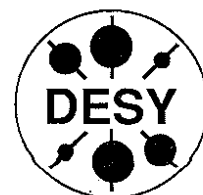


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IS A KNOT CLASSIFICATION POSSIBLE?

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Abstract

The goal of this paper is to discuss the possibility of finding an algorithm that can give all distinct knots up to a desired complexity. Two such algorithms are presented, one based on projections on a plane, the other on closed self-avoiding walks.

1. INTRODUCTION

Knot theory has been one of the most thoroughly studied and researched areas of Mathematical Physics during the last and the current decade. While work on knots has been carried out almost exclusively by mathematicians for about a century, during recent years a number of unexpected applications in science, from topological field theories in High Energy Physics [1] to DNA properties in Biology [2] has given an enormous boost to the subject and has expanded the number of disciplines related to it. In particular, it is known that Wilson lines are the observables in topological field theories. As Witten has shown in Ref. 1, when the gauge group is $SU(2)$, the correlation functions of Wilson lines obey relations similar to the ones satisfied for the Jones polynomials of the corresponding knots and links, which were initially introduced in knot theory. A knot classification would therefore be extremely useful in studying the observables of topological field theories and their correlation functions.

In spite, however, of the work done on knot theory and the recent progress that has been achieved, we have yet to come up with some method or formula that if appropriately applied would yield a complete table with all the distinct types of knot. Current knot tables that are in use have either been found by hand or through incomplete algorithms

that fail when the knot complexity becomes too large; it is hardly surprising, therefore, that such tables often differ with each other. One might think that the task of obtaining such tables should have been facilitated by the discovery of a number of knot and link characteristics ranging from the Alexander polynomials [3] to the Vassiliev invariants [4]. While such characteristics are indeed very helpful in distinguishing inequivalent knots, they fail to provide the full picture. This is not only due to the fact that it is possible for inequivalent knots to have the same polynomial, but in addition, given a polynomial, one cannot always tell which knot, if any, corresponds to this polynomial.

Until just a few years ago the best knot tables existing were based on the ones provided by Reidemeister and Seifert in the 1930's [5], which did not proceed further than the first 250 simplest knots. A few errors crept in these tables however; it was only in 1974, for example, that Perko proved that the knots 10_{161} and 10_{162} of Rolfsen's list are the same [6]. In 1985 Thistlethwaite came up with a table of about 13,000 knots [7], but the possibility of duplications remains and the problem is still far from being resolved. Similarly, a number of attempts to prove that the program is intrinsically solvable were either incomplete [8], or impractical [9].

In this paper we first provide the necessary definitions and background and review some of the relevant work carried out in the past. Then we develop two possible algorithms for knot classification. The first one is based on two-dimensional projections and bears some similarity to Thistlethwaite's work, while the other on the theory of self-avoiding walks.

2. DEFINITIONS

A knot is defined as a one-to-one homeomorphism from S^1 to some manifold \mathcal{M} .

According to the definition above, knots may not possess "double" points; the figure "8" for example is not a knot. Such figures however are sometimes called "singular knots" and play an important role in defining the Vassiliev invariants of (non-singular) knots.

In addition, the definition above is too "broad", and it obviously allows for a continuously infinite number of distinct knots; no algorithm may ever hope to classify the set of these knots. On the other side, however, if one considers as equivalent all knots isotopic to each other, one gets a set that is too "narrow", since all knots are isotopic to S^1 and therefore to each other. This is because intrinsically all knots are the same; only when

they are embedded in some target space one may possibly look for distinct classes of knots. Knot equivalence is therefore defined as follows.

Two knots K_1 and K_2 are called **ambient isotopic** and considered equivalent iff there is a homeomorphism f from $[0, 1]$ to the set of knots such that $f(0) = K_1$ and $f(1) = K_2$.

This definition may also be stated as follows. Let g_1 and g_2 be the functions from S^1 to \mathcal{M} defining K_1 and K_2 . The two knots are ambient isotopic iff a function $f(t, s)$ from $[0, 1] \times S^1$ to \mathcal{M} exists which is continuous with respect to both t and s and satisfies $f(0, s) = g_1(s)$, $f(1, s) = g_2(s)$, and $\forall t \in [0, 1] : s_1 \neq s_2 \Rightarrow f(t, s_1) \neq f(t, s_2)$.

Proving that ambient isotopy is an equivalence relation is trivial; one may easily check that $K \sim K$, $K_1 \sim K_2 \Rightarrow K_2 \sim K_1$ and $K_1 \sim K_2 \wedge K_2 \sim K_3 \Rightarrow K_1 \sim K_3$, where \sim denotes ambient isotopy. Therefore ambient isotopy divides knots into distinct classes; it is the classification and enumeration of such classes that we are going to study from now on.

Two subtle points shall be mentioned now. First, if one reverses orientation, that is replace $f(s)$ with $f(-s)$ as the defining function, one gets a different knot, even though these two knots consist of exactly the same points. The same thing happens of course by an orientation preserving reparametrization. When orientation is preserved, however, the two knots are always ambient isotopic, while when orientation is reversed, they need not be so. Similarly, when one replaces $f(s)$ with $-f(s)$ one gets the *mirror image* of $f(s)$; a knot may or may not be ambient isotopic to its mirror image. If it is, it is called **amphicheiral**, while if it is not, it is called **chiral**. In most of the paper, however, knots will be considered equivalent to both their mirror images and to inversely oriented knots, while in section 11 the problem of how to find which knots are invertible and amphicheiral will be raised once more.

Finally, before leaving this section, a few examples for various target spaces shall be provided, while from section 3 the discussion will be entirely directed towards solving the $\mathcal{M} = S^3$ case.

For $\mathcal{M} = R^1$ obviously no knots are allowed, while for $\mathcal{M} = S^1$ or $\mathcal{M} = R^2$ there are just two distinct classes of knots, one that is clockwise and one that is counterclockwise, which are related through orientation inversion. For $\mathcal{M} = S^2$ these two classes coincide. On the other side, for $\mathcal{M} = S^n$ and $\mathcal{M} = R^n$ for $n \geq 4$, all knots are equivalent to each other and just one class exists.

A more interesting case arises for the torus. $\mathcal{M} = T^2 = S^1 \times S^1$. While all homotopic knots are ambient isotopic and vice versa, not all homotopy classes are suitable to generate knots. Indeed a class (m, n) may only generate knots if the Greatest Common Denominator (GCD) of m and n is 1. If it is larger than 1, all loops belonging to such class will possess "double" points and thus do not fit the definition of knots. The set of knot classes is therefore $\{(m, n) : m, n \in Z \wedge \text{GCD}(m, n) = 1\}$.

The $\mathcal{M} = S^3$ case, however, is even more interesting since it still remains unsolved, and it is this case to be discussed from now on.

3. KNOTS ON S^3 ; THE REIDEMEISTER MOVES

While for $\mathcal{M} = T^2$ distinct knot classes exist due to the non-triviality of the fundamental group, S^3 possesses a trivial fundamental group and thus all loops are homotopic; non-trivial knots do exist, however, because although they can be continuously deformed to identity, such a process involves self-intersecting loops. It is impossible, for example, to deform the *trefoil* (Figure 1a) to the *unknot* (Figure 1b) without passing through a self-intersecting loop.

The question of knots in S^3 was drastically simplified and advanced during the 1920's when Reidemeister proved that knots may be studied through their projections in S^2 [10]. Given a certain knot, one may replace it with a two-dimensional projection such that no more than two knot points may be projected on the same point of S^2 , while the neighborhood of all double points should look as in Figure 2a (and not as in Figure 2b). The second condition can be more rigorously stated as follows. Let $f(s_1)$ and $f(s_2)$ be two points of S^3 that are projected at the same point of S^2 . The difference of the derivatives at these two points, $f'(s_2) - f'(s_1)$, should not be orthogonal to S^2 . In addition, one must keep track of overcrossings and undercrossings so that Figure 3a is considered different from 3b. Projections satisfying these conditions are called **regular**.

Once regular projections of two knots are given, the two knots are equivalent if one may continuously deform one projection to the other. Even however if such a continuous deformation is not possible, the two knots are equivalent if the projections are connected through a sequence of moves such as the ones shown in Figures 4a, 4b and 4c. These three kinds of moves are called the **Reidemeister moves** [11]. It is important to notice that no double points other than the ones shown in Figure 4 are allowed in the bold face

T A B L E I

Double Point	Overcrossing Number	Undercrossing Number
A	1	12
B	2	7
C	10	3
D	4	9
E	5	14
F	6	15
G	8	11
H	16	13

rectangulars, and subsequently the move of Figure 5 is not allowed. In addition, the part of the knot outside the bold face rectangular should not change during the move.

While one may easily notice that by performing any of these three moves, or any combination of them, one gets a knot equivalent to the initial one, the important result obtained by Reidemeister is the converse, meaning that these three kinds of moves are sufficient; if two knot projections cannot be connected through a sequence of just these three moves, the knots are definitely inequivalent. This result will be crucial when devising an algorithm to get distinct classes of inequivalent knots.

4. NAMING A KNOT

A knot, as defined earlier, is a one-to-one homeomorphism from S^1 to S^3 . No algorithm can of course give an actual knot, since knots are continuous objects while algorithms are discrete. It is therefore essential to provide a knot class with a discrete "name", that is a finite grouping of integer numbers which uniquely characterizes a knot class; a desirable algorithm is then going to be a computer program whose input corresponds to some desired knot complexity, while the output is a list of knot class "names" whose complexity does not exceed the input's value. (Such an input is necessary because the number of knot classes is infinite and without this input the program would run forever. The definition of a knot complexity will depend of course on the algorithm, and therefore two such definitions shall be given, one for each algorithm to be discussed, while an interesting point which shall be very briefly touched here is whether a knot that is more complex to another with respect to one algorithm, may be less complex with respect to another.)

Let us now consider a two-dimensional projection of a knot such as the one shown in Figure 6 satisfying the rules set in section 3. First one chooses a starting point on the projection and one of the two orientations; in Figure 6 this point is O and the orientation is clockwise at the neighborhood of O. As one travels starting from O along the chosen orientation on the knot projection, one assigns the double points encountered, successive natural numbers starting from 1 and going upwards, until one returns to O. Each double point is thus acquiring two such numbers, one for the overcrossing segment and one for the undercrossing. Thus for example the double points of Figure 6 are assigned the following numbers.

Each double point is now assigned a pair whose left and right elements are the overcrossing and the undercrossing numbers respectively. The name of the projection is the set of all such pairs. For the case of Figure 6, for example, the name of the knot is $\{(1, 12), (2, 7), (10, 3), (4, 9), (5, 14), (6, 15), (8, 11), (16, 13)\}$. The reason for such a choice is that while permuting the order of pairs does not affect the projection, permuting the order of the two numbers of a pair does.

Once a knot is assigned a name using the process above, or any other process for that reason, there are four questions one needs to answer before going ahead.

- 1) Can all knot classes be assigned such a name?
- 2) Is such a name unique for a certain class?
- 3) Is there always a knot class to be assigned for any possible name?
- 4) If so, is such a class unique?

The answers to these questions come out to be: yes, no, no, yes. As a result of the negative answers to the second and third questions, two problems arise.

- 1) How can one proceed to choose just one name for a particular class?
- 2) How can one distinguish the names that do correspond to knot classes from the ones that do not?

In the next two sections these two questions are going to be discussed. First, however, two points should be mentioned. One is the notion of knot complexity; here it is going to be the number of double points. Therefore once some natural number n is set as input, one expects to get as output all knot classes, named the way described in this section, that have at least one projection with not more than n double points.

The other is about the number of possible names. While for any natural number n this number is $\frac{(2n)!}{n!}$ since there are $(2n)!$ ways to order the numbers from 1 to $2n$ and $n!$ ways to order the n pairs, the actual number of knot classes is much smaller. Before proceeding it would be interesting to give a look at the following table.

T A B L E II

n	$(2n)! \div n!$	Actual Number	of Knots
0	1	1	1
1	2	0	0
2	12	0	0
3	120	1	1
4	1680	1	1
5	30240	2	2
6	665280	3	3
7	17297280	7	7
8	518918400	21	21
9	17643225600	49	49

5. CHOOSING A NAME

While in section 4 a procedure for naming knot projections was given, by doing so one is still far from reaching the ultimate goal. In order to get distinct knot classes one has to make sure that no knot class appears more than once under different names. To achieve now this goal one should first know which names correspond to equivalent knots and why, and then provide a criterion that will eliminate all names but one, so that the surviving name will appear in the output as the sole representative of its knot class.

There are two reasons for multiple names to be assigned to one and the same knot class. First, by varying the starting point O of section 4 and/or inverting orientation one gets an identical projection but possibly under a different name. Second, one may additionally perform Reidemeister moves on a certain projection and obtain different projections, which belong however to the same knot class. In each of these two cases a name will be affected as follows.

By varying the starting point while preserving orientation, all pairs (i, j) become $(k + i, k + j)$ for some $k \in \{0, 1, 2, \dots, 2n - 1\}$. If however orientation is reversed, all pairs (i, j) become instead $(k - i, k - j)$. In both cases the value of k depends on the location of the new starting point. One may notice that the numbers appearing in the new pairs do

not necessarily belong to $\{1, 2, \dots, 2n\}$. In such a case they have to be replaced by $\text{mod} 2n$. Such a case is very likely to occur in all name changes to be studied from now on, and the substitution of r by $r \text{ mod } 2n$ shall always be implied without further saying.

One may therefore establish the following rule. Two names N_1 and N_2 , which of course contain the same number of pairs, belong to the same projection if and only if there is a number $k \in \{0, 1, 2, \dots, 2n - 1\}$ such that either $(i, j) \in N_1 \Rightarrow (k + i, k + j) \in N_2$, or $(i, j) \in N_1 \Rightarrow (k - i, k - j) \in N_2$. Having now established the criterion above, all one has to do now is find the names of some projection which are at most $4n$, give some rule that will eliminate all names but one, and keep this one name as the sole representative of the projection. This name will later be compared to names of other projections of the same knot class so that eventually only one name should survive and appear at the output. It is important to notice here that this is only possible because the maximum possible number of equivalent names. $4n$, is finite.

In order now for some rule that compares names to be valid, one must first make sure that for any two names different from each other this rule keeps exactly one of them, if a name happens to be compared to itself it must survive, and finally if N_1 is eliminated by N_2 and N_2 by N_3 , then N_1 has to be eliminated by N_3 . While there is a number of criteria that can satisfy these demands, the one to be chosen, the "lexicographical", shall be described at the end of section 6, after the discussion of the problem of "impossible" names.

One last point will be mentioned here before moving on to the Reidemeister moves. Let N_1 and N_2 be two names containing the same number of pairs such that $(i, j) \in N_1 \Rightarrow (j, i) \in N_2$. Such names may either belong to the same knot class (Figure 7a) or to mirror images (Figure 7b). According to what was said in section 2, in either case N_1 and N_2 will be considered identical for the time being. In order to avoid duplication we shall only keep names where 1 appears at the left side of a pair. If after some procedure we get a name where 1 appears at the right, then all pairs (i, j) shall be replaced by (j, i) so that 1 appears at the left.

When one performs a "first" Reidemeister move (Figure 4a), the name of a projection acquires (if the move is done from right to left) or loses (if it is done from left to right) a pair $(i, i \pm 1)$. In addition, all other numbers that are larger or equal to i , are going to increase or decrease by 2. Since the name to be kept must possess as few pairs as possible,

an appearance of any pair of the form $(i, i \pm 1)$ in a name means automatic elimination.

Similarly by performing a "second" Reidemeister move (Figure 4b) one adds or removes two pairs of the form (i, j) and $(i \pm 1, j \pm 1)$, while increasing or decreasing some of the other numbers by 2 or by 4. Therefore the appearance of two pairs (i, j) and $(i \pm 1, j \pm 1)$ in a name will result to its elimination.

On the other side the "third" Reidemeister move (Figure 4c) does not alter the number of pairs; in fact it keeps all but three of the pairs the same, while replacing pairs (i, j) , (i', k) and (j', k') , where $|i' - i| = |j' - j| = |k' - k| = 1$, with the pairs (i, k') , (i', j) and (j, k) . Therefore the "minimum pair" criterion is not sufficient and we shall have to find some additional one.

Let N be a name that does not possess any pair $(i, i \pm 1)$, neither does it possess any two pairs (i, j) and $(i \pm 1, j \pm 1)$. Let also N_1, N_2, \dots, N_f be the names generated by performing series of third moves. (f is definitely finite since it cannot exceed $\frac{(2n)!}{n!}$). If any of the f names possesses a subset of either the form $\{(i, i \pm 1)\}$, or $\{(i, j), (i \pm 1, j \pm 1)\}$, then not only this name, but all N, N_1, N_2, \dots, N_f have to be eliminated, since an equivalent name with fewer pairs has already appeared at the output. The "tie breaker" criterion is therefore going to be the following.

Let N be some name and let $|a_i|$ denote the "companion" of i with respect to N , a_i being positive if i is at the left and negative if i is at the right; in other words, $j = a_i \Rightarrow (i, j) \in N \vee (-j, i) \in N$. Let now

$$A(N) = \min_{1 \leq i \leq 2n} |a_i - i|$$

$$B(N) = 2n - \max_{1 \leq i \leq 2n} |a_i - i|$$

$$C(N) = \min_{1 \leq i \leq 2n} |a_{i+1} - a_i|$$

$$D(N) = 2n - \max_{1 \leq i \leq 2n} |a_{i+1} - a_i|$$

Finally define $M(N) = \min(A(N), B(N), C(N), D(N))$. $M(N) = 1$ means that N can lose one or more pairs after performing a first or a second move, while $M(N_r) = 1$ for some $r \in \{1, 2, \dots, f\}$ implies that N_r can lose pairs and so can N , when third moves are performed to bring it to N_r , which are followed then by a first or a second move. It is thus desirable to keep the name whose M is the least and check whether this M is equal

to 1. If so, N is eliminated, otherwise one can be sure that neither this name nor any of the other equivalent names is reducible. In case however where the minimum value of M belongs to more than one names we shall have to resort to the lexicographical rule.

The process described until now is finite, since given some name N , it only involves names whose order does not exceed that of N . It is possible however that two projections are equivalent, but in order to connect them one has to perform moves that, initially at least, increase the number of double points and thus the order of the names involved. Such a process is more involved than what a first look may suggest, and the discussion shall be left for section 9 when the relevant algorithm is going to be described in detail.

6. TO NAME THE IMPOSSIBLE KNOT

Choosing just one name for each class and eliminating all others is only part of the problem faced while trying to devise an algorithm. An additional difficulty arises from the fact that not all possible "names" actually correspond to some knot class. In fact if one takes the first $2n$ natural numbers, makes n pairs of them and considers the set of these pairs as a name, one may not be able to draw the resulting projection. When n goes in fact to infinity, the probability of getting a drawable name tends to zero.

To understand more clearly this statement, take as an example the set $\{(1, 3), (2, 4)\}$ and try to draw the corresponding projection (Figure 8). While one can draw the two double points (1,3) and (2,4), the knot cannot close without creating a third double point, which of course is forbidden since it would alter the name: the reason for this problem is that one "leg" is inside while the other is outside a loop. While one may obviously see that closing the loop is impossible, mathematically this is a direct consequence of the **Jordan Curve Theorem** which states that all closed lines in R^2 or S^2 which do not intersect themselves divide R^2 or S^2 into two disjoint segments [12]. Although this may seem self evident, it does not occur for all two-dimensional surfaces. It does not occur for example for T^2 or RP^2 .

By using the Jordan Curve Theorem one may prove that any name containing one or more pairs whose elements are either both odd or both even, does not correspond to any actual knot class. The proof goes as follows. Let $(i, j) \in N$ as shown in Figure 9. If the number of double points between i and j is odd, it is impossible to draw the name, since the part of the knot not shown in Figure 9 must enter the loop $i \rightarrow j$ as many times as it

exists and therefore the number of double points between i and j must be even. In Figure 9 the double points between i and j are numbered consecutively along the loop, and if they are even then $j - i$ must be odd, which means that i and j cannot be both odd or both even. It may happen of course that some numbers of the loop $i \rightarrow j$ are paired together, in which case Figure 9 has to be replaced by Figure 10. Such a change does not affect the argument above, since the number of such points is even, and therefore the number of the remaining points of the loop must also be even.

Strange as it may appear, this constraint actually facilitates the devising of an algorithm. First, the number of possible names for a given number of pairs n is reduced from $\frac{(2n)!}{n!}$ to $2^n n!$, since 1 has n even numbers to choose from as a "partner". 3 has $n - 1$ choices, 5 has $n - 2$ etc, while for each pair there are two choices: (odd,even) or (even,odd). Therefore one may substitute Table II with the following one.

T A B L E III

n	$(2n)! \div n!$	$2^n n!$	Actual Number of Knots
0	1	1	1
1	2	2	0
2	12	8	0
3	120	48	1
4	1680	384	1
5	30240	3840	2
6	665280	46080	3
7	17297280	645120	7
8	518918400	10321920	21
9	17643225600	185794560	49

Second, by denoting $f(i) = j$ when $2i - i$ is paired to $2j$ and $g(i) = 0$ or 1 depending on whether the pair is $(2i - 1, j)$ or $(2j, 2i - 1)$, ordering the possible names to be studied becomes simpler, since each of them is actually a pair of an n -permutation and an n -digit binary number.

The problem of impossible names, however, is still unresolved. Take for instance the name $\{(1, 4), (3, 6), (5, 8), (7, 10), (9, 2)\}$ (Figure 11), in which every pair contains one odd and one even number. When one tries to draw it, things will go smoothly for the first seven points. The eighth point, however, cannot be paired to the fifth, since one is inside the loop A-B-C-A, while the other is outside. This is once more a consequence of the Jordan Curve Theorem. The loops 1 - 2 - 3 - 4 and 5 - 6 - 7 - 8 that should intersect at an

even number of points, intersect instead at just one point, namely the (3, 6). We therefore arrive to the following procedure.

Consider any two loops that do not have common segment. (This condition is necessary, since a common segment will add an infinite number of common points and thus the notion of an odd or even number of intersection points will be meaningless). Now consider the points where they intersect, that is all points (i, j) where i belongs to one loop and j to the other. If the number of such points is odd for at least one pair of loops, the corresponding name is impossible.

One should notice of course that not all loops are going to be of the form $i, i \pm 1, i \pm 2, \dots, j \mp 1$, where $(i, j) \in N$. Loops may also be of the form $i, i \pm 1, \dots, i' \mp 1, i' \equiv j', j' \pm 1, \dots, j \mp 1$ where $(i, j) \wedge (i', j') \in N$. Such loops possess one or more "angles". If two loops share such an "angle", this should not be counted as an intersecting point, since at such points the loops "bounce" instead of entering or exiting each other (Figure 12).

We shall conclude this section by describing the lexicographical criterion as promised before. Let $j = f_N(i)$ if $2i - 1$ and $2j$ are paired in some name N , and let $g_N(i) = 0$ if the odd number $2i - 1$ appears on the left while $g_N(i) = 1$ if it appears on the right of its pair. Let now N_1 and N_2 be two names containing the same number of double points. According to the lexicographical rule N_1 is preferred to N_2 if $\exists k \in \{1, 2, \dots, n - 1\} : \forall l < k, f_{N_1}(l) = f_{N_2}(l) \wedge f_{N_1}(k) < f_{N_2}(k)$. In case, however, when $\forall k \in \{1, 2, \dots, n - 1\}, f_{N_1}(k) = f_{N_2}(k)$, an additional tie breaker is needed. As such we choose the following. N_1 is preferable to N_2 if $\exists k \in \{1, 2, \dots, n\} : \forall l < k, g_{N_1}(l) = g_{N_2}(l) \wedge g_{N_1}(k) < g_{N_2}(k)$. No further tie breaker is needed, since if for two names N_1 and $N_2, f_{N_1}(k) = f_{N_2}(k) \wedge g_{N_1}(k) = g_{N_2}(k) \forall k$, then N_1 and N_2 are identical.

7. PRIME KNOTS - WILD KNOTS

Let M_1 and M_2 be two n -dimensional manifolds without a boundary. By removing an n -dimensional ball from each of them and by identifying the resulting boundaries one gets a manifold $M_1 \# M_2$ which is called their **connected sum**. On the other hand, a manifold that cannot be decomposed as a connected sum of other manifolds (other than itself and the n -sphere S^n) is called a **prime manifold** [13].

Let now K_1 and K_2 be two knots which are of course 1-dimensional manifolds without a boundary. Following the definition above we may construct their connected $K_1 \# K_2$ sum

as shown in Figure 13. Such knots, which are connected sums of simpler non-trivial knots, do not usually appear on knot tables, the reason being that once all prime knots are obtained, one can easily construct their connected. Let for instance N_1 be the name of K_1 having n_1 pairs and N_2 the name of K_2 with n_2 pairs. First one constructs the set N_2' made out of pairs $(i + 2n_1, j + 2n_1)$, where (i, j) are the pairs of N_2 . The name of $K_1 \# K_2$ is the set $N_1 \cup N_2'$.

This rule can also help us check whether a knot is prime or not. Let N be the name of a connected sum knot containing whose number of pairs n is minimum. Then one can find two numbers k and l , at least one of which is not 1 or $2n$, which satisfy the following property.

$$\forall (i, j) \in N : k \leq i \leq l \Leftrightarrow k \leq j \leq l$$

Therefore through a small modification of the algorithm one can avoid the appearance of connected sums at the output.

There is another category of knots that shall be briefly mentioned now, which are unobtainable: no matter how "good" is an algorithm and no matter how high the input parameter, one may never hope to reach the so called **wild knots**, defined as knots that are inequivalent to all three-dimensional polygons [14]. If one tries to assign a name to a wild knot following the rules we have established, it is going to contain an infinite number of pairs, so its turn to appear at the output will never come. In spite of this fact, the length of a wild knot is finite; the lengths between two double points of its projection can be, for example, terms of a decreasing geometric sequence.

Knots which are not wild, and which one can hope to obtain, are called **tame**.

8. INEQUIVALENT KNOTS

The discussion until now has mostly been from an "equivalence" point of view; in other words, by obtaining all projections with a maximum number of double points, showing that certain projections are equivalent and by eliminating the repeated ones, one aimed at keeping just one representative from each knot class. Such a process, however, can go on indefinitely, since one may not know *a priori* how many moves may be required to show an equivalence. No matter how many moves one allows for in the program, failing to prove an equivalence relation does not prove inequivalence.

While knot inequivalence may be proved by using a number of knot polynomials such as the **Alexander-Conway**, the **Jones** [15], the **Kauffman** [16] and the **HOMFLY** [17] ones, such methods face two problems. First, in order to find a knot's polynomial one is required to get link polynomials as well, since in the defining relations $P(a) = \lambda_1 P(b) + \lambda_2 P(c)$ where a , b and c are shown in Figure 14, when switching a crossing one still gets a knot, but when a double point is eliminated one gets a link. One would therefore have to generalize the concept of the "name" to include links in addition to knots, which would drastically change and complicate the algorithm. Second, while distinct polynomials mean inequivalent knots, the converse is not always true; it is possible that inequivalent knots possess identical polynomials.

Both of these difficulties can be overcome, however, by using **knot groups** [18]. The group of a knot is defined as the first homotopic (fundamental) group of its complement in S^3 . In order to obtain this group one considers first all loops in S^3 which do not intersect the knot, regards as equivalent any loops that can be continuously deformed to each other without intersecting the knot in the process, and thus gets the set of loop classes, which becomes a group under *loop multiplication*. One interesting point to notice is that while knots are not allowed to have multiple points in S^3 , there is no such restriction for the loops of the knot group.

Two knots that possess distinct groups are of course inequivalent, but in addition the converse is also (almost) true. Prime knots with identical groups are equivalent (up to chirality and orientation reversal) [19]. While in section 7 connected sums were already eliminated, this statement provides an additional reason to do so. Comparing knot groups could be a second method of obtaining knot classes, and were it a finite process it would be preferable to the one based on Reidemeister moves.

First however one has to find the knot groups. The best way to present a knot group is the **Wirtinger presentation** introduced by Wirtinger in 1905 [20]. According to this presentation, let K be a knot and $P(K)$ one of its regular projections on S^2 which possesses n double points. The knot group of K is a group generated by n generators g_i , each corresponding to a segment connecting two undercrossings, satisfying n relations of the form $g_{i+1} = g_j g_i g_j^{-1}$, where the generators g_{i+1} , g_i and g_j start, end and pass through some double point respectively, as shown in Figure 15. While the presentation depends on the projection chosen, the group itself does not.

Finding presentations of two knot groups does not tell us yet whether the knots are equivalent or not, unless of course the presentations are identical. Otherwise, it is possible that two knot groups are the same, even though their presentations may look quite different.

To illustrate the points above, the knot group of the trefoil shall be compared to the group of the unknot. Let us first begin with the trefoil. Let a , b and c be the generators of the trefoil group corresponding to the segments A, B and C as shown in Figure 16. The relations they satisfy are the following.

$$\text{i) } b = c^{-1}ac \quad \text{ii) } c = a^{-1}ba \quad \text{iii) } a = b^{-1}cb$$

One may easily check that one of them is redundant, which is always the case with one of the n relations of a knot group. The other two yield the relation $aba = bab$, and thus the trefoil group is the **braid group** B_3 , which happens also to be the fundamental group of the configuration space of three identical particles on R^2 [21]. This group is of course different from the unknot group which can easily be shown to be the free one-generator group, which is Z .

In general, however, a knot group may not come out to be some group already known having appeared somewhere else. A proof given by Reidemeister, although essentially based on the method shown above, has the advantage that it can be generalized to prove knot inequivalence in more involved cases.

One may show the trefoil's non-triviality by asking the following question: "Is it possible to use three distinct colors to paint a knot projection, provided that color change can only occur at undercrossings, and at the neighborhood of an undercrossing either just one color is used, or all three of them?" First one shows that the answer does not change when a Reidemeister move is performed, which is demonstrated in Figure 17. In this figure one considers some projection for which the answer to the question is negative and thus only one color is allowed which is denoted as color X. If a Reidemeister move is performed, new segments may arise, but still one may not use any color other than X and thus the answer remains negative [22].

One can clearly see the relation between Reidemeister's proof and knot groups by identifying the three colors with the permutations $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ of P_3 . The constraint that either just one or all three colors appear at the neighborhood of an undercrossing can be understood as the fulfillment of the relations of the knot group. Finally the fact that all three colors can be used to paint the trefoil but only one for the

unknot implies that trefoil group generators may be assigned to all three permutations mentioned above, which is the full $\lambda_1 = 2$ $\lambda_2 = 1$ conjugacy class of P_3 , while this assignment is not possible for the unknot group. The whole procedure does not depend on our knowledge of Z and B_3 being different from each other, not even from our ability to recognize these groups.

One can clearly see now how the "three color" method can be generalized. The conjugacy class $\lambda_1 = 2$ $\lambda_2 = 1$ of P_3 can be replaced by any other conjugacy class of P_n for any possible value of n . When comparing two knot projections, if at least one conjugacy class exists that can be generated by one knot group but not from the other, the groups are definitely distinct and thus the projections belong to inequivalent knots.

This process is infinite too, since no matter for how many conjugacy classes two knot groups respond identically, one cannot be sure that the groups are the same. The best one may do, therefore, is to run simultaneously two programs, one in order to show knot equivalence by applying Reidemeister moves and one to show inequivalence by checking various conjugacy classes of permutation groups, and wait until one of the programs reaches a verdict.

9. A SUGGESTED ALGORITHM

Based on the theory already mentioned, we are now going to suggest an algorithm whose purpose is to list all distinct knot classes which having at least one projection whose number of double points does not exceed some desired value. This value is the only input data required. This algorithm, which can be run as a Fortran program, is divided into two parts. In the first, our goal is to find permissible "shadows", where by "shadow" one means a "flat" knot projection where one does not distinguish between overcrossings and undercrossings; in other words, the Figures 3a and 3b are considered identical. Each such shadow will be named not by a set of pairs, but by a set of two-element sets, since there is no distinction between (i, j) and (j, i) . Once the permissible shadows are separated from the "forbidden" ones, we go on with the second part in order to get the actual knot classes.

Using the notation of section 6, where $j = f(i)$ means that $2i - 1$ is paired to $2j$, we start with the shadow $f(i) = i \quad \forall i \in \{1, 2, \dots, n\}$. First one has to check if this shadow is allowed (actually it is not). If it is, one proceeds to the second part where different combinations of $g(i)$ are considered, and once this is over one moves on to the next shadow.

If it is not, one moves directly on to the next shadow, according to a procedure that shall now shall be described.

Let $f(i) = j$ be a shadow just studied and from which one wants to move on. Let $f(n) < f(n-1) < \dots < f(l+1)$ and $f(l+1) > f(l)$ for some l . The shadow that follows, $f'(i) = j$, satisfies the following relations: a) $i < l \Rightarrow f'(i) = f(i)$, b) $f'(l)$ is the smallest among $f(l+1), f(l+2), \dots, f(n)$ that is larger than $f(l)$, while c) for $i > l$ one has $f'(l+1) < f'(l+2) < \dots < f'(n)$. Following this procedure, which is equivalent to taking the $n!$ shadows in a lexicographical order, one eventually arrives to the point where $l = 0$ or $f(n) < f(n-1) < \dots < f(1)$ which means that $f(i) = n+1-i \quad \forall i$. Once that point is reached and after that shadow has been studied (it is forbidden too), the running of the algorithm is over.

Once some shadow $f(i) = j$ is chosen, it is much simpler from that point on to switch to the notation $j = h(i) \Leftrightarrow i$ and j are paired. In addition, in order to satisfy the fact that our notation is mod $2n$, we shall also define $h(i \pm 2n) = h(i)$. Permissible shadows are going to be the ones satisfying the following conditions.

1) $\forall i \in \{0, 1, 2, \dots, 2n\} \quad h(i) \notin \{i-1, i+1\}$ to make sure the knot cannot be simplified through a first Reidemeister move. (The reason this property must also be satisfied by $i=0$ is to make sure that $\{1, 2n\}$ does not belong to the shadow).

2) $\nexists k, l: k < l \wedge (k \neq 1 \vee l \neq 2n) \wedge (k \leq i \leq l \Rightarrow k \leq h(i) \leq l)$, in order to avoid a connected sum.

3) If one compares $h(i)$ to $h'(i) = h(i+k) - k$ or $h(k-i) + i$, modulo $2n$ of course, then $h(i)$ must be lexicographically preferable or at least identical to all $4n$ possible values of $h'(i)$ according to the rule established at the end of section 6. This will ensure that each projection will only be considered once.

4) We finally have to check whether some name belongs to an actual shadow or whether it violates Jordan's Curve Theorem. This check is in fact the most time consuming step of the first part, and is done as follows.

We first choose a value of k that runs from 1 to $2n$. Then, for a given k , we choose numbers j_1, j_2, \dots, j_k belonging to $\{1, 2, \dots, 2n\}$ which are different from each other. These numbers define a loop $j_1 \equiv h(j_1), h(j_1) + 1, \dots, j_2 - 1, j_2 \equiv h(j_2), h(j_2) \pm 1, \dots, j_3 \mp 1, j_3 \equiv h(j_3), \dots, j_1 \pm 1, j_1 \equiv h(j_1)$. One may notice that the points j_i are the "angles" of the loop where it changes direction following $h(j_i)$ instead of j_i . Self intersecting loops are of course

excluded.

Once now one is given two such loops, one first has to make sure that they do not share any segment. For those loops that do satisfy this property one counts the number of intersecting points, that is the number of points (i, j) where i belongs to one loop and j to the other. As explained in section 6, common "angles" they may share should not be counted. If the number comes out to be even, the test is considered successful and one continues with the next pair of loops. If it comes out odd, the impossibility of the shadow has just been proved and now one moves on to the next shadow by replacing $f(i)$ with $f'(i)$. Since in fact we do not care about the actual number of common points but only whether this number is even or odd, it is simpler to consider some parameter L equal to 1, and each time we reach a common point, to replace L with $-L$. If at the end $L = 1$ the test is successful, if $L = -1$ the shadow is forbidden. Loops sharing one or more common segments possess an infinite number of points in common, so it would be meaningless to carry out this test for them; the result for such a pair is considered successful in advance.

While this process is extremely large due to the big amount of possible loops and constraints they have to satisfy, it is definitely finite so that a shadow's permissibility can be checked after a finite amount of steps. Once this is over, one moves on to the second part of the algorithm in order to find out which (if any) out of the 2^n possible combinations of overcrossings/undercrossings of a permissible shadow give rise to distinct knot classes and should thus appear at the output.

Let us assume now that a certain shadow has passed all four tests described until now and has thus been found to be permissible. One now considers the projections possessing this shadow, starting from the case $g_i = 0 \quad \forall i$, where g_i was defined in section 6. This projection is an **alternating** one since if one travels along the knot, one encounters alternatively overcrossings and undercrossings; indeed it is the only alternating projection of the shadow. Once some projection has been studied and has either been rejected or printed out at the output, one proceeds by finding the maximum i for which $g_i = 0$, changing this g_i from 0 to 1 and, if $i \neq n$, $\forall j > i$, change g_j from 1 to 0. When one is through with $g_1 = 0, g_i = 1 \quad \forall i > 1$, instead of proceeding with $g_1 = 1, g_i = 0 \quad \forall i > 1$, one leaves the shadow, returns to the first part of the algorithm and continues with the next shadow. The reason for ignoring projections where $g_1 = 1$ is that as explained in section 5, one only considers names where 1 is the left element of its pair, while according to the definition of section 6, $g_1 = 1$ would place 1 at the right. This

condition reduces the number of possible projections by a factor of 2.

For a given projection now it will be convenient for all (i, j) belonging to the name to define $p(i) = j$ and $p(j) = -i$, and of course $p(i \pm 2n) = p(i)$. First one checks whether the projection can be simplified through a second Reidemeister move. If so, one should be able to find some i for which $|p(i+1) - p(i)| = 1$, in such a case the projection for which this occurs is rejected, and one moves on with the next one.

If no such i can be found, one proceeds by performing third Reidemeister moves. First one checks for any three pairs (i, j) , (i', k) and (j', k') where $|i' - i| = |j' - j| = |k' - k| = 1$ and performs third moves as described in section 5. One also performs third moves on the new projections one gets, and keeps going until no new projections appear. On each new projection obtained one carries out the test described in section 5, by comparing $M(N_{new})$ to $M(N_{original})$. If this test is inconclusive, one settles this issue with the lexicographical test of section 6. One should be careful however to put the new projection in a lexicographically preferred form by replacing each pair (i, j) with the pair $(k \pm i, k \pm j)$ for the appropriate value of k . This may be necessary because while for the original projection the name used has been checked to be the preferable one, once the pairs (i, j) , (i', k) and (j', k') have been replaced by (i, k') , (i', j) and (j, k) , the new name may not be the preferable one.

Even when this test is carried out successfully, one is far from certain that the projection should appear at the outcome, since it might happen that two projections are equivalent but the series of Reidemeister moves that connect them begins with moves that increase the number of double points. Schematically one shows such a case in Figure 18.

The next test to be carried out is to increase the number of double points by one by performing a first move as shown in Figure 19. One then performs third moves and compares all new projections to the initial one. One may see of course that all of the new projections have a larger number of double points, unless one can reduce them by performing a first or a second move at one of the new projections. One must therefore first compare the number of double points of the initial and any subsequent projections before moving on to any further criteria. Fortunately as one can see in Figure 19, the number of the first moves with which one may begin is limited; adding a pair $(i, i+1)$ is fruitless unless $|p(i-1) + p(i)| = 1$, since it does not lead to further third moves. Once this move is performed, one is going to have $|p(i) + p(i+1)| = |\pm[(i+1) - i]| = 1$. If one uses this

result to perform an additional first move as shown in Figure 20, the net result is a second Reidemeister move as one may see in Figure 21. Since this move is going to be carried out later where it shall be discussed extensively, this test is not going to take place at this stage. Therefore the process of increasing the number of double points by performing first moves is a finite one.

Finally one increases the number of a projection's double points by performing second moves. While when performing first moves one might add an $(i, i \pm 1)$ pair for any value of i and proceed with third moves, one may not just simply add new pairs (i, j) and $(i \pm 1, j \pm 1)$ for any possible values of i and j , the reason being that by doing so one may arrive to a name that is impossible to be drawn. In Figure 22 for example, while the segments i and j can be joined to perform a second move, that is not possible with k and l . Since the ability to perform a second move depends only on the shadow of the projection, it is preferable to find the pairs of "neighboring" segments such as i and j during the first part of the algorithm. One may also have ready from the first part the shadows of the projections after the second move is performed. Once a particular projection is chosen, for each such pair of segments there are two possibilities, either i goes "above" j or it goes "below".

In order now to find the pairs of neighboring segments i and j one has to check which of the loops found before, while testing the name's possibility, leave the rest of the knot on the same side of S^2 . In other words one is looking for loops that do not intersect any other loop. All segments constituting such a loop may be paired with each other to perform a second move.

Unfortunately however the number of second moves one may carry out is infinite, and there is no way to tell in advance how many of them may be needed to connect two equivalent projections. It is at exactly this point that the algorithm becomes infinitely long. If one tries to make it finite by cutting it at some point, there is no guarantee that some knot class will not be repeated.

Using the theory discussed up to this point, we constructed a fortran program that initially considered all possible one-to-one functions $p(i)$ as defined above from $\{1, 2, \dots, 2n\}$ to $\{-2n, -2n+1, \dots, -1, 1, 2, \dots, 2n\}$, where odd numbers are mapped to even and vice versa. Subsequently it eliminated all functions corresponding to knots that could be shown to be either undrawable or equivalent to preferred knots. Were this process finite, only distinct knots would survive and appear at the output. In order to keep the process finite,

we allowed for at most one second move "up". The results we obtained may be summarized as follows.

T A B L E IV

n	Shadows	Shadows $\times 2^{n-1}$	Program's Knots	Actual Knots
0	1	1	1	1
1	0	0	0	0
2	0	0	0	0
3	1	4	1	1
4	1	8	1	1
5	2	32	2	2
6	3	96	3	3
7	10	640	7	7
8	27	3456	21	21
9	101	25856	57	49

In addition to obtaining the same number of knots to the one appearing in the literature when $n \leq 8$, we verified that these knots are exactly the same. For $n = 9$ we verified the fact that 49 of the knots we obtained are identical to the ones appearing in the literature, while the other 8 should be identical to some of the previous 49.

10. A SUGGESTED ALGORITHM - THE SEQUEL

Up to now the algorithm described in section 9 is based on starting with all possible names that have a given number of pairs and where each pair has an odd and an even element. Then one eliminates all names that are either impossible or can be shown after a certain number of steps to be equivalent to some other more preferable name. Finally one prints out all the names that survive. Such a process is however incomplete, since it fails to show the inequivalence of the knot projections it prints out. In this section the previous algorithm shall be extended by using ideas described in section 8, in order to prove the (possible) inequivalence between the surviving projections.

Let m be an integer and $\lambda_1, \lambda_2, \dots, \lambda_k$ a partition of m defined as a decreasing sequence of natural numbers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ whose sum is equal to m . As is well known from group theory, there is a one-to-one correspondence between each such partition and a conjugacy class of the m -permutation group P_m [23]. One now tries to map generators of knot groups to permutations belonging to various conjugacy classes, noticing in fact that due to the defining relations of knot groups, its generators are conjugate to each other.

We demand of course that the permutations mapped to various generators fulfill the same relations that the generators do; this constraint is going to make many such mappings impossible, so that in a number of cases the permutations one gets do not generate the whole conjugacy class.

Take for instance the conjugacy class of P_3 corresponding to the partition $\lambda_1 = 2$, $\lambda_2 = 1$, consisting of the permutations $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$. While one may map each such permutation to a distinct generator of the trefoil group, only one of these three permutations may be assigned to the (unique) generator of the unknot group Z , and this permutation by itself cannot generate of course the whole conjugacy class. The method to be therefore applied is the following. First one attempts to generate conjugacy classes in this way, and after each such attempt one assigns a "YES" or a "NO" answer for each (surviving) knot projection and each partition used. This attempt is of course finite since there is a finite amount of permutations to be mapped to a finite number of generators. Then one compares the answers for various knot projections. If there is at least one partition which gives a different response for two knot projections, these projections have been proven to be inequivalent. Our final goal is to perform enough Reidemeister moves and then to use enough partitions so that all surviving projections can be shown to be inequivalent.

The suggested algorithm, which can either be inserted in the second part of the one described in section 9, or become an additional third part, goes as follows. One begins with $m = 1$ and proceeds with successive natural numbers until one reaches some desired maximum value of m . For a given m now one starts with the partition $\lambda_1 = m$ and proceeds until the partition $\lambda_1 = \lambda_2 = \dots = \lambda_m = 1$ is reached, at which point one increases the value of m by one. For a given value of m now the process goes as follows.

Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be a partition been studied and which is to be replaced with a new one. There is definitely going to be some number σ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\sigma \geq 2$, while $\lambda_{\sigma+1} = \lambda_{\sigma+2} = \lambda_k = 1$. One keeps the first $\sigma - 1$ terms unchanged, replaces λ_σ with $\lambda_\sigma - 1$ and then makes $\lambda_{\sigma+1} = \lambda_{\sigma+2} = \dots$ equal to the new value of λ_σ . One considers as many of these λ 's as possible so that the total sum $\lambda_1 + \lambda_2 + \dots$ does not exceed m . If there is some quantity remaining which is less than the new value of λ_σ , one takes a last λ_f equal to this remaining.

Once now a partition has been chosen, all segments starting and ending at undercross-

ings are considered, which correspond to the n numbers appearing at the right of each pair. One maps them to permutations of the conjugacy class under consideration and checks whether the knot group relations are satisfied. If so, one stays with this mapping, and once one has found all such permutations p_1, p_2, \dots, p_s , one performs all possible combinations $p_i^{-1} p_j p_i$ and $p_i p_j p_i^{-1}$ in order to get new such permutations. One similarly continues with the new permutations until one gets all possible permutations. If their set is the full conjugacy class, one has reached an affirmative answer, if not, the answer is negative.

One difficulty that arises in this process and which is now going to be discussed, is the form of the knot group relations. By looking at Figure 15 one may consider some pair (i, j) , the generator g_j corresponding to the segment beginning from the undercrossing j . The generator g_i , where j' is the largest number appearing at the right of a pair which does not exceed j , and the generator $g_{i'}$ where i' is the largest number at the right of a pair which does not exceed i . These three generators should satisfy either the relation $g_j = g_i g_j g_i^{-1}$ if they look as in Figure 23a, or the relation $g_j = g_{i'}^{-1} g_j g_{i'}$ if they look as in Figure 23b. Which one however?

For a partition where $\lambda_1 = 2$ it hardly matters, since all corresponding permutations are equal to their inverses; this is in fact the reason that made Reidemeister's proof of the trefoil's non-triviality look so simple. For $\lambda_1 > 2$ however the distinction is important. While looking at the drawing of a projection it may look "obvious" which of the two is the case, when only a name is available, which is just a set of pairs of numbers, the answer is far from being so clear.

One should first notice that for some starting point one may choose any of the two relations one wishes, and then the relations for the other points will be with respect to the starting one. So let us assume that for the pair (i, j) or (j, i) , where $i < j$, one knows which one is the appropriate relation. For the next pair, which is either of the form $(i+1, j')$ or of the form $(j', i+1)$, one considers a parameter L which shall now be defined, and if L comes out to be equal to 1, then the relations for the two pairs are the same, if it comes out to be -1 , they are different. We shall now show how to calculate L .

One begins with $L = -1$. One multiplies it with a factor of -1 if i is an overcrossing and $i+1$ an undercrossing or vice versa, while one leaves it as it is if both are undercrossings or both are overcrossings. One then multiplies L by a factor of -1 for each number between $j-1$ and $j'-1$ that is paired to a number between $i+2$ and $j-1$. The value final value

of L determines the relation for $i+1$.

An example is illustrated in Figure 24. Here one begins with (i, j) and ends with $(j+3, i+1)$. Initially $L = -1$, then the presence of the pair $(i+2, j+2)$ switches L from -1 to $+1$, and finally the fact that i is an overcrossing while $i+1$ is an undercrossing switches L to each final value -1 .

11. INVERSE AND SYMMETRIC KNOTS

In this section a problem avoided until now is going to be discussed, which is how to find whether some knot is ambient isotopic to its inverse and to its mirror symmetric. Let f be the continuous one-to-one function from S^1 to S^3 which defines some knot K , such that $f(\theta) = \bar{x} \in S^3 = R^1 \cup \{\infty\}$, $0 \leq \theta \leq 2\pi$, $f(0) = f(2\pi)$. The inverse of K is defined by the function $f_{m.i.}(\theta) = f(2\pi - \theta)$, while its mirror image by the function $f_{m.i.}(\theta) = -f(\theta)$. In general a knot may or may not be ambient isotopic to its inverse or its mirror symmetric, depending on whether a series of Reidemeister moves may connect it to them. Unfortunately knot group theory is not helpful in proving possible inequivalence, since knot groups of inverse and mirror symmetric knots are always identical; the best proof to our knowledge of a knot's chirality involves HOMFLY polynomials, but does not always work; the knot 9_2 for example is chiral, even though its chirality cannot be proved through its HOMFLY polynomial.

Let now K' be a knot with a name $N(K)$ and K'' its inverse whose name is $N(K')$. These knots are oriented, so when naming them one has to choose inverse orientations. If one chooses the same starting point for both of them, then for each $(i, j) \in N(K)$, a pair $(2n+1-i, 2n-1-j)$ will belong to $N(K')$. They are equivalent if there is a series of Reidemeister moves that connects them; during the process one is allowed to change the starting points by replacing the pairs (i, j) of some name with pairs $(k+i, k+j)$ as described in section 5: one is not allowed however to change orientation by replacing them with $(k-i, k-j)$.

Unfortunately the chirality problem is not that simple. As already showed in Figures 7a and 7b, by merely inverting the order of the pairs of a name and thus substituting (i, j) with (j, i) one does not necessarily get the mirror image of a knot; similarly, once given the name of a knot one may draw either the knot or its mirror image and still be consistent with the name. The reason is that while replacing (i, j) with (j, i) implies a reflection with

respect to the xy plane, a reflection with respect to a plane passing through the z axis does not alter a name; in order to get rid of this ambiguity, the first double point of a knot shall be constrained to look like in Figure 23a. Once this has been agreed upon, there is no further ambiguity, unless the knot is a connected sum, and this is one further reason to consider only prime knots. One is now able to move on to compare the name (i, j) with the name (j, i) .

While the convention above takes care of the ambiguity, one has to be careful while performing Reidemeister moves; if either the first double point is involved, or if one has to switch to another first double point in order to reach a "preferrable" name, the new first double point may look instead like the one in Figure 23b, so one may have to change all new pairs (i, j) with (j, i) in order to avoid performing an undesirable reflection. Fortunately the rules established in section 10 can help us here too, by telling us whether a switch from Figure 23a to 23b has occurred when the first double point has changed. As far as Reidemeister moves are concerned, one had better avoid involving the first double point in such moves, if necessary by switching the first double point before and after the move.

12. KNOTS ON A LATTICE

Until now the problem of knot classification was studied by using knot projections on S^2 . Knots however are three-dimensional objects, and it would be interesting if one could study them on the full three-dimensional space. If one uses as three-dimensional space S^3 , one faces the continuity problem: the space of knots is continuous while algorithms always involve discrete quantities and thus would not be useful in such a case. In this section we shall elaborate on an idea first suggested by Pippenger, Janse van Rensburg, Soteros, Summers and Whittington [24], which is to approximate S^3 with a cubic lattice Z^3 and study knots on this lattice.

One begins with $Z^3 = \{(m, n, l) \mid m, n, l \in Z\}$ which is the cubic lattice. A tame knot may always be continuously deformed to a polygon whose vertices belong to the lattice, while successive vertices are neighboring points on the lattice. The defining function of such a knot is determined by the values of $f(\frac{2\pi k}{n})$ for some $n \in N$ and $k \in \{0, 1, 2, \dots, n-1\}$. In addition, the following relations must be satisfied.

- i) $k \neq k' \Rightarrow f(\frac{2\pi k}{n}) \neq f(\frac{2\pi k'}{n})$
- ii) $|\bar{v}_k| = |f(\frac{2\pi(k+1)}{n}) - f(\frac{2\pi k}{n})| = 1$

$$\text{iii) } f(\frac{2\pi(k+\epsilon)}{n}) = (1 - \epsilon)f(\frac{2\pi k}{n}) + \epsilon f(\frac{2\pi(k+1)}{n}) \quad \forall \epsilon \in (0, 1)$$

Therefore one may write such a knot as an ordered n -ad a_1, a_2, \dots, a_n , where $a_i \in \{1, 2, 3, 4, 5, 6\} \forall i \in \{1, 2, \dots, n\}$, the value of a_k determining the direction of \bar{v}_k . In particular, the values 1, 2 and 3 indicate that \bar{v}_k is the unit vector along the positive x , y and z axes, while values of 4, 5 and 6 indicate that it is along the negative axes $-x$, $-y$ and $-z$.

Not all such n -ads are suitable of course; there must be as many 4's, 5's and 6's as 3's, 2's and 1's in order for the knot to close, that is to have $f(2\pi) = f(0)$. This statement requires that n is even. On the other side no i and j other than 0 and n should share this property, or in other words one is not allowed to have a subsequence $a_{i+1}, a_{i+2}, \dots, a_j$ with as many 4's, 5's and 6's as 3's, 2's and 1's; if such a subsequence did exist, one would have $f(\frac{2\pi i}{n}) = f(\frac{2\pi j}{n})$ and f would not be one-to-one.

Once these two constraints are satisfied however any ordered n -ad will do, and such an n -ad will serve as the name of a knot. One should contrast the simple form of these constraints with the rather elaborate form of the constraints of the previous algorithm. What one needs now in order to proceed, is first a new definition of knot complexity to replace the number of double points, second some name changes (moves) that will play a similar role to the Reidemeister moves, and third some method that can show possible knot inequivalence.

One may easily define the "length" of the knot, n , as a measure of knot complexity, and thus provide such a number at the program's input; what is expected as output is a set of knot names a_1, a_2, \dots, a_n for $m \leq n$ such that no two knots corresponding to these names are ambient isotopic, and all names missing from the output should be equivalent to names that do. It would be interesting to see whether the two notions of complexity, namely the number of double points and the length as defined above, agree with each other; in other words, if a knot that is "simpler" to another knot with respect to one definition is also simpler with respect to the other. For the first few cases checked this is indeed the case, and the trefoil remains the simplest non-trivial knot followed by Listing's knot. This is very unlikely however to be the case for all knots.

As far as equivalence moves are concerned, one gets two such possibilities which are shown in Figures 25a and 25b; one is not allowed however to perform these moves if they result to double points, or in other words if one or more of the points to be occupied have already been occupied before. The name changes corresponding to such moves are for the

While this “looks like” a trefoil and when one projects it in two dimensions one may even rigorously prove it is a trefoil, just from the number sequence given above we have not yet obtained a method to show its nontriviality.

Lately a new set of knot characteristics, the **Vassiliev invariants** has come to our attention, which might be promising in proving knot inequivalence for three-dimensional algorithms; these invariants are defined in a larger space than the space of knots, since they include the so called **singular knots** which are polygons which do intersect themselves. Extending our concept of name in order to include singular knots is far simpler than the case of previous polynomials where multicomponent links had to be included, since now one merely has to disregard the constraint $i \neq j \Rightarrow f(\frac{2\pi i}{n}) \neq f(\frac{2\pi j}{n})$. The dependence however of the invariants on **actuality tables** makes the problem more involved than what it looks at first sight.

When comparing the two algorithms one arrives to the following conclusions. First, the lattice algorithm is much easier to be written down and there are fewer “traps” that may cause errors; the projection algorithm is however much faster to run, since the values of n involved are much smaller. The trefoil for example possesses only 3 double points but has a length of 24. In addition, the projection algorithm can obtain more reliable results since it can prove both equivalence and inequivalence.

13. MULTIKNOTS

In section 2 knots were defined as continuous one-to-one functions from S^1 to S^3 . This definition may be extended to include n -knots or **multiknots**, which are continuous one-to-one functions from S^n to S^{n+2} . While such knots cannot be visualized in a three-dimensional space, they do exist as mathematical objects and there has been some interest in their study [25]. Ambient isotopy may also be defined in a similar manner; Reidemeister moves do not exist, however, since a possible projection on S^n will possess a continuously infinite number of double points. As a result, an algorithm based on such a projection cannot work.

One may do, however, use an extension of the algorithm described in section 12 based on placing a knot on a (hyper)cubic lattice. We shall now demonstrate the case for 2-knots which are functions from S^2 to S^4 .

A suitable two-knot for such a purpose consists of unit squares whose vertices belong

first case an “exchange”;

$$a_1, \dots, a_{k-1}, a_k, a_{k+1}, a_{k+2}, \dots, a_n \leftrightarrow a_1, \dots, a_{k-1}, a_k, a_{k+2}, \dots, a_n$$

and for the second case a “pair creation/annihilation”

$$a_1, \dots, a_{k-1}, a_k, a_{k+1}, \dots, a_n \leftrightarrow a_1, \dots, a_{k-1}, a_j, a_k, \bar{l} - a_j, a_{k+1}, \dots, a_n$$

(The words “exchange” and “pair creation/annihilation” are of course used metaphorically; let us consider a knot-polygon as a set of n particles and n antiparticles where the value of v_i indicates the “flavor” of the i -th particle or antiparticle, and let a flavor $-x$, $-y$ or $-z$ denote the antiparticle of the flavor x , y and z respectively. Then the two moves just mentioned may be considered the first as an “exchange” and the second as a particle-antiparticle “creation” (from left to right) or “annihilation” (from right to left. A knot-polygon may thus be considered as a loop in the configuration space of a system of three flavor particles-antiparticles, where particles or antiparticles having the same flavor are considered indistinguishable. Equivalent knots correspond to homotopic loops, while the unknot corresponds to the trivial loop. The implications of this analogy have not yet been studied.)

Among two equivalent names now the one preferred is the shortest one, and if both are of equal length, the one with the smallest value of a_1 ; if these two are equal, the one with the smallest a_2 and so on.

By using what we has already been said until now one may write up a program which initially provides the 6^n possible names for some value of n . Then it eliminates names that do not satisfy the necessary constraints. Finally by using the two the two moves described above, one checks for possible equivalences and keeps from each class the preferred name. This process however suffers from the same deficiency with the previous algorithm, namely that it is not finite; when two names are given, one cannot know in advance how many moves may be required at most to show their equivalence, so failure to prove equivalence after a certain number of moves does not prove inequivalence. In addition, we have yet to come up with some procedure such as the one based on knot groups that directly proves inequivalence. Take for instance the 24-ad $a_1 = a_2 = a_3 = 1, a_4 = a_5 = 2, a_6 = 3, a_7 = a_8 = 6, a_9 = a_{10} = 5, a_{12} = a_{13} = 4, a_{14} = 1, a_{15} = a_{16} = 2, a_{17} = a_{18} = a_{19} = 3, a_{20} = a_{21} = 6, a_{22} = 5, a_{23} = a_{24} = 4$, which is “obviously” the trefoil (Figure 26).

to Z^4 . There are three kinds of equivalence moves which are shown in Figure 27, and which substitute either one cube face with the other five (Figure 27a), or two faces sharing a common edge with the other four (Figure 27b), or three faces sharing a common vertex with the other three (Figure 27c). (The dots in these figures are the centers of the substituted faces).

Naming a two-knot however is not so simple as naming a knot, since the unit squares do not follow one after the other in a well defined order as was the case with knots. This is because instead of functions from S^1 one has now functions from S^2 . One solution to this problem is to replace S^2 with the surface of a cube and divide each of the six faces into m^2 squares. The number m is going to be a measure of the multiknot's complexity. One may begin with some square which will be mapped to $(0, 0, 0, 0)$ of S^4 , and assign a number from 1 to 8 to each pair of neighboring squares; if one square is mapped to \vec{r}_1 and its neighbor to \vec{r}_2 , then one requires that $|\vec{r}_2 - \vec{r}_1| = 1$, while the number assigned to their pair will give the direction of $|\vec{r}_2 - \vec{r}_1|$. This assignment of numbers to each pair of neighboring squares is going to be the name of the knot. Not any name of course is acceptable, the conditions being now that any loop on the cubic surface must be mapped to a loop in S^4 and vice versa; these conditions will impose constraints on the name. In Figure 28 for example, the fact that a loop starting from the center of square A, and via the centers of B, C and D returns to A, must be mapped to a loop in S^4 , imposes the constraint $\{a_{AB}, a_{BC}\} = \{a_{AD}, a_{DC}\}$, where a_{ij} is the number assigned to the neighboring squares i and j .

While such an algorithm in principle leaves only the problem of proving multiknot inequivalence unsolved, in practice it should need an enormous amount of time to run in order to yield even the first non-trivial multiknot.

14. CONCLUSION

The question posed at the title of this paper, that is whether a complete classification of (tame) knots is possible, continues to defy resolution. While by using two at least algorithms, one based on the double points of a two-dimensional projection and one constraining a knot on a cubic lattice, one may get correctly some of the simplest knots, when running the program for knots of greater complexity one cannot be sure that the results obtained are correct and that all knot classes that appear are distinct. Only the first of these two algorithms in fact provides some help with respect to this matter. In addition

to this theoretical difficulty, the time needed for the algorithms is enormous so that it is extremely difficult for the moment to improve on the tables obtained by Thistlethwaite.

It would be interesting to extend the discussion in order to obtain algorithms that classify multicomponent links; in principle there shouldn't be any difficulty, although the algorithms would come out much more involved and the finiteness problem would remain unresolved. A bonus of obtaining such an algorithm would be its application in calculating knot and link polynomials.

By using the cubic lattice algorithm an attempt was made to extend the discussion to multiknots. In principle this should also be possible, but the finiteness problem and a number of practical difficulties remain.

We hope that in the future we may return to these questions and report further progress.

A C K N O W L E D G E M E N T S

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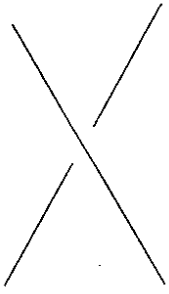


Fig. 3a

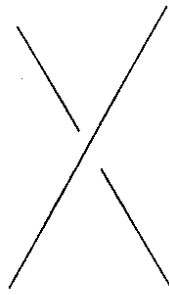


Fig. 3b

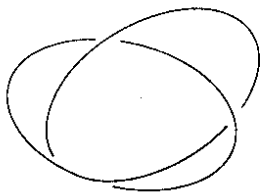


Fig. 1a

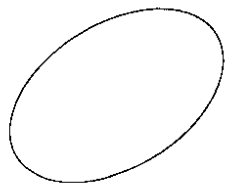


Fig. 1b

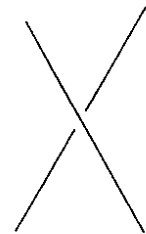


Fig. 2a

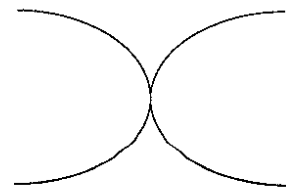


Fig. 2b

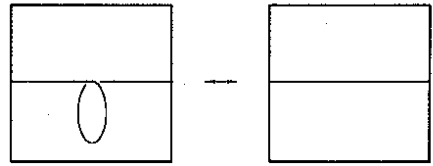


Fig. 4a

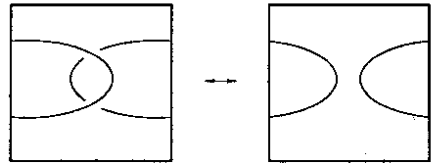


Fig. 4b

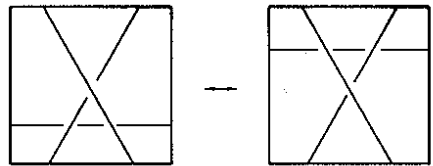


Fig. 4c

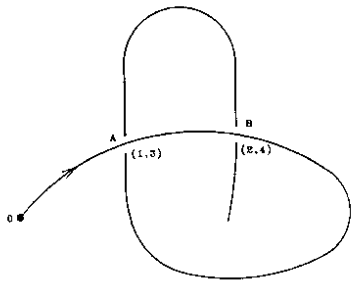


Fig. 8

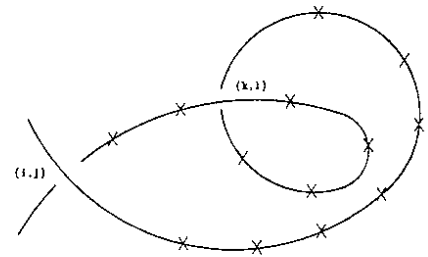


Fig. 10

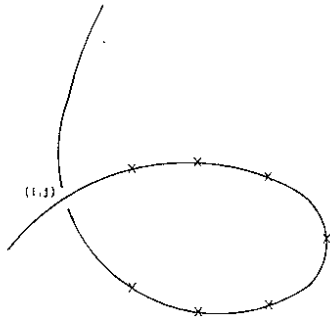


Fig. 9

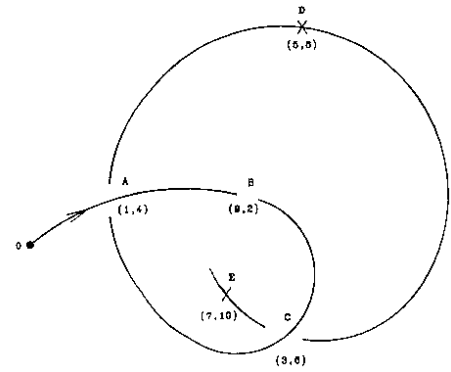


Fig. 11

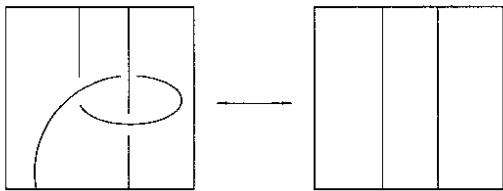
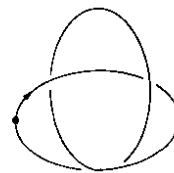


Fig. 5

$\{(1,4), (3,0), (5,2)\}$



$\{(4,1), (6,3), (5,2)\}$

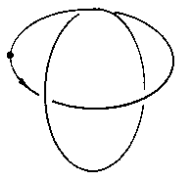


Fig. 7a

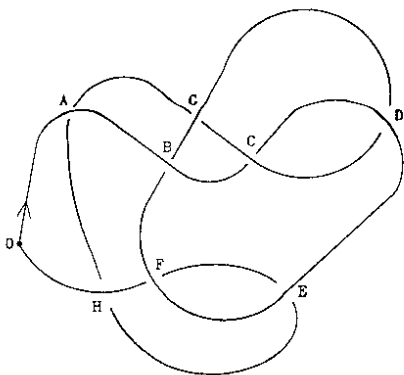
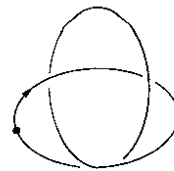


Fig. 6

$\{(1,4), (3,0), (5,2)\}$



$\{(4,1), (6,3), (5,2)\}$

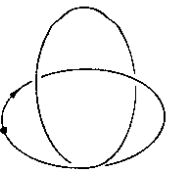


Fig. 7b

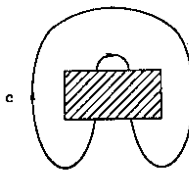
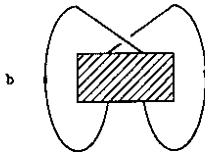
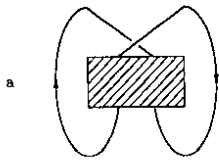


Fig. 14

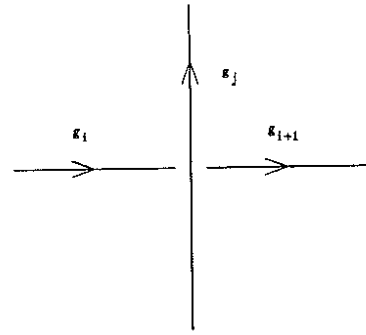


Fig. 15

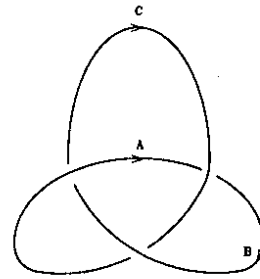


Fig. 16

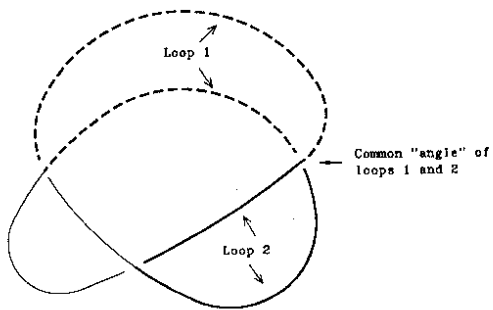
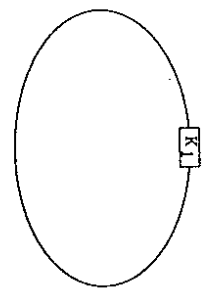
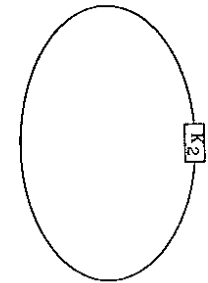


Fig. 12



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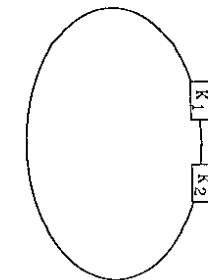


Fig. 13

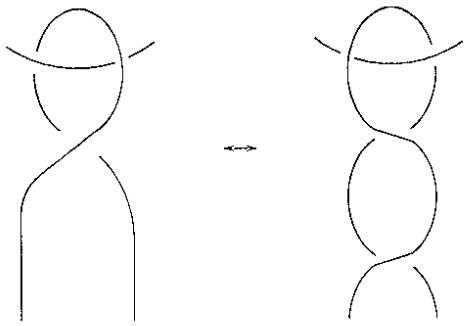


Fig. 20

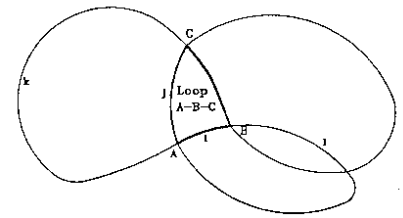


Fig. 22

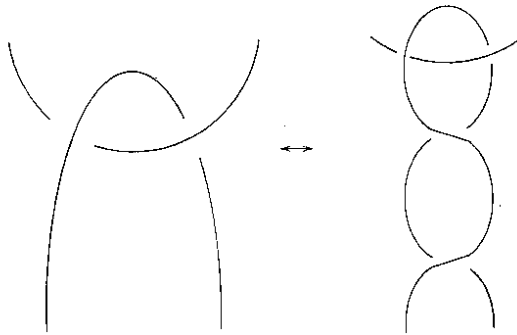


Fig. 21

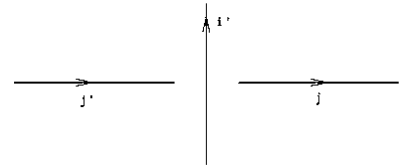


Fig. 23a

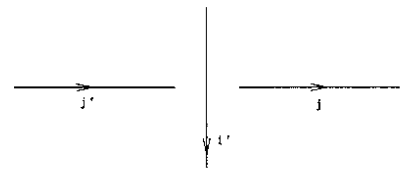


Fig. 23b

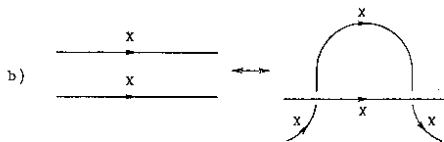
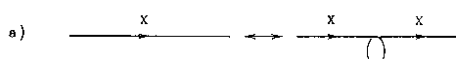


Fig. 17

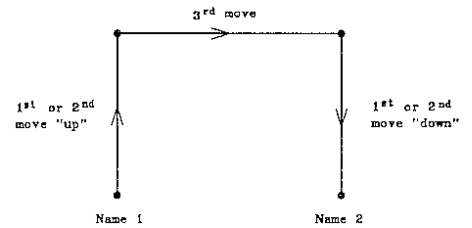


Fig. 18



Fig. 19

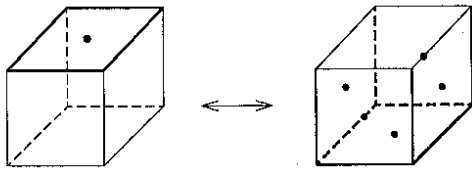


Fig. 27a

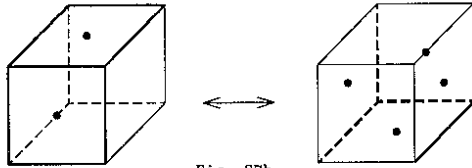


Fig. 27b

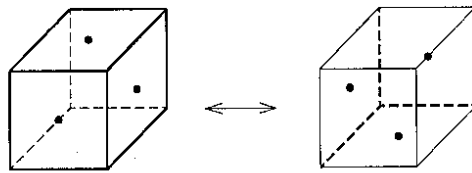


Fig. 27c

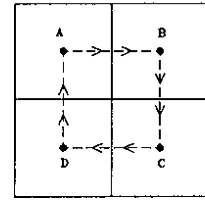


Fig. 28

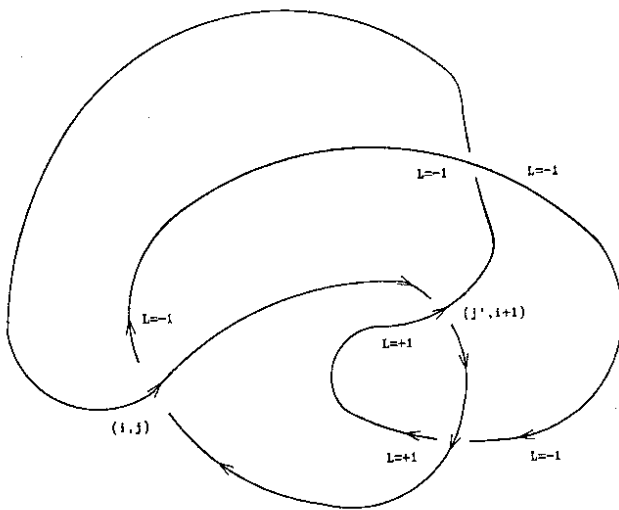


Fig. 24



Fig. 25a



Fig. 25b

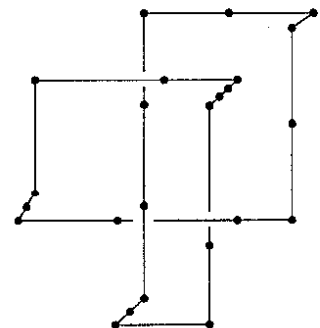


Fig. 26