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CONDITIONALLY SOLVABLE PATH INTEGRAL PROBLEMS: II. NATANZON POTENTIALS

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ABSTRACT

New classes of exactly solvable potentials are discussed within the path integral formalism. They are constructed from the hypergeometric and confluent Natanzon potentials, respectively. It is found that they allow incorporation of four free parameters, which give rise to fractional power behaviour, long range and strongly anharmonic terms. We find six different classes of such potentials.

1 Introduction.

In a previous paper [1] I have discussed a class of potentials which are known as "conditionally solvable potentials" [2]–[5]. They have the specific feature that they incorporate strongly anharmonic, fractional power behaviour, and long-range terms. They modify the usual potentials in quantum mechanics in a specific way such that they are quantum mechanically exactly solvable, however, the parameters and the couplings of the potentials are not completely free to choose.

In this second article I want to generalize the potentials of [1]–[5] to more general classes which are related to the Natanzon potentials [6], and c.f. [7]–[16]. With their six parameter structure the Natanzon potentials are designed in such a way that a wider range of shapes and potential wells is allowed in comparison to other well-known potential problems in quantum mechanics. Let us mention, e.g., the Morse potential, the radial harmonic oscillator, the Coulomb potential, and the class of hypergeometric potentials as contained in Pöschl-Teller and modified Pöschl-Teller potentials [17]. They are subject to many applications, e.g., in the study of solvable potentials in quantum mechanics in general, in the study of molecular physics for modeling a more realistic single particle electronic shell structure, in atomic physics for quark-antiquark forces, charge densities of nuclei, or in solid state physics. The two classes of Natanzon potentials cover all known potentials for which an analytic solution to the bound and continuous state problem can be found.

The class of the hypergeometric Natanzon potentials is defined by (note the different notations used in the literature)

$$V_h(r)=\frac{\hbar^2}{2m}\frac{fz(z-1)+h_0(1-z)+h_1z}{R(z)}+\frac{\hbar^2}{8m}\bigg(3\Big(\frac{z''}{z'}\Big)^2-2\frac{z'''}{z'}\bigg)\ , \qquad (1)$$

where $R(z)=a_0z^2+b_0z+c_0$, and $z=z(r)$ is implicitly defined by the differential equation $z'=2z(1-z)/\sqrt{R(z)}$. The function z varies in the interval $z\in(0,1)$. The $\hbar^2$-term is the Schwarz derivative of z with respect to r.

The class of the confluent Natanzon potentials can be obtained by the substitution [6] $a_0=\sigma_2/\tau^2$, $b_0=\sigma_1/\tau$, $f=g_2/\tau^2$, $h_1-h_0-f=g_1/\tau$, $z=h/\tau$, and taking into account the limit $\tau\to0$. This yields

$$V_c(r)=\frac{\hbar^2}{2m}\frac{g_2h^2+g_1h+\eta}{R(r)}+\frac{\hbar^2}{8m}\bigg(3\Big(\frac{h''}{h'}\Big)^2-2\frac{h'''}{h'}\bigg)\ , \qquad (2)$$

where $R(r)=\sigma_2h^2+\sigma_1h+c_0$, and $h=h(r)$ is implicitly defined by the differential equation $h'/2h=1/\sqrt{R(r)}$. The variable r and the function $h=h(r)$ are assumed to be positive. The $\hbar^2$-term is the Schwarz derivative of h with respect to r. I succeeded in [18] to calculate the path integral representations corresponding to the two potentials (1) and (2) explicitly in terms of the corresponding Green's function. The energy eigenvalue (= quantization conditions) were equations of fourth degree in the energy. In spite of the fact that the bound state energy-level conditions are rather complicated, closed form solutions in terms of the Green's function were still possible. This is quite surprising because the exact analytic form of a particular Natanzon potential is only implicitly defined and may even not be known analytically.

The two path integral representations in [18] contain all former path integral solutions which are related to the radial harmonic oscillator and the (modified) Pöschl-Teller potential, respectively. In the two latter cases at most two free parameters are free to choose.

In particular for (2), the choices $c_0 = \eta = 0$, $g_1 h \mapsto g_1 h^3$, and $\sigma_1 h \mapsto \sigma_1 h^4$, respectively $c_0 = \eta = 0$, $g_1 h \mapsto g_1 h^4$, $\sigma_1 h \mapsto \sigma_1 h^3$ produce modifications of the two kinds of conditionally solvable potentials as discussed in [1]. They have been called a modified Coulomb potential and a radial confinement potential, respectively. The new potentials which I would like to call *conditionally solvable Natanzon potentials* have four free parameters, and seem to be entirely new. Of course, a similar modification is done for the hypergeometric version, and gives four new classes of potentials. These modifications of the original Natanzon potentials are suitable for our purposes, and the path integral discussion of these four new classes of potentials is the main object of this paper.

Although exactly solvable, these potentials are complicated enough to be of serious consideration in the modeling of actual physical forces. This can be the case where one wants to study an approximation of a model, or where an exactly solvable model is used as a starting point for a comprehensive numerical investigation, c.f.e.g. the recent review [19]. By choosing a path integral approach we succeed in gaining comprehensive information of the bound-stated solutions of these potentials (if they exist), and what is often more important, about the scattering states which eventually allow for the calculation of cross-sections and phase-shifts. In this respect, the path integral provides a convenient tool for the calculation in which the proper analytic structure of the solutions is manifest.

In order to avoid unnecessary overlap I do not repeat the space-time transformation technique as sketched in [1], e.g., Refs. [20]–[34], and references therein. For the actual formulation of the path integral representations of the potentials I use the canonical path integral definition as developed in Refs. [20]–[23, 27, 35, 36]. This will not be repeated here, too.

This article is organized as follows. In the second section I present the six classes of "conditionally solvable Natanzon" potentials labeled V_1, V_2, V_3, V_4, V_5 and V_6, respectively. The well-established space-time transformation technique reduces each path integral problem to an already known one. The final result in each case includes the statement of the corresponding Green's function. The poles of the Green's functions yield in the first two classes implicit expressions (transcendental equations) for the energy eigenvalues, in the second two cases equations of fourth degree in the energy which admit an analytic solution, and in the last two classes also only transcendental equations for the energy eigenvalues can be stated. The bound state wave-functions and the scattering states are not evaluated explicitly. The last section contains a summary and a short discussion.

2 The Potentials.

2.1 A generalized Coulomb potential.

In the first class of potentials I consider a modification of (2) by changing the power-behaviour of h in the g_1-term into a $g_1 h^3$, in the σ_1-term into $\sigma_1 h^4$, and we set $c_0 = \eta = 0$. Thus we obtain the following first *conditionally solvable confluent Natanzon potential*

$$V_1(r) = \frac{\hbar^2}{2m}\frac{g_1 h^3 + g_2 h^2}{R(r)} + \frac{\hbar^2}{8m}\left(3\left(\frac{h''}{h'}\right)^2 - 2\frac{h'''}{h'}\right) , \tag{3}$$

where

$$R(r) = \sigma_1 h^4 + \sigma_2 h^2 , \qquad \frac{h'(r)}{2h(r)} = \frac{1}{\sqrt{R(r)}} . \tag{4}$$

The variable r and the function $h = h(r)$ are assumed to be positive. The special choice $\sigma_1 = 16$ and $\sigma_2 = 0$ gives $h = h(r) = \sqrt{r}$ and reproduces the modified Coulomb potential of Ref. [1], including, of course, the quantum potential $\Delta V = -3\hbar^2/32mr^2$, i.e.,

$$V(r) = \frac{\hbar^2}{32m}\left(\frac{g_2}{r} + \frac{g_1}{\sqrt{r}} - \frac{3}{r^2}\right) . \tag{5}$$

Assuming a power dependence $h = h(r) = r^\alpha$, as $r \to \infty$, we find $\alpha = 1/2$, and the potential (5) describes the asymptotic behaviour of $V_1(r)$ as well. In order to calculate the path integral representation of the potential (3) we perform the transformation $r \mapsto z$ together with the time-substitution $dt = ds/z'^2$ such that the new pseudo-time s'' can be introduced via the constraint $\int_0^{s''} ds/z'^2 = T = t'' - t'$. This space-time transformation causes the emerging Schwarz derivative to cancel with the $\hbar^2$-term and gives the path integral representation in the polar coordinate h

$$K^{(V_1)}(r'', r'; T) = \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{r}^2 - V_1(r)\right]dt\right\}$$

$$= \left(\frac{\sqrt{R(r')R(r'')}}{4h(r')h(r'')}\right)^{1/2} \int_{\mathbb{R}} \frac{dE}{2\pi\hbar}\, \mathrm{e}^{-\mathrm{i}ET/\hbar} \int_0^\infty ds''\, \mathrm{e}^{\mathrm{i}(\sigma_2 E - \hbar^2 g_2/2m)s''/4\hbar}$$

$$\times \int\limits_{h(0)=h'}^{h(s'')=h''} \mathcal{D}h(s) \exp\left[\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left(\frac{m}{2}\dot{h}^2 + \frac{E\sigma_1}{4}h^2 - \frac{\hbar^2}{8m}g_1 h\right)ds\right] \tag{6}$$

$$= \left(\frac{\sqrt{R(r')R(r'')}}{4h(r')h(r'')}\right)^{1/2} \int_{\mathbb{R}} \frac{dE}{2\pi\hbar}\, \mathrm{e}^{-\mathrm{i}ET/\hbar} \int_0^\infty ds''\, \mathrm{e}^{\mathrm{i}(\sigma_2 E - \hbar^4 g_1^2/16m^2\sigma_1 E - \hbar^2 g_2/2m)s''/4\hbar}$$

$$\times \int\limits_{v(0)=v'}^{v(s'')=v''} \mathcal{D}v(s) \exp\left[\frac{\mathrm{i}m}{2\hbar}\int_0^{s''}\left(\dot{v}^2 - \omega^2 v^2\right)ds\right] \tag{7}$$

$$\equiv \int_{\mathbb{R}} \frac{dE}{2\pi\mathrm{i}}\, \mathrm{e}^{-\mathrm{i}ET/\hbar} G^{(V_1)}(r'', r'; E) , \tag{8}$$

and I have performed the additional variable shift $v = h - \hbar^2 g_1/4m\sigma_1 E$, and set $\omega^2 = -\sigma_1 E/2m$. The path integral (6) now has exactly the form as the one in [1] for the modified Coulomb potential after the space-time transformation. There it was solved with the appropriate boundary conditions, i.e., the path integral (6) is not a path integral of a shifted harmonic oscillator in the entire $\mathbb{R}$. As pointed out in [37] the wave-function has the wrong behaviour at the origin of a singular potential, and therefore such a "solution" must be discarded as physically unacceptable [38]. Similarly, as in [1] it is not possible to extend the variable h to the entire $\mathbb{R}$, a feature which is in accordance with the one-dimensional Kustaanheimo-Stiefel transformation which maps $\mathbb{R}^+ \to \mathbb{R}^+$ [34, 39]. Therefore the path integral (6) is a radial path integral with $h > 0$ [40], and the path integral (7) is a radial path integral for a harmonic oscillator with $v > -\hbar^2/4m\sigma_1 E$, for fixed energy E. The additional linear term spoils the symmetry with respect to reflections in the variable h. In [41, 42] I have developed a procedure how to deal with such problems within the path integral. We assume that we have evaluated a path integral problem with a potential $V(x)$ in, say, the entire $\mathbb{R}$. This path integral is called $K^{(V)}(T)$. The corresponding Green's function is denoted by $G^{(V)}(E)$. Now we consider the path integral problem with the same potential V, but with Dirichlet boundary conditions at the location $x = a$ and we consider the half-space $x > a$. Then the Green's function in the half-space $x > a$ is given by [41, 42]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty dT\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}^{(D)}_{(x>a)}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2 - V(x)\right]dt\right\}$$
$$= G^{(V)}(x'',x';E) - \frac{G^{(V)}(x'',a;E)G^{(V)}(a,x';E)}{G^{(V)}(a,a;E)} \ . \tag{9}$$

The Green's function corresponding to $G^{(V)}(E)$ which we need is the Green's function of the harmonic oscillator, $G^{(\omega)}(E)$, and has the form

$$G^{(\omega)}(v'',v';E) = \sqrt{\frac{m}{\pi\hbar^3\omega}}\,\Gamma(-\nu)D_\nu\left(\sqrt{\frac{2m\omega}{\hbar}}v_>\right)D_\nu\left(-\sqrt{\frac{2m\omega}{\hbar}}v_<\right) \ . \tag{10}$$

Here $D_\nu(z)$ is a parabolic cylinder function [43, p.1064]. Inserting (10) into (7) and complying with (9) we obtain the following solution for the Green's function $G^{(V_1)}(E)$

$$\begin{aligned} G^{(V_1)}(r'',r';E) &= \left(\frac{\sqrt{R(r')R(r'')}}{4h(r')h(r'')}\right)^{1/2}\sqrt{\frac{m}{\pi\hbar^3\omega}}\,\Gamma(-\nu)\\ &\times\Bigg\{D_\nu\left[\sqrt{\frac{2m\omega}{\hbar}}\left(h_>(r)-\frac{\hbar^2 g_1}{4m\sigma_1 E}\right)\right]D_\nu\left[-\sqrt{\frac{2m\omega}{\hbar}}\left(h_<(r)-\frac{\hbar^2 g_1}{4m\sigma_1 E}\right)\right]\\ &\quad -D_\nu\left[\sqrt{\frac{2m\omega}{\hbar}}\left(h(r')-\frac{\hbar^2 g_1}{4m\sigma_1 E}\right)\right]D_\nu\left[\sqrt{\frac{2m\omega}{\hbar}}\left(h(r'')-\frac{\hbar^2 g_1}{4m\sigma_1 E}\right)\right]\\ &\quad\times D_\nu\left(\sqrt{\frac{2m\omega}{\hbar}}\frac{\hbar^2 g_1}{4m\sigma_1 E}\right)\Big/D_\nu\left(-\sqrt{\frac{2m\omega}{\hbar}}\frac{\hbar^2 g_1}{4m\sigma_1 E}\right)\Bigg\} \ . \end{aligned} \tag{11}$$

Here I have abbreviated

$$\nu = -\frac{1}{2} + \frac{1}{4\omega\hbar}\left(\sigma_2 E - \frac{\hbar^2 g_2}{2m} - \frac{\hbar^4 g_1^2}{16m^2\sigma_1 E}\right) \ . \tag{12}$$

This determines the energy-spectrum by zeros of the parabolic cylinder function, i.e.,

$$D_{\nu_n}\left(-\sqrt{\frac{2m\omega_n}{\hbar}}\frac{\hbar^2 g_1}{4m\sigma_1 E_n}\right) = 0 \ . \tag{13}$$

I have indicated by $\nu_n = \nu(E_n)$ and $\omega_n = \omega(E_n)$ the explicit dependence on E_n. The analysis in [42] showed that the poles coming form prefactor in (11) play no rôle in the corresponding boundary condition problem. Equation (13) clearly generalizes the corresponding result of [1] to the case of four parameters.

2.2 A radial confinement potential.

The second class of *conditionally solvable confluent Natanzon potentials* we want to consider has the form

$$V_2(r) = \frac{\hbar^2}{2m}\frac{g_1 h^4 + g_2 h^2}{R(r)} + \frac{\hbar^2}{8m}\left(3\left(\frac{h''}{h'}\right)^2 - 2\frac{h'''}{h'}\right) \ , \tag{14}$$

where

$$R(r) = \sigma_1 h^3 + \sigma_2 h^2 \ , \qquad \frac{h'(r)}{2h(r)} = \frac{1}{\sqrt{R(r)}} \ . \tag{15}$$

The special choice $\sigma_1 = 9$ and $\sigma_2 = 0$ gives $h = h(r) = r^{2/3}$ and reproduces the radial confinement potential of Ref. [1], including the quantum potential $\Delta V = -5\hbar^2/72mr^2$,

$$V(r) = \frac{\hbar^2}{18m}\left(g_1 r^{2/3} + g_2 r^{-2/3} - \frac{5}{4r^2}\right) \ , \tag{16}$$

and the radial confinement potential (16) is also the asymptotic solution of V_2, for $h = h(r)$ powerlike as $r \to \infty$. The necessary space-time transformation has the form $r \mapsto h$ accompanied by the time-substitution $dt = R(h)ds/4h^2$, as before. This gives the following path integral representation in the polar coordinate h

$$\begin{aligned} K^{(V_2)}(r'',r';T) &= \int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{r}^2 - V_2(r)\right]dt\right\}\\ &= \left(\frac{\sqrt{R(r')R(r'')}}{4h(r')h(r'')}\right)^{1/2}\int_{\mathbb{R}}\frac{dE}{2\pi\hbar}\mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty ds''\,\mathrm{e}^{\mathrm{i}(\sigma_2 E-\hbar^2 g_2/2m)s''/4\hbar}\\ &\quad\times\int\limits_{h(0)=h'}^{h(s'')=h''}\mathcal{D}h(s)\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left(\frac{m}{2}\dot{h}^2 - \frac{\hbar^2}{8m}g_1 h^2 + \frac{E\sigma_1}{4}h\right)ds\right] \end{aligned} \tag{17}$$

$$\begin{aligned} &= \left(\frac{\sqrt{R(r')R(r'')}}{4h(r')h(r'')}\right)^{1/2}\int_{\mathbb{R}}\frac{dE}{2\pi\hbar}\mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty ds''\,\mathrm{e}^{\mathrm{i}(\sigma_2 E+m\sigma_1^2E^2/2\hbar^2 g_1-\hbar^2 g_2/2m)s''/4\hbar}\\ &\quad\times\int\limits_{v(0)=v'}^{v(s'')=v''}\mathcal{D}v(s)\exp\left[\frac{\mathrm{i}m}{2\hbar}\int_0^{s''}\left(\dot{v}^2-\omega^2v^2\right)ds\right] \end{aligned} \tag{18}$$

$$\equiv \int_{\mathbb{R}}\frac{dE}{2\pi\mathrm{i}}\mathrm{e}^{-\mathrm{i}ET/\hbar}G^{(V_2)}(r'',r';E) \ , \tag{19}$$

and I have performed the additional variable shift $v = h - m\sigma_1 E/\hbar^2 g_1$, and set $\omega = \hbar\sqrt{g_1}/2m$. In comparison to (6) the rôles of the quadratic and the linear term are interchanged. The path integral (17) now has exactly the form as the one in [1] for the radial confinement potential. Taking the corresponding result of [1] we obtain the following solution for the Green's function $G^{(V_2)}(E)$

$$\begin{aligned} G^{(V_2)}(r'',r';E) &= \left(\frac{\sqrt{R(r')R(r'')}}{4h(r')h(r'')}\right)^{1/2}\sqrt{\frac{m}{2\pi\hbar^3\omega}}\,\Gamma(-\nu) \\ &\times\Bigg\{D_\nu\left[\sqrt{\frac{m\omega}{\hbar}}\left(h_>(r)-\frac{m\sigma_1}{\hbar^2 g_1}E\right)\right]D_\nu\left[-\sqrt{\frac{m\omega}{\hbar}}\left(h_<(r)-\frac{m\sigma_1}{\hbar^2 g_1}E\right)\right] \\ &\quad -D_\nu\left[\sqrt{\frac{m\omega}{\hbar}}\left(h(r')-\frac{m\sigma_1}{\hbar^2 g_1}E\right)\right]D_\nu\left[\sqrt{\frac{m\omega}{\hbar}}\left(h(r'')-\frac{m\sigma_1}{\hbar^2 g_1}E\right)\right] \\ &\quad \times D_\nu\left(\sqrt{\frac{m\omega}{\hbar}}\frac{m\sigma_1}{\hbar^2 g_1}E\right)\Big/D_\nu\left(-\sqrt{\frac{m\omega}{\hbar}}\frac{m\sigma_1}{\hbar^2 g_1}E\right)\Bigg\} \ . \end{aligned} \tag{20}$$

Here I have abbreviated

$$\nu = -\frac{1}{2}+\frac{1}{4\hbar\omega}\left(\sigma_2 E+\frac{m\sigma_1^2E^2}{2\hbar^2 g_1}-\frac{\hbar^2 g_2}{2m}\right) \ . \tag{21}$$

This determines the energy-levels E_n by zeros of the parabolic cylinder function, i.e.,

$$D_{\nu_n}\left(-\sqrt{\frac{m\omega}{\hbar}}\frac{m\sigma_1}{\hbar^2 g_1}E_n\right) = 0 \ . \tag{22}$$

I have indicated by $\nu_n = \nu(E_n)$ the explicit dependence on E_n. This result again generalizes the corresponding case of [1].

In the special case of $g_1 = 0$ we must consider the path integral solution of the linear potential. We obtain in this case for the Green's function $G^{(g_1=0)}(E)$, note $E' = \sigma_2 E - \hbar^2 g_2/2m$

$$\begin{aligned} G^{(V_2)}_{g_1=0}(r'',r';E) &= \frac{4}{3}\frac{m}{\hbar^2}\left(\frac{\sqrt{R(r')R(r'')}}{4h(r')h(r'')}\right)^{1/2}\left[\left(h(r')+\frac{4E'}{\sigma_1 E}\right)\left(h(r'')+\frac{4E'}{\sigma_1 E}\right)\right]^{1/2} \\ &\times\Bigg\{K_{1/3}\left[\frac{\sqrt{-2m\sigma_1 E}}{3\hbar}\left(h_>(r)+\frac{4E'}{\sigma_1 E}\right)^{3/2}\right]I_{1/3}\left[\frac{\sqrt{-2m\sigma_1 E}}{3\hbar}\left(h_<(r)+\frac{4E'}{\sigma_1 E}\right)^{3/2}\right] \\ &\quad -K_{1/3}\left[\frac{\sqrt{-2m\sigma_1 E}}{3\hbar}\left(h(r')+\frac{4E'}{\sigma_1 E}\right)^{3/2}\right]K_{1/3}\left[\frac{\sqrt{-2m\sigma_1 E}}{3\hbar}\left(h(r'')+\frac{4E'}{\sigma_1 E}\right)^{3/2}\right] \\ &\quad \times I_{1/3}\left[\frac{\sqrt{-2m}}{3\sigma_1\hbar E}\left(\sigma_2 E-\frac{\hbar^2 g_2}{2m}\right)^{3/2}\right]\Big/K_{1/3}\left[\frac{\sqrt{-2m}}{3\sigma_1\hbar E}\left(\sigma_2 E-\frac{\hbar^2 g_2}{2m}\right)^{3/2}\right]\Bigg\} \ . \end{aligned} \tag{23}$$

$I_\nu(z)$ and $K_\nu(z)$ are modified Bessel functions [43, p.958]. Possible bound states are determined by

$$K_{1/3}\left[\frac{\sqrt{-2m}}{3\sigma_1\hbar E_n}\left(\sigma_2 E_n-\frac{\hbar^2 g_2}{2m}\right)^{3/2}\right] = 0 \ . \tag{24}$$

In terms of the Airy function $\mathrm{Ai}(z)$ this gives

$$\mathrm{Ai}\left[\left(\frac{\sqrt{-2m}}{2\sigma_1\hbar E_n}\right)^{2/3}\left(\sigma_2 E_n-\frac{\hbar^2 g_2}{2m}\right)\right] = 0 \ , \tag{25}$$

which gives a cubic equation in E_n and has the form ($\alpha_n > 0, n \in \mathbb{N}_0$, are the zeros of the Airy function, i.e. $\mathrm{Ai}(-\alpha_n) = 0$ [44, p.166])

$$\left(\sigma_2 E_n-\frac{\hbar^2 g_2}{2m}\right)^3-\frac{2\sigma_1^2\hbar^2\alpha_n^3}{m}E_n^2 = 0 \ , \tag{26}$$

which can be cast into the canonical form (the case $\sigma_2 = 0$ is equivalent with the case discussed in [1] and must be treated separately)

$$E_n^3-\frac{\hbar^2}{2m\sigma_2^3}\left(3g_2\sigma_2^2+4\sigma_1^2\alpha_n^3\right)E_n^2+3\left(\frac{\hbar^2 g_2}{2m\sigma_2}\right)^2E_n-\left(\frac{\hbar^2 g_2}{2m\sigma_2}\right)^3 = 0 \ . \tag{27}$$

One *real* solution of this cubic equation is given by [50]

$$E_n = \sqrt[3]{\sqrt{D}-\frac{Q}{2}}-\sqrt[3]{\sqrt{D}+\frac{Q}{2}}-\frac{R}{3} \tag{28}$$

$$D = \left(\frac{P}{3}\right)^3+\left(\frac{Q}{2}\right)^2 , \qquad P = \frac{3S-R^2}{3} , \qquad Q = \frac{2R^3}{27}-\frac{RS}{3}+T , \tag{29}$$

$$R = -\frac{\hbar^2}{2m\sigma_2^3}\left(3g_2\sigma_2^2+4\sigma_1^2\alpha_n^3\right) , \qquad S = 3\left(\frac{\hbar^2 g_2}{2m\sigma_2}\right)^2 , \qquad T = -\left(\frac{\hbar^2 g_2}{2m\sigma_2}\right)^3 . \tag{30}$$

An asymptotic analysis of the cubic equations shows that for $\alpha_n \to \infty$, i.e. $n \to \infty$, we have a behaviour of the bound state energy-levels according to $E_n \propto -|g_2|\alpha_n^{-3/2}$, $g_2 < 0$, and the accumulation point is $E_\infty = 0$.

2.3 A modified Rosen-Morse potential I.

In the third class of potentials I want to investigate, I consider the hypergeometric Natanzon potential with the following modifications

$$V_3(r) = \frac{\hbar^2}{2m}\frac{\frac{3}{4}z(1-z)+h_0(1-z)+h_1 z^{1/2}}{R(z)}+\frac{\hbar^2}{8m}\left(3\left(\frac{z''}{z'}\right)^2-2\frac{z'''}{z'}\right) , \tag{31}$$

where $R(z) = b_0 z + c_0$, and $z = z(r)$ is implicitly defined by the differential equation $z' = 2z(1-z)/\sqrt{R(z)}$. The variable z varies in the interval $z \in (0,1)$. For $R(z) = 1$, we find $z = \frac{1}{2}(1+\tanh x)$, $x \in \mathbb{R}$, and the emerging potential has the form

$$V(r) = \frac{\hbar^2}{2m}\left(h_0+1-\frac{h_0-3/4}{1+e^{-2r}}+\frac{h_1}{\sqrt{1+e^{-2r}}}-\frac{3}{4(1+e^{-2r})^2}\right) , \tag{32}$$

which looks exactly like the first modified Rosen-Morse potential in [1]. In the original Natanzon potential V_N this choice yields the usual Rosen-Morse potential, which justifies our notion.

In order to calculate the path integral representation corresponding to the potential (31) we perform the transformation $r \mapsto z$ together with the time-substitution $dt = ds/z'^2$. This space-time transformation causes the emerging Schwarz derivative to cancel with the $\hbar^2$-term and gives the path integral representation

$$K^{(V_3)}(r'',r';T) = \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot r^2 - V_3(r)\right]dt\right\}$$
$$= \left(\frac{\sqrt{R(r')R(r'')}}{4z(r')z(r'')(1-z(r'))(1-z(r''))}\right)^{1/2} \int_{\mathbb{R}} \frac{dE}{2\pi\hbar} e^{-\mathrm{i}ET/\hbar} \int_0^\infty ds''$$
$$\times \int\limits_{z(0)=z'}^{z(s'')=z''} \mathcal{D}z(s) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot z^2 + \frac{ER(z)}{4z^2(1-z)^2}\right.\right.$$
$$\left.\left. - \frac{\hbar^2}{8m}\left(\frac{h_0}{z^2(1-z)} + \frac{h_1}{z^{3/2}(1-z)^2} - \frac{3}{4z(z-1)}\right)\right]ds\right\} . \quad (33)$$

We perform a further space-time transformation $z = z(x) = \tanh^2 x$, $x > 0$, together with the time substitution $dt = 4\tanh^2 x ds/\cosh^4 x$. The quantum potential emerging from the Schwarz derivative of z with respect to x is given by

$$\Delta V = \frac{\hbar^2}{8m}\left(4 + \frac{3}{\sinh^2 x} - \frac{3}{\cosh^2 x}\right) , \quad (34)$$

and we obtain the path integral representation for the emerging Manning-Rosen [45] potential with the solution according to [23, 25, 46]–[49]

$$K^{(V_3)}(r'',r';T) = \left(\frac{R(r')R(r'')}{z(r')z(r'')}\right)^{1/4} \int_{\mathbb{R}} \frac{dE}{2\pi\hbar} e^{-\mathrm{i}ET/\hbar} \int_0^\infty ds''\, e^{\mathrm{i}[(b_0+c_0)E-\hbar^2/2m]s''/\hbar}$$
$$\times \int\limits_{x(0)=x'}^{x(s'')=x''} \mathcal{D}x(s) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot x^2 - \frac{\hbar^2}{2m}\left(\frac{\eta^2 - 1/4}{\sinh^2 x} + h_1 \coth x\right)\right]ds\right\} \quad (35)$$
$$= \int_{\mathbb{R}} \frac{dE}{2\pi \mathrm{i}} e^{-\mathrm{i}ET/\hbar} G^{(V_3)}(r'',r';E) , \quad (36)$$

with the Green's function $G^{(V_3)}(E)$ of the third conditionally solvable Natanzon potential given by

$$G^{(V_3)}(r'',r';E) = \frac{m}{\hbar^2}\left(\frac{R(r')R(r'')}{z(r')z(r'')}\right)^{1/4} \frac{\Gamma(m_1 - L_E)\Gamma(L_E + m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\Gamma(m_1 - m_2 + 1)}$$
$$\times \left(\frac{2\sqrt{z(r')}}{1+\sqrt{z(r')}} \cdot \frac{2\sqrt{z(r'')}}{1+\sqrt{z(r'')}}\right)^{\frac{1}{2}(m_1+m_2+1)} \left(\frac{1-\sqrt{z(r')}}{1+\sqrt{z(r')}} \cdot \frac{1-\sqrt{z(r'')}}{1+\sqrt{z(r'')}}\right)^{\frac{1}{2}(m_1-m_2)}$$
$$\times {}_2F_1\left(-L_E + m_1, L_E + m_1 + 1; m_1 - m_2 + 1; \frac{1-\sqrt{z_>(r)}}{1+\sqrt{z_>(r)}}\right)$$
$$\times {}_2F_1\left(-L_E + m_1, L_E + m_1 + 1; m_1 + m_2 + 1; \frac{2\sqrt{z_<(r)}}{1+\sqrt{z_<(r)}}\right) , \quad (37)$$

where

$$L_E = \frac{1}{2}\left(\sqrt{1 - h_1 - 2m(b_0+c_0)E/\hbar^2} - 1\right) , \quad (38)$$
$$m_{1,2} = \sqrt{h_0 + 1 - 2mc_0E/\hbar^2} \pm \frac{1}{2}\sqrt{h_1 + 1 - 2m(b_0+c_0)E/\hbar^2} , \quad (39)$$

${}_2F_1(a,b;c;z)$ is the hypergeometric function [43, p.1039]. Furthermore I have abbreviated

$$\eta^2 = h_0 + 1 - \frac{2mc_0E}{\hbar^2} . \quad (40)$$

Note that the number η is a square root and the specific sign it takes may vary in different examples. From the poles of the Green's function, respectively from the spectral expansion of the Manning-Rosen potential [49] we derive the quantization condition for the bound state wave-functions ($n \in \mathbb{N}$)

$$\sqrt{h_0 + 1 - 2mc_0E_n/\hbar^2} + \frac{\hbar}{\sqrt{2m}}(n + \tfrac{1}{2})$$
$$= \frac{1}{2}\left(\sqrt{1 + h_1 - 2m(b_0+c_0)E_n/\hbar^2} - \sqrt{1 - h_1 - 2m(b_0+c_0)E_n/\hbar^2}\right) . \quad (41)$$

This equation, which is actually an equation of fourth degree in the variable E_n, is with its four parameters an obvious generalization of the corresponding case in [1]. Introducing the abbreviations $B_1 = \frac{\hbar^2}{2m}(h_1 - 1)$, $B_2 = \frac{\hbar^2}{2m}(h_1 + 1)$, $C_1 = 4\frac{\hbar^2}{2m}(h_0 + 1)$, $\alpha_1 = 4c_0$, $\alpha_2 = b_0 + c_0$, $\tilde n = \hbar(n + \frac{1}{2})/\sqrt{2m}$ it can be rewritten into

$$\sqrt{C_1 - \alpha_1 E_n} + 2\tilde n = \sqrt{B_1 - \alpha_2 E_n} - \sqrt{B_2 - \alpha_2 E_n} , \quad (42)$$

and can be cast into the canonical form $\left(C_2 = C_1 + 4\tilde n^2 - \hbar/m, b_2 = (2b_0 - c_0)\right)$

$$E_n^4 + bE_n^3 + cE_n^2 + dE_n + e = 0 , \quad (43)$$

$$\left.\begin{aligned}
a &= (b_2^2 - 4\alpha_2^2)^2 ,\\
b &= \frac{4}{a}\left[(b_2^2 - 4\alpha_2^2)(2\alpha_2(B_1+B_2) - b_2C_2 + 8\alpha_1\tilde n^2) + 64\alpha_1\alpha_2^2\tilde n^2\right] ,\\
c &= \frac{8}{a}\Big[2(4\alpha_1\tilde n^2 - \alpha_2B_1)^2 + \tfrac{1}{2}(2\alpha_2B_2 - b_2C_2)^2 + \tfrac{1}{4}b_2^2C_2^2 + B_1B_2(8\alpha_2^2 - b_2^2)\\
&\quad -\alpha_2C_2(\alpha_2C_2 + 2b_2B_1) - 4\tilde n^2(2\alpha_1b_2C_2 + b_2^2C_1 + 4\alpha_1\alpha_2B_2 + 4\alpha_2^2C_1)\Big] ,\\
d &= \frac{8}{a}\Big[(B_2\alpha_2 - \tfrac{1}{2}b_2C_2 - 4\alpha_1\tilde n^2 + \alpha_2B_1)(C_2^2 - 4B_1B_2 + 16C_1\tilde n^2)\\
&\quad +8C_2\tilde n^2(\alpha_1C_2 + 2b_2C_1)\Big] ,\\
e &= \frac{1}{a}\left[(C_2^2 - 4B_1B_2 + 16C_1\tilde n^2)^2 - 64C_1C_2^2\tilde n^2\right] .
\end{aligned}\right\} \quad (44)$$

Equation (43) can be solved [50] by considering the solutions of the quadratic equation

$$E_n^2 + (b + A)\frac{E_n}{2} + y + \frac{by - d}{A} = 0 , \quad (45)$$

where $A = \pm\sqrt{8y + b^2 - 4c}$, i.e., with the four solutions

$$E_{n_{1,2,3,4}} = -(b + A) \pm \sqrt{(b+A)^2 - 16\left(y + \frac{by - d}{A}\right)} , \quad (46)$$

and y is any of the real roots of the cubic equation

$$8y^3 - 4cy^2 + (2bd - 8e)y + e(4c - b^2) - d^2 = 0 . \quad (47)$$

2.4 A modified Rosen-Morse potential II.

In the fourth class of potentials we want to investigate, we modify the hypergeometric Natanzon potential according to

$$V_4(r) = \frac{\hbar^2}{2m}\frac{\frac{3}{4}z(1-z) + h_0(1-z) + h_1(1-z)^{1/2}}{R(z)} + \frac{\hbar^2}{8m}\left(3\left(\frac{z''}{z'}\right)^2 - 2\frac{z'''}{z'}\right) , \qquad (48)$$

where $R(z) = b_0 z + c_0$, and $z = z(r)$ is again implicitly defined by the differential equation $z' = 2z(1-z)/\sqrt{R(z)}$. Repeating the considerations of (32) gives an additional factor e^{-r} in the h_1-term, therefore reproducing the modified Rosen-Morse potential of [1]. We perform the same space-time transformation as before and obtain (η^2 as in (40))

$$\begin{aligned}
K^{(V_4)}(r'',r';T) &= \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot r^2 - V_4(r)\right]dt\right\}\\
&= \left(\frac{\sqrt{R(r')R(r'')}}{4z(r')z(r'')(1-z(r'))(1-z(r''))}\right)^{1/2}\int_{\mathbb{R}}\frac{dE}{2\pi\hbar}e^{-iET/\hbar}\int_0^\infty ds''\\
&\quad\times\int\limits_{z(0)=z'}^{z(s'')=z''}\mathcal{D}z(s)\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot z^2 + \frac{ER(z)}{4z^2(1-z)^2}\right.\right.\\
&\qquad\left.\left. - \frac{\hbar^2}{8m}\left(\frac{h_0}{z^2(1-z)} + \frac{h_1}{z^2(1-z)^{3/2}} - \frac{3}{4z(z-1)}\right)\right]ds\right\}\\
&= \left(\frac{R(r')R(r'')}{z(r')z(r'')}\right)^{1/4}\int_{\mathbb{R}}\frac{dE}{2\pi\hbar}e^{-iET/\hbar}\int_0^\infty ds''\, e^{i[b_0E+\hbar^2(h_0-1/4)/2m]s''/\hbar}\\
&\quad\times\int\limits_{u(0)=u'}^{u(s'')=u''}\mathcal{D}u(s)\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot u^2 - \frac{\hbar^2}{2m}\left(\frac{\eta^2-1/4}{\tanh^2 u} + h_1\frac{\coth u}{\sinh u}\right)\right]ds\right\} \qquad (49)\\
&= \int_{\mathbb{R}}\frac{dE}{2\pi i}e^{-iET/\hbar}G^{(V_4)}(r'',r';E) . \qquad (50)
\end{aligned}$$

Here the path integral for the hyperbolic Scarf-like potential [48, 51] has been used. The Green's function $G^{(V_4)}(E)$ of the fourth conditionally solvable Natanzon potential is thus given by

$$\begin{aligned}
G^{(V_4)}(x'',x';E) &= \frac{2m}{\hbar^2}\left(\frac{R(r')R(r'')}{z(r')z(r'')}\right)^{1/4}\frac{\Gamma(m_1-L_\eta)\Gamma(L_\eta+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}\\
&\times\left[\frac{1}{4}\left(1+\frac{1}{\sqrt{1-z(r')}}\right)\cdot\left(1+\frac{1}{\sqrt{1-z(r'')}}\right)\right]^{-(m_1-m_2)/2}\\
&\times\left(\frac{\sqrt{z(r')}}{1+\sqrt{1-z(r')}}\cdot\frac{\sqrt{z(r'')}}{1+\sqrt{1-z(r'')}}\right)^{m_1+m_2+1/2}\\
&\times{}_2F_1\left(-L_\eta+m_1, L_\eta+m_1+1; m_1-m_2+1; 2\left(1+\frac{1}{\sqrt{1-z_<(r)}}\right)^{-1}\right)\\
&\times{}_2F_1\left(-L_\eta+m_1, L_\eta+m_1+1; m_1+m_2+1; \frac{z_>(r)}{\left(1+\sqrt{1-z_>(r)}\right)^2}\right) . \qquad (51)
\end{aligned}$$

Here denote

$$\begin{aligned}
m_{1,2} &= \frac{1}{2}\sqrt{h_0+h_1+1-\frac{2m}{\hbar^2}c_0E} \pm \sqrt{1-\frac{2m}{\hbar^2}b_0E} , \qquad (52)\\
L_\eta &= \frac{1}{2}\left(\sqrt{h_0-h_1+1-\frac{2m}{\hbar^2}c_0E}-1\right) . \qquad (53)
\end{aligned}$$

With the abbreviations $B_1 = \frac{\hbar^2}{2m}(h_0+1-h_1)$, $B_2 = \frac{\hbar^2}{2m}(h_0+1+h_1)$, $C_1 = 2\hbar^2/m$, $\alpha_1 = 4b_0$ we get the following quantization condition ($n \in \mathbb{N}$)

$$\sqrt{C_1 - \alpha_1 E_n} + 2\hbar = \sqrt{B_1 - c_0E_n} - \sqrt{B_2 - c_0E_n} . \qquad (54)$$

This equation is again an equation of fourth degree in E_n. Equation (54) is an obvious generalization of the corresponding case of [1], where the eigenvalue equation was a cubic equation, and can be solved in a similar way as the previous case.

2.5 A modified Manning-Rosen potential I.

In the fifth class of potentials we want to investigate, we modify the hypergeometric Natanzon potential according to

$$V_5(r) = \frac{\hbar^2}{2m}\frac{fz(z-1) + \frac{3}{4}(1-z) + h_1 z^{3/2}}{R(z)} + \frac{\hbar^2}{8m}\left(3\left(\frac{z''}{z'}\right)^2 - 2\frac{z'''}{z'}\right) , \qquad (55)$$

where $R(z) = a_0z^2 + zb_0$, and $z = z(r)$ is again implicitly defined by the differential equation $z' = 2z(1-z)/\sqrt{R(z)}$. If we make the Ansatz $R = z^2$, we get $z = 1 - e^{-2r}$, $r > 0$. In the original Natanzon potential we obtain the Manning-Rosen potential. In the present case we get

$$V(r) = \frac{\hbar^2}{2m}\left(f + 1 - \frac{f-3/4}{1-e^{-2r}} + \frac{h_1}{\sqrt{1-e^{-2r}}} - \frac{3}{4(1-e^{-2r})^2}\right) . \qquad (56)$$

This potential may called due to its singular structure, a modified Manning-Rosen potential. Such a "conditionally solvable potential" seems to be new. We perform the same space-time transformations as in the previous two cases and obtain

$$\begin{aligned}
K^{(V_5)}(r'',r';T) &= \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot r^2 - V_5(r)\right]dt\right\}\\
&= \left(\frac{\sqrt{R(r')R(r'')}}{4z(r')z(r'')(1-z(r'))(1-z(r''))}\right)^{1/2}\int_{\mathbb{R}}\frac{dE}{2\pi\hbar}e^{-iET/\hbar}\int_0^\infty ds''\\
&\quad\times\int\limits_{z(0)=z'}^{z(s'')=z''}\mathcal{D}z(s)\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot z^2 + \frac{ER(z)}{4z^2(1-z)^2}\right.\right.
\end{aligned}$$

$$-\frac{\hbar^2}{8m}\left(\frac{f}{z(1-z)}+\frac{h_1}{z^{1/2}(1-z)^2}+\frac{3}{4z^2(1-z)}\right)\Bigg]ds\Bigg\} \tag{57}$$

$$= \left(\frac{R(r')R(r'')}{z(r')z(r'')}\right)^{1/4}\int_{\mathbb{R}}\frac{dE}{2\pi\hbar}\,e^{-iET/\hbar}\int_0^\infty ds''\,e^{i[(a_0+b_0)E-\hbar^2/2m)s''/\hbar}$$
$$\times\int\limits_{x(0)=x'}^{x(s'')=x''}\mathcal{D}x(s)\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{x}^2+\frac{\hbar^2}{2m}\left(\frac{\nu^2+1/4}{\cosh^2 x}-h_1\tanh x\right)\right]ds\right\} \tag{58}$$

$$= \int_{\mathbb{R}}\frac{dE}{2\pi i}\,e^{-iET/\hbar}G^{(V_5)}(r'',r';E)\ . \tag{59}$$

(58) corresponds to the path integral of the Rosen-Morse potential as discussed in [26, 47]. However, we encounter in the present case the same difficulty as in the discussion of the two confluent conditionally solvable Natanzon potentials. The transformation $z = z(x)$ maps $(0,1) \to \mathbb{R}^+$, and (58) is therefore a radial path integral. But the Rosen-Morse potential is defined in the entire $\mathbb{R}$ and due to the odd $\tanh x$-term we cannot continue to the entire real line. Hence, we must apply the same technique as in the first two examples and therefore the Green's function $G^{(V_5)}(E)$ of the fifth conditionally solvable Natanzon potential is given by

$$G^{(V_5)}(E)(r'',r';E) = G(r'',r';E) - \frac{G(r'',z(0);E)G(z(0),r';E)}{G(z(0),z(0);E)}\ , \tag{60}$$

with the Green's function $G(E)$ given by

$$G(r'',r';E) = \frac{m}{\hbar^2}\left(\frac{\sqrt{R(r')R(r'')}}{z(r')z(r'')}\right)^{1/2}\frac{\Gamma(m_1-L_B)\Gamma(L_B+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}$$
$$\times\left(\frac{1-\sqrt{z(r')}}{2}\cdot\frac{1-\sqrt{z(r'')}}{2}\right)^{(m_1-m_2)/2}\left(\frac{1+\sqrt{z(r')}}{2}\cdot\frac{1+\sqrt{z(r'')}}{2}\right)^{(m_1+m_2)/2}$$
$$\times{}_2F_1\left(-L_B+M_1, L_B+m_1+1; m_1+m_2+1; \frac{1+\sqrt{z_>(r)}}{2}\right)$$
$$\times{}_2F_1\left(-L_B+M_1, L_B+m_1+1; m_1-m_2+1; \frac{1-\sqrt{z_<(r)}}{2}\right)\ . \tag{61}$$

Here I have used the abbreviations

$$L_B = \sqrt{f+1-\frac{2m}{\hbar^2}a_0E}-\frac{1}{2}\ , \tag{62}$$

$$m_{1,2} = \frac{1}{2}\sqrt{-(h_1+1)-\frac{2m}{\hbar^2}(a_0+b_0)E}\pm\frac{1}{2}\sqrt{h_1-1-\frac{2m}{\hbar^2}(a_0+b_0)E}\ , \tag{63}$$

and have further set

$$\nu^2 = f+1-\frac{2ma_0E}{\hbar^2}\ . \tag{64}$$

Note that the number ν is a square root and the specific sign it takes may vary in different examples. We obtain consequently the transcendental quantization condition for

the bound state energy-levels E_n

$${}_2F_1\Big(-L_B(E_n)+m_1(E_n), L_B(E_n)+m_1(E_n)+1; m_1(E_n)+m_2(E_n)+1; \tfrac{1}{2}\Big)\ . \tag{65}$$

In the case of $R = z^2$, i.e., for the potential

$$V(r) = \frac{\hbar^2}{2m}\left(\frac{B}{1-e^{-2r}}-\frac{A}{\sqrt{1-e^{-2r}}}-\frac{3}{4(1-e^{-2r})^2}\right)\ , \tag{66}$$

we obtain the quantization condition

$${}_2F_1\Big(-\tilde{L}_B(E_n)+\tilde{m}_1(E_n), \tilde{L}_B(E_n)+\tilde{m}_1(E_n)+1; \tilde{m}_1(E_n)+\tilde{m}_2(E_n)+1; \tfrac{1}{2}\Big)\ , \tag{67}$$

where we have set

$$\tilde{L}_B(E) = \sqrt{\frac{1}{4}-\frac{2m}{\hbar^2}E}-\frac{1}{2}\ , \tag{68}$$

$$\tilde{m}_{1,2}(E) = \frac{1}{2}\sqrt{B+A-\frac{3}{4}-\frac{2m}{\hbar^2}E}\pm\frac{1}{2}\sqrt{B-A-\frac{3}{4}-\frac{2m}{\hbar^2}E}\ . \tag{69}$$

2.6 A modified Manning-Rosen potential II.

In the sixth class of potentials we want to investigate, we modify the hypergeometric Natanzon potential according to

$$V_6(r) = \frac{\hbar^2}{2m}\frac{fz(z-1)+\frac{3}{4}(1-z)+h_1z^{3/2}\sqrt{1-z}}{R(z)}+\frac{\hbar^2}{8m}\left(3\left(\frac{z''}{z'}\right)^2-2\frac{z'''}{z'}\right)\ , \tag{70}$$

where $R(z) = a_0z^2+zb_0$, and $z = z(r)$ is again implicitly defined by the differential equation $z' = 2z(1-z)/\sqrt{R(z)}$. Repeating the analysis as in V_5 we obtain an additional factor e^{-r} in the h_1-term, therefore producing another modified Manning-Rosen potential, i.e.,

$$V(r) = \frac{\hbar^2}{2m}\left(f+1-\frac{f-3/4}{1-e^{-2r}}+\frac{h_1e^{-r}}{\sqrt{1-e^{-2r}}}-\frac{3}{4(1-e^{-2r})^2}\right)\ . \tag{71}$$

We perform the same space-time transformations as in the previous cases and obtain with ν as in (64)

$$K^{(V_6)}(r'',r';T) = \int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{r}^2-V_6(r)\right]dt\right\}$$
$$= \left(\frac{\sqrt{R(r')R(r'')}}{4z(r')z(r'')(1-z(r'))(1-z(r''))}\right)^{1/2}\int_{\mathbb{R}}\frac{dE}{2\pi\hbar}\,e^{-iET/\hbar}\int_0^\infty ds''$$
$$\times\int\limits_{z(0)=z'}^{z(s'')=z''}\mathcal{D}z(s)\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{z}^2+\frac{ER(z)}{4z^2(1-z)^2}\right.\right.$$
$$\left.\left.-\frac{\hbar^2}{8m}\left(\frac{f}{z(1-z)}+\frac{h_1}{z^{1/2}(1-z)^2}+\frac{3}{4z^2(1-z)}\right)\right]ds\right\} \tag{72}$$

$$= \left(\frac{R(r')R(r'')}{z(r')z(r'')}\right)^{1/4} \int_{\mathbb{R}} \frac{dE}{2\pi\hbar} e^{-iET/\hbar} \int_0^\infty ds''\, e^{i[b_0E+\hbar^2(f-1/4)/2m)s''/\hbar}$$
$$\times \int\limits_{x(0)=x'}^{x(s'')=x''} \mathcal{D}x(s) \exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{x}^2 - \frac{\hbar^2}{2m}\left((\nu^2-\tfrac{1}{4})\tanh^2 x + h_1\frac{\tanh x}{\cosh x}\right)\right]ds\right\} \quad (73)$$
$$= \int_{\mathbb{R}} \frac{dE}{2\pi i} e^{-iET/\hbar} G^{(V_6)}(r'', r'; E) \ . \quad (74)$$

The path integral (73) is the path integral for the hyperbolic barrier potential as discussed in [48], however in the half-space $\mathbb{R}^+$. Applying the same method as for $G^{(V_5)}(E)$, the Green's function $G^{(V_6)}(E)$ of the sixth conditionally solvable Natanzon potential is therefore given by

$$G^{(V_6)}(E)(r'', r'; E) = G(r'', r'; E) - \frac{G(r'', z(0); E)G(z(0), r'; E)}{G(z(0), z(0); E)} \ . \quad (75)$$

with the Green's function $G(E)$ given by

$$G(r'', r'; E) = \frac{m}{\hbar^2}\left(\frac{\sqrt{R(r')R(r'')}}{z(r')z(r'')}\right)^{1/2} \frac{\Gamma(m_1 - L_\nu)\Gamma(L_\nu + m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\Gamma(m_1 - m_2 + 1)}$$
$$\times \left(\frac{1-\sqrt{z(r')}}{2}\cdot\frac{1-\sqrt{z(r'')}}{2}\right)^{(m_1-m_2)/2} \left(\frac{1+\sqrt{z(r')}}{2}\cdot\frac{1+\sqrt{z(r'')}}{2}\right)^{(m_1+m_2+1/2)/2}$$
$$\times {}_2F_1\left(-L_\nu + M_1, L_\nu + m_1 + 1; m_1 + m_2 + 1; \frac{1+\sqrt{z_>(r)}}{2}\right)$$
$$\times {}_2F_1\left(-L_\nu + M_1, L_\nu + m_1 + 1; m_1 - m_2 + 1; \frac{1-\sqrt{z_<(r)}}{2}\right) \ . \quad (76)$$

Here I have used the abbreviations

$$L_\nu = \frac{1}{2}\left(\sqrt{f + 1 + ih_1 - \frac{2m}{\hbar^2}a_0E} - 1\right) \ , \quad (77)$$
$$m_{1,2} = -\frac{1}{2}\sqrt{f + 1 - ih_1 - \frac{2m}{\hbar^2}a_0E} \pm \sqrt{\frac{1}{4} - f - \frac{2m}{\hbar}b_0E} \ . \quad (78)$$

Note that the minus-sign in the first term in $m_{1,2}$ is due to the reality condition of the problem, c.f. [48]. Bound states with energy E_n are determined by the equation

$${}_2F_1\Big(-L_\nu(E_n) + m_1(E_n), L_\nu(E_n) + m_1(E_n) + 1; m_1(E_n) + m_2(E_n) + 1; \tfrac{1}{2}\Big) \ . \quad (79)$$

A more detailed investigation, and of the special case $R = z^2$, is left to the reader.

3 Summary and Discussion.

In this article the path integral treatments of six classes of "conditionally solvable Natanzon" potentials have been presented. For the analysis of these problems I have used the path integral approach. In spite of the fact that the potentials are defined only implicitly and may even not be known analytically, it has been possible to calculate the Green's function in terms of the variable r, and the functions $R(r)$ and $h(r)$, respectively $z(r)$. The poles of the Green's functions have given the bound state energy-levels, i.e., we have obtained transcendental equations in terms of parabolic cylinder functions, hypergeometric functions, and equations of fourth degree, respectively, and the cuts have provided the scattering states.

The two "conditionally solvable confluent Natanzon potentials" (5) and (16) are but two simple solutions of the potential classes V_1 and V_2, respectively, where $\sigma_2 = 0$. The general structure of the potential V_1 modifies the Coulomb interaction by adding a long-range effect. V_2 instead has a confinement character. The special case σ_1 gives trivial results, and there seem to be no more other simple potentials.

It remains to consider not only the real solutions of the eigenvalue equations, but also the other possible complex solutions as well. For instance in (27), we have found an asymptotic behaviour of the bound state levels which is in accordance with the result of [1]. There only real solutions were allowed because the zeros of the Airy function are known to be real. By a suitable choice of parameters all solutions of (27) need not to be real. The existence of complex solutions, i.e. resonances, depends on how strong a potential barrier is above the energy of the lowest lying scattering states. It is known that (3) can have such resonance states [37], and this feature of locating resonances in scattering processes was one of the reasons to study potentials like (3). Therefore it should be worthwhile to make a numerical investigation along these lines, respectively a detailed Green's function analysis. This would fix the requirements which of the three solutions of the cubic equation (27) or of the four solutions of (43, 54) actually contribute to the spectrum, and which are unphysically and must be discarded. In particular, resonance states would add to the variety of potential forces in nuclear physics and elementary particle physics for states which are subject to decay.

On the other hand, the potential (14) has a confinement character, and its power-dependence describes phenomenological a $q\bar{q}$ interaction, as supported by lattice QCD, c.f. [52] and references therein. Therefore both potentials can due to their four parameter structure serve as a proper refinement of important phenomenological models.

In [53] de Souza Dutra and Girlich have presented two-dimensional extensions of the potentials (5) and (16) according to $(\omega, \lambda > 0)$

$$V_i(x, y) = \frac{m}{2}\omega^2(x^2 + y^2) + \frac{\hbar^2}{2m}\frac{1}{x^2 + y^2}\left[\lambda^2 + V_i\big(\arctan(y/x)\big) - \frac{1}{4}\right] \ . \quad (80)$$

$i = 1, 2$ and $V_1(r)$ the potential (5) and $V_2(r)$ the potential (16). With the coordinate choice $x = u\cos\gamma$, $y = u\sin\gamma$ the problem is separable. However, the same line of reasoning as sketched in section 2.1 concerning the proper boundary conditions applies as well. Therefore the *transcendental* equations for the eigenvalues $E_{i,n}^{(\gamma)}$ (13, 22), now for the variable $\gamma > 0$, determine the separation constant which enters in (80). This yields eventually a radial harmonic oscillator path integral with

$$V_i(u) = \frac{m}{2}\omega^2 u^2 + \frac{\hbar^2}{2m}\frac{\lambda^2 + E_{i,n}^{(\gamma)} - 1/4}{u^2} \quad , \tag{81}$$

which is easy to solve, e.g. [21, 23, 27, 40].

In the second set of *hypergeometric conditionally solvable Natanzon potentials* the bound-state solutions are determined by an equation of fourth degree which complicates the expressions analytically. In both cases the bound-state energy-levels with the wave-functions and the scattering solutions can be obtained in principle. However, due to the complicated structure of the equations, and the fact that a more simple case has already been discussed in an earlier paper [1], an explicit evaluation is omitted. The six path integral representations (8, 19, 35, 49, 58) and (73) together with the explicit form of the corresponding Green's functions (11, 20, 37, 51, 60) and (75) therefore contain the *former* path integral solutions as special cases, and at the same time generalize them.

In the set of potentials (31) and (48) the choice $R(z) \propto z^2$ is explicitly excluded, so one can either set $b_0 = 0$ or $c_0 = 0$. Choosing $b_0 = 0$ yields $z = \frac{1}{2}(1 + \tanh x)$, $x \in \mathbb{R}$, leading to the two modified Rosen-Morse potentials of [1]. The structure of the two potentials is such that they can describe a complicated scattering process by wells or troughs on the real line. The alternative $c_0 = 0$ gives $z = \coth^2 r$, $r > 0$, and leads to the Rosen-Morse and the hyperbolic barrier potential of [48, 51], respectively, and therefore gives no new features.

In order to include in the hypergeometric case $R \propto z^2$, where $z = 1 - e^{-2r}$, $r > 0$, which gives in the original Natanzon potential the Manning-Rosen potential, I have modified the structure of V_h once more. I have obtained two new potentials which I have called due to their singular behaviour "modified Manning-Rosen" potentials. The effect of the potentials can be interpreted as modifying a Coulomb interaction in a space of constant curvature, e.g. [54, 55] and references therein. Alternatively, they combine the effect of screening a Coulomb potential with a long range behaviour according to (5). In both cases the quantization conditions are transcendental equations involving the hypergeometric function.

I have therefore found four classes of "conditionally solvable hypergeometric Natanzon potentials" with either complicated scattering properties on the real line, or modified screening features for singular potentials. In particular, the potentials (56, 66) and (71) provide explicit solutions in terms of the radial variable r which appear to be new.

The particular features of the "conditionally solvable Natanzon potential" also clarify the origin of their solubility. The term which is proportional to $\hbar^2$ guaranties this very fact. Whereas in [2]–[5] this term seems to come in "by hand" and has to be chosen in a suitable way, its structure is fixed because it *is* a Schwarzian derivative. The incorporation of this term in the potential arises therefore in a natural way, similarly as the Schwarzian derivative appears in the space-time transformation of path integrals, and the quantum potential ΔV in the definition of path integrals on curved manifolds.

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