.

However there is a difficulty. G_0 will suffer a shift of its pole values compared to the poles in G. We will show that these shifts are of first order in ε. The relation given in figure 47 to third order in ε is

$$G = G_0 - G_0 tr_x\left(V_0^{(0,4)}G_0\right)G_0 \approx \frac{1}{G_0^{-1} + tr_x\left(V_0^{(0,4)}G_0\right)}. \quad (200)$$

Here $tr_x\left(V_0^{(0,4)}G_0\right)$ is an abbreviation for all the terms produced by contracting two legs of $V_0^{(0,4)}$ with the propagator G_0. Although these terms coming from $tr_x\left(V_0^{(0,4)}G_0\right)$ give contributions in the denominator which are suppressed by two orders in ε compared to G_0^{-1}, the resulting shift in the pole values of G compared to those of G_0 is of first order in ε. Hence these changes have to be taken into account.

$tr_x\left(V_0^{(0,4)}G_0\right)$ is assumed to be a function of p^2 which is proportional to ε. Let us therefore define $K(p^2)$ by the equation

$$tr_x\left(V^{(0,4)}G_0\right) = \varepsilon K(p^2) = \varepsilon K(-\Lambda^{*2}) + \varepsilon K'(-\Lambda^{*2})(p^2+\Lambda^{*2}) + \frac{\varepsilon}{2}K''(-\Lambda^{*2})(p^2+\Lambda^{*2})^2 + \ldots \quad (201)$$

We make an ansatz for G_0.

$$G_0 = \frac{2a\varepsilon}{(p^2+\Lambda^{*2}-c_1\varepsilon)(p^2+\Lambda^{*2}+c_2\varepsilon)} \quad (202)$$

where a, c_1 and c_2 are constants to be determined. If we insert this into equation (200) for G we find

$$G = \frac{2a\varepsilon}{(p^2+\Lambda^{*2}-c_1\varepsilon)(p^2+\Lambda^{*2}+c_2\varepsilon) + 2a\varepsilon^2 K(p^2)} \quad (203)$$

We expand the denominator around $p^2 = -\Lambda^{*2}$ calling $\tilde{p}^2 = (p^2+\Lambda^{*2})$.

$$\begin{aligned} G &= \frac{2a\varepsilon}{(\tilde{p}^2-c_1\varepsilon)(\tilde{p}^2+c_2\varepsilon) + 2a\varepsilon^2[K(-\Lambda^{*2}) + K'(-\Lambda^{*2})\tilde{p}^2 + \frac{1}{2}K''(-\Lambda^{*2})\tilde{p}^4]} \\ &= \frac{2a\varepsilon(1-a\varepsilon^2K'')}{\tilde{p}^4 + \tilde{p}^2(c_2\varepsilon - c_1\varepsilon + 2a\varepsilon^2K') - c_1c_2\varepsilon^2 + 2aK\varepsilon^2 + O(\varepsilon^3)} \end{aligned} \quad (204)$$

where we neglected higher orders in $\tilde{p}^2$. Near the poles of the denominator $\tilde{p}^2$ is assumed to be of the order ε so that we consistently expanded the denominator up to order ε^2. This has to be compared with

$$G = \frac{2\varepsilon}{\tilde{p}^4 - \varepsilon^2}. \quad (205)$$

We find $a = 1 + O(\varepsilon)$. The denominator is quadratic in $\tilde{p}^2$ and we calculate its zeros to be

$$\tilde{p}^2_{1,2} = \frac{\varepsilon}{2}\left[c_1 - c_2 \pm \sqrt{(c_1+c_2)^2 - 8K}\right] + O(\varepsilon^{\frac{3}{2}}) \tag{206}$$

The correction due to K' and K'' are of higher order in ε. Hence to the considered order in ε the shift in the zeros would be the same if the correction were $\varepsilon K(-m^2)$, i.e. a constant in $\tilde{p}^2$. Since the denominator without the correction has its zeros at $\tilde{p}^2 = \pm\varepsilon$, an addition of a constant term leaves the zeros symmetric with respect to $\tilde{p}^2 = 0$. Hence $c_1 = c_2 = c$ with

$$c = \sqrt{1 + 2K} \tag{207}$$

If $K(-\Lambda^{*2}) < -\frac{1}{2}$ then there exists no G_0 such that G has the required poles at $p^2 = -m^2$ and $p^2 = -\Lambda^2$. However $K(p^2)$ is expected to be a positive function. It is difficult to calculate the four composite field interaction even to leading order in $\frac{1}{N}$. If we set all external momenta to zero we can nevertheless state that the sign of the four composite field interaction is positive. Hence there is a chance that $K(-\Lambda^{*2})$ is also positive. To determine c we must calculate K which contains in turn G_0. Thus this is an implicit equation. The dependence of K on the locations of the poles in G_0 should be weak although this has to be checked in an actual calculation. Therefore K may be calculated approximately using the known propagator G. This yields an approximation to G_0 with which we can make a consistency check, i.e. calculating K again with G_0. If this check is passed it is easy to calculate the bare interactions to first order in ε. This is done using the second row of figure 47. We take only the first two terms on the right hand side into account. V_0 in the second term is changed into V which yields then an equation for V_0. Thus we can consistently calculate the bare quantities G_0 and V_0 from G and V at the compositeness scale to first order in ε.

6.5 Lowering the cut-off after introducing the composite fields

Here we comment on subsequent cut-off lowering steps in the presence of a composite field. The suggested procedure is a generalization of the provisional method discussed in section 4 including a probably necessary rescaling of the composite field.

We start with the effective action which has its cut-off exactly at the compositeness scale, i.e. $\Lambda = \Lambda_c$. It was constructed in section 6.2 and contains fermion fields and composite boson fields. We wish to lower the cut-off below Λ_c. We integrate out the high frequency modes of the fundamental field. If we do a small cut-off lowering this can be done to first order in the high frequency propagator.

We have seen in the toy model to leading order in $\frac{1}{N}$ (see section 5.5) that such a procedure needs a rescaling of the composite field. Therefore we must be prepared to do such a rescaling. The problem is to find the correct rescaling function. In the toy model this is easy. There we set the value of the Yukawa interaction to 1. The generalization would be to choose the rescaling function in such a way that the value of the $V^{(2,1)}_{\Lambda-\delta\Lambda}$ interaction is equal to the value of the $V^{(2,1)}_{\Lambda}$ interaction.

The second source of nonlocalities in section 5.5 are the links in the trees in figure 30. The procedure is to remove these nonlocalities by applying the above decomposition of nonlocal fundamental field amplitudes using an additional (auxiliary) composite field. The additional field has to be merged with the already existing composite field using the convolution theorem for gaussian measures. This requires an appropriate cancellation of the ghost (cut-off) pole in the already existing composite propagator against the mass pole of the additional (auxiliary) composite propagator. Instead a new ghost pole has to be introduced which is located at the new cut-off.

Without this cancellation of poles the propagator of the composite boson would have new poles below the actual cut-off of the theory. The composite propagator could then be written as a sum of propagators with masses below the cut-off but some of these propagators would have the wrong sign. A ghost propagator of this sort is forbidden in an effective theory. Hence the development of new poles below the cut-off has to avoided.

The consequence is that there is no freedom left in the choice of this additional (auxiliary) propagator once the rescaling function has been fixed. Thus the question arises whether the determination of the rescaling function by the above described procedure admits the cancellation of poles which is needed.

To clarify this question we consider the corrections to the composite propagator and the $V^{(2,1)}$ interaction to first order in the small high frequency propagator of the fundamental field. The correction to the composite field propagator from the contraction of the $V^{(2,2)}$ amplitude is given in figure 52

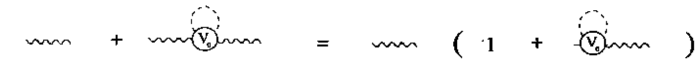


Figure 52

The correction to the $V^{(2,1)}$ involves the $V^{(4,1)}$ interaction.

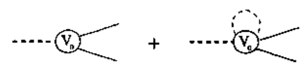


Figure 53

Thus there is in general no match between these corrections. We may assume that the long range behaviour of the correction to the $V^{(2,1)}$ interaction is like

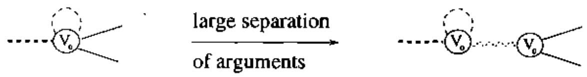


Figure 54

i.e. for large separations of the arguments the correction to $V^{(2,1)}$ factorizes. Hence to first order in the small high frequency propagator we find a similar factor as correction for the $V^{(2,1)}$ amplitude.

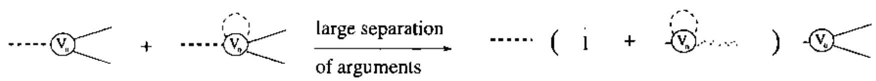


Figure 55

We have to assume in addition that the dashed wiggly propagator has a pole coinciding with the ghost pole in the solid wiggly propagator. In this case there is a chance that the cancellations between the poles can occur. Both of the conditions were satisfied in the toy model. We conjecture that it will be true more generally. This has to be checked when an actual calculation is to be done.

7 Other approaches

Here we will discuss other methods which exploit Wilson's renormalization group. We rederive flow equations and show the relation to the low energy effective action. Flow equations which contain composite fields are rederived as well.

We have seen that it is possible to derive the Polchinski renormalization group equation [19] by differentiating the low energy effective action with respect to the cut-off Λ. It was noticed by Keller, Kopper and Salmhofer [38] that a similar flow equation can be derived for W_Λ, the generating functional of connected Green functions for a theory with infrared cut-off Λ. The flow equation for W_Λ induces a similar flow equation for its Legendre transformed Γ_Λ, the generating functional of all 1PI Green functions with infrared cut-off Λ. A flow equation for the so called average action Γ_Λ was found by Wetterich [39] in a different context. The connection via a Legendre transformation to the Polchinski equation was noticed afterwards [41, 44]. We will rederive these flow equations later in a slightly more general formulation following the treatment of Morris [45].

The aim of this approach is to integrate these flow equations and to calculate from the solutions conveniently all properties of a low energy effective theory. It is possible to expand these flow equations into an infinite system of coupled integro-differential equations for the Green functions. To solve these equations requires some sort of approximation, for instance a large N expansion or the truncation of the system. These truncations are a delicate problem. Since the flow equations contain derivatives with respect to the cut-off Λ, the flow equations depend on the cut-off function. Although the complete flow equations are exact, the truncated versions are not and their solutions might depend very sensitively on the form of the actual cut-off function [45, 42]. A sharp momentum cut-off introduces nonlocalities into the effective action. In the limit of a Θ function cut-off the effective action becomes even ill defined because there are no inverse propagators. Therefore the choice of the cut-off function is crucial for the truncation of such an infinite system of coupled differential equations.

It is possible to incorporate composite degrees of freedom into this formulation and we will report on this also [42, 41, 43].

7.1 Derivation of flow equations

In this section we derive the flow equations for the low energy effective potential V_Λ, W_Λ and Γ_Λ and show the connections between these functionals. We choose a general formalism [45] which makes the comparison between the quantities very easy.

We will work in the context of a scalar field theory. The generalization to fermion fields is straightforward. The generating functional of Green functions for an action with large cut-off Λ' is defined as

$$Z[J] = \int d\mu_{u_{\Lambda'}}(\phi)\, e^{-V_{\Lambda'}(\phi)+J\phi} \tag{208}$$

where $d\mu_{u_{\Lambda'}}(\phi) = \mathcal{D}\phi\; e^{-\frac{1}{2}\phi u_{\Lambda'}^{-1}\phi}$ is a normalized gaussian measure. We perform a frequency split $u_{\Lambda'} = u_\Lambda + v_0$ which induces a field split $\phi = \Phi + \xi$.

$$\begin{aligned} Z[J] &= \int d\mu_{u_\Lambda}(\Phi) \int d\mu_{v_0}(\xi) e^{-V_{\Lambda'}(\Phi+\xi)+J(\Phi+\xi)} \\ &= \int d\mu_{u_\Lambda}(\Phi) \int \mathcal{D}\xi e^{-\frac{1}{2}\xi v_0^{-1}\xi} e^{-V_{\Lambda'}(\Phi+\xi)+J(\Phi+\xi)} \equiv \int d\mu_{u_\Lambda}(\Phi) e^{-W_\Lambda[\Phi|J]} \end{aligned} \tag{209}$$

By performing a shift in the integration variable ξ we can change the form of $W_\Lambda[\Phi|J]$. We do two different shifts to gain two expressions of $W_\Lambda[\Phi|J]$ for later convenience. One is $\xi \to (\xi + v_0 J)$ and the other is $\xi \to (\xi - \Phi)$

$$Z[J] = \int d\mu_{u_\Lambda}(\Phi) e^{\frac{1}{2}Jv_0J+J\Phi} \underbrace{\int d\mu_{v_0}(\xi) e^{-V_{\Lambda'}(\Phi+\xi+v_0J)}}_{\equiv e^{-V_\Lambda(\Phi+v_0J)}} \equiv \int d\mu_{u_\Lambda}(\Phi) e^{W_\Lambda[\Phi|J]} \tag{210}$$

$$= \int d\mu_{u_\Lambda}(\Phi) e^{-\frac{1}{2}\Phi v_0^{-1}\Phi} \int d\mu_{v_0}(\xi) e^{-V_{\Lambda'}(\xi)+\xi(J+v_0^{-1}\Phi)} \equiv \int d\mu_{u_\Lambda}(\Phi) e^{W_\Lambda[\Phi|J]} \tag{211}$$

This $W_\Lambda[\Phi|J]$ is equal to the low energy effective potential V_Λ for $J = 0$ and to a generating functional of connected Green functions with an infrared cut-off Λ for $\Phi = 0$.

$$J = 0 \quad \longrightarrow \quad W_\Lambda[\Phi|0] = -V_\Lambda(\Phi) \tag{212}$$

$$\Phi = 0 \quad \longrightarrow \quad e^{W_\Lambda[0|J]} = \int d\mu_{v_0}(\xi) e^{-V_{\Lambda'}(\xi)+J\xi} \equiv e^{W_\Lambda[J]} \tag{213}$$

There exists another relation between the low energy effective potential $V_\Lambda(\Phi)$ and the generating functional $W_\Lambda[J]$. This can be also derived immediately.

$$\begin{aligned} e^{-V_\Lambda(\Phi)} &= \int d\mu_{v_0}(\xi)\, e^{-V_{\Lambda'}(\Phi+\xi)} \\ &= \int \mathcal{D}\xi' e^{-\frac{1}{2}(\Phi v_0^{-1}\Phi)-\frac{1}{2}(\xi' v_0^{-1}\xi')-V_\Lambda(\xi')+(\xi' v_0^{-1}\Phi)} \\ &= e^{W_\Lambda[J]-\frac{1}{2}Jv_0J}\big|_{J=v_0^{-1}\Phi} \end{aligned} \tag{214}$$

By comparing the exponents we find

$$V_\Lambda(\Phi) = -W_\Lambda[J] + \frac{1}{2}Jv_0J\big|_{J=v_0^{-1}\Phi} \tag{215}$$

For completeness we derive quickly the Polchinski flow equation for V_Λ.

$$e^{-V_\Lambda(\Phi)} = \int d\mu_{v_0}(\xi)\, e^{-V_{\Lambda'}(\Phi+\xi)} = e^{\frac{1}{2}(\frac{\delta}{\delta\Phi}v_0\frac{\delta}{\delta\Phi})} e^{-V_{\Lambda'}(\Phi)} \tag{216}$$

By taking the derivative with respect to Λ we find immediately

$$\partial_\Lambda V_\Lambda(\Phi) = \frac{1}{2}(\frac{\delta}{\delta\Phi}(\partial_\Lambda v_0)\frac{\delta}{\delta\Phi})V_\Lambda(\Phi) - \frac{1}{2}(\frac{\delta V_\Lambda(\Phi)}{\delta\Phi}(\partial_\Lambda v_0)\frac{\delta V_\Lambda(\Phi)}{\delta\Phi}) \tag{217}$$

In this formulation we could replace $\partial_\Lambda v_0 = -\partial_\Lambda u_\Lambda$ because $u_{\Lambda'} = u_\Lambda + v_0$ is independent of Λ. It is possible to use this flow equation to derive a flow equation for $W_\Lambda[J]$. We choose the easier way to derive it directly from equation (209).

$$\begin{aligned} \partial_\Lambda e^{W_\Lambda[\Phi|J]} &= -\frac{1}{2}\int \mathcal{D}\xi\, (\xi(\partial_\Lambda v_0^{-1})\xi)\, e^{-\frac{1}{2}\xi v_0^{-1}\xi} e^{-V_{\Lambda'}(\Phi+\xi)+J(\Phi+\xi)} \\ &= -\frac{1}{2}\left((\frac{\delta}{\delta J}-\Phi)(\partial_\Lambda v_0^{-1})(\frac{\delta}{\delta J}-\Phi)\right) e^{W_\Lambda[\Phi|J]} \end{aligned} \tag{218}$$

We dropped a J independent term from the derivation of the Λ dependent normalization of the gaussian measure. From the above equation we derive immediately

$$\begin{aligned} \partial_\Lambda W_\Lambda[\Phi|J] &= -\frac{1}{2}\left((\underbrace{\frac{\delta W_\Lambda}{\delta J}}_{\equiv\varphi_c}-\Phi)(\partial_\Lambda v_0^{-1})(\underbrace{\frac{\delta W_\Lambda}{\delta J}}_{\equiv\varphi_c}-\Phi)\right) - \frac{1}{2}tr_x\left((\partial_\Lambda v_0^{-1})\frac{\delta^2 W_\Lambda}{\delta J\delta J}\right) \\ &= -\frac{1}{2}\left((\varphi_c-\Phi)(\partial_\Lambda v_0^{-1})(\varphi_c-\Phi)\right) - \frac{1}{2}tr_x\left((\partial_\Lambda v_0^{-1})\frac{\delta^2 W_\Lambda}{\delta J\delta J}\right) \end{aligned} \tag{219}$$

To derive from there a flow equation for $W_\Lambda[J]$ is trivial ($\Phi = 0$). We can perform a Legendre transformation on $W_\Lambda[\Phi|J]$.

$$W_\Lambda[\Phi|J] + \tilde{\Gamma}_\Lambda[\Phi|\varphi_c] = \varphi_c J \tag{220}$$

Φ is an external parameter in this equation and we have therefore

$$\frac{\delta W_\Lambda}{\delta\Phi} = -\frac{\delta\tilde{\Gamma}_\Lambda}{\delta\Phi}. \tag{221}$$

Using equation (211) we find easily the functional derivative

$$\frac{\delta W_\Lambda}{\delta\Phi} = -v_0^{-1}\Phi + v_0^{-1}\frac{\delta W_\Lambda}{\delta J} = v_0^{-1}(\varphi_c - \Phi) \tag{222}$$

This can be integrated to

$$\tilde{\Gamma}_\Lambda[\Phi|\varphi_c] = \frac{1}{2}(\varphi_c - \Phi)v_0^{-1}(\varphi_c - \Phi) + \Gamma_\Lambda[\varphi_c] \tag{223}$$

where $\Gamma_\Lambda[\varphi_c]$ is a part of the Φ independent constant of integration. The other constant of integration $\frac{1}{2}\varphi_c v_0^{-1}\varphi_c$ is written explicitly for later convenience. Observing the well known relation

$$\frac{\delta^2 W_\Lambda}{\delta J\delta J} = \left(\frac{\delta^2\tilde{\Gamma}_\Lambda}{\delta\varphi_c\delta\varphi_c}\right)^{-1} \tag{224}$$

and noting that Λ is also an external parameter in the Legendre transformation we find from equation (219)

$$\begin{aligned}\partial_\Lambda W_\Lambda[\Phi|J] &= -\partial_\Lambda \tilde{\Gamma}_\Lambda[\varphi_c] \overset{(223)}{=} -\frac{1}{2}\left((\varphi_c-\Phi)(\partial_\Lambda v_0^{-1})(\varphi_c-\Phi)\right) - \partial_\Lambda \Gamma_\Lambda[\Phi|\varphi_c] \\ &\overset{(219)}{=} -\frac{1}{2}\left((\varphi_c-\Phi)(\partial_\Lambda v_0^{-1})(\varphi_c-\Phi)\right) - \frac{1}{2} tr_x \left((\partial_\Lambda v_0^{-1}) \left\{ \frac{\delta^2 \tilde{\Gamma}_\Lambda}{\delta\varphi_c \delta\varphi_c} \right\}^{-1} \right) \end{aligned} \quad (225)$$

From there it is obvious that

$$\partial_\Lambda \Gamma_\Lambda[\varphi_c] = \frac{1}{2} tr_x \left((\partial_\Lambda v_0^{-1}) \left\{ \frac{\delta^2 \tilde{\Gamma}_\Lambda}{\delta\varphi_c \delta\varphi_c} \right\}^{-1} \right) = \frac{1}{2} tr_x \left((\partial_\Lambda v_0^{-1}) \left\{ v_0^{-1} + \frac{\delta^2 \Gamma_\Lambda}{\delta\varphi_c \delta\varphi_c} \right\}^{-1} \right) \quad (226)$$

These are the promised flow equations. These are very implicit equations and we will transform equation (226) to understand better which graphs are produced by this equation. We use the relation $\partial_\Lambda v_0^{-1} = -v_0^{-1}(\partial_\Lambda v_0) v_0^{-1}$

$$\partial_\Lambda \Gamma_\Lambda[\varphi_c] = -\frac{1}{2} tr_x \left(v_0^{-1}(\partial_\Lambda v_0) \left\{ 1 + \frac{\delta^2 \Gamma_\Lambda}{\delta\varphi_c \delta\varphi_c} v_0 \right\}^{-1} \right) = -\frac{1}{2} tr_x \left(v_0^{-1}(\partial_\Lambda v_0) \sum_{n=0}^{\infty} \left(-\frac{\delta^2 \Gamma_\Lambda}{\delta\varphi_c \delta\varphi_c} v_0 \right)^n \right) \quad (227)$$

If we denote Γ by a black dot this equation is depicted in figure 56.

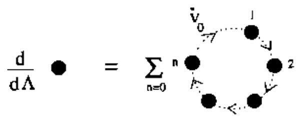


Figure 56

Since the equation is exact all 1PI diagrams are created by iterating this equation. In addition a truncation of this equation which keeps only $\Gamma^{(n)}$ with $n < 5$ still contains essential features. For instance the source of nonlocality in figure 20 are geometric series in fermion bubbles. These are still contained in the truncated version. Hence the flow equation for the average action is probably less sensitive to truncations than the Polchinski flow equation.

The low energy effective potential which appears in the Polchinski equation has a simpler interpretation as the interaction part of a low energy effective theory which is able to describe physics up to a certain momentum scale Λ. Γ_Λ is the generating functional of all 1PI Green functions with infrared cutoff. For very low momenta these 1PI Green functions with infrared cut-off Λ are essentially equal to the interactions in V_Λ. This is easy to understand in terms of diagrams. All 1PR diagrams in V_Λ are connected via v_0. Examples are given in figure 57.

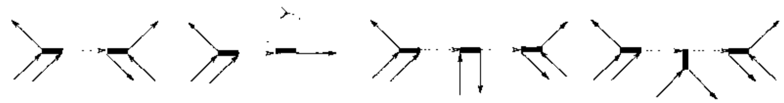


Figure 57

They are essentially zero for external momenta much smaller than Λ because v_0 has an infrared cut-off. Hence for this momentum range only the 1PI diagrams of V_Λ survive which are equal to those in Γ_Λ.

$W_\Lambda[\Phi|J]$ is the generating functional of connected Green functions of a theory with an infrared cut-off Λ and a background field Φ. Quite interesting is the connection between $W_\Lambda[J]$ and V_Λ given in (215). It shows how to find the low energy effective potential if $W_\Lambda[J]$ has been found, for example by integrating the flow equation. This relation is much simpler than constructing the low energy effective action from the 1PI Green functions.

7.2 Flow equations for theories containing composite fields

Here we will show how composite degrees of freedom are introduced into flow equations. The form of the resulting flow equations is similar to the one without composite fields. We make a comment that this may not be sufficient to guarantee locality of the corresponding effective action.

The composite degrees are essentially introduced as auxiliary fields. We derive the flow equations given in [43]. Since we use the auxiliary field method for the derivation, we can easily compare this approach with our provisional outline. This reveals deficiencies concerning locality which are even worse than in our provisional outline. The Green functions produced by these flow equations can become nonlocal for sufficiently low cut-off Λ. We work within the context of a pure bosonic theory to keep the notation simple. The argument does not depend on the choice of fields however. Flow equations for fermion and composite boson fields are given in the appendix of [43].

We start with a pure bosonic action $S[\phi]$ with no composite fields yet. We assume that the interaction is such that bound states form as we lower the cut-off Λ. Composite fields can be introduced by inserting a constant into the path integral.

$$const = \int \mathcal{D}\Delta \; e^{(\Delta-\phi\phi\lambda)\lambda^{-1}(\Delta-\phi\phi\lambda)}. \quad (228)$$

This will cancel the ϕ^4 interaction in the original interaction and introduce an auxiliary field Δ instead. For the moment we want to neglect problems like the convergence of the above gaussian integral. For fermion fields this problem does not exist because the sign of a four fermion interaction is just opposite to the sign of the ϕ^4 interaction.

We introduce currents for the fields ϕ and Δ and find the generating functional to be

$$\begin{aligned} Z[J,K] &= \int d\mu_{u_{\Lambda'}}(\phi)\mathcal{D}\Delta \; e^{\Delta\frac{1}{\lambda}\Delta - V(\phi) - 2\Delta\phi\phi + \phi\phi\lambda\phi\phi + J\phi + K\Delta} \\ &= \int d\mu_{u_{\Lambda'}}(\phi) \; e^{-V(\phi)+J\phi-\frac{1}{4}K\lambda K + K\lambda\phi\phi} \underbrace{\int \mathcal{D}\Delta \; e^{(\Delta+\frac{1}{2}K\lambda-\phi\phi\lambda)\lambda^{-1}(\Delta+\frac{1}{2}\lambda K-\lambda\phi\phi)}}_{=const} \end{aligned} \quad (229)$$

Integrating out the Δ field again does not change the generating functional

$$Z[J,K] = \int d\mu_{u_{\Lambda'}}(\phi) \; e^{-V(\phi)-\frac{1}{4}K\lambda K + K\lambda\phi\phi + J\phi}. \quad (230)$$

Hence the addition of $\frac{1}{4}K\lambda K$ changes connected Green functions for composite operators $\lambda\phi\phi$ into connected Green functions for auxiliary fields Δ. The only difference is in the two point function for the Δ field which has an additional constant $\frac{\lambda}{2}$.

The Δ propagator has no cut-off and in addition we have eliminated the ϕ^4 interaction completely. Both features are not wanted for a composite field. Instead of the auxiliary field Δ we better introduce immediately the composite field χ. This is done at the compositeness scale Λ_c with a propagator G_{Λ_c} which is equal to the nonlocal part of the ϕ^4 coupling at the compositeness scale. Hence at the compositeness scale the generating functional reads

$$\begin{aligned} Z[J,K] &= \int d\mu_{u_{\Lambda'}}(\phi)\mathcal{D}\chi \; e^{\chi G_{\Lambda_c}^{-1}\chi - V_{\Lambda_c}(\phi) - 2\chi\phi\phi + \phi\phi G_{\Lambda_c}\phi\phi + J\phi + K\chi} \\ &= \int d\mu_{u_{\Lambda'}}(\phi) \; e^{-V_{\Lambda_c}(\phi)+J\phi-\frac{1}{4}KG_{\Lambda_c}K + KG_{\Lambda_c}\phi\phi} \underbrace{\int \mathcal{D}\chi \; e^{(\chi+\frac{1}{2}KG_{\Lambda_c}-\phi\phi G_{\Lambda_c})G_{\Lambda_c}^{-1}(\chi+\frac{1}{2}G_{\Lambda_c}K-G_{\Lambda_c}\phi\phi)}}_{=const} \end{aligned} \quad (231)$$

Formally this constant term can be present in the generating functional also above the compositeness scale. This can be done by multiplying the term $G_{\Lambda_c}\phi\phi$ in the gaussian with a cut-off function which switches it on only at and below the compositeness scale. This suppresses an interaction between χ and ϕ until the compositeness scale. The χ field has no Λ dependence until below the compositeness scale and does not influence the flow of the ϕ couplings. This is cosmetic since one has to know the compositeness scale to define the right cut-off function. The flow equations until the compositeness scale are nevertheless simply the flow equations given in the previous section.

When lowering the cut-off further we have to do in general propagator splits for both fields in the action. By integrating the high frequencies of the ϕ field in the gaussian part new ϕ interactions develop. These will cancel appropriate parts in $V_\Lambda(\phi)$ and in this way certain nonlocal interactions in $V_\Lambda(\phi)$ are replaced by χ-ϕ interactions. Thus the generating functional below the compositeness scale reads

$$\begin{aligned} Z[J,K] &= \int d\mu_{u_\Lambda}(\Phi) d\mu_{G_L}(\chi) \int d\mu_{v_0}(\xi) \int d\mu_{G_H}(\Delta)\, e^{-V_{\Lambda_c}(\Phi+\xi|\chi+\Delta)+J(\Phi+\xi)+K(\chi+\Delta)} \\ &\equiv \int d\mu_{u_\Lambda}(\Phi) d\mu_{G_L}(\chi)\, e^{W_\Lambda[J,K|\Phi,\chi]} \end{aligned} \tag{232}$$

where

$$u_{\Lambda_c} = u_\Lambda + v_0 \quad ; \quad G_{\Lambda_c} = 2G_L + 2G_H \tag{233}$$

and

$$V_{\Lambda_c}(\phi|\chi) = V_{\Lambda_c}(\phi) - \phi\phi G_{\Lambda_c}\phi\phi - 2\chi\phi\phi \tag{234}$$

χ and Δ are the low and high frequency composite fields respectively. This defines a Λ dependent generating functional. The proceeding is now analogous to the one in the previous section. We take the derivative with respect to Λ. The result is the flow equation for $W_\Lambda[J,K|\Phi,\chi]$.

$$\begin{aligned} \partial_\Lambda W_\Lambda[J,K|\Phi,\chi] &= -\frac{1}{2}\left((\underbrace{\frac{\delta W_\Lambda}{\delta J}}_{\equiv\varphi_c} - \Phi)(\partial_\Lambda v_0^{-1})(\underbrace{\frac{\delta W_\Lambda}{\delta J}}_{\equiv\varphi_c} - \Phi) \right) - \frac{1}{2} tr_x \left((\partial_\Lambda v_0^{-1}) \frac{\delta^2 W_\Lambda}{\delta J \delta J} \right) \\ &\quad -\frac{1}{2}\left((\underbrace{\frac{\delta W_\Lambda}{\delta K}}_{\equiv\sigma_c} - \chi)(\partial_\Lambda G_H^{-1})(\underbrace{\frac{\delta W_\Lambda}{\delta K}}_{\equiv\sigma_c} - \chi) \right) - \frac{1}{2} tr_x \left((\partial_\Lambda G_H^{-1}) \frac{\delta^2 W_\Lambda}{\delta K \delta K} \right) \\ &= -\frac{1}{2}\left((\varphi_c - \Phi)(\partial_\Lambda v_0^{-1})(\varphi_c - \Phi) \right) - \frac{1}{2} tr_x \left((\partial_\Lambda v_0^{-1}) \frac{\delta^2 W_\Lambda}{\delta J \delta J} \right) \\ &\quad -\frac{1}{2}\left((\sigma_c - \chi)(\partial_\Lambda G_H^{-1})(\sigma_c - \chi) \right) - \frac{1}{2} tr_x \left((\partial_\Lambda G_H^{-1}) \frac{\delta^2 W_\Lambda}{\delta K \delta K} \right) \end{aligned} \tag{235}$$

We can define now a block diagonal propagator $\mathcal{G} = \begin{pmatrix} v_0 & 0 \\ 0 & G_H \end{pmatrix}$ and an appropriate current $\mathcal{J} = \begin{pmatrix} J \\ K \end{pmatrix}$ together with fields $\Xi = \begin{pmatrix} \xi \\ \Delta \end{pmatrix}$ and $\Theta = \begin{pmatrix} \Phi \\ \chi \end{pmatrix}$. This allows to write the flow equations as

$$\begin{aligned} \partial_\Lambda W_\Lambda[\mathcal{J}|\Theta] &= -\frac{1}{2}\left((\underbrace{\frac{\delta W_\Lambda}{\delta \mathcal{J}}}_{\equiv\Sigma_c} - \Theta)(\partial_\Lambda \mathcal{G}^{-1})(\underbrace{\frac{\delta W_\Lambda}{\delta \mathcal{J}}}_{\equiv\Sigma_c} - \Theta) \right) - \frac{1}{2} tr_x \left((\partial_\Lambda \mathcal{G}^{-1}) \frac{\delta^2 W_\Lambda}{\delta \mathcal{J} \delta \mathcal{J}} \right) \\ &= -\frac{1}{2}\left((\Sigma_c - \Theta)(\partial_\Lambda \mathcal{G}^{-1})(\Sigma_c - \Theta) \right) - \frac{1}{2} tr_x \left((\partial_\Lambda \mathcal{G}^{-1}) \frac{\delta^2 W_\Lambda}{\delta \mathcal{J} \delta \mathcal{J}} \right) \end{aligned} \tag{236}$$

The Legendre transformation is easily performed.

$$\tilde{\Gamma}_\Lambda[\Sigma_c|\Theta] + W_\Lambda[\mathcal{J}|\Theta] = \mathcal{J}\Sigma_c \tag{237}$$

The derivative of $\tilde{\Gamma}_\Lambda$ with respect to Θ is known because

$$\frac{\delta \tilde{\Gamma}_\Lambda}{\delta \Theta} = -\frac{\delta W_\Lambda}{\delta \Theta} = \mathcal{G}^{-1}(\Theta - \Sigma_c) \tag{238}$$

This can be integrated with the result

$$\tilde{\Gamma}_\Lambda = \frac{1}{2}(\Theta - \Sigma_c)\mathcal{G}^{-1}(\Theta - \Sigma_c) + \Gamma_\Lambda[\Sigma_c] \tag{239}$$

The flow equation for Γ can be found by using the relation

$$\frac{\delta^2 W}{\delta \mathcal{J} \delta \mathcal{J}} = \left(\frac{\delta^2 \tilde{\Gamma}}{\delta \Sigma_c \delta \Sigma_c} \right)^{-1} \tag{240}$$

$$\partial_\Lambda \Gamma_\Lambda[\Sigma_c] = \frac{1}{2} tr_x \left((\partial_\Lambda \mathcal{G}^{-1}) \left\{ \frac{\delta^2 \tilde{\Gamma}_\Lambda}{\delta \Sigma_c \delta \Sigma_c} \right\}^{-1} \right) = \frac{1}{2} tr_x \left((\partial_\Lambda \mathcal{G}^{-1}) \left\{ \mathcal{G}^{-1} + \frac{\delta^2 \Gamma_\Lambda}{\delta \Sigma_c \delta \Sigma_c} \right\}^{-1} \right) \tag{241}$$

This together with equation (226) are essentially the flow equation given in [43]. These equations are exact. Concerning the flow equation for composite fields we could ask in addition whether the equation describes the flow of a local average action. The above approach is equivalent to introducing a composite field at the compositeness scale. Afterwards both fields are treated as if they were fundamental. In the context of an $\frac{1}{N}$ expansion we have shown that such a treatment may fail to keep the action local for scales below Λ_c. To remove the appearing nonlocal terms which arise from the high frequency fundamental field integration requires an integration with a negative propagator which is forbidden. If we would do this anyway the resulting interactions from this composite field integration should cancel the nonlocalities. In the framework of flow equations this would mean that the derivative ∂G_H depends on ∂v_0 and has a different sign compared to fundamental fields.

Hence we argue that this form the flow equations need at least some additional specification concerning the relation between ∂v_0 and ∂G_H.

This difficulty is hard to perceive in truncated versions of the flow equation where the remaining four fermion amplitude is neglected. This remaining four fermion amplitude creates the new nonlocalities. This flow equation is quite deceptive because the composite field propagator does change due to the integration of the high frequency fundamental field. The appropriate diagram which is responsible for this change is given in figure 58.

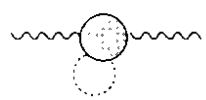


Figure 58

In the flow equation this diagram is hidden in the relation between the composite and the fundamental fields through the inversion of the matrix $\frac{\delta^2 \Gamma_\Lambda}{\delta \Sigma_c \delta \Sigma_c}$. Thus it seems that the effects of the high frequency fundamental fields are taken into account.

However, to incorporate the demand for locality of the low energy effective action into the flow equations requires at least some additional constraints. Nonlocal parts of the interactions are continuously absorbed into interactions mediated by composite fields. The resulting changes of the composite fields determine the flow of the composite fields. It depends on the flow of the fundamental fields. An example of a local flow equation has been given in equation (129).

This consideration will be especially important in prospective applications to gauge theories. New four fermion interactions (and higher ones) will be continuously produced through the integration of high frequency gauge fields. After Fierz transformation (see section 3) they will very likely increase the appearing nonlocalities. Hence they have an impact on the formation of the composite boson fields. In these circumstances it is no longer legitimate to imagine that the boson field has been introduced once and for all by bosonization of the original theory. But this is implicitly assumed in the flow equations approach [43].

Thus at the present time flow equations are not in a suitable shape to calculate low energy effective actions which could serve as a starting point for numerical simulations.

8 Conclusions

In this work we presented a method to calculate low energy effective actions containing composite fields. This method works in coordinate space. Hence the resulting low energy effective actions are useful for subsequent Monte Carlo simulations when a lattice cut-off is used.

We found that the interactions produced by lowering the cut-off develop exponential tails. Eventually the rate of decay becomes smaller than the cut-off, so that the action S_Λ becomes nonlocal. This sets the compositeness scale. We have given a systematical method for reformulating this nonlocal interactions in terms of composite fields. The nonlocal fundamental field interactions can be decomposed into interactions which are mediated by composite fields and remaining local interactions. The quantum numbers in the nonlocal channels of fundamental field interactions give the quantum numbers of the composite fields. In this way it is possible to construct a local effective theory which now contains composite fields.

The composite field is first introduced at the compositeness scale. Nevertheless it is not sufficient to *bosonize* only at the compositeness scale once and for all. We have argued that nonlocal term can appear again as the cut-off is lowered further. This requires a *bosonization* step for each renormalization group step. This is done by introducing an auxiliary composite field which is then merged with the already existing composite field. Thereby we lower the mass and the cut-off of the composite field. We have shown in the framework of an $\frac{1}{N}$ expansion that this is already sufficient to eliminate the high frequency modes of the composite propagator. Hence an integration over the high frequency modes of the composite field is not necessary.

Thus the composite field is treated different in the present real space renormalization group compared to a fundamental field. The flow of the composite field propagator is determined by the corrections from the high frequency fundamental field integration and the absorption of nonlocal interactions into those mediated by the composite field.

Flow equations for theories containing composite fields must take into account this feature if locality of the resulting low energy effective action is demanded. In the leading order $\frac{1}{N}$ expansion we have presented a flow equation which keeps an effective action local to leading order in $\frac{1}{N}$.

The concept of locality has been proven to be useful in selecting effective actions. On the other hand the concept does not determine the blockspin field completely. The remaining freedom has been used to introduce the composite fields smoothly into the effective action. In this way we avoid technical problems with nonconverging series. The remaining freedom concerns only the local behaviour of the terms in the action and therefore the low energy predictions of the theory are not touched.

9 Outlook

An application of this method to a numerical simulation or to a multigrid algorithm would require finite cut-off lowering steps instead of the very small cut-off lowerings we have studied here. As a consequence it will probably no longer be sufficient to do the expansions only to first order in the cut-off lowering. In this work this would mean that we have to include higher orders in v_0 in the calculations. A finite cut-off lowering step corresponds to an integration of flow equations over a small but finite range. This can be used to check how many orders of v_0 should be taken into account. This is work in progress [11]

Another problem is of course that the toy model is only a crude approximation of the situation in gauge theories. An important feature, namely the continuous addition of a local four fermion interaction which arises from an integration of high frequency gluons, has been left out in the study of the toy model. It can be expected that this accelerates the production of composite bosons.

The inclusion of these terms from the high frequency gluon integration requires of course that this integration can be done. To integrate out the high frequency gluons in a gauge invariant manner is a highly nontrivial problem. A solution for lattice regularized gauge theory has been given by Bałaban [3]. The procedure is applicable to the nonabelian case in principle, but it needs to be made practical. This is also future work.

Another problem is to control the improvement one has made with an approximation of a perfect action. The extrapolation to the continuum results requires a careful investigation of the remaining influence of the regulator or the neglected infrared irrelevant interactions.

Thus perfect actions for nonabelian gauge theories are still beyond reach. Control over results from simulations even further. However efforts in this direction seem worthwhile as the very encouraging example of Hasenfratz [15] has shown.

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A Calculation of integrals I_n

In this appendix we want to solve certain integrals, which are used in the $\frac{1}{N}$ expansion.

$$\int \frac{d^n l}{(2\pi)^n}\, tr_\gamma(v(l+p)w(l)) = I_n(p^2, \Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3) \tag{242}$$

for $n = 2, 4$. Using the explicit representation of the propagators v, w

$$\begin{aligned} v(p) &= \frac{i\not p}{p^2}\{e^{-\frac{p^2}{\Lambda_0^2}} - e^{-\frac{p^2}{\Lambda_1^2}}\} = \frac{i\not p}{p^2}\{e^{-c_0p^2} - e^{-c_1p^2}\} = -i\not p \int_{c_1}^{c_0} da\, e^{-ap^2} \\ w(p) &= \frac{i\not p}{p^2}\{e^{-\frac{p^2}{\Lambda_2^2}} - e^{-\frac{p^2}{\Lambda_3^2}}\} = -i\not p \int_{c_3}^{c_2} db\, e^{-bp^2} \end{aligned} \tag{243}$$

where $\Lambda_0 > \Lambda_1$ and $\Lambda_2 > \Lambda_3$ the integral can be written as follows

$$\begin{aligned} \int \frac{d^n l}{(2\pi)^n}\, tr_\gamma(v(l+p)w(l)) &= -n \int \frac{d^n l}{(2\pi)^n}\, l\cdot(l+p) \int_{c_1}^{c_0}\int_{c_3}^{c_2} e^{-a(l+p)^2 - bl^2}\, da\, db \\ &= \frac{n}{2}\int_{c_1}^{c_0}\int_{c_3}^{c_2}\int \frac{d^n l}{(2\pi)^n}\{\frac{d}{da} + \frac{d}{db} + p^2\}\, e^{-(a+b)l^2 - 2ap\cdot l - ap^2}\, da\, db \\ &= \frac{n}{2}\int_{c_1}^{c_0}\int_{c_3}^{c_2}\{\frac{d}{da} + \frac{d}{db} + p^2\}\frac{\pi^{\frac{n}{2}}}{(2\pi)^n}\frac{1}{(a+b)^{\frac{n}{2}}}\, e^{-\frac{ab}{a+b}p^2}\, da\, db \\ &= \frac{n}{2}\frac{\pi^{\frac{n}{2}}}{(2\pi)^n}\int_{c_1}^{c_0}\left\{\frac{1}{(a+c_2)^{\frac{n}{2}}}\, e^{-\frac{ac_2}{a+c_2}p^2} - \frac{1}{(a+c_3)^{\frac{n}{2}}}\, e^{-\frac{ac_3}{a+c_3}p^2}\right\} da \\ &+ \frac{n}{2}\frac{\pi^{\frac{n}{2}}}{(2\pi)^n}\int_{c_3}^{c_2}\left\{\frac{1}{(b+c_0)^{\frac{n}{2}}}\, e^{-\frac{bc_0}{b+c_0}p^2} - \frac{1}{(b+c_1)^{\frac{n}{2}}}\, e^{-\frac{bc_1}{b+c_1}p^2}\right\} db \\ &+ \frac{n}{2}\frac{\pi^{\frac{n}{2}}}{(2\pi)^n}p^2\int_{c_1}^{c_0}\int_{c_3}^{c_2}\frac{1}{(a+b)^{\frac{n}{2}}}\, e^{-\frac{ab}{a+b}p^2}\, da\, db \end{aligned} \tag{244}$$

We calculate the case $n = 2$ first. The first integral we have to solve is

$$\begin{aligned} \int_{c_1}^{c_0}\frac{1}{(a+b)}\, e^{-\frac{ab}{a+b}p^2}\, da &= -\int_{\frac{1}{c_1+b}}^{\frac{1}{c_0+b}}\frac{1}{u^2}\, u\, e^{-(1-bu)bp^2}\, du \\ &= -e^{-bp^2}\int_{\frac{1}{c_1+b}}^{\frac{1}{c_0+b}}\frac{1}{u}\, e^{ub^2p^2}\, du \\ &= e^{-bp^2}\left(Ei(\frac{b^2}{c_1+b}p^2) - Ei(\frac{b^2}{c_0+b}p^2)\right) \end{aligned} \tag{245}$$

$Ei(x)$ is the exponential integral. It is defined as [27]

$$\begin{aligned} Ei(x) &= \int_{-\infty}^{x}\frac{e^t}{t}dt \qquad \text{for} \qquad x < 0 \\ Ei(x) &= \lim_{\epsilon\to+0}\left[\int_{-\infty}^{-\epsilon}\frac{e^t}{t}dt + \int_{\epsilon}^{x}\frac{e^t}{t}dt\right] \qquad \text{for} \qquad x > 0 \end{aligned} \tag{246}$$

It has the general expansion

$$Ei(x) = \gamma + \log(|x|) + \sum_{k=1}^{\infty}\frac{x^k}{k\cdot k!} \tag{247}$$

where $\gamma = 0.5772156649$ is Eulers constant. For large $|x|$ Ei has the expansion [25]

$$Ei(x) = \frac{e^x}{x}\left[\sum_{k=0}^{n}\frac{k!}{x^k} + r_n(x)\right] \qquad \text{where} \qquad |r_n(x)| \leq \frac{(n+1)!}{|x|^{n+1}} \tag{248}$$

We still have to do the b integration. First we do a partial integration

$$\int_{c_3}^{c_2} e^{-bp^2}\, Ei(\frac{b^2}{c+b}p^2)\, db \;=\; -\frac{1}{p^2}\left(e^{-c_2p^2} Ei(\frac{c_2^2}{c+c_2}p^2) - e^{-c_3p^2} Ei(\frac{c_3^2}{c+c_3}p^2)\right) + \frac{1}{p^2}\int_{c_3}^{c_2} e^{-bp^2}\frac{d}{db} Ei(\frac{b^2}{c+b}p^2)\, db \tag{249}$$

The remaining integral can be calculated easily.

$$\begin{aligned}\frac{1}{p^2}\int_{c_3}^{c_2} e^{-bp^2}\frac{d}{db}Ei(\frac{b^2}{c+b}p^2)\, db &= \frac{1}{p^2}\int_{c_3}^{c_2} e^{-\frac{cb}{c+b}p^2}\,\frac{1}{b}\,\frac{b+2c}{b+c}\, db\\ &= -\frac{1}{p^2}e^{-cp^2}\int_{\frac{1}{c+c_3}}^{\frac{1}{c+c_2}} e^{uc^2p^2}\,\frac{1}{u}\,\frac{1+cu}{1-cu}\, du\\ &= +\frac{1}{p^2}e^{-cp^2}\{Ei(\frac{c^2}{c_3+c}p^2) - Ei(\frac{c^2}{c_2+c}p^2)\}\\ &\quad +\frac{2}{p^2}\{Ei(-\frac{c_2c}{c_2+c}p^2) - Ei(-\frac{c_3c}{c_3+c}p^2)\}\end{aligned} \tag{250}$$

Collecting all pieces the double integral in equation (244) becomes

$$\begin{aligned}\int_{c_1}^{c_0}\int_{c_3}^{c_2}\frac{1}{(a+b)}\, e^{-\frac{ab}{a+b}p^2}\, da\, db \;=\; &-\frac{1}{p^2}\left(e^{-c_2p^2}Ei(\frac{c_2^2}{c_1+c_2}p^2) - e^{-c_3p^2}Ei(\frac{c_3^2}{c_1+c_3}p^2)\right)\\ &+\frac{1}{p^2}e^{-c_1p^2}\left(Ei(\frac{c_1^2}{c_3+c_1}p^2) - Ei(\frac{c_1^2}{c_2+c_1}p^2)\right)\\ &+\frac{2}{p^2}\left(Ei(-\frac{c_2c_1}{c_2+c_1}p^2) - Ei(-\frac{c_3c_1}{c_3+c_1}p^2)\right)\\ &+\frac{1}{p^2}\left(e^{-c_2p^2}Ei(\frac{c_2^2}{c_0+c_2}p^2) - e^{-c_3p^2}Ei(\frac{c_3^2}{c_0+c_3}p^2)\right)\\ &-\frac{1}{p^2}e^{-c_0p^2}\left(Ei(\frac{c_0^2}{c_3+c_0}p^2) - Ei(\frac{c_0^2}{c_2+c_0}p^2)\right)\\ &-\frac{2}{p^2}\left(Ei(-\frac{c_2c_0}{c_2+c_0}p^2) - Ei(-\frac{c_3c_0}{c_3+c_0}p^2)\right).\end{aligned} \tag{251}$$

Finally we get for the integral after a lot of cancellations

$$\int\frac{d^2l}{(2\pi)^2}\, tr_\gamma(v(l+p)w(l)) = \frac{1}{2\pi}\left\{Ei(-\frac{c_0c_3}{c_0+c_3}p^2) - Ei(-\frac{c_1c_3}{c_1+c_3}p^2) - Ei(-\frac{c_0c_2}{c_0+c_2}p^2) + Ei(-\frac{c_1c_2}{c_1+c_2}p^2)\right\} \tag{252}$$

For the special case $v = w$ there are even more cancellations.

$$\int\frac{d^2l}{(2\pi)^2}\, tr_\gamma(v(l+p)v(l)) = \frac{1}{2\pi}\{2Ei(-\frac{c_0c_1}{c_0+c_1}p^2) - Ei(-\frac{c_0}{2}p^2) - Ei(-\frac{c_1}{2}p^2)\} \tag{253}$$

We give the values of these integrals for $p = 0$.

$$\int\frac{d^2l}{(2\pi)^2}\, tr_\gamma(v(l)w(l)) = \frac{1}{2\pi}\ln\left[\frac{(c_1+c_3)(c_0+c_2)}{(c_0+c_3)(c_1+c_2)}\right] \tag{254}$$

which reduces for the special case $v = w$ to:

$$\int\frac{d^2l}{(2\pi)^2}\, tr_\gamma(v(l)v(l)) = \frac{1}{2\pi}\ln[\frac{4c_0c_1}{(c_0+c_1)^2}]. \tag{255}$$

We can ask for the limit of small cut-off lowering $\Lambda_0 - \Lambda_1 = \delta\Lambda \ll \Lambda$

$$\lim_{\delta\Lambda\to 0}\int\frac{d^2l}{(2\pi)^2}\, tr_\gamma(v(l)v(l)) = -\frac{1}{2\pi}\frac{\delta\Lambda^2}{\Lambda^2} \tag{256}$$

Another approximation needed is the following. Take $\Lambda_0 = \Lambda' > \Lambda_2 = \Lambda + \delta\Lambda > \Lambda_1 = \Lambda_3 = \Lambda$. This is the approximation where only the propagator w is small and the propagator v is finite.

$$\int\frac{d^2l}{(2\pi)^2}\, tr_\gamma(v(l+p)w(l)) = \frac{1}{2\pi}\left\{Ei(-\frac{-p^2}{\Lambda'^2+\Lambda^2}) - Ei(-\frac{-p^2}{2\Lambda^2}) - Ei(-\frac{-p^2}{\Lambda'^2+(\Lambda+\delta\Lambda)^2}) + Ei(-\frac{-p^2}{\Lambda+(\Lambda+\delta\Lambda)^2})\right\} \tag{257}$$

We expand this result to first order in $\delta\Lambda$.

$$\int\frac{d^2l}{(2\pi)^2}\, tr_\gamma(v(l+p)w(l)) = \frac{\delta\Lambda}{2\pi}\left\{\frac{e^{-\frac{p^2}{\Lambda'^2+\Lambda^2}}}{\Lambda'^2+\Lambda^2} - \frac{e^{-\frac{p^2}{2\Lambda^2}}}{2\Lambda^2}\right\}2\Lambda \tag{258}$$

This shows that the integral is proportional to $\delta\Lambda$ if it contains only one small propagator. Now for the case $n = 4$. From equation (244) we find

$$\begin{aligned}\int\frac{d^4l}{(2\pi)^4}\, tr_\gamma(v(l+p)w(l)) \;=\; &\frac{1}{8\pi^2}\int_{c_3}^{c_2}\left\{\frac{1}{(b+c_0)^2}\, e^{-\frac{bc_0}{b+c_0}p^2} - \frac{1}{(b+c_1)^2}\, e^{-\frac{bc_1}{b+c_1}p^2}\right\} db\\ &\frac{1}{8\pi^2}\int_{c_1}^{c_0}\left\{\frac{1}{(a+c_2)^2}\, e^{-\frac{ac_2}{a+c_2}p^2} - \frac{1}{(a+c_3)^2}\, e^{-\frac{ac_3}{a+c_3}p^2}\right\} da\\ &+\frac{p^2}{8\pi^2}\int_{c_1}^{c_0}\int_{c_1}^{c_0}\frac{1}{(a+b)^2}\, e^{-\frac{ab}{a+b}p^2}\, da\, db.\end{aligned} \tag{259}$$

And because

$$\frac{d}{db}\left(\frac{ab}{a+b}\right) = \frac{a^2}{(a+b)^2} \tag{260}$$

we can write the integral as follows

$$\begin{aligned}\int\frac{d^4l}{(2\pi)^4}\, tr_\gamma(v(l+p)w(l)) \;=\; &-\frac{1}{8\pi^2}\int_{c_3}^{c_2}\left\{\frac{1}{c_0^2p^2}\frac{d}{db}e^{-\frac{bc_0}{b+c_0}p^2} - \frac{1}{c_1^2p^2}\frac{d}{db}e^{-\frac{bc_1}{b+c_1}p^2}\right\} db\\ &-\frac{1}{8\pi^2}\int_{c_1}^{c_0}\left\{\frac{1}{c_2^2p^2}\frac{d}{da}e^{-\frac{ac_2}{a+c_2}p^2} - \frac{1}{c_3^2p^2}\frac{d}{db}e^{-\frac{ac_3}{a+c_3}p^2}\right\} da\\ &-\frac{1}{8\pi^2}\int_{c_1}^{c_0}\int_{c_3}^{c_2}\frac{1}{a^2}\frac{d}{db}e^{-\frac{ab}{a+b}p^2}\, da\, db\\ =\; &-\frac{1}{8\pi^2}\left\{\frac{1}{c_0^2p^2}\left(e^{-\frac{c_2c_0}{c_2+c_0}p^2} - e^{-\frac{c_0c_3}{c_0+c_3}p^2}\right) - \frac{1}{c_1^2p^2}\left(e^{-\frac{c_2c_1}{c_2+c_1}p^2} - e^{-\frac{c_3c_1}{c_3+c_1}p^2}\right)\right\}\\ &-\frac{1}{8\pi^2}\left\{\frac{1}{c_2^2p^2}\left(e^{-\frac{c_0c_2}{c_0+c_2}p^2} - e^{-\frac{c_1c_2}{c_1+c_2}p^2}\right) - \frac{1}{c_3^2p^2}\left(e^{-\frac{c_0c_3}{c_0+c_3}p^2} - e^{-\frac{c_1c_3}{c_1+c_3}p^2}\right)\right\}\\ &-\frac{1}{8\pi^2}\int_{c_1}^{c_0}\frac{1}{a^2}\left(e^{-\frac{ac_2}{a+c_2}p^2} - e^{-\frac{ac_3}{a+c_3}p^2}\right) da.\end{aligned} \tag{261}$$

Therefore we have to calculate the integral

$$\begin{aligned}\int_{c_1}^{c_0}\frac{1}{a^2}\, e^{-\frac{ac}{a+c}p^2}\, da &= \frac{1}{c}\int_{\frac{c_1}{c+c_1}}^{\frac{c_0}{c+c_0}}\frac{1}{x^2}\, e^{-xcp^2}\, dx\\ &= -\frac{c_0+c}{c_0c}e^{-\frac{cc_0}{c+c_0}p^2} + \frac{c_1+c}{c_1c}e^{-\frac{cc_1}{c+c_1}p^2} - p^2\left(Ei(-\frac{cc_0}{c+c_0}p^2) - Ei(-\frac{cc_1}{c+c_1}p^2)\right)\end{aligned} \tag{262}$$

where we used

$$\int \frac{1}{x^2}\, e^{ax}\, dx = -\frac{e^{ax}}{x} + a Ei(ax). \tag{263}$$

Picking up all contributions we find for the integral

$$\begin{aligned}\int \frac{d^4l}{(2\pi)^4}\, tr_\gamma(v(l+p)w(l)) &= \\ -\frac{1}{8\pi^2}\Bigg\{ &\left(\frac{c_0^2+c_2^2}{c_0^2c_2^2}\frac{1}{p^2} - \frac{c_0+c_2}{c_2c_0}\right) e^{-\frac{c_2c_0}{c_2+c_0}p^2} - \left(\frac{c_1^2+c_2^2}{c_1^2c_2^2}\frac{1}{p^2} - \frac{c_1+c_2}{c_2c_1}\right) e^{-\frac{c_2c_1}{c_2+c_1}p^2} \\ &- \left(\frac{c_0^2+c_3^2}{c_0^2c_3^2}\frac{1}{p^2} - \frac{c_0+c_3}{c_3c_0}\right) e^{-\frac{c_3c_0}{c_3+c_0}p^2} + \left(\frac{c_1^2+c_3^2}{c_1^2c_3^2}\frac{1}{p^2} - \frac{c_1+c_3}{c_3c_1}\right) e^{-\frac{c_3c_1}{c_3+c_1}p^2} \\ &-p^2\left[Ei(-\frac{c_2c_0}{c_2+c_0}p^2) + Ei(-\frac{c_3c_1}{c_3+c_1}p^2) - Ei(-\frac{c_2c_1}{c_2+c_1}p^2) - Ei(-\frac{c_3c_0}{c_3+c_0}p^2)\right]\Bigg\} \end{aligned} \tag{264}$$

This leads for the special case $v = w$ to

$$\begin{aligned}\int \frac{d^4l}{(2\pi)^4}\, tr_\gamma(v(l+p)v(l)) = -\frac{1}{4\pi^2}\Bigg\{ &\left(\frac{1}{c_0^2p^2} - \frac{1}{c_0}\right) e^{-\frac{c_0}{2}p^2} + \left(\frac{1}{c_1^2p^2} - \frac{1}{c_1}\right) e^{-\frac{c_1}{2}p^2} \\ &- \left(\frac{c_1^2+c_0^2}{c_1^2c_0^2p^2} - \frac{c_1+c_0}{c_1c_0}\right) e^{-\frac{c_0c_1}{c_0+c_1}p^2} \\ &-\frac{p^2}{2}\left(Ei(-\frac{c_0}{2}p^2) + Ei(-\frac{c_1}{2}p^2) - 2Ei(-\frac{c_1c_0}{c_1+c_0}p^2)\right)\Bigg\} \end{aligned} \tag{265}$$

We give the $p = 0$ values for this integrals, too.

$$\begin{aligned}\int \frac{d^4l}{(2\pi)^4}\, tr_\gamma(v(l)w(l)) &= \\ \frac{1}{8\pi^2}&\left\{\frac{c_0^2+c_2^2}{c_0c_2(c_0+c_2)} - \frac{c_1^2+c_2^2}{c_1c_2(c_1+c_2)} - \frac{c_0^2+c_3^2}{c_0c_3(c_0+c_3)} + \frac{c_1^2+c_3^2}{c_1c_3(c_1+c_3)}\right\} \end{aligned} \tag{266}$$

This yields for the special case $v = w$ the value

$$\int \frac{d^4l}{(2\pi)^4}\, tr_\gamma(v(l)v(l)) = -\frac{1}{8\pi^2}\left\{\frac{(c_1-c_0)^2}{c_0c_1(c_0+c_1)}\right\} \tag{267}$$

For the case $\Lambda_0 - \Lambda_1 = \delta\Lambda \ll \Lambda$

$$\lim_{\delta\Lambda\to 0} \int \frac{d^4l}{(2\pi)^4}\, tr_\gamma(v(l)v(l)) = -\frac{1}{4\pi^2}\delta\Lambda^2 \tag{268}$$

And finally we give the approximation for the case that w is small, i.e. $\Lambda_0 = \Lambda' > \Lambda_2 = \Lambda + \delta\Lambda > \Lambda_1 = \Lambda_3 = \Lambda$.

$$\begin{aligned}\int \frac{d^4l}{(2\pi)^4}\, tr_\gamma(v(l+p)w(l)) &= \\ -\frac{1}{8\pi^2}\Bigg\{ &\left(\frac{\Lambda'^4 + (\Lambda+\delta\Lambda)^4}{p^2} - (\Lambda'^2 + (\Lambda+\delta\Lambda)^2)\right) e^{-\frac{p^2}{\Lambda'^2+(\Lambda+\delta\Lambda)^2}} \\ &- \left(\frac{\Lambda^4 + (\Lambda+\delta\Lambda)^4}{p^2} - (\Lambda^2 + (\Lambda+\delta\Lambda)^2)\right) e^{-\frac{p^2}{\Lambda^2+(\Lambda+\delta\Lambda)^2}} \\ &- \left(\frac{\Lambda'^4 + \Lambda^4}{p^2} - (\Lambda'^2 + \Lambda^2)\right) e^{-\frac{p^2}{\Lambda'^2+\Lambda^2}} + \left(\frac{2\Lambda^4}{p^2} - 2\Lambda^2\right) e^{-\frac{p^2}{2\Lambda^2}} \\ &+p^2 2\pi I_2(p^2, \Lambda', \Lambda, \Lambda+\delta\Lambda, \Lambda)\Bigg\} \end{aligned} \tag{269}$$

Expanding the integral to first order in $\delta\Lambda$ we find

$$\int \frac{d^4l}{(2\pi)^4}\, tr_\gamma(v(l+p)w(l)) = -\frac{\delta\Lambda}{8\pi^2}\left\{4\Lambda^3\left(\frac{1}{p^2} - \frac{\Lambda'2}{(\Lambda'^2+\lambda^2)^2}\right) e^{-\frac{p^2}{\Lambda'^2+\lambda^2}} + \left(\frac{4\Lambda^3}{p^2} - \Lambda\right) e^{-\frac{p^2}{2\Lambda^2}}\right\} \tag{270}$$

B Performing consecutive cut-off lowerings for a bosonized action

In this appendix we want to show that a second cut-off lowering in the bosonized action of our toy model does not change the form of the action. We start with the action given in equation 108. For notational convenience we set $N = 1$.

$$S = \bar\psi u_\Lambda^{-1}\psi + \Delta\frac{1}{\lambda}\Delta - 2\Delta\bar\psi\psi - 2\Delta\bar\psi\{v_0^{-1} - 2\Delta\}^{-1}\psi 2\Delta - Tr\ln(1 - 2\Delta v_0). \tag{271}$$

Next we do a propagator split for the fermion propagator.

$$u_\Lambda = u_{\Lambda_1} + v_1 \qquad \text{where} \qquad \Lambda_1 < \Lambda \tag{272}$$

This induces a field split $\psi = \Psi + \zeta$. If we insert this split into the action we get

$$\begin{aligned} S &= \bar\Psi u_{\Lambda_1}^{-1}\Psi + \bar\zeta v_1^{-1}\zeta + \Delta\frac{1}{\lambda}\Delta \\ &\quad -2\Delta(\bar\Psi + \bar\zeta)(\Psi + \zeta) - 2\Delta(\bar\Psi + \bar\zeta)\{v_0^{-1} - 2\Delta\}^{-1}(\psi + \zeta)2\Delta - Tr\ln(1 - 2\Delta v_0) \\ &= \bar\Psi u_{\Lambda_1}^{-1}\Psi + \Delta\frac{1}{\lambda}\Delta + \bar\zeta\left(v_1^{-1} - 2\Delta - 2\Delta\{v_0^{-1} - 2\Delta\}^{-1}2\Delta\right)\zeta - 2\Delta\bar\Psi\Psi \\ &\quad -2\Delta\bar\Psi\{v_0^{-1} - 2\Delta\}^{-1}\Psi 2\Delta - 2\Delta\bar\Psi(1 + \{v_0^{-1} - 2\Delta\}^{-1}2\Delta)\zeta - 2\Delta\bar\zeta(1 + \{v_0^{-1} - 2\Delta\}^{-1}2\Delta)\Psi \\ &\quad -Tr\ln(1 - 2\Delta v_0) \\ &= \bar\Psi u_{\Lambda_1}^{-1}\Psi + \Delta\frac{1}{\lambda}\Delta + \bar\zeta\left(v_1^{-1} - \frac{2\Delta}{1 - 2\Delta v_0}\right)\zeta - 2\Delta\bar\Psi\Psi \\ &\quad -2\Delta\bar\Psi\{v_0^{-1} - 2\Delta\}^{-1}\Psi 2\Delta - 2\Delta\bar\Psi\frac{1}{1 - 2\Delta v_0}\zeta - \bar\zeta\frac{1}{1 - 2\Delta v_0}\Psi 2\Delta \\ &\quad -Tr\ln(1 - 2\Delta v_0) \end{aligned} \tag{273}$$

We do the ζ integration and neglect a field independent constant.

$$\begin{aligned} S &= \bar\Psi u_{\Lambda_1}^{-1}\Psi + \Delta\frac{1}{\lambda}\Delta - 2\Delta\bar\Psi\Psi - 2\Delta\bar\Psi\{v_0^{-1} - 2\Delta\}^{-1}\Psi 2\Delta \\ &\quad -2\Delta\bar\Psi\frac{1}{1 - 2\Delta v_0}\left(v_1^{-1} - \frac{2\Delta}{1 - 2\Delta v_0}\right)^{-1}\frac{1}{1 - 2\Delta v_0}\Psi 2\Delta \\ &\quad -Tr\ln(1 - 2\Delta v_0) - Tr\ln\left(1 - \frac{2\Delta v_1}{1 - 2\Delta v_0}\right) \\ &= \bar\Psi u_{\Lambda_1}^{-1}\Psi + \Delta\frac{1}{\lambda}\Delta - 2\Delta\bar\Psi\Psi \\ &\quad -2\Delta\bar\Psi\frac{1}{1 - 2\Delta v_0}\left(\frac{1}{v_1^{-1} - \frac{2\Delta}{1-2\Delta v_0}}\frac{1}{1 - 2\Delta v_0} + v_0\right)\Psi 2\Delta \\ &\quad -Tr\ln(1 - 2\Delta(v_0 + v_1)) \\ &= \bar\Psi u_{\Lambda_1}^{-1}\Psi + \Delta\frac{1}{\lambda}\Delta - 2\Delta\bar\Psi\Psi \\ &\quad -2\Delta\bar\Psi\frac{1}{1 - 2\Delta v_0}\left(v_1\frac{1}{1 - 2\Delta(v_1 + v_0)} + v_0\right)\Psi 2\Delta \\ &\quad -Tr\ln(1 - 2\Delta(v_0 + v_1)) \\ &= \bar\Psi u_{\Lambda_1}^{-1}\Psi + \Delta\frac{1}{\lambda}\Delta - 2\Delta\bar\Psi\Psi \\ &\quad -2\Delta\bar\Psi\frac{1}{1 - 2\Delta v_0}(v_1 + v_0 - v_0 2\Delta(v_1 + v_0))\frac{1}{1 - 2\Delta(v_1 + v_0)}\Psi 2\Delta \\ &\quad -Tr\ln(1 - 2\Delta(v_0 + v_1)) \\ &= \bar\Psi u_{\Lambda_1}^{-1}\Psi + \Delta\frac{1}{\lambda}\Delta - 2\Delta\bar\Psi\Psi \\ &\quad -2\Delta\bar\Psi\frac{v_1 + v_0}{1 - 2\Delta(v_1 + v_0)}\Psi 2\Delta - Tr\ln(1 - 2\Delta(v_0 + v_1)) \end{aligned} \tag{274}$$

Therefore the action has the same form as before with v_0 changed to $v_0 + v_1$.

The result can be used to calculate the effect of a high frequency fermion field integration on the potential $\tilde{V}_\Lambda$.

$$\begin{aligned} \tilde{V}_\Lambda(\Delta|\bar{\Psi},\Psi) &= -2\Delta\bar{\Psi}\Psi - 2\Delta\bar{\Psi}\{v_0^{-1} - 2\Delta\}^{-1}\Psi 2\Delta + \sum_{n\geq 2}\frac{1}{2n}Tr\,[(2\Delta v_0 2\Delta v_0)^n] \\ &= -2\Delta\bar{\Psi}\Psi - 2\Delta\bar{\Psi}\{v_0^{-1} - 2\Delta\}^{-1}\Psi 2\Delta - 2\Delta tr_\gamma(v_0\bar{v}_0)\Delta - Tr\ln(1 - 2\Delta v_0) \end{aligned} \quad (275)$$

This integration is needed in section 5.5. The main difference to the previous case is that $\tilde{V}_\Lambda$ has no kinetic term for the boson. Nevertheless the corrections to the kinetic term are produced by the high frequency fermion field integration The result can be given without calculation by comparison with the previous calculation.

$$e^{-\left(\frac{\delta}{\delta\Psi}v_1\frac{\delta}{\delta\bar{\Psi}}\right)}\, e^{-\tilde{V}_\Lambda(\Delta|\bar{\Psi},\Psi)} = e^{-\Delta[4tr_\gamma(v_0\bar{v}_1)+2tr_\gamma(v_1\bar{v}_1)]\Delta}\, e^{-\tilde{V}_{\Lambda_1}(\Delta|\bar{\Psi},\Psi)} \quad (276)$$

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