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Hadamard Vacua in Curved Spacetime and the Principle of Local Definiteness*

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Abstract

The problem of fixing the physically relevant states for quantum fields on arbitrary globally hyperbolic spacetimes is discussed. We report on a recent proof of a long-standing conjecture that the Hadamard vacua for free scalar fields fulfill the principle of local definiteness and thereby determine (locally) the folium of physical states. Some results on the detailed algebraic structure of the observables in representations induced by Hadamard states (split-property, Haag-duality etc.) are also presented.

1. This talk is about structural properties of quantum field theory in curved spacetime (QFT in CST, for short). Let me therefore recall a few general features of QFT in CST. QFT in CST means that quantum fields are considered which propagate in the background of a (classically described) curved spacetime manifold $(\mathcal{M}, g)$.[1] The non-trivial geometrical or topological structure of the underlying spacetime manifold affects the quantum field propagating on it; the most prominent examples of this are the Hawking- and the Fulling-Unruh effects [2,3,4] and the Casimir effect [5] (see also [6]).

A generic spacetime need not admit any symmetries (isometries). But we know that in the usual quantum field theory in Minkowski spacetime the invariance of a quantum field state under spacetime symmetries (Poincaré transformations in this case) and its spectral behaviour is of utmost importance for the notions of "vacuum" and "particle" and hence for the characterization of physically relevant states of a quantum field. So the absence of spacetime symmetries poses some questions concerning the conceptual and mathematical structure of

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[1] This is a 4-dimensional, time-orientable, Lorentzian manifold with metric signature $(+ - - -)$, cf. [1]

QFT in CST. These questions can be discussed best in the framework of algebraic quantum field theory. Let us now turn to the basic features of this subject.

2. Suppose that we are given a spacetime manifold $(\mathcal{M}, g)$. The Lorentzian metric g induces a causal structure: We can distinguish causally related pairs of subsets $\mathcal{O}_1, \mathcal{O}_2$ of $\mathcal{M}$ as those for which there is a causal curve[2] intersecting both $\mathcal{O}_1$ and $\mathcal{O}_2$. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are not causally related, we say that $\mathcal{O}_1$ is contained in the causal complement of $\mathcal{O}_2$ (or vice versa) and write this as $\mathcal{O}_1 \subset \mathcal{O}_2^\perp$ (resp., $\mathcal{O}_2 \subset \mathcal{O}_1^\perp$). Now let $\mathsf{B}(\mathcal{M})$ be the set of all open, relatively compact subsets of $\mathcal{M}$. A map $\mathcal{O} \to \mathcal{A}(\mathcal{O})$, $\mathcal{O} \in \mathsf{B}(\mathcal{M})$ which assigns to each $\mathcal{O} \in \mathsf{B}(\mathcal{M})$ a C^*-algebra,[3] is called a local net of observable algebras if all $\mathcal{A}(\mathcal{O})$ have the same unit element and if there holds:[4]

(1) $$\textit{Isotony}: \quad \mathcal{O}_1 \subset \mathcal{O}_2 \;\Rightarrow\; \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$$

(2) $$\textit{Locality}: \quad \mathcal{O}_1 \subset \mathcal{O}_2^\perp \;\Rightarrow\; [\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = \{0\}$$

Recall that these two conditons are motivated by the idea that each $\mathcal{A}(\mathcal{O})$ is the C^*-algebra formed by the (bounded) observables which can be measured within the spacetime region $\mathcal{O}$ on the system (here: a quantum field propagating in $(\mathcal{M}, g)$). We refer to [7] for further discussion.

$\mathsf{B}(\mathcal{M})$ is a directed set with respect to set-inclusion and so we can by (1) form the smallest C^*-algebra $\mathcal{A} = \overline{\bigcup_{\mathcal{O} \in \mathsf{B}(\mathcal{M})} \mathcal{A}(\mathcal{O})}^{\|\cdot\|}$ which contains all local algebras $\mathcal{A}(\mathcal{O})$.

For the description of a system we need not only observables but also states. The set $\mathcal{A}_1^{*+}$ of all positive, normalized, linear functionals on $\mathcal{A}$ is mathematically referred to as the set of *states* on $\mathcal{A}$, but not all elements of $\mathcal{A}_1^{*+}$ represent physically realizable states of the system. Therefore, given a local net of observable algebras $\mathcal{O} \to \mathcal{A}(\mathcal{O})$ for a physical system, one must specify the set of physically relevant states $\mathcal{S}$, which is a suitable subset of $\mathcal{A}_1^{*+}$. Now, every $\omega \in \mathcal{A}_1^{*+}$ determines a triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$, called the GNS-representation of ω (see e.g. [8]), characterized by the following properties: $\mathcal{H}_\omega$ is a complex Hilbertspace, π_ω is a representation of $\mathcal{A}$ by bounded linear operators on $\mathcal{H}_\omega$ with cyclic vector $\Omega_\omega \in \mathcal{H}_\omega$, and $\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle$ for all $A \in \mathcal{A}$. Whence one can define the net of local von Neumann algebras induced by ω, $\mathcal{O} \to \mathcal{R}_\omega(\mathcal{O}) := \pi_\omega(\mathcal{A}(\mathcal{O}))^-$, where the bar means taking the closure in the weak operator topology in the set of bounded linear operators on $\mathcal{H}_\omega$.

Some of the mathematical properties of the GNS-representations, or of the induced nets of von Neumann algebras, of states ω on $\mathcal{A}$ can be interpreted physically; this leads to constraints on the set of physical states. In this line of

[2] The definition of causal curve is analogous to that in Minkowski spacetime, cf. [1]

[3] In this talk we assume that all C^*-algebras are unital, i.e. possess a unit element, denoted by $\mathbf{1}$

[4] where $[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = \{A_1 A_2 - A_2 A_1 \;:\; A_j \in \mathcal{A}(\mathcal{O}_j) \;\; (j = 1, 2)\}$

thought Haag, Narnhofer and Stein [9] formulated the "principle of local definiteness", consisting of the following three conditions to be obeyed by any collection $\mathcal{S}$ of physical states on $\mathcal{A}$.

(I) Local definiteness: $\bigcap_{\mathcal{O}\ni p} \mathcal{R}_\omega(\mathcal{O}) = \mathbb{C}1$ for all $\omega \in \mathcal{S}$ and all $p \in \mathcal{M}$.

(II) Local primarity: For each $\omega \in \mathcal{S}$, $\mathcal{R}_\omega(\mathcal{O})$ is a factor.

(III) Local quasiequivalence: For each pair $\omega_1, \omega_2 \in \mathcal{S}$ and each $\mathcal{O} \in \mathsf{B}(\mathcal{M})$, the representations $\pi_{\omega_1}|\mathcal{A}(\mathcal{O})$ and $\pi_{\omega_2}|\mathcal{A}(\mathcal{O})$ of $\mathcal{A}(\mathcal{O})$ are quasiequivalent.

Remarks. (i) $\mathcal{R}_\omega(\mathcal{O})$ is a factor if $\mathcal{R}_\omega(\mathcal{O}) \cap \mathcal{R}_\omega(\mathcal{O})' = \mathbb{C}1$ where the prime means taking the commutant. We have not stated in the formulation of local primarity for which regions $\mathcal{O}$ the algebra $\mathcal{R}_\omega(\mathcal{O})$ is required to be a factor. The regions $\mathcal{O}$ should be taken from a class of subsets of $\mathcal{M}$ which forms a base for the topology.

(ii) Quasiequivalence of representations means unitary equivalence up to multiplicity. Another characterization of quasiequivalence is to say that the folia of the representations coincide, where the folium of a representation π is defined as the set of density matrix states in the representation, i.e. the set of all $\omega \in \mathcal{A}_1^{*+}$ which can be realized as $\omega(A) = tr(\rho\pi(A))$ with a density matrix ρ on the representation Hilbertspace of π.

(iii) Conditions (I) and (III) together express that physical states have finite (spatio-temporal) energy-density with respect to each other, and (II) and (III) rule out local macroscopic observables and local superselection rules. We refer to [7] for further discussion and background material.

In quantum field theory in Minkowski spacetime where one is given a vacuum state ω_0, one can define the the set of physical states $\mathcal{S}$ simply as the set of all states on $\mathcal{A}$ which are locally quasiequivalent to ω_0. (Using the spectral condition, ω_0 satisfies (I) [10]. General realistic conditions entailing that the $\mathcal{R}_{\omega_0}(\mathcal{O})$ are factors appear to be unknown. But for many models the $\mathcal{R}_{\omega_0}(\mathcal{O})$ are known to be factors.)

In QFT in CST one does in general not know what a vacuum state is and so $\mathcal{S}$ cannot be determined in the same way. But in some cases (for some models) there may be a set $\mathcal{S}_0 \subset \mathcal{A}_1^{*+}$ of distinguished states, and if this class of states satisfies the conditons (I), (II) and (III), then one may define $\mathcal{S}$ as the set of all $\omega_1 \in \mathcal{A}_1^{*+}$ which are locally quasiequivalent to any (and hence all) $\omega \in \mathcal{S}_0$. Notice that this ensures that $\mathcal{S}$ is a convex subset of $\mathcal{A}_1^{*+}$ even if $\mathcal{S}_0$ is not. The said situation, i.e. that there is a collection $\mathcal{S}_0$ of candidates for physical states, arises for the Klein-Gordon field in globally hyperbolic spacetimes, which is considered next.

3. Recall that a globally hyperbolic spacetime $(\mathcal{M}, g)$ is a spacetime which can be smoothly foliated in Cauchy-surfaces, where a Cauchy-surface is a smooth hypersurface in $\mathcal{M}$ which is intersected exactly once by each inextendible g-causal curve in $\mathcal{M}$. Global hyperbolicity of $(\mathcal{M}, g)$ entails the well-posedness of the

Cauchy-problem for the Klein-Gordon (KG) equation on $(\mathcal{M}, g)$,

$$(3) \qquad (\nabla^a\nabla_a + m^2)\varphi = 0$$

(for smooth, realvalued φ) where ∇ is the metric connection and m is a positive constant. This implies that there exist uniquely determined advanced and retarded fundamental solutions of (3), $E^\pm : C_0^\infty(\mathcal{M}, \mathbb{R}) \to C^\infty(\mathcal{M}, \mathbb{R})$. Their difference $E := E^+ - E^-$ is called the propagator of the KG equation. Denoting $C_0^\infty(\mathcal{M}, \mathbb{R})/\ker(E)$ by K and the quotient map $C_0^\infty(\mathcal{M}, \mathbb{R}) \to K$ by $f \mapsto [f]$, one can show that

$$\kappa([f],[h]) := \int_{\mathcal{M}} f\, Eh\, d\eta\,, \quad f, h \in C_0^\infty(\mathcal{M}, \mathbb{R}),$$

where $d\eta$ is the g-induced volume form on $\mathcal{M}$, is a symplectic form on K. So (K, κ) is a symplectic space. To this symplectic space one can associate its Weylalgebra $\mathcal{A}[K, \kappa]$, which is generated by a family of unitaries $W(x)$, $x \in K$, satisfying the CCR,

$$W(x)W(y) = \mathrm{e}^{-i\kappa(x,y)/2}W(x+y).$$

We call this the Weylalgebra of the KG field in $(\mathcal{M}, g)$. If one defines $\mathcal{A}(\mathcal{O})$ as the C^*-subalgebra of $\mathcal{A}[K, \kappa]$ generated by all $W([f])$, $\mathrm{supp}(f) \subset \mathcal{O}$, then $\mathcal{O} \to \mathcal{A}(\mathcal{O})$ is a local net of observable algebras over $(\mathcal{M}, g)$ (called local net of the KG field over $(\mathcal{M}, g)$). The just sketched constructions are presented in detail in [11] (see also [12]).

A state ω on $\mathcal{A}[K, \kappa]$ is called *quasifree* if its two-point function

$$\lambda_\omega(x, y) := \frac{\partial}{\partial t}\frac{\partial}{\partial \tau}\bigg|_{t=\tau=0} \omega(W(tx)W(\tau y))$$

exists for all $x, y \in K$, and if ω is determined by λ_ω through

$$\omega(W(x)) = \mathrm{e}^{-\lambda_\omega(x,x)/2}\,, \quad x \in K\,.$$

A state ω on $\mathcal{A}[K, \kappa]$ is called a quasifree *Hadamard state* if it is quasifree and if λ_ω is of Hadamard form: This means roughly that

$$\lambda_\omega([f],[h]) = \lim_{\epsilon \to 0} \int_{\mathcal{M}\times\mathcal{M}} (G_\epsilon(p,q) + H_\omega(p,q))\, f(p)h(q)\, d\eta(p)\, d\eta(q)$$

where H_ω is a smooth integral kernel depending on the state ω, while the singular part of λ_ω is given as the limit of a family of integral kernels G_ϵ, $\epsilon > 0$, which are determined by the metric g and the KG equation (through the "Hadamard recursion relations"). Thus a quasifree Hadamard state has a short-distance behaviour of a specified form. We refer to [13] for a discussion of Hadamard form and to [6] for further background material.

Remarks. (i) If ω is an Hadamard state, then $\langle T_{\mu\nu}\rangle_\omega$, the expectation value of the energy-momentum tensor of the (quantized) KG field in the state ω, can be defined and shown to possess reasonable properties. This provides motivation to view Hadamard states as physically relevant states of the quantized KG field. Moreover, the usual vacuum state of the KG field in Minkowski spacetime (and more generally, in ultrastatic spacetimes) is a quasifree Hadamard state. See [6] and literature cited there for details.

(ii) It was shown in [14] (see also [13]) that there exist pure, quasifree Hadamard states on $\mathcal{A}[K,\kappa]$, and one can also show that the set of quasifree Hadamard states is infinite dimensional [12].

When $\mathcal{S}_0$ = *set of quasifree Hadamard states* is taken as a collection of candidates for physical states of the KG field in $(\mathcal{M}, g)$, the question is now if it obeys the conditions (I,II,III) of the principle of local definiteness. It has been conjectured for some time that (III) holds [5,15], and we actually have the following result.

Theorem 1. *The quasifree Hadamard states of the KG field in an arbitrary globally hyperbolic spacetime $(\mathcal{M}, g)$ fulfill the principle of local definiteness. That is, the conditions (I), (II) and (III) hold for all $\omega, \omega_1, \omega_2 \in \mathcal{S}_0$ = set of all quasifree Hadamard states on the Weylalgebra of the KG field in $(\mathcal{M}, g)$.*

Remarks. (i) Local primarity holds for those regions $\mathcal{O}$ (cf. our prior remark) which are of the form $\mathcal{O} = \text{int}\, D(\mathcal{C}_\flat)$[5] where $\mathcal{C}_\flat$ is an open, relatively compact subset of any Cauchy-surface $\mathcal{C}$ in $\mathcal{M}$ whose boundary $\partial\mathcal{C}_\flat$ is contained in the union of finitely many closed, two-dimensional submanifolds of $\mathcal{C}$. We refer to the regions $\mathcal{O}$ of this form as *regular diamonds.*

(ii) In [12] local quasiequivalence of quasifree Hadamard states was proved for arbitrary globally hyperbolic spacetimes, but properties (I) and (II) could only be established under the additional assumption that the spacetimes are ultrastatic. A recent result shows that for quasifree Hadamard states (I) and (II) hold also in arbitrary globally hyperbolic spacetimes [16].

(iii) Making use of the type III_1 property of the local von Neumann algebras $\mathcal{R}_\omega(\mathcal{O})$ induced by quasifree Hadamard states (see Theorem 2 below), we even have local unitary equivalence of the GNS representations of quasifree Hadamard states on the Weylalgebra of the KG field in arbitrary globally hyperbolic spacetimes.

(iv) While certainly the field theoretical model — the KG field — is fixed, one should notice that the spacetimes in which the KG field propagates are only abstractly characterized (as being globally hyperbolic), i.e. the explicit form of the metric or of the spacetime manifold is not of importance for the theorem. So the result is "spacetime-model independent" within the class of globally hyperbolic spacetimes.

(v) Lüders and Roberts [17] have obtained a similar result for the class of adiabatic vacuum states of the KG field in Robertson-Walker spacetimes.

[5] $D(\mathcal{C}_\flat)$ is the domain of dependence of $\mathcal{C}_\flat$, see [1]

This result shows that the set of quasifree Hadamard states (once accepted as physical states) may be used to single out the set of physical states of the KG field in globally hyperbolic spacetimes. It is *precisely in this sense*[6] that the quasifree Hadamard states play a role analogous to the vacuum state in Minkowski-spacetime quantum field theories: their GNS-representations lie locally in the same quasiequivalence class and hence determine locally a unique folium of states.

4. Actually many of the detailed properties known for the vacuum state-induced nets of von Neumann algebras of quantum field theories in Minkowski spacetime can be established for the nets of von Neumann algebras induced by quasifree Hadamard states of the KG field in globally hyperbolic spacetimes. We list some recent results in this direction.

Theorem 2. *Let ω be a quasifree Hadamard state on the Weylalgebra of the KG field in a globally hyperbolic spacetime. Then one has:*

(a) The split property holds for the net $\mathcal{O} \to \mathcal{R}_\omega(\mathcal{O})$. This means that for any pair of regular diamonds $\mathcal{O}_1, \mathcal{O}_2$ with $\overline{\mathcal{O}_1} \subset \mathcal{O}_2$ there is a type I_∞ factor $\mathcal{N}$, acting on $\mathcal{H}_\omega$, such that

$$\mathcal{R}_\omega(\mathcal{O}_1) \subset \mathcal{N} \subset \mathcal{R}_\omega(\mathcal{O}_2).$$

(b) If ω is pure, then Haag-duality

$$\mathcal{R}_\omega(\mathcal{O})' = \mathcal{R}_\omega(\mathcal{O}^\perp)$$

holds for all regular diamonds $\mathcal{O}$.

(c) For all sufficiently small regular diamonds $\mathcal{O}$, $\mathcal{R}_\omega(\mathcal{O})$ is isomorphic to the unique hyperfinite type III_1 factor.

Remarks. (i) (a) is proved in [18]. The split property expresses a strong form of statistical independence, see the review [19]. If the spacetime is ultrastatic, one can prove a strong version of nuclearity for the ultrastatic vacuum [18].

(ii) (b) and (c) are proved in [18] for the ultrastatic, and in [16] for the general globally hyperbolic case.

(iii) Reeh-Schlieder-like properties of the $\mathcal{R}_\omega(\mathcal{O})$ for quasifree Hadamard states can also be established. See [20].

Concluding we may say that the results presented here are in support of the suggestion to view quasifree Hadamard states as physically relevant states of the KG field in globally hyperbolic spacetimes. In terms of the (algebraic) properties of their induced representations and nets of von Neumann algebras they behave as would be expected from the examples on Minkowski spacetime for physically

[6] Notice that there are quasifree Hadamard states which are e.g. thermal equilibrium states

reasonable states. Finally we remark that in the proofs of the results above, use is made at several technical points of the strict positivity of the mass term m in the KG equation (3). It would be interesting to see that the results can also be obtained in the massless case.

As a last point it should be mentioned here that an interesting connection between the Hadamard form of the two-point function λ_ω and the wave front set of λ_ω has recently been established by Radzikowski, see [21] and also [22].

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