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Beyond Affine Kac-Moody Algebras in String Theory

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Reinhold Willigis Gebert
aus Hanau am Main

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Gutachter der Dissertation:	Prof. Dr. H. Nicolai Prof. Dr. F. Steiner
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Zusammenfassung

Diese Arbeit ist dem Studium gewisser unendlich-dimensionaler Lie-Algebren gewidmet, die in der Stringtheorie auftauchen. Die zwei untersuchten Modelle beschreiben einen chiralen Sektor eines vollständig kompaktifizierten, geschlossenen bosonischen Strings, der sich auf einem subkritischen 10-dimensionalen bzw. auf einem kritischen 26-dimensionalen Torus als Raumzeit bewegt. Um die zugehörige Lie-Algebra der physikalischen Zustände zu analysieren, wird eine diskrete Version der DDF-Konstruktion im Rahmen des Vertexalgebren-Formalismus entwickelt. Angewandt auf das subkritische Beispiel verschafft die Methode neue Einsichten in die komplizierte Struktur der hyperbolischen Kac-Moody-Algebra E_{10} in bezug auf transversale und longitudinale Zustände. Aufgrund des No-ghost-Theorems treten in 26 Dimensionen lediglich die transversalen physikalischen Zustände auf, die die sogenannte Fake-Monster-Lie-Algebra formen. Letztere stellt ein Beispiel für eine Borcherds-Algebra dar, die eine verallgemeinerte Kac-Moody-Algebra in dem Sinne ist, daß imaginäre einfache Wurzeln in den definierenden Relationen zugelassen werden. Es wird für das Beispiel demonstriert, wie diese Besonderheit ebenfalls mit Hilfe der DDF-Operatoren verstanden werden kann. Schließlich wird ein neues Resultat über die Darstellungstheorie dieser Algebren bewiesen, das im Hinblick auf mögliche Anwendungen in der Physik analysiert wird.

Abstract

This work is devoted to the study of certain infinite-dimensional Lie algebras arising in string theory. The two investigated models describe a chiral sector of a fully compactified closed bosonic string moving on a subcritical 10-dimensional or a critical 26-dimensional spacetime torus, respectively. To analyze the corresponding Lie algebra of physical states, a discrete version of the DDF construction in the framework of vertex algebras is developed. When applied to the subcritical example, the method yields some new insights into the complicated structure of the hyperbolic Kac–Moody algebra E_{10} in terms of transversal and longitudinal states. Due to the no-ghost theorem, in 26 dimensions only transversal physical states appear which make up the so-called fake monster Lie algebra. The latter represents an example of a Borcherds algebra, which is a generalized Kac–Moody algebras in the sense that imaginary simple roots are allowed for in the defining relations. It is demonstrated for the example, that this feature can be understood by means of the DDF operators, too. Finally, a new result about representation theory of these algebras is proved which is analyzed in view of possible applications to physics.

Meinen Kindern
Maximilian, Theresa und Katharina
gewidmet

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INTRODUCTION

String theory, which still represents the only serious candidate for a consistent unification of the known fundamental forces in nature, is admittedly poorly understood in terms of a conceptual framework so that it is reasonable to ask again and again the question: What is string theory? (see [69] and references therein) Almost all our knowledge relies on the perturbative formulation as sum over Riemann surfaces although there are some attempts to gain deeper insight into the theory. Of course, one might follow the strategy of "staying on the ground", that is, not to lose view of the standard model. This amounts to searching for the true ground state of string theory. On the other hand, one must also not neglect the more conceptual issues, in particular the question for the fundamental symmetry of the theory. It is widely believed that discrete symmtries such as S-, T-, or U-duality (cf. [50]) will come to play an important role in the final formulation (whatever this might be!) of string theory; but one should not be too confident about that idea for it could very well be the case that the discreteness is just an artefact caused by the specific models. Yet, there are indications for continous symmetries as well. For certain string models it turns out that the physical string states form a closed Lie algebra. The interpretation of these Lie algebras of physical states has to be discussed in the context of both physics and mathematics, since the algebras as such represent an unexplored territory of mathematics. This may be regarded as a promising sign, because we know from many examples in the past that new physics often involves new mathematics.

As regards the interpretation in string theory, it is at first sight rather puzzling that the physical states always form a Lie algebra. Although this is known to be an inherent feature of any meromorphic conformal field theory, it provides nevertheless a first hint to some deeper symmetry of the theory. In fact, it turns out that physical states generate one-parameter groups of inner automorphisms of the underlying conformal field theory. In the space of conformal field theories describing solutions of the string equations of motion, the physical states therefore determine an infinite-dimensional submanifold of physically equivalent theories on which the noncommutability of two flows is encoded in the Lie bracket of the corresponding physical states. One might object that this observation is irrelevant for string theory because the action of inner automorphisms should be divided out from the outset. But things are not as trivial as they look like. There are other strong indications for the claim that the Lie algebra of physical states does play some role for the (still hypothetical) fundamental symmetry algebra of string theory.

Gross [47] has established the interesting result that in the ultrahigh-energy limit of string theory, where the Planck mass goes to zero (i.e. this is the antipodal limit considered to make contact with the standard model!), the scattering amplitudes for the bosonic string only depend on the number of external legs, irrespective of the nature of the inserted vertex operators. It may seem unreasonable to study such a queer limit but, in fact, it is a very conservative approach. Namely, we all know that if we had not known the spontaneously broken symmetry of the electroweak interactions, we could have in principle discovered it at high energies where all gauge particles become massless again. In agreement with this analogy it is indeed a reasonable hope to get grasp of the unbroken string gauge algebra by studying the relations between high energy scattering amplitudes. But since the latter, according to Gross' result, are independent of the choice of scattered physical states, it is tempting to regard the Lie algebra of physical states itself as part of some universal gauge algebra. Note that we obtain, by construction, different Lie algebras of physical states when we consider inequivalent string backgrounds. Moreover, due to the presence of a whole plethora of massive physical states, each Lie algebra would have to be almost completely spontaneously broken. If we take the above appealing picture for granted then our task will be to make a clever choice for the specific string background in order to find a Lie algebra of physical states as large as possible. 'Clever' here apparently means 'as symmetric as possible', and one is therefore naturally led to Minkowskian torus compactifications where *all* spacetime dimensions are chosen to be periodic (hence "finite in all directions" [64]). More specifically, for the 26-dimensional bosonic string there is a unique choice of maximal symmetry, namely the even selfdual Lorentzian lattice $II_{25,1}$ which indeed provides a 'large' algebra — the infinite rank fake monster Lie algebra introduced by Borcherds [9].

The above, to some extent heuristic arguments were recently put on solid ground in a paper by West [79]. He was able to show that the fake monster Lie algebra *is* a symmetry of string theory in the sense that every physical state leads to a symmetry of the string scattering amplitudes. In view of this result one could now pose the question to which extent the vertices are already fixed by stipulating the fake monster Lie algebra as symmetry algebra. The degree of uniqueness would then give us a clue of how small the algebra is in comparison with the universal string gauge algebra. Certainly, they cannot be the same. For on the one hand it is clear that the string vertices describe the string field theory, on the other hand we know (see [64]) that the fake monster Lie algebra does not contain all Lie algebras arising from other string backgrounds. It is worth mentioning that the calculations were carried out in the so-called group theoretical approach to string theory which seems to be a powerful formalism to analyze the issue of string symmetries.

After all, we still lack a clear understanding of the symmetry principle that underlies string theory. So we might turn to the purely mathematical aspects and ask of what kind these Lie algebras of physical states are. For Euclidean compactifications nothing spectacular happens and we encounter finite dimensional Lie algebras. For the Minkowskian case, however, things dramatically change. Hyperbolic Kac–Moody algebras occur, but only as proper subalgebras since in general we end up with so-called Borcherds algebras. The latter are distingiushed from the ordinary Kac–Moody algebras by allowing for imaginary (≡ nonpositive norm) simple roots.

So one could attempt to gain some deeper insight into string theory by interpreting the general structure of the Lie algebras that show up. Unfortunately, at present this plan immediately wrecks. It is surprising how little is known about Kac–Moody algebras of indefinite type and in particular about the subclass of hyperbolic Kac–Moody algebras. Essentially, we comprehend the strucure of their root systems and have the remarkable result that the Wey–Kac character formula continues to hold for them (see e.g. [63] for a concise review). Of course, the latter provides us in principle with the root multiplicties (and can be put in a modified version on the computer to determine recursively some low level multiplicities), but not for a single Kac–Moody algebra of indefinite type all root multiplicities are known. In fact, the explicit multiplicity formulas we are aware of apply to roots of level two at most, whereas these algebras are always made up of pieces with any integer level. In contrast to this, the other two types of Kac–Moody algebras are thoroughly understood and have also found widespread applications in physics. Finite Lie algebras, whose roots always have multiplicity 1, emerge as symmetry algebras in many branches of physics. Affine Kac–Moody algebras, which first appeared in quantum field theory in the guise of two-dimensional current algebras [2], contain roots with multiplicities 1 or rank − 2 only. Indefinite Kac–Moody algebras have so far withstood any truly satisfactory explanation. They do appear, however, in the above mentioned compactified string theories via the vertex operator construction which leads to the Lie algebra of physical states.

Hence it might not be random that we are stuck with both the question of symmetry in string theory and the issue of indefinite Kac–Moody algebras (not to speak about the even more complicated Borcherds algebras). It is conceivable that they can only be understood in terms of each other. This would entail that progress can be solely made in little steps by applying the techniques of string theory, say, to the analysis of the algebras and vice versa. This is precisely the strategy I shall follow in this work.

I will now give a brief survey of the five chapters of this thesis which is based on the publications [34], [35] and [36].

In Chap. I, the necessary mathematical background is carefully reviewed. We shall begin with a presentation of the machinery of formal calculus. Many results are similar to those obtained in complex analysis but here we deal solely with formal variables and formal power series. We have collected all relevant formulas occurring in the literature (especially in [30]). The next two sections are devoted to the axiomatic setup of vertex algebras and their fundamental properties, with special emphasis on the connection to chiral algebras in physics and conformal field theory. Then the Jacobi identity for vertex algebras is analyzed and interpreted so that it will become clear that it represents the main axiom of the theory. As an outcome we will discuss the important notions of locality and duality. In the following section, the algebra of primary states of conformal weight 1 will be presented which in string theory corresponds to the Lie algebra of physical states — the main object of my investigations in this work. We shall see that the residue of the three-point function for physical states is the corresponding structure constant for the Lie algebra of physical states. For completeness it will be exposed how the algebra of weight 2 states under certain circumstances leads to the Griess algebra, which has the famous monster group as symmetry group. Finally, a natural definition for the normal-ordered product of vertex operators is proposed which enables us to interpret nicely the finiteness condition for vertex operator algebras by Zhu [81]. A surprising resemblance with the results of Lian and Zuckerman [58] about the Gerstenhaber algebra structure in string theory is exhibited.

Vertex algebras associated with even lattices have their origin in toroidal compactifications of bosonic strings. In the first two sections of Chap. II we will review the construction of this important class of examples of vertex algebras (cf. [11]) and present the proof of the vertex algebra axioms. Then a detailed account to the Lie algebra of physical states follows, including an exposition of how Kac–Moody algebras in general emerge as subalgebras in this context. We will also explain the relation between the invariant and the contravariant bilinear forms on the Fock space.

The new idea pursued in this work is the application of the DDF construction from string theory to the analysis of the Lie algbra of physical states. For this purpose, in Chap. III a discrete DDF construction will be developed which is appropriate for the toroidally compactified string. While the transversal DDF operators can be straightforwardly constructed from photonic states and make up, for each consistent choice of tachyonic and photonic states, a Heisenberg algebra, the definition of the longitudinal states is more subtle. Namely, it will turn out that the longitudinal DDF operators represent non-summable operators in the framework of vertex algebras which have not been considered in this context before. We shall demonstrate that this problem can be resolved by introducing the notion of generalized vertex algebra. In the next section, the appearance of a whole family of Virasoro algebras, made up by the longitudinal DDF operators, will be stressed. Finally, the consequences of the no-ghost theorem for the discrete scenario will be discussed.

In Chap. IV, the formalism will be applied to the analysis of the maximal hyperbolic Kac–Moody algebra E_{10}, after recalling some known results. Then it is defined what we call the DDF decomposition of a given root vector of arbitrary level ℓ; this is the point where we will be forced to admit fractional momenta $\frac{1}{\ell}\mathbf{r}$ with $\mathbf{r}$ in the root lattice. For the level-one root space elements, which are known to form the so-called basic representation of E_9 [54], a simple and explicit realization in terms of transversal DDF states will be exhibited. Finally, we will perform a complete analysis of a non-trivial level-two root space and construct an explicit basis for it, whereas only its dimension was known so far [55]. This example displays the appearance of longitudinal and the disappearance of certain transversal states, which I conjecture to be generic for higher level root spaces of hyperbolic Kac–Moody algebras. Of course, the results should be regarded only as a tiny step into the *terra incognita* of hyperbolic Kac–Moody algebras, but hopefully they will give the reader at least a flavor of their complexity.

Chapter V deals with Borcherds algebras, which generalize ordinary Kac–Moody algebras by allowing for imaginary simple roots in the defining relations. In the first section the celebrated fake monster Lie algebra [10], which is the first generic example of a Borcherds algebra, is worked out in detail. We shall write down the necessary commutators to verify that the Lie algebra indeed fulfills the definition of a Borcherds algebra in terms of generators and relations; whereas so far only a proof had been given that was based on a more abstract definition of a Borcherds algebra [9]. It will turn out that the positive multiples of the lightlike Weyl vector have to be included as imaginary simple roots. It will be demonstrated that this feature can be nicely understood by means of the DDF construction. In the next section, we present the definition of Borcherds algebras via generators and relations and summarize the basic properties of these generalized Kac-Moody algebras [6],[7]. Then Slansky's investigation [74] of the simplest nontrivial examples of Borcherds algebras are reviewed. Inspired by his results we will prove that the basic imaginary representation of Borcherds algebras with a single imaginary simple root is given by the full tensor algebra over some integrable highest weight module for the underlying ordinary Kac–Moody algebra. We conclude with some speculations about possible realizations of Borcherds algebras in physics.

Explicit formulas for transversal and longitudinal DDF states as well as a number of Lie algebra commutators giving level-two root space elements are collected in the appendices. They were checked with a REDUCE program.

Chapter I:

Formal Calculus and Vertex Algebras

In this chapter we shall present the mathematical background and the tools needed throughout the rest of this work. We have decided to employ formal calculus and the framework of vertex algebras for the following reasons: Firstly, formal calculus provides quick elegant calculational methods which circumvent almost all the annoying aspects of complex analysis. In fact, complex coordinates and Laurent series in conformal field theory are replaced by their formal counterparts. The reason for this formal approach to work is simply that most calculations in conformal field theory are genuinely of algebraic nature disguised by contour integrals, residues etc. The second reason for choosing this approach is because theoretical physicists generally prefer the formulation of a theory where a certain number of physical principles are combined with a well-developed mathematical machinery.[1] This allows one to investigate both the physical and the mathematical origins of certain aspects of the theory and to distinguish them. Although this did not happen to be the case in the original formulation of conformal field theory (see [4]), nowadays the mathematical theory of vertex (operator) algebras is fairly well-developed and constitutes a rigorous formulation of chiral algebras in physics.

1. Formal calculus

A nice exposition of vertex operator formal calculus can be found in [31]. We shall closely follow [30] where the subject is treated thoroughly.

In contrast to conformal field theory (see e.g. [4], [38] or [65]), in the vertex algebra approach we use *formal* variables $z, z_0, z_1, z_2, \ldots$. As already mentioned, the great advantage of formal calculus is that we perform purely algebraic manipulations instead of bothering about contour integrals, single-valuedness, complex analysis etc.

The objects we will work with are formal power series. For a vector space W, we set

$$W\{z\} = \left\{ \sum_{n\in\mathbb{C}} w_n z^n \mid w_n \in W \right\},$$

$$W[\![z, z^{-1}]\!] = \left\{ \sum_{n\in\mathbb{Z}} w_n z^n \mid w_n \in W \right\},$$

$$W[\![z]\!] = \left\{ \sum_{n\in\mathbb{N}} w_n z^n \mid w_n \in W \right\},$$

$$W[z, z^{-1}] = \left\{ \sum_{n\in\mathbb{Z}} w_n z^n \mid w_n \in W, \text{ almost all } w_n = 0 \right\} \quad \text{(Laurent polynomials)},$$

$$W[z] = \left\{ \sum_{n\in\mathbb{N}} w_n z^n \mid w_n \in W, \text{ almost all } w_n = 0 \right\} \quad \text{(polynomials)},$$

where "almost all" means "all but finitely many".

Note that these sets are $\mathbb{C}$ vector spaces under obvious pointwise operations. We can generalize the above spaces in a straightforward way to the case of several commuting formal variables, e.g. $W[\![z_1, z_2^{-1}]\!] = \{\sum_{m,n\in\mathbb{N}} w_{mn} z_1^m z_2^{-n} | w_{mn} \in W\}$. Though $W\{z\}$ may look strange at first sight due to the sum over the complex numbers it is just another way of writing the elements of $W^{\mathbb{C}} \equiv \{f : \mathbb{C} \to W\}$, the space of W-valued functions over $\mathbb{C}$.

Since we will often multiply formal series or add up an infinite number of series it is necessary to introduce the notion of algebraic summability. Let $(x_i)_{i\in I}$ be a family in End W, the vector space of endomorphisms of W (I is an index set). We say that $(x_i)_{i\in I}$ is **summable** if for every $w \in W$, $x_i w = 0$ for all but a finite number of $i \in I$. Then the operator $\sum_{i\in I} x_i$ is well defined. In general an algebraic limit or a product of formal series is defined if and only if the coefficient of *every* monomial in the formal variables in the formal expression is summable.

An example of a nonexistent product is $(\sum_{n\in\mathbb{N}} z^n)(\sum_{m\in\mathbb{N}} z^{-m})$, where even the coefficient of any monomial z^l is not summable (because it would be $\mathbb{N}$ times the identity $1 \equiv \mathrm{id}_W$).

Recall that for $x \in \mathbb{C}$

$$(1-x)^{-1} = \begin{cases} \sum_{k\in\mathbb{N}} x^k & \text{if } |x| < 1 \\ -x^{-1} \sum_{k\in\mathbb{N}} x^{-k} & \text{if } |x| > 1 \end{cases}.$$

[1] The example par excellence is general relativity where the theory was built by combining general covariance and the principle of equivalence with powerful differential geometry.

If we define

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n \quad \in \mathbb{C}[\![z, z^{-1}]\!] \tag{1.1.1}$$

then, formally, this is the Laurent expansion of the classical δ function at $z = 1$. Indeed, $\delta(z)$ enjoys the following fundamental properties:

Proposition 1

1. *Let $w(z) \in W[z, z^{-1}]$, $a \in \mathbb{C}^\times$. Then*

$$w(z)\delta(az) = w(a^{-1})\delta(az). \tag{1.1.2}$$

2. *Let $X(z_1, z_2) \in (\mathrm{End}\, W)[\![z_1, z_1^{-1}, z_2, z_2^{-1}]\!]$ be such that $\lim_{z_1\to z_2} X(z_1, z_2)$ exists (algebraically) and let $a \in \mathbb{C}^\times$. Then*

$$X(z_1, z_2)\delta\left(a\frac{z_1}{z_2}\right) = X(a^{-1}z_2, z_2)\delta\left(a\frac{z_1}{z_2}\right) = X(z_1, az_1)\delta\left(a\frac{z_1}{z_2}\right). \tag{1.1.3}$$

Proof
Write $w(z) = \sum_{n\in\mathbb{Z}} w_n z^n$, $X(z_1, z_2) = \sum_{m,n\in\mathbb{Z}} x_{mn} z_1^m z_2^n$, use the definition of $\delta(z)$ and shift summation indices.

Note that $w(z)$ must be a Laurent polynomial to ensure existence of the product with the δ series. For explicit calculations it is useful to keep in mind that the substitutions in the arguments correspond formally to $az = 1$ and $az_1/z_2 = 1$, respectively. We want to stress that in analogy with the theory of distributions an expression like $\delta(z)\delta(z)$ does not exist. Moreover, in (1.1.2) and (1.1.3) integral powers of z, z_1 and z_2 are required so that, for example, $z^{1/2}\delta(z) \neq 1^{1/2}\delta(z)$, $z_1^{1/2}z_2^{1/2}\delta(\frac{z_1}{z_2}) \neq z_2\delta(\frac{z_1}{z_2})$.

Since we can always (formally) differentiate formal power series, it is interesting to study the properties of higher derivations of $\delta(z)$. For this purpose we consider a generating function for all the higher drivatives,

$$\delta(z + z_0) \equiv e^{z_0\frac{d}{dz}}\delta(z) = \sum_{n\in\mathbb{N}} \frac{1}{n!} z_0^n \delta^{(n)}(z), \tag{1.1.4}$$

where $(z + z_0)^n$, $n \in \mathbb{Z}$, is to be expanded in nonnegative powers of z_0. We will come back to this convention later. It turns out that a generalization of the formula $f(x)\delta^{(n)}(x) = (-1)^n f^{(n)}(0)\delta^{(n)}(x)$ for the classical δ function is valid for the formal series $\delta(z)$.

Proposition 2

1. *Let $p(z) \in \mathbb{C}[z, z^{-1}]$ and consider the derivation $D = p(z)\frac{d}{dz}$ of $\mathbb{C}[z, z^{-1}]$. Let $w(z) \in W[z, z^{-1}]$, $a \in \mathbb{C}^\times$, $y \in z_0\mathbb{C}[\![z_0]\!]$. Then*

$$w(z)e^{yD}\delta(az) = \left(e^{-yD}w\right)(a^{-1})e^{yD}\delta(az). \tag{1.1.5}$$

2. *Let $p(z_1, z_2) \in \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]$ and consider the derivations $D_i = p(z_1, z_2)\frac{\partial}{\partial z_i}$, $i = 1, 2$, of $\mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]$. Let $a \in \mathbb{C}^\times$, $y \in z_0\mathbb{C}[\![z_0]\!]$ and let $X(z_1, z_2) \in (\mathrm{End}\, W)[\![z_1, z_1^{-1}, z_2, z_2^{-1}]\!]$ be such that $\lim_{z_1\to z_2} X(z_1, z_2)$ exists. Then*

$$X(z_1, z_2)e^{yD_1}\delta\left(a\frac{z_1}{z_2}\right) = \left(e^{-yD_1}X\right)(a^{-1}z_2, z_2)e^{yD_1}\delta\left(a\frac{z_1}{z_2}\right),$$
$$X(z_1, z_2)e^{yD_2}\delta\left(a\frac{z_1}{z_2}\right) = \left(e^{-yD_2}X\right)(z_1, az_1)e^{yD_2}\delta\left(a\frac{z_1}{z_2}\right). \tag{1.1.6}$$

Proof
Since y has no constant term, e^{yD} is well defined. We have the Leibniz rule for D, $w(z) \in W[z, z^{-1}]$, $v(z) \in W\{z\}$, $n \geq 0$,

$$D^n[w(z)v(z)] = \sum_{k=0}^{n} \binom{n}{k} \left[D^k w(z)\right]\left[D^{n-k}v(z)\right].$$

Hence

$$e^{yD}[w(z)v(z)] = \left[e^{yD}w(z)\right]\left[e^{yD}v(z)\right].$$

Apply e^{yD} to $\left[e^{-yD}w(z)\right]\delta(az) = (e^{-yD}w)(a^{-1})\delta(az)$ and invoke the above formula to obtain (1.1.5). An obvious extension of (1.1.5) to two variables together with (1.1.3) gives (1.1.6).

If we read off the coefficients of y^n for $n \geq 0$ in the above formulas, we find that

$$w(z)D^n\delta(az) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(D^k w\right)(a^{-1})D^{n-k}\delta(az),$$
$$X(z_1, z_2)D_1^n\delta\left(a\frac{z_1}{z_2}\right) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(D_1^k X\right)(a^{-1}z_2, z_2)D_1^{n-k}\delta\left(a\frac{z_1}{z_2}\right),$$
$$X(z_1, z_2)D_2^n\delta\left(a\frac{z_1}{z_2}\right) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(D_2^k X\right)(z_1, az_1)D_2^{n-k}\delta\left(a\frac{z_1}{z_2}\right),$$

all expressions existing.

It is no restriction to consider only derivations of the form $D = p(z)\frac{d}{dz}$, $p(z) \in \mathbb{C}[z, z^{-1}]$, since the derivations of $\mathbb{C}[z, z^{-1}]$ are precisely the endomorphisms of that form. To see this let $d \in (\mathrm{End}\,\mathbb{C})[z, z^{-1}]$ be a derivation. Set $p(z) := d(z)$, so that $D(z) = d(z)$. We have $D(1) = 0 = d(1)$ because of $d(1) = d(1\cdot 1) = d(1) + d(1)$, and $d(z^{-1}) = -z^{-2}d(z)$ because of $0 = d(1) = d(z\cdot z^{-1}) = d(z)z^{-1} + d(z^{-1})z$. This shows that D and d agree on all powers of z.

Now we want to introduce the tools for formal calculus which correspond to contour integrals and residues for complex variables. Define

$$\begin{aligned}\mathbb{C}(z) = \mathbb{C}(z^{-1}) &= \{p(z)/q(z)\,|\,p(z), q(z) \in \mathbb{C}[z],\ q \neq 0\} \quad \text{(rational functions)},\\ \mathbb{C}((z)) &= \{p(z)/q(z)\,|\,p(z), q(z) \in \mathbb{C}[[z]],\ q \neq 0\},\\ \mathbb{C}((z^{-1})) &= \{p(z^{-1})/q(z^{-1})\,|\,p(z^{-1}), q(z^{-1}) \in \mathbb{C}[[z^{-1}]],\ q \neq 0\}.\end{aligned}$$

Elements of the latter spaces will often be expressed by analytic functions of z and z^{-1}, respectively. They are understood as formal Taylor or Laurent expansions. For example,

$$(1+z)^a = \sum_{n\in\mathbb{N}} \binom{a}{n} z^n \in \mathbb{C}[[z]], \tag{1.1.7}$$

$$(1+z^{-1})^a = \sum_{n\in\mathbb{N}} \binom{a}{n} z^{-n} \in \mathbb{C}[[z^{-1}]]. \tag{1.1.8}$$

In the following we will always (though sometimes not explicitly stated) refer to the **binomial convention** which says that *all binomial expressions are to be expanded in nonnegative integral powers of the second variable.* This is the only point in explicit calculations at which one must not be too sloppy. For example, for $a \in \mathbb{C}$ the following expressions are in general not the same:

$$\left(\frac{z_1 - z_2}{z_0}\right)^a = \sum_{n\in\mathbb{N}} \binom{a}{n} (-1)^n z_0^{-a} z_1^{a-n} z_2^n, \tag{1.1.9}$$

$$\left(\frac{-z_2 + z_1}{z_0}\right)^a = \sum_{n\in\mathbb{N}} \binom{a}{n} (-1)^{a-n} z_0^{-a} z_1^n z_2^{a-n}. \tag{1.1.10}$$

With the binomial convention we can rewrite the generating function for the derivatives of the δ series as

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) = z_0^{-1} e^{-z_2\frac{\partial}{\partial z_1}} \delta\left(\frac{z_1}{z_0}\right). \tag{1.1.11}$$

By the use of the above examples it is not difficult to prove subsequent identities which will be extremely useful for vertex operator calculus.

Proposition 3

1.

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) = z_1^{-1}\delta\left(\frac{z_0 + z_2}{z_1}\right). \tag{1.1.12}$$

2.

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right), \tag{1.1.13}$$

where all binomial expressions are expanded in nonnegative integral powers of the second variable.

Now let us define two canonical embeddings of the rational functions

$$\iota_+ : \mathbb{C}(z) \hookrightarrow \mathbb{C}((z)), \tag{1.1.14}$$

$$\iota_- : \mathbb{C}(z^{-1}) \hookrightarrow \mathbb{C}((z^{-1})), \tag{1.1.15}$$

which denote the formal Laurent expansion in z and z^{-1}, respectively. For example,

$$\iota_+(1-z)^{-1} = (1-z)^{-1} = \sum_{n\in\mathbb{N}} z^n, \tag{1.1.16}$$

$$\iota_-(1-z)^{-1} = \iota_-\left[-z^{-1}(1-z^{-1})^{-1}\right] = -z^{-1}\sum_{n\in\mathbb{N}} z^{-n}, \tag{1.1.17}$$

so that

$$\delta(z) = (\iota_+ - \iota_-)(1-z)^{-1}. \tag{1.1.18}$$

We introduce a linear map Θ by

$$\Theta \equiv \Theta_z : \mathbb{C}(z) \to \mathbb{C}[[z, z^{-1}]], \quad f \mapsto \iota_+ f - \iota_- f. \tag{1.1.19}$$

We observe that $\ker\Theta = \mathbb{C}[z, z^{-1}]$, i.e. the Laurent polynomials are precisely those formal series for which the expansions in z and z^{-1}, respectively, agree. Moreover, by the partial fraction decomposition of a rational function, we see that the family $\{(1-az)^{-n-1}|n \geq 0,\ a \in \mathbb{C}^\times\}$ spans a linear complement of $\mathbb{C}[z, z^{-1}]$ in $\mathbb{C}(z)$. The elements of $\mathrm{im}\,\Theta$ are called **expansions of zero**. The most prominent examples of expansions of zero are given by the δ series and its derivations. In fact, the following statement shows that the set $\{\delta^{(n)}(az)|n \in \mathbb{N},\ a \in \mathbb{C}^\times\}$ is a basis of the space $\mathrm{im}\,\Theta$ of expansions of zero.

Proposition 4

For $n \in \mathbb{N}$, $a \in \mathbb{C}^\times$

$$\frac{1}{n!}\delta^{(n)}(az) = \Theta\left[(1-az)^{-n-1}\right] = (1-az)^{-n-1} - (-az)^{-n-1}(1-a^{-1}z^{-1})^{-n-1}, \tag{1.1.20}$$

i.e.

$$\delta(z+z_0) \equiv e^{z_0 \frac{d}{dz}}\delta(z) = \sum_{n\in\mathbb{N}} \Theta\left[(1-z)^{-n-1}\right] z_0^n. \tag{1.1.21}$$

Proof
The case $n = 0$ being clear, use the fact that Θ commutes with $\frac{d}{dz}$ together with the formula $\frac{1}{n!}\left(\frac{d}{dz}\right)^n (1-z)^{-1} = (1-z)^{-n-1}$ to obtain the case $n > 0$.

Next we shall generalize the map Θ to the case of two formal variables. Let S denote the set of nonzero linear polynomials in variables z_1 and z_2:

$$S = \{az_1 + bz_2 \mid a, b \in \mathbb{C},\ |a| + |b| \neq 0\} \subset \mathbb{C}[z_1, z_2]. \tag{1.1.22}$$

Consider the subring $\mathbb{C}[z_1, z_2]_S$ of the field of rational functions $\mathbb{C}(z_1, z_2)$ obtained by inverting the product of elements of S. We can write any $f(z_1, z_2) \in \mathbb{C}[z_1, z_2]_S$ in the form

$$f(z_1, z_2) = \frac{g(z_1, z_2)}{z_{i_2}^s \prod_{l=1}^r (a_l z_{i_1} + b_l z_{i_2})}, \tag{1.1.23}$$

where $g(z_1, z_2) \in \mathbb{C}[z_1, z_2]$, $r, s \in \mathbb{N}$, $a_l \neq 0$ for $l = 1, \ldots, r$.
For a permutation $(i_1\ i_2)$ of $(1\ 2)$ we define the map

$$\iota_{z_{i_1} z_{i_2}} \equiv \iota_{i_1 i_2} : \mathbb{C}[z_1, z_2]_S \to \mathbb{C}[\![z_1, z_1^{-1}, z_2, z_2^{-1}]\!] \tag{1.1.24}$$

such that each factor $(a_l z_{i_1} + b_l z_{i_2})^{-1}$ in $f \in \mathbb{C}[z_1, z_2]_S$ is expanded in nonnegative integral powers of z_{i_2}. Clearly the maps $\iota_{i_1 i_2}$ are injective.
To obtain "expansions of zero" in the variables z_1, z_2 we set

$$\Theta_{i_1 i_2} \equiv \Theta_{z_{i_1} z_{i_2}} : \mathbb{C}[z_1, z_2]_S \to \mathbb{C}[\![z_1, z_1^{-1}, z_2, z_2^{-1}]\!], \quad f \mapsto \iota_{i_1 i_2} f - \iota_{i_2 i_1} f. \tag{1.1.25}$$

Then $\ker \Theta_{i_1 i_2} = \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]$.
We shall use the following residue notation. For a formal series

$$w(z) = \sum_{n\in\mathbb{C}} w_n z^n \in W\{z\} \tag{1.1.26}$$

we write

$$\mathrm{Res}_z\,[w(z)] = w_{-1}, \tag{1.1.27}$$

so that we may think of $\mathrm{Res}_z[\ldots]$ as the formal analog of the operation $\oint_0 \frac{dz}{2\pi i}[\ldots]$ in complex analysis. Indeed, the formal residue enjoys some properties of contour integration:

Proposition 5

1. *Let* $w(z) = \sum_{n\in\mathbb{C}} w_n z^n \in W\{z\}$. *For* $n \in \mathbb{C}$
$$w_n = \mathrm{Res}_z\left[z^{-n-1} w(z)\right]. \tag{1.1.28}$$
2. **(Integration by parts)** *Let* $v(z), w(z) \in W\{z\}$. *Then*
$$\mathrm{Res}_z\left[v(z)\frac{d}{dz}w(z)\right] = -\mathrm{Res}_z\left[w(z)\frac{d}{dz}v(z)\right]. \tag{1.1.29}$$
3. **(Cauchy theorem)**
$$\begin{aligned}\mathrm{Res}_{z_1-z_2}\left[\iota_{z_1, z_1-z_2} f(z_1, z_2)\right] &= \mathrm{Res}_{z_2}\left[(\iota_{z_1 z_2} - \iota_{z_2 z_1}) f(z_1, z_2)\right] \\ &\equiv \mathrm{Res}_{z_2}\left[\Theta_{z_1 z_2} f(z_1, z_2)\right],\end{aligned} \tag{1.1.30}$$
for $f(z_1, z_2) = g(z_1, z_2)/z_1^r z_2^s (z_1 - z_2)^t$, $r, s, t \in \mathbb{Z}$, $g(z_1, z_2) \in \mathbb{C}[z_1, z_2]$.

Proof

1. Clear.
2. Use the fact that $\mathrm{Res}_z\left[\frac{d}{dz}u(z)\right] = 0$ for all $u(z) \in W\{z\}$.
3. It is sufficient to consider $h(z_1, z_2) = z_1^l z_2^m (z_1 - z_2)^n$, $l, m, n \in \mathbb{Z}$. Then
$$\iota_{21} h = \sum_{j\in\mathbb{N}} (-1)^{n-j}\binom{n}{j} z_1^{l+j} z_2^{m+n-j},$$
$$\iota_{12} h = \sum_{j\in\mathbb{N}} (-1)^{j}\binom{n}{j} z_1^{l+n-j} z_2^{m+j},$$
$$\iota_{1,1-2} h = \sum_{j\in\mathbb{N}} (-1)^{j}\binom{m}{j} z_1^{l+m-j} (z_1 - z_2)^{n+j}.$$
This implies that
$$\mathrm{Res}_{z_2}\left[\iota_{21} h\right] = (-1)^{-m-1}\binom{n}{n+m+1} z_1^{l+m+n+1},$$
$$\mathrm{Res}_{z_2}\left[\iota_{12} h\right] = (-1)^{-m-1}\binom{n}{-m-1} z_1^{l+m+n+1},$$
$$\mathrm{Res}_{z_1-z_2}\left[\iota_{1,1-2} h\right] = (-1)^{-n-1}\binom{m}{-n-1} z_1^{l+m+n+1},$$

i.e. we have to show that

$$-(-1)^{m+1}\binom{n}{n+m+1}+(-1)^{m+1}\binom{n}{-m-1}=(-1)^{n+1}\binom{m}{-n-1}$$

for all $m,n \in \mathbb{Z}$. If $m,n<0$ then $\binom{n}{n+m+1}=0$ and the above equation holds. If $n+1\leq 0\leq m$ then $\binom{n}{-m-1}=0$ and the above equation holds. If $0\leq -m-1\leq n$ then $\binom{m}{-n-1}=0$ and the above equation holds. In all other cases the binomial coefficients vanish identically.

Note that Cauchy's theorem is equivalent to (1.1.13): Just multiply (1.1.30) for the special case $f(z_1,z_2)=z_2^m(z_1-z_2)^n$ with $z_2^{-m-1}z_0^{-n-1}$ and sum over $n,m\in\mathbb{Z}$ to obtain (1.1.13).

We have already used exponentials of derivatives like $e^{z_0\frac{d}{dz}}$ in deriving formulas for the higher derivatives of $\delta(z)$. However, one might also expect $e^{z_0\frac{d}{dz}}$ to act somehow as a one-parameter group of automorphisms (parametrized by z_0). This turns out to be true in the following sense.

Proposition 6

Let $w(z)=\sum_{m\in\mathbb{C}}w_m z^m\in W\{z\}$, $y\in z_0\mathbb{C}[\![z_0]\!]$ *and write* $D_n=-z^{n+1}\frac{d}{dz}$, $n\in\mathbb{N}$. *Then we have*

1. **(Translation)**

$$e^{-yD_{-1}}w(z)\equiv e^{y\frac{d}{dz}}w(z)=w(z+y). \tag{1.1.31}$$

2. **(Scaling)**

$$(e^y)^{-D_0}w(z)\equiv e^{yz\frac{d}{dz}}w(z)=w(e^y z). \tag{1.1.32}$$

3. **(Projective change)**

$$e^{yD_n}w(z)=w\left[(z^{-n}+ny)^{-1/n}\right]\quad \textit{for } n\neq 0 \tag{1.1.33}$$

with binomial convention.

Proof

1. Write out the expressions as sums.
2. Write out the expressions as sums.
3. We have $D_n=n\frac{d}{d(z^{-n})}$ for $n\neq 0$. Thus, by (1.1.31),

$$e^{yD_n}\left(\sum_{m\in\mathbb{C}}w_m(z^{-n})^{-m/n}\right)=\sum_{m\in\mathbb{C}}w_m(z^{-n}+ny)^{-m/n}=w\left[(z^{-n}+ny)^{-1/n}\right].$$

Note that we have already made use of (1.1.31) symbolically in (1.1.4) and Prop. 4.

For later discussion of meromorphic conformal field theory (see also [39]) it is important to observe that $\{D_{-1},D_0,D_1\}$ generates a representation of the group of Möbius transformations by

$$z\mapsto\frac{az+b}{cz+d},\quad \begin{pmatrix}a&b\\c&d\end{pmatrix}=e^{\frac{b}{d}\rho(D_{-1})}d^{-2\rho(D_0)}e^{-\frac{c}{d}\rho(D_1)},\quad ad-bc=1, \tag{1.1.34}$$

where the identification is given by

$$\rho:\operatorname{span}\{D_{-1},D_0,D_1\}\xrightarrow{\cong}\mathfrak{su}(1,1),$$

$$D_{-1}\mapsto\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad D_0\mapsto\begin{pmatrix}\frac12&0\\0&-\frac12\end{pmatrix},\quad D_1\mapsto\begin{pmatrix}0&0\\-1&0\end{pmatrix}, \tag{1.1.35}$$

so that

$$e^{z_0\rho(D_{-1})}=\begin{pmatrix}1&z_0\\0&1\end{pmatrix},$$

$$e^{z_0\rho(D_0)}=\begin{pmatrix}e^{z_0/2}&0\\0&e^{-z_0/2}\end{pmatrix},$$

$$e^{z_0\rho(D_1)}=\begin{pmatrix}1&0\\-z_0&1\end{pmatrix}.$$

The full set of D_n's, however, establishes a representation of the **Witt algebra**,

$$[D_m,D_n]=(m-n)D_{m+n},$$

the central extension of which is the essential ingredient of two-dimensional conformal field theory.

2. Axiomatics of vertex algebras

We shall give a definition of vertex (operator) algebra (cf. [26]) using the notation of [39], which we believe is more accessible to physicists.

Definition 1

A **vertex algebra** *is a* $\mathbb{Z}$*-graded vector space*

$$\mathcal{F}=\bigoplus_{n\in\mathbb{Z}}\mathcal{F}_{(n)}, \tag{1.2.1}$$

equipped with a linear map $\mathcal{V}:\mathcal{F}\to(\operatorname{End}\mathcal{F})[\![z,z^{-1}]\!]$, *which assigns to each state* $\psi\in\mathcal{F}$ *a* **vertex operator** $\mathcal{V}(\psi,z)$, *and the vertex operators satisfy the following axioms:*

1. **(Regularity)** *If $\psi, \varphi \in \mathcal{F}$ then*

$$\mathrm{Res}_z\,[z^n \mathcal{V}(\psi, z)\varphi] = 0 \quad \text{for } n \text{ sufficiently large} \tag{1.2.2}$$

and n depending on ψ and φ.

2. **(Vacuum)** *There is a preferred state $\mathbf{1} \in \mathcal{F}$, called the vacuum, satisfying*

$$\mathcal{V}(\mathbf{1}, z) = \mathrm{id}_{\mathcal{F}}. \tag{1.2.3}$$

3. **(Injectivity)** *There is a one-to-one correspondence between states and vertex operators:*

$$\mathcal{V}(\psi, z) = 0 \quad \Longleftrightarrow \quad \psi = 0. \tag{1.2.4}$$

4. **(Conformal vector)** *There is a preferred state $\omega \in \mathcal{F}$, called the conformal vector, such that its vertex operator,*

$$\mathcal{V}(\boldsymbol{\omega}, z) = \sum_{n\in\mathbb{Z}} \mathrm{L}_{(n)} z^{-n-2}, \tag{1.2.5}$$

(a) gives the **Virasoro algebra** *with some central charge $c \in \mathbb{R}$,*

$$[\mathrm{L}_{(m)}, \mathrm{L}_{(n)}] = (m-n)\mathrm{L}_{(m+n)} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}; \tag{1.2.6}$$

(b) provides a **translation generator,** $\mathrm{L}_{(-1)}$,

$$\mathcal{V}(\mathrm{L}_{(-1)}\psi, z) = \frac{d}{dz}\mathcal{V}(\psi, z) \quad \text{for every } \psi \in \mathcal{F}; \tag{1.2.7}$$

(c) gives the grading of $\mathcal{F}$ via the eigenvalues of $\mathrm{L}_{(0)}$,

$$\mathrm{L}_{(0)}\psi = n\psi \equiv \Delta_\psi \psi \quad \text{for every } \psi \in \mathcal{F}_{(n)}, n \in \mathbb{Z}; \tag{1.2.8}$$

the eigenvalue Δ_ψ is called the **(conformal) weight** *of ψ.*

5. **(Jacobi identity)** *For every $\psi, \varphi \in \mathcal{F}$,*

$$\begin{aligned} z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) \mathcal{V}(\psi, z_1)\mathcal{V}(\varphi, z_2) - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) \mathcal{V}(\varphi, z_2)\mathcal{V}(\psi, z_1) \\ = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) \mathcal{V}(\mathcal{V}(\psi, z_0)\varphi, z_2), \end{aligned} \tag{1.2.9}$$

where binomial expressions have to be expanded in nonnegative integral powers of the second variable.

We denote the vertex algebra just defined by $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \boldsymbol{\omega})$.

We may think of $\mathcal{F}$ as the space of finite occupation number states in a Fock space so that $\mathcal{F}$ is a dense subspace of the Hilbert space $\mathcal{H}$ of states. Note, however, that for a general vertex algebra a priori no inner product is assumed to exist. The regularity axiom states that, given $\psi, \varphi \in \mathcal{F}$, there is always a high enough power z^n such that $z^n\mathcal{V}(\psi, z)\varphi$ is (at "$z = 0$") a regular formal series. In other words, the regularity axiom ensures that any $\mathcal{V}(\psi, z)\varphi$ contains only a *finite* number of singular (at "$z = 0$") expressions. In terms of creation and annihilation operators it reflects the fact that any finite occupation number state φ is killed by a finite but large enough number of annihilation operators contained in (the normal-ordered expression) ψ_n. We also note that in physical applications the vertex operator of the conformal vector corresponds to the stress–energy tensor of the field theory.

Definition 2

A **vertex operator algebra** *is a vertex algebra with the additional assumptions that*

1. *the spectrum of $\mathrm{L}_{(0)}$ is bounded below;*
2. *the eigenspaces $\mathcal{F}_{(n)}$ of $\mathrm{L}_{(0)}$ are finite-dimensional.*

The first condition is an immediate consequence of a physical postulate. As we will see, $\mathrm{L}_{(0)}$ generates scale transformations. Recalling that the variable z in conformal field theory has its origin in e^{t+ix} (cf. [38]), one finds that $\mathrm{L}_{(0)}$ corresponds to time translations. Thus it may be identified with the energy which should be bounded below in any sensible quantum field theory. In fact, vertex operator algebras can be regarded as a rigorous mathematical definition of chiral algebras in physics [65]. Then the formal variable z can be thought of as a local complex coordinate and the formulas in Prop. 5 can be realized by contour integrals. The vertex operators $\mathcal{V}(\psi, z)$ correspond to holomorphic chiral fields, i.e. they can be viewed as operator-valued distributions on a local coordinate chart of a Riemannn surface. In this context the three terms of the Jacobi identity are geometrically interpreted as the three ways of cutting the four-punctured Riemann sphere into two three-punctured spheres (cf. [32], [81]).

Since vertex operators are operator-valued formal Laurent series, we can give an alternative formulation (see e.g. [10]) of the axioms of a vertex algebra using the mode expansion

$$\mathcal{V}(\psi, z) = \sum_{n\in\mathbb{Z}} \psi_n z^{-n-1}. \tag{1.2.10}$$

One has

1. (Regularity)

$$\psi_n\varphi = 0 \quad \text{for } n \text{ sufficiently large}, \tag{1.2.11}$$

2. (Vacuum)

$$\mathbf{1}_n\psi = \delta_{n+1,0}\psi, \tag{1.2.12}$$

3. (Injectivity)

$$\psi_n = 0 \quad \forall n \in \mathbb{Z} \quad \Longleftrightarrow \quad \psi = 0, \tag{1.2.13}$$

4. (Conformal vector)

$$\omega_{n+1} = \mathrm{L}_{(n)}, \tag{1.2.14}$$

5. (Jacobi identity)

$$\sum_{i\geq 0}(-1)^i\binom{l}{i}\left[\psi_{l+m-i}(\varphi_{n+i}\xi) - (-1)^l\varphi_{l+n-i}(\psi_{m+i}\xi)\right] = \sum_{i\geq 0}\binom{m}{i}(\psi_{l+i}\varphi)_{m+n-i}\xi, \tag{1.2.15}$$

for all $\psi, \varphi, \xi \in \mathcal{F}$, $l, m, n \in \mathbb{Z}$.

To see the equivalence of the two formulations of the Jacobi identity evaluate $\mathrm{Res}_{z_2}\big[\mathrm{Res}_{z_1}\big[\mathrm{Res}_{z_0}\big[z_2^n z_1^m z_0^l$ (1.2.9) $\big]\big]\big]$. As an intermediate result one finds, by using (1.1.12), yet another version of the Jacobi identity which occurs in the literature [81], [32]:

$$\begin{aligned}\mathrm{Res}_{z_1}\left[\mathcal{V}(\psi, z_1)\mathcal{V}(\varphi, z_2)\iota_{12}\left((z_1-z_2)^l z_1^m\right) - \mathcal{V}(\varphi, z_2)\mathcal{V}(\psi, z_1)\iota_{21}\left((z_1-z_2)^l z_1^m\right)\right] \\ = \mathrm{Res}_{z_0}\left[\mathcal{V}(\mathcal{V}(\psi, z_0)\varphi, z_2)\iota_{20}\left(z_0^l(z_2+z_0)^m\right)\right].\end{aligned} \tag{1.2.16}$$

As one might suspect, the Jacobi identity contains most information about a vertex algebra. In fact we will see that one can derive from it important properties such as locality and duality in conformal field theory (cf. [39]). If we put $l = m = n = 0$ in (1.2.15) we get $\psi_0(\varphi_0\xi) - \varphi_0(\psi_0\xi) = (\psi_0\varphi)_0\xi$. Later (see Sect. 5) we will define an antisymmetric product on the subspace $\mathcal{F}/\mathrm{L}_{(-1)}\mathcal{F}$ of $\mathcal{F}$ by $[\psi, \varphi] := \psi_0\varphi$. Then the above formula indeed establishes the classical Jacobi identity for Lie algebras. On the other hand, choosing $\psi = \varphi = \mathbf{1}$ in the Jacobi identity for vertex algebras and using the vacuum axiom we recover (1.1.13) and thus the Cauchy theorem (1.1.30). Hence the Jacobi identity for vertex algebras may be regarded as a combination of the classical Jacobi identity for Lie algebras and the Cauchy residue formula for meromorphic functions.

In what follows we will frequently make use of two important formulas which are the special cases $m = 0$ and $l = 0$, respectively, of (1.2.15):

(Associativity formula)

$$(\psi_l\varphi)_n = \sum_{i\geq 0}(-1)^i\binom{l}{i}\left[\psi_{l-i}\varphi_{n+i} - (-1)^l\varphi_{l+n-i}\psi_i\right], \tag{1.2.17}$$

(Commutator formula)

$$[\psi_m, \varphi_n] = \sum_{i\geq 0}\binom{m}{i}(\psi_i\varphi)_{m+n-i}, \tag{1.2.18}$$

for all $\psi, \varphi \in \mathcal{F}$, $l, m, n \in \mathbb{Z}$.

Actually, we observe that the associativity formula is equivalent to the Jacobi identity. From the commutator formula we can immediately infer the interesting result that in any vertex algebra the zero mode operators, ψ_0, act as derivations on the products $\varphi_n\chi$, viz

$$\psi_0(\varphi_n\chi) = (\psi_0\varphi)_n\chi + \varphi_n(\psi_0\chi). \tag{1.2.19}$$

In particular we conclude that the Jacobi identity alone already implies the remarkable fact that the zero mode operators always form a closed Lie algebra:

$$[\psi_0, \varphi_0] = (\psi_0\varphi)_0 \qquad \forall \psi, \varphi \in \mathcal{F}. \tag{1.2.20}$$

3. Basic properties of vertex algebras

To become familiar with the formalism and the axioms it is instructive to derive some important properties of vertex algebras. Iterating (1.2.7) and using the translation (1.1.31), we find that $\mathrm{L}_{(-1)}$ indeed generates translations,

$$\mathcal{V}\left(e^{z_0\mathrm{L}_{(-1)}}\psi, z\right) = \mathcal{V}(\psi, z+z_0). \tag{1.3.1}$$

Moreover, the vacuum is translation invariant because (1.2.7) for $\psi = \mathbf{1}$, together with the vacuum axiom (1.2.3) and the injectivity (1.2.4), gives

$$\mathrm{L}_{(-1)}\mathbf{1} = 0. \tag{1.3.2}$$

Take $\mathrm{Res}_{z_0}\left[\mathrm{Res}_{z_1}\left[z_1^n \text{ (1.2.9) }\right]\right]$,

$$[\psi_n, \mathcal{V}(\varphi, z)] = \sum_{i\geq 0}\binom{n}{i}\mathcal{V}(\psi_i\varphi, z)z^{n-i}. \tag{1.3.3}$$

In the special case $\psi = \omega$ we obtain

$$[\mathrm{L}_{(n)}, \mathcal{V}(\varphi, z)] = \sum_{i \geq -1} \binom{n+1}{i+1} \mathcal{V}(\mathrm{L}_{(i)}\varphi, z) z^{n-i}; \tag{1.3.4}$$

in particular,

$$[\mathrm{L}_{(-1)}, \mathcal{V}(\varphi, z)] = \frac{d}{dz}\mathcal{V}(\varphi, z), \tag{1.3.5}$$

$$[\mathrm{L}_{(0)}, \mathcal{V}(\varphi, z)] = \left(z\frac{d}{dz} + \Delta_\varphi\right) \mathcal{V}(\varphi, z) \quad \text{if } \varphi \in \mathcal{F}_{(\Delta_\varphi)}. \tag{1.3.6}$$

By the use of the well-known formula

$$\mathrm{e}^A B \mathrm{e}^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathrm{ad}_A)^n B \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{[A, [A, \ldots [A,}_{n} B]] \ldots] \tag{1.3.7}$$

and Eqs. (1.1.31), (1.1.32), the above equations give, respectively,

(Translation property)

$$\mathrm{e}^{y\mathrm{L}_{(-1)}} \mathcal{V}(\varphi, z) \mathrm{e}^{-y\mathrm{L}_{(-1)}} = \mathcal{V}(\varphi, z+y), \tag{1.3.8}$$

(Scaling property)

$$\mathrm{e}^{y\mathrm{L}_{(0)}} \mathcal{V}(\varphi, z) \mathrm{e}^{-y\mathrm{L}_{(0)}} = \mathrm{e}^{y\Delta_\varphi} \mathcal{V}(\varphi, \mathrm{e}^y z) \quad \text{if } \varphi \in \mathcal{F}_{(\Delta_\varphi)}, \tag{1.3.9}$$

for every $y \in z_0\mathbb{C}[\![z_0]\!]$ by Prop. 6.

Thus $\mathrm{L}_{(0)}$ generates scale transformations. Note that (1.3.6) also implies that

$$\varphi_n \mathcal{F}_{(m)} \subset \mathcal{F}_{(\Delta_\varphi + m - n - 1)} \quad \text{if } \varphi \in \mathcal{F}_{(\Delta_\varphi)}, \tag{1.3.10}$$

which means that the operator φ_n shifts the grading by $\Delta_\varphi - n - 1$, i.e. it can be assigned "degree" $\Delta_\varphi - n - 1$. In view of this relation the reader might wonder again why we use subscripts in round brackets for the grading of $\mathcal{F}$ and for the Virasoro generators, in contrast to the naked subscripts occurring in the mode expansion (1.2.10) of a vertex operator. This possibly causes some confusion but stems from the fact that we employ two different mode expansions. In conformal field theory we are acquainted with the expansion

$$\psi(z) \equiv \mathcal{V}(\psi, z) = \sum_{n \in \mathbb{Z}} \psi_{(n)} z^{-n-\Delta_\psi}, \tag{1.3.11}$$

which depends on the conformal weight of the "chiral field" $\psi(z)$. To exhibit explicitly the Virasoro algebra in the definition of a vertex algebra, we used this expansion for the vertex operator associated with the conformal vector (stress–energy tensor!) in (1.2.5). It is quite easy to convert results obtained in one expansion into the other formalism — simply by shifting the grading:

$$\psi_n \equiv \psi_{(n+1-\Delta_\psi)}, \tag{1.3.12}$$

$$\psi_{(n)} \equiv \psi_{n-1+\Delta_\psi}, \tag{1.3.13}$$

for any homogeneous element $\psi \in \mathcal{F}$. For example, we can rewrite (1.3.10) as

$$\varphi_{(n)} \mathcal{F}_{(m)} \subset \mathcal{F}_{(m-n)}, \tag{1.3.14}$$

so that $\varphi_{(n)}$ always has "degree" $-n$ irrespective of φ. The mode expansion (1.3.11) is therefore the more natural one because it respects the grading of $\mathcal{F}$. On the other hand, for formal calculus it is more useful to stick to an expansion which does not refer to the conformal weight of a (in general not homogeneous!) state. Hence we shall almost everywhere in the formulas assume the mode expansion (1.2.10).

Let us exploit the fact that the Jacobi identity is obviously invariant under $(\psi, z_1, z_0) \leftrightarrow (\varphi, z_2, -z_0)$:

$$\begin{aligned} z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) \mathcal{V}(\mathcal{V}(\psi, z_0)\varphi, z_2) &= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) \mathcal{V}(\mathcal{V}(\varphi, -z_0)\psi, z_1) \\ &= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) \mathcal{V}(\mathcal{V}(\varphi, -z_0)\psi, z_2 + z_0) \\ &= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) \mathcal{V}(\mathcal{V}(\varphi, -z_0)\psi, z_2 + z_0) \end{aligned}$$

by (1.1.6) and (1.1.12), respectively. Taking $\mathrm{Res}_{z_1}[\ldots]$ we get

$$\begin{aligned} \mathcal{V}(\mathcal{V}(\psi, z_0)\varphi, z_2) &= \mathcal{V}(\mathcal{V}(\varphi, -z_0)\psi, z_2 + z_0) \\ &= \mathcal{V}\left(\mathrm{e}^{z_0\mathrm{L}_{(-1)}} \mathcal{V}(\varphi, -z_0)\psi, z_2\right) \quad \text{by (1.3.1)}. \end{aligned} \tag{1.3.15}$$

The injectivity (1.2.4) finally yields

(Skew symmetry)

$$\mathcal{V}(\psi, z_0)\varphi = \mathrm{e}^{z_0\mathrm{L}_{(-1)}} \mathcal{V}(\varphi, -z_0)\psi \tag{1.3.16}$$

or, in components,

$$\psi_n\varphi = -(-1)^n \varphi_n\psi + \sum_{i \geq 1} \frac{1}{i!}(-1)^{i+n+1} \left(\mathrm{L}_{(-1)}\right)^i (\varphi_{n+i}\psi). \tag{1.3.17}$$

In particular, we observe that the vertex operator $\mathcal{V}(\psi, z)$ "creates" the state $\psi \in \mathcal{F}$ when applied to the vacuum:

$$\mathcal{V}(\psi, z)\mathbf{1} = e^{z\mathrm{L}_{(-1)}}\psi \tag{1.3.18}$$

by (1.2.3). In components,

$$\psi_n \mathbf{1} = \begin{cases} 0 & \text{for } n \geq 0, \\ \psi & \text{for } n = -1, \\ \frac{1}{(-n-1)!}\left(\mathrm{L}_{(-1)}\right)^{-n-1}\psi & \text{for } n \leq -2. \end{cases} \tag{1.3.19}$$

Hence the vacuum satisfies

$$\mathrm{L}_{(n)}\mathbf{1} = 0 \qquad \forall n \geq -1. \tag{1.3.20}$$

We shall denote by $\mathcal{P}_{(\Delta)}$ the space of **(conformal) highest weight vectors** or **primary states** satisfying

$$\begin{aligned} \mathrm{L}_{(0)}\psi &= \Delta\psi \qquad \text{i.e. } \psi \in \mathcal{F}_{(\Delta)}, \\ \mathrm{L}_{(n)}\psi &= 0 \qquad \forall n > 0. \end{aligned} \tag{1.3.21}$$

Thus in any vertex algebra the vacuum is a primary state of weight 0 and therefore $\mathcal{F}_{(0)}$ is always at least one-dimensional. We immediately infer from (1.3.4) that, for $\psi \in \mathcal{P}_{(\Delta)}$,

$$[\mathrm{L}_{(n)}, \mathcal{V}(\psi, z)] = z^n \left\{ z\frac{d}{dz} + (n+1)\Delta \right\} \mathcal{V}(\psi, z) \qquad \forall n \in \mathbb{Z} \tag{1.3.22}$$

or

$$[\mathrm{L}_{(n)}, \psi_m] = \{(\Delta - 1)(n+1) - m\}\, \psi_{m+n} \qquad \forall m, n \in \mathbb{Z}, \tag{1.3.23}$$

i.e. $\mathcal{V}(\psi, z)$ is a so-called **(conformal) primary field** of weight Δ. We can rewrite (1.3.22) as

$$[\mathrm{L}_{(n)}, z^{\Delta(n+1)}\mathcal{V}(\psi, z)] = z^{n+1}\frac{d}{dz}\left\{ z^{\Delta(n+1)}\mathcal{V}(\psi, z) \right\}, \tag{1.3.24}$$

so that, by (1.1.33),

$$e^{y\mathrm{L}_{(n)}}\mathcal{V}(\psi, z)e^{-y\mathrm{L}_{(n)}} = \left(\frac{\partial z_1}{\partial z}\right)^{\Delta} \mathcal{V}(\psi, z_1) \qquad \forall n \neq 0, \tag{1.3.25}$$

for every $y \in z_0\mathbb{C}[\![z_0]\!]$ where $z_1 = (z^{-n} - ny)^{-1/n} = z(1 - nyz^n)^{-1/n}$.

The operators $\{\mathrm{L}_{(-1)}, \mathrm{L}_{(0)}, \mathrm{L}_{(1)}\}$ satisfy the $\mathfrak{su}(1,1)$ Lie algebra

$$[\mathrm{L}_{(0)}, \mathrm{L}_{(1)}] = -\mathrm{L}_{(1)}, \qquad [\mathrm{L}_{(0)}, \mathrm{L}_{(-1)}] = \mathrm{L}_{(-1)}, \qquad [\mathrm{L}_{(1)}, \mathrm{L}_{(-1)}] = 2\mathrm{L}_{(0)}. \tag{1.3.26}$$

Hence we have established the following Möbius transformation properties of the vertex operators (see also [39]): If $\psi \in \mathcal{F}$ is a **quasiprimary state** of weight Δ, i.e. ψ satisfies $\mathrm{L}_{(0)}\psi = \Delta\psi$ and $\mathrm{L}_{(1)}\psi = 0$, then

$$D_\gamma \mathcal{V}(\psi, z) D_\gamma^{-1} = \left(\frac{d\gamma(z)}{dz}\right)^{\Delta} \mathcal{V}(\psi, \gamma(z)), \tag{1.3.27}$$

where

$$\gamma(z) = \frac{az + b}{cz + d}, \qquad D_\gamma = e^{\frac{b}{d}\mathrm{L}_{(-1)}} \left(\frac{\sqrt{ad - bc}}{d}\right)^{2\mathrm{L}_{(0)}} e^{-\frac{c}{d}\mathrm{L}_{(1)}}, \tag{1.3.28}$$

for $a, b, c, d \in z_0\mathbb{C}[\![z_0]\!]$ (cf. end of Sect. 1).

Now Eq. (1.3.20) tells us that the vacuum vector is $SU(1,1)$-invariant and the question arises whether the state $\mathbf{1}$ is uniquely (up to scalar multiples) characterized by this property. In general the answer is no, but $SU(1,1)$-invariant states come quite close to the properties of the vacuum. To see this suppose that $\kappa \in \mathcal{F}$ satisfies $\mathrm{L}_{(n)}\kappa = 0,\ n = 0, \pm 1$. Then it follows from Eqs. (1.2.7), (1.2.10) that $\mathrm{L}_{(-1)}\kappa = 0$ is equivalent to $\kappa_n = 0$ for $n \neq -1$, in agreement with (1.2.12). However, as regards κ_{-1} the associativity formula (1.2.17) and the commutator formula (1.2.18) yield $(\kappa_{-1}\varphi)_n = \kappa_{-1}\varphi_n = \varphi_n\kappa_{-1}\ \forall\varphi \in \mathcal{F},\ n \in \mathbb{Z}$. For general vertex algebras this does *not* force κ_{-1} to be a scalar multiple of the identity but rather states that κ_{-1} may be regarded as a Casimir operator of the vertex algebra. In the case of a *simple* vertex algebra (i.e. the vertex algebra constitutes an irreducible module for itself; cf. [26]), we can apply Schur's lemma to infer that κ_{-1} is indeed a scalar multiple of the identity. We conclude that for simple vertex algebras the vacuum vector is the unique $SU(1,1)$-invariant state. In concrete examples, however, uniqueness of the vacuum vector is often established by proving that $\mathcal{F}_{(0)}$ or $\mathcal{F}_{(0)} \cap \ker \mathrm{L}_{(-1)}$ (translation invariant states of weight 0) are one-dimensional.

Finally, using the Virasoro algebra (1.2.6) and (1.3.19) we find that

$$\begin{aligned} \omega_l\omega &= \omega_l(\omega_{-1}\mathbf{1}) \\ &= [\omega_l, \omega_{-1}]\mathbf{1} + \omega_{-1}(\omega_l\mathbf{1}) \\ &= (l+1)\omega_{l-2}\mathbf{1} + \frac{c}{2}\delta_{l,3}\mathbf{1} + \omega_{-1}(\omega_l\mathbf{1}), \end{aligned} \tag{1.3.29}$$

i.e. (cf. [10])

$$\omega_1\omega = 2\omega, \tag{1.3.30}$$

$$\omega_2\omega = 0, \tag{1.3.31}$$

$$\omega_3\omega = \tfrac{c}{2}, \tag{1.3.32}$$

$$\omega_l\omega = 0 \quad \text{for } l > 3. \tag{1.3.33}$$

In particular, ω is a quasiprimary state of conformal weight 2. Note that ω is characterized by the above relations together with (1.2.8) and $\omega_0\psi = \psi_{-2}\mathbf{1}$ for $\psi \in \mathcal{F}$.

4. Locality and duality for vertex algebras

To complete the relation between vertex algebras and conformal field theory, we consider matrix elements of products of vertex operators. Define the **restricted dual** of $\mathcal{F}$,

$$\mathcal{F}' \equiv \bigoplus_{n\in\mathbb{Z}} \mathcal{F}^*_{(n)}, \tag{1.4.1}$$

the direct sum of the dual spaces of the homogeneous subspaces $\mathcal{F}_{(n)}$, i.e. the space of linear functionals on the vertex algebra $\mathcal{F}$ vanishing on all but finitely many $\mathcal{F}_{(n)}$. We shall use $\langle\_|\_\rangle$ for the natural pairing between $\mathcal{F}$ and $\mathcal{F}'$. From the regularity axiom and (1.3.10) it is clear that any matrix element of a vertex operator is a Laurent polynomial in z,

$$\langle\chi^*|\mathcal{V}(\psi,z)\varphi\rangle \in \mathbb{C}[z,z^{-1}] \quad \text{for all } \chi^* \in \mathcal{F}',\ \psi,\varphi\in\mathcal{F}, \tag{1.4.2}$$

and in this sense these three-point correlation functions may be regarded as meromorphic functions of z. Of course, we identify formally χ^* with an "out-state" inserted at $z=\infty$ and φ with an "in-state" inserted at $z=0$. We have the following important theorem due to [30].

Theorem 1

1. **(Locality $\equiv$ rationality of products + commutativity)**
For $\chi^* \in \mathcal{F}'$, $\psi,\varphi,\xi\in\mathcal{F}$, the formal series $\langle\chi^*|\mathcal{V}(\psi,z_1)\mathcal{V}(\varphi,z_2)\xi\rangle$ which involves only finitely many negative powers of z_2 and only finitely many positive powers of z_1, lies in the image of the map ι_{12}:

$$\langle\chi^*|\mathcal{V}(\psi,z_1)\mathcal{V}(\varphi,z_2)\xi\rangle = \iota_{12}f(z_1,z_2), \tag{1.4.3}$$

where the (uniquely determined) element $f\in\mathbb{C}[z_1,z_2]_S$ is of the form

$$f(z_1,z_2) = \frac{g(z_1,z_2)}{z_1^r z_2^s (z_1-z_2)^t} \tag{1.4.4}$$

for some polynomial $g(z_1,z_2)\in\mathbb{C}[z_1,z_2]$ and $r,s,t\in\mathbb{Z}$. We also have

$$\langle\chi^*|\mathcal{V}(\varphi,z_2)\mathcal{V}(\psi,z_1)\xi\rangle = \iota_{21}f(z_1,z_2), \tag{1.4.5}$$

i.e. $\mathcal{V}(\psi,z_1)\mathcal{V}(\varphi,z_2)$ agrees with $\mathcal{V}(\varphi,z_2)\mathcal{V}(\psi,z_1)$ as operator-valued rational functions.

2. **(Duality $\equiv$ rationality of iterates + associativity)**
For $\chi^*\in\mathcal{F}'$, $\psi,\varphi,\xi\in\mathcal{F}$, the formal series $\langle\chi^*|\mathcal{V}(\mathcal{V}(\psi,z_0)\varphi,z_2)\xi\rangle$ which involves only finitely many negative powers of z_0 and only finitely many positive powers of z_2, lies in the image of the map ι_{20}:

$$\langle\chi^*|\mathcal{V}(\mathcal{V}(\psi,z_0)\varphi,z_2)\xi\rangle = \iota_{20}f(z_0+z_2,z_2), \tag{1.4.6}$$

with the same f as above, and

$$\langle\chi^*|\mathcal{V}(\psi,z_0+z_2)\mathcal{V}(\varphi,z_2)\xi\rangle = \iota_{02}f(z_0+z_2,z_2), \tag{1.4.7}$$

i.e. $\mathcal{V}(\psi,z_1)\mathcal{V}(\varphi,z_2)$ agrees with $\mathcal{V}(\mathcal{V}(\psi,z_1-z_2)\varphi,z_2)$ as operator-valued rational functions, where the right hand expression is to be expanded as a Laurent series in z_1-z_2.

Proof
(taken from [30], [26])

1. Using (1.1.13) we can rewrite the Jacobi identity in the form

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\mathcal{V}(\psi,z_1)\mathcal{V}(\varphi,z_2) - z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)\mathcal{V}(\varphi,z_2)\mathcal{V}(\psi,z_1) \\ &= \mathcal{V}\left(\left[z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\mathcal{V}(\psi,z_1-z_2) - z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)\mathcal{V}(\psi,-z_2+z_1)\right]\varphi,z_2\right).\end{aligned}$$

Taking $\mathrm{Res}_{z_0}[\ldots]$ leads to the commutator formula

$$[\mathcal{V}(\psi,z_1),\mathcal{V}(\varphi,z_2)] = \mathcal{V}\left([\mathcal{V}(\psi,z_1-z_2)-\mathcal{V}(\psi,-z_2+z_1)]\varphi,z_2\right). \tag{1.4.8}$$

By (1.4.2), the matrix element $\langle\chi^*|\ldots\xi\rangle$ of the right hand side is clearly an expansion of zero in the variables z_1, z_2 of the form

$$\langle\chi^*|\mathcal{V}\left([\mathcal{V}(\psi,z_1-z_2)-\mathcal{V}(\psi,-z_2+z_1)]\varphi,z_2\right)\xi\rangle = \Theta_{12}\left[\frac{g(z_1,z_2)}{z_2^s(z_1-z_2)^t}\right],$$

with $g(z_1,z_2)$, s,t as stated above. Thus

$$\begin{aligned}&\langle\chi^*|\mathcal{V}(\psi,z_1)\mathcal{V}(\varphi,z_2)\xi\rangle - \iota_{12}\left[\frac{g(z_1,z_2)}{z_2^s(z_1-z_2)^t}\right] \\ &\quad = \langle\chi^*|\mathcal{V}(\varphi,z_2)\mathcal{V}(\psi,z_1)\xi\rangle - \iota_{21}\left[\frac{g(z_1,z_2)}{z_2^s(z_1-z_2)^t}\right].\end{aligned}$$

But the left hand side involves only finitely many positive powers of z_1, by (1.3.10), and the right hand side involves only finitely many negative powers of z_1, by the regularity axiom.

If we further take into account that, by (1.4.2), the coefficient of each power of z_1 on either side is a Laurent polynomial in z_2, then

$$f(z_1, z_2) := \frac{g(z_1, z_2)}{z_2^s (z_1 - z_2)^t} + h(z_1, z_2),$$

for some $h(z_1, z_2) \in \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]$, satisfies the desired conditions.

2. Using (1.1.12) and (1.1.6) we can rewrite the Jacobi identity in the form

$$z_1^{-1}\delta\left(\frac{z_0+z_2}{z_1}\right)\mathcal{V}(\psi, z_0+z_2)\mathcal{V}(\varphi, z_2) - z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)\mathcal{V}(\varphi, z_2)\mathcal{V}(\psi, z_1)$$
$$= z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)\mathcal{V}(\mathcal{V}(\psi, z_0)\varphi, z_2).$$

Thus

$$z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)\mathcal{V}(\mathcal{V}(\psi, z_0)\varphi, z_2) - z_1^{-1}\delta\left(\frac{z_0+z_2}{z_1}\right)\mathcal{V}(\psi, z_0+z_2)\mathcal{V}(\varphi, z_2)$$
$$= \left[z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) - z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\right]\mathcal{V}(\varphi, z_2)\mathcal{V}(\psi, z_1) \quad \text{by (1.1.13)}$$
$$= \mathcal{V}(\varphi, z_2)\left[z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)\mathcal{V}(\psi, z_2+z_0) - z_1^{-1}\delta\left(\frac{z_0+z_2}{z_1}\right)\mathcal{V}(\psi, z_0+z_2)\right]$$
by (1.1.12), (1.1.6).

Taking $\mathrm{Res}_{z_1}[\ldots]$ leads to

$$\mathcal{V}(\mathcal{V}(\psi, z_0)\varphi, z_2) - \mathcal{V}(\psi, z_0+z_2)\mathcal{V}(\varphi, z_2)$$
$$= \mathcal{V}(\varphi, z_2)\left[\mathcal{V}(\psi, z_2+z_0) - \mathcal{V}(\psi, z_0+z_2)\right].$$

We use this formula in place of the above commutator formula and apply the same arguments as in part one to obtain the desired result. To get the last statement put formally $z_0 = z_1 - z_2$.

The first part of Theorem 1 in particular states that these matrix elements may be viewed as meromorphic functions of the formal variables. Thus vertex algebras can be seen as a rigorous formulation of meromorphic conformal field theories. Note that the second part of the theorem should be interpreted as "duality (crossing symmetry) of the four-point correlation function on the Riemann sphere." It establishes a precise formulation of an operator product expansion in two-dimensional conformal field theory (see e.g. [4], [38], [39]) in the sense that $\mathcal{V}(\psi, z_1)\mathcal{V}(\varphi, z_2)$ agrees with $\sum_{n\in\mathbb{Z}}(z_1 - z_2)^{-n-1}\mathcal{V}(\psi_n\varphi, z_2)$ as operator-valued rational functions.The theorem then also ensures that this operator product expansion involves only finitely many singular (at "$z_1 = z_2$") terms. It is worth mentioning that Theorem 1 contains the full information about the Jacobi identity, i.e. one can derive the latter starting from the principles of locality and duality. Even more is true. By the use of products of three vertex operators, duality follows from locality, (1.3.5), (1.3.6) and the axioms for vertex algebras except for the Jacobi identity. In particular, in the definition of a vertex algebra the Jacobi identity may be replaced by the principle of locality, (1.3.5) and (1.3.6). Proofs of these statements can be found in [26], where the generalization of the above notion of duality to arbitrary n-point functions is also presented.

If we regard our formal variables as *complex* variables, then the formal expansions of rational functions that we have been discussing converge in suitable domains. The matrix elements in (1.4.3) and (1.4.5) converge to a common rational function in the disjoint domains $|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$, respectively. The matrix elements in (1.4.6) and (1.4.7) for $z_0 = z_1 - z_2$ converge to a common rational function in the domains $|z_2| > |z_1 - z_2| > 0$ and $|z_1| > |z_2| > 0$, respectively, and in the common domain $|z_1| > |z_2| > |z_1 - z_2| > 0$ these two series converge to the common function.

Finally, we shall discuss the space $\mathcal{F}'$ and the pairing $\langle \_|\_ \rangle$ in more detail. We can assign to each state $\psi \in \mathcal{F}$ a **contragredient vertex operator** $\mathcal{V}^*(\psi, z) = \sum \psi_n^* z^{-n-1} \in (\mathrm{End}\,\mathcal{F}')[\![z, z^{-1}]\!]$ by the condition

$$\langle \mathcal{V}^*(\psi, z)\chi^* | \varphi \rangle = \langle \chi^* | \mathcal{V}\left(e^{z\mathrm{L}_{(1)}}(-z^{-2})^{\mathrm{L}_{(0)}}\psi, z^{-1}\right)\varphi \rangle . \tag{1.4.9}$$

It is crucial to observe that the formal sum on the right hand side in general does not exist (check!) unless $(\mathrm{L}_{(1)})^n\psi = 0$ for n large enough, which is assured if the spectrum of $\mathrm{L}_{(0)}$ is bounded below. For the remainder of this section we will therefore assume that $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \boldsymbol{\omega})$ is a vertex operator algebra. Without giving the precise definition of a module for a vertex operator algebra, we just state that with this definition $(\mathcal{F}', \mathcal{V}^*)$ and $(\mathcal{F}, \mathcal{V})$ become $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \boldsymbol{\omega})$ modules (For a proof and the relevant definitions see [26].) If we furthermore have a grading-preserving linear isomorphism $F : \mathcal{F} \to \mathcal{F}'$, $\psi \mapsto F(\psi) \equiv \psi^*$, satisfying $F(\psi_n\varphi) = F(\psi)_n F(\varphi)$ (i.e. $F \circ \psi_n = \psi_n^* \circ F$), then this amounts to choosing a nondegenerate bilinear form $(\_|\_)$ on $\mathcal{F}$ as $(\chi|\varphi) := \langle F(\chi)|\varphi\rangle$ with the **adjoint vertex operator** defined by

$$\mathcal{V}^\dagger(\psi, z) = \sum_{n\in\mathbb{Z}} \psi_n^\dagger z^{-n-1}$$
$$:= \mathcal{V}\left(e^{z\mathrm{L}_{(1)}}(-z^{-2})^{\mathrm{L}_{(0)}}\psi, z^{-1}\right) \in (\mathrm{End}\,\mathcal{F})[\![z, z^{-1}]\!] \tag{1.4.10}$$

such that $(\mathcal{V}(\psi, z)\chi|\varphi) = (\chi|\mathcal{V}^\dagger(\psi, z)\varphi)$.

This definition of an adjoint vertex operator is quite close to the one familiar to physicists, as one immediately sees by calculating explicitly the adjoint vertex operator associated with a quasiprimary state ψ :

$$\mathcal{V}^\dagger(\psi, z) = (-1)^{\Delta_\psi} z^{-2\Delta_\psi}\mathcal{V}(\psi, z^{-1}) \qquad \text{(since } \psi \text{ is quasiprimary)}$$
$$= (-1)^{\Delta_\psi}\sum_{n\in\mathbb{Z}} \psi_n z^{n+1-2\Delta_\psi}, \tag{1.4.11}$$

i.e.

$$\psi_n^\dagger = (-1)^{\Delta_\psi}\psi_{-n+2\Delta_\psi-2} \qquad \forall n \in \mathbb{Z}. \tag{1.4.12}$$

With the shifted grading $\psi_{(n)} \equiv \psi_{n+\Delta_\psi-1}$ this reads

$$\psi_{(n)}^\dagger = (-1)^{\Delta_\psi}\psi_{(-n)} \qquad \forall n \in \mathbb{Z}. \tag{1.4.13}$$

In particular, we observe that the vacuum vertex operator (the identity) is self-adjoint and that the Virasoro generators satisfy the well-known relation $L_{(n)}^\dagger = L_{(-n)}$ or, in terms of the "stress–energy tensor", $\mathcal{V}^\dagger(\omega, z) = \frac{1}{z^4}\mathcal{V}(\omega, z^{-1})$ [1]. Hence we obtain for any two homogeneous elements $\chi \in \mathcal{F}_{(m)}, \varphi \in \mathcal{F}_{(n)}$,

$$(m-n)(\chi|\varphi) = (L_{(0)}\chi|\varphi) - (\chi|L_{(0)}\varphi) = 0,$$

i.e. the homogeneous subspaces $\mathcal{F}_{(n)}$, $n \in \mathbb{Z}$, are orthogonal to each other with respect to this bilinear form,

$$(\mathcal{F}_{(m)}|\mathcal{F}_{(n)}) = 0 \quad \text{if } m \neq n. \tag{1.4.14}$$

A straightforward calculation shows that for an $\mathfrak{su}(1,1)$**-descendant state**

$$\psi^{(N)} \equiv \frac{1}{N!}(L_{(-1)})^N\psi = \psi_{-N-1}\mathbf{1} \tag{1.4.15}$$

the adjoint is given by

$$(\psi_n^{(N)})^\dagger = \sum_{i=0}^{N}(-1)^{\Delta_\psi+i}\binom{2\Delta_\psi+i}{N-i}\psi^{(i)}_{-n+N+i+2\Delta_\psi-2}, \tag{1.4.16}$$

in agreement with (1.4.12) for $N = 0$.

To check whether adjointness satisfies the involution property $\mathcal{V}^{\dagger\dagger} = \mathcal{V}$, we note that the commutation relation $[L_{(0)}, L_{(n)}] = -nL_{(n)}$ yields $z_0^{L_{(0)}}L_{(n)}z_0^{-L_{(0)}} = z_0^{-n}L_{(n)}$, which by iteration yields the conjugation formula

$$z_0^{L_{(0)}}e^{zL_{(n)}}z_0^{-L_{(0)}} = e^{z_0^{-n}zL_{(n)}}, \tag{1.4.17}$$

so that we have indeed

$$\mathcal{V}\left(e^{z^{-1}L_{(1)}}(-z^2)^{L_{(0)}}e^{zL_{(1)}}(-z^{-2})^{L_{(0)}}\psi, z\right) = \mathcal{V}(\psi, z) \qquad \forall\psi \in \mathcal{F}. \tag{1.4.18}$$

It is by no means obvious from the definition that the bilinear form is symmetric. To establish symmetry we first note that $(\chi|\varphi) = \mathrm{Res}_z\left[z^{-1}(\mathcal{V}(\chi, z)\mathbf{1}|\varphi)\right]$ by (1.3.18). Therefore it is sufficient to prove that $(\mathcal{V}(\chi, z)\mathbf{1}|\varphi) = (\varphi|\mathcal{V}(\chi, z)\mathbf{1})$. Now,

$$\begin{aligned}
(\mathcal{V}(\chi, z)\mathbf{1}|\varphi) &= (\mathbf{1}\,|\,\mathcal{V}\left(e^{zL_{(1)}}(-z^{-2})^{L_{(0)}}\chi, z^{-1}\right)\varphi) && \text{by definition}\\
&= (\mathbf{1}\,|\,e^{z^{-1}L_{(-1)}}\mathcal{V}(\varphi, -z^{-1})e^{zL_{(1)}}(-z^{-2})^{L_{(0)}}\chi) && \text{by skew symmetry}\\
&= (\mathcal{V}\left(e^{-z^{-1}L_{(1)}}(-z^2)^{L_{(0)}}\varphi, -z\right)\mathbf{1}\,|\,e^{zL_{(1)}}(-z^{-2})^{L_{(0)}}\chi) && \text{by involution}\\
&= (\mathcal{V}(\mathbf{1}, z)e^{-z^{-1}L_{(1)}}(-z^2)^{L_{(0)}}\varphi\,|\,(-z^{-2})^{L_{(0)}}\chi) && \text{by skew symmetry}\\
&= (\varphi\,|\,(-z^2)^{L_{(0)}}e^{-z^{-1}L_{(-1)}}(-z^{-2})^{L_{(0)}}\chi) && \text{by definition}\\
&= (\varphi|\mathcal{V}(\chi, z)\mathbf{1}) && \text{by conjugation,}
\end{aligned}$$

so that the bilinear form is indeed symmetric.

5. Algebras of primary fields of weight 1

We shall provide a certain subspace of the Fock space $\mathcal{F}$ with the structure of a Lie algebra (cf. [11], [10], [30]). We define a bilinear product on $\mathcal{F}$ by

$$[\psi, \varphi] := \psi_0\varphi, \tag{1.5.1}$$

which is antisymmetric on the quotient space $\mathcal{F}/L_{(-1)}\mathcal{F}$ due to the skew symmetry property (1.3.17). Putting $l = m = n = 0$ in the Jacobi identity (1.2.15) we get $\psi_0(\varphi_0\xi) - \varphi_0(\psi_0\xi) = (\psi_0\varphi)_0\xi$. But this equation translates precisely into the classical Jacobi identity for Lie algebras,

$$[[\psi, \varphi], \xi] + [[\varphi, \xi], \psi] + [[\xi, \psi], \varphi] = 0, \tag{1.5.2}$$

on $\mathcal{F}/L_{(-1)}\mathcal{F}$.

If we take a closer look at the above definition of the Lie bracket we might be tempted to identify ψ_0 with the adjoint action of ψ on $\mathcal{F}/L_{(-1)}\mathcal{F}$,

$$\mathrm{ad}_\psi(\varphi) = [\psi, \varphi] = \psi_0(\varphi). \tag{1.5.3}$$

Indeed, it is not difficult to show that the Lie algebra of zero mode operators, $\{\psi_0\,|\,\psi \in \mathcal{F}\}$, is the adjoint representation of $\mathcal{F}/L_{(-1)}\mathcal{F}$. First note that the linear map $\mathcal{F} \to \mathrm{End}\,\mathcal{F}$, $\psi \mapsto \psi_0$, reduces to a well-defined map on $\mathcal{F}/L_{(-1)}\mathcal{F}$; for if $\psi = L_{(-1)}\varphi \in L_{(-1)}\mathcal{F}$ for some $\varphi \in \mathcal{F}$, then $\psi_0 = \mathrm{Res}_z\left[\mathcal{V}(L_{(-1)}\varphi, z)\right] = \mathrm{Res}_z\left[\frac{d}{dz}\mathcal{V}(\varphi, z)\right] = 0$ by (1.2.7) and (1.1.29). In other words, dividing out the subspace $L_{(-1)}\mathcal{F}$ reflects the fact that the zero mode ψ_0 of a vertex operator $\mathcal{V}(\psi, z)$ remains unchanged when a total derivative is added to $\mathcal{V}(\psi, z)$. Finally, recall

Eq. (1.2.20) which was used to show that zero mode operators always form a closed Lie algebra. Now it establishes the desired homomorphism property, viz

$$\mathrm{ad}_{[\psi,\varphi]} = ([\psi,\varphi])_0 = [\psi_0,\varphi_0] = [\mathrm{ad}_\psi, \mathrm{ad}_\varphi]. \tag{1.5.4}$$

Another glimpse at skew symmetry shows that the Lie algebra $\mathcal{F}/\mathrm{L}_{(-1)}\mathcal{F}$ is also equipped with a symmetric product by

$$(\psi|\varphi) := \psi_1\varphi. \tag{1.5.5}$$

To investigate possible $\mathcal{F}/\mathrm{L}_{(-1)}\mathcal{F}$ invariance of $(\_|\_)$, we note that

$$\begin{aligned}([\psi,\varphi]|\xi) &\equiv (\psi_0\varphi)_1\xi \\ &= \psi_0(\varphi_1\xi) - \varphi_1(\psi_0\xi) \quad \text{by (1.2.15)} \\ &\equiv [\psi,(\varphi|\xi)] - (\varphi|[\psi,\xi]).\end{aligned} \tag{1.5.6}$$

Hence the product $(\_|\_)$ is in general *not* $\mathcal{F}/\mathrm{L}_{(-1)}\mathcal{F}$ - invariant unless we make further assumptions.

For that purpose let us restrict our attention to the piece of conformal weight 1, i.e. to $\mathcal{F}_{(1)}$. Then the space

$$\mathcal{F}_{(1)}/(\mathrm{L}_{(-1)}\mathcal{F}\cap\mathcal{F}_{(1)}) = \mathcal{F}_{(1)}/\mathrm{L}_{(-1)}\mathcal{F}_{(0)} \tag{1.5.7}$$

is a subalgebra of the Lie algebra $\mathcal{F}/\mathrm{L}_{(-1)}\mathcal{F}$ and, by (1.3.10),

$$(\varphi|\xi) \in \mathcal{F}_{(0)} \quad \text{for all } \varphi,\xi \in \mathcal{F}_{(1)}. \tag{1.5.8}$$

If we now assume that $\mathcal{F}_{(0)}$ is one-dimensional then $(\varphi|\xi)$ for $\varphi,\xi \in \mathcal{F}_{(1)}$ is a scalar multiple of the vacuum and $\mathrm{L}_{(-1)}\mathcal{F}_{(0)} = 0$ by (1.3.2). Thus $[\psi,(\varphi|\xi)]$ is proportional to $\psi_0\mathbf{1}$, which vanishes because of (1.3.19) and we have indeed established invariance of the scalar product.

In some cases we can say even more. Let us look again at the commutator formula (1.2.18):

$$\begin{aligned}[\psi_m,\varphi_n] &= \sum_{i\geq 0}\binom{m}{i}(\psi_i\varphi)_{m+n-i} \\ &= (\psi_0\varphi)_{m+n} + m(\psi_1\varphi)_{m+n-1} + \sum_{i\geq 2}\binom{m}{i}(\psi_i\varphi)_{m+n-i} \\ &= ([\psi,\varphi])_{m+n} + m(\psi|\varphi)\mathbf{1}_{m+n-1} + \sum_{i\geq 2}\binom{m}{i}(\psi_i\varphi)_{m+n-i} \\ &= ([\psi,\varphi])_{m+n} + m(\psi|\varphi)\delta_{m+n,0}\mathrm{id}_\mathcal{F} + \sum_{i\geq 2}\binom{m}{i}(\psi_i\varphi)_{m+n-i} \quad \text{by (1.2.12),}\end{aligned}$$

for $\psi,\varphi \in \mathcal{F}_{(1)}$. Since all states $\psi_i\varphi$, $i \geq 2$, which occur in the sum on the right hand side have negative conformal weight we conclude that:

Theorem 2

If the weight 0 piece, $\mathcal{F}_{(0)}$, of a vertex algebra $(\mathcal{F},\mathcal{V},\mathbf{1},\boldsymbol{\omega})$ is one-dimensional and the spectrum of the operator $\mathrm{L}_{(0)}$ is nonnegative, then the weight 1 piece, $\mathcal{F}_{(1)}$, is a Lie algebra with antisymmetric product $[\psi,\varphi] := \psi_0\varphi$ and $\mathcal{F}_{(1)}$-invariant bilinear form $(\psi|\varphi) := \psi_1\varphi$. The space

$$\widehat{\mathcal{F}_{(1)}} := \{\psi_n | \psi \in \mathcal{F}_{(1)}, n \in \mathbb{Z}\} \oplus \mathrm{span}\{\mathrm{id}_\mathcal{F}\} \tag{1.5.9}$$

provides a representation of the affinization of $\mathcal{F}_{(1)}$ on $\mathcal{F}$ by

$$[\psi_m,\varphi_n] = ([\psi,\varphi])_{m+n} + m(\psi|\varphi)\delta_{m+n,0}\mathrm{id}_\mathcal{F}. \tag{1.5.10}$$

In particular, $\mathcal{F}_{(1)}$ may be identified with the Lie algebra of operators $\{\psi_0|\psi \in \mathcal{F}_{(1)}\}$ on $\mathcal{F}$.

It is quite interesting that in physical applications such as string theory, two-dimensional statistical systems and two-dimensional quantum field theories *physical* considerations lead to the same condition on the spectrum of $\mathrm{L}_{(0)}$ as in Theorem 2. In such theories $\mathrm{L}_{(0)}$ is identified with the Hamiltonian so that the above condition immediately translates into the postulate of the positivity of the energy (see e.g. [43]). Then the condition $\dim\mathcal{F}_{(0)} = 1$ ensures that the vacuum is (up to scalar multiples) the unique zero energy state and, a fortiori, it forces the vacuum to be the unique $SU(1,1)$-invariant state in the Fock space (cf. the discussion in Sect. 3). From the assumption that the conformal weights are bounded below, we infer that $\mathrm{L}_{(1)}^N\varphi = 0$ for N sufficiently large, where φ is an arbitrary state of some definite conformal weight. This shows that in this case $\mathcal{F}$ splits up into a direct sum of $\mathfrak{su}(1,1)$ highest weight representations generated by some basis of the quasiprimary states. Since for any quasiprimary state $\psi \in \mathcal{F}$ the $\mathfrak{su}(1,1)$ representation space $(\psi) \subset \mathcal{F}$ generated from ψ is spanned by the $\mathfrak{su}(1,1)$-descendant states $\{\mathrm{L}_{(-1)}^N\psi \mid N \in \mathbb{N}\}$, we may identify $\mathcal{F}/\mathrm{L}_{(-1)}\mathcal{F}$ with the set of quasiprimary states in the Fock space $\mathcal{F}$.

The Lie algebra $\mathcal{F}/\mathrm{L}_{(-1)}\mathcal{F}$ is too large for further investigations. In string theory, for example, a distinguished role is played by the primary states of weight $\Delta = 1$, which we shall call **physical states** from now on. In fact, we learn from Eq. (1.3.23) that for a physical state ψ the corresponding zero mode operator ψ_0 commutes with the Virasoro algebra thereby preserving all subspaces $\mathcal{P}_{(n)}$ of primary states of weight n. In particular, it maps physical states into physical states, i.e. $[\mathcal{P}_{(1)},\mathcal{P}_{(1)}] \subset \mathcal{P}_{(1)} \bmod \mathrm{L}_{(-1)}\mathcal{P}_{(0)}$. Hence it is natural to look in detail at the **Lie algebra of physical states,**

$$\mathfrak{g}_\mathcal{F} := \mathcal{P}_{(1)}/\mathrm{L}_{(-1)}\mathcal{P}_{(0)}\,, \tag{1.5.11}$$

where we used the fact that

$$\mathrm{L}_{(-1)}\mathcal{F}_{(0)} \cap \mathcal{P}_{(1)} = \mathrm{L}_{(-1)}\mathcal{P}_{(0)} \tag{1.5.12}$$

in any vertex algebra. To see this we start from the identity

$$\mathrm{L}_{(n)}\mathrm{L}_{(-1)}\psi = (n+1)\mathrm{L}_{(n-1)}\psi + \mathrm{L}_{(-1)}\mathrm{L}_{(n)}\psi \qquad \forall \psi \in \mathcal{F}, n \in \mathbb{Z}. \tag{1.5.13}$$

Then the inclusion "$\supseteq$" in (1.5.12) obviously holds. On the other hand, let $\psi \in \mathcal{F}_{(0)}$ and demand that $\mathrm{L}_{(n)}\mathrm{L}_{(-1)}\psi \overset{!}{=} 0\ \forall n \geq 1$. Hence

$$\mathrm{L}_{(n-1)}\psi = -\tfrac{1}{n+1}\mathrm{L}_{(-1)}\mathrm{L}_{(n)}\psi \qquad \forall n \geq 1, \tag{1.5.14}$$

which by induction yields the inclusion "$\subseteq$" in (1.5.12), when the regularity axiom (1.2.11) is applied to the right hand side.

The problem of finding an invariant bilinear form for the Lie algebra of physical states is resolved quite elegantly; for it turns out that we get one for free, i.e. we may drop the above uniqueness condition on the vacuum ($\dim \mathcal{F}_{(0)} = 1$), which was necessary to establish invariance. In fact, we shall see below that the pairing $\langle \_|\_\rangle$ between $\mathcal{F}$ and its restricted dual, $\mathcal{F}'$, quite generally gives rise to an invariant bilinear form on $\mathfrak{g}_{\mathcal{F}}$. Assuming that we may identify $\mathcal{F}$ with $\mathcal{F}'$ via an isomorphism F, we first have to convince ourself that the bilinear form $(\chi|\varphi) := \langle F(\chi)|\varphi\rangle$ on $\mathcal{F}$ projects down to a well-defined form on $\mathfrak{g}_{\mathcal{F}}$. Indeed, $(\mathrm{L}_{(-1)}\chi|\varphi) = (\chi|\mathrm{L}_{(1)}\varphi) = 0$ for any quasi primary state φ, i.e. the space $\mathrm{L}_{(-1)}\mathcal{F}$ is orthogonal to all quasiprimary states and thus deserves to be called null space. In particular, $\mathrm{L}_{(-1)}\mathcal{P}_{(0)}$ consists of **null physical states**, physical states orthogonal to all physical states including themselves. Hence the Lie algebra $\mathfrak{g}_{\mathcal{F}}$ is obtained from $\mathcal{P}_{(1)}$ by dividing out (unwanted) null physical states. Recall that when defining the Lie algebra $\mathcal{F}/\mathrm{L}_{(-1)}\mathcal{F}$ we had to divide out the space $\mathrm{L}_{(-1)}\mathcal{F}$ for mathematical reasons. But with the bilinear form at hand we are now led to a physical interpretation of that maneuver.

It is well known that there are additional null physical states in $\mathcal{P}_{(1)}$ if and only if the central charge takes the critical value $c = 26$, namely the space $(\mathrm{L}_{(-2)} + \frac{3}{2}\mathrm{L}_{(-1)}^2)\mathcal{P}_{(-1)}$ (see [45] for the calculations). The existence of these additional null physical states is used in the proof of the no-ghost theorem [44] which we shall refer to in Chap. III when we discuss the DDF construction.

Coming back to our original task of constructing an invariant bilinear form for $\mathfrak{g}_{\mathcal{F}}$, we notice next that the adjoint vertex operator in Eq. (1.4.10) is summable for any quasiprimary state irrespective of the spectrum of $\mathrm{L}_{(0)}$. This implies that the form always satisfies $(\chi|\psi_n\varphi) = -(\psi_{-n}\chi|\varphi)$ for physical states χ, ψ, φ. Spezializing to $n = 0$ we indeed recover the invariance property:

$$([\chi,\psi]|\varphi) = (\chi|[\psi,\varphi]) \qquad \forall \chi,\psi,\varphi \in \mathfrak{g}_{\mathcal{F}}. \tag{1.5.15}$$

If we put $n = 1$ and $\chi = \mathbf{1}$, then we obtain

$$(\psi|\varphi) = (\mathbf{1}| - \psi_1\varphi) \qquad \forall \psi,\varphi \in \mathfrak{g}_{\mathcal{F}}, \tag{1.5.16}$$

so that we can still regard $-\psi_1\varphi$ as the bilinear form like in the above theorem.

Now suppose that $\Omega := \{\chi^a \mid a \in I\}$ constitutes some complete countable basis for $\mathfrak{g}_{\mathcal{F}}$, i.e.

$$\sum_{a\in I}\chi^a(\chi^a)^* = \mathrm{id}_{\mathfrak{g}_{\mathcal{F}}}. \tag{1.5.17}$$

Then we define the **structure constants** C_c^{ab} (w.r.t. Ω) of $\mathfrak{g}_{\mathcal{F}}$ by

$$\mathrm{C}_c^{ab} := (\chi^c|[\chi^a,\chi^b]) \tag{1.5.18}$$

so that

$$[\chi^a;\chi^b] = \sum_{c\in I}\mathrm{C}_c^{ab}\chi^c. \tag{1.5.19}$$

But since

$$(\chi^c|[\chi^a,\chi^b]) = \mathrm{Res}_z\left\langle(\chi^c)^* \mid \mathcal{V}(\chi^a,z)\chi^b\right\rangle, \tag{1.5.20}$$

we conclude that *the residue of the three-point function for physical states is precisely the corresponding structure constant for the Lie algebra of physical states.*

At first sight it is quite astonishing that the physical states in a vertex algebra always form a Lie algebra. However, we can get an intuitive understanding of this feature from a more conceptual point of view. We shall see in a moment that the physical states generate one-parameter groups of automorphisms of the vertex algebra; but since the latter naturally form a group under composition, the former can be somehow expected to carry a Lie algebra structure. To make this statement precise we have to introduce an important notion from the category of vertex algebras, namely the idea of an automorphism. Roughly speaking, an automorphism of a vertex algebra is a map which preserves all the axioms of the vertex algebra. Mathematicians usually remove all redundancy from that definition and use the following version [26].

Definition 3

An **automorphism** *of a vertex algebra* $(\mathcal{F},\mathcal{V},\mathbf{1},\omega)$ *is a linear isomorphism* $f : \mathcal{F} \to \mathcal{F}$ *such that*

$$f \circ \mathcal{V}(\psi,z)\circ f^{-1} = \mathcal{V}(f(\psi),z) \qquad \forall \psi \in \mathcal{F}, \tag{1.5.21}$$

$$f(\omega) = \omega. \tag{1.5.22}$$

Putting $\psi = \mathbf{1}$ in (1.5.21) immediately yields $f(\mathbf{1}) = \mathbf{1}$, whereas $\psi = \mathrm{L}_{(0)}\varphi$ in (1.5.21), together with (1.5.22), shows that f is automatically grading-preserving, i.e. it commutes with $\mathrm{L}_{(0)}$.

To any physical state ψ we now associate a family of automorphisms $\sigma_{t\psi}$, $t \in \mathbb{R}$, by regarding ψ as an "infinitesimal generator" for an one-parameter group of inner automorphisms, viz

$$\sigma_{t\psi} := \mathrm{e}^{t\psi_0} \equiv \sum_{n=0}^{\infty}\frac{t^n}{n!}\psi_0^n. \tag{1.5.23}$$

To check whether $\sigma_{t\psi}$ is indeed an automorphism, first note that $\sigma_{t\psi}$ has inverse $\sigma_{-t\psi}$ and is linear because ψ_0 is.[2] Furthermore we find that

$$\begin{aligned} e^{t\psi_0}\mathcal{V}(\varphi,z)e^{-t\psi_0} &= \sum_{n=0}^{\infty}\frac{t^n}{n!}[\psi_0,[\psi_0,\ldots[\psi_0,\mathcal{V}(\varphi,z)]]\ldots] \quad \text{by (1.3.7)} \\ &= \sum_{n=0}^{\infty}\frac{t^n}{n!}\mathcal{V}(\psi_0^n\varphi,z) \quad \text{by (1.3.3)} \\ &= \mathcal{V}(e^{t\psi_0}\varphi,z). \end{aligned} \tag{1.5.24}$$

We observe that we established the homomorphism property (1.5.21) by the mere fact that ψ_0 is the zero mode of a vertex operator. In contrast to this, the preservation of the conformal vector relies on the physical state conditions:

$$\psi_0\omega = [\psi_0, \mathrm{L}_{(-2)}]\mathbf{1} = 0 \quad \text{for } \psi \in \mathfrak{g}_{\mathcal{F}}, \tag{1.5.25}$$

by (1.3.19) and (1.3.23).

The alert reader will object that the above formal manipulations remain meaningless as long as we do not prove that $e^{t\psi_0}$ makes sense. It is immediately clear, however, that $\sigma_{t\psi}$ does have a well-defined action on $\mathcal{F}$ if ψ_0 is **locally nilpotent** on $\mathcal{F}$, i.e. for any $\varphi \in \mathcal{F}$ there exists some $N \in \mathbb{N}$ such that $\psi_0^N\varphi = 0$. Strictly speaking therefore only locally nilpotent physical states generate automorphisms of the vertex algebra.[3]

The most natural question to ask at this point is, what structure the group of physical inner automorphisms has. And indeed the first naive thing we can do is to consider its commutator subgroup, which measures the failure of a group to be commutative. Let $\sigma_{t\psi}$ and $\sigma_{s\varphi}$ be two flows generated by physical states ψ and φ, respectively, and expand them up to second order in the parameters t and s, respectively, which are assumed to be small. Then it is not difficult to see that

$$\sigma_{t\psi}\circ\sigma_{s\varphi}\circ\sigma_{t\psi}^{-1}\circ\sigma_{s\varphi}^{-1} = \mathrm{id}_{\mathcal{F}} + st[\psi_0,\varphi_0] + \mathcal{O}(s^2t^2). \tag{1.5.26}$$

We conclude that the zero modes of physical states should form a closed Lie algebra and that the commutator measures the failure of flows to commute with each other.

Finally, we can also bring the notion of symmetry into the game by going back to the vertex algebra itself. Having a physical system with state space $\mathcal{H}$ we usually call a symmetry a represenation of a group on Aut $\mathcal{H}$. But in the above discussion we were dealing with a subgroup of the automorphism group of a vertex algebra. Thus we have arrived at the interpretation of the Lie algebra of physical states in a vertex algebra as a symmetry algebra for the physical system encoded by the vertex algebra.

We end this section with a historical remark. Borcherds (see [11], [7], [10]) was led to his definition of generalized Kac-Moody algebras (see Chap. V) precisely by Lie algebras of type $\mathcal{P}_{(1)}/\mathrm{L}_{(-1)}\mathcal{P}_{(0)}$ for vertex algebras associated with even Lorentzian lattices, which we shall discuss in detail in the following chapters.

6. Cross-bracket and the algebra of fields of weight 2

Let us investigate the case where the vertex algebra $(\mathcal{F},\mathcal{V},\mathbf{1},\omega)$ contains *no* states of weight 1 (see [30]). Define a product by

$$\psi\times\varphi := \psi_1\varphi \quad \text{for } \psi,\varphi\in\mathcal{F}_{(2)}, \tag{1.6.1}$$

which is symmetric but nonassociative on the piece $\mathcal{F}_{(2)}$ in view of (1.3.17) with $\mathrm{L}_{(-1)}\mathcal{F}_{(1)} = 0$. Note that $\frac{1}{2}\omega$ provides an identity element on $\mathcal{F}_{(2)}$,

$$\tfrac{1}{2}\omega\times\psi = \tfrac{1}{2}\omega_1\psi = \tfrac{1}{2}\mathrm{L}_{(0)}\psi = \psi \qquad \forall\psi\in\mathcal{F}_{(2)}. \tag{1.6.2}$$

If we assume that $\mathcal{F}_{(0)}$ is one-dimensional then $\psi_3\varphi$ for $\psi,\varphi\in\mathcal{F}_{(2)}$ is a scalar multiple of the vacuum by (1.3.10) so that

$$\langle\psi,\varphi\rangle := \psi_3\varphi \tag{1.6.3}$$

gives us a symmetric bilinear form on $\mathcal{F}_{(2)}$. Moreover, this form is associative in the sense that

$$\langle\varphi,\psi\times\xi\rangle = \langle\varphi\times\psi,\xi\rangle \quad \text{for } \psi,\varphi,\xi\in\mathcal{F}_{(2)}. \tag{1.6.4}$$

This can be seen most easily by setting $l = m = n = 1$ in (1.2.15):

$$\psi_2(\varphi_2\xi) - \varphi_2(\psi_2\xi) - \psi_1(\varphi_3\xi) + \varphi_3(\psi_1\xi) = (\psi_1\varphi)_3\xi + (\psi_2\varphi)_2\xi. \tag{1.6.5}$$

Since $\mathcal{F}_{(1)} = 0$ the first two terms on the left hand side and the last term on the right hand side vanish while $\psi_1(\varphi_3\xi)$ is proportional to $\psi_1\mathbf{1}$, which is zero because of (1.3.19).

We define the **cross-bracket** as follows:

$$[\psi_m \times_1 \varphi_n] \equiv [\psi\times_1\varphi]_{mn} := [\psi_{m+1},\varphi_n] - [\psi_m,\varphi_{n+1}] \quad \text{for } \psi,\varphi\in\mathcal{F}_{(2)}. \tag{1.6.6}$$

This looks kind of awkward but turns out to be quite interesting as soon as one recalls the Jacobi identity in components, (1.2.15), which gives for $l = 1$, $\psi,\varphi\in\mathcal{F}_{(2)}$,

$$[\psi\times_1\varphi]_{mn} = \sum_{i\geq 0}\binom{m}{i}(\psi_{i+1}\varphi)_{m+n-i}$$

[2] Together with $\sigma_0 = \mathrm{id}_{\mathcal{F}}$ and $\sigma_{s\psi}\circ\sigma_{t\psi} = \sigma_{(s+t)\psi}$, this establishes that $\{\sigma_{t\psi} \mid t\in\mathbb{R}\}$ is indeed a one-parameter group.

[3] In the framework of formal calculus we would obtain a well-defined automorphism associated to any physical state ψ by considering $e^{y\psi_0}$, where $y\in z_0\mathbb{C}[\![z_0]\!]$; but introducing an extra formal variable to make the series summable is not appropriate for our heuristic argumentation.

$$= (\psi_1\varphi)_{m+n} + m\underbrace{(\psi_2\varphi)_{m+n-1}}_{=0} + \frac{1}{2}m(m-1)(\psi_3\varphi)_{m+n-2} + \sum_{i\geq 3}\binom{m}{i}(\psi_{i+1}\varphi)_{m+n-i}$$
$$= (\psi\times\varphi)_{m+n} + \frac{1}{2}m(m-1)\,\langle\psi,\varphi\rangle\,\delta_{m+n,1}\mathrm{id}_{\mathcal{F}} + \sum_{i\geq 3}\binom{m}{i}(\psi_{i+1}\varphi)_{m+n-i}.$$

Since the sum on the right hand side involves only terms with negative conformal weight, we have arrived at the following result:

Theorem 3

Let $(\mathcal{F},\mathcal{V},\mathbf{1},\omega)$ be a vertex algebra. If the weight 0 piece is one-dimensional, the weight 1 piece is empty and the spectrum of the operator $\mathrm{L}_{(0)}$ is nonnegative, then the weight 2 piece, $\mathcal{F}_{(2)}$, is a nonassociative algebra with symmetric product $\psi\times\varphi := \psi_1\varphi$ and associative bilinear form $\langle\psi,\varphi\rangle := \psi_3\varphi$. The space

$$\widehat{\mathcal{F}_{(2)}} := \{\psi_n|\psi\in\mathcal{F}_{(2)}, n\in\mathbb{Z}\}\oplus\mathrm{span}\{\mathrm{id}_{\mathcal{F}}\} \tag{1.6.7}$$

provides a representation of the commutative affinization of $\mathcal{F}_{(2)}$ on $\mathcal{F}$ by the cross-bracket,

$$[\psi_m \times_1 \varphi_n] = (\psi\times\varphi)_{m+n} + \frac{1}{2}m(m-1)\,\langle\psi,\varphi\rangle\,\delta_{m+n,1}\mathrm{id}_{\mathcal{F}}. \tag{1.6.8}$$

Of course, this cross-bracket is quite a nice algebraic structure in our vertex algebra, but immediately the question arises whether such a commutative non-associative algebra really exists. In fact, the celebrated moonshine module constructed by Frenkel, Lepowsky, and Meurman *is* a vertex operator algebra that satisfies all the assumptions of Theorem 3, and it turns out then that $\mathcal{F}_{(2)}$ is precisely the 196,884-dimensional **Griess algebra**, which possesses the monster group, F_1, as its full automorphism group (cf. [46], [29], [28], [11]).

7. Symmetry products

The commutator and the cross-bracket of the last two subsections can be embedded in an infinite family of symmetry products by using the Jacobi identity in the following way [30]: Take $\mathrm{Res}_{z_0}[z_0^n\,(1.2.9)\,]$ and define for $l\in\mathbb{Z}$

$$\begin{aligned}[\mathcal{V}(\psi,z_1)\times_l\mathcal{V}(\varphi,z_2)] &:= \mathrm{Res}_{z_0}\left[z_0^l z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\mathcal{V}(\mathcal{V}(\psi,z_0)\varphi,z_2)\right]\\ &= (z_1-z_2)^l\mathcal{V}(\psi,z_1)\mathcal{V}(\varphi,z_2) - (-z_2+z_1)^l\mathcal{V}(\varphi,z_2)\mathcal{V}(\psi,z_1),\end{aligned} \tag{1.7.1}$$

which is expressed in modes as

$$\begin{aligned}[\psi\times_l\varphi]_{mn} &:= \sum_{i\geq 0}(-1)^i\binom{l}{i}\left[\psi_{m+l-i}\varphi_{n+i} - (-1)^l\varphi_{n+l-i}\psi_{m+i}\right]\\ &= \sum_{i\geq 0}\binom{m}{i}(\psi_{l+i}\varphi)_{m+n-i}.\end{aligned} \tag{1.7.2}$$

It is clear that these products are symmetric for odd l and alternating for even l. The symmetric product $\times_1$ is precisely the cross-bracket used to construct the affinization of the Griess algebra in Theorem 3 while the alternating product $\times_0$ yields nothing but the commutator for the modes in Theorem 2, $[\psi\times_0\varphi]_{mn} = [\psi_m,\varphi_n]$. As regards the interpretation of the other symmetry products, so far we can only associate with the product $\times_{-1}$ a well-known feature of conformal field theory. Recall that the commutator of two fields is completely determined by the singular part of the operator product expansion. The regular part of the latter encodes the normal-ordered product of two fields. Thus the product $\times_{-1}$, which differs from $\times_0$ by a factor $(z_1-z_2)^{-1}$ (more precisely, by a factor z_0^{-1} in $\mathrm{Res}_{z_0}[\ldots]$), might be a good guess for defining normal-ordered products in a vertex algebra. For this purpose one would usually consider the (algebraic) limit $z_1\to z_2$ of $[\mathcal{V}(\psi,z_1)\times_{-1}\mathcal{V}(\varphi,z_2)]$, which unfortunately does not exist. Hence we start with the following definition of a **normal-ordered product**:

$$\begin{aligned}{}^{\times}_{\times}\mathcal{V}(\psi,z)\mathcal{V}(\varphi,z)^{\times}_{\times} &:= \sum_{n\in\mathbb{Z}}[\psi\times_{-1}\varphi]_{0n}z^{-n-1}\\ &= \sum_{n\in\mathbb{Z}}(\psi_{-1}\varphi)_n z^{-n-1}\\ &= \mathcal{V}(\psi_{-1}\varphi,z).\end{aligned} \tag{1.7.3}$$

In the following we will also refer to the product $\psi_{-1}\varphi$ as normal-ordered product of states. At first sight this does not look like the standard normal-ordered product of fields in conformal field theory. However, using the mode expansion (1.7.2) we can rewrite the definition as

$$\begin{aligned}{}^{\times}_{\times}\mathcal{V}(\psi,z)\mathcal{V}(\varphi,z)^{\times}_{\times} &= \sum_{n\in\mathbb{Z}}\sum_{i\geq 0}(\psi_{-i-1}\varphi_{n+i} + \varphi_{n-i-1}\psi_i)z^{-n-1}\\ &= \sum_{n\in\mathbb{Z}}\sum_{i\geq 0}{:}\psi_{-i-1}\varphi_{n+i}{:}z^{-n-1},\end{aligned} \tag{1.7.4}$$

where we have introduced the normal-ordering of the modes,

$$:\psi_{-i-1}\varphi_{n+i}: := \begin{cases} \psi_{-i-1}\varphi_{n+i} & \text{if } i \geq 0 \\ \varphi_{n+i}\psi_{-i-1} & \text{if } i < 0 \end{cases} . \tag{1.7.5}$$

And this is indeed the familiar normal-ordered product of modes [39] if we employ the standard shifted grading $\psi_{(n)} \equiv \psi_{n+\Delta_\psi-1}$.

In general one expects the normal-ordered product of bosonic fields to be commutative. Since the skew symmetry (1.3.17) forces $\psi_{-1}\varphi = \varphi_{-1}\psi$ on $\mathcal{F}/\mathrm{L}_{(-1)}\mathcal{F}$, we therefore should restrict the above definition of normal-ordered product to the quotient space $\mathcal{F}/\mathrm{L}_{(-1)}\mathcal{F}$. This has the nice effect of automatically projecting the normal-ordered products of quasiprimary fields onto the space of quasiprimary fields. The idea is to subtract expressions of the form $(\mathrm{L}_{(-1)})^i(\psi_{i-1}\varphi)$, $i \geq 1$, from $\psi_{-1}\varphi$ such that one ends up with a quasiprimary state. Indeed, we found that the projected normal-ordered product of two quasiprimary states ψ, φ is given by

$$[\psi *_{-1} \varphi] := \psi_{-1}\varphi + \sum_{i\geq 1} \frac{(-1)^i}{i!} \binom{2\Delta_\psi - 1}{i} \binom{2(\Delta_\psi + \Delta_\varphi - 1)}{i}^{-1} (\mathrm{L}_{(-1)})^i(\psi_{i-1}\varphi), \tag{1.7.6}$$

i.e. $[\psi *_{-1} \varphi]$ is a quasiprimary state of weight $\Delta_\psi + \Delta_\varphi$. This formula is similar to those given in [13] and [5]. For the sake of completeness we note that this projection onto quasiprimary states generalizes to any product $\psi_n\varphi$, $n \in \mathbb{Z}$, of two quasiprimary states, i.e.

$$[\psi *_l \varphi] := \sum_{i\geq 0} \frac{(-1)^i}{i!} p_i^{(l)}(\Delta_\psi, \Delta_\varphi)(\mathrm{L}_{(-1)})^i(\psi_{i+l}\varphi), \tag{1.7.7}$$

where

$$p_i^{(l)}(\Delta_\psi, \Delta_\varphi) := \binom{2\Delta_\psi - 2 - l}{i} \binom{2(\Delta_\psi + \Delta_\varphi - 2 - l)}{i}^{-1}, \tag{1.7.8}$$

is a quasiprimary state of weight $\Delta_\psi + \Delta_\varphi - l - 1$. If we calculate the corresponding vertex operator using the translation axiom (1.2.7), we find that

$$\mathcal{V}([\psi *_l \varphi], z) = \sum_{m\in\mathbb{Z}} \sum_{i\geq 0} p_i^{(l)}(\Delta_\psi, \Delta_\varphi) \binom{m}{i} (\psi_{l+i}\varphi)_{m-i}. \tag{1.7.9}$$

Surprisingly, we observe that these projected products are just the "old" symmetry products (1.7.2) for $n = 0$, where each term in the sum is weighted with an additional polynomial factor $p_i^{(l)}(\Delta_\psi, \Delta_\varphi)$. Moreover we can prove that the projected products also inherit the symmetry properties from the original ones, i.e. $[\psi *_l \varphi] = (-1)^{l+1}[\varphi *_l \psi]$.

It is quite interesting that the normal-ordered product (1.7.3) turns out to be also associative if we consider the quotient space $\mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F}$. To see this we first note that $\mathrm{L}_{(-1)}\mathcal{F} = \mathcal{F}_{(-2)}\mathbf{1} \subset \mathcal{F}_{(-2)}\mathcal{F}$ by (1.3.19). Additionally we have for $n \leq -2$

$$\begin{aligned}
\psi_n\varphi &= \psi_n(\varphi_{-1}\mathbf{1}) \\
&= [\psi_n, \varphi_{-1}]\mathbf{1} + \varphi_{-1}(\psi_n\mathbf{1}) \\
&= \sum_{i\geq 0} \binom{n}{i} (\psi_i\varphi)_{n-1-i}\mathbf{1} + \frac{1}{(-n-1)!}\varphi_{-1}\left[(\mathrm{L}_{(-1)})^{-n-1}\psi\right] \\
&= \underbrace{\sum_{i\geq 0} \frac{1}{(i-n)!}\binom{n}{i} \left[(\mathrm{L}_{(-1)})^{i-n-1}(\psi_i\varphi)\right]_{-2}\mathbf{1}}_{\in \mathcal{F}_{(-2)}\mathbf{1}} \\
&\quad + \underbrace{\frac{1}{(-n-1)!}\varphi_{-1}\left\{\left[(\mathrm{L}_{(-1)})^{-n-2}\psi\right]_{-2}\mathbf{1}\right\}}_{\in \mathcal{F}_{(-2)}\mathcal{F}},
\end{aligned} \tag{1.7.10}$$

where the last term lies in $\mathcal{F}_{(-2)}\mathcal{F}$ because of

$$\varphi_{-1}(\xi_{-2}\mathbf{1}) = \xi_{-2}\varphi + \sum_{i\geq 0} \frac{(-1)^i}{(i+2)!} \left[(\mathrm{L}_{(-1)})^{i+1}(\varphi_i\xi)\right]_{-2}\mathbf{1} \qquad \forall \varphi, \xi \in \mathcal{F}. \tag{1.7.11}$$

This gives us associativity of the normal-ordered product:

$$\begin{aligned}
(\psi_{-1}\varphi)_{-1}\chi - \psi_{-1}(\varphi_{-1}\chi) &= \sum_{i\geq 0} [\psi_{-1-i}(\varphi_{-1+i}\chi) + \varphi_{-2-i}(\psi_i\chi)] - \psi_{-1}(\varphi_{-1}\chi) \\
&= 0 \qquad \text{on } \mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F}.
\end{aligned} \tag{1.7.12}$$

The fact that $\psi_n\varphi \in \mathcal{F}_{(-2)}\mathcal{F}$ for $n \leq -2$ in particular implies that $\mathrm{L}_{(n)}\mathcal{F} \subset \mathcal{F}_{(-2)}\mathcal{F}$ for $n \leq -3$, which allows us to give a nice interpretation of the quotient space $\mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F}$. Consider the conformal family $[\psi]$ associated with a primary state $\psi \in \mathcal{F}$, which is by definition the Virasoro Verma module built on ψ. It is spanned by elements of the form [4]

$$(\mathrm{L}_{(-1)})^{i_1}(\mathrm{L}_{(-2)})^{i_2}\ldots(\mathrm{L}_{(-n)})^{i_n}\psi, \qquad n \geq 1,\ i_1, \ldots, i_n \geq 0. \tag{1.7.13}$$

Hence only the subspace spanned by the states $(\mathrm{L}_{(-2)})^N\psi$, $N \in \mathbb{N}$, survives when the conformal family is projected onto $\mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F}$. Moreover, $\mathrm{L}_{(-2)} \equiv \boldsymbol{\omega}_{-1}$ so that these states are just multiple normal-ordered products of the Virasoro vector with the conformal highest weight vector ψ.

Let us summarize: The quotient space $\mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F}$ with the induced normal-ordered product $\psi_{-1}\varphi$ carries the structure of a commutative associative algebra. If the Fock space $\mathcal{F}$ splits up into a direct sum of highest weight representations of the Virasoro algebra, then $\mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F}$ can be identified with the span of primary states, Virasoro vector, and multiple normal-ordered products of the latter with the primary states.

The above quotient space plays a special role when vertex algebras on the torus are considered, i.e. when the notion of modular invariance is built into the framework of vertex algebras [81]. A vertex operator algebra $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \boldsymbol{\omega})$ is said to satisfy the **finiteness condition** if the quotient space $\mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F}$ is finite-dimensional. A vertex operator algebra is called **rational** if it has only finitely many irreducible representations and every finitely generated representation is completely reducible. In fact, Zhu [81] proved that if a rational vertex operator algebra satisfies the finiteness condition then the linear span of the characters $\operatorname{tr} q^{\mathrm{L}_{(0)}-\frac{c}{24}}$ of its irreducible representations is modular invariant with respect to SL(2,$\mathbb{Z}$). It is conjectured that rational vertex operator algebras automatically satisfy the finiteness condition.

Finally, we note that the above discussion of the quotient space $\mathcal{F}/\mathrm{L}_{(-1)}\mathcal{F}$, equipped with the product $\psi_{-1}\varphi$, might be useful for explaining how Gerstenhaber algebras arise in the context of super vertex algebras. To see this, one supposes that an odd operator Q satisfying $Q^2 = 0$ acts on the vertex algebra $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \boldsymbol{\omega})$, and that Q can be represented as the zero mode of a vertex operator $\mathcal{V}(\boldsymbol{\sigma}, z)$ associated with an odd "ghost" state $\boldsymbol{\sigma}$ of weight -1, i.e. $Q = \boldsymbol{\sigma}_0$. Furthermore one assumes that there is an odd "antighost" state $\boldsymbol{\beta}$ of weight 2 such that $\mathrm{L}_{(n)} \equiv \boldsymbol{\omega}_{n+1} = (Q\boldsymbol{\beta})_{n+1} \equiv (\boldsymbol{\sigma}_0\boldsymbol{\beta})_{n+1}\ \forall n$. It is proved in [58](see also [68]) that the superspace $\ker Q/\operatorname{im} Q$, the cohomology of Q, can be equipped with the structure of a Gerstenhaber algebra.

Definition 4

A **Gerstenhaber algebra** G^* *is a $\mathbb{Z}$-graded vector space equipped with two bilinear multiplication operations, denoted by $\psi\cdot\varphi$ and $\{\psi,\varphi\}$, respectively, and satisfying the following assumptions:*

(i) If ψ and φ are homogeneous elements of degree $|\psi|$ and $|\varphi|$, respectively, then $\psi\cdot\varphi$ is homogeneous of degree $|\psi|+|\varphi|$ and $\{\psi,\varphi\}$ is homogeneous of degree $|\psi|+|\varphi|-1$;

(ii) $\psi\cdot\varphi = (-1)^{|\psi||\varphi|}\varphi\cdot\psi$,

(iii) $(\psi\cdot\varphi)\cdot\xi = \psi\cdot(\varphi\cdot\xi)$,

(iv) $\{\psi,\varphi\} = -(-1)^{(|\psi|-1)(|\varphi|-1)}\{\varphi,\psi\}$,

(v) $(-1)^{(|\psi|-1)(|\xi|-1)}\{\{\psi,\varphi\},\xi\} + (-1)^{(|\varphi|-1)(|\psi|-1)}\{\{\varphi,\xi\},\psi\}$
$+(-1)^{(|\xi|-1)(|\varphi|-1)}\{\{\xi,\psi\},\varphi\} = 0$,

(vi) $\{\psi,\varphi\cdot\xi\} = \{\psi,\varphi\}\cdot\xi + (-1)^{(|\varphi|-1)|\psi|}\varphi\cdot\{\psi,\xi\}$,

for any triple of homogeneous elements ψ, φ and ξ in G^*.

Let us make some remarks. In physical applications the $\mathbb{Z}$-grading of G^* is nothing else but the ghost number. The product $\psi\cdot\varphi$ is usually called the dot product, and the product $\{\psi,\varphi\}$ is referred to as the Gerstenhaber bracket product. The space G^0 is a strict commutative algebra and has been dubbed ground ring by physicists. Note that G^1 is closed under the Gerstenhaber bracket and hence is an ordinary Lie algebra. In general, every G^n is a G^1 module via the Gerstenhaber bracket. A special case of this is property *(vi)*, from which we learn that the Lie algebra G^1 acts as derivations on the commutative algebra G^0.

Theorem 4

Let $H^* \equiv \ker Q/\operatorname{im} Q$ denote the cohomology of Q. Define the dot product by $\psi\cdot\varphi := \psi_{-1}\varphi$ and the bracket by $\{\psi,\varphi\} := (-1)^{|\psi|}(\beta_0\psi)_0\varphi$ for all $\psi,\varphi\in H^*$. Then H^*, equipped with this algebraic structure, is a Gerstenhaber algebra graded by the ghost number.

Obviously, the structure of $\mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F}$ is closely related to that of the cohomology of the nilpotent operator Q, if we collect the relevant formulas for the product $\psi_{-1}\varphi$ and add the properties of the bracket operation (1.5.2), (1.2.19), (1.3.10) to obtain

Theorem 5

The dot product $\psi\cdot\varphi := \psi_{-1}\varphi$ and the bracket $[\psi,\varphi] := \psi_0\varphi$ enjoy the following properties on the quotient space $\mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F}$:

(i) If ψ and φ are homogeneous elements of conformal weight Δ_ψ and Δ_φ, respectively, then $\psi\cdot\varphi$ is homogeneous of weight $\Delta_\psi+\Delta_\varphi$ and $[\psi,\varphi]$ is homogeneous of weight $\Delta_\psi+\Delta_\varphi-1$.

(ii) $\psi\cdot\varphi = \varphi\cdot\psi$,

(iii) $(\psi\cdot\varphi)\cdot\xi = \psi\cdot(\varphi\cdot\xi)$,

(iv) $[\psi,\varphi] = -[\varphi,\psi]$,

(v) $[[\psi,\varphi],\xi] + [[\varphi,\xi],\psi] + [[\xi,\psi],\varphi] = 0$,

(vi) $[\psi,\varphi\cdot\xi] = [\psi,\varphi]\cdot\xi + \varphi\cdot[\psi,\xi]$,

for any triple of homogeneous elements ψ, φ and ξ in $\mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F}$.

We observe that the dot products in both algebras are really the same, while the bracket products are defined slightly different. Moreover, Lian and Zuckerman established a Lie algebra homomorphism $\mathfrak{g}_{\mathcal{F}} \to H^1$, i.e. the Lie algebra of physical states can be mapped homomorphically to the ghost number 1 part of H^*. Thus the quotient space $\mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F}$ with its algebraic structure should be regarded as the origin of the Gerstenhaber algebra structure of the cohomology of Q. One has to keep in mind, however, that $\mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F}$ is graded by the conformal weight, whereas H^* is graded by the ghost number. The exact relation between the cohomology of Q and the quotient space $\mathcal{F}/\mathcal{F}_{(-2)}\mathcal{F}$ has not been explored in detail yet.

Chapter II:

Toroidal Compactification of the Bosonic String

It is by no means obvious that nontrivial examples of vertex (operator) algebras exist. However, an important class of vertex algebras is provided by the following result, which was announced in [11] and was proved in [30].

Theorem 6

Associated with each nondegenerate even lattice Λ *is a vertex algebra* $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \omega)$. *If in addition* Λ *is positive definite then* $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \omega)$ *has the structure of a vertex operator algebra.*

In fact, the above examples of vertex algebras gave rise to the very notion and the abstract definition of vertex algebras. As we shall see below, the physics described by these vertex algebras is the chiral sector of a first quantized closed bosonic string moving (partly) on a spacetime torus. Let us briefly indicate the physical motivation for studying such string models.

It is well-known that imposing Lorentz invariance on bosonic strings inevitably leads to the "critical" number of 26 spacetime dimensions, which apparently contradicts our experimental observations. Physicists found a potential resolution of this dilemma by reviving the old idea of Kaluza-Klein theories, that the extra dimensions should be ultramiroscopically compactified to an internal space. Hence one is led to the investigation of string models where some spatial dimensions are compactified. The most simple-minded compactification is an Euclidean torus as internal manifold. Together with the canonical Heisenberg commutation relations for the center of mass position and momentum operators, this immediately requires discrete values for the components of the compactified momenta. More precisely, in the case of a closed bosonic string with d spatial dimensions compactified on a torus, one finds that the left- and the right-moving momenta build an even selfdual $2d$-dimensional Lorentzian lattice with signature $\big((+)^d, (-)^d\big)$ (see e.g. [60]).

The theory considerably simplifies for so-called rational torus compactifications in which left and right momenta separately form even Euclidean lattices. Treating left- and right-moving compactified coordinates as completely independent is of utmost importance when it comes to the heterotic string. For the latter is a hybrid construction of a left-moving 26-dimensional bosonic string together with a right-moving 10-dimensional superstring such that 16 left-moving spatial coordinates are compactified on a torus, whereas no internal space occurs in the right-moving sector. Thus, although the closed bosonic string as such suffers from phenomenogically unacceptable features like the appearance of a tachyon and the non-existence of spacetime fermions, it is nevertheless reasonable to study the chiral part of a closed bosonic string compactified on a torus, because precisely that is needed as a building block of the more realistic heterotic string.

The above described toroidally compactified bosonic string models are fairly well-understood in the sense that one has the clear picture that the Euclidean left- (or right-) momentum lattice serves as the weight lattice of some finite-dimensional internal gauge group. Translated into the vertex algebra language this means that for the vertex operator algebra associated to an even Euclidean lattice, the Lie algebra of physical states is finite-dimensional.

The first two sections will be concerned with the explicit construction of the vertex algebra stated above. Then we shall discuss in detail the Lie algebra of physical states and finally introduce the invariant and covariant bilinear forms.

1. Fock space and vertex operators

For further details of the construction below, the reader may also wish to consult the articles [42], [40] and [43] or the comprehensive review [57].

Let Λ be an even lattice of rank $d < \infty$ with a symmetric nondegenerate $\mathbb{Z}$-valued $\mathbb{Z}$-bilinear form $\_\cdot\_$ and corresponding metric tensor $\eta^{\mu\nu}$, $1 \le \mu, \nu \le d$ (Λ even means that $\mathbf{r}^2 \in 2\mathbb{Z}$ for all $\mathbf{r} \in \Lambda$). The vertex algebra $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \omega)$ which we shall construct can be thought of as a chiral sector of a first quantized closed bosonic string theory with d spacetime dimensions compactified on a torus. Thus Λ represents the allowed momentum vectors of the theory.

Introduce **oscillators** α_m^μ, $m \in \mathbb{Z}, 1 \le \mu \le d$, satisfying the commutation relations of a d-fold **Heisenberg algebra**,

$$[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}, \tag{2.1.1}$$

and **zero mode states** $\Psi_\mathbf{r}$, $\mathbf{r} \in \Lambda$. We want the latter to carry momentum $\mathbf{r}$ and to be annihilated by the positive oscillator modes, i.e.

$$\alpha_m^\mu \Psi_\mathbf{r} = 0 \quad \text{if } m > 0, \tag{2.1.2}$$

$$p^\mu \Psi_\mathbf{r} = r^\mu \Psi_\mathbf{r}, \tag{2.1.3}$$

where $p^\mu \equiv \alpha_0^\mu$ denotes the center of mass momentum operator for the string and r^μ are the components of $\mathbf{r} \in \Lambda$. While the operators α_m^μ for $m > 0$ by definition act as annihilation operators, the operators α_m^μ for $m < 0$ will be called creation operators, since they generate an irreducible Heisenberg module $\mathcal{F}^{(\mathbf{r})}$ with highest weight $\mathbf{r} \in \Lambda$ from any ground state $\Psi_\mathbf{r}$.

We take $\Lambda_\mathbb{R} := \mathbb{R} \otimes_\mathbb{Z} \Lambda$ to be the real vector space in which Λ is embedded, and for notational convenience we define

$$\mathbf{r}(m) := \sum_{\mu=1}^{d} r_\mu \alpha_m^\mu \equiv \mathbf{r} \cdot \boldsymbol{\alpha}_m \tag{2.1.4}$$

for $\mathbf{r} \in \Lambda_\mathbb{R}$, $m \in \mathbb{Z}$, such that

$$[\mathbf{r}(m), \mathbf{s}(n)] = m(\mathbf{r} \cdot \mathbf{s})\delta_{m+n,0}, \tag{2.1.5}$$

with the $\mathbb{Z}$-bilinear form on Λ to be extended to an $\mathbb{R}$-bilinear form on $\Lambda_\mathbb{R}$. We denote the d-fold Heisenberg algebra spanned by the oscillators by

$$\hat{\mathbf{h}} := \{\mathbf{r}(m) \,|\, \mathbf{r} \in \Lambda_\mathbb{R}, m \in \mathbb{Z}\} \oplus \mathbb{R} \cdot 1, \tag{2.1.6}$$

and for the vector space of finite products of creation operators ($\equiv$ algebra of polynomials on the negative oscillator modes) we write

$$S(\hat{\mathbf{h}}^-) := \bigoplus_{N \in \mathbb{N}} \left\{ \prod_{i=1}^{N} \mathbf{r}_i(-m_i) \,|\, \mathbf{r}_i \in \Lambda_\mathbb{R}, m_i > 0 \text{ for } 1 \le i \le N \right\}, \tag{2.1.7}$$

where "S" stands for "symmetric" because of the fact that the creation operators commute with each other. Hence the Heisenberg module built on some ground state $\Psi_\mathbf{r}$ is given by $\mathcal{F}^{(\mathbf{r})} = S(\hat{\mathbf{h}}^-)\Psi_\mathbf{r}$.

If we formally introduce center of mass position operators q^μ, $1 \le \mu \le d$, commuting with α_m^μ for $m \ne 0$ and satisfying

$$[q^\nu, p^\mu] = i\eta^{\mu\nu}, \tag{2.1.8}$$

then we find that

$$e^{i\mathbf{r}\cdot\mathbf{q}}\Psi_\mathbf{s} = \Psi_{\mathbf{r}+\mathbf{s}}, \tag{2.1.9}$$

i.e. the zero mode states can be generated from the vacuum $\Psi_\mathbf{0}$:

$$\Psi_\mathbf{r} = e^{i\mathbf{r}\cdot\mathbf{q}}\Psi_\mathbf{0}. \tag{2.1.10}$$

Thus the operators $e^{i\mathbf{r}\cdot\mathbf{q}}$, $\mathbf{r} \in \Lambda$, may be identified with the zero mode states and form an Abelian group which is called the **group algebra** of the lattice Λ and is denoted by $\mathbb{R}[\Lambda]$. Collecting all the Heisenberg modules $\mathcal{F}^{(\mathbf{r})}$ one might expect the full Fock space $\mathcal{F}$ of the vertex algebra to be $S(\hat{\mathbf{h}}^-) \otimes \mathbb{R}[\Lambda]$. However, it is well-known that we need to replace the group algebra $\mathbb{R}[\Lambda]$ by something more delicate in order to adjust the signs in the Jacobi identity for the vertex algebra. We will multiply $e^{i\mathbf{r}\cdot\mathbf{q}}$ by a so-called **cocycle factor** $c_\mathbf{r}$ which is a function of momentum $\mathbf{p}$. This means that it commutes with all oscillators α_m^μ and satisfies the eigenvalue equations

$$c_\mathbf{r}\Psi_\mathbf{s} = \epsilon(\mathbf{r}, \mathbf{s})\Psi_\mathbf{s}. \tag{2.1.11}$$

More specifically, we define operators $e^\mathbf{r} := e^{i\mathbf{r}\cdot\mathbf{q}}c_\mathbf{r}$ and impose the conditions

$$e^\mathbf{r}e^\mathbf{s} = \epsilon(\mathbf{r}, \mathbf{s})e^{\mathbf{r}+\mathbf{s}}, \tag{2.1.12}$$
$$e^\mathbf{r}e^\mathbf{s} = (-1)^{\mathbf{r}\cdot\mathbf{s}}e^\mathbf{s}e^\mathbf{r}, \tag{2.1.13}$$
$$e^\mathbf{r}e^{-\mathbf{r}} = (-1)^{\frac{1}{2}\mathbf{r}^2}, \tag{2.1.14}$$
$$e^\mathbf{0} = 1, \tag{2.1.15}$$

which are equivalent to requiring, respectively,

$$\epsilon(\mathbf{r}, \mathbf{s})\epsilon(\mathbf{r}+\mathbf{s}, \mathbf{t}) = \epsilon(\mathbf{r}, \mathbf{s}+\mathbf{t})\epsilon(\mathbf{s}, \mathbf{t}), \tag{2.1.16}$$
$$\epsilon(\mathbf{r}, \mathbf{s}) = (-1)^{\mathbf{r}\cdot\mathbf{s}}\epsilon(\mathbf{s}, \mathbf{r}), \tag{2.1.17}$$
$$\epsilon(\mathbf{r}, -\mathbf{r}) = (-1)^{\frac{1}{2}\mathbf{r}^2}, \tag{2.1.18}$$
$$\epsilon(\mathbf{0}, \mathbf{0}) = 1. \tag{2.1.19}$$

For example, associativity of the product $e^\mathbf{r}e^\mathbf{s}e^\mathbf{t}$ and (2.1.12) yield (2.1.16). Note that the cocycle condition (2.1.16) implies $\epsilon(\mathbf{0}, \mathbf{0}) = \epsilon(\mathbf{0}, \mathbf{r}) = \epsilon(\mathbf{r}, \mathbf{0})\ \forall \mathbf{r}$. It is not difficult to show that it is always possible to construct cocycles with these properties (see [43], e.g.). In fact, without loss of generality we can assume that the function ϵ is bimultiplicative, i.e. $\epsilon(\mathbf{r}+\mathbf{s}, \mathbf{t}) = \epsilon(\mathbf{r}, \mathbf{t})\epsilon(\mathbf{s}, \mathbf{t})$ and $\epsilon(\mathbf{r}, \mathbf{s}+\mathbf{t}) = \epsilon(\mathbf{r}, \mathbf{s})\epsilon(\mathbf{r}, \mathbf{t})\ \forall \mathbf{r}, \mathbf{s}, \mathbf{t}$. Together with (2.1.18) and the normalization condition (2.1.19), this then implies that $\epsilon(m\mathbf{r}, n\mathbf{r}) = [\epsilon(\mathbf{r}, \mathbf{r})]^{mn} = (-1)^{\frac{1}{2}mn\mathbf{r}^2}\ \forall \mathbf{r},\ m, n \in \mathbb{Z}$. Also note that every 2-cocycle $\epsilon : \Lambda \times \Lambda \longrightarrow \{\pm 1\}$ corresponds to a central extension $\hat{\Lambda}$ of Λ by $\{\pm 1\}$:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \hat{\Lambda} \longrightarrow \Lambda \longrightarrow 1, \tag{2.1.20}$$

where we put $\hat{\Lambda} = \{\pm 1\} \times \Lambda$ as a set and define a multiplication in $\hat{\Lambda}$ by

$$(\rho, \mathbf{r}) * (\sigma, \mathbf{s}) := (\epsilon(\mathbf{r}, \mathbf{s})\rho\sigma, \mathbf{r}+\mathbf{s}) \quad \text{for } \rho, \sigma \in \{\pm 1\},\ \mathbf{r}, \mathbf{s} \in \Lambda. \tag{2.1.21}$$

We will take the **twisted group algebra** $\mathbb{R}\{\Lambda\}$ consisting of the operators $e^\mathbf{r}$, $\mathbf{r} \in \Lambda$, instead of $\mathbb{R}[\Lambda]$. This means nothing but working with a certain section in the double cover $\hat{\Lambda}$ of the lattice Λ.

To summarize: The Fock space associated with the lattice Λ is defined to be

$$\mathcal{F} := S(\hat{\mathbf{h}}^-) \otimes \mathbb{R}\{\Lambda\}. \tag{2.1.22}$$

Note that the oscillators $\mathbf{r}(m)$, $m \neq 0$, act only on the first tensor factor, namely, creation operators as multiplication operators and annihilation operators via the adjoint representation, i.e. by (2.1.5). The zero mode operators α_0^μ, however, are only sensible for the twisted group algebra, viz

$$\mathbf{r}(0)e^{\mathbf{s}} = (\mathbf{r}\cdot\alpha_0)e^{\mathbf{s}} = (\mathbf{r}\cdot\mathbf{s})e^{\mathbf{s}} \qquad \forall \mathbf{r} \in \Lambda_{\mathbb{R}}, \mathbf{s} \in \Lambda, \tag{2.1.23}$$

while the action of $e^{\mathbf{r}}$ on $\mathbb{R}\{\Lambda\}$ is given by (2.1.12).

We shall define next the **vertex operators** $\mathcal{V}(\psi, z)$ for $\psi \in \mathcal{F}$. For $\mathbf{r} \in \Lambda_{\mathbb{R}}$ we introduce the formal sum

$$\mathbf{r}(z) := \sum_{m\in\mathbb{Z}} \mathbf{r}(m) z^{-m-1}, \tag{2.1.24}$$

which is an element of $\hat{\mathbf{h}}[\![z, z^{-1}]\!]$ and may be regarded as a generating function for the operators $\mathbf{r}(m)$, $m \in \mathbb{Z}$, or as a "current" in contrast to the "states" in $\mathcal{F}$. It is convenient to split the current $\mathbf{r}(z)$ into three parts:

$$\mathbf{r}(z) = \mathbf{r}_<(z) + \mathbf{r}(0) + \mathbf{r}_>(z), \tag{2.1.25}$$

where

$$\mathbf{r}_<(z) := \sum_{m>0} \mathbf{r}(-m) z^{m-1}, \qquad \mathbf{r}_>(z) := \sum_{m>0} \mathbf{r}(m) z^{-m-1} \tag{2.1.26}$$

We will employ the usual normal-ordering procedure, i.e. colons indicate that in the enclosed expressions, q^ν is written to the left of p^μ, as well as the creation operators are to be placed to the left of the annihilation operators:

$$\begin{aligned} :\mathbf{r}(m)\mathbf{s}(n): &= \begin{cases} \mathbf{r}(m)\mathbf{s}(n) & \text{if } m \leq n, \\ \mathbf{s}(n)\mathbf{r}(m) & \text{if } m > n, \end{cases} \\ :q^\nu p^\mu: &= :p^\mu q^\nu: = q^\nu p^\mu. \end{aligned} \tag{2.1.27}$$

For $e^{\mathbf{r}} \in \mathbb{R}\{\Lambda\}$, we set

$$\mathcal{V}(e^{\mathbf{r}}, z) := e^{\int \mathbf{r}_<(z)dz} e^{\mathbf{r}} z^{\mathbf{r}(0)} e^{\int \mathbf{r}_>(z)dz}, \tag{2.1.28}$$

using an obvious formal integration notation, viz

$$\int \mathbf{r}_<(z)dz := \sum_{m>0} \frac{1}{m} \mathbf{r}(-m) z^m, \tag{2.1.29}$$

$$\int \mathbf{r}_>(z)dz := -\sum_{m>0} \frac{1}{m} \mathbf{r}(m) z^{-m}. \tag{2.1.30}$$

This can be written in a way more familiar to physicists by introducing the **Fubini–Veneziano coordinate field**,

$$Q^\mu(z) \equiv q^\mu - ip^\mu \ln z + i \sum_{m\in\mathbb{Z}} \frac{1}{m} \alpha_m^\mu z^{-m}, \tag{2.1.31}$$

which really only has a meaning when exponentiated. We find that the vertex operator in (2.1.28) takes the familiar form

$$\mathcal{V}(e^{\mathbf{r}}, z) = :e^{i\mathbf{r}\cdot\mathbf{Q}(z)}: c_{\mathbf{r}}, \tag{2.1.32}$$

and the current $\mathbf{r}(z)$ becomes

$$\mathbf{r}(z) = i\frac{d}{dz}[\mathbf{r}\cdot\mathbf{Q}(z)]. \tag{2.1.33}$$

This shows that the vertex operators in (2.1.28) are indeed already normal-ordered and carry a cocycle factor hidden in the elements of the twisted group algebra $\mathbb{R}\{\Lambda\}$.

Let $\psi = \left[\prod_{j=1}^N \mathbf{s}_j(-n_j)\right] \otimes e^{\mathbf{r}}$ be a typical homogeneous element of $\mathcal{F}$ and define

$$\begin{aligned} \mathcal{V}(\psi, z) &:= :\mathcal{V}(e^{\mathbf{r}}, z) \prod_{j=1}^N \frac{1}{(n_j-1)!} \left(\frac{d}{dz}\right)^{n_j-1} \mathbf{s}_j(z): \\ &\equiv i :e^{i\mathbf{r}\cdot\mathbf{Q}(z)} \prod_{j=1}^N \frac{1}{(n_j-1)!)} \left(\frac{d}{dz}\right)^{n_j} (\mathbf{s}_j\cdot\mathbf{Q}(z)) : c_{\mathbf{r}}. \end{aligned} \tag{2.1.34}$$

Extending this definition by linearity we finally obtain a well-defined map

$$\begin{aligned} \mathcal{V} : \mathcal{F} &\to (\operatorname{End}\mathcal{F})[\![z, z^{-1}]\!], \\ \psi &\mapsto \mathcal{V}(\psi, z) = \sum_{n\in\mathbb{Z}} \psi_n z^{-n-1}. \end{aligned} \tag{2.1.35}$$

2. Proof of the vertex algebra axioms

The first four axioms in the definition of a vertex algebra can be proved straightforwardly.

1. **(Regularity)** Note that $\mathcal{F}$ contains only states with a finite occupation number of creation operators and the vertex operators are normal-ordered expressions. Having this in mind it is clear that $\psi_n\varphi = 0$ for n large enough (depending on $\psi, \varphi \in \mathcal{F}$) since annihilation operators are always attached to negative powers of the formal variables.

2. **(Vacuum)** We choose the vacuum to be the zero mode state with no momentum and without any creation operators, i.e.

$$\mathbf{1} := 1 \otimes \mathrm{e}^{\mathbf{0}}, \tag{2.2.1}$$

so that

$$\mathcal{V}(\mathbf{1}, z) = {:}\mathrm{e}^{i\mathbf{0}\cdot \mathbf{Q}(z)}{:}c_{\mathbf{0}} = \mathrm{id}_{\mathcal{F}} \tag{2.2.2}$$

by the normalization condition (2.1.15).

3. **(Injectivity)** Observe that, when acting on the vacuum, only terms involving creation operators survive in the expression for a vertex operator. Then it is obvious that

$$\psi_{-1}\mathbf{1} = \mathrm{Res}_z\left[z^{-1}\mathcal{V}(\psi, z)\mathbf{1}\right] = \psi \qquad \forall \psi \in \mathcal{F}. \tag{2.2.3}$$

In particular, $\mathcal{V}(\psi, z) = 0$ implies $\psi = 0$.

4. **(Conformal vector)** We claim that the element

$$\boldsymbol{\omega} := \tfrac{1}{2}\sum_{\mu=1}^{d} \mathrm{e}^{(\mu)}(-1)\mathrm{e}_{(\mu)}(-1)(\otimes \mathrm{e}^{\mathbf{0}}) \tag{2.2.4}$$

provides a conformal vector of dimension d which is independent of the choice of the basis $\{\mathrm{e}_{(\mu)}\}$ of $\Lambda_{\mathbb{R}}$ with dual basis $\{\mathrm{e}^{(\mu)}\}$ (w.r.t. $\eta^{\mu\nu}$). By (2.1.34) and (2.1.24), we have

$$\begin{aligned}\mathcal{V}(\boldsymbol{\omega}, z) &= \tfrac{1}{2}\sum_{\mu=1}^{d} {:}\mathrm{e}^{(\mu)}(z)\mathrm{e}_{(\mu)}(z){:} \\ &= \tfrac{1}{2}\sum_{m,n\in\mathbb{Z}} {:}\boldsymbol{\alpha}_m \cdot \boldsymbol{\alpha}_n{:}\, z^{-m-n-2}.\end{aligned} \tag{2.2.5}$$

(Note that in the last step we had to rely on nondegeneracy of the lattice, i.e. we used the completeness relation $\sum_{\mu=1}^{d}(\mathrm{e}^{(\mu)})_\rho(\mathrm{e}_{(\mu)})_\sigma = \eta_{\rho\sigma}$.) Thus

$$\mathrm{L}_{(n)} \equiv \boldsymbol{\omega}_{n+1} = \tfrac{1}{2}\sum_{m\in\mathbb{Z}} {:}\boldsymbol{\alpha}_m \cdot \boldsymbol{\alpha}_{n-m}{:}, \tag{2.2.6}$$

in agreement with the well-known expression from string theory. Using the oscillator commutation relations one indeed finds that the $\mathrm{L}_{(n)}$'s obey (1.2.6) with central charge $c = d$ (see e.g. [45] for the calculation). To establish the translation property of $\mathrm{L}_{(-1)}$ we find that

$$\mathrm{L}_{(-1)}\mathrm{e}^{\mathbf{r}} = \mathbf{r}(-1)\mathrm{e}^{\mathbf{r}}, \tag{2.2.7}$$

$$\mathrm{L}_{(-1)}\mathbf{r}(-m) = m\mathbf{r}(-m-1) \quad \text{for } m > 0, \tag{2.2.8}$$

by (2.1.23) and (2.1.5); but, on the other hand,

$$\frac{d}{dz}\mathcal{V}(\mathrm{e}^{\mathbf{r}}, z) = {:}\mathbf{r}(z)\mathcal{V}(\mathrm{e}^{\mathbf{r}}, z){:} = \mathcal{V}(\mathbf{r}(-1)\mathrm{e}^{\mathbf{r}}, z), \tag{2.2.9}$$

$$\frac{d}{dz}\mathcal{V}(\mathbf{r}(-m), z) = \frac{1}{(m-1)!}\left(\frac{d}{dz}\right)^m \mathbf{r}(z) = \mathcal{V}(m\mathbf{r}(-m-1), z), \tag{2.2.10}$$

by (2.1.28), (2.1.34) and (2.1.24). Together with the derivation property of $\mathrm{L}_{(-1)}$ and $\frac{d}{dz}$, this proves (1.2.7). Finally, let $\psi = \left[\prod_{j=1}^{N}\mathbf{s}_j(-n_j)\right] \otimes \mathrm{e}^{\mathbf{r}}$ be a typical homogeneous element of $\mathcal{F}$. Then

$$\begin{aligned}\mathrm{L}_{(0)}\psi &= \left(\tfrac{1}{2}\mathbf{p}^2 + \sum_{m\geq 1}\boldsymbol{\alpha}_{-m}\cdot\boldsymbol{\alpha}_m\right)\left\{\left[\prod_{j=1}^{N}\mathbf{s}_j(-n_j)\right]\otimes \mathrm{e}^{\mathbf{r}}\right\} \\ &= \left(\tfrac{1}{2}\mathbf{r}^2 + \sum_{j=1}^{N} n_j\right)\psi\end{aligned} \tag{2.2.11}$$

yields the desired grading of $\mathcal{F}$. Furthermore we observe that the spectrum of $\mathrm{L}_{(0)}$ is nonnegative and the eigenspaces of $\mathrm{L}_{(0)}$ are finite-dimensional provided that Λ is a positive definite lattice; while if Λ is Lorentzian then $\mathbf{r}^2$ can be arbitrarily negative so that the spectrum of $\mathrm{L}_{(0)}$ is unbounded from above as well as from below.

It is not surprising that by far the hardest axiom to prove is the Jacobi identity because it contains most information about a vertex algebra. We will not go much into details; we will only mention the important steps and crucial ideas (see [30]).

Step 1:
We make a change of variables. Let $\mathbf{r}_1, \ldots, \mathbf{r}_M, \mathbf{s}_1, \ldots, \mathbf{s}_N \in \Lambda$ and consider the formal sums

$$\begin{aligned}R &\equiv \prod_{i=1}^{M}\left(\mathrm{e}^{\sum_{m>0}\frac{1}{m}\mathbf{r}_i(-m)x_i^m}\mathrm{e}^{\mathbf{r}_i}\right) \\ &= \prod_{i=1}^{M}\left[\sum_{m\geq 0}\mathcal{S}_m(\mathbf{r}_i(-1), \ldots, \mathbf{r}_i(-m))x_i^m \mathrm{e}^{\mathbf{r}_i}\right] \quad \in \mathcal{F}[\![x_1, \ldots, x_M]\!],\end{aligned} \tag{2.2.12}$$

$$\begin{aligned}S &\equiv \prod_{j=1}^{N}\left(\mathrm{e}^{\sum_{n>0}\frac{1}{n}\mathbf{s}_j(-n)y_j^n}\mathrm{e}^{\mathbf{s}_j}\right) \\ &= \prod_{j=1}^{N}\left[\sum_{n\geq 0}\mathcal{S}_n(\mathbf{s}_j(-1), \ldots, \mathbf{s}_j(-n))y_j^n \mathrm{e}^{\mathbf{s}_j}\right] \quad \in \mathcal{F}[\![y_1, \ldots, y_N]\!],\end{aligned} \tag{2.2.13}$$

where $\mathcal{S}_n$ denotes the n^{th} Schur polynomial (see also (2.3.7)). We note that the coefficients of the monomials in the formal variables *span* $\mathcal{F}$ as M and the $\mathbf{r}_i$'s and N and the $\mathbf{s}_j$'s vary, respectively. Hence it suffices to prove the Jacobi identity with ψ and φ replaced by R and S, respectively.

Using (2.1.28) and (2.1.23) we can immediately rewrite R and S as

$$R = :\prod_{i=1}^{M} \mathcal{V}(e^{\mathbf{r}_i}, x_i):\mathbf{1}\,, \tag{2.2.14}$$

$$S = :\prod_{j=1}^{N} \mathcal{V}(e^{\mathbf{s}_j}, y_j):\mathbf{1}\,. \tag{2.2.15}$$

Step 2:
A lengthy but straightforward calculation which uses normal-ordering properties and (1.1.31) shows that

$$\mathcal{V}(R, z_1) = :e^{\sum_{i=1}^{M}\sum_{m\geq 1}\frac{1}{m!}\left(\frac{d}{dz_1}\right)^{m-1}\mathbf{r}_i(z_1)x_i^m}\,\mathcal{V}\Big(\prod_{i=1}^{M} e^{\mathbf{r}_i}, z_1\Big):$$

$$= :\prod_{i=1}^{M}\mathcal{V}(e^{\mathbf{r}_i}, z_1 + x_i):\,, \tag{2.2.16}$$

$$\mathcal{V}(S, z_2) = :e^{\sum_{j=1}^{N}\sum_{n\geq 1}\frac{1}{n!}\left(\frac{d}{dz_2}\right)^{n-1}\mathbf{s}_j(z_2)y_j^n}\,\mathcal{V}\Big(\prod_{j=1}^{N} e^{\mathbf{s}_j}, z_2\Big):$$

$$= :\prod_{j=1}^{N}\mathcal{V}(e^{\mathbf{s}_j}, z_2 + y_j):\,. \tag{2.2.17}$$

Hence

$$:\mathcal{V}(R, z_1)\mathcal{V}(S, z_2): = \prod_{\substack{1\leq i\leq M\\ 1\leq j\leq N}} (-1)^{\mathbf{r}_i\cdot\mathbf{s}_j} :\mathcal{V}(S, z_2)\mathcal{V}(R, z_1): \tag{2.2.18}$$

and

$$\mathcal{V}(R, z_1)\mathcal{V}(S, z_2) = \prod_{\substack{1\leq i\leq M\\ 1\leq j\leq N}} [z_1 + (-z_2 + x_i - y_j)]^{\mathbf{r}_i\cdot\mathbf{s}_j} :\mathcal{V}(R, z_1)\mathcal{V}(S, z_2):\,, \tag{2.2.19}$$

$$\mathcal{V}(S, z_2)\mathcal{V}(R, z_1) = \prod_{\substack{1\leq i\leq M\\ 1\leq j\leq N}} [-z_2 + (z_1 + x_i - y_j)]^{\mathbf{r}_i\cdot\mathbf{s}_j} :\mathcal{V}(R, z_1)\mathcal{V}(S, z_2): \tag{2.2.20}$$

(all binomial expressions to be expanded in the second term!).

Step 3:
Fix $k \in \mathbb{Z}$ and a monomial $q = \prod_{i=1}^{M}\prod_{j=1}^{N} x_i^{m_i} y_j^{n_j}$, $m_i, n_j \geq 0\ \forall i, j$. Choose $K \geq 0$ such that $K + k \geq 0$ and $K + k \geq \deg q - \sum_{i=1}^{M}\sum_{j=1}^{N}\mathbf{r}_i\cdot\mathbf{s}_j$. Thus the coefficient of q and of each monomial of lower total degree than q in

$$F_K \equiv (z_1 - z_2)^{K+k} \prod_{\substack{1\leq i\leq M\\ 1\leq j\leq N}} [z_1 + (-z_2 + x_i - y_j)]^{\mathbf{r}_i\cdot\mathbf{s}_j}$$

$$= (z_1 - z_2)^{K+k} \prod_{\substack{1\leq i\leq M\\ 1\leq j\leq N}} \sum_{\substack{m_i\geq 0\\ n_j\geq 0}} \binom{\mathbf{r}_i\cdot\mathbf{s}_j}{m_i}\binom{\mathbf{r}_i\cdot\mathbf{s}_j - m_i}{n_j}(-1)^{n_j}(z_1 - z_2)^{\mathbf{r}_i\cdot\mathbf{s}_j - m_i - n_j} x_i^{m_i} y_j^{n_j}$$

is a *polynomial* in $z_1 - z_2$.

Let $V_q(z_1, z_2)$ denote the coefficient of q in

$$(z_1 - z_2)^{K+k}\mathcal{V}(R, z_1)\mathcal{V}(S, z_2) = F_K :\mathcal{V}(R, z_1)\mathcal{V}(S, z_2):\,.$$

Then the coefficient of q in $(z_1 - z_2)^k\mathcal{V}(R, z_1)\mathcal{V}(S, z_2)$ is $(z_1 - z_2)^{-K}V_q(z_1, z_2)$. Similarly we find that the coefficient of q in $(-z_2 + z_1)^k\mathcal{V}(S, z_2)\mathcal{V}(R, z_1)$ is $(-z_2 + z_1)^{-K}V_q(z_1, z_2)$. It follows that the coefficient of q in

$$(z_1 - z_2)^k\mathcal{V}(R, z_1)\mathcal{V}(S, z_2) - (-z_2 + z_1)^k\mathcal{V}(S, z_2)\mathcal{V}(R, z_1)$$

is, by (1.1.20),

$$-(-z_2)^{-K}\Theta\left[\left(1 - \frac{z_1}{z_2}\right)^{-K}\right] V_q(z_1, z_2),$$

which is the coefficient of z_0^{K-1} in

$$z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) V_q(z_1, z_2) = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) V_q(z_2 + z_0, z_2)$$

by (1.1.20),(1.1.12) and (1.1.6). But $V_q(z_2 + z_0, z_2)$ is the coefficient of q in

$$:\mathcal{V}(R, z_2 + z_0)\mathcal{V}(S, z_2): z_0^{K+k} \prod_{\substack{1\leq i\leq M\\ 1\leq j\leq N}} [z_0 + (x_i - y_j)]^{\mathbf{r}_i\cdot\mathbf{s}_j}\,.$$

Hence $(z_1 - z_2)^k\mathcal{V}(R, z_1)\mathcal{V}(S, z_2) - (-z_2 + z_1)^k\mathcal{V}(S, z_2)\mathcal{V}(R, z_1)$ is the coefficient of z_0^{-k-1} in

$$z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) :\mathcal{V}(R, z_2 + z_0)\mathcal{V}(S, z_2): \prod_{\substack{1\leq i\leq M\\ 1\leq j\leq N}} [z_0 + (x_i - y_j)]^{\mathbf{r}_i\cdot\mathbf{s}_j}\,.$$

Note that the last expression is independent of q and K! We conclude that

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\mathcal{V}(R,z_1)\mathcal{V}(S,z_2)-z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)\mathcal{V}(S,z_2)\mathcal{V}(R,z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) : \mathcal{V}(R,z_2+z_0)\mathcal{V}(S,z_2) : \prod_{\substack{1\le i\le M\\1\le j\le N}}[z_0+(x_i-y_j)]^{\mathbf{r}_i\cdot\mathbf{s}_j}.$$

Step 4:
On the other hand, by the use of Step 2,

$$\mathcal{V}(R,z_0)S = \prod_{\substack{1\le i\le M\\1\le j\le N}}[z_0+(x_i-y_j)]^{\mathbf{r}_i\cdot\mathbf{s}_j} : \prod_{i=1}^{M}\mathcal{V}(\mathrm{e}^{\mathbf{r}_i},z_0+x_i)\prod_{j=1}^{N}\mathcal{V}(\mathrm{e}^{\mathbf{s}_j},y_j) : \mathbf{1}.$$

Thus

$$\mathcal{V}(\mathcal{V}(R,z_0)S,z_2)$$
$$= \prod_{\substack{1\le i\le M\\1\le j\le N}}[z_0+(x_i-y_j)]^{\mathbf{r}_i\cdot\mathbf{s}_j} : \prod_{i=1}^{M}\mathcal{V}(\mathrm{e}^{\mathbf{r}_i},z_2+z_0+x_i)\prod_{j=1}^{N}\mathcal{V}(\mathrm{e}^{\mathbf{s}_j},z_2+y_j) :$$
$$= : \mathcal{V}(R,z_2+z_0)\mathcal{V}(S,z_2) : \prod_{\substack{1\le i\le M\\1\le j\le N}}[z_0+(x_i-y_j)]^{\mathbf{r}_i\cdot\mathbf{s}_j}.$$

This completes the proof of the Jacobi identity.

3. Lie algebra of physical states

We turn now to the analysis of the Lie algebra of physical states, $\mathfrak{g}_\Lambda$, and work out some of its commutators. A closed formula for the commutator of zero mode operators associated to general weight one states and a related investigation of the Lie algebra of quasiprimary states of weight one can be found in [61] and [62].

Let us first list the simplest physical states:

1. **Tachyonic states:**

$$\mathfrak{g}_\Lambda^{[0]} := \{\mathrm{e}^{\mathbf{r}} \,|\, \mathbf{r}\in\Lambda_2\}; \tag{2.3.1}$$

2. **Photonic states:**

$$\mathfrak{g}_\Lambda^{[1]} := \{\mathrm{s}(-1)\otimes\mathrm{e}^{\mathbf{r}} \,|\, \mathbf{r}\cdot\mathbf{s}=0, \mathbf{s}\in\Lambda_{\mathbb{R}}, \mathbf{r}\in\Lambda_0\}; \tag{2.3.2}$$

3. **Massive spin 2 states:**

$$\mathfrak{g}_\Lambda^{[2]} := \{\ [(\mathbf{s}\cdot\mathbf{r})\mathbf{t}(-2)+(\mathbf{t}\cdot\mathbf{r})\mathbf{s}(-2)-2\mathbf{s}(-1)\mathbf{t}(-1)]\otimes\mathrm{e}^{\mathbf{r}} \,|\, \mathbf{s}\cdot\mathbf{t}=2(\mathbf{s}\cdot\mathbf{r})(\mathbf{t}\cdot\mathbf{r}),\ \mathbf{s},\mathbf{t}\in\Lambda_{\mathbb{R}},\ \mathbf{r}\in\Lambda_{-2}\ \}; \tag{2.3.3}$$

where $\Lambda_n := \{\mathbf{r}\in\Lambda \,|\, \mathbf{r}^2=n\ (\in 2\mathbb{Z})\}$ denotes the set of lattice vectors of squared length n and the superscript of $\mathfrak{g}_\Lambda$ counts the oscillator excitations. The relevant physical state conditions for the above polarization vectors $\mathbf{s},\mathbf{t}\in\Lambda_{\mathbb{R}}$ follow immediately from (2.2.11) and

$$\begin{aligned}\mathrm{L}_{(m)}\psi = &\sum_{\substack{k=1\\n_k>m}}^{N} n_k\Big[\mathbf{s}_k(m-n_k)\prod_{\substack{j=1\\j\ne k}}^{N}\mathbf{s}_j(-n_j)\Big]\otimes\mathrm{e}^{\mathbf{r}}\\ &+m\sum_{k=1}^{N}\delta_{m,n_k}(\mathbf{s}_k\cdot\mathbf{r})\Big[\prod_{\substack{j=1\\j\ne k}}^{N}\mathbf{s}_j(-n_j)\Big]\otimes\mathrm{e}^{\mathbf{r}}\\ &+\sum_{k<k'}^{N} n_k n_{k'}\delta_{m,n_k+n_{k'}}(\mathbf{s}_k\cdot\mathbf{s}_{k'})\Big[\prod_{\substack{j=1\\j\ne k,k'}}^{N}\mathbf{s}_j(-n_j)\Big]\otimes\mathrm{e}^{\mathbf{r}}\end{aligned} \tag{2.3.4}$$

for $\psi = \big[\prod_{j=1}^{N}\mathbf{s}_j(-n_j)\big]\otimes\mathrm{e}^{\mathbf{r}}\in\mathcal{F}$. The above formula also exhibits an explicit example for the regularity axiom (1.2.11), namely that $\mathrm{L}_{(m)}\psi=0$ for $m>\max_{j\ne k}(n_j+n_k)$. We want to stress again that the physical states in $\mathfrak{g}_\Lambda$ are only defined modulo $\mathrm{L}_{(-1)}\mathcal{P}_{(0)}$, which means for example that $\mathbf{r}(-1)\otimes\mathrm{e}^{\mathbf{r}}=\mathrm{L}_{(-1)}(\mathrm{e}^{\mathbf{r}})\equiv 0$ in $\mathfrak{g}_\Lambda$ for $\mathbf{r}\in\Lambda_0$.

For the antisymmetric product (1.5.1) on $\mathcal{P}_{(1)}/\mathrm{L}_{(-1)}\mathcal{P}_{(0)}$ we obtain

$$\begin{aligned}[\mathrm{e}^{\mathbf{r}},\mathrm{e}^{\mathbf{s}}] &:= \mathrm{e}^{\mathbf{r}}_0\mathrm{e}^{\mathbf{s}}\\ &= \mathrm{Res}_z\Big[\mathrm{e}^{\int\mathbf{r}_<(z)dz}\mathrm{e}^{\mathbf{r}}z^{\mathbf{r}(0)}\mathrm{e}^{\int\mathbf{r}_>(z)dz}(1\otimes\mathrm{e}^{\mathbf{s}})\Big]\\ &= \mathrm{Res}_z\Big[\sum_{m\ge0}\mathcal{S}_m(\mathbf{r})z^{m+\mathbf{r}\cdot\mathbf{s}}\mathrm{e}^{\mathbf{r}}\mathrm{e}^{\mathbf{s}}\Big]\\ &= \begin{cases}0 & \text{if } \mathbf{r}\cdot\mathbf{s}\ge 0,\\ \epsilon(\mathbf{r},\mathbf{s})\mathcal{S}_{-1-\mathbf{r}\cdot\mathbf{s}}(\mathbf{r})\otimes\mathrm{e}^{\mathbf{r}+\mathbf{s}} & \text{if } \mathbf{r}\cdot\mathbf{s}<0,\end{cases}\end{aligned} \tag{2.3.5}$$

where we used the Schur polynomials $\mathcal{S}_m(\mathbf{r}) \equiv \mathcal{S}_m(\mathbf{r}(-1),\mathbf{r}(-2),\ldots,\mathbf{r}(-m))$, which are defined via the generating function

$$\mathrm{e}^{\sum_{n>0}\frac{1}{n}\mathbf{r}(-n)z^n} = \sum_{m\ge0}\mathcal{S}_m(\mathbf{r}(-1),\mathbf{r}(-2),\ldots,\mathbf{r}(-m))z^m; \tag{2.3.6}$$

for example,

$$\begin{aligned} \mathcal{S}_0(\mathbf{r}) &= 1, \\ \mathcal{S}_1(\mathbf{r}) &= \mathbf{r}(-1), \\ \mathcal{S}_2(\mathbf{r}) &= \tfrac{1}{2!}\left[\mathbf{r}(-1)^2 + \mathbf{r}(-2)\right], \\ \mathcal{S}_3(\mathbf{r}) &= \tfrac{1}{3!}\left[\mathbf{r}(-1)^3 + 3\mathbf{r}(-2)\mathbf{r}(-1) + 2\mathbf{r}(-3)\right]. \end{aligned} \tag{2.3.7}$$

For notational convenience we put $\mathcal{S}_m(\mathbf{r}) := 0$ for $m < 0, \mathbf{r} \in \Lambda$. We also find that

$$\begin{aligned} &[\mathbf{s}(-1) \otimes \mathrm{e}^{\mathbf{r}}, \mathrm{e}^{\mathbf{t}}] \\ &= \mathrm{Res}_z \left[:\mathrm{e}^{\int \mathbf{r}_<(z)dz} \mathrm{e}^{\mathbf{r}} z^{\mathbf{r}(0)} \mathrm{e}^{\int \mathbf{r}_>(z)dz} \mathbf{s}(z):(1 \otimes \mathrm{e}^{\mathbf{t}}) \right] \\ &= \mathrm{Res}_z \left\{ \sum_{m\geq 0} \mathcal{S}_m(\mathbf{r}) z^{m+\mathbf{r}\cdot\mathbf{t}} \left[(\mathbf{s}\cdot\mathbf{t}) z^{-1} + \sum_{n>0} \mathbf{s}(-n) z^{n-1} \right] \otimes \mathrm{e}^{\mathbf{r}}\mathrm{e}^{\mathbf{t}} \right\} \\ &= \begin{cases} 0 & \text{if } \mathbf{r}\cdot\mathbf{t} \geq 1, \\ \epsilon(\mathbf{r},\mathbf{t}) \left[(\mathbf{s}\cdot\mathbf{t}) \mathcal{S}_{-\mathbf{r}\cdot\mathbf{t}}(\mathbf{r}) + \sum_{m=0}^{-1-\mathbf{r}\cdot\mathbf{t}} \mathcal{S}_m(\mathbf{r}) \mathbf{s}(m+\mathbf{r}\cdot\mathbf{t}) \right] \otimes \mathrm{e}^{\mathbf{r}+\mathbf{t}} & \text{if } \mathbf{r}\cdot\mathbf{t} \leq 0; \end{cases} \end{aligned} \tag{2.3.8}$$

and

$$\begin{aligned} &[\mathbf{s}(-1) \otimes \mathrm{e}^{\mathbf{r}}, \mathbf{u}(-1) \otimes \mathrm{e}^{\mathbf{t}}] \\ &= \mathrm{Res}_z \left[:\mathrm{e}^{\int \mathbf{r}_<(z)dz} \mathrm{e}^{\mathbf{r}} z^{\mathbf{r}(0)} \mathrm{e}^{\int \mathbf{r}_>(z)dz} \mathbf{s}(z):(\mathbf{u}(-1) \otimes \mathrm{e}^{\mathbf{t}}) \right] \\ &= \mathrm{Res}_z \left\{ \sum_{m\geq 0} \mathcal{S}_m(\mathbf{r}) z^{m+\mathbf{r}\cdot\mathbf{t}} \left[[\mathbf{s}\cdot\mathbf{u} - (\mathbf{r}\cdot\mathbf{u})(\mathbf{s}\cdot\mathbf{t})] z^{-2} + ((\mathbf{s}\cdot\mathbf{t})\mathbf{u} - (\mathbf{r}\cdot\mathbf{u})\mathbf{s})(-1) z^{-1} \right.\right. \\ &\qquad \left.\left. + \sum_{n>0} [\mathbf{s}(-n)\mathbf{u}(-1) - (\mathbf{r}\cdot\mathbf{u})\mathbf{s}(-n-1)] z^{n-1} \right] \otimes \mathrm{e}^{\mathbf{r}}\mathrm{e}^{\mathbf{t}} \right\} \\ &= \begin{cases} 0 & \text{if } \mathbf{r}\cdot\mathbf{t} \geq 2, \\ \epsilon(\mathbf{r},\mathbf{t}) \Big[[\mathbf{s}\cdot\mathbf{u} - (\mathbf{r}\cdot\mathbf{u})(\mathbf{s}\cdot\mathbf{t})] \mathcal{S}_{1-\mathbf{r}\cdot\mathbf{t}}(\mathbf{r}) & \\ \quad + ((\mathbf{s}\cdot\mathbf{t})\mathbf{u} - (\mathbf{r}\cdot\mathbf{u})\mathbf{s})(-1) \mathcal{S}_{-\mathbf{r}\cdot\mathbf{t}}(\mathbf{r}) & \\ \quad + \sum_{m=0}^{-1-\mathbf{r}\cdot\mathbf{t}} [\mathbf{s}(-m-\mathbf{r}\cdot\mathbf{t})\mathbf{u}(-1) & \\ \quad - (\mathbf{r}\cdot\mathbf{u})\mathbf{s}(-m-1-\mathbf{r}\cdot\mathbf{t})] \mathcal{S}_m(\mathbf{r}) \Big] \otimes \mathrm{e}^{\mathbf{r}+\mathbf{t}} & \text{if } \mathbf{r}\cdot\mathbf{t} \leq 1. \end{cases} \end{aligned} \tag{2.3.9}$$

These formulas simplify drastically in the special case where Λ is a positive definite even lattice. Obviously, $\mathcal{F}_{(0)} = \mathbb{R}\mathbf{1}$ and the spectrum of $\mathrm{L}_{(0)}$ is nonnegative so that $\mathfrak{g}_\Lambda = \mathcal{P}_{(1)} = \mathcal{F}_{(1)}$. Its elements are easy to describe:

$$\mathfrak{g}_\Lambda = \mathrm{span}_{\mathbb{R}}\{\mathrm{e}^{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \Lambda_2\} \oplus \{\boldsymbol{\xi}(-1) \mid \boldsymbol{\xi} \in \Lambda_{\mathbb{R}}\}. \tag{2.3.10}$$

The commutators become

$$[\boldsymbol{\xi}(-1), \boldsymbol{\eta}(-1)] = 0, \tag{2.3.11}$$

$$[\boldsymbol{\xi}(-1), \mathrm{e}^{\boldsymbol{\alpha}}] = (\boldsymbol{\xi}\cdot\boldsymbol{\alpha})\mathrm{e}^{\boldsymbol{\alpha}}, \tag{2.3.12}$$

$$[\mathrm{e}^{\boldsymbol{\alpha}}, \mathrm{e}^{\boldsymbol{\beta}}] = \begin{cases} 0 & \text{if } \boldsymbol{\alpha}\cdot\boldsymbol{\beta} \geq 0, \\ \epsilon(\boldsymbol{\alpha},\boldsymbol{\beta})\mathrm{e}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} & \text{if } \boldsymbol{\alpha}\cdot\boldsymbol{\beta} = -1, \\ -\boldsymbol{\alpha}(-1) & \text{if } \boldsymbol{\alpha}\cdot\boldsymbol{\beta} = -2, \end{cases} \tag{2.3.13}$$

for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Lambda_2$ and $\boldsymbol{\xi}, \boldsymbol{\eta} \in \Lambda_{\mathbb{R}}$. Note that here, in the special case of a Euclidean lattice, the Schwarz inequality yields $|\boldsymbol{\alpha}\cdot\boldsymbol{\beta}| \leq 2$. Moreover, $\boldsymbol{\alpha}\cdot\boldsymbol{\beta} = -1 \iff \boldsymbol{\alpha}+\boldsymbol{\beta} \in \Lambda_2$ and $\boldsymbol{\alpha}\cdot\boldsymbol{\beta} = -2 \iff \boldsymbol{\alpha}+\boldsymbol{\beta} = 0$ for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Lambda_2$. Thus we have arrived at a root space decomposition of the finite-dimensional[1] real Lie algebra $\mathfrak{g}_\Lambda$, where the root lattice is precisely the lattice Λ, the set of roots is given by Λ_2 and the Cartan subalgebra is just $\{\boldsymbol{\xi}(-1) \mid \boldsymbol{\xi} \in \Lambda_{\mathbb{R}}\}$.

For the affine Lie algebra $\widehat{\mathfrak{g}_\Lambda} = \widehat{\mathcal{F}_{(1)}}$ in Theorem 2 we find the formulas

$$[\boldsymbol{\xi}(-1)_m, \boldsymbol{\eta}(-1)_n] = m(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\delta_{m+n,0}\mathrm{id}_{\mathcal{F}}, \tag{2.3.14}$$

$$[\boldsymbol{\xi}(-1)_m, \mathrm{e}^{\boldsymbol{\alpha}}_n] = (\boldsymbol{\xi}\cdot\boldsymbol{\alpha})\mathrm{e}^{\boldsymbol{\alpha}}_{m+n}, \tag{2.3.15}$$

$$[\mathrm{e}^{\boldsymbol{\alpha}}_m, \mathrm{e}^{\boldsymbol{\beta}}_n] = \begin{cases} 0 & \text{if } \boldsymbol{\alpha}\cdot\boldsymbol{\beta} \geq 0, \\ \epsilon(\boldsymbol{\alpha},\boldsymbol{\beta})\mathrm{e}^{\boldsymbol{\alpha}+\boldsymbol{\beta}}_{m+n} & \text{if } \boldsymbol{\alpha}\cdot\boldsymbol{\beta} = -1, \\ -\boldsymbol{\alpha}(-1)_{m+n} - m\delta_{m+n,0}\mathrm{id}_{\mathcal{F}} & \text{if } \boldsymbol{\alpha}\cdot\boldsymbol{\beta} = -2. \end{cases} \tag{2.3.16}$$

The first commutator is no surprise, since the operators $\boldsymbol{\xi}(-1)_m$ are nothing but the oscillators we have started with, viz

$$\boldsymbol{\xi}(-1)_m = \mathrm{Res}_z\left[z^m \boldsymbol{\xi}(z)\right] = \boldsymbol{\xi}(m), \tag{2.3.17}$$

while the operators $\mathrm{e}^{\boldsymbol{\alpha}}_m$ are those occurring in the Frenkel–Kac construction [27] of affine Lie algebras:

$$\mathrm{e}^{\boldsymbol{\alpha}}_m = \mathrm{Res}_z\left[z^m :\mathrm{e}^{i\boldsymbol{\alpha}\cdot\mathbf{Q}(z)}:\right] c_{\boldsymbol{\alpha}}. \tag{2.3.18}$$

In the physics literature this construction of affine Lie algebras is presented pedagogically in Refs. [73], [56], [42], [43], [40].

[1] Since Λ is positive definite, the set Λ_2 of roots must be finite!

We will not pursue further the well-understood case of positive definite Λ, which leads to a finite-dimensional Lie algebra $\mathfrak{g}_\Lambda$ and its affinization, but rather turn to the case of Lorentzian lattices, which is of course far more complicated.

We have seen that a special role is played by the norm 2 vectors of Λ which we call **real roots** of the lattice. The **reflection** $w_{\mathbf{r}}$ associated with a real root $\mathbf{r}$ is defined as $w_{\mathbf{r}}(\mathbf{x}) = \mathbf{x} - (\mathbf{x}\cdot\mathbf{r})\mathbf{r}$ for $\mathbf{x} \in \Lambda_{\mathbb{R}}$. It is easy to see that a reflection in a real root is an automorphism of the lattice. The hyperplanes perpendicular to these real roots divide the vector space $\Lambda_{\mathbb{R}}$ into regions called **Weyl chambers**. The reflections in the real roots of Λ generate a group called the **Weyl group** W of Λ, which acts simply transitively on the Weyl chambers of Λ. This means that if we fix one Weyl chamber C once and for all, then any real root from the interior of another Weyl chamber can be transported via Weyl reflection to a unique real root in C. The real roots $\mathbf{r}_i$ that are perpendicular to the faces of C and have inner product at most 0 with the elements of C are called the **simple roots** of C. The **Coxeter–Dynkin diagram** G of C is the set of simple roots of C, drawn as a graph with one vertex for each simple root of C and two vertices corresponding to the distinct roots $\mathbf{r}_i, \mathbf{r}_j$ are joined by $-\mathbf{r}_i\cdot\mathbf{r}_j$ lines.

Let us denote the group of graph automorphisms of the Coxeter–Dynkin diagram by $\mathrm{Aut}\,(\mathrm{G})$. Note that an automorphism $\sigma \in \mathrm{Aut}\,(\mathrm{G})$ induces an automorphism of Λ by $\sigma(\mathbf{r}_i) := \mathbf{r}_{\sigma(i)}\ \forall i$. Hence $\mathrm{Aut}\,(\mathrm{G})$ may be identified with the group of automorphisms of Λ fixing C. Furthermore, one can show that $\sigma w_{\mathbf{r}_i}\sigma^{-1} = w_{\mathbf{r}_{\sigma(i)}}$ and that $W \cap \mathrm{Aut}\,(\mathrm{G}) = 1$. Then the group of all autochronous automorphisms of the lattice Λ is a split extension of its Weyl group by $\mathrm{Aut}\,(\mathrm{G})$, viz

$$0 \longrightarrow W \overset{\iota}{\longrightarrow} \mathrm{Aut}\,(\Lambda)^+ \overset{\pi}{\longrightarrow} \mathrm{Aut}\,(\mathrm{G}) \longrightarrow 0, \qquad \mathrm{im}\,\iota = \ker\pi,$$

i.e. it is equivalent to a semidirect product of the Weyl group and the group of graph automorphisms:

$$\mathrm{Aut}\,(\Lambda)^+ = W \rtimes \mathrm{Aut}\,(\mathrm{G}).$$

The full automorphism group of Λ is just the autochronous subgroup extended by the negative of the identity operation (which interchanges the forward and backward light cones).

Returning to the vertex algebra $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \boldsymbol{\omega})$ associated with the even Lorentzian lattice Λ, we immediately infer from (2.3.1) and (2.3.2) that, for any simple root $\mathbf{r}_i$, the elements $\mathrm{e}^{\mathbf{r}_i}$, $\mathrm{e}^{-\mathbf{r}_i}$, and $\mathbf{r}_i(-1)$ describe physical states, i.e. they lie in $\mathcal{P}_{(1)}$. Define generators for a Lie algebra $\mathfrak{g}(A)$ by

$$e_i \mapsto \mathrm{e}^{\mathbf{r}_i}, \tag{2.3.19}$$

$$f_i \mapsto -\mathrm{e}^{-\mathbf{r}_i}, \tag{2.3.20}$$

$$h_i \mapsto \mathbf{r}_i(-1). \tag{2.3.21}$$

Then, by (2.3.11) – (2.3.13), we find the following relations to hold:

$$[h_i, h_j] = 0, \tag{2.3.22}$$

$$[h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \tag{2.3.23}$$

$$[e_i, f_j] = \delta_{ij} h_i, \tag{2.3.24}$$

where we defined the **Cartan matrix** $A = (a_{ij})$ associated with C by $a_{ij} := \mathbf{r}_i\cdot\mathbf{r}_j$. The elements h_i obviously form a basis for an abelian subalgebra of $\mathfrak{g}(A)$ called the **Cartan subalgebra** $\mathfrak{h}(A)$. In technical terms, from the above commutators we learn that the elements $\{e_i, f_i, h_i \,|\, i\}$ generate the so-called free Lie algebra associated with A. But even more is true; for we can show that the **Serre relations**

$$(\mathrm{ad}\, e_i)^{1-a_{ij}} e_j = 0, \quad (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j = 0, \tag{2.3.25}$$

are also fulfilled for all i, j. To see this we recall that $\mathcal{F}$ is Λ-graded by construction,

$$\mathcal{F} = \bigoplus_{\mathbf{x}\in\Lambda} S(\hat{\mathbf{h}}^-) \otimes \mathrm{e}^{\mathbf{x}} \equiv \bigoplus_{\mathbf{x}\in\Lambda} \mathcal{F}^{(\mathbf{x})}. \tag{2.3.26}$$

Then the Lie algebra of physical states inherits a natural Λ-gradation from $\mathcal{F}$ by defining

$$\mathfrak{g}_\Lambda^{(\mathbf{x})} := \mathfrak{g}_\Lambda \cap \left[S(\hat{\mathbf{h}}^-) \otimes \mathrm{e}^{\mathbf{x}}\right], \tag{2.3.27}$$

so that indeed

$$[\mathfrak{g}_\Lambda^{(\mathbf{x})}, \mathfrak{g}_\Lambda^{(\mathbf{y})}] \subset \mathfrak{g}_\Lambda^{(\mathbf{x}+\mathbf{y})} \tag{2.3.28}$$

for $\mathbf{x}, \mathbf{y} \in \Lambda$. Of course, some of the subspaces $\mathfrak{g}_\Lambda^{(\mathbf{x})}$ may be empty, e.g. for $\mathbf{x}^2 > 2$; but if $\mathfrak{g}_\Lambda^{(\mathbf{x})}$ is nonempty we shall refer to $\mathbf{x} \in \Lambda$ as a **root** of $\mathfrak{g}_\Lambda$ with **root space** $\mathfrak{g}_\Lambda^{(\mathbf{x})}$ and **multiplicity** $\dim \mathfrak{g}_\Lambda^{(\mathbf{x})}$. Hence the number of linearly independent polarization vectors for a physical state with certain momentum $\mathbf{x}$ accounts for the multiplicity of $\mathbf{x}$ as a root for the Lie algebra of physical states. The Λ-gradation of $\mathfrak{g}_\Lambda$ now in particular yields

$$(\mathrm{ad}\, \mathrm{e}^{\mathbf{r}})^j \mathrm{e}^{\mathbf{s}} \in \mathfrak{g}_\Lambda^{(j\mathbf{r}+\mathbf{s})} \quad \forall j \geq 0,\ \mathbf{r}, \mathbf{s} \in \Lambda_2. \tag{2.3.29}$$

From (2.2.11) we infer that the element $(\mathrm{ad}\, \mathrm{e}^{\mathbf{r}})^j \mathrm{e}^{\mathbf{s}}$ has an $\mathrm{L}_{(0)}$ eigenvalue of at least $\frac{1}{2}(j\mathbf{r}+\mathbf{s})^2 = 1 + j(j + \mathbf{r}\cdot\mathbf{s})$. Comparing this with the physical state condition $\mathrm{L}_{(0)}\psi = \psi$ we conclude that

$$(\mathrm{ad}\, \mathrm{e}^{\mathbf{r}})^j \mathrm{e}^{\mathbf{s}} = 0 \quad \text{for } j \geq 1 - \mathbf{r}\cdot\mathbf{s}. \tag{2.3.30}$$

Having established the Serre relations, the Gabber–Kac theorem [54, Theorem 9.11] tells us that the Lie algebra $\mathfrak{g}(A)$ generated by the elements $\{e_i, f_i, h_i \,|\, i\}$ is just the **Kac–Moody algebra** associated with the Cartan matrix A. Namely, the latter is defined as the above free Lie algebra divided by the maximal ideal intersecting $\mathfrak{h}(A)$ trivially, and the theorem states that this maximal ideal is generated by the elements $\{(\mathrm{ad}\, e_i)^{1-a_{ij}} e_j,\ (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j \,|\, i \neq j\}$.

We emphasize the remarkable fact that the physical state condition $\mathrm{L}_{(0)}\psi = \psi$ accounts for all Serre relations which are usually very difficult to deal with in the theory of Kac–Moody algebras; or, in string theory language, the absence of particles with squared mass below the tachyon reflects the validity of the Serre relations for the Lie algebra $\mathfrak{g}(A)$.

To summarize (cf. [11]): The physical states $\{\mathrm{e}^{\mathbf{r}_i}, \mathrm{e}^{-\mathbf{r}_i}, \mathbf{r}_i(-1) \,|\, i\}$ generate via multiple commutators the Kac–Moody algebra $\mathfrak{g}(A)$ associated with the Cartan matrix $A = (\mathbf{r}_i \cdot \mathbf{r}_j)$ which is a subalgebra of the Lie algebra of physical states, $\mathfrak{g}_\Lambda$.

Only in the Euclidean case these two Lie algebras coincide. In general, we have a *proper* inclusion

$$\mathfrak{g}(A) \subset \mathfrak{g}_\Lambda, \tag{2.3.31}$$

and the characterization of the elements of $\mathfrak{g}_\Lambda$ not contained in the Lie algebra $\mathfrak{g}(A)$ is the key problem for the vertex operator construction of hyperbolic Kac–Moody algebras. The special feature of (2.3.31) is that the root system of the Kac–Moody algebra $\mathfrak{g}(A)$ is well understood though its root multiplicities are not completely known for a single example; whereas the root system of $\mathfrak{g}_\Lambda$ is a priori not related to that of a Kac–Moody algebra although the root multiplicities are always known. Thus a complete understanding of (2.3.31) requires a "mechanism" which tells us how $\mathfrak{g}(A)$ has to be filled up with physical states to reach the complete Lie algebra of physical states. For the special case of the unique self-dual Lorentzian lattice $I\!I_{25,1}$, this was accomplished in [9] by adding *imaginary simple roots*, or, equivalently, by adjoining new generators to the Kac–Moody algebra $L_\infty (= \mathfrak{g}(A)$, where the infinite matrix A corresponds to the Coxeter–Dynkin diagram built up from the Leech roots), thereby furnishing the transition to the "fake monster" Lie algebra $\mathfrak{g}_{I\!I_{25,1}}$. We shall work out this example in detail in Sect. V.1. See also [66] for an attempt to determine the structure constants of this algebra and to relate them with three-point functions (cf. Sect. I.5 for the general statement!).

4. Bilinear forms on the Fock space

Finally we turn to the construction of a nondegenerate bilinear from $(\_|\_)$ on $\mathcal{F}$ satisfying the condition $(\mathcal{V}(\psi, z)\chi|\varphi) = (\chi|\mathcal{V}^\dagger(\psi, z)\varphi)$ with the adjoint vertex operator as defined in (1.4.10). As we shall see, this "invariance" condition is strong enough to determine the bilinear form uniquely up to a normalization.

Recall Eq. (2.3.17) which puts any oscillator $\mathbf{r}(n)$ into the form of a mode operator. On the other hand, Eq. (1.4.12) leads to $\mathbf{r}(-1)^\dagger_n = -\mathbf{r}(-1)_{-n}$ so that we infer that

$$\mathbf{r}(-n)^\dagger = -\mathbf{r}(n) \tag{2.4.1}$$

for $\mathbf{r} \in \Lambda_{\mathbb{R}}$, $n \in \mathbb{Z}$. Thus the oscillator part of the bilinear form is uniquely fixed and it remains to calculate the zero modes. To do so, we evaluate $(\mathbf{r}(0)\mathrm{e}^{\mathbf{r}}|\mathrm{e}^{\mathbf{s}})$ and $(\mathrm{e}^{\mathbf{r}}|\mathbf{s}(0)\mathrm{e}^{\mathbf{s}})$ in both ways using (2.1.23) and $\mathbf{r}(0)^\dagger = -\mathbf{r}(0)$. We obtain

$$\mathbf{r}\cdot(\mathbf{r}+\mathbf{s})(\mathrm{e}^{\mathbf{r}}|\mathrm{e}^{\mathbf{s}}) = \mathbf{s}\cdot(\mathbf{r}+\mathbf{s})(\mathrm{e}^{\mathbf{r}}|\mathrm{e}^{\mathbf{s}}) = 0 \tag{2.4.2}$$

for all $\mathbf{r}, \mathbf{s} \in \Lambda$, which means that $(\mathrm{e}^{\mathbf{r}}|\mathrm{e}^{\mathbf{s}})$ vanishes unless $\mathbf{r} = -\mathbf{s}$. Now,

$$\begin{aligned}(\mathrm{e}^{\mathbf{r}}|\mathrm{e}^{-\mathbf{r}}) &= (\mathrm{e}^{\mathbf{r}}_{-1}\mathbf{1}|\mathrm{e}^{-\mathbf{r}}) \\ &= (-1)^{\frac{1}{2}\mathbf{r}^2}(\mathbf{1}|(\mathrm{e}^{\mathbf{r}})_{\mathbf{r}^2-1}\mathrm{e}^{-\mathbf{r}}) \quad \text{by (1.4.12)} \\ &= (-1)^{\frac{1}{2}\mathbf{r}^2}\epsilon(\mathbf{r},-\mathbf{r})(\mathbf{1}|\mathcal{S}_0(\mathbf{r})\mathrm{e}^0) \\ &= (\mathbf{1}|\mathbf{1})\end{aligned} \tag{2.4.3}$$

for all $\mathbf{r} \in \Lambda$, where we used a generalization of (2.3.5),

$$\mathrm{e}^{\mathbf{r}}_n\mathrm{e}^{\mathbf{s}} = \epsilon(\mathbf{r},\mathbf{s})\mathcal{S}_{-1-n-\mathbf{r}\cdot\mathbf{s}}(\mathbf{r}) \otimes \mathrm{e}^{\mathbf{r}+\mathbf{s}}. \tag{2.4.4}$$

Thus we have

$$(\mathrm{e}^{\mathbf{r}}|\mathrm{e}^{\mathbf{s}}) = \delta_{\mathbf{r},-\mathbf{s}}(\mathbf{1}|\mathbf{1}), \tag{2.4.5}$$

which, together with (2.4.1), indeed uniquely fixes the bilinear form up to the normalization of $(\mathbf{1}|\mathbf{1})$. For reasons which become clear in a moment, we shall choose the awkward normalization

$$(\mathbf{1}|\mathbf{1}) := -1 \tag{2.4.6}$$

so that

$$\begin{aligned}(\mathrm{e}^{\mathbf{r}}|-\mathrm{e}^{-\mathbf{s}}) &= \delta_{\mathbf{r},\mathbf{s}}, \\ (\mathbf{r}(-1)|\mathbf{s}(-1)) &= \mathbf{r}\cdot\mathbf{s}\end{aligned} \tag{2.4.7}$$

for $\mathbf{r}, \mathbf{s} \in \Lambda$. When we go over to the induced form on $\mathfrak{g}_\Lambda$ these relations respectively give

$$\begin{aligned}(e_i|f_j) &= \delta_{ij}, \\ (h_i|h_j) &= a_{ij},\end{aligned} \tag{2.4.8}$$

for the generators of the Kac–Moody algebra $\mathfrak{g}(A)$. Together with Eq. (1.5.15) this shows that $(\_|\_)$ induces on $\mathfrak{g}(A)$ a **standard invariant bilinear form** [54]. This justifies our choice of normalization.

Whereas the above defined bilinear form nondegenerately pairs $\mathrm{e}^{\mathbf{r}}$ with $\mathrm{e}^{-\mathbf{r}}$, in physics we prefer a bilinear form which nondegenerately pairs $\mathrm{e}^{\mathbf{r}}$ with itself, because ultimately we are looking for a symmetric bilinear form[2] which does not lead to physical states of negative norm. Hence we introduce a **contravariant bilinear form** by defining

$$(\psi|\varphi)_0 := -(\psi|\theta(\varphi)) \tag{2.4.9}$$

[2] If we worked over the complex numbers, as one usually does in a Hilbert space, we would want an Hermitian form.

for all $\psi, \varphi \in \mathcal{F}$, where the **Chevalley involution** θ is given by

$$\begin{aligned} \theta(\mathrm{e}^{\mathbf{r}}) &:= \mathrm{e}^{-\mathbf{r}}, \\ \theta(\mathbf{r}(-n)) &:= -\mathbf{r}(-n). \end{aligned} \tag{2.4.10}$$

Note that this definition of θ consistently provides us with the Chevalley involution on the Kac–Moody algebra $\mathfrak{g}(A)$, viz

$$\begin{aligned} \theta(e_i) &= -f_i, \\ \theta(f_i) &= -e_i, \\ \theta(h_i) &= -h_i. \end{aligned} \tag{2.4.11}$$

We easily see, that, with respect to $(\_|\_)_0$, the zero mode states are orthonormal to each other and $\mathbf{r}(n)$ is the adjoint of $\mathbf{r}(-n)$. Hence the contravariant bilinear form $(\_|\_)_0$ is the one we are familiar with in string theory and to which the no-ghost theorem applies.

CHAPTER III:

DISCRETE DDF CONSTRUCTION

As can be seen from Eqs. (2.3.1) – (2.3.3) and (2.3.4), the Virasoro conditions $(\mathrm{L}_{(n)} - \delta_{n0})\psi = 0$, $n \geq 0$, which should be obeyed by physical states ψ, become increasingly complicated at higher excitations. In fact, we cannot hope to arrive at a general description of the physical states by this method of calculating polarization vectors. However, there is an elegant resolution of this problem by Del Giudice, Di Vecchia and Fubini [21] which allows an explicit construction of all the physical excited states. The idea is to find a set of operators that commute with the Virasoro operators and which, when applied successively to the tachyonic ground states, give all possible physical states. These operators form a closed algebra called the spectrum generating algebra. It turns out that the latter consists of transversal DDF operators A_n^i, $1 \leq i \leq d-2$, $n \in \mathbb{Z}$, describing the transversal modes of the string, and of longitudinal DDF operators $\mathfrak{L}_n$, $n \in \mathbb{Z}$, for the longitudinal excitations. We shall now introduce the discrete version of these operators taking into account that the momenta lie on the even lattice Λ so that we are not allowed to use Lorentz transformations to rotate them into convenient frames. Apparently, the longitudinal DDF operators have so far not been considered in this discrete context.

1. DDF vertex operators

Let $\mathbf{k}$ be a primitive lightlike lattice vector, i.e. $\mathbf{k} \in \Lambda_0$ and $\frac{1}{n}\mathbf{k} \notin \Lambda_0$ $\forall n > 1$. Using (2.3.9) we can immediately write down the commutator of physical states $\boldsymbol{\xi}(-1)\mathrm{e}^{m\mathbf{k}}$ and $\boldsymbol{\eta}(-1)\mathrm{e}^{n\mathbf{k}}$, $m, n \in \mathbb{Z}$:

$$\begin{aligned}\left[\boldsymbol{\xi}(-1)\mathrm{e}^{m\mathbf{k}}, \boldsymbol{\eta}(-1)\mathrm{e}^{n\mathbf{k}}\right] &= \epsilon(m\mathbf{k}, n\mathbf{k})(\boldsymbol{\xi}\cdot\boldsymbol{\eta})m\mathbf{k}(-1)\mathrm{e}^{(m+n)\mathbf{k}} \\ &= m(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\delta_{m+n,0}\mathbf{k}(-1),\end{aligned} \tag{3.1.1}$$

since $\boldsymbol{\xi}\cdot\mathbf{k} = \boldsymbol{\eta}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{k} = 0$ and $\mathbf{k}(-1)\mathrm{e}^{n\mathbf{k}} = \frac{1}{n}\mathrm{L}_{(-1)}(\mathrm{e}^{n\mathbf{k}}) \equiv 0$ for $n \neq 0$. Recall that we assumed the cocycle ϵ to be bimultiplicative so that $\epsilon(m\mathbf{k}, n\mathbf{k}) = (-1)^{\frac{1}{2}mn\mathbf{k}^2} = 1$.

We define the **transversal DDF operator** $A_m^{\boldsymbol{\xi}} = A_m(\boldsymbol{\xi}, \mathbf{k})$ as the zero mode operator corresponding to the physical state $\boldsymbol{\xi}(-1)\mathrm{e}^{m\mathbf{k}}$:

$$\begin{aligned}A_m^{\boldsymbol{\xi}} &:= \left(\boldsymbol{\xi}(-1)\mathrm{e}^{m\mathbf{k}}\right)_0 \\ &= \mathrm{Res}_z\left[\mathcal{V}(\boldsymbol{\xi}(-1)\mathrm{e}^{m\mathbf{k}}, z)\right] \\ &= \mathrm{Res}_z\left[\boldsymbol{\xi}(z)\mathcal{V}(\mathrm{e}^{m\mathbf{k}}, z)\right],\end{aligned} \tag{3.1.2}$$

where normal-ordering in the last line is unnecessary due to $\boldsymbol{\xi}\cdot\mathbf{k} = 0$. According to (1.5.4) the above commutator then translates into

$$\begin{aligned}\left[A_m^{\boldsymbol{\xi}}, A_n^{\boldsymbol{\eta}}\right] &= m(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\delta_{m+n,0}(\mathbf{k}(-1))_0 \\ &= m(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\delta_{m+n,0}\mathbf{k}(0).\end{aligned} \tag{3.1.3}$$

We observe that, apart from the operator $\mathbf{k}(0) = \mathbf{k}\cdot\boldsymbol{\alpha}_0$, this is just an oscillator commutation relation like (2.1.5) but now for $d-2$ oscillators, since the space $\{\boldsymbol{\xi} \in \Lambda_{\mathbb{R}} \mid \boldsymbol{\xi}\cdot\mathbf{k} = 0, \boldsymbol{\xi} \equiv \boldsymbol{\xi} \bmod \mathbb{R}\mathbf{k}\}$ has indeed dimension $d-2$. Moreover, it is clear from (1.3.23) that these operators commute with the Virasoro algebra,

$$\left[\mathrm{L}_{(n)}, A_m^{\boldsymbol{\xi}}\right] = 0 \qquad \forall n, m \in \mathbb{Z}. \tag{3.1.4}$$

Since we shall encounter the DDF operators only when acting on physical states with certain momenta $\mathbf{r} - n\mathbf{k}$ for $\mathbf{r} \in \Lambda$ and $n \in \mathbb{N}$, the operator $\mathbf{k}(0)$ can be thought of as an integer $\mathbf{k}\cdot\mathbf{r}$. The crucial feature of the DDF construction is then that, for given momentum $\mathbf{r}$, one has to find a lightlike vector $\mathbf{k} = \mathbf{k}(\mathbf{r})$ such that $\mathbf{k}\cdot\mathbf{r} = 1$. In this case the transversal DDF operators $A_m^{\boldsymbol{\xi}}(\mathbf{k})$ realize precisely the algebra of $d-2$ transversal oscillators on the ground state $\mathrm{e}^{\mathbf{r}}$. Indeed, we learn from (2.3.8) that the DDF operators $A_m^{\boldsymbol{\xi}}(\mathbf{k})$ for positive m annihilate the tachyonic ground state $\mathrm{e}^{\mathbf{r}}$,

$$A_m^{\boldsymbol{\xi}}(\mathbf{k})|\mathbf{r}\rangle = 0 \qquad \forall m > 0, \tag{3.1.5}$$

the operator $A_0^{\boldsymbol{\xi}}(\mathbf{k}) = \boldsymbol{\xi}(0)$ acts diagonally with eigenvalue $\boldsymbol{\xi}\cdot\mathbf{r}$, while the operators $A_m^{\boldsymbol{\xi}}(\mathbf{k})$ for negative m, when applied to the ground state, generate new physical states called **transversal DDF states**,

$$A_{-m_1}^{\boldsymbol{\xi}_1} \cdots A_{-m_N}^{\boldsymbol{\xi}_N}|\mathbf{r}\rangle \equiv \left[\boldsymbol{\xi}_1(-1)\mathrm{e}^{-m_1\mathbf{k}}, \left[\ldots, \left[\boldsymbol{\xi}_N(-1)\mathrm{e}^{-m_N\mathbf{k}}, \mathrm{e}^{\mathbf{r}}\right]\ldots\right]\right], \tag{3.1.6}$$

where we wrote $\mathrm{e}^{\mathbf{r}} \equiv |\mathbf{r}\rangle$ to make contact with the standard physics notation. For later purposes we denote the $d-2$-fold Heisenberg algebra spanned by the transversal DDF operators by

$$\hat{\mathfrak{t}}_{(\mathbf{r},\mathbf{k})} := \left\{A_m^{\boldsymbol{\xi}} \mid \boldsymbol{\xi} \in \Lambda_{\mathbb{R}},\ \boldsymbol{\xi}\cdot\mathbf{k} = \boldsymbol{\xi}\cdot\mathbf{r} = 0,\ m \in \mathbb{Z}\right\} \oplus \mathbb{R}\cdot 1, \tag{3.1.7}$$

and the vector space of finite products of creation operators ($\equiv$ algebra of polynomials on the transversal oscillators) is written as

$$S(\hat{\mathbf{t}}^-_{(\mathbf{r},\mathbf{k})}) := \bigoplus_{N\in\mathbb{N}} \left\{ \prod_{i=1}^{N} A^{\boldsymbol{\xi}_i}_{-m_i} \,\middle|\, \boldsymbol{\xi}_i \in \Lambda_{\mathbb{E}},\ \boldsymbol{\xi}_i\cdot\mathbf{k} = \boldsymbol{\xi}_i\cdot\mathbf{r} = 0,\ m_i > 0\ \forall i \right\}, \qquad (3.1.8)$$

where "S" stands for "symmetric" because of the fact that the creation operators commute with each other. Hence the irreducible transversal Heisenberg module built on some (tachyonic) ground state $|\mathbf{r}\rangle$ is given by $S(\hat{\mathbf{t}}^-_{(\mathbf{r},\mathbf{k})})|\mathbf{r}\rangle$. Note that we now have to deal with a whole family of different, though mutually isomorphic, transversal Heisenberg algebras, $\{\hat{\mathbf{t}}^-_{(\mathbf{r},\mathbf{k})} \,|\, \mathbf{r} \in \Lambda_2, \mathbf{k} \in \Lambda_0, \mathbf{r}\cdot\mathbf{k} = 1\}$, with corresponding modules $S(\hat{\mathbf{t}}^-_{(\mathbf{r},\mathbf{k})})|\mathbf{r}\rangle$, whereas the previous Heisenberg modules $S(\hat{\mathbf{h}}^-)|\mathbf{r}\rangle$ were built from a single set of primitive oscillators. The fundamental difference between these two types of oscillators is that the transversal DDF operators, in addition to primitive oscillator excitations, also shift the ground state momentum $\mathbf{r}$ by multiples of the lightlike vector $\mathbf{k}$ due to the zero mode terms in the expressions for the vertex operators. The price we have to pay for the convenient, systematic DDF method of writing down physical states, is the unpleasant occurrence of infinitely (but at least countably) many Heisenberg algebras $\hat{\mathbf{t}}_{(\mathbf{r},\mathbf{k})}$. In general, there are no simple commutation relations amongst elements from different Heisenberg algebras, and we shall see that precisely this feature makes hyperbolic Kac–Moody algebras so complicated.

The above identification of DDF physical states with multiple commutators in the Lie algebra $\mathfrak{g}_\Lambda$ will be our main guide in the analysis of hyperbolic Lie algebras; for the DDF construction allows us to write down elements of the Kac–Moody algebra $\mathfrak{g}(A)$ *explicitly* and to introduce the notion of *polarization* into the framework of these algebras.

Recall that the photonic physical states in (2.3.2) deserve the attribute "transversal" in the sense that the polarization vector $\mathbf{s}$ in $\mathbf{s}(-1)\mathrm{e}^{\mathbf{r}}$ has to be orthogonal to the momentum vector $\mathbf{r}$. Thus, we cannot expect to obtain "longitudinal" physical states in a straightforward way. Nevertheless, there is a "dirty trick" presented in [14]. Let $\mathbf{r} \in \Lambda$, $\mathbf{k} \in \Lambda_0$ and suppose that $\mathbf{k}\cdot\mathbf{r} \neq 0$. Then Eq. (1.3.4) yields

$$\begin{aligned}[\mathrm{L}_{(n)}, \mathcal{V}(\mathbf{r}(-1)\mathrm{e}^{\mathbf{k}}, z)] &= z^n \left\{ z\frac{d}{dz} + n + 1 \right\} \mathcal{V}(\mathbf{r}(-1)\mathrm{e}^{\mathbf{k}}, z) \\ &\quad + \tfrac{1}{2} n(n+1)(\mathbf{k}\cdot\mathbf{r})\mathcal{V}(\mathrm{e}^{\mathbf{k}}, z) z^{n-1}. \end{aligned} \qquad (3.1.9)$$

The unwanted term on the right hand side which destroys the conformal transformation properties (1.3.25) can be removed by the following trick: Introduce the formal series

$$\mathbf{k}^\times(z) := z\frac{\mathbf{k}(z)}{\mathbf{r}\cdot\mathbf{k}} - 1 = \frac{1}{\mathbf{r}\cdot\mathbf{k}} \sum_{n\neq 0} \mathbf{k}(n) z^{-n} + \left[\frac{\mathbf{k}(0)}{\mathbf{r}\cdot\mathbf{k}} - 1 \right], \qquad (3.1.10)$$

and define

$$\begin{aligned}\log\left[z\frac{\mathbf{k}(z)}{\mathbf{r}\cdot\mathbf{k}} \right] &= \log[1 + \mathbf{k}^\times(z)] \\ &:= \sum_{i\geq 1} \frac{(-1)^{i+1}}{i} [\mathbf{k}^\times(z)]^i, \end{aligned} \qquad (3.1.11)$$

which is only defined on states with momentum $\mathbf{s}$ *such that* $\mathbf{s}\cdot\mathbf{k} = \mathbf{r}\cdot\mathbf{k}$; if the second term on the right hand side of (3.1.10) does not vanish on a given state, an infinite number of terms will contribute when (3.1.11) is applied to it. This means that the above series is not (algebraically) summable on the whole space $\mathcal{F}$. In particular, it is not summable on the vacuum state $\mathbf{1} \equiv |\mathbf{0}\rangle$ which, in view of (1.3.19), makes it impossible to recover *the* state corresponding to the log series: there does not exist a universal state whose vertex operator is given by $\log[1 + \mathbf{k}^\times(z)]$. Luckily, however, we shall only need the action of this log series on states with momentum $\mathbf{r} - n\mathbf{k}$ (with $n \in \mathbb{N}$), so that the resulting series will be well-defined. And if this is the case then we may indeed find a state whose vertex operator has the same action as the log series. Thus, the log series should be interpreted as some sort of generating series for a class of genuine vertex operators which can be revealed by acting on states. Keeping in mind this subtlety let us perform some calculations in connection with the log series.

$$[\mathrm{L}_{(n)}, \mathbf{k}^\times(z)] = z^n \left\{ z\frac{d}{dz} + n \right\} \mathbf{k}^\times(z) + nz^n, \qquad (3.1.12)$$

since the current $\mathbf{k}(z)$ is a primary field of weight 1. For the formal series $\log[1 + \mathbf{k}^\times(z)]$ we therefore obtain

$$\begin{aligned}[\mathrm{L}_{(n)}, \log[1 + \mathbf{k}^\times(z)]] &= \sum_{i\geq 1} (-1)^{i+1} [\mathbf{k}^\times(z)]^{i-1} [\mathrm{L}_{(n)}, \mathbf{k}^\times(z)] \qquad (\text{since } \mathbf{k} \in \Lambda_0) \\ &= z^{n+1} \frac{d}{dz} \log[1 + \mathbf{k}^\times(z)] + nz^n, \end{aligned} \qquad (3.1.13)$$

so that

$$\left[\mathrm{L}_{(n)}, \frac{d}{dz} \log[1 + \mathbf{k}^\times(z)] \right] = z^n \left\{ z\frac{d}{dz} + n + 1 \right\} \frac{d}{dz} \log[1 + \mathbf{k}^\times(z)] + n^2 z^{n-1}. \qquad (3.1.14)$$

Using this formula and the fact that

$$\frac{d}{dz} \log\frac{\mathbf{k}(z)}{\mathbf{r}\cdot\mathbf{k}} = \frac{d}{dz} \log[1 + \mathbf{k}^\times(z)] - z^{-1}, \qquad (3.1.15)$$

we deduce that

$$\left[\mathrm{L}_{(n)}, \frac{d}{dz} \log\frac{\mathbf{k}(z)}{\mathbf{r}\cdot\mathbf{k}} \right] = z^n \left\{ z\frac{d}{dz} + n + 1 \right\} \left(\frac{d}{dz} \log\frac{\mathbf{k}(z)}{\mathbf{r}\cdot\mathbf{k}} \right) + n(n+1) z^{n-1}. \qquad (3.1.16)$$

Putting everything together we conclude that the **DDF vertex operators**

$$\mathcal{Y}_{\mathbf{k}}(\mathbf{r},z) := \mathcal{V}(\mathbf{r}(-1)\mathrm{e}^{\mathbf{k}},z) - \tfrac{1}{2}(\mathbf{r}\cdot\mathbf{k})\frac{d}{dz}\log\left(\frac{\mathbf{k}(z)}{\mathbf{r}\cdot\mathbf{k}}\right)\mathcal{V}(\mathrm{e}^{\mathbf{k}},z), \tag{3.1.17}$$

enjoy the correct conformal transformation properties for primary fields of weight 1:[1]

$$[\mathrm{L}_{(n)}, \mathcal{Y}_{\mathbf{k}}(\mathbf{r},z)] = z^n\left\{z\frac{d}{dz}+n+1\right\}\mathcal{Y}_{\mathbf{k}}(\mathbf{r},z) \qquad \forall n\in\mathbb{Z}. \tag{3.1.18}$$

For $\mathbf{r}\cdot\mathbf{k}\neq 0$ we call $\mathcal{Y}_{\mathbf{k}}(\mathbf{r},z)$ **longitudinal vertex operator** since otherwise we recover the transversal vertex operator $\mathcal{V}(\mathbf{r}(-1)\mathrm{e}^{\mathbf{k}},z)$.[2] Also note that the log term in (3.1.17) does not require normal-ordering because of $\mathbf{k}\in\Lambda_0$.

2. Longitudinal Virasoro operators

We define the **longitudinal Virasoro operator** $\mathcal{L}_m$ as the zero mode operator of the longitudinal vertex operator $\mathcal{Y}_{m\mathbf{k}}(\mathbf{r},z)$:

$$\begin{aligned}\mathcal{L}_m &:= -\mathrm{Res}_z\left[\mathcal{Y}_{m\mathbf{k}}(\mathbf{r},z)\right]\\ &\equiv \mathrm{Res}_z\left[-\mathcal{V}(\mathbf{r}(-1)\mathrm{e}^{m\mathbf{k}},z)+\frac{m}{2}(\mathbf{r}\cdot\mathbf{k})\frac{d}{dz}\log\left(\frac{\mathbf{k}(z)}{\mathbf{r}\cdot\mathbf{k}}\right)\mathcal{V}(\mathrm{e}^{m\mathbf{k}},z)\right].\end{aligned} \tag{3.2.1}$$

These operators satisfy the commutation relations of a Virasoro algebra with central charge $c=24$. To see this, we first note that

$$\begin{aligned}&\left[\mathbf{r}(-1)\mathrm{e}^{m\mathbf{k}},\mathbf{r}(-1)\mathrm{e}^{n\mathbf{k}}\right]\\ &\quad=\epsilon(m\mathbf{k},n\mathbf{k})\left[m[\mathbf{r}^2-mn(\mathbf{r}\cdot\mathbf{k})]\mathbf{k}(-1)+(n-m)(\mathbf{r}\cdot\mathbf{k})\mathbf{r}(-1)\right]\mathrm{e}^{(m+n)\mathbf{k}}\\ &\quad=(n-m)(\mathbf{r}\cdot\mathbf{k})\mathbf{r}(-1)\mathrm{e}^{(m+n)\mathbf{k}}+m[\mathbf{r}^2+m^2(\mathbf{r}\cdot\mathbf{k})]\delta_{m+n,0}\mathbf{k}(-1)\end{aligned}$$

by (2.3.9) so that

$$\begin{aligned}&\left[\mathrm{Res}_{z_1}\left[\mathcal{V}(\mathbf{r}(-1)\mathrm{e}^{m\mathbf{k}},z_1)\right],\mathrm{Res}_{z_2}\left[\mathcal{V}(\mathbf{r}(-1)\mathrm{e}^{n\mathbf{k}},z_2)\right]\right]\\ &\quad=(m-n)(\mathbf{r}\cdot\mathbf{k})\mathrm{Res}_z\left[-\mathcal{V}(\mathbf{r}(-1)\mathrm{e}^{(m+n)\mathbf{k}},z)\right]+m[\mathbf{r}^2+m^2(\mathbf{r}\cdot\mathbf{k})]\delta_{m+n,0}\mathbf{k}(0).\end{aligned} \tag{3.2.2}$$

It is also clear that the commutator of two log terms vanishes due to lightlikeness of $\mathbf{k}$. Finally, we have to calculate the cross commutator:

$$\begin{aligned}&\left[\mathrm{Res}_{z_1}\left[\mathcal{V}(\mathbf{r}(-1)\mathrm{e}^{m\mathbf{k}},z_1)\right],\mathrm{Res}_{z_2}\left[\frac{d}{dz_2}\log\left(\frac{\mathbf{k}(z_2)}{\mathbf{r}\cdot\mathbf{k}}\right)\mathcal{V}(\mathrm{e}^{n\mathbf{k}},z_2)\right]\right]\\ &\quad=\mathrm{Res}_{z_2}\left\{\frac{d}{dz_2}\log\left(\frac{\mathbf{k}(z_2)}{\mathbf{r}\cdot\mathbf{k}}\right)\left[\mathrm{Res}_{z_1}\left[\mathcal{V}(\mathbf{r}(-1)\mathrm{e}^{m\mathbf{k}},z_1)\right],\mathcal{V}(\mathrm{e}^{n\mathbf{k}},z_2)\right]\right.\\ &\qquad\left.+\left[\mathrm{Res}_{z_1}\left[\mathcal{V}(\mathbf{r}(-1)\mathrm{e}^{m\mathbf{k}},z_1)\right],\frac{d}{dz_2}\log\left(\frac{\mathbf{k}(z_2)}{\mathbf{r}\cdot\mathbf{k}}\right)\right]\mathcal{V}(\mathrm{e}^{n\mathbf{k}},z_2)\right\}.\end{aligned} \tag{3.2.3}$$

To calculate these two commutators we first recall the following version of the commutator formula (1.2.18):

$$\begin{aligned}\left[\mathrm{Res}_{z_1}\left[\mathcal{V}(\psi,z_1)\right],\mathcal{V}(\varphi,z_2)\right]&=\mathcal{V}(\psi_0\varphi,z_2)\\ &\equiv\mathcal{V}([\psi,\varphi],z_2).\end{aligned} \tag{3.2.4}$$

From (2.3.8) and (2.3.9) we therefore deduce that

$$\left[\mathrm{Res}_{z_1}\left[\mathcal{V}(\mathbf{r}(-1)\mathrm{e}^{m\mathbf{k}},z_1)\right],\mathcal{V}(\mathrm{e}^{n\mathbf{k}},z_2)\right]=n(\mathbf{r}\cdot\mathbf{k})\mathcal{V}(\mathrm{e}^{(m+n)\mathbf{k}},z_2), \tag{3.2.5}$$

and

$$\left[\mathrm{Res}_{z_1}\left[\mathcal{V}(\mathbf{r}(-1)\mathrm{e}^{m\mathbf{k}},z_1)\right],\mathcal{V}(\mathbf{k}(-1),z_2)\right]=m(\mathbf{r}\cdot\mathbf{k})\mathcal{V}(\mathbf{k}(-1)\mathrm{e}^{m\mathbf{k}},z_2), \tag{3.2.6}$$

respectively. The last formula then yields

$$\begin{aligned}&\left[\mathrm{Res}_{z_1}\left[\mathcal{V}(\mathbf{r}(-1)\mathrm{e}^{m\mathbf{k}},z_1)\right],\frac{d}{dz_2}\log\left(\frac{\mathbf{k}(z_2)}{\mathbf{r}\cdot\mathbf{k}}\right)\right]\\ &\quad=\frac{d}{dz_2}\left[\sum_{i\geq 1}(-1)^{i+1}\left(\frac{\mathbf{k}(z_2)}{\mathbf{r}\cdot\mathbf{k}}-1\right)^{i-1}m(\mathbf{r}\cdot\mathbf{k})\mathcal{V}(\mathbf{k}(-1)\mathrm{e}^{m\mathbf{k}},z_2)\right]\\ &\quad=\frac{d}{dz_2}\left[m(\mathbf{r}\cdot\mathbf{k})\mathcal{V}(\mathrm{e}^{m\mathbf{k}},z_2)\right]\\ &\quad=m^2(\mathbf{r}\cdot\mathbf{k})\mathcal{V}(\mathbf{k}(-1)\mathrm{e}^{m\mathbf{k}},z_2).\end{aligned} \tag{3.2.7}$$

[1]This is in perfect agreement with [14] since we employ a different normal-ordering prescription for p^μ and q^ν; we use $:q^\nu p^\mu: = q^\nu p^\mu$ in contrast to the "standard" symmetric normal-ordering ${}^\times_\times q^\nu p^\mu{}^\times_\times = \frac{1}{2}(q^\nu p^\mu + p^\mu q^\nu) = :q^\nu p^\mu: - \frac{i}{2}\eta^{\mu\nu}$ which leads to

$$\mathcal{V}_{sy.}(\mathbf{r}(-1)\mathrm{e}^{\mathbf{k}},z)=\mathcal{V}(\mathbf{r}(-1)\mathrm{e}^{\mathbf{k}},z)+\tfrac{1}{2}(\mathbf{k}\cdot\mathbf{r})\mathcal{V}(\mathrm{e}^{\mathbf{k}},z)z^{-1},$$

so that indeed

$$\mathcal{V}_{sy.}(\mathbf{r}(-1)\mathrm{e}^{\mathbf{k}},z)-\tfrac{1}{2}(\mathbf{k}\cdot\mathbf{r})\frac{d}{dz}\log[1+\mathbf{k}^\times(z)]\mathcal{V}(\mathrm{e}^{\mathbf{k}},z)=\mathcal{V}(\mathbf{r}(-1)\mathrm{e}^{\mathbf{k}},z)-\tfrac{1}{2}(\mathbf{k}\cdot\mathbf{r})\frac{d}{dz}\log\left(\frac{\mathbf{k}(z)}{\mathbf{r}\cdot\mathbf{k}}\right)\mathcal{V}(\mathrm{e}^{\mathbf{k}},z).$$

[2]Apparently, the essential log term was missed in [25].

Collecting the above commutators we finally get

$$\begin{aligned}
[\mathcal{L}_m,\mathcal{L}_n] &= (m-n)(\mathbf{r}\cdot\mathbf{k})\mathrm{Res}_z\left[-\mathcal{V}(\mathbf{r}(-1)\mathrm{e}^{(m+n)\mathbf{k}},z)\right] + m[\mathbf{r}^2+m^2(\mathbf{r}\cdot\mathbf{k})]\delta_{m+n,0}\mathbf{k}(0)\\
&\quad -\frac{n^2}{2}(\mathbf{r}\cdot\mathbf{k})^2\mathrm{Res}_z\left[\frac{d}{dz}\log\left(\frac{\mathbf{k}(z)}{\mathbf{r}\cdot\mathbf{k}}\right)\mathcal{V}(\mathrm{e}^{(m+n)\mathbf{k}},z)\right]\\
&\quad -\frac{nm^2}{2}(\mathbf{r}\cdot\mathbf{k})^2\mathrm{Res}_z\left[\mathcal{V}(\mathbf{k}(-1)\mathrm{e}^{(m+n)\mathbf{k}},z)\right]\\
&\quad +\frac{m^2}{2}(\mathbf{r}\cdot\mathbf{k})^2\mathrm{Res}_z\left[\frac{d}{dz}\log\left(\frac{\mathbf{k}(z)}{\mathbf{r}\cdot\mathbf{k}}\right)\mathcal{V}(\mathrm{e}^{(m+n)\mathbf{k}},z)\right]\\
&\quad +\frac{mn^2}{2}(\mathbf{r}\cdot\mathbf{k})^2\mathrm{Res}_z\left[\mathcal{V}(\mathbf{k}(-1)\mathrm{e}^{(m+n)\mathbf{k}},z)\right]\\
&= (m-n)\mathcal{L}_{m+n} + \{[(\mathbf{r}\cdot\mathbf{k})^2+(\mathbf{r}\cdot\mathbf{k})]m^3+\mathbf{r}^2m\}\delta_{m+n,0}\mathbf{k}(0).
\end{aligned} \tag{3.2.8}$$

As for the central term, we shall assume from now on that $\mathbf{r}\cdot\mathbf{k}=1$ so that the factor in the central term reads $2m^3+\mathbf{r}^2m$. The standard form $\frac{c}{12}(m^3-m)$ can be obtained by redefining $\mathcal{L}_0\to\mathcal{L}_0+(1+\frac{1}{2}\mathbf{r}^2)\mathbf{k}(0)$ so that we end up with

$$[\mathcal{L}_m,\mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}+2(m^3-m)\delta_{m+n,0}\mathbf{k}(0). \tag{3.2.9}$$

We conclude that the longitudinal Virasoro operators $\mathcal{L}_m$, when applied to physical states with momentum $\mathbf{r}-n\mathbf{k}$, realize a Virasoro algebra, $\mathrm{Vir}_{\mathcal{L}}$, with central charge $c_{\mathcal{L}}=24$. Remarkably, this Virasoro algebra is universal in the sense that its central charge does *not* depend on the dimension of the lattice. On the other hand, however, we encounter the same situation as in the case of the transversal DDF operators; namely that the longitudinal Virasoro operators are by construction untearably bound up with the pair $(\mathbf{r},\mathbf{k})$ so that we consequently have to deal with an infinite family of longitudinal Virasoro algebras, $\{\mathrm{Vir}_{\mathcal{L}}^{(\mathbf{r},\mathbf{k})}\,|\,\mathbf{r}\in\Lambda_2,\ \mathbf{k}\in\Lambda_0,\ \mathbf{r}\cdot\mathbf{k}=1\}$.

Let us proceed with determining the commutator of the transversal DDF operators and the longitudinal Virasoro operators:[3]

$$\begin{aligned}
[\mathcal{L}_m,A_n^{\boldsymbol{\xi}}] &= \left[\mathrm{Res}_{z_1}\left[\mathcal{V}(\boldsymbol{\xi}(-1)\mathrm{e}^{n\mathbf{k}},z_1)\right],\mathrm{Res}_{z_2}\left[\mathcal{V}(\mathbf{r}(-1)\mathrm{e}^{m\mathbf{k}},z_2)\right]\right]\\
&= \mathrm{Res}_z\left[\mathcal{V}\left([\boldsymbol{\xi}(-1)\mathrm{e}^{n\mathbf{k}},\mathbf{r}(-1)\mathrm{e}^{m\mathbf{k}}],z\right)\right]\\
&= \mathrm{Res}_z\left[n(\boldsymbol{\xi}\cdot\mathbf{r})\mathcal{V}(\mathbf{k}(-1)\mathrm{e}^{(m+n)\mathbf{k}},z)-n(\mathbf{r}\cdot\mathbf{k})\mathcal{V}(\boldsymbol{\xi}(-1)\mathrm{e}^{(m+n)\mathbf{k}},z)\right]\\
&= -n(\mathbf{r}\cdot\mathbf{k})A^{\boldsymbol{\xi}}_{n+m}+n(\mathbf{r}\cdot\boldsymbol{\xi})\delta_{m+n,0}\mathbf{k}(0)
\end{aligned} \tag{3.2.10}$$

by (2.3.9). Obviously, we can remove the second term by choosing $\boldsymbol{\xi}$ orthogonal to $\mathbf{r}$; and if we make our standard assumption that $\mathbf{r}\cdot\mathbf{k}=1$ we arrive at the important formula

$$[\mathcal{L}_m,A_n^{\boldsymbol{\xi}}] = -nA^{\boldsymbol{\xi}}_{n+m}, \tag{3.2.11}$$

[3] When the $(\mathbf{r},\mathbf{k})$ dependance is suppressed in the notation for operators and states, then a certain pair $(\mathbf{r},\mathbf{k})$ is tacitly assumed to be fixed once and for all.

which reveals that the two algebras together form a semidirect product $\hat{\mathfrak{t}}\rtimes\mathrm{Vir}_{\mathcal{L}}$.

We claim that the tachyonic ground state $\mathrm{e}^{\mathbf{r}}$ is annihilated by the longitudinal Virasoro operators $\mathcal{L}_m$ for nonnegative m,

$$\mathcal{L}_m|\mathbf{r}\rangle = 0 \qquad \forall m\geq 0. \tag{3.2.12}$$

First note that the operator $\mathcal{L}_0=-\mathbf{r}(0)+(1+\frac{1}{2}\mathbf{r}^2)\mathbf{k}(0)$ acts diagonally with eigenvalue $(1-\frac{1}{2}\mathbf{r}^2)$ which indeed vanishes because $\mathbf{r}\in\Lambda_2$. Next, using the Λ-gradation (2.3.28) of $\mathfrak{g}_\Lambda$ we observe that the state $\mathcal{L}_m|\mathbf{r}\rangle$ carries momentum $\mathbf{r}+m\mathbf{k}$. But $\frac{1}{2}(\mathbf{r}+m\mathbf{k})^2=1+m$ contradicts the physical state condition $\mathrm{L}_{(0)}\psi=\psi$ for positive m in view of (2.2.11) unless the state itself vanishes. We conclude that only the operators $\mathcal{L}_m$ for negative m generate new physical states when applied to the ground state:

$$\mathcal{L}_{-n_1}\ldots\mathcal{L}_{-n_P}|\mathbf{r}\rangle \quad \in\mathcal{P}_{(1)} \tag{3.2.13}$$

for $n_1,\ldots,n_P\geq 1$, thereby making up a Virasoro Verma module $V(24,0)$, where $V(c,h)$ denotes the irreducible highest weight module for the Virasoro algebra with central charge c and highest weight h. Further, we can verify that the state $\mathcal{L}_{-1}|\mathbf{r}\rangle$ is a null physical state, i.e. the action of the operator $\mathcal{L}_{-1}$ is essentially the same as the action of $\mathrm{L}_{(-1)}$:

$$\mathcal{L}_{-1}|\mathbf{r}\rangle = \epsilon(\mathbf{r},\mathbf{k})\mathrm{L}_{(-1)}|\mathbf{r}-\mathbf{k}\rangle, \tag{3.2.14}$$

which vanishes as an element of $\mathfrak{g}_\Lambda$! To prove this equation we first deduce from (2.3.8) that

$$\begin{aligned}
\mathrm{Res}_z\left[-\mathcal{V}(\mathbf{r}(-1)\mathrm{e}^{-\mathbf{k}},z)\right](\mathrm{e}^{\mathbf{r}}) &= -\left[\mathbf{r}(-1)\mathrm{e}^{-\mathbf{k}},\mathrm{e}^{\mathbf{r}}\right]\\
&= \epsilon(\mathbf{r},\mathbf{k})(\mathbf{r}-2\mathbf{k})(-1)\otimes\mathrm{e}^{\mathbf{r}-\mathbf{k}}.
\end{aligned} \tag{3.2.15}$$

The calculations for the log term have to be performed explicitly:

$$\begin{aligned}
&\mathrm{Res}_z\left[-\tfrac{1}{2}\frac{d}{dz}\log[\mathbf{k}(z)]\mathcal{V}(\mathrm{e}^{-\mathbf{k}},z)\right](\mathrm{e}^{\mathbf{r}})\\
&\quad= \mathrm{Res}_z\left[-\tfrac{1}{2}\frac{d}{dz}\log[\mathbf{k}(z)]\mathcal{V}(\mathrm{e}^{-\mathbf{k}},z)\right](\mathrm{e}^{\mathbf{r}})\\
&\quad= -\tfrac{1}{2}\mathrm{Res}_z\left\{\left[\frac{d}{dz}\log[1+\mathbf{k}^\times(z)]-z^{-1}\right]\mathcal{V}(\mathrm{e}^{-\mathbf{k}},z)\right\}(\mathrm{e}^{\mathbf{r}})\\
&\quad= -\tfrac{1}{2}\mathrm{Res}_z\left[\mathrm{e}^{-\mathbf{k}}\sum_{i\geq 1}\frac{(-1)^{i+1}}{i}[\mathbf{k}^\times(z)]^i\mathbf{k}(z)\sum_{m\geq 0}\mathcal{S}_m(-\mathbf{k})z^{m-1}\right](\mathrm{e}^{\mathbf{r}})\\
&\qquad+\tfrac{1}{2}\mathrm{Res}_z\left[z^{-1}\sum_{m\geq 0}\mathcal{S}_m(-\mathbf{k})z^{m-1}\mathrm{e}^{-\mathbf{k}}\mathrm{e}^{\mathbf{r}}\right]
\end{aligned}$$

$$= -\tfrac{1}{2}\mathrm{Res}_z\left\{\sum_{i\geq 1}\frac{(-1)^{i+1}}{i}\left[\sum_{n>0}\mathbf{k}(-n)z^n\right]^i\left[z^{-1}+\sum_{n>0}\mathbf{k}(-n)z^{n-1}\right]\times\right.$$
$$\left.\times\sum_{m\geq 0}\mathcal{S}_m(-\mathbf{k})z^{m-1}\mathrm{e}^{-\mathbf{k}}\mathrm{e}^{\mathbf{r}}\right\} - \tfrac{1}{2}\epsilon(-\mathbf{k},\mathbf{r})\mathbf{k}(-1)\mathrm{e}^{\mathbf{r}-\mathbf{k}}$$
$$= -\epsilon(-\mathbf{k},\mathbf{r})\mathbf{k}(-1)\mathrm{e}^{\mathbf{r}-\mathbf{k}}. \tag{3.2.16}$$

Collecting the above results we obtain $\epsilon(\mathbf{r},\mathbf{k})(\mathbf{r}-\mathbf{k})(-1)|\mathbf{r}-\mathbf{k}\rangle$ as desired. Thus, by the use of the commutation relations and (3.2.14), we can rewrite any state of the form (3.2.13) in $\mathfrak{g}_\Lambda$ as a linear combination of states not containing $\mathcal{L}_{-1}$. As a basis for states of the form (3.2.13) in $\mathfrak{g}_\Lambda$ we may therefore choose

$$\mathcal{L}_{-n_1}\ldots\mathcal{L}_{-n_P}|\mathbf{r}\rangle, \tag{3.2.17}$$

with fixed ordering $n_1 \geq \ldots \geq n_P \geq 2$.

We turn now to the no-ghost theorem applied to our discrete construction. We fix a tachyonic ground state $\mathrm{e}^{\mathbf{r}} \equiv |\mathbf{r}\rangle$, $\mathbf{r}\in\Lambda_2$, and suppose that there exists a lightlike vector $\mathbf{k}=\mathbf{k}(\mathbf{r})\in\Lambda_0$ such that $\mathbf{r}\cdot\mathbf{k}=1$. Then we can always find $d-2$ orthonormal lattice vectors $\boldsymbol{\xi}_i$, $1\leq i\leq d-2$, orthogonal to both $\mathbf{r}$ and $\mathbf{k}$. If we put $A^i_m \equiv A^{\boldsymbol{\xi}_i}_m$, then the no-ghost theorem [44] tells us that the states

$$A^{i_1}_{-m_1}\ldots A^{i_N}_{-m_N}\mathcal{L}_{-n_1}\ldots\mathcal{L}_{-n_P}|\mathbf{r}\rangle \tag{3.2.18}$$

for $i_1,\ldots,i_N\in\{1,\ldots,d-2\}$, $m_1,\ldots,m_N\geq 1$ and $n_1\geq\ldots\geq n_P\geq 1$, account for *all* physical states (including null physical states!) with momentum

$$\mathbf{r}-\left(\sum_{a=1}^{N}m_a+\sum_{b=1}^{P}n_b\right)\mathbf{k}. \tag{3.2.19}$$

Reformulated in the language of the Lie algebra $\mathfrak{g}_\Lambda$: The subspace

$$\mathfrak{g}_\Lambda(\mathbf{r}) := \bigoplus_{n\in\mathbb{N}}\mathfrak{g}_\Lambda^{(\mathbf{r}-n\mathbf{k})}, \qquad \mathbf{r}\in\Lambda_2, \tag{3.2.20}$$

is spanned by elements of the form

$$A^{i_1}_{-m_1}\ldots A^{i_N}_{-m_N}\mathcal{L}_{-n_1}\ldots\mathcal{L}_{-n_P}|\mathbf{r}\rangle, \tag{3.2.21}$$

where $i_1,\ldots,i_N\in\{1,\ldots,d-2\}$, $m_1,\ldots,m_N\geq 1$ and $n_1\geq\ldots\geq n_P\geq 2$.

3. Spectrum generating algebra

Note that due to (3.2.11) we had to fix some ordering of the operators in (3.2.18). Historically, this was the reason for replacing the longitudinal Virasoro operators $\mathcal{L}_m$ by **longitudinal DDF operators** A^-_m which commute with the transversal DDF operators. To see this, we define the standard normal-ordering of the transversal DDF operators by

$$:A^i_m A^j_n: := \begin{cases} A^i_m A^j_n & \text{if } m\leq n,\\ A^j_n A^i_m & \text{if } m>n,\end{cases} \tag{3.3.1}$$

and define

$$\mathrm{L}_n := \tfrac{1}{2}\sum_{i=1}^{d-2}\sum_{m\in\mathbb{Z}} :A^i_m A^i_{n-m}:. \tag{3.3.2}$$

Comparing this with (2.2.6) we immediately infer that the L_n's obey a Virasoro algebra, $\mathrm{Vir}_{\mathrm{L}}$, with central charge $c_{\mathrm{L}}=d-2$. Furthermore, it is straightforward to show that

$$[\mathrm{L}_m, A^i_n] = -nA^i_{n+m}. \tag{3.3.3}$$

Hence, if we define

$$A^-_n := \mathcal{L}_n - \mathrm{L}_n = \mathcal{L}_n - \tfrac{1}{2}\sum_{i=1}^{d-2}\sum_{m\in\mathbb{Z}} :A^i_m A^i_{n-m}:, \tag{3.3.4}$$

we get

$$[A^i_m, A^-_n] = 0 \qquad \forall m,n\in\mathbb{Z},\ 1\leq i\leq d-2. \tag{3.3.5}$$

The last equation can be used to show that longitudinal DDF operators form a "coset" Virasoro algebra, Vir_{A^-}, with central charge $c_{A^-}=c_{\mathcal{L}}-c_{\mathrm{L}}=26-d$ (cf. [41]):

$$\begin{aligned}[A^-_m, A^-_n] &= [\mathcal{L}_m-\mathrm{L}_m, \mathcal{L}_n-\mathrm{L}_n]\\ &= [\mathcal{L}_m,\mathcal{L}_n]-[\mathrm{L}_m,\mathrm{L}_n]\\ &= (m-n)A^-_{m+n}+\tfrac{26-d}{12}(m^3-m)\delta_{m+n,0}.\end{aligned} \tag{3.3.6}$$

Thus we may rewrite the basis of all physical states (including null states!) as

$$A^{i_1}_{-m_1}\ldots A^{i_N}_{-m_N}A^-_{-n_1}\ldots A^-_{-n_P}|\mathbf{r}\rangle \tag{3.3.7}$$

where $i_1,\ldots,i_N\in\{1,\ldots,d-2\}$, $m_1,\ldots,m_N\geq 1$ and $n_1\geq\ldots\geq n_P\geq 1$, which exhibits explicitly how the space of physical states with momentum $\mathbf{r}-n\mathbf{k}$, $n\geq 0$, splits into a tensor product of the algebra of polynomials in the transversal oscillators with a Virasoro Verma module:

$$\mathcal{P}_{(1)}(\mathbf{r}) := \bigoplus_{n\geq 0}\mathcal{P}_{(1)}^{(\mathbf{r}-n\mathbf{k})} = S(\hat{\mathbf{t}}^-)\otimes V(26-d,0). \tag{3.3.8}$$

In other words, we may regard the associative algebra

$$S(\hat{\mathbf{t}}^-) \otimes \mathfrak{U}(\mathrm{Vir}^-_{A^-}), \tag{3.3.9}$$

where $\mathfrak{U}(\mathrm{Vir}^-_{A^-})$ denotes the universal enveloping algebra of the negative modes of the longitudinal DDF operators, as the **spectrum generating algebra** associated with $\mathbf{r}$, since it generates all physical states with momentum $\mathbf{r} - n\mathbf{k}$, $n \in \mathbb{N}$, by acting on the fixed tachyonic ground state $|\mathbf{r}\rangle$. In particular, we observe how the critical dimension $d = 26$ arises: In 26 dimensions the longitudinal and the transversal modes decouple because the coset Virasoro module $V(26-d, 0)$ becomes trivial. Moreover, (3.3.7) enables us to write down a formula for the dimension of the physical subspaces with momentum $\mathbf{r} - n\mathbf{k}$, $\mathbf{r} \in \Lambda_2$:

$$\dim \mathfrak{g}_\Lambda^{(\mathbf{r}-n\mathbf{k})} = p_{d-1}(n) - p_{d-1}(n-1), \tag{3.3.10}$$

where $p_{d-1}(n)$ counts the partitions of $n \in \mathbb{N}$ into "parts" of $d-1$ "colours", i.e.

$$\begin{aligned} \phi(q)^{1-d} &:= \prod_{l=1}^{\infty}(1-q^l)^{1-d} \\ &= \sum_{n\in\mathbb{N}} p_{d-1}(n)q^n \\ &= 1 + (d-1)q + \tfrac{1}{2}(d-1)(d+2)q^2 + \tfrac{1}{6}(d-1)d(d+7)q^3 + \dots, \end{aligned} \tag{3.3.11}$$

in terms of the generating Euler function $\phi(q)$. Hence

$$\begin{aligned} p_{d-1}(n) - p_{d-1}(n-1) &= \sum_{l=0}^{n} p_{d-2}(l)p_1(n-l) - \sum_{l=0}^{n-1} p_{d-2}(l)p_1(n-l-1) \\ &= p_{d-2}(n) + \Delta_{d-2}(n), \end{aligned} \tag{3.3.12}$$

with the function $\Delta_{d-2}(n) := \sum_{l=0}^{n-1} p_{d-2}(l)[p_1(n-l) - p_1(n-l-1)]$ counting the deviation of $\dim \mathfrak{g}_\Lambda^{(\mathbf{r}-n\mathbf{k})}$ from the number of n excitations by $d-2$ transversal oscillators, $p_{d-2}(n)$; for example:

$$\begin{aligned} \Delta_{d-2}(0) &= 0, \\ \Delta_{d-2}(1) &= 0, \\ \Delta_{d-2}(2) &= 1, \\ \Delta_{d-2}(3) &= d-1, \\ \Delta_{d-2}(4) &= \tfrac{1}{2}(d-1)(d+2). \end{aligned}$$

Explicitly,

$$\sum_{n=0}^{\infty} \dim \mathfrak{g}_\Lambda^{(\mathbf{r}-n\mathbf{k})} q^n = 1 + (d-2)q + \tfrac{1}{2}(d-1)dq^2 + \tfrac{1}{6}(d-1)(d^2+4d-6)q^3 + \dots$$

Note that the null physical states account for the second term in (3.3.10), for they are given by $S(\hat{\mathbf{t}}^-)A^-_{-1}|\mathbf{r}\rangle$.

Since we will mainly focus on a deeper understanding of the Kac–Moody algebra $\mathfrak{g}(A)$, the question arises how to make contact between the elegant DDF formulation of $\mathfrak{g}_\Lambda$ and the construction of $\mathfrak{g}(A)$ in terms of generators and relations. In other words, we have to face the problem of how to separate the DDF states contained in $\mathfrak{g}(A)$ from those which cannot be generated by the set $\{e_i, f_i, h_i \,|\, i\}$ via multiple commutators. Note that a special case of (2.3.5) gives us a recipe for writing physical states $\boldsymbol{\xi}(-1)\mathrm{e}^{\mathbf{k}}$ as Lie algebra commutators:

$$\begin{aligned} [\mathrm{e}^{\mathbf{s}}, \mathrm{e}^{\mathbf{t}}] &= \epsilon(\mathbf{s},\mathbf{t})\mathbf{s}(-1)\mathrm{e}^{\mathbf{s}+\mathbf{t}} \\ &= \tfrac{1}{2}\epsilon(\mathbf{s},\mathbf{t})(\mathbf{s}-\mathbf{t})(-1)\mathrm{e}^{\mathbf{s}+\mathbf{t}} \end{aligned} \tag{3.3.13}$$

for $\mathbf{s}, \mathbf{t} \in \Lambda_2$ such that $\mathbf{s}\cdot\mathbf{t} = -2$. The last equality is obtained by subtracting the null physical state ("total derivative") $\frac{1}{2}\epsilon(\mathbf{s},\mathbf{t})\mathrm{L}_{(-1)}\mathrm{e}^{\mathbf{s}+\mathbf{t}}$. Hence we may put $\boldsymbol{\xi} = \mathbf{s} - \mathbf{t}$ and $\mathbf{k} = \mathbf{s} + \mathbf{t}$. This observation will be useful later.

We conclude with a comment that will be crucial for the discrete DDF construction of E_{10}. So far, we have tacitly assumed the DDF vectors $\mathbf{k}$ and $\mathbf{r}$ to be elements of the root lattice. However, inspection of the above computations shows that all arguments remain valid if only $\mathbf{k}^2 = 0, \mathbf{r}^2 = 2, \mathbf{r}\cdot\mathbf{k} = 1$ and $\boldsymbol{\xi}\cdot\mathbf{r} = \boldsymbol{\xi}\cdot\mathbf{k} = 0$. Thus, there is actually no need to assume the vectors $\mathbf{k}$ and $\mathbf{r}$ to be on the root lattice as long as these conditions are satisfied. In particular, under these circumstances we may choose $\mathbf{k}$ and $\mathbf{r}$ on the rational extension $\Lambda_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda$, and the discrete DDF construction still works. All our formulas will continue to make sense, whereas the interpretation of physical states and the identification of Lie algebra elements need some care. This subtlety arises because, rigorously speaking, we are dealing with a **generalized vertex algebra** associated with $\Lambda_{\mathbb{Q}}$, into which the original vertex algebra (associated with Λ) can be embedded. The generalized vertex operators are then defined as in (2.1.28) and (2.1.34), but are no longer elements of $(\mathrm{End}\,\mathcal{F})[\![z, z^{-1}]\!]$; rather, the generalized vertex operator associated with a typical homogeneous element $\psi = [\prod_{j=1}^{N} \mathbf{s}_j(-n_j)] \otimes \mathrm{e}^{\mathbf{r}}$ (where now $\mathbf{r} \in \Lambda_{\mathbb{Q}}$) is an element of of $(\mathrm{End}\,\mathcal{D}_{\mathbf{r}})[\![z, z^{-1}]\!]$ with

$$\mathcal{D}_{\mathbf{r}} := \bigoplus_{\substack{\mathbf{s}\in\Lambda_{\mathbb{Q}} \\ \mathbf{r}\cdot\mathbf{s}\in\mathbb{Z}}} S(\hat{\mathbf{h}}^-) \otimes \mathrm{e}^{\mathbf{s}} \equiv \bigoplus_{\substack{\mathbf{s}\in\Lambda_{\mathbb{Q}} \\ \mathbf{r}\cdot\mathbf{s}\in\mathbb{Z}}} \mathcal{F}^{(\mathbf{s})}. \tag{3.3.14}$$

This means that the modes of the generalized vertex operators are *not* well-defined operators on the whole Fock space $\mathcal{F}$ but only on certain of its subspaces. In physics language we would say that we have to give up mutual locality of the vertex operators, i.e. we enlarge our original set of vertex operators so that the momenta of any two of them not necessarily have integer scalar product. Thus we are in danger of encountering fractional powers of the formal variable z, which in complex analysis would lead to branch cuts. But this problem can be resolved by the above restriction of the domains for the vertex operators. For example, if $\psi \in \mathcal{F}^{(\mathbf{r})}$, $\varphi \in \mathcal{F}^{(\mathbf{s})}$ and

$\mathbf{r}\cdot\mathbf{s}$ is not an integer, then φ is obviously not an element of $\mathcal{D}_{\mathbf{r}}$ and hence $\mathcal{V}(\psi, z_1)\mathcal{V}(\varphi, z_2)\mathbf{1}$ by definition does not exist.

CHAPTER IV:

THE HYPERBOLIC ALGEBRA E_{10} AND THE DDF CONSTRUCTION

We now want to apply the concepts developed in the foregoing chapters to the study of Kac–Moody algebras $\mathfrak{g}(A)$ whose Cartan matrix A is of hyperbolic type, choosing the hyperbolic Kac–Moody algebra E_{10} as our example. We remind the reader that hyperbolic algebras are distinguished from the more general algebras based on arbitrary indefinite Cartan matrices by the additional requirement that the removal of any point from the Dynkin diagram leaves a Kac–Moody algebra which is either of affine or finite type (for a review of hyperbolic root systems, see [63]). As shown in [54], the rank can then be 10 at most, and the root lattice must be Minkowskian, i.e. with metric signature $(+\ldots+|-)$. There are altogether three hyperbolic algebras of maximal rank. Of these, E_{10} is not only the most interesting, containing E_8 and its affine extension E_9 as subalgebras, but also distinguished by the fact that it has only *one* affine subalgebra that can be obtained by removing a point from the E_{10} Dynkin diagram, while the other two rank 10 algebras contain at least two regular affine subalgebras, (see e.g. [70]). Furthermore, the root lattice $Q(E_{10})$ coincides with the (unique) 10-dimensional even unimodular Lorentzian lattice $II_{9,1}$ [16], whereas the root lattices of the other two maximal rank hyperbolic algebras are not self-dual.

Overall, our knowledge about Kac–Moody algebras of hyperbolic type is rather limited. As already explained in Sect. II.3, they are generally defined in terms of multiple commutators of the basic generators e_i, f_i, h_i and the multilinear Serre relations (2.3.25). In contradistinction to the finite and affine cases, a manageable representation of all the Lie algebra elements obtained in this way has not yet been found. In principle, the string vertex operator construction provides a more concrete realization with the additional advantage that the Serre relations (2.3.25) are built in from the outset (see the discussion at the end of Sect. II.3), but the problem of characterizing the missing elements belonging to $\mathfrak{g}_\Lambda$ and not to $\mathfrak{g}(A)$ in (2.3.31) remains. We emphasize that we face essentially the same problem if instead we want to define the Borcherds algebra (see Chap. V) based on $II_{9,1}$, because we then would have to supply the missing generators "by hand" by adding extra imaginary simple roots, which again presupposes knowledge of what the missing Lie algebra elements are (not to mention the potential arbitrariness as to the number of ways in which this can be consistently done).

As already mentioned, our analysis makes use of a discretized version of the DDF construction and relies in a crucial way on the identification of Lie algebra elements with physical Fock space states. In the previous chapter we have seen that a central role is played the tachyon momentum $\mathbf{a}$ of the ground state (so $\mathbf{a}^2 = 2$) and the null vector $\mathbf{k}$, subject to the condition $\mathbf{k}\cdot\mathbf{a} = 1$. For continous momenta $\mathbf{a}$, we can always find suitable $\mathbf{k} = \mathbf{k}(\mathbf{a})$; moreover, we can rotate these vectors into a convenient frame by means of a Lorentz transformation [71]. On the lattice, however, the full Lorentz invariance is broken to a discrete subgroup (containing the Weyl group generated by the fundamental Weyl reflections), and for generic roots $\mathbf{\Lambda}$, the vectors $\mathbf{a}$ and $\mathbf{k}$ arising in the "DDF decomposition" $\mathbf{\Lambda} = \mathbf{a} - n\mathbf{k}$ (for some $n \in \mathbb{N}$) will *not* be elements of the root lattice $II_{9,1}$ in general[1]. Nevertheless, we employ these vectors in our analysis because we can still use the associated (transversal and longitudinal) DDF operators to construct a complete basis for any root space of the Lie algebra of physical states $\mathfrak{g}_{II_{9,1}}$. The corresponding root space of the Kac–Moody algebra $\mathfrak{g}(A)$ is then a proper subspace thereof. As we will see, longitudinal states are absent only for level 0 and level 1; this accounts for the comparative simplicity of the corresponding multiplicity formulas.

In Sect. 1 we will summarize the pertinent results about E_{10}. In Sect. 2 we apply the discrete DDF construction to level 0 and level 1 elements of $\mathfrak{g}(A)$ thereby recovering some known results. Our method for dealing with higher level states is presented in Sect. 3. In Sect. 4, we turn to the level 2 states, analyzing one example in complete detail.

1. Basic results about E_{10}

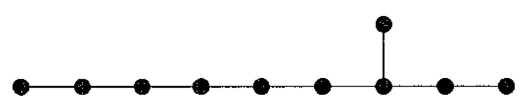
The hyperbolic Kac–Moody algebra E_{10} is defined via its Coxeter-Dynkin diagram and the Serre relations following from it. As already mentioned, the root lattice $Q(E_{10})$ coincides with the unique 10-dimensional even unimodular Lorentzian lattice $II_{9,1}$. The latter can be defined as the lattice of all points $\mathbf{x} = (x_1, \ldots, x_9|x_0)$ for which the x_μ's are all in $\mathbb{Z}$ or all in $\mathbb{Z} + \frac{1}{2}$ and which have integer inner product with the vector $\mathbf{l} = (\frac{1}{2}, \ldots, \frac{1}{2} \,|\, \frac{1}{2})$, all norms and inner products being evaluated in the Minkowskian metric $\mathbf{x}^2 = x_1^2 + \ldots + x_9^2 - x_0^2$ (cf. [72]). In more physical parlance, we are dealing with a subcritical open bosonic string moving in 10-dimensional space-time fully compactified on a torus (hence "finite in all directions" [64]), so that the momenta

[1] To make this explicit in the notation, we designate the tachyon momentum by $\mathbf{a}$ rather than $\mathbf{r}$ as in Chap. III.

lie on $II_{9,1}$. According to [16], a set of positive norm simple roots for $II_{9,1}$ is given by the ten vectors $\mathbf{r}_{-1}, \mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_8$ in $II_{9,1}$ for which $\mathbf{r}_i^2 = 2$ and $\mathbf{r}_i \cdot \boldsymbol{\rho} = -1$ where the **Weyl vector** is $\boldsymbol{\rho} = (0, 1, 2, \ldots, 8|38)$ with $\boldsymbol{\rho}^2 = -1240$. Explicitly,

$$\begin{aligned}
\mathbf{r}_{-1} &= (\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,-1\,|\,0),\\
\mathbf{r}_{0} &= (\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,-1,\ 0\,|\,0),\\
\mathbf{r}_{1} &= (\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,-1,\ 0,\ 0\,|\,0),\\
\mathbf{r}_{2} &= (\ 0,\ 0,\ 0,\ 0,\ 1,-1,\ 0,\ 0,\ 0\,|\,0),\\
\mathbf{r}_{3} &= (\ 0,\ 0,\ 0,\ 1,-1,\ 0,\ 0,\ 0,\ 0\,|\,0),\\
\mathbf{r}_{4} &= (\ 0,\ 0,\ 1,-1,\ 0,\ 0,\ 0,\ 0,\ 0\,|\,0),\\
\mathbf{r}_{5} &= (\ 0,\ 1,-1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\,|\,0),\\
\mathbf{r}_{6} &= (-1,-1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\,|\,0),\\
\mathbf{r}_{7} &= (\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2}\,|\,\tfrac{1}{2}),\\
\mathbf{r}_{8} &= (\ 1,-1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\,|\,0).
\end{aligned}$$

These simple roots indeed generate the reflection group of $II_{9,1}$. The corresponding Coxeter-Dynkin diagram looks as follows:



(4.1.1)

and is associated with the Cartan matrix

$$A \equiv (a_{ij}) = (\mathbf{r}_i \cdot \mathbf{r}_j) = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2
\end{pmatrix},$$

whose inverse

$$A^{-1} = -\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 4 & 2 & 3\\
1 & 2 & 4 & 6 & 8 & 10 & 12 & 8 & 4 & 6\\
2 & 4 & 6 & 9 & 12 & 15 & 18 & 12 & 6 & 9\\
3 & 6 & 9 & 12 & 16 & 20 & 24 & 16 & 8 & 12\\
4 & 8 & 12 & 16 & 20 & 25 & 30 & 20 & 10 & 15\\
5 & 10 & 15 & 20 & 25 & 30 & 36 & 24 & 12 & 18\\
6 & 12 & 18 & 24 & 30 & 36 & 42 & 28 & 14 & 21\\
4 & 8 & 12 & 16 & 20 & 24 & 28 & 18 & 9 & 14\\
2 & 4 & 6 & 8 & 10 & 12 & 14 & 9 & 4 & 7\\
3 & 6 & 9 & 12 & 15 & 18 & 21 & 14 & 7 & 10
\end{pmatrix}$$

we shall need below. Since there is obviously no nontrivial graph automorphism, the group of autochronous automorphisms of $II_{9,1}$ is already given by the Weyl group of E_{10}. The E_9 **null root** is

$$\boldsymbol{\delta} = \sum_{i=0}^{8} n_i \mathbf{r}_i = (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1\,|\,1), \tag{4.1.2}$$

where the coefficients n_i (called **marks** of E_9) can be read off from

$$\begin{bmatrix} & & & & & & 3 & & \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 4 & 2 \end{bmatrix}. \tag{4.1.3}$$

The fundamental Weyl chamber C of E_{10} is the convex cone generated by the **fundamental weights** $\boldsymbol{\Lambda}_i$,

$$\boldsymbol{\Lambda}_i = -\sum_{j=-1}^{8} (A^{-1})_{ij} \mathbf{r}_j \quad \text{for } i = -1, 0, 1, \ldots 8, \tag{4.1.4}$$

with the inverse Cartan matrix from above.[2] Thus,

$$\boldsymbol{\Lambda} \in \mathrm{C} \quad \Longleftrightarrow \quad \boldsymbol{\Lambda} = \sum_{i=-1}^{8} k_i \boldsymbol{\Lambda}_i \quad \text{for } k_i \in \mathbb{Z}_+. \tag{4.1.5}$$

A special feature of E_{10} is that we need not distinguish between root and weight lattice, since these are the same for selfdual root lattices[3]. Since Weyl transformations preserve multiplicities

[2] Notice that our convention is opposite to the one adopted in [55]. The fundamental weights here are *positive* and satisfy $\boldsymbol{\Lambda}_i \cdot \mathbf{r}_j = -\delta_{ij}$ Thus, we will be dealing with "lowest weight" rather than "highest weight" representations in accordance with physics usage.

[3] In the remainder, we will consequently denote arbitrary roots by $\boldsymbol{\Lambda}$ and reserve the letter $\mathbf{r}$ for real roots (i.e. $\mathbf{r}^2 = 2$).

and since every root can be brought into C by means of a Weyl transformation, the structure of the algebra is completely understood once the root spaces for roots belonging to C are under control. Note also that the null root plays a special role: The first fundamental weight is just $\mathbf{\Lambda}_{-1} = \boldsymbol{\delta}$, and all null vectors in C must be multiples of $\mathbf{\Lambda}_{-1}$, since $\mathbf{\Lambda}_i^2 < 0$ for all other fundamental weights.

As described in Sect. II.3, the algebra $\mathfrak{g}(A) = E_{10}$ is spanned by all multiple commutators of the Chevalley generators e_i, f_i, h_i with $i = -1, 0, 1, \ldots, 8$. It is a standard result [54] that this algebra can be written as a direct sum

$$\mathfrak{g}(A) = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-, \tag{4.1.6}$$

where the subalgebras $\mathfrak{n}_-$ and $\mathfrak{n}_+$ are spanned by multiple commutators of the form $[f_{i_1}, [f_{i_2}, \ldots [f_{i_{n-1}}, f_{i_n}] \ldots]]$ and $[e_{i_1}, [e_{i_2}, \ldots [e_{i_{n-1}}, e_{i_n}] \ldots]]$, respectively, modulo the multilinear Serre relations (2.3.25). Since $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are conjugate and thus enjoy analogous properties, it is enough in practice to consider only multiple commutators made out of e_i generators (corresponding to positive roots). To classify such commutators one introduces the **level** $\ell \in \mathbb{Z}$ of a root, such that positive ℓ counts the number of e_{-1} generators in $[e_{i_1}, [e_{i_2}, \ldots [e_{i_{n-1}}, e_{i_n}] \ldots]]$ (similarly, if ℓ is negative, $-\ell$ counts the number of f_{-1} generators in $[f_{i_1}, [f_{i_2}, \ldots [f_{i_{n-1}}, f_{i_n}] \ldots]]$). In terms of the corresponding root $\mathbf{\Lambda} = \mathbf{r}_{i_1} + \ldots + \mathbf{r}_{i_n}$, ℓ is defined by

$$\ell := -\mathbf{\Lambda} \cdot \boldsymbol{\delta}. \tag{4.1.7}$$

Observe that ℓ is not preserved under arbitrary E_{10} Weyl transformations, but only under the subgroup $W(E_9)$ corresponding to the E_9 subalgebra. Therefore, we can freely use this notion also for roots $\mathbf{\Lambda}$ which are not in C, but can be brought into C by an E_9 Weyl transformation.

The level derives its importance from the fact that it grades the algebra E_{10} with respect to its affine subalgebra E_9 [23]. The subspaces belonging to a fixed level can be decomposed into irreducible representations of E_9, the level being equal to the eigenvalue of the central term of the E_9 algebra on this representation (the full E_{10} algebra contains E_9 representations of *all* integer levels!). Let us emphasize that for general hyperbolic algebras there would be a separate grading associated with every regular affine subalgebra, and therefore the graded structure would no longer be unique.

An important result is the following [24].

Theorem 7

Suppose that x is an element of E_{10} at level n. Then it can be represented as a linear combination of n-fold commutators of level 1 elements, viz.

$$x = [x_1, [x_2, \ldots [x_{n-1}, x_n] \ldots]], \tag{4.1.8}$$

where each x_i contains exactly one generator e_{-1} in the right-most position[4], *i.e.*

$$x_1 = [e_{i_1}, [e_{i_2}, \ldots [e_{i_k}, e_{-1}] \ldots]], \tag{4.1.9}$$

with $i_\nu \in \{0, 1, \ldots, 8\}$, and similarly for the other x_i.

We are going to make use of this result in the next section in order to effectively construct higher level elements.

As already mentioned, little is known about the general structure of E_{10}. Partial progress has been made in determining the multiplicity of certain roots, i.e. the number of linearly independent Lie algebra elements in the corresponding root space. Although the general form of the multiplicity formulas for arbitrary levels appears to be beyond reach for the moment, the following results for levels $|\ell| \leq 2$ have been established.[5] For $\ell = 0$ and $\ell = 1$, we have [54]

$$\text{mult}(\mathbf{\Lambda}) = p_8(1 - \tfrac{1}{2}\mathbf{\Lambda}^2), \tag{4.1.10}$$

i.e. the multiplicities are just given by the number of transversal states; we will see in the next section that the corresponding states are indeed transversal. At higher levels we cannot expect simple multiplicity formulas since, a priori, the multiplicity of a root will not only depend on its *length* but also on its very *direction* (see [54, Table H_3] for an example). Surprisingly enough, however, for $\ell = 2$, it was shown in [55] that[6]

$$\text{mult}(\mathbf{\Lambda}) = \xi(3 - \tfrac{1}{2}\mathbf{\Lambda}^2), \tag{4.1.11}$$

where

$$\sum_{n \geq 0} \xi(n) q^n = \frac{1}{\phi(q)^8}\left[1 - \frac{\phi(q^2)}{\phi(q^4)}\right], \tag{4.1.12}$$

and the Euler function $\phi(q)$ is defined in (3.3.11). For sufficiently large (negative) $\mathbf{\Lambda}^2$, one can check from this formula that there are roots $\mathbf{\Lambda}$ such that $\text{mult}(\mathbf{\Lambda}) > p_8(1 - \frac{1}{2}\mathbf{\Lambda}^2)$; this clearly

[4] All level 1 elements can be cast into this form by use of the Jacobi identity and by taking appropriate linear combinations.

[5] We use the pedantic notation $|\ell|$ at this point to remind the reader that the subalgebras $\mathfrak{n}_-$ and $\mathfrak{n}_+$ are conjugate. Thus any statement about a certain level of the algebra is also true for the negative level (modulo signs, of course).

[6] The derivation of this result in is based on the E_9 decomposition

$$L(\mathbf{\Lambda}_0) \wedge L(\mathbf{\Lambda}_0) \cong L(\mathbf{\Lambda}_1) \otimes V\left(\tfrac{1}{2}, \tfrac{1}{16}\right)$$

where $L(\mathbf{\Lambda}_i)$ denotes the irreducible E_9 module with lowest weight $\mathbf{\Lambda}_i$ and $V\left(\frac{1}{2}, \frac{1}{16}\right)$ the irreducible Virasoro module with $c = \frac{1}{2}$ and $h = \frac{1}{16}$ (by abuse of notation, we use the same labels for the E_9 weights as for the E_{10} weights). Observe that the module $L(\mathbf{\Lambda}_1)$ precisely corresponds to the ideal generated by the double commutator $[[e_0, e_{-1}], e_{-1}]$. For higher levels, analogous decompositions contain more than one term on the right hand side, and it seems no longer possible to divide out the Serre relations by this method.

implies the presence of longitudinal states. Beyond $\ell = 2$, no general formula seems to be known although the multiplicities can be determined recursively from the Peterson formula (see e.g. [56]), which might be useful, too; for it would be worth checking by brute force whether there are some indications for further unexpected simplifications in the pattern of the root multiplicities for E_{10}. Such a result could give a clue how to tackle higher level root spaces of the algebra.

In the physical interpretation, the multiplicity of a root $\mathbf{\Lambda}$ for E_{10} is nothing but the number of linearly independent polarization states of the associated vertex operator of momentum $\mathbf{\Lambda}$, which can be generated by multiple commutators (recall that not all physical states can be obtained in this way, cf. (2.3.31)). Given a root $\mathbf{\Lambda} \in \mathsf{C}$, we call a polarization vector $\boldsymbol{\xi}$ **transversal** if $\boldsymbol{\xi}\cdot\mathbf{\Lambda} = \boldsymbol{\xi}\cdot\boldsymbol{\delta} = 0$, and **longitudinal** otherwise. This terminology is, of course, physically motivated. We also define the **little group** $W(\mathbf{\Lambda}, \boldsymbol{\delta})$ to be that subgroup of the full Weyl group of E_{10} which leaves the vectors $\mathbf{\Lambda}$ and $\boldsymbol{\delta}$ invariant. Unless $\mathbf{\Lambda}$ is collinear with $\boldsymbol{\delta}$ (corresponding to $\ell = 0$), $W(\mathbf{\Lambda}, \boldsymbol{\delta})$ is a *finite* subgroup of $W(E_{10})$, as well as a discrete subgroup of $SO(8)$. As an example consider $\ell = 1$; then $\mathbf{\Lambda} = \mathbf{\Lambda}_0 = \mathbf{r}_{-1} + 2\boldsymbol{\delta}$ and $W(\mathbf{\Lambda}, \boldsymbol{\delta})$ is isomorphic to the Weyl group of E_8. In fact, for $\mathbf{\Lambda} \in \mathsf{C}$, it is known ([54, Prop. 3.12]) that $W(\mathbf{\Lambda}, \boldsymbol{\delta})$ is generated by the reflections w_i corresponding to those simple roots $\mathbf{r}_i$ for which $\mathbf{\Lambda}\cdot\mathbf{r}_i = \boldsymbol{\delta}\cdot\mathbf{r}_i = 0$. This indicates that the little group will not be quite as useful in this context as it is in conventional quantum field theory, because it becomes trivial for sufficiently high levels. However, at low levels, this problem does not yet arise, and the polarization states and hence the elements belonging to the root space $\mathfrak{g}^{(\Lambda)}_{II_{9,1}}$ can be classified as representations of $W(\mathbf{\Lambda}, \boldsymbol{\delta})$.

Any root $\mathbf{\Lambda} \in \mathsf{C}$ can be represented in the form

$$\mathbf{\Lambda} = \ell\mathbf{r}_{-1} + M\boldsymbol{\delta} - \mathbf{b} \tag{4.1.13}$$

where ℓ is the level of $\mathbf{\Lambda}$ and $\mathbf{b}$ an element of the E_8-root lattice $Q(E_8)$ ($\mathbf{b}$ need not be negative by itself as only $M\boldsymbol{\delta} - \mathbf{b}$ must be positive). We now define the **DDF decomposition** of $\mathbf{\Lambda}$ by

$$\mathbf{\Lambda} = \mathbf{a} - n\mathbf{k}(\mathbf{a}), \tag{4.1.14}$$

where

$$\mathbf{k}(\mathbf{a}) := -\tfrac{1}{\ell}\boldsymbol{\delta} \tag{4.1.15}$$

and

$$n := 1 - \tfrac{1}{2}\mathbf{\Lambda}^2 = 1 + (M-\ell)\ell - \tfrac{1}{2}\mathbf{b}^2 \tag{4.1.16}$$

so that

$$\mathbf{a} = \ell\mathbf{r}_{-1} + \tfrac{1}{\ell}\left(\ell^2 - 1 + \tfrac{1}{2}\mathbf{b}^2\right)\boldsymbol{\delta} - \mathbf{b}. \tag{4.1.17}$$

By construction, $\mathbf{a}$ obeys $\mathbf{a}^2 = 2$ and is therefore associated with a tachyon state, and n is the number of steps required to reach the root $\mathbf{\Lambda}$ by starting from $\mathbf{a}$ and decreasing the momentum by $\mathbf{k}$ at each step (n is nonnegative because $\mathbf{\Lambda}^2 \leq 2$). Obviously, for $\ell > 1$, neither $\mathbf{k}$ nor $\mathbf{a}$ belong to the lattice in general. As a consequence, the intermediate DDF states associated with momenta $\mathbf{a} - m\mathbf{k}$ not on the lattice will not correspond to elements of the algebra. On the other hand, states associated with the root $\mathbf{\Lambda}$ do belong to the algebra of physical states, and the DDF decomposition enables us to write down all possible polarization states associated with the root $\mathbf{\Lambda} \in \mathsf{C}$ in terms of transversal and longitudinal DDF states; the totality of these states constitutes the complete set of elements in the root space $\mathfrak{g}^{(\Lambda)}_{II_{9,1}}$.

Of course, we could also try to apply the DDF decomposition to roots $\mathbf{\Lambda}$ not $W(E_9)$-equivalent to roots in C. Whenever we succeed in finding a suitable null vector $\mathbf{k}$ on the lattice obeying $\mathbf{\Lambda}\cdot\mathbf{k} = 1$, we can also find a Weyl transformation $w \in W(E_{10})$ such that $w(\mathbf{k}) = -\boldsymbol{\delta}$ because $\boldsymbol{\delta}$ is the only primitive null vector in C. Since $\mathbf{\Lambda}\cdot\mathbf{k} = -w(\mathbf{\Lambda})\cdot\boldsymbol{\delta}$ is just the level, it follows that $w(\mathbf{\Lambda}) = \mathbf{a} + n\boldsymbol{\delta}$ is a level 1 root with tachyon momentum

$$\mathbf{a} = \mathbf{r}_{-1} + \left(\tfrac{1}{2}\mathbf{b}^2\right)\boldsymbol{\delta} - \mathbf{b}, \tag{4.1.18}$$

for some $\mathbf{b} \in Q(E_8)$. Therefore, nothing is gained by searching for DDF vectors outside the $W(E_9)$ transforms of the fundamental Weyl chamber. Note that the elements of the form (4.1.18) constitute the E_9 translation orbit of $\mathbf{r}_{-1}$. To see this, recall that the Weyl group of an affine Kac–Moody algebra has the structure of a semidirect product of the Weyl group of the underlying finite Lie algebra with the group of translations, here

$$W(E_9) = \mathsf{T} \rtimes W(E_8), \tag{4.1.19}$$

where T is isomorphic to $Q(E_8)$ as an Abelian group and acts on the real vector space $\mathbb{R}\otimes_{\mathbb{Z}} II_{9,1}$ (which is nothing but the dual space of the affine Cartan subalgebra, $\mathfrak{h}^*_{\mathbb{R}}(E_9)$) by the formula

$$\mathsf{t}_{\mathbf{b}}(\mathbf{x}) := \mathbf{x} + (\mathbf{x}\cdot\boldsymbol{\delta})\mathbf{b} - [(\mathbf{x}\cdot\mathbf{b}) + \tfrac{1}{2}(\mathbf{x}\cdot\boldsymbol{\delta})\mathbf{b}^2]\boldsymbol{\delta} \tag{4.1.20}$$

for $\mathbf{b} \in Q(E_8)$ and $\mathbf{x} \in \mathbb{R}\otimes_{\mathbb{Z}} II_{9,1}$. Indeed, one can easily deduce that the additivity property, $\mathsf{t}_{\mathbf{a}}\mathsf{t}_{\mathbf{b}} = \mathsf{t}_{\mathbf{a}+\mathbf{b}}$, and the normal subgroup property, $\mathsf{t}_{w(\mathbf{a})} = w\mathsf{t}_{\mathbf{a}}w^{-1}$ for $w \in W(E_8)$, are valid (see [54, §6.5]). For $\mathbf{x} = \mathbf{r}_{-1}$ we immediately verify that the orbit $\mathsf{T}\mathbf{r}_{-1}$ is made up by elements of the above form.

2. The DDF states at levels 0 and 1

Although the multiplicity formulas for levels $\ell = 0$ and $\ell = 1$ are understood [54], we here derive them once more, because our explicit DDF representation of the level 1 elements has apparently not been exhibited in the literature so far. The level 0 elements make up the E_9 subalgebra of E_{10}. The real roots are all $\mathbf{r} \in II_{9,1}$ obeying $\mathbf{r}^2 = 2$ and $\mathbf{r}\cdot\boldsymbol{\delta} = 0$ (hence having no $\mathbf{r}_{-1}$

component), the imaginary roots are all nonzero multiples of the null root, i.e. $m\boldsymbol{\delta}$ for $m \in \mathbb{Z}^\times$. These correspond to the tachyonic and photonic states with multiplicities 1 and 8, respectively:

$$|\mathbf{r}\rangle \equiv \mathrm{e}^{\mathbf{r}} \quad \text{for } \mathbf{r}^2 = 2,\ \mathbf{r}\cdot\boldsymbol{\delta} = 0, \tag{4.2.1}$$

$$\boldsymbol{\xi}_i(-1)|m\boldsymbol{\delta}\rangle \equiv \boldsymbol{\xi}_i(-1)\mathrm{e}^{m\boldsymbol{\delta}}, \tag{4.2.2}$$

where $\boldsymbol{\xi}_i\cdot\boldsymbol{\delta} = 0$ and $\boldsymbol{\xi}_i$ has no component along $\boldsymbol{\delta}$ (i.e. $\mathbf{r}_{-1}\cdot\boldsymbol{\xi}_i = 0$). The Cartan subalgebra of E_9 is spanned by the states

$$\boldsymbol{\delta}(-1)|\mathbf{0}\rangle =: K, \tag{4.2.3}$$

$$(\mathbf{r}_{-1}+\boldsymbol{\delta})(-1)|\mathbf{0}\rangle =: d, \tag{4.2.4}$$

$$\boldsymbol{\xi}_i(-1)|\mathbf{0}\rangle \qquad \text{for } i = 1,\ldots,8\,, \tag{4.2.5}$$

where K represents the central element, d is the derivation of E_9, and $\{\boldsymbol{\xi}_i(-1)|\mathbf{0}\rangle \mid i = 1,\ldots,8\}$ span the Cartan subalgebra of E_8. This is the standard "light-cone" basis of $\mathfrak{h}(E_9)$ in the sense that K and d are lightlike. As for the commutators we rewrite (2.3.9) and (2.3.11) – (2.3.13) as

$$\left[\boldsymbol{\eta}(-1)|\mathbf{0}\rangle, \boldsymbol{\zeta}(-1)|\mathbf{0}\rangle\right] = 0, \tag{4.2.6}$$

$$\left[\boldsymbol{\eta}(-1)|\mathbf{0}\rangle, \boldsymbol{\xi}_i(-1)|m\boldsymbol{\delta}\rangle\right] = m(\boldsymbol{\eta}\cdot\boldsymbol{\delta})\boldsymbol{\xi}_i(-1)|m\boldsymbol{\delta}\rangle, \tag{4.2.7}$$

$$\left[\boldsymbol{\eta}(-1)|\mathbf{0}\rangle, |\mathbf{r}\rangle\right] = (\boldsymbol{\eta}\cdot\mathbf{r})|\mathbf{r}\rangle, \tag{4.2.8}$$

$$\left[\boldsymbol{\xi}_i(-1)|m\boldsymbol{\delta}\rangle, \boldsymbol{\xi}_j(-1)|n\boldsymbol{\delta}\rangle\right] = m\delta_{m+n,0}(\boldsymbol{\xi}_i\cdot\boldsymbol{\xi}_j)\boldsymbol{\delta}(-1)|\mathbf{0}\rangle, \tag{4.2.9}$$

$$\left[\boldsymbol{\xi}_i(-1)|m\boldsymbol{\delta}\rangle, |\mathbf{r}\rangle\right] = (\boldsymbol{\xi}_i\cdot\mathbf{r})|\mathbf{r}+m\boldsymbol{\delta}\rangle, \tag{4.2.10}$$

$$\left[|\mathbf{r}\rangle, |\mathbf{s}\rangle\right] = \begin{cases} 0 & \text{if } \mathbf{r}\cdot\mathbf{s} \geq 0, \\ \epsilon(\mathbf{r},\mathbf{s})|\mathbf{r}+\mathbf{s}\rangle & \text{if } \mathbf{r}\cdot\mathbf{s} = -1, \\ -\mathbf{r}(-1)|k\boldsymbol{\delta}\rangle & \text{if } \mathbf{r}+\mathbf{s} = k\boldsymbol{\delta}, \end{cases} \tag{4.2.11}$$

for $\boldsymbol{\eta}, \boldsymbol{\zeta} \in \mathfrak{h}(E_9)$ and E_9 roots $\mathbf{r}, \mathbf{s}$. To see that photonic states with all required transversal polarizations can be generated by commuting tachyonic states, we recall (3.3.13) (a special case of (2.3.5)): choosing $\mathbf{s} = \mathbf{r}_i$ and $\mathbf{t} = m\boldsymbol{\delta} - \mathbf{r}_i$ (where $\mathbf{r}_i$ is any simple root of E_9), we obtain all transversal polarizations. There is obviously no way to generate longitudinal states, because the polarization vectors $\boldsymbol{\xi}_i$ would then have to have components along $\mathbf{r}_{-1}$, which we cannot generate by commuting tachyonic states belonging to E_9 roots only. Since we can ignore null physical states (for which $\boldsymbol{\xi} \propto \boldsymbol{\delta}$), we can in addition impose the requirement $\boldsymbol{\xi}\cdot\mathbf{r}_{-1} = 0$, so $\boldsymbol{\xi} \in \mathrm{span}_{\mathbb{R}}\{\mathbf{r}_1,\ldots,\mathbf{r}_8\}$, so that by taking appropriate linear combinations we can arrange that $\boldsymbol{\xi}_i\cdot\boldsymbol{\xi}_j = \delta_{ij}$ with $\boldsymbol{\xi}_i\cdot\boldsymbol{\delta} = \boldsymbol{\xi}_i\cdot\mathbf{r}_{-1} = 0$ for $i,j = 1,\ldots,8$. It is clear that an infinity of conjugate E_9 subalgebras in E_{10} can be obtained by Weyl conjugation of these states with elements of $W(E_{10})$ not in $W(E_9)$.

Recall that we presented in Sect. II.3 the Frenkel–Kac construction of affine Lie algebras, which seemingly provides a completely different realization of E_9, because there we considered modes of physical states for the root lattice of E_8, $Q(E_8) = \mathrm{span}_{\mathbb{Z}}\{\mathbf{r}_1,\ldots,\mathbf{r}_8\}$, while above we stayed inside the Lie algebra of level 0 physical states for the lattice $II_{9,1} = \mathrm{span}_{\mathbb{Z}}\{\mathbf{r}_{-1},\mathbf{r}_0,\mathbf{r}_1,\ldots,\mathbf{r}_8\}$. In fact, as shown in [42], the mode structure in the first approach corresponds to the integral shifts by the lightlike vector $\mathbf{k}$ in the second one. Explicitly, since any real root of E_9 can be decomposed into $\mathbf{r} = \boldsymbol{\alpha} + m\boldsymbol{\delta}$ for some $\boldsymbol{\alpha} \in \Delta(E_8)$, $m \in \mathbb{Z}$, we have the correspondence

$$|\mathbf{r}\rangle = |\boldsymbol{\alpha}+m\boldsymbol{\delta}\rangle \quad\leftrightarrow\quad \mathrm{e}^{\boldsymbol{\alpha}}_m, \tag{4.2.12}$$

while for the imaginary roots and the Cartan subalgebra we put

$$\boldsymbol{\xi}(-1)|m\boldsymbol{\delta}\rangle \quad\leftrightarrow\quad \boldsymbol{\xi}(-1)_m, \tag{4.2.13}$$

where $m \in \mathbb{Z}$ and $\boldsymbol{\xi} \in \mathrm{span}_{\mathbb{R}}\{\mathbf{r}_1,\ldots,\mathbf{r}_8\}$ (so that $\boldsymbol{\xi}\cdot\boldsymbol{\delta} = \boldsymbol{\xi}\cdot\mathbf{r}_{-1} = 0$ as required). The only subtlety arises for the central element. To see this, let $\mathbf{r} = \boldsymbol{\alpha} + m\boldsymbol{\delta}$ and $\mathbf{s} = \boldsymbol{\beta} + n\boldsymbol{\delta}$ be two real roots of E_9. Then $\mathbf{r}\cdot\mathbf{s} = \boldsymbol{\alpha}\cdot\boldsymbol{\beta}$, and for the case $\mathbf{r}\cdot\mathbf{s} = -2$, i.e. $\boldsymbol{\alpha} = -\boldsymbol{\beta}$, we obtain $[|\mathbf{r}\rangle, |\mathbf{s}\rangle] = -\boldsymbol{\alpha}(-1)|(m+n)\boldsymbol{\delta}\rangle - m\delta_{m+n,0}K$, because $\boldsymbol{\delta}(-1)|k\boldsymbol{\delta}\rangle$ is a null physical state unless $k = 0$. We conclude that the Frenkel–Kac construction is bound to give a level 1 representation of E_9, whereas in the above construction the value of the central element is not specified as it is appropriate for E_{10}.

Let us now turn to the level 1 roots. Inspection of the inverse Cartan matrix shows that the only such roots in C are of the form

$$\boldsymbol{\Lambda} = k_{-1}\boldsymbol{\Lambda}_{-1} + \boldsymbol{\Lambda}_0 = \mathbf{r}_{-1} + (2+k_{-1})\boldsymbol{\delta}, \tag{4.2.14}$$

where $k_{-1} \in \mathbb{Z}_+$, corresponding to the DDF decomposition (4.1.14) with $\mathbf{a} = \mathbf{r}_{-1}$, $\mathbf{k} = -\boldsymbol{\delta}$ and $n = 2 + k_{-1}$. Since all these vectors are elements of the lattice, we can straightforwardly apply the DDF construction to obtain the physical states

$$A^{i_1}_{-m_1}\cdots A^{i_N}_{-m_N}|\mathbf{r}_{-1}\rangle, \tag{4.2.15}$$

where $m_1 + \ldots + m_N = 2 + k_{-1}$ and with the polarization vectors chosen as above. Recall that $A^i_{-m} \equiv \left(\boldsymbol{\xi}_i(-1)\mathrm{e}^{m\boldsymbol{\delta}}\right)_0$. Hence the above states correspond to the multiple commutators

$$\left[\boldsymbol{\xi}_{i_1}(-1)|m_1\boldsymbol{\delta}\rangle, \left[\ldots, \left[\boldsymbol{\xi}_{i_N}(-1)|m_N\boldsymbol{\delta}\rangle, |\mathbf{r}_{-1}\rangle\right]\ldots\right]\right] \quad \in E_{10}^{(A)}, \tag{4.2.16}$$

as we have already shown. Moreover, we can explicitly verify that they form part of the basic representation of E_9 with lowest weight vector $|\mathbf{r}_{-1}\rangle$. To see this we have to work out the commutators of the E_9 elements (4.2.1) – (4.2.5) with the level 1 states (4.2.15):

$$\begin{aligned}&\left[\boldsymbol{\eta}(-1)|\mathbf{0}\rangle, A^{i_1}_{-m_1}\cdots A^{i_N}_{-m_N}|\mathbf{r}_{-1}\rangle\right] \\ &\quad = \left[(m_{i_1} + \ldots + m_{i_N})\boldsymbol{\delta}\cdot\boldsymbol{\eta} + \mathbf{r}_{-1}\cdot\boldsymbol{\eta}\right]A^{i_1}_{-m_1}\cdots A^{i_N}_{-m_N}|\mathbf{r}_{-1}\rangle,\end{aligned} \tag{4.2.17}$$

$$\begin{aligned}
&\big[\boldsymbol{\xi}_j(-1)|n\boldsymbol{\delta}\rangle, A^{i_1}_{-m_1}\cdots A^{i_N}_{-m_N}|\mathbf{r}_{-1}\rangle\big] \\
&\quad = A^j_{-n}\left(A^{i_1}_{-m_1}\cdots A^{i_N}_{-m_N}|\mathbf{r}_{-1}\rangle\right) \\
&\quad = \begin{cases} -\sum_{k=1}^{N} n\delta_{j,i_k}\delta_{n,m_k}\prod_{l\neq k} A^{i_l}_{-m_l}|\mathbf{r}_{-1}\rangle & \text{if } n<0, \\ A^j_{-n}A^{i_1}_{-m_1}\cdots A^{i_N}_{-m_N}|\mathbf{r}_{-1}\rangle & \text{if } n>0, \end{cases}
\end{aligned} \tag{4.2.18}$$

$$\begin{aligned}
&\big[|\mathbf{s}\rangle, A^{i_1}_{-m_1}\cdots A^{i_N}_{-m_N}|\mathbf{r}_{-1}\rangle\big] \\
&\quad = -\sum_{k=1}^{N}(\mathbf{s}\cdot\boldsymbol{\xi}_{i_k})A^{i_1}_{-m_1}\cdots A^{i_{k-1}}_{-m_{k-1}}\big[|\mathbf{s}+m_k\boldsymbol{\delta}\rangle, A^{i_{k+1}}_{-m_{k+1}}\cdots A^{i_N}_{-m_N}|\mathbf{r}_{-1}\rangle\big] \\
&\quad\quad + A^{i_1}_{-m_1}\cdots A^{i_N}_{-m_N}\big[|\mathbf{s}\rangle, |\mathbf{r}_{-1}\rangle\big].
\end{aligned} \tag{4.2.19}$$

The first commutator tells us that the Cartan subalgebra of E_9 acts diagonally on the DDF states, giving the components $\boldsymbol{\eta}\cdot\boldsymbol{\Lambda}$ of a weight of the representation. In particular, we can verify that $|\mathbf{r}_{-1}\rangle$ has components $\mathbf{r}_{-1}\cdot\boldsymbol{\delta}=-1$, $\mathbf{r}_{-1}\cdot\boldsymbol{\xi}_i=0\ \forall i$, i.e. it is indeed basic. The second commutator which directly follows from the definitions (1.5.1) and (3.1.2), reveals that the E_9 elements corresponding to multiples of the null root $\boldsymbol{\delta}$ act by multiplication with a DDF operator. The last commutator is obtained by rewriting the DDF states (4.2.15) as multiple commutators and repeated application of the following version of the Jacobi identity (1.2.15):

$$\begin{aligned}
\big[|\mathbf{s}\rangle, A^i_{-m}\psi\big] &\equiv \big[|\mathbf{s}\rangle, [\boldsymbol{\xi}_i(-1)|m\boldsymbol{\delta}\rangle, \psi]\big] \\
&= \big[[|\mathbf{s}\rangle, \boldsymbol{\xi}_i(-1)|m\boldsymbol{\delta}\rangle], \psi\big] + \big[\boldsymbol{\xi}_i(-1)|m\boldsymbol{\delta}\rangle, [|\mathbf{s}\rangle, \psi]\big] \\
&= -m(\mathbf{s}\cdot\boldsymbol{\xi}_i)\big[|\mathbf{s}+m\boldsymbol{\delta}\rangle, \psi\big] + A^i_{-m}\big[|\mathbf{s}\rangle, \psi\big]
\end{aligned} \tag{4.2.20}$$

for any state ψ. Note that the above commutator $\big[|\mathbf{s}\rangle, |\mathbf{r}_{-1}\rangle\big]$ can be evaluated using (2.3.5). For example, it vanishes whenever s is a negative root of E_9; furthermore, we always get a level 1 state because s is a E_9 root. Since, according to (4.2.19), the state $|\mathbf{r}_{-1}\rangle$ commutes with all generators $f_i=-|-\mathbf{r}_i\rangle$, $i=-1,0,1,\ldots,8$, we conclude that it indeed represents a lowest weight vector for the basic representation of E_9. Weyl-equivalent level 1 states can be generated by Weyl conjugation with elements $w\in W$ leaving the level fixed, i.e. $w\in W(E_9)$. The tachyonic momentum $\mathbf{r}_{-1}$ is then mapped to a vector of the form (4.1.18) with $\mathbf{a}=w(\mathbf{r}_{-1})$, while $\boldsymbol{\delta}$ is left invariant. The polarizations used above must be replaced by the rotated polarization vectors $\boldsymbol{\xi}_{w(i)}:=w(\boldsymbol{\xi}_i)$ with corresponding changes in the DDF vectors. Denoting the rotated DDF operators by $A^{w(i)}_{-m}\equiv A^{w(\boldsymbol{\xi}_i)}_{-m}$, we obtain the new states

$$A^{w(i_1)}_{-m_1}\cdots A^{w(i_N)}_{-m_N}|w(\mathbf{r}_{-1})\rangle. \tag{4.2.21}$$

The so-called basic representation is then spanned by all elements of this form. Notice that although we are using transversal indices these now transform under different little groups (which are all conjugate to $W(E_8)$). The multiplicity formula for the level 0 and level 1 roots [54],

$$\mathrm{mult}(\boldsymbol{\Lambda}) = p_8(n) = p_8(1-\tfrac{1}{2}\boldsymbol{\Lambda}^2), \tag{4.2.22}$$

can be read off immediately from (4.2.1), (4.2.2) and (4.2.15). This multiplicity formula holds likewise for roots related by an arbitrary Weyl rotation to a level 1 root.

As already mentioned before, the states (4.2.15) transform covariantly under the corresponding little group $W(\mathbf{r}_{-1},\boldsymbol{\delta})$, which is just the Weyl group of E_8. Now it is known that $W(E_8)=D_4(2)\times(\mathbb{Z}_2)^2$, where $D_4(2)$ is the Chevalley group of order $2^{12}3^5 5^2 7$, or, equivalently, the set of $SO(8)$ matrices with entries in the field $\mathbb{Z}_2$ (see e.g. [15]). Since it is the maximal discrete subgroup of $SO(8)$ of this type in the sense that the little groups of all higher level roots will be much smaller, this also explains why the states (4.2.15) look "$SO(8)$ covariant" (although the polarization indices $i,j,\ldots$ should by no means be regarded as $SO(8)$ indices!). As we will see, the higher level root spaces will exhibit much less symmetry.

3. Higher level: generalities

Before turning to the discussion of an explicit example of a level 2 root space, we will explain the general ideas underlying the description of higher level elements in terms of the DDF construction. As we have already mentioned, the DDF states constitute a complete basis of physical states for any allowed momentum on the root lattice. Consequently, the root space $E^{(\Lambda)}_{10}$ is a (proper for $\ell\geq 1$) subspace of $\mathfrak{g}^{(\Lambda)}_{\Pi_{9,1}}$ for any root $\boldsymbol{\Lambda}$ (this inclusion is a special case of (2.3.31)). The physical states are explicitly given by (3.3.7) or, equivalently, by (3.2.18). Anticipating that the final results are somewhat simpler in terms of (3.2.18), we will use the basis

$$A^{i_1}_{-m_1}(\mathbf{a})\ldots A^{i_M}_{-m_M}(\mathbf{a})\mathcal{L}_{-n_1}(\mathbf{a})\ldots\mathcal{L}_{-n_N}(\mathbf{a})|\mathbf{a}\rangle, \tag{4.3.1}$$

explicitly indicating the dependence of the DDF operators and their polarizations on the tachyon momentum $\mathbf{a}$ and the associated lightlike vector $\mathbf{k}(\mathbf{a})=-\frac{1}{\ell}\boldsymbol{\delta}$, and assuming $n_i\geq 2$ from now on to exclude null states. For $\ell>1$, we have

$$A^i_{-m}(\mathbf{a}) \equiv \Big[\big(\boldsymbol{\xi}_i(\mathbf{a})\big)(-1)e^{\frac{m}{\ell}\boldsymbol{\delta}}\Big]_0 \tag{4.3.2}$$

with an extra factor of $\frac{1}{\ell}$ in the exponent, as appropriate for level ℓ by (4.1.15). In accordance with the DDF decomposition $\boldsymbol{\Lambda}=\mathbf{a}-n\mathbf{k}(\mathbf{a})$, the indices obey the sum rule $m_1+\ldots+m_M+n_1+\ldots+n_N=n$. We emphasize once more that neither $\mathbf{a}$ nor $\mathbf{k}(\mathbf{a})$ need be on the root lattice for $\ell>1$ any more. The problem of characterizing the root spaces of the hyperbolic Kac–Moody algebra is now no longer one of dividing out the Serre relations (2.3.25) (these are automatically taken care of by the vertex operator formalism as we pointed out already), but rather one of identifying the missing states which cannot be generated by multiple commutators of the Chevalley generators e_i or f_i. The above representation immediately yields the following upper bound on the root multiplicities [11]

$$\text{mult}(\mathbf{\Lambda}) \le p_9(1-\tfrac{1}{2}\mathbf{\Lambda}^2) - p_9(-\tfrac{1}{2}\mathbf{\Lambda}^2). \tag{4.3.3}$$

To effectively construct higher level elements we invoke Theorem 7 of Sect. 1. For instance, given a level 2 root $\mathbf{\Lambda}$ in the fundamental Weyl chamber C, we write

$$\mathbf{\Lambda} = \mathbf{r} + \mathbf{s} + m\boldsymbol{\delta}, \tag{4.3.4}$$

where $\mathbf{r}$ and $\mathbf{s}$ are real positive level 1 roots (i.e. containing the simple root $\mathbf{r}_{-1}$ exactly once and obeying $\mathbf{r}^2 = \mathbf{s}^2 = 2$). In general, there will be many different ways to split $\mathbf{\Lambda}$ in this manner, as well as different integers m. For fixed value of m, these decompositions are related by the little group, which leaves $\mathbf{\Lambda}$ and $\boldsymbol{\delta}$ fixed, but varies $\mathbf{r}$ and $\mathbf{s}$. Thus, we work with a fixed decomposition and then act on the resulting commutator states with the little group so as to obtain all possible states with the same value of m. The commutator to be computed is

$$\Big[A^{i_1}_{-m_1}(\mathbf{s})\dots A^{i_M}_{-m_M}(\mathbf{s})|\mathbf{s}\rangle\,,\,A^{j_1}_{-n_1}(\mathbf{r})\dots A^{j_N}_{-n_N}(\mathbf{r})|\mathbf{r}\rangle\Big]\,, \tag{4.3.5}$$

where $m_1 + \dots m_M + n_1 + \dots + n_N = m$. For the special example to be discussed below, this expression can be evaluated with the help of the formulas given in the appendices. Expanding it in terms of the basis (4.3.1), we arrive at

$$(4.3.5) = \sum_{\substack{p_1+\dots+q_Q=n\\ k_1,\dots,k_P}} c^{i_1\dots i_M j_1\dots j_N}_{k_1\dots k_P} A^{k_1}_{-p_1}(\mathbf{a})\dots A^{k_P}_{-p_P}(\mathbf{a})\mathcal{L}_{-q_1}(\mathbf{a})\dots\mathcal{L}_{-q_Q}(\mathbf{a})|\mathbf{a}\rangle \tag{4.3.6}$$

with the "Clebsch Gordan coefficients" $c^{i_1\dots i_M j_1\dots j_N}_{k_1\dots k_P}$, into which all the information about the missing states is encoded. For the Fock space states, this equality holds of course only modulo terms $\mathrm{L}_{(-1)}(\dots)$, which can however be ignored for the Lie algebra, as they are factored out by (1.5.11). (4.3.6) is the crucial formula, containing both transversal and longitudinal excitations[7]. For the calculations, we note that the polarization vectors $\boldsymbol{\xi}_i(\mathbf{r})$ and $\boldsymbol{\xi}_i(\mathbf{s})$ can always be chosen orthonormal and such that they agree for $i = 1,\dots,7$; from (4.3.4) we then see that $\boldsymbol{\xi}_i(\mathbf{a}) = \boldsymbol{\xi}_i(\mathbf{r})$ as well for these values of the indices. As for the remaining components $\boldsymbol{\xi}_8(\mathbf{r})$, $\boldsymbol{\xi}_8(\mathbf{s})$ and $\boldsymbol{\xi}_8(\mathbf{a})$, one can convince oneself that their differences are proportional to the null vector $\boldsymbol{\delta}$. Since such contributions drop out in the non-zero mode parts of the DDF operators (cf. the discussion after (1.5.4)), the respective DDF operators are really the same except for their zero mode parts and the crucial fact that their photon momenta depend on the level. We stress that this would not be true if the Weyl chamber contained more than one null direction.

Just as for the level 1 states, one can determine how the states (4.3.6) transform under E_9. Suppressing the label (a) on the DDF operators to make the formulas less cumbersome, this calculation requires the commutators

$$\begin{aligned}&\Big[\boldsymbol{\eta}(-1)|0\rangle, A^{i_1}_{-m_1}\dots A^{i_M}_{-m_M}\mathcal{L}_{-n_1}\dots\mathcal{L}_{-n_N}|\mathbf{a}\rangle\Big]\\ &\quad= [(\tfrac{1}{\ell}(m_1+\dots+n_N)\boldsymbol{\delta}\cdot\boldsymbol{\eta} + \mathbf{a}\cdot\boldsymbol{\eta}]A^{i_1}_{-m_1}\dots A^{i_M}_{-m_M}\mathcal{L}_{-n_1}\dots\mathcal{L}_{-n_N}|\mathbf{a}\rangle.\end{aligned} \tag{4.3.7}$$

The scalar product in parantheses is easily seen to reduce to $\boldsymbol{\eta}\cdot\mathbf{\Lambda}$, giving the components of a weight of the representation. Furthermore,

$$\begin{aligned}&\Big[\boldsymbol{\xi}_j(-1)|n\boldsymbol{\delta}\rangle, A^{i_1}_{-m_1}\dots A^{i_M}_{-m_M}\mathcal{L}_{-n_1}\dots\mathcal{L}_{-n_N}|\mathbf{a}\rangle\Big]\\ &\quad= A^{j}_{-\ell n}\left(A^{i_1}_{-m_1}\dots A^{i_M}_{-m_M}\mathcal{L}_{-n_1}\dots\mathcal{L}_{-n_N}|\mathbf{a}\rangle\right),\end{aligned} \tag{4.3.8}$$

(notice that the index on the first operator is $(-\ell n)$ rather than $(-n)$!) and

$$\begin{aligned}&\Big[|\mathbf{s}\rangle, A^{i_1}_{-m_1}\dots A^{i_M}_{-m_M}\mathcal{L}_{-n_1}\dots\mathcal{L}_{-n_N}|\mathbf{a}\rangle\Big]\\ &\quad= -\sum_{k=1}^{M}(\mathbf{s}\cdot\boldsymbol{\xi}_{i_k})A^{i_1}_{-m_1}\dots A^{i_{k-1}}_{m_{k-1}}\Big[|\mathbf{s}+\tfrac{1}{\ell}m_k\boldsymbol{\delta}\rangle, A^{i_{k+1}}_{m_{k+1}}\dots A^{i_M}_{m_M}\mathcal{L}_{-n_1}\dots\mathcal{L}_{-n_N}|\mathbf{a}\rangle\Big]\\ &\qquad+\sum_{l=1}^{N}(\mathbf{s}\cdot\mathbf{a})A^{i_1}_{-m_1}\dots A^{i_M}_{-m_M}\mathcal{L}_{-n_1}\dots\mathcal{L}_{n_{l-1}}\Big[|\mathbf{s}+\tfrac{1}{\ell}n_l\boldsymbol{\delta}\rangle, \mathcal{L}_{n_{l+1}}\dots\mathcal{L}_{n_N}|\mathbf{a}\rangle\Big]\\ &\qquad+A^{i_1}_{-m_1}\dots A^{i_M}_{-m_M}\mathcal{L}_{-n_1}\dots\mathcal{L}_{-n_N}\big[|\mathbf{s}\rangle, |\mathbf{a}\rangle\big].\end{aligned} \tag{4.3.9}$$

Observe that there are no contributions from the logarithmic terms in $\mathcal{L}_{-m}$ to the last commutator because $\mathbf{s}\cdot\boldsymbol{\delta} = 0$ for any E_9 root $\mathbf{s}$. The proof of these formulas is analogous to the proof of the corresponding formulas for the level 1 states in the previous section, save for the following important caveat. When building up the states from the tachyonic groundstate $|\mathbf{a}\rangle$ by successive application of the DDF operators, the intermediate states, whose momenta are not on the root lattice, do *not* belong to the Kac–Moody algebra, because the Lie bracket with arbitrary elements is in general not defined due to branch cuts in the relevant operator product expansions.[8] Therefore, the "commutators" in (4.3.9) are neither commutators in E_{10} nor even in the Lie algebra of physical states $\mathfrak{g}_\Lambda$; nevertheless, the above calculation does make sense because all relevant products of momenta are integer, and therefore the generic branch cuts are absent. So we must keep in mind that only the final result including summation according to (4.3.6) is an element

[7] This formula also shows why the fake monster Lie algebra $\mathfrak{g}_{II_{25,1}}$ (see Sect. V.1) is, in a certain sense, much simpler (though bigger) than E_{10}. The longitudinal components generated by commuting two transversal DDF states decouple in 26 dimensions, and therefore only the terms *without* longitudinal states survive in the expansion (4.3.6). To be sure, one must still prove that indeed *all* transversal states can be generated in this way if one takes into account the imaginary simple roots.

[8] We note that the cocycle conditions (2.1.12)–(2.1.19) can be solved on a rational extension of the root lattice [30].

of E_{10} again. The fact that the direct construction of the DDF states has no Lie algebra analog beyond level 1 explains the emergence of longitudinal states as well as the disappearance of certain transversal states.

Before turning to a concrete example, we sketch our method for dealing with general higher level elements. Given a level ℓ root $\mathbf{\Lambda}$ in the fundamental Weyl chamber C, we write

$$\mathbf{\Lambda} = \sum_{i=1}^{\ell} \mathbf{s}_i + m\boldsymbol{\delta}, \tag{4.3.10}$$

where the $\mathbf{s}_i$'s are real positive level 1 roots. Again, there will be many different ways to split $\mathbf{\Lambda}$ in this manner, but for fixed m, these decompositions are related by the little group which leaves $\mathbf{\Lambda}$ and $\boldsymbol{\delta}$ invariant, but varies the $\mathbf{s}_i$'s. The relevant commutators are

$$\begin{aligned}&\Big[A^{i_{1,1}}_{-m_{1,1}}(\mathbf{s}_1)\cdots A^{i_{1,M_1}}_{-m_{1,M_1}}(\mathbf{s}_1)|\mathbf{s}_1\rangle, \ldots \\ &\quad \ldots\Big[A^{i_{\ell-1,1}}_{-m_{\ell-1,1}}(\mathbf{s}_{\ell-1})\cdots A^{i_{\ell-1,M_{\ell-1}}}_{-m_{\ell-1,M_{\ell-1}}}(\mathbf{s}_{\ell-1})|\mathbf{s}_{\ell-1}\rangle, A^{i_{\ell,1}}_{-m_{\ell,1}}(\mathbf{s}_\ell)\cdots A^{i_{\ell,M_\ell}}_{-m_{\ell,M_\ell}}(\mathbf{s}_\ell)|\mathbf{s}_\ell\rangle\Big]\Big]\ldots\Big] \\ &= \sum_{\substack{p_1+\ldots+q_Q=n\\ k_1,\ldots,k_P}} c^{i_{1,1}\ldots i_{\ell,M_\ell}}_{k_1\ldots k_P} A^{k_1}_{-p_1}(\mathbf{a})\cdots A^{k_P}_{-p_P}(\mathbf{a})\mathcal{L}_{-q_1}(\mathbf{a})\cdots\mathcal{L}_{-q_Q}(\mathbf{a})|\mathbf{a}\rangle,\end{aligned} \tag{4.3.11}$$

where $\sum_{a=1}^{\ell}\sum_{\mu_a=1}^{M_a} m_{a,\mu_a} = m$. Expanding the multiple commutator in terms of the basis (4.3.1) appropriate to the DDF decomposition $\mathbf{\Lambda} = \mathbf{a} - n\mathbf{k}$, $\mathbf{k} = -\frac{1}{\ell}\boldsymbol{\delta}$, with $p_1 + \ldots + p_P + q_1 + \ldots + q_Q = n$, we arrive at the right hand side with the "level ℓ Clebsch Gordan coefficients" $c^{i_{1,1}\ldots i_{\ell,M_\ell}}_{k_1\ldots k_P}$, into which all the information about the missing states is encoded. In order to perform the actual computation of this $\ell - 1$-fold commutator, it is clear that we have to start from the most inner commutator and then successively work out the outer commutators. But since already the inner (level 2) commutator in general leads to longitudinal DDF states (see (4.3.6)), we shall effectively need to compute commutators between level 1 transversal DDF states and higher level general (transversal and longitudinal) DDF states. This is surely an additional unpleasant technical complication when we come to deal with elements of level $\ell > 2$.

In the next section, we shall apply the above considerations to a specific example and work out the "Clebsch–Gordan coefficients" in (4.3.6) for one non-trivial level 2 root, arriving at a complete description of its root space in terms of DDF states, which decompose into irreducible representations of the little group $W(\Lambda, \delta)$; as a by-product, we verify the multiplicity formula of [55] for a concrete example. The comparative simplicity of the representation obtained in this manner is perhaps best appreciated by noting that the number of Lie brackets needed to represent any of its elements in terms of Chevalley generators is equal to $(-\boldsymbol{\rho}\cdot\mathbf{\Lambda} - 1)$, where $\boldsymbol{\rho}$ is the Weyl vector.

4. A level 2 example: $\Lambda = \Lambda_7$

Any level 2 root in C must be of the form $\mathbf{\Lambda}_1 + n\boldsymbol{\delta}$ or $\mathbf{\Lambda}_7 + n\boldsymbol{\delta}$ or $2\mathbf{\Lambda}_0 + n\boldsymbol{\delta}$ for some $n \in \mathbb{N}$. We will here only discuss the root $\mathbf{\Lambda} = \mathbf{\Lambda}_7$, dual to the simple root $\mathbf{r}_7$. Explicitly, $\mathbf{\Lambda}_7$ is given by

$$\mathbf{\Lambda}_7 = \begin{bmatrix} & & & & & & 7 & & \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 9 & 4 \end{bmatrix} = (0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0\,|\,2), \tag{4.4.1}$$

so $\mathbf{\Lambda}_7^2 = -4$. Its decomposition into two level 1 tachyonic roots is $\mathbf{\Lambda}_7 = \mathbf{r} + \mathbf{s} + 2\boldsymbol{\delta}$, where

$$\mathbf{r} := \mathbf{r}_{-1} = \begin{bmatrix} & & & & & & 0 & & \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = (0,\,0,\,0,\,0,\,0,\,0,\,0,\,1,-1\,|\,0),$$

$$\mathbf{s} := \begin{bmatrix} & & & & & & 1 & & \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 0 \end{bmatrix} = (0,\,0,\,0,\,0,\,0,\,0,\,0,-1,-1\,|\,0).$$

Since $n = 1 - \frac{1}{2}\mathbf{\Lambda}_7^2 = 3$ we have the DDF decomposition $\mathbf{\Lambda}_7 = \mathbf{a} - 3\mathbf{k}$ where $\mathbf{k} := -\frac{1}{2}\boldsymbol{\delta}$ and

$$\mathbf{a} := \mathbf{r} + \mathbf{s} - \mathbf{k} = (0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,-\tfrac{3}{2}\,|\,\tfrac{1}{2}).$$

As expected, neither $\mathbf{k}$ nor $\mathbf{a}$ are elements of $II_{9,1}$. Nevertheless, since $\mathbf{a}\cdot\mathbf{k} = 1$, the action of the DDF operators $A^i_{-n}(\mathbf{k})$ on the tachyonic ground-state $|\mathbf{a}\rangle$ is perfectly well-defined as we already pointed out. As for the three sets of polarization vectors associated with the tachyon momenta $|\mathbf{r}\rangle$, $|\mathbf{s}\rangle$ and $|\mathbf{a}\rangle$, respectively, a convenient choice is

$$\begin{aligned}\boldsymbol{\xi}_\alpha &\equiv \boldsymbol{\xi}_\alpha(\mathbf{r}) = \boldsymbol{\xi}_\alpha(\mathbf{s}) = \boldsymbol{\xi}_\alpha(\mathbf{a}) \quad \text{for } \alpha = 1,\ldots,7\,, \\ \boldsymbol{\xi}_1 &:= (1,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0\,|\,0), \\ &\;\;\vdots \\ \boldsymbol{\xi}_7 &:= (0,\,0,\,0,\,0,\,0,\,0,\,1,\,0,\,0\,|\,0); \\ \boldsymbol{\xi}_8(\mathbf{r}) &:= (0,\,0,\,0,\,0,\,0,\,0,\,0,\,1,\,1\,|\,1), \\ \boldsymbol{\xi}_8(\mathbf{s}) &:= (0,\,0,\,0,\,0,\,0,\,0,\,0,-1,\,1\,|\,1), \\ \boldsymbol{\xi}_8 \equiv \boldsymbol{\xi}_8(\mathbf{a}) &:= (0,\,0,\,0,\,0,\,0,\,0,\,0,\,1,\,0\,|\,0).\end{aligned} \tag{4.4.2}$$

The little group is $W(\mathbf{\Lambda}_7, \boldsymbol{\delta}) = W(D_8) = S_8 \ltimes (\mathbb{Z}_2)^7$ of order $2^{14}3^15^17^1$. This group is generated by the fundamental reflections $\{w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_8\}$. On the polarization vectors $\boldsymbol{\xi}_i(\mathbf{a})$ it acts as follows:

$$\begin{aligned} w_0(\boldsymbol{\xi}_7) &= \boldsymbol{\xi}_8 \quad, & w_0(\boldsymbol{\xi}_8) &= \boldsymbol{\xi}_7, \\ w_1(\boldsymbol{\xi}_6) &= \boldsymbol{\xi}_7 \quad, & w_1(\boldsymbol{\xi}_7) &= \boldsymbol{\xi}_6, \\ w_2(\boldsymbol{\xi}_5) &= \boldsymbol{\xi}_6 \quad, & w_2(\boldsymbol{\xi}_6) &= \boldsymbol{\xi}_5,\end{aligned}$$

$$\begin{aligned} w_3(\xi_4) &= \xi_5 \quad , \quad w_3(\xi_5) = \xi_4, \\ w_4(\xi_3) &= \xi_4 \quad , \quad w_4(\xi_4) = \xi_3, \\ w_5(\xi_2) &= \xi_3 \quad , \quad w_5(\xi_3) = \xi_2, \\ w_6(\xi_1) &= -\xi_2 \quad , \quad w_6(\xi_2) = -\xi_1, \\ w_8(\xi_1) &= \xi_2 \quad , \quad w_8(\xi_2) = \xi_1, \end{aligned} \tag{4.4.3}$$

and as the identity on all those that have not been listed. Furthermore, $\{w_1, \ldots, w_6, w_8\}$ leave $\mathbf{r}$ and $\mathbf{s}$ invariant, whereas

$$w_0(\mathbf{r}) = \mathbf{r} + \mathbf{r}_0 \quad , \quad w_0(\mathbf{s}) = \mathbf{s} - \mathbf{r}_0 .$$

The Weyl group element $w = w_0 w_1 w_2 w_3 w_4 w_5 w_8 w_6 w_5 w_4 w_3 w_2 w_1 w_0$ interchanges $\mathbf{r}$ and $\mathbf{s}$.

There are three sets of DDF operators acting on different tachyonic ground states $|\mathbf{r}\rangle$, $|\mathbf{s}\rangle$, and $|\mathbf{a}\rangle$, respectively. Now, since $\mathfrak{g}^{(\Lambda_7)}_{II_{9,1}}$ is spanned by the 192 transversal and the 9 longitudinal DDF states

$$\begin{aligned} A^i_{-1}(\mathbf{a})A^j_{-1}(\mathbf{a})A^k_{-1}(\mathbf{a})|\mathbf{a}\rangle \ , \\ A^i_{-2}(\mathbf{a})A^j_{-1}(\mathbf{a})|\mathbf{a}\rangle \ , \\ A^i_{-3}(\mathbf{a})|\mathbf{a}\rangle \ , \\ A^i_{-1}(\mathbf{a})\mathcal{L}_{-2}(\mathbf{a})|\mathbf{a}\rangle \ , \\ \mathcal{L}_{-3}(\mathbf{a})|\mathbf{a}\rangle \ , \end{aligned}$$

we can express any element of the root space $E_{10}^{(\Lambda_7)}$ as a linear combination of the above elements modulo $\mathrm{L}_{(-1)}$ terms. This is done by using the formulas from the appendices and solving the resulting (overdetermined!) systems of linear equations for the coefficients. We will suppress in our notation the dependence of the DDF operators on the tachyon momenta. *In the following we adopt the convention that DDF operators are always understood to be associated with the tachyons on which they act.* Hence the DDF operators occurring on the left hand side and on the right hand side of the formulas below are *not* the same. In Eq. (4.4.4), for example, we have $A^\alpha_{-1} \equiv A^\alpha_{-1}(\mathbf{r})$ on the left hand side but $A^\alpha_{-1} \equiv A^\alpha_{-1}(\mathbf{a})$ on the right hand side. Here are our results:

$$\begin{aligned} \big[|\mathbf{s}\rangle, A^\alpha_{-1}A^\beta_{-1}|\mathbf{r}\rangle\big] = \epsilon\Big\{ &-\tfrac12 A^\alpha_{-2}A^\beta_{-1} - \tfrac12 A^\beta_{-2}A^\alpha_{-1} - A^\alpha_{-1}A^\beta_{-1}A^8_{-1} \\ &+ \tfrac{1}{24}\delta^{\alpha\beta}\big[A^8_{-3} + 3A^8_{-1}\mathcal{L}_{-2} - 4A^8_{-1}A^8_{-1}A^8_{-1}\big]\Big\}|\mathbf{a}\rangle, \end{aligned} \tag{4.4.4}$$

$$\big[|\mathbf{s}\rangle, A^\alpha_{-1}A^8_{-1}|\mathbf{r}\rangle\big] = \epsilon\Big\{\tfrac14 A^\alpha_{-3} + \tfrac12 A^\alpha_{-2}A^8_{-1} - \tfrac12 A^8_{-2}A^\alpha_{-1} - \tfrac14 A^\alpha_{-1}\mathcal{L}_{-2}\Big\}|\mathbf{a}\rangle, \tag{4.4.5}$$

$$\big[|\mathbf{s}\rangle, A^8_{-1}A^8_{-1}|\mathbf{r}\rangle\big] = \epsilon\Big\{\tfrac{17}{24} A^8_{-3} + A^8_{-2}A^8_{-1} + \tfrac18 A^8_{-1}\mathcal{L}_{-2} + \tfrac16 A^8_{-1}A^8_{-1}A^8_{-1}\Big\}|\mathbf{a}\rangle, \tag{4.4.6}$$

$$\big[|\mathbf{s}\rangle, A^\alpha_{-2}|\mathbf{r}\rangle\big] = \epsilon\Big\{-\tfrac34 A^\alpha_{-3} - \tfrac14 A^\alpha_{-1}\mathcal{L}_{-2} + A^\alpha_{-1}A^8_{-1}A^8_{-1}\Big\}|\mathbf{a}\rangle, \tag{4.4.7}$$

$$\big[|\mathbf{s}\rangle, A^8_{-2}|\mathbf{r}\rangle\big] = \epsilon\Big\{-\tfrac12 A^8_{-3} + \tfrac12 A^8_{-1}\mathcal{L}_{-2}\Big\}|\mathbf{a}\rangle, \tag{4.4.8}$$

$$\begin{aligned} \big[A^\alpha_{-1}|\mathbf{s}\rangle, A^\beta_{-1}|\mathbf{r}\rangle\big] = \epsilon\Big\{ &-\tfrac12 A^\alpha_{-2}A^\beta_{-1} + \tfrac12 A^\beta_{-2}A^\alpha_{-1} + A^\alpha_{-1}A^\beta_{-1}A^8_{-1} \\ &- \tfrac{1}{24}\delta^{\alpha\beta}\big[A^8_{-3} + 3A^8_{-1}\mathcal{L}_{-2} - 4A^8_{-1}A^8_{-1}A^8_{-1}\big]\Big\}|\mathbf{a}\rangle, \end{aligned} \tag{4.4.9}$$

$$\big[A^\alpha_{-1}|\mathbf{s}\rangle, A^8_{-1}|\mathbf{r}\rangle\big] = \epsilon\Big\{-\tfrac14 A^\alpha_{-3} + \tfrac12 A^\alpha_{-2}A^8_{-1} + \tfrac12 A^8_{-2}A^\alpha_{-1} + \tfrac14 A^\alpha_{-1}\mathcal{L}_{-2}\Big\}|\mathbf{a}\rangle, \tag{4.4.10}$$

$$\big[A^8_{-1}|\mathbf{s}\rangle, A^8_{-1}|\mathbf{r}\rangle\big] = \epsilon\Big\{\tfrac{17}{24} A^8_{-3} + \tfrac18 A^8_{-1}\mathcal{L}_{-2} + \tfrac16 A^8_{-1}A^8_{-1}A^8_{-1}\Big\}|\mathbf{a}\rangle, \tag{4.4.11}$$

for $\alpha, \beta = 1, \ldots, 7$ and with $\epsilon \equiv \epsilon(\mathbf{s},\mathbf{r})\epsilon(\mathbf{k},\mathbf{a})$; contributions involving $\mathrm{L}_{(-1)}(\ldots)$ have been neglected in accordance with (1.5.11) (these extra terms are listed in Appendix C). Let us also record the following simple formula, which is an immediate consequence:

$$-(-1)^{\delta_{j8}+\delta_{i8}}\big[A^i_{-1}|\mathbf{s}\rangle, A^j_{-1}|\mathbf{r}\rangle\big] - (-1)^{\delta_{i8}}\big[|\mathbf{s}\rangle, A^i_{-1}A^j_{-1}|\mathbf{r}\rangle\big] = A^i_{-2}A^j_{-1}|\mathbf{a}\rangle. \tag{4.4.12}$$

Further careful analysis of the above results and use of the little Weyl group action (4.4.3) finally reveals that the following states form a complete basis of the root space $E_{10}^{(\Lambda_7)}$ (no summation convention!):

$$\begin{aligned} A^i_{-2}A^j_{-1}|\mathbf{a}\rangle \quad & \text{for } i, j \text{ arbitrary}, \\ A^i_{-1}A^j_{-1}A^k_{-1}|\mathbf{a}\rangle \quad & \text{for } i \neq j \neq k \neq i, \\ \big(A^i_{-3} - A^i_{-1}A^j_{-1}A^j_{-1}\big)|\mathbf{a}\rangle \quad & \text{for } i \neq j, \\ \big(5A^i_{-3} + A^i_{-1}A^i_{-1}A^i_{-1}\big)|\mathbf{a}\rangle \quad & \text{for } i \text{ arbitrary}, \\ \big(A^i_{-3} - A^i_{-1}\mathcal{L}_{-2}\big)|\mathbf{a}\rangle \quad & \text{for } i \text{ arbitrary}. \end{aligned} \tag{4.4.13}$$

Remarkably, this choice is consistent with the above eight commutator equations and their Weyl rotated analogs thereby proving the viability of our method. Altogether, we get $64 + 2{\cdot}56 + 2{\cdot}8 = 192$ states in agreement with the formula (4.1.11) predicting $\xi(3) = 192$ [55]. Despite the fact that this number coincides with the number of transversal states, our result explicitly shows the appearance of longitudinal as well as the disappearance of some transversal states. The above states form irreducible representations of the little group, whose action on the polarizations can be determined from (4.4.3) in a straightforward fashion; in particular, the longitudinal DDF operator is inert under the little Weyl group. We note that the states (4.4.13) do not even look "$SO(8)$ covariant" any more, unlike the level 1 states (4.2.15).

Having a complete description of the root space $E_{10}^{(\Lambda_7)}$, we can now in principle explore root spaces associated with other level 2 roots of the form $\mathbf{\Lambda} = \mathbf{\Lambda}_7 + n\delta$ (i.e. the **root string** associated with $\mathbf{\Lambda}_7$) by commuting the states (4.4.13) with the E_9 elements (4.2.2). From (4.3.8) it is evident that all states obtained by acting with a product $A^{i_1}_{-2m_1}\cdots A^{i_M}_{-2m_M}$ on any of the states (4.4.13) belong to the root space of $\mathbf{\Lambda} = \mathbf{\Lambda}_7 + (m_1 + \ldots m_M)\delta$ (note that each operator $A^i_{-2m}(\mathbf{a})$ shifts the momentum by $m\delta$!). However, it is also clear that we cannot obtain all root space elements in this way. For this, it is necessary to calculate DDF commutators of the form (4.3.5).

An alternative, more elucidating way might be to consider the action of the Sugawara generators defined by

$$\mathrm{L}_m^{\mathrm{Sug.}} := \frac{1}{2(\ell + h^\vee)} \left\{ \sum_{n\in\mathbb{Z}} \sum_{i=1}^{8} :A_n^i A_{m-n}^i: + \sum_{\mathbf{s}\in\Delta^{\mathrm{real}}(E_9)} {}^\times_\times \mathrm{ad}_{|\mathbf{s}\rangle}\, \mathrm{ad}_{|-\mathbf{s}-m\delta\rangle} {}^\times_\times \right\} \qquad (4.4.14)$$

on the states (4.4.13); here, $h^\vee = 30$ is the dual Coxeter number of E_8, the level is $\ell = 2$, and the normal-ordering of the operators $\mathrm{ad}_{|\mathbf{r}\rangle} \equiv (\mathrm{e}^{\mathbf{r}})_0$ is chosen as

$${}^\times_\times \mathrm{ad}_{|\mathbf{s}+m\delta\rangle}\, \mathrm{ad}_{|\mathbf{t}+n\delta\rangle} {}^\times_\times := \begin{cases} \mathrm{ad}_{|\mathbf{s}+m\delta\rangle}\, \mathrm{ad}_{|\mathbf{t}+n\delta\rangle} & \text{if } m \geq n, \\ \mathrm{ad}_{|\mathbf{t}+n\delta\rangle}\, \mathrm{ad}_{|\mathbf{s}+m\delta\rangle} & \text{if } m < n, \end{cases} \qquad (4.4.15)$$

for E_8 roots $\mathbf{s}, \mathbf{t}$ and $m, n \in \mathbb{Z}$. It is now not difficult to check that

$$\mathrm{L}_m^{\mathrm{Sug.}} |\mathbf{a}\rangle = 0 \qquad (4.4.16)$$

for $m \geq 1$. Furthermore, when evaluating $\mathrm{L}_0^{\mathrm{Sug.}}$ on the ground state $|\mathbf{a}\rangle$, only the term with $A_0^8 A_0^8$ contributes in the sum due to our choice of polarization vectors. With $A_0^8|\mathbf{a}\rangle = -2|\mathbf{a}\rangle$, we thus obtain

$$\mathrm{L}_0^{\mathrm{Sug.}} |\mathbf{a}\rangle = \tfrac{1}{16} |\mathbf{a}\rangle, \qquad (4.4.17)$$

showing that the state $|\mathbf{a}\rangle$ is a highest weight vector of weight $h = \frac{1}{16}$ for the level 2 Sugawara generators. In accordance with the remarks in the footnote on page 41, we therefore expect these states to belong to the irreducible Virasoro module with $c = \frac{1}{2}$ and $h = \frac{1}{16}$. The problem that remains is to relate the Sugawara generators to the longitudinal DDF operators. If this can be done, a completely explicit description of *all* level 2 root spaces is within reach.

5. Discussion

At this point we want to address the issue of missing states in more detail. Due to Eq. (3.3.12), tachyonic and photonic physical states are necessarily transversal, so that

$$\mathfrak{g}_{II_{9,1}}^{(\Lambda)} \equiv E_{10}^{(\Lambda)} \quad \text{for } \mathbf{\Lambda}^2 \geq 0 \qquad (4.5.1)$$

(of course, for $\mathbf{\Lambda}^2 > 2$, both spaces are empty). This means that there are no missing states for $\mathbf{\Lambda}^2 \geq 0$. But already for the massive spin 2 states, we encounter one longitudinal physical state that surely does not belong to the Kac–Moody algebra E_{10}. It is clear that there is only one weight in C of norm -2, namely the fundamental weight $\mathbf{\Lambda}_0 = \mathbf{r}_{-1} + 2\delta$. Since the latter is a level 1 element, which we know to occur in E_{10} just with transversal polarizations, we infer that the longitudinal state $\mathcal{L}_{-2}|\mathbf{r}_{-1}\rangle$ is not contained in the root space $E_{10}^{(\Lambda_0)}$ and thus represents a missing state, so

$$\mathfrak{g}_{II_{9,1}}^{(\Lambda_0)} \equiv E_{10}^{(\Lambda_0)} \oplus \mathbb{R} \cdot \mathcal{L}_{-2}|\mathbf{r}_{-1}\rangle. \qquad (4.5.2)$$

Acting with the full Weyl group on the missing state, we obtain the associated orbit of missing states in E_{10}. Our detailed analysis of the root space for $\mathbf{\Lambda}_7$ in the last section enables us to discuss the case of norm -4, for $\mathbf{\Lambda}_7$ is the only weight in the fundamental Weyl chamber with this property. From the multiplicity formula we learn that there have to be $201 - 192 = 9$ missing states, and in view of our DDF basis (4.4.13) we write

$$\mathfrak{g}_{II_{9,1}}^{(\Lambda_7)} \equiv E_{10}^{(\Lambda_7)} + \mathrm{span}_{\mathbb{R}}\{\mathcal{L}_{-3}|\mathbf{a}\rangle;\ A^i_{-3}|\mathbf{a}\rangle, i = 1, \ldots, 8\}, \qquad (4.5.3)$$

which can be also acted on with $W(E_{10})$ to find its analogue in other chambers.

The above formulas naturally suggest two ways of how to proceed. If we are primarily interested in E_{10}, we could try to systematize the way of splitting of $\mathfrak{g}_{II_{9,1}}$ into E_{10} states and missing states. In other words, we are seeking a mechanism which satisfactorily answers the following question: How do the missing states decouple from the E_{10} states? That this idea is not far-fetched shows the example of the 26-dimensional bosonic string. There we separate the longitudinal physical states from the transversal ones by introducing a positive semidefintite contravariant bilinear form which renders the former states to be null physical states. If one prefers the modern cohomological treatment then the decoupling is furnished by the nilpotent BRST operator and its cohomology. Thus we may rephrase the above question as: *Is there a cohomology describing how E_{10} sits inside the Lie algebra of physical states, $\mathfrak{g}_{II_{9,1}}$?*

The other point of view, as advocated by Borcherds, involves a generalization of the framework of Kac–Moody algebras. We know how a large part of $\mathfrak{g}_{II_{9,1}}$, namely the E_{10} part, can be formulated in terms of generators and relations. The idea then is to extend this approach to the whole Lie algebra. We would have to find an additional set of Chevalley generators which, when adjoined to the generators for E_{10}, produce all physical states as multiple commutators.

For example, we certainly have to add $\mathcal{L}_{-2}|\mathbf{r}_{-1}\rangle$ as such a new generator. This amounts to saying that $\mathbf{\Lambda}_0$ constitutes an imaginary simple root with multiplicity 1. Kac–Moody algebras allowing for imaginary ($\equiv$ nonpositive norm) simple roots were invented by Borcherds [7]. We will discuss them in the following chapter. So far, the introduction of this new generator seems to be very natural and appealing, but the second step of the procedure is subtle and becomes cumbersome when repeatedly done. In order to decide which missing states for the case of $\mathbf{\Lambda}_7$ have to be chosen as new generators, we need to take into account the previous additional generator $\mathcal{L}_{-2}|\mathbf{r}_{-1}\rangle$. Thus we ought to calculate the commutator $[|\mathbf{s}\rangle, \mathcal{L}_{-2}|\mathbf{r}\rangle]$ and express it in terms of the DDF basis for $\mathfrak{g}_{II_{9,1}}^{(\Lambda_7)}$ to see which missing states now do appear. We have not completed this calculation yet, for we are mainly interested in E_{10} itself and hence focus on the first approach. Alternatively, it is also possible to determine recursively the imaginary simple roots by anticipating $\mathfrak{g}_{II_{9,1}}$ as a Borcherds algebra and then plug its well-known root multiplicities, $p_9(1-\frac{1}{2}\mathbf{r}^2) - p_9(-\frac{1}{2}\mathbf{r}^2)$, into the Weyl–Kac–Borcherds denominator formula [7].

Although it is possible in principle (with some effort) to extend our discussion to other hyperbolic Kac–Moody algebras, the following points must be kept in mind. Our method may not apply to strictly hyperbolic algebras, which by definition have no affine, but only finite subalgebras, because their associated Weyl chambers contain no null vectors (i.e. they lie entirely within the light-cone), so the DDF operators cannot be defined. On the other hand, the Weyl chambers of arbitrary Kac–Moody algebras of indefinite type generically contain several linearly independent null directions, a feature that will greatly complicate (if not vitiate) the application of our method, because one must then deal with at least two different sets of photon momenta for the DDF operators. Moreover, if the algebra contains more than one regular affine subalgebra, the level of a root is no longer uniquely defined; for indefinite algebras, which are not hyperbolic, it is not even clear whether this notion can be sensibly defined at all. We thus begin to understand the possible significance of the fact that the fundamental Weyl chamber of E_{10} touches the light-cone at precisely one edge.

After all, what did we learn from our analysis of the root space $E_{10}^{(\Lambda_7)}$ and how may it be relevant for other hyperbolic Kac–Moody algebras? Our approach suggests that root spaces of E_{10} and other algebras of that type carry an additional structure called polarization; this differs from the conventional point of view that a root space is essentially, up to its dimension, a black box. The DDF framework, as developed here, provides adequate tools for the analysis of the complicated structure of hyperbolic algebras.

In particular, we now have a deeper understanding why Frenkel's conjecture [25] is wrong. Inspired by the example of the 26-dimensional bosonic string and the results about the canonical hyperbolic extension of $\mathfrak{su}(2)$ [23], he conjectured that for every hyperbolic algebra $\mathfrak{g}$ of rank d one has, for any root $\mathbf{r}$, $\dim \mathfrak{g}^{(\mathbf{r})} \leq p_{d-2}(1-\frac{1}{2}\mathbf{r}^2)$ as an upper bound. This conjecture was disproved in [55] by establishing the level 2 multiplicity formula for E_{10} as a counterexample. We argue that, firstly, the 26-dimensional bosonic string represents a rather untypical example, because there the longitudinal states span the radical of the contravariant bilinear form which is divided out. Hence only transversal states survive and we end up indeed with the exact multiplicity formula $p_{24}(1-\frac{1}{2}\mathbf{r}^2)$. In the generic case, on the other hand, the longitudinal states do appear as Lie algebra elements.

In terms of the DDF realization the following picture emerges. At level 0 and level 1 we naturally obtain all transversal states giving the affine subalgebra and its basic representation, respectively. By commuting transversal level 1 states, which is necessary for generating higher level elements, we cannot escape from producing longitudinal states, too. Hence there is no reason to expect a connection between higher level root multiplicities and the formula $p_{d-2}(1-\frac{1}{2}\mathbf{r}^2)$, which just counts the number of transversal states. Of course, we start off from the transversal level 1 states, but the more commutators we take between them the more subtle the mixture of longitudinal and transversal states becomes.

For example, look at the canonical hyperbolic extension of $\mathfrak{su}(2)$ whose level 2 root multiplicities coincide with the number of transversal states, $p_1(1-\frac{1}{2}\mathbf{r}^2)$, up to $\mathbf{r}^2 \leq -36$ (see [54, Table H_3]) and then drop below this bound. We conjecture that, when we perform the DDF construction for this example, we shall see at level 2 from the very beginning longitudinal states to appear and transversal states to be missed, even though the multiplicity superficially suggests the existence of transversal states alone. For higher levels we predict an increasing mixing of longitudinal and transversal states which manifests itself in an increasing deviation of the multiplicities from the number of transversal states. Thus the DDF analysis of a single level 2 root space E_{10} allows us to make some reasonable predictions for the stucture of other hyperbolic algebras of that type.

CHAPTER V:

THE FAKE MONSTER LIE ALGEBRA AND OTHER BORCHERDS ALGEBRAS

When we start to proceed along the lines in the last chapter with the analysis of the Lie algebra of physical states for the vertex algebra associated to the unique 26-dimensional even unimodular lattice $II_{25,1}$, then a few things dramatically change. First, it is well-known that only in 26 dimensions the longitudinal states all decouple as they become null physical states. Secondly, an infinite number of simple roots is required in order to generate the full reflection group of the lattice, i.e. the fundamental Weyl chamber possesses infinitely many walls. Moreover, it turns out that the set of simple roots as such is isometric to the famous Leech lattice. Finally, in 26 dimensions the Weyl vector becomes lightlike so that it cannot be written as a linear combination of the simple roots. Thus the critical dimension for the toroidally compactified bosonic string manifests itself in several surprising ways.

The Kac–Moody algebra $\mathfrak{g}(A)$ we shall encounter here is the Lie algebra L_∞ of infinite rank, which is a proper subalgebra of the celebrated fake monster Lie algebra $\mathfrak{g}_{II_{25,1}}$. The latter is obtained from the former by adjoining the positive multiples of the Weyl vector as imaginary simple roots [9]. Historically, this was the first example of a Borcherds algebra. Hence these generalized Kac–Moody algebras naturally emerge in string theory and may ultimately be relevant for the (hypothetical) fundamental string symmetry algebra.

After discussing the fake monster Lie algebra in Sect. 1, we shall present some formal aspects of the theory of Borcherds algebras in Sect. 2. Section 3 will be devoted to the study of some simple examples, and in Sect. 4 we shall prove a remarkable theorem about certain representations for a class of Borcherds algebras. In Sect. 5 some speculations concerning possible applications in physics will be presented.

1. The fake monster Lie algebra

Let us now consider the unique 26-dimensional even unimodular Lorentzian lattice $II_{25,1}$, which can be taken to be the lattice of all points $\mathbf{x} = (x_1, \ldots, x_{25}|x_0)$ for which the x_μ's are all in $\mathbb{Z}$ or all in $\mathbb{Z} + \frac{1}{2}$ and which have integer inner product with the vector $\mathbf{l} = (\frac{1}{2}, \ldots, \frac{1}{2}|\frac{1}{2})$, all norms and inner products being evaluated in the Minkowskian metric $\mathbf{x}^2 = x_1^2 + \ldots + x_{25}^2 - x_0^2$ (cf. [72]). In physics this corresponds to an open bosonic string moving in 26-dimensional space–time compactified on a torus so that the momenta lie on a lattice.

Calculations in connection with the automorphism group of $II_{25,1}$ show, that a set of positive norm simple roots for $II_{25,1}$ is given by the subset of vectors $\mathbf{r}$ in $II_{25,1}$ for which $\mathbf{r}^2 = 2$ and $\mathbf{r}\cdot\boldsymbol{\rho} = -1$, where the Weyl vector is $\boldsymbol{\rho} = (0, 1, 2, \ldots, 24|70)$ with $\boldsymbol{\rho}^2 = 0$ (cf. [18], [16]). These simple roots generate the reflection group of $II_{25,1}$, where the reflection $\sigma_\mathbf{r}$ associated with a root $\mathbf{r}$ is defined as usual by $\sigma_\mathbf{r}(\mathbf{x}) = \mathbf{x} - \frac{2}{\mathbf{r}^2}(\mathbf{x}\cdot\mathbf{r})\mathbf{r}$. Unfortunately we cannot provide a complete list of the simple roots, for it turns out that there are infinitely many! This is in striking contrast to the 10-dimensional case, for example, where only 10 simple roots were required. Here the fundamental Weyl chamber has infinitely many walls (and edges) which is quite hard to visualize. But this is not the only peculiarity about the set of simple roots. We shall call the positive norm simple roots of $II_{25,1}$ **Leech roots**, since Conway has shown that this subset is indeed isometric to the Leech lattice, the unique 24-dimensional even unimodular Euclidean lattice with no vectors of square length 2. Consequently, we shall therefore write Λ_{Leech} for the set of Leech roots in $II_{25,1}$. For further informations about the Leech lattice which is one of the 24 so-called Niemeier lattices, the reader may wish to consult [67], [17], [19] and [8]. So we encounter here the strange feature that the simple roots themselves carry a lattice structure, which has nothing to do with their $\mathbb{Z}$ span, i.e. the root lattice they span.

With a set of simple roots at hand, we can nevertheless proceed along the lines of Sect. II.3 and introduce the Kac–Moody algebra $\mathfrak{g}(A)$ associated with the set of Leech roots. We define a Kac–Moody algebra L_∞, of infinite dimension and rank, as follows (see [12]): L_∞ has three generators $e(\mathbf{r})$, $f(\mathbf{r})$, $h(\mathbf{r})$ for each Leech root $\mathbf{r}$, and is presented by the relations

$$[h(\mathbf{r}), h(\mathbf{s})] = 0, \tag{5.1.1}$$

$$[h(\mathbf{r}), e(\mathbf{s})] = (\mathbf{r}\cdot\mathbf{s})e(\mathbf{s}), \tag{5.1.2}$$

$$[h(\mathbf{r}), f(\mathbf{s})] = -(\mathbf{r}\cdot\mathbf{s})f(\mathbf{s}), \tag{5.1.3}$$

$$[e(\mathbf{r}), f(\mathbf{s})] = \delta_{\mathbf{r},\mathbf{s}}h(\mathbf{r}), \tag{5.1.4}$$

$$(\mathrm{ad}\, e(\mathbf{s}))^{1-\mathbf{r}\cdot\mathbf{s}}e(\mathbf{r}) = 0 \quad \text{for } \mathbf{r} \neq \mathbf{s}, \tag{5.1.5}$$

$$(\mathrm{ad}\, f(\mathbf{s}))^{1-\mathbf{r}\cdot\mathbf{s}}f(\mathbf{r}) = 0 \quad \text{for } \mathbf{r} \neq \mathbf{s}, \tag{5.1.6}$$

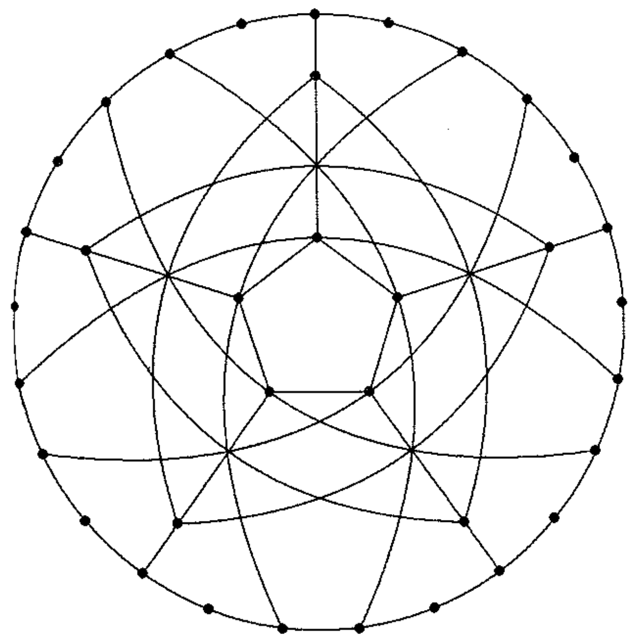


Fig. 1: Part of the Dynkin diagram of L_∞

where $\mathbf{r}$ and $\mathbf{s}$ are Leech roots. In the Coxeter–Dynkin diagram for L_∞ two nodes $\mathbf{r}$,$\mathbf{s}$ are joined by $-\mathbf{r}\cdot\mathbf{s}$ lines and a portion of the (infinite) graph looks like Fig. 1 (cf. [16]).

The full diagram has one node for each Leech lattice vector so that it will exhibit a huge symmetry, namely the goup of Leech lattice automorphisms. More precisely, the group of graph automorphisms for L_∞ is given by the Conway goup $\mathrm{Co}_\infty = \cdot\infty$, which is the group of affine automorphisms of the Leech lattice, i.e. the infinite group of all distance-preserving transformations that take the lattice to itself. Clearly, the latter is obtained by adjoining the translations in lattice vectors to the Conway group $\mathrm{Co}_0 = \cdot 0 = \mathrm{Aut}\,(\Lambda_{\mathrm{Leech}})$, the (finite) group of isometries that fix the origin and take the lattice to itself. Hence the transformations in Co_0 may be represented by orthogonal matrices. We conclude that the group of autochronous automorphisms of the lattice $\mathit{II}_{25,1}$ is given by the semidirect product (cf. [20])

$$\mathrm{Aut}\,(\mathit{II}_{25,1})^+ = W(L_\infty)\rtimes\mathrm{Co}_\infty . \tag{5.1.7}$$

Note that the group of graph automorphisms, Co_∞, is infinite and acts transitively on the set of simple roots, while in the case of E_{10}, for example, not a single nontrivial graph automorphism exists.

Let us return to the vertex algebra $(\mathcal{F}, \mathcal{V}, \mathbf{1}, \boldsymbol{\omega})$ associated with the lattice $\mathit{II}_{25,1}$. As already mentioned in Sect. I.5, there are additional null physical states besides the space $\mathrm{L}_{(-1)}\mathcal{P}_{(0)}$, if and only if the central charge takes the critical value $c = 26$, namely the space $(\mathrm{L}_{(-2)}+\frac{3}{2}\mathrm{L}^2_{(-1)})\mathcal{P}_{(-1)}$. Hence this space has to be divided out, too, so that the Lie algebra of physical states is given by

$$\mathfrak{g}_{\mathit{II}_{25,1}} := \mathcal{P}_{(1)}\big/\mathrm{rad}(\_|\_)_0 , \tag{5.1.8}$$

where $\mathrm{rad}(\_|\_)_0$ denotes the radical ($\equiv$ null space) of the contravariant bilinear form (2.4.9). The no–ghost theorem now tells us, that the dimension of a subspace of momentum $\mathbf{x} \in \mathit{II}_{25,1}$ is exactly given by the number of transversal states, i.e.

$$\dim \mathfrak{g}^{(\mathbf{x})}_{\mathit{II}_{25,1}} = p_{24}(1 - \tfrac{1}{2}\mathbf{x}^2). \tag{5.1.9}$$

Moreover, the bilinear form $(\_|\_)_0$ is positive definite on any subspace of nonzero momentum.

We can now immediately infer form Sect. II.3 that the Kac–Moody algebra L_∞ is mapped into $\mathfrak{g}_{\mathit{II}_{25,1}}$ by

$$e(\mathbf{r}) \mapsto \mathrm{e}^{\mathbf{r}}, \tag{5.1.10}$$

$$f(\mathbf{r}) \mapsto -\mathrm{e}^{-\mathbf{r}}, \tag{5.1.11}$$

$$h(\mathbf{r}) \mapsto \mathbf{r}(-1), \tag{5.1.12}$$

for all $\mathbf{r} \in \Lambda_{\mathrm{Leech}}$.

Apart from the above upper bound for its root multiplicities, essentially nothing is known about L_∞. Indeed, if we compare the Dynkin diagrams for L_∞ and E_{10} and recall how complicated even the "tiny" hyperbolic algebra E_{10} was, then it seems rather hopeless to gain insight into the structure of the algebra. Furthermore, it appears even more immodest to investigate how the algebra sits inside $\mathfrak{g}_{\mathit{II}_{25,1}}$. Are there missing states at all? At this point, the Weyl vector saves the day, for it allows us to prove easily, without calculating any commutator, that L_∞ is a proper subalgebra of $\mathfrak{g}_{\mathit{II}_{25,1}}$. Recall that the Weyl vector $\boldsymbol{\rho}$ has scalar product -1 with all Leech roots. Hence $\boldsymbol{\rho}$ can certainly never be written as a positive or negative linear combination of simple roots, i.e. $\boldsymbol{\rho}$ and its nonzero integer multiples are not roots for L_∞:

$$\mathfrak{g}^{(m\boldsymbol{\rho})}_{\mathit{II}_{25,1}} \not\subset L_\infty \qquad \forall m \in \mathbb{Z}^\times . \tag{5.1.13}$$

On the other hand, these root spaces are nonempty: lightlikeness of the Weyl vector immediately implies that they have dimension 24. Explicitly,

$$\mathfrak{g}^{(m\boldsymbol{\rho})}_{\mathit{II}_{25,1}} = \{\boldsymbol{\xi}(-1)\mathrm{e}^{m\boldsymbol{\rho}} \mid \boldsymbol{\xi} \in \mathbb{R}\otimes_{\mathbb{Z}} \mathit{II}_{25,1}, \boldsymbol{\xi}\cdot\boldsymbol{\rho} = 0, \boldsymbol{\xi} \equiv \boldsymbol{\xi} \bmod \mathbb{R}\boldsymbol{\rho}, m \in \mathbb{Z}^\times\}. \tag{5.1.14}$$

Therefore we encounter for L_∞ the special situation that there are certain momentum vectors in $\mathit{II}_{25,1}$ which cannot be reached by adding up Leech roots, even though they are infinite in number.

If we wish to describe $\mathfrak{g}_{II_{25,1}}$ in terms of generators and relations we are thus forced to adjoin to L_∞ at least the above photonic states as new Lie algebra generators. But this means that we have to admit a set of lightlike vectors as simple roots with multiplicity 24, thereby transcending the framework of ordinary Kac–Moody algebras.

It was Borcherds' great achievement to observe that one can consistently define generalized Kac–Moody algebras, also called Borcherds algebras, by allowing for imaginary simple roots in the defining relations for ordinary Kac–Moody algebras (see [7]). As regards the Lie algebra $\mathfrak{g}_{II_{25,1}}$, which for historical reasons has been dubbed **fake monster Lie algebra**, Borcherds was able to prove the astonishing result that the above photonic "Weyl generators", together with the generators for L_∞, already constitute a *complete* set of generators for the spectrum of physical states.

Theorem 8

The fake monster Lie algebra $\mathfrak{g}_{II_{25,1}}$ is the Lie algebra with root lattice $II_{25,1}$, whose simple roots are the simple roots of the Kac–Moody algebra L_∞, together with the positive integer multiples of the Weyl vector ρ, each with multiplicity 24. Then any nonzero root $\mathbf{x} \in II_{25,1}$ of $\mathfrak{g}_{II_{25,1}}$ has multiplicity $p_{24}(1-\frac{1}{2}\mathbf{x}^2)$.

For a proof see [9]. Note that the set of simple roots is characterized by the condition $\mathbf{r}\cdot\rho = -\frac{1}{2}\mathbf{r}^2$.

To complete our example of a Borcherds algebra, we introduce a set of 24 orthonormal transversal polarization vectors $\boldsymbol{\xi}_i \in \mathbb{R}\otimes_{\mathbb{Z}} II_{25,1}$, i.e. $\boldsymbol{\xi}_i\cdot\rho = 0$, $\boldsymbol{\xi}_i\cdot\boldsymbol{\xi}_j = \delta_{ij}$ for $1 \le i,j \le 24$, and consider the generators

$$e_i(m\rho) \mapsto \boldsymbol{\xi}_i(-1)e^{m\rho}, \tag{5.1.15}$$

$$f_i(m\rho) \mapsto \boldsymbol{\xi}_i(-1)e^{-m\rho}, \tag{5.1.16}$$

$$h_i(m\rho) \mapsto m\rho(-1), \tag{5.1.17}$$

for $1 \le i \le 24$ and $m \in \mathbb{Z}_+$. In addition to the relations for L_∞ we get

$$[h(\mathbf{r}), h_i(m\rho)] = 0, \tag{5.1.18}$$

$$[h_i(m\rho), h_j(n\rho)] = 0, \tag{5.1.19}$$

$$[h(\mathbf{r}), e_i(m\rho)] = -me_i(m\rho), \tag{5.1.20}$$

$$[h(\mathbf{r}), f_i(m\rho)] = mf_i(m\rho), \tag{5.1.21}$$

$$[h_i(m\rho), e(\mathbf{r})] = -me(\mathbf{r}), \tag{5.1.22}$$

$$[h_i(m\rho), f(\mathbf{r})] = mf(\mathbf{r}), \tag{5.1.23}$$

$$[h_i(m\rho), e_j(n\rho)] = 0, \tag{5.1.24}$$

$$[h_i(m\rho), f_j(n\rho)] = 0, \tag{5.1.25}$$

$$[e(\mathbf{r}), f_i(m\rho)] = 0, \tag{5.1.26}$$

$$[f(\mathbf{r}), e_i(m\rho)] = 0, \tag{5.1.27}$$

$$[e_i(m\rho), f_j(n\rho)] = \delta_{mn}\delta_{ij}h_i(m\rho) \tag{5.1.28}$$

$$[e_i(m\rho), e_j(n\rho)] = 0, \tag{5.1.29}$$

$$[f_i(m\rho), f_j(n\rho)] = 0, \tag{5.1.30}$$

$$[e(\mathbf{r}), [e(\mathbf{r}), e_i(m\rho)]] = 0, \tag{5.1.31}$$

$$[f(\mathbf{r}), [f(\mathbf{r}), f_i(m\rho)]] = 0, \tag{5.1.32}$$

for all $\mathbf{r} \in \Lambda_{\text{Leech}}$; the above relations easily follow from (2.3.8), (2.3.9) and the "lattice gradation argument" in Sect. II.3, respectively. The Cartan matrix looks thus:

$$\hat{A}_{II_{25,1}} := \left(\begin{array}{cc|ccc|ccc|c} \multicolumn{2}{c|}{L_{(\infty)}} & & \vdots & & & \vdots & & \vdots \\ & & -1 & \dots & -1 & -2 & \dots & -2 & \dots \\ \hline & -1 & 0 & \dots & 0 & 0 & \dots & 0 & \\ \dots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \dots \\ & -1 & 0 & \dots & 0 & 0 & \dots & 0 & \\ \hline & -2 & 0 & \dots & 0 & 0 & \dots & 0 & \\ \dots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \dots \\ & -2 & 0 & \dots & 0 & 0 & \dots & 0 & \\ \hline \dots & \vdots & & \vdots & & & \vdots & & \ddots \end{array}\right) \tag{5.1.33}$$

where the block matrices are 24×24 zero matrices and $L_{(\infty)}$ denotes the infinite-dimensional Cartan matrix for the Leech roots, i.e. with entry $\mathbf{r}\cdot\mathbf{s}$ in the $\mathbf{r}$th row and $\mathbf{s}$th column for Leech roots $\mathbf{r}$, $\mathbf{s}$.

Let us briefly discuss the commutation relations for the fake monster Lie algebra. Equations (5.1.1), (5.1.18) and (5.1.19) show that the set $\{h(\mathbf{r}), \mathbf{r} \in \Lambda_{\text{Leech}}; h_i(m\rho), 1 \le i \le 24, m \in \mathbb{Z}_+\}$ is an Abelian subalgebra. It is the Cartan subalgebra $\mathfrak{h}(\mathfrak{g}_{II_{25,1}})$ for the chosen presentation since, according to Eqs. (5.1.2), (5.1.3), and (5.1.20)–(5.1.25), the h's act diagonally in the adjoint representation on the Chevalley generators. Equations (5.1.4) and (5.1.26)–(5.1.28) additionally confirm our definition of e's and f's because any subset $\{e(\mathbf{r}), f(\mathbf{r}), h(\mathbf{r})\}$ or $\{e_i(m\rho), f_i(m\rho), h_i(m\rho)\}$ of generators with fixed $\mathbf{r} \in \Lambda_{\text{Leech}}$ or $m \in \mathbb{Z}_+$, $i \in \{1,\dots,24\}$, respectively, indeed makes up a closed $\mathfrak{su}(2)$ subalgebra of $\mathfrak{g}_{II_{25,1}}$. Finally, Eqs. (5.1.5), (5.1.6) and (5.1.29)–(5.1.32) represent the Serre relations for the fake monster Lie algebra. We observe that, even though the Cartan matrix is symmetric, the last two Serre relations are "asymmetric" in the sense that there are *no* relations of the form $[e_i(m\boldsymbol{\rho})[e_i(m\boldsymbol{\rho}), e(\mathbf{r})]] = 0$, that one might have expected. As we shall see in the next section, this is a general feature of Borcherds algebras, namely by definition Serre relations involving multiple commutators of imaginary generators do not occur.

The absence of these relations seems strange and one may wonder whether there is a deeper reason for it, at least in the example of the fake monster Lie algebra. Indeed, we shall see that it can be beautifully explained by means of DDF operators. Let us choose any Leech root $\mathbf{r}$. In order to perform the DDF construction we need to find a lightlike vector $\mathbf{k}(\mathbf{r})$ such that $\mathbf{r}\cdot\mathbf{k} = 1$. Surprisingly enough, the lightlike Weyl vector provides an universal vector of that type for all Leech roots if we define

$$\mathbf{k} \equiv \mathbf{k}(\mathbf{r}) := -\boldsymbol{\rho} \qquad \forall \mathbf{r} \in \Lambda_{\text{Leech}}. \tag{5.1.34}$$

With the pair $(\mathbf{r}, -\boldsymbol{\rho})$ at hand we can always select 24 orthonormal vectors $\boldsymbol{\xi}_i \in \mathbb{R} \otimes_{\mathbb{Z}} II_{25,1}$ such that $\boldsymbol{\xi}_i\cdot\mathbf{r} = \boldsymbol{\xi}_i\cdot\boldsymbol{\rho} = 0\,\forall i$. Exactly as in Sect. III.1 we then introduce the transversal DDF operators

$$A^i_m := \left(\boldsymbol{\xi}_i(-1)\mathrm{e}^{-m\boldsymbol{\rho}}\right)_0, \tag{5.1.35}$$

which span a 24-fold Heisenberg algebra $\hat{\mathfrak{t}}_{(\mathbf{r},-\boldsymbol{\rho})}$ so that the irreducible transversal Heisenberg module built on the tachyonic ground state $|\mathbf{r}\rangle$, is given by $S(\hat{\mathfrak{t}}^-_{(\mathbf{r},-\boldsymbol{\rho})}|\mathbf{r}\rangle$. But now a very simple connection between the transversal DDF operators and the imaginary generators emerges, viz

$$\begin{aligned} A^{i_1}_{-m_1}\dots A^{i_M}_{-m_M}|\mathbf{r}\rangle &\equiv \left[\boldsymbol{\xi}_{i_1}(-1)\mathrm{e}^{m_1\boldsymbol{\rho}}, [\dots, [\boldsymbol{\xi}_{i_M}(-1)\mathrm{e}^{m_M\boldsymbol{\rho}}, \mathrm{e}^{\mathbf{r}}]\dots]\right] \\ &= [e_{i_1}(m_1\boldsymbol{\rho}), [\dots, [e_{i_M}(m_M\boldsymbol{\rho}), e(\mathbf{r})]\dots]] . \end{aligned} \tag{5.1.36}$$

Thus

$$A^i_{-m} = \mathrm{ad}\, e_i(m\boldsymbol{\rho}), \tag{5.1.37}$$

and we infer from the DDF construction that an expression like $(\mathrm{ad}\, e_i(m\boldsymbol{\rho}))^2 e(\mathbf{r})$ must not vanish as it is a fully fledged member of the Heisenberg module built on $|\mathbf{r}\rangle$.

But we can exploit the DDF formulation even further to verify partly Borcherds' theorem. On the one hand, the transversal DDF states provide the correct multiplicity formula, on the other hand, we know that they represent multiple Lie algebra commutators of photonic states and a tachyonic. Hence any root for $\mathfrak{g}_{II_{25,1}}$ of the form $\mathbf{r} + n\boldsymbol{\rho}$ for $\mathbf{r} \in \Lambda_{\text{Leech}}$, $n \in \mathbb{N}$, is seen to have multiplicity $p_{24}(n)$, just by adjoining the positive multiples of the Weyl vector as extra imaginary simple roots. No other Lie algebra generators are required. It is conceivable that an alternative, illuminating proof of Borcherds' theorem may be obtained by combining the DDF formulation with other string theory techniques (see e.g. [79]).

2. Definition and properties of Borcherds algebras

The purpose of this section is to develop the formal aspects of Borcherds algebras as presented in [7], [6], [10]. Other recent references on the subject which also clarify some points, are [54], [48], [52] and [53].

Definition 5

Let I be a (finite or) countable index set and let $\hat{A} = (a_{ij})_{i,j\in I}$ be a symmetric real matrix with no zero columns, satisfying the following conditions:

(i) either $a_{ii} = 2$ or $a_{ii} \le 0$,

(ii) $a_{ij} \le 0$ if $i \ne j$,

(iii) $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$.

Then the **universal generalized Kac–Moody algebra** *associated with $\hat{A}$ is defined to be the Lie algebra $\hat{\mathfrak{g}}(\hat{A})$ given by the following generators and relations.*
Generators: Elements $e_i,\ f_i,\ h_{ij}$ for $i, j \in I$.
Relations:

(1) $[e_i, f_j] = h_{ij}$,

(2) $[h_{ij}, e_k] = \delta_{ij}a_{jk}e_k,\ [h_{ij}, f_k] = -\delta_{ij}a_{jk}f_k$,

(3) $e_{ij} := (\mathrm{ad}\, e_i)^{1-a_{ij}}e_j = 0,\quad f_{ij} := (\mathrm{ad}\, f_i)^{1-a_{ij}}f_j = 0$ *if $a_{ii} = 2$ and $i \ne j$,*

(4) $[e_i, e_j] = 0,\ [f_i, f_j] = 0$ *if $a_{ii} \le 0, a_{jj} \le 0$ and $a_{ij} = 0$*

Let us make some remarks and list important properties of universal generalized Kac–Moody algebras.

1. There is a unique invariant bilinear form $(\_,\_)$ on $\hat{\mathfrak{g}}(\hat{A})$ such that $(e_i, f_j) = \delta_{ij}$; invariance and relations (1), (2) then imply $(h_{ii}, h_{jj}) = a_{ij}$.

2. If $a_{ii} = 2$ for all $i \in I$ then $\hat{\mathfrak{g}}(\hat{A})$ is the same as the ordinary Kac–Moody algebra with symmetrized Cartan matrix $\hat{A}$. In general, $\hat{\mathfrak{g}}(\hat{A})$ has almost all the properties that ordinary Kac–Moody algebras have, and the only major difference is that generalized Kac–Moody algebras are allowed to have imaginary simple roots. Note the apparent asymmetry in the Serre relations (3), namely that $a_{ii} = 2$ is necessary whereas a_{jj} is arbitrary.

3. The Jacobi identity applied to the elements h_{ij}, e_k, f_l yields

$$[h_{ij}, h_{kl}] = \delta_{ij}(a_{jk} - a_{jl})h_{kl}$$

so that

(a) h_{ij} lies in the centre of $\hat{\mathfrak{g}}(\hat{A})$ if $i \ne j$,

(b) all the h's commute with each other,

(c) $h_{ij} = 0$ if the ith and the jth columns of $\hat{A}$ are not equal

The elements h_{ij} for which the ith and the jth columns of $\hat{A}$ are equal form a basis for an Abelian subalgebra of $\hat{\mathfrak{g}}(\hat{A})$, called its **Cartan subalgebra** $\hat{\mathfrak{h}}$. In the case of ordinary Kac–Moody algebras, the ith and the jth columns of $\hat{A}$ cannot be equal unless $i = j$, and so the only nonzero elements h_{ij} are those of the form h_{ii} which are usually denoted by h_i. The reason why we need the elements h_{ij} for $i \neq j$ is that $\hat{\mathfrak{g}}(\hat{A})$, so defined, is equal to its own universal central extension.

4. We can define a $\mathbb{Z}$ gradation of $\hat{\mathfrak{g}}(\hat{A})$ by $\deg(e_i) = -\deg(f_i) = n_i$ where $\{n_i | i \in I\}$ is a collection of positive integers with finite repetitions. The degree 0 piece of $\hat{\mathfrak{g}}(\hat{A})$ is the Cartan subalgebra $\hat{\mathfrak{h}}$.

5. $\hat{\mathfrak{g}}(\hat{A})$ has a triangular decomposition

$$\hat{\mathfrak{g}}(\hat{A}) = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+, \tag{5.2.1}$$

where $\hat{\mathfrak{n}}_-$ (resp. $\hat{\mathfrak{n}}_+$) denotes the algebra obtained by dividing the free algebra $\tilde{\mathfrak{n}}_-$ (resp. $\tilde{\mathfrak{n}}_+$) generated by the f_i's (resp. e_i's) by the ideal $\mathfrak{r}_-$ (resp. $\mathfrak{r}_+$) generated by the Serre relations f_{ij} (resp. e_{ij}).

6. $\hat{\mathfrak{g}}(\hat{A})$ has an automorphism θ of order 2 which satisfies $\theta(e_i) = -f_i, \theta(f_i) = -e_i, \theta(h_{ij}) = -h_{ji}$, and is called the **Chevalley involution**.

7. The contravariant form $(x, y)_0 := -(\theta(x), y)$ is "almost positive definite" on $\hat{\mathfrak{g}}(\hat{A})$ which means that $(x, x)_0 > 0$ whenever x is a homogeneous element of nonzero degree in $\hat{\mathfrak{g}}(\hat{A})$.

8. The **root lattice** Λ_R is defined to be the free Abelian group generated by elements $\mathbf{r}_i$ for $i \in I$, with the bilinear form given by $\mathbf{r}_i \cdot \mathbf{r}_j := a_{ij}$. The elements $\mathbf{r}_i$ are called the **simple roots**. The universal generalized Kac–Moody algebra $\hat{\mathfrak{g}}(\hat{A})$ is Λ_R-graded by letting $\hat{\mathfrak{h}}$ have degree zero, e_i have degree $\mathbf{r}_i$ and f_i have degree $-\mathbf{r}_i$. The root space of an element $\mathbf{r} \in \Lambda_R$ is the vector space of elements of $\hat{\mathfrak{g}}(\hat{A})$ of that degree; if $\mathbf{r}$ is nonzero and has a nonzero root space then $\mathbf{r}$ is called a **root** of $\hat{\mathfrak{g}}(\hat{A})$. A root $\mathbf{r}$ is called **positive** if it is a sum of simple roots, and **negative** if $-\mathbf{r}$ is positive. Every root is either positive or negative. A root $\mathbf{r}$ is called **real** if $\mathbf{r}^2 > 0$ and **imaginary** otherwise.

9. There is a **denominator formula** for universal generalized Kac–Moody algebras. This states that

$$e^{\rho} \prod_{\mathbf{r}>0} (1 - e^{\mathbf{r}})^{\mathrm{mult}(\mathbf{r})} = \sum_{w \in W} \det(w) w \left(e^{\rho} \sum_{\mathbf{r}} \epsilon(\mathbf{r}) e^{\mathbf{r}} \right) \tag{5.2.2}$$

Here ρ is the Weyl vector ($\equiv$ vector with $\rho \cdot \mathbf{r} = -\frac{1}{2}\mathbf{r}^2$ for all simple roots), $\mathbf{r} > 0$ means that $\mathbf{r}$ is a positive root, W is the Weyl group.[$\equiv$ group of isometries of Λ_R generated by the reflections $\sigma_i(\mathbf{r}) := \mathbf{r} - (\mathbf{r} \cdot \mathbf{r}_i)\mathbf{r}_i$ corresponding to the *real* simple roots $\mathbf{r}_i$]; $\det(w)$ is defined to be $+1$ or -1 depending on whether w is the product of an even or odd number of reflections and $\epsilon(\mathbf{r})$ is $(-1)^n$ if $\mathbf{r}$ is the sum of n distinct pairwise orthogonal imaginary simple roots, and zero otherwise.
Note that the Weyl vector ρ may be replaced by any vector having inner product $-\frac{1}{2}\mathbf{r}^2$ with all *real* simple roots $\mathbf{r}$ since $e^{w(\rho)-\rho}$ only involves inner products of ρ with the real simple roots.
For ordinary Kac–Moody algebras there are no imaginary simple roots, so the sum over $\mathbf{r}$ equals 1 and we end up with the well-known denominator formula.

10. The above denominator formula follows from a character formula which has been proved for **standard modules**. A weight $\mathbf{\Lambda}$ is called **dominant integral** if $\mathbf{\Lambda} \cdot \mathbf{r}_i \geq 0$ for all $i \in I$ and $\mathbf{\Lambda} \cdot \mathbf{r}_i \in \mathbb{N}$ for i such that $a_{ii} = 2$. Then the standard module $L(\mathbf{\Lambda})$ for $\hat{\mathfrak{g}}(\hat{A})$ is defined to be a (irreducible) highest weight module with highest weight $\mathbf{\Lambda}$ and highest weight vector v_{Λ} such that:

$$f_i^{1+\Lambda \cdot \mathbf{r}_i} \cdot v_{\Lambda} = 0 \quad \text{if } a_{ii} = 2, \tag{5.2.3}$$
$$f_i \cdot v_{\Lambda} = 0 \quad \text{if } a_{ii} \leq 0 \text{ and } \mathbf{\Lambda} \cdot \mathbf{r}_i = 0. \tag{5.2.4}$$

11. There is a natural homomorphism of Abelian groups from the root lattice Λ_R to the Cartan subalgebra $\hat{\mathfrak{h}}$ taking $\mathbf{r}_i$ to $h_{ii} \equiv h_i$ which preserves the bilinear form. This map is not usually injective. It is possible for n imaginary simple roots to have the same image h_i in $\hat{\mathfrak{h}}$ in which case we say, by abuse of language, that $\mathbf{r}_i$ is a simple root "of multiplicity n".

If we take the quotient of a universal generalized Kac–Moody algebra we obtain a Borcherds algebra (generalized Kac–Moody algebra).

Definition 6

A **Borcherds algebra (generalized Kac–Moody algebra)** *is a Lie algebra $\mathfrak{g}$ with an almost positive definite contravariant form, which means that $\mathfrak{g}$ has the following properties:*

(1) $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ *is $\mathbb{Z}$-graded with* $\dim \mathfrak{g}_i < \infty$,

(2) $\mathfrak{g}$ has an involution θ which acts as -1 on $\mathfrak{g}_0$ and maps $\mathfrak{g}_i$ to $\mathfrak{g}_{-i}$,

(3) $\mathfrak{g}$ carries an invariant bilinear form $(\_,\_)$ invariant under θ such that $(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ unless $i + j = 0$,

(4) The contravariant form $(x, y)_0 := -(\theta(x), y)$ is positive definite on $\mathfrak{g}_i$ if $i \neq 0$,

(5) $\mathfrak{g}_0 \subset [\mathfrak{g}, \mathfrak{g}]$.

If the last condition is omitted we may add an Abelian algebra of outer derivations to a generalized Kac–Moody algebra. If the ith and the jth column of $\hat{A}$ are equal, then $\hat{\mathfrak{g}}(\hat{A})$ has an outer derivation d defined by $[d, e_i] = e_j$, $[d, e_j] = -e_i$, $[d, f_i] = f_j$, $[d, f_j] = -f_i$, and $[d, e_k] = [d, f_k] = 0$ if $k \neq i, j$. These outer derivations do not always commute with the elements of the Cartan subalgebra $\hat{\mathfrak{h}}$.

The main theorem about Borcherds algebras states that we can construct any Borcherds algebra from some universal generalized Kac–Moody algebra by factoring out some of the centre and adding a commuting algebra of outer derivations.

It is now easy to see that the fake monster Lie algebra $\mathfrak{g}_{II_{25,1}}$ is a Borcherds algebra in the sense of Def. 5 for the Cartan matrix $\hat{A}_{II_{25,1}}$, because the commutators from the last section establish all the defining relations. The only slight subtlety arises from the centre of the fake monster Lie algebra. Since the Cartan matrix has many identical columns, the universal generalized Kac–Moody algebra associated with $\hat{A}_{II_{25,1}}$, $\hat{\mathfrak{g}}(\hat{A}_{II_{25,1}})$, contains in its centre the subalgebra spanned by the set $\{h_{ij}(m\rho) \,|\, 1 \leq i,j \leq 24, i \neq j, m \in \mathbb{Z}_+\}$ which has to be factored out in order to get the fake monster Lie algebra. Hence the precise relation is

$$\mathfrak{g}_{II_{25,1}} = \hat{\mathfrak{g}}(\hat{A}_{II_{25,1}}) \Big/ \mathrm{span}_{\mathbb{R}}\{h_{ij}(m\rho) \,|\, 1 \leq i,j \leq 24, i \neq j, m \in \mathbb{Z}_+\}. \tag{5.2.5}$$

3. Simple examples of Borcherds algebras

Let us consider two simple examples of Borcherds algebras, namely the Borcherds extensions of $\mathfrak{su}(2)$ and $\widehat{\mathfrak{su}(2)}$ with one lightlike simple root (see [74]).

We start with the following generalized Cartan matrix:

$$\hat{A} = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}. \tag{5.3.1}$$

The simple roots $\mathbf{r}_1$ and $\mathbf{r}_2$ are imaginary and real, respectively. The scalar product of two roots $\mathbf{r} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2$, $\mathbf{s} = n_1\mathbf{r}_1 + n_2\mathbf{r}_2$ is given by $\mathbf{r}\cdot\mathbf{s} = -m_1n_2 - m_2n_1 + 2m_2n_2$. A presentation of the corresponding Borcherds algebra $\hat{\mathfrak{g}}(\hat{A})$ is given in terms of six generators e_1, e_2, f_1, f_2, h_1, h_2, and the relations

$$[e_i, f_j] = \delta_{ij} h_i, \tag{5.3.2}$$

$$[h_i, e_j] = a_{ij} e_j, \tag{5.3.3}$$

$$[h_i, f_j] = -a_{ij} f_j, \tag{5.3.4}$$

$$[e_2, [e_2, e_1]] = 0, \tag{5.3.5}$$

$$[f_2, [f_2, f_1]] = 0. \tag{5.3.6}$$

We define the **fundamental dominant weights** $\mathbf{\Lambda}_1, \mathbf{\Lambda}_2$ relative to the simple roots $\mathbf{r}_1, \mathbf{r}_2$ by the dual base condition

$$\mathbf{\Lambda}_i \cdot \mathbf{r}_j \stackrel{!}{=} \delta_{ij}, \qquad i,j = 1,2, \tag{5.3.7}$$

so that $\mathbf{\Lambda}_1 = -2\mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{\Lambda}_2 = -\mathbf{r}_1$. An important result of the representation theory of Borcherds algebras (see the previous remark) then tells us that any highest weight associated with an irreducible highest weight representation can be written as

$$\mathbf{\Lambda} = l_1\mathbf{\Lambda}_1 + l_2\mathbf{\Lambda}_2 \quad \text{for } l_1 \in \mathbb{R}_+,\ l_2 \in \mathbb{N}. \tag{5.3.8}$$

To actually compute the weight multiplicities for a highest weight representation of a Borcherds algebra, one derives from the denominator identity the Peterson recursion formula which can be put on a computer. A part of the result for the $(1,0)$ fundamental representation (i.e. with $\mathbf{\Lambda}_1$ as highest weight) of $\hat{\mathfrak{g}}(\hat{A})$ is listed in the following table:

n_1	Multiplicity of weight $\mathbf{\lambda} = n_1\mathbf{r}_1 + [n_2]\mathbf{r}_2$	$\mathfrak{su}(2)$ content	As tensor products
0	$[0]$	(0)	(0)
1	$[1] + 1[0]$	(1)	(1)
2	$[2] + 2[1] + 1[0]$	$(2) + 1(0)$	$(1) \otimes (1)$
3	$[3] + 3[2] + 3[1] + 1[0]$	$(3) + 2(1)$	$(1) \otimes (1) \otimes (1)$
4	$[4] + 4[3] + 6[2] + 4[1] + 1[0]$	$(4) + 3(2) + 2(0)$	$(1) \otimes (1) \otimes (1) \otimes (1)$
⋮	⋮	⋮	⋮

Note that we slice the representation with the imaginary simple root $\mathbf{r}_1$, which means that we regard n_1 as a number operator eigenvalue. At each level n_1 we can rewrite the portion of the representation in terms of $\mathfrak{su}(2)$ representations with highest weight $\mathbf{\Lambda}(h_2) = 2l$. For example, at slice $n_1 = 3$ we find weights $\mathbf{\lambda} = 3\mathbf{r}_1 + n_2\mathbf{r}_2$ with $n_2 = 0, 1, 2, 3$ and multiplicity $1, 3, 3, 1$, respectively. Thus we have one 4-dimensional (iso)spin $\frac{3}{2}$ and two 2-dimensional (iso)spin $\frac{1}{2}$ representations indicated by $2(1) + 1(3)$ in the table. And this is nothing but the tensor product of three (iso)spin $\frac{1}{2}$ (see also the review [75]).

We observe that the multiplicity of the weight $\mathbf{\lambda} = n_1\mathbf{r}_1 + n_2\mathbf{r}_2$ equals $\binom{n_1}{n_2}$, the total number of states at slice n_1 being $\sum_{n_2=0}^{n_1} \binom{n_1}{n_2} = 2^{n_1}$ which is the coefficient of q^{n_1} in the partition function $P(q) = \frac{1}{1-2q}$. This suggests that the $\mathfrak{su}(2)$ structure at slice n_1 is the tensor product of the 2-dimensional (iso)spin $\frac{1}{2}$ representation with itself n_1 times with no symmetry or antisymmetry constraints.

The number operator, N, and the diagonalized operator of $\mathfrak{su}(2)$, I_3, can be expressed in terms of the $\{h_1, h_2\}$ basis of the Cartan subalgebra as $N = -2h_1 - h_2$ and $2I_3 = h_2$, respectively, which shows that the operator N counting the number of e_1 operators lies in the Cartan subalgebra and corresponds to the root $\mathbf{r}_N = \mathbf{\Lambda}_1 = -2\mathbf{r}_1 - \mathbf{r}_2$.

In the case of extending the affine Lie algebra $\widehat{\mathfrak{su}(2)}$, we consider the following Cartan matrix:

$$\hat{A} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}. \qquad (5.3.9)$$

The roots are of the form $\mathbf{r} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3$. It is most natural to break the $\widehat{(1,0,0)}$ representation of the corresponding Borcherds algebra $\hat{\mathfrak{g}}(\hat{A})$ into representations of $\widehat{\mathfrak{su}(2)}$, i.e. again we slice the representation with the imaginary simple root $\mathbf{r}_1$. Computer calculations for the first values of n_1 show that the $\widehat{\mathfrak{su}(2)}$ structure at slice n_1 is the tensor product of the $(1,0)$ fundamental representation of $\widehat{\mathfrak{su}(2)}$ with itself n_1 times. In other words, starting from two-dimensional current algebra, the fundamental $(1,0,0)$ representation of the simplest Borcherds extension contains a vacuum at $n_1 = 0$, single particles at $n_1 = 1$, two particle states at $n_2 = 2$, and so on. The surprising result is that the full multiparticle space of states is included in this single representation. Moreover both number operator and Hamilton operator are members of the Cartan subalgebra: $N = -h_2 - h_3$, $\mathrm{L}_{(0)} = -h_1 - h_2 - \frac{1}{2}h_3$, $2I_3 = h_2$.

4. A theorem about representations of Borcherds algebras

So far we have presented the result of Slansky's computer calculations of the first few weight multiplicities of the "basic imaginary" representations for some simple Borcherds algebras, which suggested that these represenations might be written as the tensor algebra over some module for the underlying nonextended Kac–Moody algebras. In the following we shall prove that this is true for any Kac–Moody algebra extended by an arbitrary imaginary simple root.

Let $\hat{\mathfrak{g}}(\hat{A})$ be a Borcherds algebra with one imaginary simple root (so that $h_{ij} = \delta_{ij} h_i \; \forall i,j$). It is clear that if we delete in $\hat{A}$ the row and the column corresponding to the imaginary root then the resulting submatrix A is a generalized Cartan matrix in the sense of Kac [54] with associated Kac–Moody algebra $\mathfrak{g}(A)$.

Recall the triangular decomposition

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \qquad (5.4.1)$$

and the induced decomposition of the universal enveloping algebra:

$$\mathfrak{U}(\mathfrak{g}(A)) = \mathfrak{U}(\mathfrak{n}_-) \otimes \mathfrak{U}(\mathfrak{h}) \otimes \mathfrak{U}(\mathfrak{n}_+). \qquad (5.4.2)$$

An irreducible $\mathfrak{g}(A)$-module $\mathcal{F}_\lambda$ is called **integrable highest weight module** if there exists a dominant integral weight $\lambda \in \mathfrak{h}^*$ and a nonzero vector $\omega \in \mathcal{F}_\lambda$ such that

$$h(\omega) = \lambda(h)\omega \quad \text{for } h \in \mathfrak{h}, \qquad (5.4.3)$$
$$\mathfrak{n}_+(\omega) = 0, \qquad (5.4.4)$$
$$\mathfrak{U}(\mathfrak{n}_-)(\omega) = \mathcal{F}_\lambda. \qquad (5.4.5)$$

We denote by $\mathfrak{T}(\mathcal{F}_\lambda)$ the tensor algebra over $\mathcal{F}_\lambda$,

$$\mathfrak{T}(\mathcal{F}_\lambda) := \bigoplus_{n=0}^{\infty} \mathcal{F}_\lambda^n \equiv \mathbb{C}\cdot\mathbf{1} \oplus \mathcal{F}_\lambda \oplus (\mathcal{F}_\lambda \otimes \mathcal{F}_\lambda) \oplus (\mathcal{F}_\lambda \otimes \mathcal{F}_\lambda \otimes \mathcal{F}_\lambda) \oplus \dots \qquad (5.4.6)$$

Now we are ready to state our result [36].

Theorem 9

Let $\hat{A} = (a_{ij})$, $0 \le i,j \le n$, be a symmetric integer matrix satisfying the following properties:

(i) $a_{00} \le 0$, $a_{ii} = 2$ for $1 \le i \le n$,

(ii) $a_{ij} \le 0$ if $i \ne j$.

Let $\mathcal{F}_\lambda$ be the integrable highest weight module over the Kac–Moody algebra $\mathfrak{g}(A)$ associated to the Cartan matrix $A = (a_{ij})$, $1 \le i,j \le n$, with highest weight λ defined by $\lambda(h_i) := -a_{0i}$, $1 \le i \le n$, and highest weight vector ω. Then the tensor algebra $\mathfrak{T}(\mathcal{F}_\lambda)$ over $\mathcal{F}_\lambda$ is $\hat{\mathfrak{g}}(\hat{A})$-module isomorphic to the highest weight module $L(\Lambda)$, $\Lambda(h_i) = \delta_{i0}$, $0 \le i \le n$, of $\hat{\mathfrak{g}}(\hat{A})$.

Proof
We define an action of the generators of $\hat{\mathfrak{g}}(\hat{A})$ on the tensor algebra $\mathfrak{T}(\mathcal{F}_\lambda)$. Our convention for indices will be that i, j, k run from 1 to n unless otherwise stated!

The Kac–Moody generators e_i, h_i, f_i act trivially on the "vacuum" vector $\mathbf{1}$ and as highest weight representation on $\mathcal{F}_\lambda$. We extend this action to the tensor algebra $\mathfrak{T}(\mathcal{F}_\lambda)$ by Leibnitz' rule. The generator h_0 acts diagonally, viz

$$h_0(\mathbf{1}) := \mathbf{1}, \qquad (5.4.7)$$
$$h_0(\omega) := (1 - a_{00})\omega, \qquad (5.4.8)$$
$$h_0(f_k\omega) := -a_{0k} f_k\omega + f_k h_0(\varphi) \quad \text{for } \varphi \in \mathcal{F}_\lambda, \qquad (5.4.9)$$
$$h_0(\Phi \otimes \Psi) := h_0(\Phi) \otimes \Psi + \Phi \otimes h_0(\Psi) - \Phi \otimes \Psi \quad \text{for } \Phi, \Psi \in \mathfrak{T}(\mathcal{F}_\lambda). \qquad (5.4.10)$$

The "imaginary" generator f_0 adjoins one tensor factor of the highest weight vector ω, i.e.

$$f_0(\Psi) := \omega \otimes \Psi \quad \text{for } \Psi \in \mathfrak{T}(\mathcal{F}_\lambda). \qquad (5.4.11)$$

For e_0 we put

$$e_0(\mathbf{1}) := 0, \qquad (5.4.12)$$

while for the definition on $\mathcal{F}_\lambda^n$, $n \ge 1$, we observe that $\mathcal{F}_\lambda^n = \mathfrak{U}(\mathfrak{n}_-)(\omega \otimes \mathcal{F}_\lambda^{n-1})$, so that it is sufficient to require, inductively,

$$e_0(f_i(\Psi)) := f_i(e_0(\Psi)), \qquad (5.4.13)$$
$$e_0(\omega \otimes \Psi) := h_0(\Psi) + \omega \otimes e_0(\Psi), \qquad (5.4.14)$$

for $\Psi \in \mathcal{F}_\lambda^n$, $n \geq 0$.

Having defined the action of the generators on the tensor algebra, we will now check that $\mathfrak{T}(\mathcal{F}_\lambda)$ carries the claimed $\hat{\mathfrak{g}}(\hat{A})$-module structure. First we note that h_0 and the h_i's are defined to act diagonally on the tensor algebra. Hence the h's commute with each other. Secondly, all commutation relations involving only Kac–Moody generators e_i, h_i, f_i are valid by assumption. Next, we have a look at those commutation relations which are more or less trivial since they can be checked immediately on the whole tensor algebra, viz

$$(e_0 f_0 - f_0 e_0)(\Psi) = e_0(\omega \otimes \Psi) - \omega \otimes e_0(\Psi) = h_0(\Psi), \tag{5.4.15}$$

$$(e_0 f_i - f_i e_0)(\Psi) = 0, \tag{5.4.16}$$

$$(e_i f_0 - f_0 e_i)(\Psi) = e_i(\omega \otimes \Psi) - \omega \otimes e_i(\Psi) = 0, \tag{5.4.17}$$

$$\begin{aligned}(h_0 f_0 - f_0 h_0)(\Psi) &= h_0(\omega \otimes \Psi) - \omega \otimes h_0(\Psi)\\ &= (h_0 - 1)(\omega) \otimes \Psi\\ &= -a_{00} f_0(\Psi),\end{aligned} \tag{5.4.18}$$

$$\begin{aligned}(h_i f_0 - f_0 h_i)(\Psi) &= h_i(\omega \otimes \Psi) - \omega \otimes h_i(\Psi)\\ &= h_i(\omega) \otimes \Psi\\ &= -a_{0i} f_0(\Psi).\end{aligned} \tag{5.4.19}$$

Finally, we check the remaining four types of commutators:

$$(h_0 f_i - f_i h_0)(\mathbf{1}) = -f_i(\mathbf{1}) = 0 = -a_{0i} f_i(\mathbf{1}), \tag{5.4.20}$$

$$(h_0 f_i - f_i h_0)(\varphi) = -a_{0i} f_i \varphi, \tag{5.4.21}$$

$$\begin{aligned}(h_0 f_i - f_i h_0)(\Phi \otimes \Psi) &= h_0(f_i(\Phi) \otimes \Psi + \Phi \otimes f_i(\Psi))\\ &\quad - f_i(h_0(\Phi) \otimes \Psi + \Phi \otimes h_0(\Psi) - \Phi \otimes \Psi)\\ &= (h_0 f_i - f_i h_0)(\Phi) \otimes \Psi + \Phi \otimes (h_0 f_i - f_i h_0)(\Psi)\\ &= -a_{0i} f_i(\Phi \otimes \Psi) \quad \text{by induction,}\end{aligned} \tag{5.4.22}$$

$$(h_0 e_i - e_i h_0)(\mathbf{1}) = 0 = a_{0i} e_i(\mathbf{1}), \tag{5.4.23}$$

$$(h_0 e_i - e_i h_0)(\omega) = 0 = a_{0i} e_i(\omega), \tag{5.4.24}$$

$$\begin{aligned}(h_0 e_i - e_i h_0)(f_k \omega) &= h_0(\delta_{ik} h_i(\varphi) + f_k e_i(\varphi)) - e_i(-a_{0k} f_k \omega + f_k h_0(\varphi))\\ &= a_{0k}(e_i f_k - f_k e_i)(\varphi) + f_k(h_0 e_i - e_i h_0)(\varphi)\\ &= a_{0k}\delta_{ik} h_i(\varphi) + a_{0i} f_k e_i(\varphi) \quad \text{by induction}\\ &= a_{0i} e_i(f_k \omega),\end{aligned} \tag{5.4.25}$$

$$\begin{aligned}(h_0 e_i - e_i h_0)(\Phi \otimes \Psi) &= h_0(e_i(\Phi) \otimes \Psi + \Phi \otimes e_i(\Psi))\\ &\quad - e_i(h_0(\Phi) \otimes \Psi + \Phi \otimes h_0(\Psi) - \Phi \otimes \Psi)\\ &= (h_0 e_i - e_i h_0)(\Phi) \otimes \Psi + \Phi \otimes (h_0 e_i - e_i h_0)(\Psi)\\ &= a_{0i} e_i(\Phi \otimes \Psi) \quad \text{by induction,}\end{aligned} \tag{5.4.26}$$

$$(h_i e_0 - e_0 h_i)(\mathbf{1}) = 0 = a_{i0} e_0(\mathbf{1}), \tag{5.4.27}$$

$$(h_i e_0 - e_0 h_i)(\omega) = h_i(\mathbf{1}) + a_{0i} e_0(\omega) = a_{0i}\mathbf{1} = a_{i0} e_0(\omega), \tag{5.4.28}$$

$$(h_i e_0 - e_0 h_i)(f_k \omega) = 0 = a_{i0} e_0(f_k \omega), \tag{5.4.29}$$

$$\begin{aligned}(h_i e_0 - e_0 h_i)(\omega \otimes \Psi) &= h_i(h_0(\Psi) + \omega \otimes e_0(\Psi)) - e_0(h_i(\omega) \otimes \Psi + \omega \otimes h_i(\Psi))\\ &= a_{0i} h_0(\Psi) + \omega \otimes (h_i e_0 - e_0 h_i)(\Psi) + (h_i h_0 - h_0 h_i)(\Psi)\\ &= a_{i0} e_0(\omega \otimes \Psi),\end{aligned} \tag{5.4.30}$$

$$\begin{aligned}(h_i e_0 - e_0 h_i)(f_k \omega \otimes \Psi) &= h_i(-e_0(\varphi \otimes f_k(\Psi)) + f_k(e_0(\varphi \otimes \Psi)))\\ &\quad - e_0(h_i(f_k \omega) \otimes \Psi + f_k \omega \otimes h_i(\Psi))\\ &= -h_i(e_0(\varphi \otimes f_k(\Psi))) + h_i(f_k(e_0(\varphi \otimes \Psi)))\\ &\quad - a_{ik} e_0(\varphi \otimes f_k(\Psi)) + a_{ik} f_k(e_0(\varphi \otimes \Psi))\\ &\quad + e_0(h_i(\varphi) \otimes f_k(\Psi)) - f_k(e_0(h_i(\varphi) \otimes \Psi))\\ &\quad + e_0(\varphi \otimes f_k(h_i(\Psi))) - f_k(e_0(\varphi \otimes h_i(\Psi)))\\ &= (e_0 h_i - h_i e_0)(\varphi \otimes f_k(\Psi))\\ &\quad + f_k((h_i e_0 - e_0 h_i)(\varphi \otimes \Psi))\\ &= a_{i0}(-e_0(\varphi \otimes f_k(\Psi)) + f_k(e_0(\varphi \otimes \Psi))) \quad \text{by induction}\\ &= a_{i0} e_0(f_k \omega \otimes \Psi),\end{aligned} \tag{5.4.31}$$

$$(h_0 e_0 - e_0 h_0)(\mathbf{1}) = 0 = a_{00} e_0(\mathbf{1}), \tag{5.4.32}$$

$$(h_0 e_0 - e_0 h_0)(\omega) = h_0(\mathbf{1}) - (1 - a_{00}) e_0(\omega) = a_{00}\mathbf{1} = a_{00} e_0(\omega), \tag{5.4.33}$$

$$(h_0 e_0 - e_0 h_0)(f_k \omega) = 0 = a_{00} e_0(f_k \omega), \tag{5.4.34}$$

$$\begin{aligned}(h_0 e_0 - e_0 h_0)(\omega \otimes \Psi) &= h_0(h_0(\Psi) + \omega \otimes e_0(\Psi)) - e_0(-a_{00}\omega \otimes \Psi + \omega \otimes h_0(\Psi))\\ &= a_{00} h_0(\Psi) + \omega \otimes (h_0 e_0 - e_0 h_0)(\Psi)\\ &= a_{00} e_0(\omega \otimes \Psi) \quad \text{by induction,}\end{aligned} \tag{5.4.35}$$

$$\begin{aligned}(h_0 e_0 - e_0 h_0)(f_k \omega \otimes \Psi) &= h_0(-e_0(\varphi \otimes f_k(\Psi)) + f_k(e_0(\varphi \otimes \Psi)))\\ &\quad - e_0(h_0(f_k \omega) \otimes \Psi + f_k \omega \otimes h_0(\Psi) - f_k \omega \otimes \Psi)\\ &= -h_0(e_0(\varphi \otimes f_k(\Psi))) + h_0(f_k(e_0(\varphi \otimes \Psi)))\\ &\quad - a_{0k} e_0(\varphi \otimes f_k(\Psi)) + a_{0k} f_k(e_0(\varphi \otimes \Psi))\end{aligned}$$

$$\begin{aligned}
&+ e_0(h_0(\varphi) \otimes f_k(\Psi)) - f_k(e_0(h_0(\varphi) \otimes \Psi)) \\
&+ e_0(\varphi \otimes f_k(h_0(\Psi))) - f_k(e_0(\varphi \otimes h_0(\Psi))) \\
&+ e_0(\varphi \otimes f_k(\Psi)) - f_k(e_0(\varphi \otimes \Psi)) \\
&= (e_0 h_0 - h_0 e_0)(\varphi \otimes f_k(\Psi)) \\
&\quad + f_k((h_0 e_0 - e_0 h_0)(\varphi \otimes \Psi)) \\
&= a_{00}(-e_0(\varphi \otimes f_k(\Psi)) + f_k(e_0(\varphi \otimes \Psi))) \quad \text{by induction} \\
&= a_{00} e_0(f_k \omega \otimes \Psi),
\end{aligned} \tag{5.4.36}$$

for all $\varphi \in \mathcal{F}_\lambda$ and $\Phi, \Psi \in \mathfrak{T}(\mathcal{F}_\lambda)$.

Now we shall prove that $\mathfrak{T}(\mathcal{F}_\lambda)$ is indeed isomorphic to $L(\Lambda)$ as a $\hat{\mathfrak{g}}(\hat{A})$ module. Denote the highest weight vector of $L(\Lambda)$ by v_Λ. Define a map $\nu : \mathfrak{U}(\tilde{\mathfrak{n}}_-)v_\Lambda \to \mathfrak{T}(\mathcal{F}_\lambda)$ by putting

$$\nu(f_{i_1} \cdots f_{i_n} v_\Lambda) := f_{i_1} \cdots f_{i_n}(\mathbf{1}), \tag{5.4.37}$$

where $i_1, \ldots, i_n \in \{0, \ldots, n\}$, and then extending by linearity. To prove that ν reduces to a well defined $\hat{\mathfrak{g}}(\hat{A})$ module homomorphism $\nu' : \mathfrak{U}(\hat{\mathfrak{n}}_-)v_\Lambda \to \mathfrak{T}(\mathcal{F}_\lambda)$, one has to check that the action of elements of $\mathfrak{r}_-$ on $\mathfrak{T}(\mathcal{F}_\lambda)$ vanishes, i.e. that the Serre relations are valid. For $f_{ij}, i, j = 1, \ldots, n$, this is part of the definition. To check the remaining ones, observe that

$$[(\mathrm{ad}\, f_i)^m f_0](\Psi) = f_i^m \omega \otimes \Psi, \tag{5.4.38}$$

so that, for $i = 1, \ldots, n$,

$$f_{i0}(\Psi) = f_i^{1+\lambda(h_i)} \omega \otimes \Psi = 0 \tag{5.4.39}$$

due to [54, Lemma 10.1]. According to [7] (see also [48] or [53]) the standard irreducible module $L(\Lambda)$ is obtained from the generalized Verma module $M(\Lambda)$ by dividing out the subspace generated by the primitive vectors $f_i^{1+\Lambda(h_i)} v_\Lambda, i = 1, \ldots, n$ (cf. (5.2.3)). Because of $f_i^{1+\Lambda(h_i)}(\mathbf{1}) = f_i(\mathbf{1}) = 0$, ν' reduces further to a map $\nu'' : L(\Lambda) \to \mathfrak{T}(\mathcal{F}_\lambda)$. ν'' is injective because the kernel of ν'' would be a proper submodule of $L(\Lambda)$, and surjective because $\mathfrak{T}(\mathcal{F}_\lambda)$ is spanned by vectors of the form

$$u_1\omega \otimes \ldots \otimes u_n\omega = \nu(u_1(f_0) \ldots u_n(f_0)v_\Lambda), \tag{5.4.40}$$

where

$$\begin{aligned}
u_i &= F_{n_1(i)} \cdots F_{n_{k(i)}(i)}, \qquad F_{n_j(i)} \in \mathfrak{g}(A), \\
u_i(f_0) &= \left[F_{n_1(i)}, \left[\ldots \left[F_{n_{k(i)}(i)}, f_0\right] \ldots\right]\right].
\end{aligned} \tag{5.4.41}$$

This completes the proof of the theorem.

We observe that the theorem is not altered if we replace a_{00} by any nonpositive real number or $\Lambda(h_0)$ by any positive real number. This shows that $L(k\Lambda_0) \cong L(\Lambda_0)$ for all $k \in \mathbb{R}_+$, where Λ_0 denotes the "basic imaginary" highest weight with components $(1, 0, \ldots, 0)$.

5. Speculations

To give an interpretation and a possible physical application of the above theorem, we make a short digression to the question of symmetry in quantum theory. We will follow [3] and [80].

Consider the differential operator $W = i\frac{\partial}{\partial t} - H$ where H denotes the Hamilton operator of some nonrelativistic quantum system. We define wavefunctions ψ of the system as solutions of the Schrödinger equation $W\psi = 0$. If there are operators $G_j, j = 1, \ldots, n$, forming a Lie algebra $\mathfrak{g}$ and satisfying, on the space Φ of solutions, $[W, G_j]\psi = 0$ then Φ is a representation space for the **dynamical Lie algebra** $\mathfrak{g}$ of the quantum system. In general, $\mathfrak{g}$ contains time-dependent operators $G(t)$ satisfying the Heisenberg equation

$$[i\frac{\partial}{\partial t}, G(t)]\psi = [H, G(t)]\psi \quad \text{for } \psi \in \Phi. \tag{5.5.1}$$

The subalgebra $\mathfrak{g}' := \{G_j \in \mathfrak{g} | [G_j, H] = 0\}$ is a more narrow definition of symmetry. The **maximal symmetry algebra of** H is defined to be the subalgebra $\mathfrak{g}'' \subset \mathfrak{g}'$ of time-independent operators commuting with the Hamilton operator.

The Heisenberg equation has the solution $G(t) = \mathrm{e}^{itH} G(0) \mathrm{e}^{-itH}$ where the evolution operator e^{itH} clearly commutes with the energyoperator H. Hence the time-dependent dynamical Lie algebra $\mathfrak{g} = \{G_j(t) | j = 1, \ldots, n\}$ and the time-independent dynamical Lie algebra $\{G_j(0) | j = 1, \ldots, n\}$ are unitarily equivalent which allows us to restrict ourselves in concrete problems to the analysis of time-independent dynamical Lie algebras.

For stationary solutions of the Schrödinger equation of the form $\psi(t) = \mathrm{e}^{-iEt}u$ we obtain the eigenvalue equation $Hu = Eu$. An eigenspace of H for a fixed value E of the energy is already a representation space of the maximal symmetry algebra $\mathfrak{g}''$ of H. Hence $\mathfrak{g}''$ should be rather called "algebra of degeneracy of the energy". In order to solve the quantum mechanical problem completely, we still have to determine the spectrum of H.

As an example let us analyze the spectrum of the nonrelativistic hydrogen atom. Rotational symmetry of the Hamilton operator, i.e. $[H, L_i] = 0$, $i = 1, 2, 3$, suggests $\mathfrak{so}(3)$ as symmetry algebra leading to a $2l + 1$-fold degeneracy for each energy level. However, this is just the kinematical symmetry algebra of the hydrogen atom. It turns out that each energy eigenvalue E_n has multiplicity n^2 (neglecting spin) independent of the angular momentum quantum number l. For a given principal quantum number n, the eigenspace $\mathcal{H}_n$ of E_n can be decomposed into $2l + 1$-dimensional irreducible representations $\mathcal{D}(l)$ of $\mathfrak{so}(3)$:

$$\mathcal{H}_n = \bigoplus_{l=0}^{n-1} \mathcal{D}(l), \tag{5.5.2}$$

$$\dim \mathcal{H}_n = \sum_{l=0}^{n-1} (2l + 1) = n^2. \tag{5.5.3}$$

This additional degeneracy is surprising and can only be explained in terms of a higher ("hidden") symmetry of the Hamilton operator. In fact, Pauli showed that the classical Runge-Lenz vector which occurs as a constant of motion in the classical Kepler problem, leads to three hermitian quantum-mechanical operators commuting with the Hamilton operator. We conclude that the maximal symmetry algebra of H is the Lie algebra $\mathfrak{so}(4)$, i.e. for a given n, the eigenspace $\mathcal{H}_n$ of $E_n < 0$ (bound states) carries a single n^2-dimensional irreducible representation of $\mathfrak{so}(4)$ (For $E_n > 0$, the continuum states, we have $\mathfrak{so}(3,1)$). Thus $\mathfrak{so}(4)$ may be interpreted as the degeneracy algebra of the nonrelativistic hydrogen atom.

If the states of the hydrogen atom are labeled by the traditional quantum numbers $|nlm\rangle$ associated with the solution in spherical coordinates then, in constructing the full dynamical algebra, we must find an algebra which contains $\mathfrak{so}(4)$ as a subalgebra and includes operators that ladder n and l. A careful analysis exhibits the 15-dimensional Lie algebra $\mathfrak{so}(4,2)$ as a dynamical algebra for the hydrogen atom which means that its operators permit us to pass from any hydrogenic state $|nlm\rangle$ to any other state $|n'l'm'\rangle$. Hence there is a *single* irreducible representation of $\mathfrak{so}(4,2)$ that covers *all* the states of the hydrogen atom. Of course this representation must be infinite-dimensional.

It is worth mentioning that already $\mathfrak{so}(4,1)$ possesses a single irreducible representation which covers the complete set of quantum numbers n, l, m, and may therefore be regarded as the quantum-number algebra of the hydrogen atom. The enlargement of $\mathfrak{so}(4,1)$ to $\mathfrak{so}(4,2)$ introduces no additional quantum numbers and leaves the representation space unchanged but it requires additional operators that can be identified with interaction operators. Thus, in principle, the calculation of electromagnetic transition amplitudes and the Stark effect has been reduced to an algebraic calculation without any need to compute integrals. It is also remarkable that the generators of $\mathfrak{so}(4,2)$ may be realized in terms of the four-dimensional Dirac γ-matrices.

After this digression on symmetries it is tempting to speculate about applications of Borcherds algebras in physics. It might be possible to construct quantum field theories in which a Borcherds algebra plays the role of a sort of dynamical Lie algebra. One would expect to find all quantum states within a single representation. In particular, the dynamical algebra should comprise the Hamilton operator as well as operators that change number of particles. The underlying Lie algebra without Borcherds extension then could determine the maximal symmetry algebra of the Hamilton operator.

According to a conjecture of Ginsparg [74], the special class of Borcherds algebras considered in the theorem might play a role in second quantization of a single particle theory. In this interpretation we regard the module $\mathcal{F}_\lambda$ from above as one-particle Fock space so that $\mathfrak{T}(\mathcal{F}_\lambda)$ comprises all multiparticle states. In other words, within a single irreducible represenation of the Borcherds algebra we encounter all possible multiparticle excitations. Thus the "imaginary" generators f_0 and e_0 act as particle creation and particle annihilation operators, respectively, whereas the vector $\mathbf{1}$ indeed deserves the name "true vacuum" in contrast to the "ground state" $\omega \in \mathcal{F}_\lambda$.

Applying this idea to string theory one should think about the underlying Kac–Moody algebra $\mathfrak{g}(A)$ as spectrum generating algebra for the physical states of the bosonic string. Consequently, the tensor algebra $\mathfrak{T}(\mathcal{F}_\lambda)$ would be intimately related to a string field theory. Note that in the special case of an underlying affine Lie algebra $\mathfrak{g}(A)$, we would end up with a string field theory on the group manifold associated to $\mathfrak{g}(A)$ (cf. [37]).

It is clear that the emergence of Borcherds algebras in quantum field theory is just a naive speculation since up to now at least one important point in dealing with particles is missing. The tensor algebra $\mathfrak{T}(\mathcal{F}_\lambda)$ carries no symmetry or antisymmetry constraints at all, which means that the concept of statistics is absent. This point is still to be clarified, for it is conceivable that it might be even desirable to have no definite statistics a priori.

In view of these possible realizations of Borcherds algebras in physics we shall finish with the useful construction of a "number operator" which counts the number of f_0's (number of particles/strings) occurring in the expression for a homogeneous state vector $\Psi \in \mathfrak{T}(\mathcal{F}_\lambda)$. We are looking for an element N in the Cartan subalgebra $\hat{\mathfrak{h}}(\hat{A})$ satisfying

$$N(\Psi) \stackrel{!}{=} n\Psi \qquad \forall \Psi \in \mathcal{F}_\lambda^n, n \geq 1, \tag{5.5.4}$$

or, equivalently,

$$[N, f_j] \stackrel{!}{=} \delta_{j0} f_j \quad \text{for } 0 \leq j \leq n. \tag{5.5.5}$$

The ansatz $N = \sum_{i=0}^n N_i h_i$ yields the following system of linear equations for the rational coefficients N_i:

$$\sum_{i=0}^n a_{ij} N_i \stackrel{!}{=} -\delta_{j0} \quad \text{for } 0 \leq j \leq n. \tag{5.5.6}$$

If $\hat{A}$ is invertible (if it is not we have to extend the Cartan subalgebra appropriately, as we are used to for affine Kac–Moody algebras) we obtain a unique solution for the number operator N. Note, however, that the eigenvalues of N give us the number of f_0's shifted by N_0 since we have $N(\mathbf{1}) = N_0\mathbf{1}$ instead of $N(\mathbf{1}) = 0$. This annoying constant may be removed by defining the "renormalized" number operator $\hat{N} := N - N_0$, which is the one we have already encountered in the examples in Sect. 3.

Even though the above remarks are just naive speculations about the application of Borcherds algebras in physics, we have seen in the example of the fake monster Lie algebra that these algebras do appear in physics. It would not be too surprising if it turned out that Borcherds algebras lead to the discovery of new exciting symmetries in nature.

APPENDIX A:

DDF STATES

We here list the transversal and longitudinal DDF states, required in Sect. IV.4, up to oscillator number 4. For the special example discussed there, we must only evaluate them for the following scalar products: $\mathbf{r}^2 = \mathbf{s}^2 = 2$, $\mathbf{k}^2 = 0$, $\mathbf{r}\cdot\mathbf{k} = \mathbf{s}\cdot\mathbf{k} = 1$, $\boldsymbol{\xi}_i\cdot\boldsymbol{\xi}_j = \delta_{ij}$, $\boldsymbol{\eta}_{i'}\cdot\boldsymbol{\eta}_{j'} = \delta_{i'j'}$, $\boldsymbol{\xi}_i\cdot\mathbf{r} = \boldsymbol{\xi}_i\cdot\mathbf{k} = \boldsymbol{\eta}_{i'}\cdot\mathbf{s} = \boldsymbol{\eta}_{i'}\cdot\mathbf{k} = 0$, $\boldsymbol{\xi}_i\cdot\mathbf{s} =: \delta_{is}$, $\boldsymbol{\eta}_{i'}\cdot\mathbf{r} =: \delta_{i'r}$, $\boldsymbol{\eta}_{i'}\cdot\boldsymbol{\xi}_j =: g_{i'j}$. Also put $\epsilon \equiv \epsilon(\mathbf{k},\mathbf{r})$, $\epsilon' \equiv \epsilon(\mathbf{s},\mathbf{r})$, $\epsilon'' \equiv \epsilon(\mathbf{s}-\mathbf{k},\mathbf{r})$.

The transversal states are:

$$A^i_{-1}|\mathbf{r}\rangle = \epsilon\boldsymbol{\xi}_i(-1)|\mathbf{r}-\mathbf{k}\rangle, \tag{A.1}$$

$$A^i_{-1}A^j_{-1}|\mathbf{r}\rangle = \Big\{\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1) + \tfrac{1}{2}\delta_{ij}\left[\mathbf{k}(-1)^2 - \mathbf{k}(-2)\right]\Big\}|\mathbf{r}-2\mathbf{k}\rangle, \tag{A.2}$$

$$A^i_{-2}|\mathbf{r}\rangle = \Big\{\boldsymbol{\xi}_i(-2) - 2\boldsymbol{\xi}_i(-1)\mathbf{k}(-1)\Big\}|\mathbf{r}-2\mathbf{k}\rangle, \tag{A.3}$$

$$\begin{aligned} A^i_{-1}A^j_{-1}A^k_{-1}|\mathbf{r}\rangle = \epsilon\Big\{&\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1)\boldsymbol{\xi}_k(-1) + \tfrac{1}{2}\left[\delta_{ij}\boldsymbol{\xi}_k(-1) + \delta_{jk}\boldsymbol{\xi}_i(-1)\right.\\ &\left.+ \delta_{ki}\boldsymbol{\xi}_j(-1)\right]\left[\mathbf{k}(-1)^2 - \mathbf{k}(-2)\right]\Big\}|\mathbf{r}-3\mathbf{k}\rangle, \end{aligned} \tag{A.4}$$

$$\begin{aligned} A^i_{-2}A^j_{-1}|\mathbf{r}\rangle = \epsilon\Big\{&\boldsymbol{\xi}_i(-2)\boldsymbol{\xi}_j(-1) - 2\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1)\mathbf{k}(-1)\\ &- \tfrac{2}{3}\delta_{ij}\left[2\mathbf{k}(-1)^3 - 3\mathbf{k}(-2)\mathbf{k}(-1) + \mathbf{k}(-3)\right]\Big\}|\mathbf{r}-3\mathbf{k}\rangle, \end{aligned} \tag{A.5}$$

$$A^i_{-3}|\mathbf{r}\rangle = \epsilon\Big\{\boldsymbol{\xi}_i(-3) - 3\boldsymbol{\xi}_i(-2)\mathbf{k}(-1) + \tfrac{3}{2}\boldsymbol{\xi}_i(-1)\left[3\mathbf{k}(-1)^2 - \mathbf{k}(-2)\right]\Big\}|\mathbf{r}-3\mathbf{k}\rangle, \tag{A.6}$$

$$\begin{aligned} A^i_{-1}A^j_{-1}A^k_{-1}A^l_{-1}|\mathbf{r}\rangle = \Big\{&\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1)\boldsymbol{\xi}_k(-1)\boldsymbol{\xi}_l(-1)\\ &+ \tfrac{1}{2}\left[\delta_{ij}\boldsymbol{\xi}_k(-1)\boldsymbol{\xi}_l(-1) + \delta_{ik}\boldsymbol{\xi}_j(-1)\boldsymbol{\xi}_l(-1)\right.\\ &+ \delta_{il}\boldsymbol{\xi}_j(-1)\boldsymbol{\xi}_k(-1) + \delta_{jk}\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_l(-1)\\ &\left.+ \delta_{jl}\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_k(-1) + \delta_{kl}\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1)\right]\left[\mathbf{k}(-1)^2 - \mathbf{k}(-2)\right]\\ &+ \tfrac{1}{4}\left[\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right]\left[\mathbf{k}(-1)^2 - \mathbf{k}(-2)\right]^2\Big\}|\mathbf{r}-4\mathbf{k}\rangle, \end{aligned} \tag{A.7}$$

$$\begin{aligned} A^i_{-2}A^j_{-1}A^k_{-1}|\mathbf{r}\rangle = \Big\{&\boldsymbol{\xi}_i(-2)\boldsymbol{\xi}_j(-1)\boldsymbol{\xi}_k(-1) - 2\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1)\boldsymbol{\xi}_k(-1)\mathbf{k}(-1)\\ &+ \tfrac{1}{2}\delta_{jk}\boldsymbol{\xi}_i(-2)\left[\mathbf{k}(-1)^2 - \mathbf{k}(-2)\right]\\ &- \tfrac{2}{3}\left[\delta_{ij}\boldsymbol{\xi}_k(-1) + \delta_{ik}\boldsymbol{\xi}_j(-1)\right]\left[2\mathbf{k}(-1)^3 - 3\mathbf{k}(-1)\mathbf{k}(-2) + \mathbf{k}(-3)\right]\\ &- \delta_{jk}\boldsymbol{\xi}_i(-1)\left[\mathbf{k}(-1)^3 - \mathbf{k}(-1)\mathbf{k}(-2)\right]\Big\}|\mathbf{r}-4\mathbf{k}\rangle, \end{aligned} \tag{A.8}$$

$$\begin{aligned} A^i_{-3}A^j_{-1}|\mathbf{r}\rangle = \Big\{&\boldsymbol{\xi}_i(-3)\boldsymbol{\xi}_j(-1) - 3\boldsymbol{\xi}_i(-2)\boldsymbol{\xi}_j(-1)\mathbf{k}(-1)\\ &+ \tfrac{3}{2}\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1)\left[3\mathbf{k}(-1)^2 - \mathbf{k}(-2)\right]\\ &+ \tfrac{3}{8}\delta_{ij}\left[9\mathbf{k}(-1)^4 - 18\mathbf{k}(-1)^2\mathbf{k}(-2) + 8\mathbf{k}(-1)\mathbf{k}(-3)\right.\\ &\left.+ 3\mathbf{k}(-2)^2 - 2\mathbf{k}(-4)\right]\Big\}|\mathbf{r}-4\mathbf{k}\rangle, \end{aligned} \tag{A.9}$$

$$\begin{aligned} A^i_{-2}A^j_{-2}|\mathbf{r}\rangle = \Big\{&\boldsymbol{\xi}_i(-2)\boldsymbol{\xi}_j(-2) - 2\left[\boldsymbol{\xi}_i(-2)\boldsymbol{\xi}_j(-1) + \boldsymbol{\xi}_j(-2)\boldsymbol{\xi}_i(-1)\right]\mathbf{k}(-1)\\ &+ 4\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1)\mathbf{k}(-1)^2 + \delta_{ij}\left[4\mathbf{k}(-1)^4 - 8\mathbf{k}(-1)^2\mathbf{k}(-2)\right.\\ &\left.+ 4\mathbf{k}(-1)\mathbf{k}(-3) + \mathbf{k}(-2)^2 - \mathbf{k}(-4)\right]\Big\}|\mathbf{r}-4\mathbf{k}\rangle, \end{aligned} \tag{A.10}$$

$$\begin{aligned} A^i_{-4}|\mathbf{r}\rangle = \Big\{&\boldsymbol{\xi}_i(-4) - 4\boldsymbol{\xi}_i(-3)\mathbf{k}(-1) + 2\boldsymbol{\xi}_i(-2)\left[4\mathbf{k}(-1)^2 - \mathbf{k}(-2)\right]\\ &- \tfrac{4}{3}\boldsymbol{\xi}_i(-1)\left[8\mathbf{k}(-1)^3 - 6\mathbf{k}(-1)\mathbf{k}(-2) + \mathbf{k}(-3)\right]\Big\}|\mathbf{r}-4\mathbf{k}\rangle. \end{aligned} \tag{A.11}$$

The longitudinal states are:

$$A^-_{-1}|\mathbf{r}\rangle = \epsilon\Big\{ -\mathbf{r}(-1) + \mathbf{k}(-1)\Big\}|\mathbf{r}-\mathbf{k}\rangle = -\epsilon \mathrm{L}_{(-1)}|\mathbf{r}-\mathbf{k}\rangle, \tag{A.12}$$

$$A^-_{-2}|\mathbf{r}\rangle = \Big\{ -\mathbf{r}(-2) + \tfrac{d-6}{4}\mathbf{k}(-2) + 2\mathbf{r}(-1)\mathbf{k}(-1) - \tfrac{1}{2}\sum_{i=1}^{d-2}\boldsymbol{\xi}_i(-1)^2 + \tfrac{6-d}{4}\mathbf{k}(-1)^2\Big\}|\mathbf{r}-2\mathbf{k}\rangle, \tag{A.13}$$

$$\begin{aligned} A^i_{-1}A^-_{-2}|\mathbf{r}\rangle = \epsilon\Big\{ & -\boldsymbol{\xi}_i(-3) - \mathbf{r}(-2)\boldsymbol{\xi}_i(-1) + 3\boldsymbol{\xi}_i(-2)\mathbf{k}(-1) + \tfrac{d-2}{4}\mathbf{k}(-2)\boldsymbol{\xi}_i(-1) \\ & + 2\mathbf{r}(-1)\boldsymbol{\xi}_i(-1)\mathbf{k}(-1) - \tfrac{1}{2}\sum_{j=1}^{d-2}\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1)^2 \\ & - \tfrac{d+6}{4}\boldsymbol{\xi}_i(-1)\mathbf{k}(-1)^2\Big\}|\mathbf{r}-3\mathbf{k}\rangle, \end{aligned} \tag{A.14}$$

$$\begin{aligned} A^-_{-3}|\mathbf{r}\rangle = \epsilon\Big\{ & -\mathbf{r}(-3) + \tfrac{2d-16}{3}\mathbf{k}(-3) + 3\mathbf{r}(-2)\mathbf{k}(-1) - \sum_{i=1}^{d-2}\boldsymbol{\xi}_i(-2)\boldsymbol{\xi}_i(-1) + \tfrac{3}{2}\mathbf{k}(-2)\mathbf{r}(-1) \\ & + \tfrac{35-4d}{2}\mathbf{k}(-2)\mathbf{k}(-1) - \tfrac{9}{2}\mathbf{r}(-1)\mathbf{k}(-1)^2 + 2\sum_{i=1}^{d-2}\boldsymbol{\xi}_i(-1)^2\mathbf{k}(-1) \\ & + \tfrac{8d-79}{6}\mathbf{k}(-1)^3\Big\}|\mathbf{r}-3\mathbf{k}\rangle, \end{aligned} \tag{A.15}$$

$$\begin{aligned} A^i_{-1}A^j_{-1}A^-_{-2}|\mathbf{r}\rangle = \Big\{ & -\boldsymbol{\xi}_i(-3)\boldsymbol{\xi}_j(-1) - \boldsymbol{\xi}_j(-3)\boldsymbol{\xi}_i(-1) \\ & + 3\big[\boldsymbol{\xi}_i(-2)\boldsymbol{\xi}_j(-1) + \boldsymbol{\xi}_j(-2)\boldsymbol{\xi}_i(-1)\big]\mathbf{k}(-1) \\ & - \mathbf{r}(-2)\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1) + \tfrac{d+2}{4}\mathbf{k}(-2)\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1) \\ & + 2\mathbf{r}(-1)\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1)\mathbf{k}(-1) - \tfrac{d+18}{4}\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1)\mathbf{k}(-1)^2 \\ & - \tfrac{1}{2}\sum_{k=1}^{d-2}\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1)\boldsymbol{\xi}_k(-1)^2 \\ & + \delta_{ij}\Big[\tfrac{1}{4}\sum_{k=1}^{d-2}\boldsymbol{\xi}_k(-1)^2\big[\mathbf{k}(-2) - \mathbf{k}(-1)^2\big] + \tfrac{1}{2}\mathbf{r}(-2)\big[\mathbf{k}(-2) - \mathbf{k}(-1)^2\big] \\ & + \mathbf{r}(-1)\big[\mathbf{k}(-1)^3 - \mathbf{k}(-1)\mathbf{k}(-2)\big] - \tfrac{d+18}{8}\mathbf{k}(-1)^4 \\ & + \tfrac{d+22}{4}\mathbf{k}(-1)^2\mathbf{k}(-2) - 4\mathbf{k}(-1)\mathbf{k}(-3) \\ & - \tfrac{d+2}{8}\mathbf{k}(-2)^2 + \mathbf{k}(-4)\Big]\Big\}|\mathbf{r}-4\mathbf{k}\rangle, \end{aligned} \tag{A.16}$$

$$\begin{aligned} A^i_{-2}A^-_{-2}|\mathbf{r}\rangle = \Big\{ & -2\boldsymbol{\xi}_i(-4) + 8\boldsymbol{\xi}_i(-3)\mathbf{k}(-1) + 2\mathbf{k}(-3)\boldsymbol{\xi}_i(-1) - \mathbf{r}(-2)\boldsymbol{\xi}_i(-2) \\ & + 2\mathbf{r}(-2)\boldsymbol{\xi}_i(-1)\mathbf{k}(-1) + \tfrac{d+2}{4}\boldsymbol{\xi}_i(-2)\mathbf{k}(-2) + 2\boldsymbol{\xi}_i(-2)\mathbf{r}(-1)\mathbf{k}(-1) \\ & - \tfrac{1}{2}\sum_{j=1}^{d-2}\boldsymbol{\xi}_i(-2)\boldsymbol{\xi}_j(-1)^2 - \tfrac{d+42}{4}\boldsymbol{\xi}_i(-2)\mathbf{k}(-1)^2 - \tfrac{d+14}{2}\mathbf{k}(-2)\boldsymbol{\xi}_i(-1)\mathbf{k}(-1) \\ & - 4\mathbf{r}(-1)\boldsymbol{\xi}_i(-1)\mathbf{k}(-1)^2 + \sum_{j=1}^{d-2}\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1)^2\mathbf{k}(-1) \\ & + \tfrac{d+18}{2}\boldsymbol{\xi}_i(-1)\mathbf{k}(-1)^3\Big\}|\mathbf{r}-4\mathbf{k}\rangle, \end{aligned} \tag{A.17}$$

$$\begin{aligned} A^i_{-1}A^-_{-3}|\mathbf{r}\rangle = \Big\{ & -\boldsymbol{\xi}_i(-4) + 4\boldsymbol{\xi}_i(-3)\mathbf{k}(-1) - \mathbf{r}(-3)\boldsymbol{\xi}_i(-1) + \tfrac{2d-13}{3}\mathbf{k}(-3)\boldsymbol{\xi}_i(-1) \\ & + \tfrac{5}{2}\boldsymbol{\xi}_i(-2)\mathbf{k}(-2) + 3\mathbf{r}(-2)\boldsymbol{\xi}_i(-1)\mathbf{k}(-1) - \sum_{j=1}^{d-2}\boldsymbol{\xi}_j(-2)\boldsymbol{\xi}_j(-1)\boldsymbol{\xi}_i(-1) \\ & - \tfrac{17}{2}\boldsymbol{\xi}_i(-2)\mathbf{k}(-1)^2 + \tfrac{3}{2}\mathbf{k}(-2)\mathbf{r}(-1)\boldsymbol{\xi}_i(-1) + (11-2d)\mathbf{k}(-2)\boldsymbol{\xi}_i(-1)\mathbf{k}(-1) \\ & - \tfrac{9}{2}\mathbf{r}(-1)\boldsymbol{\xi}_i(-1)\mathbf{k}(-1)^2 + 2\sum_{j=1}^{d-2}\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1)^2\mathbf{k}(-1) \\ & + \tfrac{4d-14}{3}\boldsymbol{\xi}_i(-1)\mathbf{k}(-1)^3\Big\}|\mathbf{r}-4\mathbf{k}\rangle, \end{aligned} \tag{A.18}$$

$$A^-_{-4}|\mathbf{r}\rangle = \Big\{ -\mathbf{r}(-4) + \tfrac{5d-42}{4}\mathbf{k}(-4) + 4\mathbf{r}(-3)\mathbf{k}(-1) - \sum_{i=1}^{d-2}\boldsymbol{\xi}_i(-3)\boldsymbol{\xi}_i(-1) + \tfrac{4}{3}\mathbf{k}(-3)\mathbf{r}(-1)$$

$$
\begin{aligned}
&+ \tfrac{134-15d}{3}\mathbf{k}(-3)\mathbf{k}(-1) + 2\mathbf{r}(-2)\mathbf{k}(-2) - \tfrac{1}{2}\sum_{i=1}^{d-2}\xi_i(-2)^2 + \tfrac{122-13d}{8}\mathbf{k}(-2)^2 \\
&- 8\mathbf{r}(-2)\mathbf{k}(-1)^2 + 5\sum_{i=1}^{d-2}\boldsymbol{\xi}_i(-2)\boldsymbol{\xi}_i(-1)\mathbf{k}(-1) - 8\mathbf{k}(-2)\mathbf{r}(-1)\mathbf{k}(-1) \\
&+ \tfrac{3}{2}\sum_{i=1}^{d-2}\mathbf{k}(-2)\boldsymbol{\xi}_i(-1)^2 + \tfrac{43d-422}{4}\mathbf{k}(-2)\mathbf{k}(-1)^2 + \tfrac{32}{3}\mathbf{r}(-1)\mathbf{k}(-1)^3 \\
&- \tfrac{13}{2}\sum_{i=1}^{d-2}\boldsymbol{\xi}_i(-1)^2\mathbf{k}(-1)^2 + \tfrac{1394-129d}{24}\mathbf{k}(-1)^4\Big\}|\mathbf{r}-4\mathbf{k}\rangle, \qquad \text{(A.19)}
\end{aligned}
$$

$$
\begin{aligned}
A^{-}_{-2}A^{-}_{-2}|\mathbf{r}\rangle = \Big\{&2\mathbf{r}(-4) + \tfrac{26-d}{4}\mathbf{k}(-4) - 8\mathbf{r}(-3)\mathbf{k}(-1) + \sum_{i=1}^{d-2}\boldsymbol{\xi}_i(-3)\boldsymbol{\xi}_i(-1) \\
&+ \tfrac{3d-78}{3}\mathbf{k}(-3)\mathbf{k}(-1) + \mathbf{r}(-2)^2 + \tfrac{6-d}{2}\mathbf{r}(-2)\mathbf{k}(-2) + \tfrac{d^2-16d-20}{16}\mathbf{k}(-2)^2 \\
&- 4\mathbf{r}(-2)\mathbf{r}(-1)\mathbf{k}(-1) + \sum_{i=1}^{d-2}\mathbf{r}(-2)\boldsymbol{\xi}_i(-1)^2 + \tfrac{d+10}{2}\mathbf{r}(-2)\mathbf{k}(-1)^2 \\
&- 3\sum_{i=1}^{d-2}\boldsymbol{\xi}_i(-2)\boldsymbol{\xi}_i(-1)\mathbf{k}(-1) + (d-6)\mathbf{k}(-2)\mathbf{r}(-1)\mathbf{k}(-1) \\
&+ \tfrac{6-d}{4}\sum_{i=1}^{d-2}\mathbf{k}(-2)\boldsymbol{\xi}_i(-1)^2 + \tfrac{-d^2+12d+284}{8}\mathbf{k}(-2)\mathbf{k}(-1)^2 \\
&+ 4\mathbf{r}(-1)^2\mathbf{k}(-1)^2 - 2\sum_{i=1}^{d-2}\mathbf{r}(-1)\boldsymbol{\xi}_i(-1)^2\mathbf{k}(-1) + (6-d)\mathbf{r}(-1)\mathbf{k}(-1)^3 \\
&+ \tfrac{1}{4}\sum_{i,j=1}^{d-2}\boldsymbol{\xi}_i(-1)^2\boldsymbol{\xi}_j(-1)^2 + \tfrac{d-2}{4}\sum_{i=1}^{d-2}\boldsymbol{\xi}_i(-1)^2\mathbf{k}(-1)^2 \\
&+ \tfrac{d^2-20d-236}{16}\mathbf{k}(-1)^4\Big\}|\mathbf{r}-4\mathbf{k}\rangle. \qquad \text{(A.20)}
\end{aligned}
$$

Appendix B:

DDF commutators

Some commutators for $\mathbf{r}\cdot\mathbf{s} = 0$:

$$
\begin{aligned}
\big[|\mathbf{s}\rangle, A^i_{-1}A^j_{-1}|\mathbf{r}\rangle\big] = \epsilon'\Big\{&\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1)\mathrm{s}(-1) - \tfrac{1}{2}\big[\delta_{i\mathbf{s}}\boldsymbol{\xi}_j(-1) + \delta_{j\mathbf{s}}\boldsymbol{\xi}_i(-1)\big]\big[\mathrm{s}(-1)^2 + \mathrm{s}(-2)\big] \\
&+ \tfrac{1}{6}\delta_{ij}\big[\mathrm{s}(-1)^3 - 3\mathrm{s}(-1)^2\mathbf{k}(-1) + 3\mathrm{s}(-1)\mathbf{k}(-1)^2 - 3\mathrm{s}(-1)\mathbf{k}(-2) \\
&\quad + 3\mathrm{s}(-1)\mathrm{s}(-2) - 3\mathrm{s}(-2)\mathbf{k}(-1) + 2\mathrm{s}(-3)\big] \\
&+ \tfrac{1}{6}\delta_{i\mathbf{s}}\delta_{j\mathbf{s}}\big[\mathrm{s}(-1)^3 + 3\mathrm{s}(-1)\mathrm{s}(-2) + 2\mathrm{s}(-3)\big]\Big\}|\mathbf{r} - 2\mathbf{k} + \mathbf{s}\rangle, \qquad \text{(B.1)}
\end{aligned}
$$

$$
\begin{aligned}
\big[|\mathbf{s}\rangle, A^i_{-2}|\mathbf{r}\rangle\big] = \epsilon'\Big\{&\boldsymbol{\xi}_i(-2)\mathrm{s}(-1) + \boldsymbol{\xi}_i(-1)\big[\mathrm{s}(-1)^2 - 2\mathrm{s}(-1)\mathbf{k}(-1) + \mathrm{s}(-2)\big] \\
&- \tfrac{1}{2}\delta_{i\mathbf{s}}\big[\mathrm{s}(-1)^3 - 2\mathrm{s}(-1)^2\mathbf{k}(-1) + 3\mathrm{s}(-1)\mathrm{s}(-2) \\
&\quad - 2\mathbf{k}(-1)\mathrm{s}(-2) + 2\mathrm{s}(-3)\big]\Big\}|\mathbf{r} - 2\mathbf{k} + \mathbf{s}\rangle, \qquad \text{(B.2)}
\end{aligned}
$$

$$
\begin{aligned}
\big[A^{i'}_{-1}|\mathbf{s}\rangle, A^j_{-1}|\mathbf{r}\rangle\big] = \epsilon'\Big\{&\delta_{j\mathbf{s}}\boldsymbol{\eta}_{i'}(-3) - \boldsymbol{\eta}_{i'}(-2)\boldsymbol{\xi}_j(-1) + \delta_{j\mathbf{s}}\boldsymbol{\eta}_{i'}(-2)\big[\mathrm{s}(-1) - \mathbf{k}(-1)\big] \\
&- \boldsymbol{\eta}_{i'}(-1)\boldsymbol{\xi}_j(-1)\big[\mathrm{s}(-1) - \mathbf{k}(-1)\big] \\
&- \tfrac{1}{2}\big[\delta_{i'\mathbf{r}}\boldsymbol{\xi}_j(-1) - \delta_{j\mathbf{s}}\boldsymbol{\eta}_{i'}(-1)\big]\big[\mathrm{s}(-1)^2 - 2\mathrm{s}(-1)\mathbf{k}(-1) + \mathbf{k}(-1)^2 \\
&\qquad + \mathrm{s}(-2) - \mathbf{k}(-2)\big] \\
&+ \tfrac{1}{6}\big[\delta_{i'\mathbf{r}}\delta_{j\mathbf{s}} - g_{i'j}\big]\big[\mathrm{s}(-1)^3 - 3\mathrm{s}(-1)^2\mathbf{k}(-1) + 3\mathrm{s}(-1)\mathbf{k}(-1)^2 \\
&\qquad - \mathbf{k}(-1)^3 + 3\mathrm{s}(-2)\mathrm{s}(-1) - 3\mathrm{s}(-2)\mathbf{k}(-1) \\
&\qquad - 3\mathbf{k}(-2)\mathrm{s}(-1) + 3\mathbf{k}(-2)\mathbf{k}(-1) \\
&\qquad + 2\mathrm{s}(-3) - 2\mathbf{k}(-3)\big]\Big\}|\mathbf{r} - 2\mathbf{k} + \mathbf{s}\rangle. \qquad \text{(B.3)}
\end{aligned}
$$

Some commutators for $\mathbf{r}\cdot\mathbf{s} = 1$:

$$
\begin{aligned}
&\big[|\mathbf{s}\rangle, A^i_{-1}A^j_{-1}A^k_{-1}|\mathbf{r}\rangle\big] \\
&\quad = \epsilon''\Big\{\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1)\boldsymbol{\xi}_k(-1)\mathrm{s}(-1) - \tfrac{1}{2}\big[\delta_{i\mathbf{s}}\boldsymbol{\xi}_j(-1)\boldsymbol{\xi}_k(-1) + \delta_{j\mathbf{s}}\boldsymbol{\xi}_k(-1)\boldsymbol{\xi}_i(-1) \\
&\qquad + \delta_{k\mathbf{s}}\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1)\big]\big[\mathrm{s}(-1)^2 + \mathrm{s}(-2)\big] \\
&\quad + \tfrac{1}{6}\big[\delta_{ij}\boldsymbol{\xi}_k(-1) + \delta_{jk}\boldsymbol{\xi}_i(-1) + \delta_{ki}\boldsymbol{\xi}_j(-1)\big]\big[\mathrm{s}(-1)^3 - 3\mathrm{s}(-1)^2\mathbf{k}(-1) \\
&\qquad + 3\mathrm{s}(-1)\mathbf{k}(-1)^2 - 3\mathrm{s}(-1)\mathbf{k}(-2) + 3\mathrm{s}(-1)\mathrm{s}(-2) \\
&\qquad - 3\mathrm{s}(-2)\mathbf{k}(-1) + 2\mathrm{s}(-3)\big] \\
&\quad + \tfrac{1}{6}\big[\delta_{i\mathbf{s}}\delta_{j\mathbf{s}}\boldsymbol{\xi}_k(-1) + \delta_{j\mathbf{s}}\delta_{k\mathbf{s}}\boldsymbol{\xi}_i(-1) \\
&\qquad + \delta_{k\mathbf{s}}\delta_{i\mathbf{s}}\boldsymbol{\xi}_j(-1)\big]\big[\mathrm{s}(-1)^3 + 3\mathrm{s}(-1)\mathrm{s}(-2) + 2\mathrm{s}(-3)\big] \\
&\quad - \tfrac{1}{24}\delta_{i\mathbf{s}}\delta_{j\mathbf{s}}\delta_{k\mathbf{s}}\big[\mathrm{s}(-1)^4 + 6\mathrm{s}(-1)^2\mathrm{s}(-2) + 3\mathrm{s}(-2)^2 \\
&\qquad\qquad + 8\mathrm{s}(-1)\mathrm{s}(-3) + 6\mathrm{s}(-4)\big] \\
&\quad - \tfrac{1}{24}\big[\delta_{i\mathbf{s}}\delta_{jk} + \delta_{j\mathbf{s}}\delta_{ki} + \delta_{k\mathbf{s}}\delta_{ij}\big]\big[\mathrm{s}(-1)^4 - 4\mathrm{s}(-1)^3\mathbf{k}(-1) - 6\mathrm{s}(-1)^2\mathbf{k}(-2) \\
&\qquad + 6\mathrm{s}(-1)^2\mathbf{k}(-1)^2 + 6\mathrm{s}(-1)^2\mathrm{s}(-2) - 12\mathrm{s}(-1)\mathbf{k}(-1)\mathrm{s}(-2) \\
&\qquad + 6\mathbf{k}(-1)^2\mathrm{s}(-2) - 6\mathrm{s}(-2)\mathbf{k}(-2) + 3\mathrm{s}(-2)^2 \\
&\qquad + 8\mathrm{s}(-1)\mathrm{s}(-3) - 8\mathbf{k}(-1)\mathrm{s}(-3) + 6\mathrm{s}(-4)\big]\Big\}|\mathbf{r} - 3\mathbf{k} + \mathbf{s}\rangle, \qquad \text{(B.4)}
\end{aligned}
$$

$$
\begin{aligned}
&\big[|\mathbf{s}\rangle, A^i_{-2}A^j_{-1}|\mathbf{r}\rangle\big] \\
&\quad = \epsilon''\Big\{\boldsymbol{\xi}_i(-2)\boldsymbol{\xi}_j(-1)\mathrm{s}(-1) - \tfrac{1}{2}\delta_{j\mathbf{s}}\boldsymbol{\xi}_i(-2)\big[\mathrm{s}(-1)^2 + \mathrm{s}(-2)\big] \\
&\qquad + \boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1)\big[\mathrm{s}(-1)^2 - 2\mathrm{s}(-1)\mathbf{k}(-1) + \mathrm{s}(-2)\big]
\end{aligned}
$$

$$- \tfrac{1}{3}\delta_{js}\boldsymbol{\xi}_i(-1)\big[\mathrm{s}(-1)^3 - 3\mathrm{s}(-1)^2\mathbf{k}(-1) + 3\mathrm{s}(-1)\mathrm{s}(-2)$$
$$- 3\mathbf{k}(-1)\mathrm{s}(-2) + 2\mathrm{s}(-3)\big]$$
$$- \tfrac{1}{2}\delta_{is}\boldsymbol{\xi}_j(-1)\big[\mathrm{s}(-1)^3 - 2\mathrm{s}(-1)^2\mathbf{k}(-1) + 3\mathrm{s}(-1)\mathrm{s}(-2)$$
$$- 2\mathbf{k}(-1)\mathrm{s}(-2) + 2\mathrm{s}(-3)\big]$$
$$+ \tfrac{1}{24}\delta_{is}\delta_{js}\big[3\mathrm{s}(-1)^4 - 8\mathrm{s}(-1)^3\mathbf{k}(-1) + 18\mathrm{s}(-1)^2\mathrm{s}(-2)$$
$$+ 9\mathrm{s}(-2)^2 - 24\mathrm{s}(-1)\mathbf{k}(-1)\mathrm{s}(-2) - 16\mathbf{k}(-1)\mathrm{s}(-3)$$
$$+ 24\mathrm{s}(-1)\mathrm{s}(-3) + 18\mathrm{s}(-4)\big]$$
$$+ \tfrac{1}{6}\delta_{ij}\big[\mathrm{s}(-1)^4 - 6\mathrm{s}(-1)^3\mathbf{k}(-1) + 12\mathrm{s}(-1)^2\mathbf{k}(-1)^2 - 8\mathrm{s}(-1)\mathbf{k}(-1)^3$$
$$+ 12\mathrm{s}(-1)\mathbf{k}(-1)\mathbf{k}(-2) - 6\mathrm{s}(-1)^2\mathbf{k}(-2) + 12\mathbf{k}(-1)^2\mathrm{s}(-2)$$
$$- 18\mathrm{s}(-1)\mathbf{k}(-1)\mathrm{s}(-2) + 6\mathrm{s}(-1)^2\mathrm{s}(-2) - 6\mathrm{s}(-2)\mathbf{k}(-2)$$
$$+ 3\mathrm{s}(-2)^2 - 4\mathrm{s}(-1)\mathbf{k}(-3) + 8\mathrm{s}(-1)\mathrm{s}(-3)$$
$$- 12\mathbf{k}(-1)\mathrm{s}(-3) + 6\mathrm{s}(-4)\big]\Big\}|\mathbf{r} - 3\mathbf{k} + \mathrm{s}\rangle, \tag{B.5}$$

$$\big[|\mathrm{s}\rangle, A^i_{-3}|\mathbf{r}\rangle\big]$$
$$= \epsilon''\Big\{\boldsymbol{\xi}_i(-3)\mathrm{s}(-1) + \tfrac{3}{2}\boldsymbol{\xi}_i(-2)\big[\mathrm{s}(-1)^2 - 2\mathrm{s}(-1)\mathbf{k}(-1) + \mathrm{s}(-2)\big]$$
$$+ \tfrac{1}{2}\boldsymbol{\xi}_i(-1)\big[2\mathrm{s}(-1)^3 - 9\mathrm{s}(-1)^2\mathbf{k}(-1) + 9\mathrm{s}(-1)\mathbf{k}(-1)^2 + 6\mathrm{s}(-1)\mathrm{s}(-2)$$
$$- 3\mathrm{s}(-1)\mathbf{k}(-2) - 9\mathbf{k}(-1)\mathrm{s}(-2) + 4\mathrm{s}(-3)\big]$$
$$- \tfrac{1}{12}\delta_{is}\big[5\mathrm{s}(-1)^4 - 24\mathrm{s}(-1)^3\mathbf{k}(-1) + 27\mathrm{s}(-1)^2\mathbf{k}(-1)^2 - 9\mathrm{s}(-1)^2\mathbf{k}(-2)$$
$$+ 30\mathrm{s}(-1)^2\mathrm{s}(-2) - 72\mathrm{s}(-1)\mathbf{k}(-1)\mathrm{s}(-2) + 27\mathbf{k}(-1)^2\mathrm{s}(-2)$$
$$+ 15\mathrm{s}(-2)^2 - 9\mathrm{s}(-2)\mathbf{k}(-2) + 40\mathrm{s}(-1)\mathrm{s}(-3)$$
$$- 48\mathbf{k}(-1)\mathrm{s}(-3) + 30\mathrm{s}(-4)\big]\Big\}|\mathbf{r} - 3\mathbf{k} + \mathrm{s}\rangle, \tag{B.6}$$

$$\big[A^{i'}_{-1}|\mathrm{s}\rangle, A^j_{-1}A^k_{-1}|\mathbf{r}\rangle\big]$$

$$= \epsilon''\Big\{\big[\delta_{js}\delta_{ks} + \delta_{jk}\big]\boldsymbol{\eta}_{i'}(-4) + \delta_{jk}\boldsymbol{\eta}_{i'}(-3)\big[\mathrm{s}(-1) - 2\mathbf{k}(-1)\big]$$
$$+ \delta_{js}\delta_{ks}\boldsymbol{\eta}_{i'}(-3)\big[\mathrm{s}(-1) - \mathbf{k}(-1)\big] - \boldsymbol{\eta}_{i'}(-3)\big[\delta_{js}\boldsymbol{\xi}_k(-1) + \delta_{ks}\boldsymbol{\xi}_j(-1)\big]$$
$$+ \boldsymbol{\eta}_{i'}(-2)\boldsymbol{\xi}_j(-1)\boldsymbol{\xi}_k(-1) + \boldsymbol{\eta}_{i'}(-1)\boldsymbol{\xi}_j(-1)\boldsymbol{\xi}_k(-1)\big[\mathrm{s}(-1) - \mathbf{k}(-1)\big]$$
$$- \boldsymbol{\eta}_{i'}(-2)\big[\delta_{js}\boldsymbol{\xi}_k(-1) + \delta_{ks}\boldsymbol{\xi}_j(-1)\big]\big[\mathrm{s}(-1) - \mathbf{k}(-1)\big]$$
$$+ \tfrac{1}{2}\delta_{jk}\boldsymbol{\eta}_{i'}(-2)\big[\mathrm{s}(-1)^2 - 4\mathrm{s}(-1)\mathbf{k}(-1) + \mathrm{s}(-2) + 4\mathbf{k}(-1)^2 - 2\mathbf{k}(-2)\big]$$
$$+ \tfrac{1}{2}\big[\delta_{js}\delta_{ks}\boldsymbol{\eta}_{i'}(-2) + \delta_{i'\mathbf{r}}\boldsymbol{\xi}_j(-1)\boldsymbol{\xi}_k(-1)$$
$$- \delta_{js}\boldsymbol{\eta}_{i'}(-1)\boldsymbol{\xi}_k(-1) - \delta_{ks}\boldsymbol{\eta}_{i'}(-1)\boldsymbol{\xi}_j(-1)\big]$$
$$\big[\mathrm{s}(-1)^2 - 2\mathrm{s}(-1)\mathbf{k}(-1) + \mathrm{s}(-2) + \mathbf{k}(-1)^2 - \mathbf{k}(-2)\big]$$
$$+ \tfrac{1}{6}\big[\delta_{js}\delta_{ks}\boldsymbol{\eta}_{i'}(-1) + [\delta_{i'j} - \delta_{i'\mathbf{r}}\delta_{js}]\boldsymbol{\xi}_k(-1) + [\delta_{i'k} - \delta_{i'\mathbf{r}}\delta_{ks}]\boldsymbol{\xi}_j(-1)\big]$$
$$\big[\mathrm{s}(-1)^3 - 3\mathrm{s}(-1)^2\mathbf{k}(-1) + 3\mathrm{s}(-1)\mathbf{k}(-1)^2 - \mathbf{k}(-1)^3$$
$$+ 3\mathrm{s}(-2)\mathrm{s}(-1) - 3\mathrm{s}(-2)\mathbf{k}(-1) - 3\mathbf{k}(-2)\mathrm{s}(-1)$$
$$+ 3\mathbf{k}(-2)\mathbf{k}(-1) + 2\mathrm{s}(-3) - 2\mathbf{k}(-3)\big]$$
$$+ \tfrac{1}{6}\delta_{jk}\boldsymbol{\eta}_{i'}(-1)\big[\mathrm{s}(-1)^3 - 6\mathrm{s}(-1)^2\mathbf{k}(-1) + 12\mathrm{s}(-1)\mathbf{k}(-1)^2 - 7\mathbf{k}(-1)^3$$
$$+ 3\mathrm{s}(-2)\mathrm{s}(-1) - 6\mathrm{s}(-2)\mathbf{k}(-1) - 6\mathbf{k}(-2)\mathrm{s}(-1)$$
$$+ 9\mathbf{k}(-2)\mathbf{k}(-1) + 2\mathrm{s}(-3) - 2\mathbf{k}(-3)\big]$$
$$+ \tfrac{1}{24}\big[\delta_{i'\mathbf{r}}\delta_{js}\delta_{ks} - \delta_{i'j}\delta_{ks} - \delta_{i'k}\delta_{js}\big] \times$$
$$\big[\mathrm{s}(-1)^4 - 4\mathrm{s}(-1)^3\mathbf{k}(-1) + 6\mathrm{s}(-1)^2\mathrm{s}(-2) + 6\mathrm{s}(-1)^2\mathbf{k}(-1)^2$$
$$- 6\mathrm{s}(-1)^2\mathbf{k}(-2) - 12\mathrm{s}(-1)\mathrm{s}(-2)\mathbf{k}(-1) + 8\mathrm{s}(-1)\mathrm{s}(-3)$$
$$- 4\mathrm{s}(-1)\mathbf{k}(-1)^3 + 12\mathrm{s}(-1)\mathbf{k}(-1)\mathbf{k}(-2) - 8\mathrm{s}(-1)\mathbf{k}(-3)$$
$$+ 3\mathrm{s}(-2)^2 + 6\mathrm{s}(-2)\mathbf{k}(-1)^2 - 6\mathrm{s}(-2)\mathbf{k}(-2) - 8\mathrm{s}(-3)\mathbf{k}(-1)$$
$$+ 6\mathrm{s}(-4) + \mathbf{k}(-1)^4 - 6\mathbf{k}(-1)^2\mathbf{k}(-2) + 8\mathbf{k}(-1)\mathbf{k}(-3)$$
$$+ 3\mathbf{k}(-2)^2 - 6\mathbf{k}(-4)\big]$$

$$\begin{aligned}
&+\tfrac{1}{24}\delta_{i'\mathbf{r}}\delta_{jk}\big[\mathrm{s}(-1)^4-8\mathrm{s}(-1)^3\mathbf{k}(-1)+6\mathrm{s}(-1)^2\mathrm{s}(-2)\\
&\qquad+24\mathrm{s}(-1)^2\mathbf{k}(-1)^2-12\mathrm{s}(-1)^2\mathbf{k}(-2)\\
&\qquad-24\mathrm{s}(-1)\mathrm{s}(-2)\mathbf{k}(-1)+8\mathrm{s}(-1)\mathrm{s}(-3)\\
&\qquad-28\mathrm{s}(-1)\mathbf{k}(-1)^3+36\mathrm{s}(-1)\mathbf{k}(-1)\mathbf{k}(-2)\\
&\qquad-8\mathrm{s}(-1)\mathbf{k}(-3)+3\mathrm{s}(-2)^2+24\mathrm{s}(-2)\mathbf{k}(-1)^2\\
&\qquad-12\mathrm{s}(-2)\mathbf{k}(-2)-16\mathrm{s}(-3)\mathbf{k}(-1)+6\mathrm{s}(-4)+11\mathbf{k}(-1)^4\\
&\qquad-30\mathbf{k}(-1)^2\mathbf{k}(-2)+16\mathbf{k}(-1)\mathbf{k}(-3)\\
&\qquad+9\mathbf{k}(-2)^2-6\mathbf{k}(-4)\big]\Big\}|\mathbf{r}-4\mathbf{k}+\mathrm{s}\rangle, \qquad \text{(B.7)}
\end{aligned}$$

$$\begin{aligned}
&[A^{i'}_{-1}|\mathrm{s}\rangle, A^{j}_{-2}|\mathbf{r}\rangle]\\
&\quad=\epsilon''\Big\{-3\delta_{j\mathbf{s}}\boldsymbol{\eta}_{i'}(-4)+2\boldsymbol{\eta}_{i'}(-3)\boldsymbol{\xi}_j(-1)+\delta_{j\mathbf{s}}\boldsymbol{\eta}_{i'}(-3)\big[5\mathbf{k}(-1)-3\mathrm{s}(-1)\big]\\
&\qquad+\boldsymbol{\eta}_{i'}(-2)\boldsymbol{\xi}_j(-2)+2\boldsymbol{\eta}_{i'}(-2)\boldsymbol{\xi}_j(-1)\big[\mathrm{s}(-1)-2\mathbf{k}(-1)\big]\\
&\qquad+\boldsymbol{\xi}_j(-2)\boldsymbol{\eta}_{i'}(-1)\big[\mathrm{s}(-1)-\mathbf{k}(-1)\big]\\
&\qquad+\tfrac{1}{2}\delta_{j\mathbf{s}}\boldsymbol{\eta}_{i'}(-2)\big[3\mathbf{k}(-2)-7\mathbf{k}(-1)^2-3\mathrm{s}(-2)+10\mathrm{s}(-1)\mathbf{k}(-1)-3\mathrm{s}(-1)^2\big]\\
&\qquad+\tfrac{1}{2}\delta_{i'\mathbf{r}}\boldsymbol{\xi}_j(-2)\big[-\mathbf{k}(-2)+\mathbf{k}(-1)^2+\mathrm{s}(-2)-2\mathrm{s}(-1)\mathbf{k}(-1)+\mathrm{s}(-1)^2\big]\\
&\qquad+\boldsymbol{\eta}_{i'}(-1)\boldsymbol{\xi}_j(-1)\big[-\mathbf{k}(-2)+3\mathbf{k}(-1)^2+\mathrm{s}(-2)-4\mathrm{s}(-1)\mathbf{k}(-1)+\mathrm{s}(-1)^2\big]\\
&\qquad+\tfrac{1}{2}\delta_{j\mathbf{s}}\boldsymbol{\eta}_{i'}(-1)\big[2\mathbf{k}(-3)-5\mathbf{k}(-1)\mathbf{k}(-2)+3\mathbf{k}(-1)^3-2\mathrm{s}(-3)\\
&\qquad\qquad+5\mathrm{s}(-2)\mathbf{k}(-1)+3\mathrm{s}(-1)\mathbf{k}(-2)-7\mathrm{s}(-1)\mathbf{k}(-1)^2\\
&\qquad\qquad-3\mathrm{s}(-1)\mathrm{s}(-2)+5\mathrm{s}(-1)^2\mathbf{k}(-1)-\mathrm{s}(-1)^3\big]\\
&\qquad+\tfrac{1}{3}\delta_{i'\mathbf{r}}\boldsymbol{\xi}_j(-1)\big[-2\mathbf{k}(-3)+6\mathbf{k}(-1)\mathbf{k}(-2)-4\mathbf{k}(-1)^3+2\mathrm{s}(-3)\\
&\qquad\qquad-6\mathrm{s}(-2)\mathbf{k}(-1)-3\mathrm{s}(-1)\mathbf{k}(-2)+9\mathrm{s}(-1)\mathbf{k}(-1)^2\\
&\qquad\qquad+3\mathrm{s}(-1)\mathrm{s}(-2)-6\mathrm{s}(-1)^2\mathbf{k}(-1)+\mathrm{s}(-1)^3\big]\\
&\qquad+\tfrac{1}{24}\delta_{i'\mathbf{r}}\delta_{j\mathbf{s}}\big[-3\mathrm{s}(-1)^4+20\mathrm{s}(-1)^3\mathbf{k}(-1)-18\mathrm{s}(-1)^2\mathrm{s}(-2)\\
&\qquad\qquad-42\mathrm{s}(-1)^2\mathbf{k}(-1)^2+18\mathrm{s}(-1)^2\mathbf{k}(-2)+60\mathrm{s}(-1)\mathrm{s}(-2)\mathbf{k}(-1)\\
&\qquad\qquad-24\mathrm{s}(-1)\mathrm{s}(-3)+36\mathrm{s}(-1)\mathbf{k}(-1)^3-60\mathrm{s}(-1)\mathbf{k}(-1)\mathbf{k}(-2)\\
&\qquad\qquad+24\mathrm{s}(-1)\mathbf{k}(-3)-9\mathrm{s}(-2)^2-42\mathrm{s}(-2)\mathbf{k}(-1)^2\\
&\qquad\qquad+18\mathrm{s}(-2)\mathbf{k}(-2)+40\mathrm{s}(-3)\mathbf{k}(-1)-18\mathrm{s}(-4)-11\mathbf{k}(-1)^4\\
&\qquad\qquad+42\mathbf{k}(-1)^2\mathbf{k}(-2)-40\mathbf{k}(-1)\mathbf{k}(-3)-9\mathbf{k}(-2)^2+18\mathbf{k}(-4)\big]\\
&\qquad+\tfrac{1}{6}\delta_{i'j}\big[\mathrm{s}(-1)^4-6\mathrm{s}(-1)^3\mathbf{k}(-1)+6\mathrm{s}(-1)^2\mathrm{s}(-2)\\
&\qquad\qquad+12\mathrm{s}(-1)^2\mathbf{k}(-1)^2-6\mathrm{s}(-1)^2\mathbf{k}(-2)-18\mathrm{s}(-1)\mathrm{s}(-2)\mathbf{k}(-1)\\
&\qquad\qquad+8\mathrm{s}(-1)\mathrm{s}(-3)-10\mathrm{s}(-1)\mathbf{k}(-1)^3+18\mathrm{s}(-1)\mathbf{k}(-1)\mathbf{k}(-2)\\
&\qquad\qquad-8\mathrm{s}(-1)\mathbf{k}(-3)+3\mathrm{s}(-2)^2+12\mathrm{s}(-2)\mathbf{k}(-1)^2-6\mathrm{s}(-2)\mathbf{k}(-2)\\
&\qquad\qquad-12\mathrm{s}(-3)\mathbf{k}(-1)+6\mathrm{s}(-4)+3\mathbf{k}(-1)^4-12\mathbf{k}(-1)^2\mathbf{k}(-2)\\
&\qquad\qquad+12\mathbf{k}(-1)\mathbf{k}(-3)+3\mathbf{k}(-2)^2-6\mathbf{k}(-4)\big]\Big\}|\mathbf{r}-3\mathbf{k}+\mathrm{s}\rangle. \qquad \text{(B.8)}
\end{aligned}$$

APPENDIX C:

COMMUTATORS FOR Λ_7

The following commutators are written in terms of the basis (3.3.7) rather than (3.2.18) as in the main text. Because the contributions $\mathrm{L}_{(-1)}(\ldots)$ are needed to find the correct results in Sect. IV.4, we list them here. We put $\epsilon \equiv \epsilon(\mathbf{s},\mathbf{r})\epsilon(\mathbf{k},\mathbf{a})$ and stress again that we are dealing with different sets of DDF operators depending on the tachyon momenta they are acting on (see remarks before Eq. (4.4.4)).

$$
\begin{aligned}
\big[|\mathbf{s}\rangle, A^\alpha_{-1}A^\beta_{-1}|\mathbf{r}\rangle\big] = \epsilon\Big\{ &-\tfrac{1}{2}A^\alpha_{-2}A^\beta_{-1} - \tfrac{1}{2}A^\beta_{-2}A^\alpha_{-1} - A^\alpha_{-1}A^\beta_{-1}A^8_{-1} \\
&+ \delta^{\alpha\beta}\Big[\tfrac{1}{24}A^8_{-3} + \tfrac{1}{8}A^8_{-1}A^-_{-2} + \tfrac{1}{16}\sum_{\gamma=1}^{7} A^8_{-1}A^\gamma_{-1}A^\gamma_{-1} \\
&- \tfrac{5}{48}A^8_{-1}A^8_{-1}A^8_{-1}\Big]\Big\}|\mathbf{a}\rangle \\
&+ \mathrm{L}_{(-1)}\Big\{\tfrac{1}{2}\boldsymbol{\xi}_\alpha(-1)\boldsymbol{\xi}_\beta(-1) + \delta^{\alpha\beta}\Big[-\tfrac{1}{8}\boldsymbol{\xi}_8(-2) + \tfrac{1}{12}\boldsymbol{\Lambda}(-2) \\
&+ \tfrac{1}{4}\boldsymbol{\xi}_8(-1)^2 + \tfrac{1}{48}\boldsymbol{\Lambda}(-1)^2 \\
&- \tfrac{1}{8}\boldsymbol{\xi}_8(-1)\big[\boldsymbol{\Lambda}(-1) - \boldsymbol{\delta}(-1)\big]\Big]\Big\}|\boldsymbol{\Lambda}_7\rangle,
\end{aligned}
\tag{C.1}
$$

$$
\begin{aligned}
\big[|\mathbf{s}\rangle, A^\alpha_{-1}A^8_{-1}|\mathbf{r}\rangle\big] = \epsilon\Big\{ &\tfrac{1}{4}A^\alpha_{-3} + \tfrac{1}{2}A^\alpha_{-2}A^8_{-1} - \tfrac{1}{2}A^8_{-2}A^\alpha_{-1} - \tfrac{1}{4}A^\alpha_{-1}A^-_{-2} \\
&- \tfrac{1}{8}\sum_{\gamma=1}^{7} A^\alpha_{-1}A^\gamma_{-1}A^\gamma_{-1} - \tfrac{1}{8}A^\alpha_{-1}A^8_{-1}A^8_{-1}\Big\}|\mathbf{a}\rangle \\
&+ \mathrm{L}_{(-1)}\Big\{-\tfrac{1}{4}\boldsymbol{\xi}_\alpha(-2) - \tfrac{1}{4}\boldsymbol{\xi}_\alpha(-1)\big[2\boldsymbol{\xi}_8(-1) - \boldsymbol{\Lambda}(-1) \\
&+ 3\boldsymbol{\delta}(-1)\big]\Big\}|\boldsymbol{\Lambda}_7\rangle,
\end{aligned}
\tag{C.2}
$$

$$
\begin{aligned}
\big[|\mathbf{s}\rangle, A^8_{-1}A^8_{-1}|\mathbf{r}\rangle\big] = \epsilon\Big\{ &\tfrac{17}{24}A^8_{-3} + A^8_{-2}A^8_{-1} + \tfrac{1}{8}A^8_{-1}A^-_{-2} \\
&+ \tfrac{1}{16}\sum_{\gamma=1}^{7} A^8_{-1}A^\gamma_{-1}A^\gamma_{-1} + \tfrac{11}{48}A^8_{-1}A^8_{-1}A^8_{-1}\Big\}|\mathbf{a}\rangle \\
&+ \mathrm{L}_{(-1)}\Big\{-\tfrac{9}{8}\boldsymbol{\xi}_8(-2) + \tfrac{5}{12}\boldsymbol{\Lambda}(-2) - \boldsymbol{\delta}(-2) - \tfrac{1}{4}\boldsymbol{\xi}_8(-1)^2 + \tfrac{5}{48}\boldsymbol{\Lambda}(-1)^2 \\
&- \tfrac{1}{2}\boldsymbol{\delta}(-1)^2 - \tfrac{1}{8}\boldsymbol{\xi}_8(-1)\big[\boldsymbol{\Lambda}(-1) + 7\boldsymbol{\delta}(-1)\big]\Big\}|\boldsymbol{\Lambda}_7\rangle,
\end{aligned}
\tag{C.3}
$$

$$
\begin{aligned}
\big[|\mathbf{s}\rangle, A^\alpha_{-2}|\mathbf{r}\rangle\big] = \epsilon\Big\{ &-\tfrac{3}{4}A^\alpha_{-3} - \tfrac{1}{4}A^\alpha_{-1}A^-_{-2} - \tfrac{1}{8}\sum_{\gamma=1}^{7} A^\alpha_{-1}A^\gamma_{-1}A^\gamma_{-1} + \tfrac{7}{8}A^\alpha_{-1}A^8_{-1}A^8_{-1}\Big\}|\mathbf{a}\rangle \\
&+ \mathrm{L}_{(-1)}\Big\{\tfrac{1}{4}\boldsymbol{\xi}_\alpha(-2) - \tfrac{1}{4}\boldsymbol{\xi}_\alpha(-1)\big[4\boldsymbol{\xi}_8(-1) - \boldsymbol{\Lambda}(-1) + \boldsymbol{\delta}(-1)\big]\Big\}|\boldsymbol{\Lambda}_7\rangle,
\end{aligned}
\tag{C.4}
$$

$$
\begin{aligned}
\big[|\mathbf{s}\rangle, A^8_{-2}|\mathbf{r}\rangle\big] = \epsilon\Big\{ &-\tfrac{1}{2}A^8_{-3} + \tfrac{1}{2}A^8_{-1}A^-_{-2} + \tfrac{1}{4}\sum_{\gamma=1}^{7} A^8_{-1}A^\gamma_{-1}A^\gamma_{-1} + \tfrac{1}{4}A^8_{-1}A^8_{-1}A^8_{-1}\Big\}|\mathbf{a}\rangle \\
&+ \mathrm{L}_{(-1)}\Big\{-\tfrac{1}{2}\boldsymbol{\xi}_8(-2) + \tfrac{1}{2}\boldsymbol{\Lambda}(-2) - \boldsymbol{\delta}(-2) + \tfrac{1}{2}\boldsymbol{\xi}_8(-1)^2 + \tfrac{1}{8}\boldsymbol{\Lambda}(-1)^2 \\
&- \tfrac{1}{2}\boldsymbol{\delta}(-1)^2 - \tfrac{1}{2}\boldsymbol{\xi}_8(-1)\big[\boldsymbol{\Lambda}(-1) - \boldsymbol{\delta}(-1)\big]\Big\}|\boldsymbol{\Lambda}_7\rangle,
\end{aligned}
\tag{C.5}
$$

$$
\begin{aligned}
\big[A^\alpha_{-1}|\mathbf{s}\rangle, A^\beta_{-1}|\mathbf{r}\rangle\big] = \epsilon\Big\{ &-\tfrac{1}{2}A^\alpha_{-2}A^\beta_{-1} + \tfrac{1}{2}A^\beta_{-2}A^\alpha_{-1} + A^\alpha_{-1}A^\beta_{-1}A^8_{-1} \\
&- \delta^{\alpha\beta}\Big[\tfrac{1}{24}A^8_{-3} + \tfrac{1}{8}A^8_{-1}A^-_{-2} + \tfrac{1}{16}\sum_{\gamma=1}^{7} A^8_{-1}A^\gamma_{-1}A^\gamma_{-1}
\end{aligned}
$$

$$- \tfrac{5}{48} A^8_{-1} A^8_{-1} A^8_{-1} \Big] \Big\} |\mathbf{a}\rangle$$

$$+ \mathrm{L}_{(-1)} \Big\{ - \tfrac{1}{2} \boldsymbol{\xi}_\alpha(-1) \boldsymbol{\xi}_\beta(-1) - \delta^{\alpha\beta} \Big[- \tfrac{1}{8} \boldsymbol{\xi}_8(-2) + \tfrac{1}{12} \mathbf{\Lambda}(-2)$$

$$+ \tfrac{1}{4} \boldsymbol{\xi}_8(-1)^2 + \tfrac{1}{48} \mathbf{\Lambda}(-1)^2$$

$$- \tfrac{1}{8} \boldsymbol{\xi}_8(-1) \big[\mathbf{\Lambda}(-1) - \boldsymbol{\delta}(-1) \big] \Big] \Big\} |\mathbf{\Lambda}_7\rangle, \tag{C.6}$$

$$\big[A^\alpha_{-1} |\mathbf{s}\rangle, A^8_{-1} |\mathbf{r}\rangle \big] = \epsilon \Big\{ - \tfrac{1}{4} A^\alpha_{-3} + \tfrac{1}{2} A^\alpha_{-2} A^8_{-1} + \tfrac{1}{2} A^8_{-2} A^\alpha_{-1} + \tfrac{1}{4} A^\alpha_{-1} A^-_{-2}$$

$$+ \tfrac{1}{8} \sum_{\gamma=1}^{7} A^\alpha_{-1} A^\gamma_{-1} A^\gamma_{-1} + \tfrac{1}{8} A^\alpha_{-1} A^8_{-1} A^8_{-1} \Big\} |\mathbf{a}\rangle$$

$$+ \mathrm{L}_{(-1)} \Big\{ - \tfrac{3}{4} \boldsymbol{\xi}_\alpha(-2) - \tfrac{1}{4} \boldsymbol{\xi}_\alpha(-1) \big[- 2 \boldsymbol{\xi}_8(-1) + \mathbf{\Lambda}(-1)$$

$$+ \boldsymbol{\delta}(-1) \big] \Big\} |\mathbf{\Lambda}_7\rangle, \tag{C.7}$$

$$\big[A^8_{-1} |\mathbf{s}\rangle, A^8_{-1} |\mathbf{r}\rangle \big] = \epsilon \Big\{ \tfrac{17}{24} A^8_{-3} + \tfrac{1}{8} A^8_{-1} A^-_{-2} + \tfrac{1}{16} \sum_{\gamma=1}^{7} A^8_{-1} A^\gamma_{-1} A^\gamma_{-1} + \tfrac{11}{48} A^8_{-1} A^8_{-1} A^8_{-1} \Big\} |\mathbf{a}\rangle$$

$$+ \mathrm{L}_{(-1)} \Big\{ - \tfrac{9}{8} \boldsymbol{\xi}_8(-2) + \tfrac{5}{12} \mathbf{\Lambda}(-2) - \boldsymbol{\delta}(-2) - \tfrac{1}{4} \boldsymbol{\xi}_8(-1)^2 + \tfrac{5}{48} \mathbf{\Lambda}(-1)^2$$

$$- \tfrac{1}{2} \boldsymbol{\delta}(-1)^2 - \tfrac{1}{8} \boldsymbol{\xi}_8(-1) \big[\mathbf{\Lambda}(-1) - \boldsymbol{\delta}(-1) \big] \Big\} |\mathbf{\Lambda}_7\rangle. \tag{C.8}$$

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Zur Person	Name: Reinhold Willigis Gebert Geboren am/in: 12.12.1965, Hanau am Main Familienstand: Verheiratet, drei Kinder (*13.5.92, *6.7.93, *28.8.94)	
Schulausbildung	08/72 – 07/78	Grundschule und Förderstufe in Hanau
	09/78 – 06/85	Karl-Rehbein-Gymnasium in Hanau *Allgemeine Hochschulreife*
Wehrdienst	07/85 – 09/86	Gebirgsjägerbataillon 234 in Mittenwald
Studium	11/86 – 09/90	Studium der Physik an der TU München
	10/90 – 11/91	M.Sc.-Programm am King's College London, England *Master of Science in Mathematics* (mit Auszeichnung) Thema der Thesis: *New variables formalism in general relativity* (betreut von Dr. D.C. Robinson) Ehrung: *Lionel Cooper Memorial Prize 1991* (für den besten M.Sc.-Abschluß in Mathematik der Universität London)
Promotion	04/92 – 11/94	Doktorarbeit am II. Institut für Theoretische Physik der Universität Hamburg Thema: *Beyond Affine Kac-Moody Algebras in String Theory* (betreut von Prof. Dr. H. Nicolai)
Stipendien	11/86 – 09/94	Begabtenförderung der Konrad-Adenauer-Stiftung
	10/94 – 12/94	Graduiertenkolleg der Deutschen Forschungsgemeinschaft