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ISSN 0418-9833

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ISSN 0418-9833

EXACT SOLUTION OF MASTER EQUATIONS FOR A SIMPLE MODEL OF SELF-ORGANIZED BIOLOGICAL EVOLUTION

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ABSTRACT

The master equations for the random neighbour Bak-Sneppen model are solved explicitly.

1 Introduction

Recently, Bak and Sneppen proposed a model of biological evolution [1,2]. It is a dynamical system describing mutation and natural selection of interacting species. The most attractive point in this model is the phenomenon of self-organized criticality, which is studied intensively by means of several numerical and analytical metods [3-6]. One can hope that the Bak-Sneppen Model(BSM) represents an important universal type of critical behaviour wich is realized in many dynamical systems independent of their detailed properties. It would be natural to expect this from the point of view of present experience in the studies of critical phenomena.

The formulation of BSM is simple. It describes an ecosystem of N species which state is defined by the set $\{x_1, ..., x_N\}$ of Nnumbers, $0 \leq x_i \leq 1$. The state of the ith species x_i is called barrier. It characterizes the effective barrier towards further evolution of the species. The BSM dynamics is the following. Initially, each x_i is set to a randomly chosen value. At each time step the barrier x_i with minimal value and $K-1$ other barriers are replaced by K new random numbers. In the random neighbour model (RNBSM) $K-1$ replaced non-minimal bariers are chosen at random. In the local or nearest neighbour model (LBSM) these are the barriers of the nearest neigbours to the species with minimal barrier. In this version of the BSM the nearest neighbours of the species are assumed to be defined.

The study of LBSM is more complex compared to the RNBSM because of non-trivial topology of the interspecies interaction in the LBSM. The analytical study of the BSM is based mainly on the mean field approximation [2,3,6]. Some exact results are obtained for the RNBSM [3] only. In this paper we obtain the expilcit solution of the master equation used in [3] for the study of the RNBSM. We consider these equations for the case of an infinite ecosystem with an infinite number of species (in the thermodynamic limit) only. The exact stationary solution of the RNBSM master equations obtained in [3] is the asymptotic of the time dependent solution which is constructed by us.

2 Statement of problem

The basic quantity used in [3] to study the RNBSM is the probability that at time t the number n of barriers have values less then a fixed value λ. It is denoted in [3] as $P_n(t)$ and fullfils the following master equation representing the dynamics of RNBSM

$$P_n(t+1) = \lambda^2 P_{n-1}(t) + 2\lambda(1-\lambda)P_n(t) + (1-\lambda)^2 P_{n+1}(t), \; n \geq 3,$$

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$$P_2(t+1) = \lambda^2 P_1(t) + 2\lambda(1-\lambda)P_2(t) + (1-\lambda)^2 P_3(t) + \lambda^2 P_0(t), \tag{1}$$
$$P_1(t+1) = 2\lambda(1-\lambda)P_1(t) + (1-\lambda)^2 P_2(t) + 2\lambda(1-\lambda)P_0(t),$$
$$P_0(t+1) = (1-\lambda)^2 P_1(t) + (1-\lambda)^2 P_0(t).$$

The initial values $P_n(0)$ at $t=0$ are assumed to be given. The equations (1) define the values of probabilities $P_n(t)$ for $t > 0$. It can be considered as the complete specification of the RNBSM dynamical rules.

It is convenient for us to rewrite the system of equations (1) in the following form:

$$P_n(t+1) = aP_n(t) + bP_{n+1}(t) + cP_{n-1}(t)(1-\delta_{n,0}) + \tag{2}$$
$$(\alpha\delta_{n,0} + \beta\delta_{n,1} + \gamma\delta_{n,2})P_0(t),$$

where

$$a = 2\lambda(1-\lambda),\ b = (1-\lambda)^2,\ c = \lambda^2, \tag{3}$$
$$\alpha = (1-\lambda)(1-3\lambda), \beta = \lambda(2-3\lambda), \gamma = \lambda^2.$$

In virtue of (3)

$$a+b+c=1,\ \alpha+\beta+\gamma-b=0. \tag{4}$$

Hence, it follows from (2) the common necessary condition for probabilities $P_n(t)$:

$$\sum_n P_n(t+1) = (a+b+c)\sum_n P_n(t) + (\alpha+\beta+\gamma-b)P_0(t) = \sum_n P_n(t) = 1$$

In the more general case, when there are not representations of the form (3) for the parameters of dynamical system (2), but the relations (4) are fulfilled, $P_n(t)$ could be treated as a probability.

By introducing the operator

$$D_{m,n}(t,t') = \delta_{t+1,t'}\delta_{m,n} - \delta_{t,t'}L_{m,n}, \tag{5}$$

where

$$L_{m,n} \equiv a\delta_{m,n} + b\delta_{m+1,n} + c\delta_{m-1,n}(1-\delta_{m,0}) + (\alpha\delta_{m,0} + \beta\delta_{m,1} + \gamma\delta_{m,2})\delta_{0,n}$$

equation (2) can be rewritten in the following compact form:

$$(DP)_m(t) \equiv \sum_{t',n=0}^{\infty} D_{m,n}(t,t')P_n(t') = \delta_{t,0}P_m(0). \tag{6}$$

Thus, if the solution $\Delta_{m,n}(t,t')$ of the operator equation

$$D\Delta = 1 \tag{7}$$

is found, the solution of equation (6) can be constructed in the following way:

$$(P)_m(t) = \sum_{n=0}^{\infty} \Delta_{m,n}(t,0)P_n(0). \tag{8}$$

Hence, the problem of solving of master equations is reduced to an invertion of the linear operators D (5). There is not an obvious transformation, which diagonalizes D directly. In this case the inversion of D would be trivial. However, the important property of operator Δ, which makes the problem of its construction solvable, is that $\Delta_{m,n}$ for $n \neq 0$ can be expressed in a simple way through $\Delta_{0,n}$ and the equation for $\Delta_{0,n}$ can be solved explicitly.

3 Modified equations for Δ

Let us define the auxiliary operator

$$\Delta^0_{n,m}(t,t') \equiv \frac{1}{2\pi i}\oint \frac{dz(a+bz^{-1}+cz)^{t-t'-1}}{z^{m-n+1}}\theta(t-t'). \tag{9}$$

Here, by definition,

$$\theta(t) \equiv 0\ for\ t \leq 0,\ \ \theta(t) \equiv 1\ for\ t > 0.$$

Multiplying the both sides of equation (7) whith Δ^0, one obtains the equation of the form

$$\Delta^0 D\Delta = \Delta^0. \tag{10}$$

If the operator V is defined as follows

$$\Delta^0 D = 1 - V, \tag{11}$$

one can to verify directly that

$$V_{m,n}(t,t') = V_m(t,t')\delta_{0,n}, \tag{12}$$

where

$$V_m \equiv \frac{1}{2\pi i}\oint \frac{dz(a+bz^{-1}+cz)^{t-t'-1}}{z^{m+1}}(\alpha+\beta z+\gamma z^2 - bz^{-1})\theta(t-t'). \tag{13}$$

Thus, in virtue of (11) and (12) the equation (10) can be rewritten in the following form:

$$\Delta_{m,n} - V_m\Delta_{0,n} = \Delta^0_{m,n}. \tag{14}$$

Setting $m = 0$ in (14), one obtains the equation for $\Delta_{0,n}$:

$$\Delta_{0,n} - V_0 \Delta_{0,n} = \Delta^0_{0,n}. \tag{15}$$

The main result is that for solving of equation (14) for the matrix operator $\Delta_{m,n}$ it is enough to find the solution of equation (15) for its part $\Delta_{0,m}$. If it is known, the matrix elements $\Delta_{m,n}$ for $m \neq 0$ can be constructed in the following way:

$$\Delta_{m,n} = V_m \Delta_{0,n} + \Delta^0_{m,n}. \tag{16}$$

4 Useful integral transformation

We introduce an special integral transformation (SIT) for the functions of the complex variables, which will be necessary to solve the equation (15). Its definition for the function $f(z)$ is written in the following form:

$$\mathbf{I}^\nu_r\{f\}(t) \equiv \oint_r \frac{dz}{2\pi i z}(\nu + z + z^{-1})^{t-1} f(z)\theta(t). \tag{17}$$

Here, the integration contour is the circle $|z| = r$ of the radius r with the center at $z = 0$, ν is a parameter. It is assumed that the integral in the right hand side of (17) exists. We shall consider the SITs as the functions of integer variables. For the SITs we define a contruction operation in the following way.

$$\mathbf{I}^\nu_r\{f\} * \mathbf{I}^\nu_r\{g\}(t,t') \equiv \sum_{s=0}^{\infty} \mathbf{I}^\nu_r\{f\}(t-s)\mathbf{I}^\nu_r\{g\}(s-t').$$

For such defined SIT the following simple statments can be proven.

Lemma 1 If the function $f(z)$ is analytical for $z \neq 0$, $z \neq \infty$ and

$$f_R(z) \equiv f(z^{-1}),\ \bar{f} \equiv \frac{1}{2}(f + f_R).$$

then

$$\mathbf{I}^\nu_r\{f\} = \mathbf{I}^\nu_r\{f_R\} = \mathbf{I}^\nu_r\{\bar{f}\}.$$

Lemma 2 If the functions $f(z)$, $g(z)$ are analytical for $|z| \leq r$, then

$$\mathbf{I}^\nu_r\{f\} * \mathbf{I}^\nu_r\{g\} = \mathbf{I}^\nu_r\{f * g\},$$

where

$$f * g\,(z) \equiv \frac{z f(z) g(z)}{1 - z^2}.$$

With help of the SITs the equation (15) for $\Delta_{0,n}$ can be solved explicitly. They are convenient also for constructing of the matrix elements of operator Δ from (16).

5 Exact solution of equations for Δ

It follows from the definition (9) of operator Δ^0 that it is proportional to the SIT of power function:

$$\Delta^0_{m,n}(t,t') = U_{m,n}(t,t')\mathbf{I}^\nu_r\{z^{n-m}\}, \tag{18}$$

where

$$U_{m,n}(t,t') \equiv c^{\frac{m-n+t-t'-1}{2}} b^{\frac{n-m+t-t'-1}{2}},\ \nu \equiv \frac{a}{\sqrt{bc}} \tag{19}$$

From (13) one obtains the similar representation for the operator V_m:

$$V_m = U_{m,0}\mathbf{I}^\nu_r\{z^{-m}Q(z)\},\ Q(z) \equiv \alpha' + \beta' z + \gamma' z^2 - b' z^{-1}. \tag{20}$$

Here

$$\alpha' = \alpha,\ \beta' = \sqrt{\frac{b}{c}}\beta,\ \gamma' = \frac{b}{c}\gamma,\ b' = \sqrt{\frac{c}{b}}b = \sqrt{bc}$$

By virtue of lemma 1 the operator V_0 is represented by the SIT of a polynomial function:

$$V_0 = U_{0,0}\mathbf{I}^\nu_r\{Q(z)\} = U_{0,0}\mathbf{I}^\nu_r\{q(z)\},\ q(z) \equiv \alpha' + (\beta' - b')z + \gamma' z^2 \tag{21}$$

Thus, it is naturally to suppose that the operator $\Delta_{0,n}$ has the representation of the form:

$$\Delta_{0,n} = U_{0,n}\mathbf{I}^\nu_r\{f_n(z)\}. \tag{22}$$

It is assumed here that the function $f_n(z)$ is analytical for $|z| \leq r$. By using of (18), (19), (21), (22) the equation (15) can be writen in the form

$$\mathbf{I}^\nu_r\{f_n(z)\} - \frac{1}{\rho}\mathbf{I}^\nu_r\{q(z)\} * \mathbf{I}^\nu_r\{f_n(z)\} = \mathbf{I}^\nu_r\{z^n\},$$

$$\rho \equiv \sqrt{bc}$$

It is fulfilled, if

$$f_n(z) - \frac{f_n(z)q(z)z}{\rho(1-z^2)} = z^n$$

The solution of this equation

$$f_n(z) = \frac{\rho z^n(1-z^2)}{\rho(1-z^2) - q(z)z}$$

is obviously an analytical function for $|z| \leq r$ if the parameter r of SIT is chosen sufficiently small. Thus, we obtain the solution of equation (15) in the following form:

$$\Delta_{0,n} = U_{0,n}\mathbf{I}^\nu_r\{\frac{\rho z^n(1-z^2)}{\rho(1-z^2) - q(z)z}\} = U_{0,n}\mathbf{I}^\nu_r\{z^n + \frac{z^{n+1}q(z)}{\rho(1-z^2) - q(z)z}\} \tag{23}$$

Substituting (20),(23) in (16) gives by using of Lemma 1 and lemma 2 the following representations for the other matrix elements of Δ:

$$\Delta_{1,n} = U_{1,n}\mathbf{I}_r^{\nu}\{z^{n-1} + \frac{z^{n+1}q_1(z)}{\rho(1-z^2) - q(z)z}\}$$

$$\Delta_{m,n} = U_{m,n}\mathbf{I}_r^{\nu}\{z^{n-m} + \frac{z^{m+n-1}q_2(z)}{\rho(1-z^2) - q(z)z}\}$$

Here $q_1(z)$, $q_2(z)$ are the polynoms:

$$q_1(z) \equiv \beta' + (\alpha' + \gamma')z - b'z^2$$

$$q_2(z) \equiv \gamma' + \beta' z + \alpha' z^2 - b'z^3$$

and the parameter r of SIT is defined in the following way:

$$for |z| < r \quad \rho(1-z^2) - q(z)z \neq 0.$$

By using of (3) the operator Δ for the original master equations (1) can be represented as follows:

$$\Delta_{m,n}(t,t') =$$

$$\lambda^{m-n+t-t'-1}(1-\lambda)^{n-m+t-t'-1}\oint_r \frac{dz(1+z)^{2(t-t')-2}}{2\pi i z^{t-t'+m-n}}(1 + z^{2m-1}\frac{1-\lambda-\lambda z}{\lambda-(1-\lambda)z})\theta(t-t')$$

$$\Delta_{1,n}(t,t') =$$

$$\lambda^{t-t'-n}(1-\lambda)^{n+t-t'-2}\oint_r \frac{dz(1+z)^{2(t-t')-2}}{2\pi i z^{t-t'-n+1}}(1 + \frac{z^2(2-3\lambda-\lambda z)}{(z+1)(\lambda-(1-\lambda)z})\theta(t-t') =$$

$$\lambda^{t-t'-n}(1-\lambda)^{n+t-t'-2}\oint_r \frac{dz(1+z)^{2(t-t')-3}(1-z)(\lambda+(3\lambda-1)z+\lambda z^2)}{2\pi i z^{t-t'-n+1}(\lambda-(1-\lambda)z)}\theta(t-t')$$

$$\Delta_{0,n}(t,t') =$$

$$\lambda^{t-t'-n-1}(1-\lambda)^{t-t'+n-1}\oint_r \frac{dz(1+z)^{2(t-t')-2}}{2\pi i z^{t-t'-n}}(1 + \frac{z((1-\lambda)z+1-3\lambda)}{(z+1)(\lambda-(1-\lambda)z})\theta(t-t') =$$

$$\lambda^{t-t'-n}(1-\lambda)^{t-t'+n-1}\oint_r \frac{dz(1+z)^{2(t-t')-3}(1-z)}{2\pi i z^{t-t'-n}(\lambda-(1-\lambda)z)}\theta(t-t')$$

Here the integration contour is the circle $|z| = r$, $r < \lambda/(1-\lambda)$.

6 Conclusion

The obtained solution for Δ can be verified by its direct substitution in the equation (7). The asymptotic form of $P_n(t)$, for $t \to \infty$ which can be calculated with help of (8) is in agreement with results of [3]. In our paper we considered only the main mathematical problem arising for the description of the the self-organized criticality in the framework of RNBSM. However, our results allow to calculate all the quantities of interest for understanding the critical phenomena in RNBSM. We hope that they will be usefull for analytical and numerical studies of self-organized criticality in more komplex models. They could be also helpful for elaborating of renormalization group methods [7,8] for the BSM.

Acknowledgments

I would like to thank Prof. G. Mack, B.Mikeska and Y.Xylander for fruitful discussions and II.Institut for Theoretical Physics of Hamburg University for hospitality.

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