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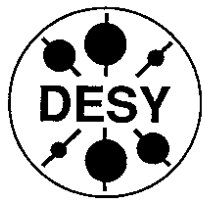


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# Symplectic Thin - Lens Transfer Maps for SIXTRACK: Treatment of Bending Magnets in Terms of the Exact Hamiltonian

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# Symplectic Thin - Lens Transfer Maps for SIXTRACK: Treatment of Bending Magnets in Terms of the Exact Hamiltonian

D.P. Barber, K. Heinemann, G. Ripken, F. Schmidt*

August 19, 1996

**Abstract**

We extend two earlier papers [1, 2] on the determination of symplectic six - dimensional thin - lens maps to show how to construct a six - dimensional symplectic thin–lens transport map for a bending magnet by using the "unexpanded" square root

$$\left\{1 - \frac{p_x^2 + p_z^2}{[1 + f(p_\sigma)]^2}\right\}^{1/2}$$

of the exact Hamiltonian as was already done in Ref. [2] for quadupoles, skew quadrupoles, sextupoles, and octupoles. Thus by combining this paper with Ref. [2], one can treat the whole ring in the thin - lens approximation by using the exact Hamiltonian.

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# Contents

# 1 Introduction

In two earlier papers [1, 2] (which we refer to as I and II) we showed how to construct six - dimensional symplectic thin–lens transport maps for the tracking program SIXTRACK [3]. Whereas in paper I we used an approximate Hamiltonian obtained by a series expansion of the square root

$$\left\{1 - \frac{p_x^2 + p_z^2}{[1 + f(p_\sigma)]^2}\right\}^{1/2}$$

up to first order in terms of the quantity

$$\frac{p_x^2 + p_z^2}{[1 + f(p_\sigma)]^2} ,$$

in II an improved Hamiltonian was introduced by using the unexpanded square root for various kinds of magnets (quadrupoles, skew quadrupoles, sextupoles, and octupoles) appearing in a straight section of a storage ring. The outcome was that the thin - lens maps remained unchanged and the corrections due to the new Hamiltonian were fully absorbed into the drift spaces. In II we also presented a symplectic treatment of the nonlinear crossing terms of the Hamiltonian resulting from the curvature in bending magnets, but only took their lowest order into account.

In this report we now demonstrate how to treat the bending magnets within a symplectic thin - lens approximation, taking into account the exact Hamiltonian also.

We achieve this by introducing a generating function in analogy to the method applied by E. Forest and K. Ohmi for the symplectic integration of complex wigglers [4]. The analysis used in this report can easily be modified for application to a thin - lens synchrotron magnet.

Combining this paper with paper II, we are thus in a position to treat the whole ring without further approximation.

The equations derived are valid for arbitrary particle velocity, i.e. below and above transition energy, and shall be incorporated into the tracking code SIXTRACK [3].

The paper is organized as follows :

In section 2 the general canonical equations of motion for a bending magnet in terms of the exact Hamiltonian are specified. In section 3 we solve the equations of motion by splitting the Hamiltonian as in paper II into two parts. The "unsplit" Hamiltonian is treated in section 4, solving the equations of motion in one step. As a byproduct, it is shown in Appendix A how to construct symplectic thin lens maps for quadrupoles, skew quadrupoles, sextupoles, and octupoles, using the new method described above. Finally, a summary of the results is presented in section 5.

# 2 The Canonical Equations of Motion

## 2.1 Notation

The formalism and notation in this paper is similar to that of Ref. [1]. Thus we will begin by simply stating the canonical equations of motion for a bending magnet already used in this earlier paper and refer the reader to the latter for details.

## 2.2 The Hamiltonian

For a bending magnet with

$$K_x^2 + K_z^2 \neq 0 \; ; \quad K_x \cdot K_z = 0$$

the exact Hamiltonian reads as:

$$\begin{aligned}
\mathcal{H}_{bend} &= p_\sigma - [1+f(p_\sigma)] \cdot [1 + K_x \cdot x + K_z \cdot z] \cdot \left\{ 1 - \frac{p_x^2 + p_z^2}{[1+f(p_\sigma)]^2} \right\}^{1/2} \\
&\quad + \frac{1}{2} \cdot [1 + K_x \cdot x + K_z \cdot z]^2 - \frac{1}{2} \\
&= p_\sigma - [1+f(p_\sigma)] \cdot \left\{ 1 - \frac{p_x^2 + p_z^2}{[1+f(p_\sigma]^2} \right\}^{1/2} \\
&\quad - [1+f(p_\sigma)] \cdot [K_x \cdot x + K_z \cdot z] \cdot \left\{ 1 - \frac{p_x^2 + p_z^2}{[1+f(p_\sigma)]^2} \right\}^{1/2} \\
&\quad + [K_x \cdot x + K_z \cdot z] + \frac{1}{2} K_x^2 \cdot x^2 + \frac{1}{2} K_z^2 \cdot z^2 \\
&= p_\sigma - [1+f(p_\sigma)] \cdot \left\{ 1 - \frac{p_x^2 + p_z^2}{[1+f(p_\sigma]^2} \right\}^{1/2} \\
&\quad - [1+f(p_\sigma)] \cdot [K_x \cdot x + K_z \cdot z] \cdot \left( \left\{ 1 - \frac{p_x^2 + p_z^2}{[1+f(p_\sigma)]^2} \right\}^{1/2} - 1 \right) \\
&\quad - f(p_\sigma) \cdot [K_x \cdot x + K_z \cdot z] + \frac{1}{2} K_x^2 \cdot x^2 + \frac{1}{2} K_z^2 \cdot z^2 \\
&= \mathcal{H}_I + \mathcal{H}_{II} \qquad (2.1)
\end{aligned}$$

with

$$\mathcal{H}_I = -f(p_\sigma) \cdot [K_x \cdot x + K_z \cdot z] + \frac{1}{2} K_x^2 \cdot x^2 + \frac{1}{2} K_z^2 \cdot z^2 \; ; \qquad (2.2a)$$

$$\mathcal{H}_{II} = \mathcal{H}_{cross} + \mathcal{H}_{drift} \qquad (2.2b)$$

where

$$\mathcal{H}_{cross} = -[1+f(p_\sigma)] \cdot [K_x \cdot x + K_z \cdot z] \cdot \left( \left\{ 1 - \frac{p_x^2 + p_z^2}{[1+f(p_\sigma)]^2} \right\}^{1/2} - 1 \right) \; ; \qquad (2.3a)$$

$$\mathcal{H}_{drift} = p_\sigma - [1+f(p_\sigma)] \cdot \left\{ 1 - \frac{p_x^2 + p_z^2}{[1+f(p_\sigma]^2} \right\}^{1/2} , \qquad (2.3b)$$

and where $f(p_\sigma)$ is given by:

$$f(p_\sigma) = \frac{1}{\beta_0} \sqrt{(1 + \beta_0^2 \cdot p_\sigma)^2 - \left( \frac{m_0 c^2}{E_0} \right)^2} - 1 \qquad (2.4)$$

($K_x$, and $K_z$ are defined in Ref. [1]).

# 3 Thin-Lens Approximation for a Bending Magnet

In order to represent a bending magnet we divide each lens into a sufficient number of thin slices of length $\Delta s$. Furthermore, we modify the Hamiltonian (2.1) by writing

$$\begin{aligned}\hat{\mathcal{H}}_I &= \mathcal{H}_I(\vec{y};\, s_0)\cdot\Delta s\cdot\delta(s-s_0)\\ &= \Delta s\cdot\delta(s-s_0)\times\\ &\quad\left\{-f(p_\sigma)\cdot[K_x(s_0)\cdot x+K_z(s_0)\cdot z]+\frac{1}{2}[K_x(s_0)]^2\cdot x^2+\frac{1}{2}[K_z(s_0)]^2\cdot z^2\right\}\end{aligned}\tag{3.1}$$

and

$$\hat{\mathcal{H}}_{bend} = \hat{\mathcal{H}}_I+\mathcal{H}_{II}\,,\tag{3.2}$$

whereby $\hat{\mathcal{H}}_I$ represents a symplectic kick effective in the region

$$s_0\le s\le s_0+\epsilon\quad\text{(region I)}$$

and $\mathcal{H}_{II}$ contains nonlinear crossing terms and the drift terms and is effective in the region

$$s_0+\epsilon\le s\le s_0+\Delta s\quad\text{(region II)}\,.$$

On approximating $\hat{\mathcal{H}}_{bend}$ by "expanding the square root" one obtains the Hamiltonian of Ref. [2].

In section 4 it is shown how to treat the unsplit Hamiltonian $\mathcal{H}_{bend}$.

## 3.1 The Term $\hat{\mathcal{H}}_I$

### 3.1.1 Canonical Equations of Motion

The canonical equations of motion due to the Hamiltonian $\hat{\mathcal{H}}_I$ read as:

$$\begin{aligned}\frac{d}{ds}x &= +\frac{\partial\hat{\mathcal{H}}_I}{\partial p_x}\\ &= 0\,;\end{aligned}\tag{3.3a}$$

$$\begin{aligned}\frac{d}{ds}p_x &= -\frac{\partial\hat{\mathcal{H}}_I}{\partial x}\\ &= -[K_x(s_0)]^2\cdot\Delta s\cdot\delta(s-s_0)\cdot x+K_x(s_0)\cdot\Delta s\cdot\delta(s-s_0)\cdot f(p_\sigma)\,;\end{aligned}\tag{3.3b}$$

$$\begin{aligned}\frac{d}{ds}z &= +\frac{\partial\hat{\mathcal{H}}_I}{\partial p_z}\\ &= 0\,;\end{aligned}\tag{3.3c}$$

$$\frac{d}{ds}\, p_z = -\frac{\partial \hat{\mathcal{H}}_I}{\partial z}$$

$$= -[K_z(s_0)]^2 \cdot \Delta s \cdot \delta(s - s_0) \cdot z + K_z(s_0) \cdot \Delta s \cdot \delta(s - s_0) \cdot f(p_\sigma) \; ; \qquad (3.3d)$$

$$\frac{d}{ds}\, \sigma = +\frac{\partial \hat{\mathcal{H}}_I}{\partial p_\sigma}$$

$$= -[K_x(s_0) \cdot x + K_z(s_0) \cdot z] \cdot \Delta s \cdot \delta(s - s_0) \cdot f'(p_\sigma) \qquad (3.3e)$$

$$\frac{d}{ds}\, p_\sigma = -\frac{\partial \hat{\mathcal{H}}_I}{\partial \sigma}$$

$$= 0 \; . \qquad (3.3f)$$

### 3.1.2 Solution of the Equations of Motion

Equations (3.3a-f) can be solved by integrating both sides from

$$s_0 \quad \text{to} \quad s_0 + \epsilon$$

with

$$0 < \epsilon \longrightarrow 0$$

leading to [1] :

$$x(s_0 + \epsilon) = x(s_0) \; ; \qquad (3.4a)$$

$$p_x(s_0 + \epsilon) = p_x(s_0) - [K_x(s_0)]^2 \cdot \Delta s \cdot x(s_0) + K_x(s_0) \cdot \Delta s \cdot f[(p_\sigma(s_0)] \; ; \qquad (3.4b)$$

$$z(s_0 + \epsilon) = z(s_0) \; ; \qquad (3.4c)$$

$$p_z(s_0 + \epsilon) = p_z(s_0) - [K_z(s_0)]^2 \cdot \Delta s \cdot z(s_0) + K_z(s_0) \cdot \Delta s \cdot f[(p_\sigma(s_0)] \; ; \qquad (3.4d)$$

$$\sigma(s_0 + \epsilon) = \sigma(s_0) - [K_x(s_0) \cdot x(s_0) + K_z(s_0) \cdot z(s_0)] \cdot \Delta s \cdot f[(p_\sigma(s_0)] \; ; \qquad (3.4e)$$

$$p_\sigma(s_0 + \epsilon) = p_\sigma(s_0) \; . \qquad (3.4f)$$

These relations which are symplectic were already derived in Refs. [1, 2].

[1] Note that the factors in (3.1b, d, e) which multiply the $\delta$–function are continuous functions of $s$ at $s_0$.

## 3.2 The Term $\mathcal{H}_{II}$

For a horizontal bending magnet [2] :

$$K_x \neq 0 \; ; \quad K_z = 0$$

we get from (2.2b) and (2.3a, b) the Hamiltonian:

$$\begin{aligned}
\mathcal{H}_{II} &= -[1+f(p_\sigma)]\cdot K_x \cdot x \cdot \left( \left\{ 1 - \frac{p_x^2+p_z^2}{[1+f(p_\sigma)]^2} \right\}^{1/2} - 1 \right) \\
&\quad -[1+f(p_\sigma)]\cdot \left\{ 1 - \frac{p_x^2+p_z^2}{[1+f(p_\sigma]^2} \right\}^{1/2} + p_\sigma \\
&= -[1+f(p_\sigma)]\cdot [\, K_x \cdot x + 1\,] \cdot \left\{ 1 - \frac{p_x^2+p_z^2}{[1+f(p_\sigma)]^2} \right\}^{1/2} \\
&\quad +[1+f(p_\sigma)]\cdot K_x \cdot x + p_\sigma \; . \qquad (3.5)
\end{aligned}$$

The canonical equations corresponding to the Hamiltonian (3.5) take the form:

$$\frac{d}{ds}\,\vec{y} = -\underline{S}\cdot \frac{\partial \mathcal{H}_{II}}{\partial \vec{y}} \qquad (3.6)$$

with

$$\begin{aligned}
\vec{y}^{\,T} &= (y_1, y_2, y_3, y_4, y_5, y_6) \\
&\equiv (x, p_x, z, p_z, \sigma, p_\sigma)
\end{aligned}$$

and

$$\underline{S} = \begin{pmatrix} \underline{S}_2 & \underline{0} & \underline{0} \\ \underline{0} & \underline{S}_2 & \underline{0} \\ \underline{0} & \underline{0} & \underline{S}_2 \end{pmatrix} \; ; \quad \underline{S}_2 = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} , \qquad (3.7)$$

or, written in components:

$$\begin{aligned}
\frac{d}{ds}\,x &= +\frac{\partial \mathcal{H}_{II}}{\partial p_x} \\
&= -[1+f(p_\sigma)]\cdot [\, K_x \cdot x + 1\,] \cdot \frac{1}{2} \left\{ 1 - \frac{p_x^2+p_z^2}{[1+f(p_\sigma]^2} \right\}^{-1/2} \cdot \frac{(-2\,p_x)}{[1+f(p_\sigma)]^2} \\
&= +[\, K_x \cdot x + 1\,] \cdot \left\{ 1 - \frac{p_x^2+p_z^2}{[1+f(p_\sigma)]^2} \right\}^{-1/2} \cdot \frac{p_x}{[1+f(p_\sigma)]} \; ; \qquad (3.8a) \\
\frac{d}{ds}\,p_x &= -\frac{\partial \mathcal{H}_{II}}{\partial x} \\
&= +[1+f(p_\sigma)]\cdot K_x \cdot \left( \left\{ 1 - \frac{p_x^2+p_z^2}{[1+f(p_\sigma)]^2} \right\}^{1/2} - 1 \right) \; ; \qquad (3.8b)
\end{aligned}$$

[2] A magnet bending in the $z$-direction can be treated in a similar way.

$$\frac{d}{ds}z = +\frac{\partial \mathcal{H}_{II}}{\partial p_z}$$

$$= +[K_x \cdot x + 1] \cdot \left\{1 - \frac{p_x^2 + p_z^2}{[1+f(p_\sigma)]^2}\right\}^{-1/2} \cdot \frac{p_z}{[1+f(p_\sigma)]}\ ; \tag{3.8c}$$

$$\frac{d}{ds}p_z = -\frac{\partial \mathcal{H}_{II}}{\partial z}$$

$$= 0\ ; \tag{3.8d}$$

$$\frac{d}{ds}\sigma = +\frac{\partial \mathcal{H}_{II}}{\partial p_\sigma}$$

$$= -f'(p_\sigma) \cdot [K_x \cdot x + 1] \cdot \left\{1 - \frac{p_x^2 + p_z^2}{[1+f(p_\sigma)]^2}\right\}^{1/2} + 1 + f'(p_\sigma) \cdot K_x \cdot x$$

$$-[1+f(p_\sigma)] \cdot [K_x \cdot x + 1] \cdot \frac{1}{2}\left\{1 - \frac{p_x^2 + p_z^2}{[1+f(p_\sigma)]^2}\right\}^{-1/2} \cdot \frac{2[p_x^2 + p_z^2]}{[1+f(p_\sigma)]^3} \cdot f'(p_\sigma)$$

$$= 1 + f'(p_\sigma) \cdot K_x \cdot x$$

$$-f'(p_\sigma) \cdot [K_x \cdot x + 1] \cdot \left\{1 - \frac{p_x^2 + p_z^2}{[1+f(p_\sigma)]^2}\right\}^{-1/2} \cdot \left\{1 - \frac{p_x^2 + p_z^2}{[1+f(p_\sigma)]^2}\right\}$$

$$-f'(p_\sigma) \cdot [K_x \cdot x + 1] \cdot \left\{1 - \frac{p_x^2 + p_z^2}{[1+f(p_\sigma)]^2}\right\}^{-1/2} \cdot \frac{[p_x^2 + p_z^2]}{[1+f(p_\sigma)]^2}$$

$$= -f'(p_\sigma) \cdot [K_x \cdot x + 1] \cdot \left\{1 - \frac{p_x^2 + p_z^2}{[1+f(p_\sigma)]^2}\right\}^{1/2} + 1 + f'(p_\sigma) \cdot K_x \cdot x$$

$$= -f'(p_\sigma) \cdot [K_x \cdot x + 1] \cdot \left(\left\{1 - \frac{p_x^2 + p_z^2}{[1+f(p_\sigma)]^2}\right\}^{1/2} - 1\right) + [1 - f'(p_\sigma)]\ ; \tag{3.8e}$$

$$\frac{d}{ds}p_\sigma = -\frac{\partial \mathcal{H}_{II}}{\partial \sigma}$$

$$= 0\,. \tag{3.8f}$$

Expanding the solution of the equations of motion (3.8) up to first order in $\Delta s$ we can write:

$$\bar{x} = x + \Delta s \cdot \frac{\partial \mathcal{H}_{II}}{\partial p_x}$$

$$= x + \Delta s \cdot [K_x \cdot x + 1] \cdot \left\{1 - \frac{p_x^2 + p_z^2}{[1+f(p_\sigma)]^2}\right\}^{-1/2} \cdot \frac{p_x}{[1+f(p_\sigma)]}\ ; \tag{3.9a}$$

$$\begin{aligned}
\bar{p}_x &= p_x - \Delta s \cdot \frac{\partial \mathcal{H}_{II}}{\partial x} \\
&= p_x + \Delta s \cdot [1 + f(p_\sigma)] \cdot K_x \cdot \left( \left\{ 1 - \frac{p_x^2 + p_z^2}{[1 + f(p_\sigma)]^2} \right\}^{1/2} - 1 \right) \; ; \qquad (3.9b) \\
\bar{z} &= z + \Delta s \cdot \frac{\partial \mathcal{H}_{II}}{\partial p_z} \\
&= z + \Delta s \cdot [\, K_x \cdot x + 1 \,] \cdot \left\{ 1 - \frac{p_x^2 + p_z^2}{[1 + f(p_\sigma)]^2} \right\}^{-1/2} \cdot \frac{p_z}{[1 + f(p_\sigma)]} \; ; \qquad (3.9c) \\
\bar{p}_z &= p_z - \Delta s \cdot \frac{\partial \mathcal{H}_{II}}{\partial z} \\
&= p_z \; ; \qquad (3.9d) \\
\sigma &= \sigma + \Delta s \cdot \frac{\partial \mathcal{H}_{II}}{\partial p_\sigma} \\
&= \sigma + \Delta s \cdot [\, 1 - f'(p_\sigma) \,] \\
&\quad - \Delta s \cdot f'(p_\sigma) \cdot [\, K_x \cdot x + 1 \,] \left( \left\{ 1 - \frac{p_x^2 + p_z^2}{[1 + f(p_\sigma)]^2} \right\}^{-1/2} - 1 \right) \; ; \qquad (3.9e) \\
\bar{p}_\sigma &= p_\sigma - \Delta s \cdot \frac{\partial \mathcal{H}_{II}}{\partial \sigma} \\
&= p_\sigma \qquad (3.9f)
\end{aligned}$$

with

$$\begin{aligned}
y &\equiv y(s_0 + \epsilon) \; ; \\
\bar{y} &\equiv y(s_0 + \Delta s) \; ;
\end{aligned}$$

$$(y \equiv (\, x,\, p_x,\, z,\, p_z,\, \sigma,\, p_\sigma \,)$$

and

$$0 < \epsilon \longrightarrow 0 \, .$$

The map

$$(x,\, p_x,\, z,\, p_z,\, \sigma,\, p_\sigma) \longrightarrow (\bar{x},\, \bar{p}_x,\, \bar{z},\, \bar{p}_z,\, \bar{\sigma},\, \bar{p}_\sigma) \qquad (3.10)$$

defined by (3.9) is not symplectic. In order to symplectify eqn. (3.9), higher order terms in $\Delta s$ have to be added on the r.h.s of (3.9). This can be achieved by defining the generating function:

$$F(x,\, \bar{p}_x;\, z,\, \bar{p}_z;\, \sigma,\, \bar{p}_\sigma) = x \cdot \bar{p}_x + z \cdot \bar{p}_z + \sigma \cdot \bar{p}_\sigma + \Delta s \cdot \mathcal{H}_{II}(x,\, \bar{p}_x;\, z,\, \bar{p}_z;\, \sigma,\, \bar{p}_\sigma)$$

so that (3.10) becomes a canonical transformation [3] .

Then:

$$
\begin{aligned}
F &= x\cdot\bar{p}_x + z\cdot\bar{p}_z + \sigma\cdot\bar{p}_\sigma \\
&\quad - \Delta s\cdot[1+f(\bar{p}_\sigma)]\cdot K_x\cdot x\cdot\left(\left\{1-\frac{\bar{p}_x^2+\bar{p}_z^2}{[1+f(\bar{p}_\sigma)]^2}\right\}^{1/2}-1\right) \\
&\quad - \Delta s\cdot[1+f(\bar{p}_\sigma)]\cdot\left\{1-\frac{\bar{p}_x^2+\bar{p}_z^2}{[1+f(\bar{p}_\sigma]^2}\right\}^{1/2} + \Delta s\cdot p_\sigma \\
&= x\cdot\bar{p}_x + z\cdot\bar{p}_z + \sigma\cdot\bar{p}_\sigma + \Delta s\cdot[1+f(\bar{p}_\sigma)]\cdot K_x\cdot x + \Delta s\cdot\bar{p}_\sigma \\
&\quad - \Delta s\cdot[1+f(\bar{p}_\sigma)]\cdot[\,K_x\cdot x+1\,]\cdot\left\{1-\frac{\bar{p}_x^2+\bar{p}_z^2}{[1+f(\bar{p}_\sigma]^2}\right\}^{1/2}, \qquad (3.11)
\end{aligned}
$$

This method has been applied by E. Forest and K. Ohmi to obtain symplectic transfer maps for wigglers [4].

With the generating function (3.11) the transfer map (3.10) reads as:

$$
\begin{aligned}
\bar{x} &= \frac{\partial F}{\partial\bar{p}_x} \\
&= x - \Delta s\cdot[1+f(\bar{p}_\sigma)]\cdot[\,K_x\cdot x+1\,]\cdot\frac{1}{2}\left\{1-\frac{\bar{p}_x^2+\bar{p}_z^2}{[1+f(\bar{p}_\sigma]^2}\right\}^{-1/2}\cdot\frac{(-2\,\bar{p}_x)}{[1+f(\bar{p}_\sigma)]^2} \\
&= x + \Delta s\cdot[\,K_x\cdot x+1\,]\cdot\left\{1-\frac{\bar{p}_x^2+\bar{p}_z^2}{[1+f(\bar{p}_\sigma)]^2}\right\}^{-1/2}\cdot\frac{\bar{p}_x}{[1+f(\bar{p}_\sigma)]}\;; \qquad (3.12a) \\
p_x &= \frac{\partial F}{\partial x} \\
&= \bar{p}_x - \Delta s\cdot[1+f(\bar{p}_\sigma)]\cdot K_x\cdot\left(\left\{1-\frac{\bar{p}_x^2+\bar{p}_z^2}{[1+f(\bar{p}_\sigma)]^2}\right\}^{1/2}-1\right)\;; \qquad (3.12b)
\end{aligned}
$$

[3] The transformation equations due to the generating function $F$ are:

$$
\begin{aligned}
\bar{y} &= \frac{\partial F}{\partial\bar{p}_y} = y + \Delta s\cdot\frac{\partial\mathcal{H}_{II}}{\partial\bar{p}_y}\;; \\
p_y &= \frac{\partial F}{\partial y} = \bar{p}_y + \Delta s\cdot\frac{\partial\mathcal{H}_{II}}{\partial y}\;;
\end{aligned}
$$

$$
y \equiv x, z, \sigma\;.
$$

This transformation is canonical and approximates the real symplectic motion due to eqn. (3.8) up to first order in $\Delta s$ [4].

$$
\begin{aligned}
\bar{z} &= \frac{\partial F}{\partial \bar{p}_z} \\
&= z + \Delta s \cdot [\, K_x \cdot x + 1\,] \cdot \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(\bar{p}_\sigma)]^2} \right\}^{-1/2} \cdot \frac{\bar{p}_z}{[1 + f(\bar{p}_\sigma)]} \; ; \qquad (3.12c) \\
p_z &= \frac{\partial F}{\partial z} \\
&= \bar{p}_z \; ; \qquad (3.12d) \\
\bar{\sigma} &= \frac{\partial F}{\partial \bar{p}_\sigma} \\
&= \sigma + \Delta s \cdot f'(\bar{p}_\sigma) \cdot K_x \cdot x + \Delta s \\
&\quad - \Delta s \cdot f'(\bar{p}_\sigma) \cdot [\, K_x \cdot x + 1\,] \cdot \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(p_{\bar{\sigma}})]^2} \right\}^{1/2} \\
&\quad - \Delta s \cdot [1 + f(\bar{p}_\sigma)] \cdot [\, K_x \cdot x + 1\,] \cdot \frac{1}{2} \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(\bar{p}_\sigma)]^2} \right\}^{-1/2} \cdot \frac{2\,[\bar{p}_x^2 + \bar{p}_z^2\,]}{[1 + f(\bar{p}_\sigma)]^3} \cdot f'(\bar{p}_\sigma) \\
&= \sigma + \Delta s + \Delta s \cdot f'(\bar{p}_\sigma) \cdot K_x \cdot x \\
&\quad - \Delta s \cdot f'(\bar{p}_\sigma) \cdot [\, K_x \cdot x + 1\,] \cdot \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(\bar{p}_\sigma)]^2} \right\}^{-1/2} \cdot \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(\bar{p}_\sigma)]^2} \right\} \\
&\quad - \Delta s \cdot f'(\bar{p}_\sigma) \cdot [\, K_x \cdot x + 1\,] \cdot \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(\bar{p}_\sigma)]^2} \right\}^{-1/2} \cdot \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(\bar{p}_\sigma)]^2} \\
&= \sigma + \Delta s \cdot [\, 1 - f'(\bar{p}_\sigma)\,] \\
&\quad - \Delta s \cdot f'(\bar{p}_\sigma) \cdot [\, K_x \cdot x + 1\,] \cdot \left( \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(\bar{p}_\sigma)]^2} \right\}^{-1/2} - 1 \right) \; ; \qquad (3.12e) \\
p_\sigma &= \frac{\partial F}{\partial \sigma} \\
&= \bar{p}_\sigma \; . \qquad (3.12f)
\end{aligned}
$$

Using the relations

$$
\begin{aligned}
\bar{p}_z &= p_z \; ; \qquad (3.13a) \\
\bar{p}_\sigma &= p_\sigma \qquad (3.13b)
\end{aligned}
$$

resulting from (3.12 d, f), eqn. (3.12b) takes the form:

$$p_x = \bar{p}_x - \Delta s \cdot [1 + f(p_\sigma)] \cdot K_x \cdot \left( \left\{ 1 - \frac{\bar{p}_x^2 + p_z^2}{[1 + f(p_\sigma)]^2} \right\}^{1/2} - 1 \right)$$

or

$$\{\bar{p}_x - p_x + \Delta s \cdot [1 + f(p_\sigma)] \cdot K_x\}^2$$

$$= \{\Delta s \cdot [1 + f(p_\sigma)] \cdot K_x\}^2 \cdot \left\{ 1 - \frac{p_z^2}{[1 + f(p_\sigma)]^2} - \frac{\bar{p}_x^2}{[1 + f(p_\sigma)]^2} \right\}$$

representing a quadratic equation in $\bar{p}_x$ the solution of which reads as:

$$\bar{p}_x = \frac{1}{1 + [K_x \cdot \Delta s]^2} \times$$

$$\left\{ p_x - \Delta s \cdot [1 + f(p_\sigma)] \cdot K_x \pm \Delta s \cdot [1 + f(p_\sigma)] \cdot K_x \cdot \sqrt{\left[ 1 - \frac{p_x^2 + p_z^2}{[1 + f(p_\sigma)]^2} \right] + \frac{C}{[1 + f(p_\sigma)]^2}} \right\} \tag{3.14}$$

with

$$C = -[K_x \cdot \Delta s]^2 \cdot p_z^2 + 2\,[K_x \cdot \Delta s] \cdot [1 + f(p_\sigma)] \cdot p_x \,. \tag{3.15}$$

Comparing eqn. (3.14) with (3.9b), it can be seen that we have to take the positive sign of the square root in (3.14), so that we may write:

$$\bar{p}_x = \frac{1}{1 + [K_x \cdot \Delta s]^2} \times$$

$$\left\{ p_x + \Delta s \cdot [1 + f(p_\sigma)] \cdot K_x \cdot \left( \sqrt{\left[ 1 - \frac{p_x^2 + p_z^2}{[1 + f(p_\sigma)]^2} \right] + \frac{C}{[1 + f(p_\sigma)]^2}} - 1 \right) \right\} . \tag{3.16}$$

The quantities $\bar{x}$, $\bar{z}$, and $\bar{\sigma}$ are then to be obtained from eqns. (3.12 a, c, e) by taking into account eqns. (3.13) and (3.16).

## 3.3 The Whole Region $s_0 \leq s \leq s_0 + \Delta s$

For the transfer map $T$ of the whole region $s_0 \leq s \leq s_0 + \Delta s$ we now have:

$$T = T_{II} \circ T_I \,, \tag{3.17}$$

where $T_I$ corresponds to region I and $T_{II}$ to region II.

If we denote (as in Refs. [2, 1]) the initial vector by $\vec{y}^{\,i}$ and the final vector by $\vec{y}^{\,f}$, the map $T_I$ is described by:

$$x^f = x^i \; ; \tag{3.18a}$$

$$p_x^f = p_x^i - [K_x(s_0)]^2 \cdot \Delta s \cdot x^i + K_x(s_0) \cdot \Delta s \cdot f(p_\sigma^i) \; ; \tag{3.18b}$$

$$z^f = z^i \; ; \tag{3.18c}$$

$$p_z^f = p_z^i - [K_z(s_0)]^2 \cdot \Delta s \cdot z^i + K_z(s_0) \cdot \Delta s \cdot f(p_\sigma^i) \; ; \tag{3.18d}$$

$$\sigma^f = \sigma^i - [\, K_x(s_0) \cdot x^i + K_z(s_0) \cdot z^i \,] \cdot \Delta s \cdot f'(p_\sigma^i) \; ; \tag{3.18e}$$

$$p_\sigma^f = p_\sigma^i \tag{3.18f}$$

(see eqn. (3.4)) and the map $T_{II}$ by:

$$p_x^f = \frac{1}{1 + [\, K_x \cdot \Delta s \,]^2} \times$$

$$\left\{ p_x^i + [\, K_x \cdot \Delta s \,] \cdot \left[ 1 + f(p_\sigma^i) \right] \cdot \left( \sqrt{\left[ 1 - \frac{(p_x^i)^2 + (p_z^i)^2}{[1 + f(p_\sigma^i)]^2} \right] + \frac{C}{[1 + f(p_\sigma^i)]^2}} - 1 \right) \right\} \; ; \tag{3.19a}$$

$$p_z^f = p_z^i \; ; \tag{3.19b}$$

$$p_\sigma^f = p_\sigma^i \; ; \tag{3.19c}$$

$$x^f = x^i + \Delta s \cdot \left[ K_x \cdot x^i + 1 \right] \cdot \left\{ 1 - \frac{(p_x^f)^2 + (p_z^f)^2}{[1 + f(p_\sigma^f)]^2} \right\}^{-1/2} \cdot \frac{p_x^f}{[1 + f(p_\sigma^f)]} \; ; \tag{3.19d}$$

$$z^f = z^i + \Delta s \cdot \left[ K_x \cdot x^i + 1 \right] \cdot \left\{ 1 - \frac{(p_x^f)^2 + (p_z^f)^2}{[1 + f(p_\sigma^f)]^2} \right\}^{-1/2} \cdot \frac{p_z^f}{[1 + f(p_\sigma^f)]} \; ; \tag{3.19e}$$

$$\sigma^f = \sigma^i + \Delta s \cdot \left[ 1 - f'(p_\sigma^f) \right]$$

$$- \Delta s \cdot f'(p_\sigma^f) \cdot \left[ K_x \cdot x^i + 1 \right] \cdot \left( \left\{ 1 - \frac{(p_x^f)^2 + (p_z^f)^2}{[1 + f(p_\sigma^f)]^2} \right\}^{-1/2} - 1 \right) \tag{3.19f}$$

with

$$C = - [\, K_x \cdot \Delta s \,]^2 \cdot (p_z^i)^2 + 2 \, [\, K_x \cdot \Delta s \,] \cdot \left[ 1 + f(p_\sigma^i) \right] \cdot p_x^i \tag{3.20}$$

(see (3.13), (3.15), and (3.16) combined with (3.12 a, c, e)).

Remarks:

1) One could also use the more symmetric decomposition

$$T \;=\; T_{II}(\Delta s/2) \circ T_I \circ T_{II}(\Delta s/2)\;. \tag{3.21}$$

2) The transfer map $T$ is symplectic for an arbitrary $\Delta s$, since $T_1$ and $T_2$ are symplectic by construction. In the limit

$$\Delta s \;\longrightarrow\; 0$$

one obtains the exact solution of the canonical equations of motion corresponding to the starting Hamiltonian (2.1).

3) The calculation of the transport map for a bending magnet can easily be extended to synchrotron magnets. In this case, the terms

$$\frac{1}{2}\,[K_x(s_0)]^2 \cdot x^2 \;\text{ and }\; \frac{1}{2}\,[K_z(s_0)]^2 \cdot z^2$$

appearing in the Hamiltonian $\mathcal{H}_I$ and $\hat{\mathcal{H}}_I$ (see eqns. (2.2a) and (3.1)), have to be replaced by

$$\frac{1}{2}\,[\,K_x^2 + g\,] \cdot x^2 \;\text{ and }\; \frac{1}{2}\,[\,K_z^2 - g\,] \cdot z^2$$

($g$ is defined in Refs. [1, 2]), while $\mathcal{H}_{II}$ (see eqn. (2.2b)) and $T_{II}$ in eqn. (3.19) remain unchanged.

The canonical equations of motion corresponding to the modified Hamiltonian $\hat{\mathcal{H}}_I$ may be integrated in the same way as in section 3.1.1 (see also Refs. [1, 2]).

# 4 A Symplectic Treatment of Bending Magnets without Splitting the Hamiltonian

In this section we treat the "unsplit" Hamiltonian $\mathcal{H}_{bend}$, using a generating function in analogy to eqn. (3.11) for the whole region $s_0 \le s \le s_0 + \Delta s$.

## 4.1 The Hamiltonian

From eqn. (2.1) we obtain for a horizontal ($K_z = 0$) bending magnet:

$$\begin{aligned}\mathcal{H}_{bend} \;=\;& p_\sigma - [1 + f(p_\sigma)] \cdot \left\{1 - \frac{p_x^2 + p_z^2}{[1 + f(p_\sigma]^2}\right\}^{1/2} \\ & - [1 + f(p_\sigma)] \cdot K_x \cdot x \cdot \left(\left\{1 - \frac{p_x^2 + p_z^2}{[1 + f(p_\sigma)]^2}\right\}^{1/2} - 1\right) \\ & - f(p_\sigma) \cdot K_x \cdot x + \frac{1}{2}\,K_x^2 \cdot x^2\;.\end{aligned} \tag{4.22}$$

## 4.2 The Generating Function

In analogy to eqn. (3.11) we define:

$$
\begin{aligned}
& F(x,\,\bar{p}_x;\,z,\,\bar{p}_z;\,\sigma,\,\bar{p}_\sigma) \\
= \;& x\cdot\bar{p}_x + z\cdot\bar{p}_z + \sigma\cdot\bar{p}_\sigma + \Delta s\cdot\mathcal{H}_{bend}(x,\,\bar{p}_x;\,z,\,\bar{p}_z;\,\sigma,\,\bar{p}_\sigma) \\
= \;& x\cdot\bar{p}_x + z\cdot\bar{p}_z + \sigma\cdot\bar{p}_\sigma \\
& - \Delta s\cdot[1+f(\bar{p}_\sigma)]\cdot K_x\cdot x\cdot\left(\left\{1-\frac{\bar{p}_x^2+\bar{p}_z^2}{[1+f(\bar{p}_\sigma)]^2}\right\}^{1/2}-1\right) \\
& - \Delta s\cdot[1+f(\bar{p}_\sigma)]\cdot\left\{1-\frac{\bar{p}_x^2+\bar{p}_z^2}{[1+f(\bar{p}_\sigma]^2}\right\}^{1/2} + \Delta s\cdot p_\sigma \\
& + \Delta s\cdot\left\{-f(\bar{p}_\sigma)\cdot K_x\cdot x+\frac{1}{2}K_x^2\cdot x^2\right\} \\
= \;& x\cdot\bar{p}_x + z\cdot\bar{p}_z + \sigma\cdot\bar{p}_\sigma + \Delta s\cdot[1+f(\bar{p}_\sigma)]\cdot K_x\cdot x \\
& - \Delta s\cdot[1+f(\bar{p}_\sigma)]\cdot[\,K_x\cdot x+1\,]\cdot\left\{1-\frac{\bar{p}_x^2+\bar{p}_z^2}{[1+f(\bar{p}_\sigma]^2}\right\}^{1/2} + \Delta s\cdot\bar{p}_\sigma \\
& + \Delta s\cdot\left\{-f(\bar{p}_\sigma)\cdot K_x\cdot x+\frac{1}{2}K_x^2\cdot x^2\right\}\;. \qquad (4.23)
\end{aligned}
$$

## 4.3 The Transfer Map

With the generating function (4.21) the transfer map

$$(x,\,p_x,\,z,\,p_z,\,\sigma,\,p_\sigma) \;\longrightarrow\; (\bar{x},\,\bar{p}_x,\,\bar{z},\,\bar{p}_z,\,\bar{\sigma},\,\bar{p}_\sigma)$$

reads as:

$$
\begin{aligned}
\bar{x} &= \frac{\partial F}{\partial\bar{p}_x} \\
&= x - \Delta s\cdot[1+f(\bar{p}_\sigma)]\cdot[\,K_x\cdot x+1\,]\cdot\frac{1}{2}\left\{1-\frac{\bar{p}_x^2+\bar{p}_z^2}{[1+f(\bar{p}_\sigma]^2}\right\}^{-1/2}\cdot\frac{(-2\,\bar{p}_x)}{[1+f(\bar{p}_\sigma)]^2} \\
&= x + \Delta s\cdot[\,K_x\cdot x+1\,]\cdot\left\{1-\frac{\bar{p}_x^2+\bar{p}_z^2}{[1+f(\bar{p}_\sigma)]^2}\right\}^{-1/2}\cdot\frac{\bar{p}_x}{[1+f(\bar{p}_\sigma)]}\;; \qquad (4.24a) \\
p_x &= \frac{\partial F}{\partial x}
\end{aligned}
$$

$$= \bar{p}_x - \Delta s \cdot [1 + f(\bar{p}_\sigma)] \cdot K_x \cdot \left( \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(\bar{p}_\sigma)]^2} \right\}^{1/2} - 1 \right)$$

$$+ \Delta s \cdot \left\{ -f(\bar{p}_\sigma) \cdot K_x + K_x^2 \cdot x \right\} \; ; \tag{4.24b}$$

$$\bar{z} = \frac{\partial F}{\partial \bar{p}_z}$$

$$= z + \Delta s \cdot [\, K_x \cdot x + 1\,] \cdot \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(\bar{p}_\sigma)]^2} \right\}^{-1/2} \cdot \frac{\bar{p}_z}{[1 + f(\bar{p}_\sigma)]} \; ; \tag{4.24c}$$

$$p_z = \frac{\partial F}{\partial z}$$

$$= \bar{p}_z \; ; \tag{4.24d}$$

$$\bar{\sigma} = \frac{\partial F}{\partial \bar{p}_\sigma}$$

$$= \sigma + \Delta s$$

$$- \Delta s \cdot f'(\bar{p}_\sigma) \cdot [\, K_x \cdot x + 1\,] \cdot \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(p_{\bar{\sigma}})]^2} \right\}^{1/2}$$

$$-\Delta s \cdot [1 + f(\bar{p}_\sigma)] \cdot [\, K_x \cdot x + 1\,] \cdot \frac{1}{2} \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(\bar{p}_\sigma)]^2} \right\}^{-1/2} \cdot \frac{2\,[\,\bar{p}_x^2 + \bar{p}_z^2\,]}{[1 + f(\bar{p}_\sigma)]^3} \cdot f'(\bar{p}_\sigma)$$

$$= \sigma + \Delta s$$

$$-\Delta s \cdot f'(\bar{p}_\sigma) \cdot [\, K_x \cdot x + 1\,] \cdot \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(\bar{p}_\sigma)]^2} \right\}^{-1/2} \cdot \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(\bar{p}_\sigma)]^2} \right\}$$

$$-\Delta s \cdot f'(\bar{p}_\sigma) \cdot [\, K_x \cdot x + 1\,] \cdot \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(\bar{p}_\sigma)]^2} \right\}^{-1/2} \cdot \frac{[p_x^2 + p_z^2]}{[1 + f(p_\sigma)]^2}$$

$$= \sigma + \Delta s - \Delta s \cdot f'(\bar{p}_\sigma) \cdot [\, K_x \cdot x + 1\,] \cdot \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(\bar{p}_\sigma)]^2} \right\}^{-1/2} \; ; \tag{4.24e}$$

$$p_\sigma = \frac{\partial F}{\partial \sigma}$$

$$= \bar{p}_\sigma \; . \tag{4.24f}$$

Using the relations

$$\bar{p}_z = p_z \; ; \tag{4.25a}$$

$$\bar{p}_\sigma = p_\sigma \tag{4.25b}$$

resulting from (4.22 d, f), eqn. (4.22 b) takes the form:

$$p_x = \bar{p}_x - \Delta s \cdot [1 + f(p_\sigma)] \cdot K_x \cdot \left( \left\{ 1 - \frac{\bar{p}_x^2 + p_z^2}{[1 + f(p_\sigma)]^2} \right\}^{1/2} - 1 \right) + \Delta s \cdot \left\{ -f(p_\sigma) \cdot K_x + K_x^2 \cdot x \right\}$$

or

$$\{ \bar{p}_x - \tilde{p}_x + \Delta s \cdot [1 + f(p_\sigma)] \cdot K_x \}^2 = \{ \Delta s \cdot [1 + f(p_\sigma)] \cdot K_x \}^2 \cdot \left\{ 1 - \frac{p_z^2}{[1 + f(p_\sigma)]^2} - \frac{\bar{p}_x^2}{[1 + f(p_\sigma)]^2} \right\}$$

with

$$\tilde{p}_x = p_x - \Delta s \cdot \left\{ -f(p_\sigma) \cdot K_x + K_x^2 \cdot x \right\} . \tag{4.26}$$

This represents a quadratic equation in $\bar{p}_x$ the solution of which reads as:

$$\bar{p}_x = \frac{1}{1 + [K_x \cdot \Delta s]^2} \times \left\{ \tilde{p}_x - \Delta s \cdot [1 + f(p_\sigma)] \cdot K_x \pm \Delta s \cdot [1 + f(p_\sigma)] \cdot K_x \cdot \sqrt{\left[ 1 - \frac{\tilde{p}_x^2 + p_z^2}{[1 + f(p_\sigma)]^2} \right] + \frac{\tilde{C}}{[1 + f(p_\sigma)]^2}} \right\} \tag{4.27}$$

with

$$\tilde{C} = -[K_x \cdot \Delta s]^2 \cdot p_z^2 + 2 \cdot [K_x \cdot \Delta s] \cdot [1 + f(p_\sigma)] \cdot \tilde{p}_x . \tag{4.28}$$

As in section 3, it can be seen by comparing eqn. (4.25) with the linear approximation of the equations of motion, that we have to take the positive sign of the square root in (4.25), so that we may write:

$$\bar{p}_x = \frac{1}{1 + [K_x \cdot \Delta s]^2} \times \left\{ \tilde{p}_x + \Delta s \cdot [1 + f(p_\sigma)] \cdot K_x \cdot \left( \sqrt{\left[ 1 - \frac{\tilde{p}_x^2 + p_z^2}{[1 + f(p_\sigma)]^2} \right] + \frac{\tilde{C}}{[1 + f(p_\sigma)]^2}} - 1 \right) \right\} \tag{4.29}$$

or, after some analysis:

$$\bar{p}_x = \frac{1}{1 + [K_x \cdot \Delta s]^2} \times \left\{ p_x - [K_x \cdot \Delta s] \cdot [1 + K_x \cdot x] + [K_x \cdot \Delta s] \cdot [1 + f(p_\sigma)] \cdot \sqrt{1 - \frac{[p_x^2 + p_z^2] + \hat{C}}{[1 + f(p_\sigma)]^2}} \right\} \tag{4.30}$$

with

$$\hat{C} = -[K_x \cdot \Delta s]^2 \cdot p_z^2 + 2 \cdot [K_x \cdot \Delta s] \cdot [1 + K_x \cdot x] \cdot p_x$$
$$- [K_x \cdot \Delta s]^2 \cdot [-f(p_\sigma) + K_x \cdot x] \cdot \{2 + f(p_\sigma) + K_x \cdot x\} \ . \qquad (4.31)$$

The quantities $\bar{x}$, $\bar{z}$, and $\bar{\sigma}$ are then to be obtained from eqns. (4.22 a, c, e) by taking into account eqns. (4.23) and (4.27).

Denoting again the initial vector by $\vec{y}^{\,i}$ and the final vector by $\vec{y}^{\,f}$, we thus have for the whole region $s_0 \leq s \leq s_0 + \Delta s$:

$$p_x^f = \frac{1}{1 + [K_x \cdot \Delta s]^2} \times$$
$$\left\{ p_x^i - [K_x \cdot \Delta s] \cdot \left( \left[ K_x \cdot x^i + 1 \right] + \left[ 1 + f(p_\sigma^i) \right] \cdot \sqrt{1 - \frac{[(p_x^i)^2 + (p_z^i)^2] + \hat{C}^i}{[1 + f(p_\sigma^i)]^2}} \right) \right\} \ ; \qquad (4.32a)$$

$$p_z^f = p_z^i \ ; \qquad (4.32b)$$

$$p_\sigma^f = p_\sigma^i \ ; \qquad (4.32c)$$

$$x^f = x^i + \Delta s \cdot \left[ K_x \cdot x^i + 1 \right] \cdot \left\{ 1 - \frac{(p_x^f)^2 + (p_z^f)^2}{[1 + f(p_\sigma^f)]^2} \right\}^{-1/2} \cdot \frac{p_x^f}{[1 + f(p_\sigma^f)]} \ ; \qquad (4.32d)$$

$$z^f = z^i + \Delta s \cdot \left[ K_x \cdot x^i + 1 \right] \cdot \left\{ 1 - \frac{(p_x^f)^2 + (p_z^f)^2}{[1 + f(p_\sigma^f)]^2} \right\}^{-1/2} \cdot \frac{p_z^f}{[1 + f(p_\sigma^f)]} \ ; \qquad (4.32e)$$

$$\sigma^f = \sigma^i + \Delta s - \Delta s \cdot f'(p_\sigma^f) \cdot \left[ K_x \cdot x^i + 1 \right] \cdot \left\{ 1 - \frac{(p_x^f)^2 + (p_z^f)^2}{[1 + f(p_\sigma^f)]^2} \right\}^{-1/2} \qquad (4.32f)$$

with

$$\hat{C}^i = -[K_x \cdot \Delta s]^2 \cdot (p_z^i)^2 + 2\,[K_x \cdot \Delta s] \cdot \left[ 1 + K_x \cdot x^i \right] \cdot p_x^i$$
$$- [K_x \cdot \Delta s]^2 \cdot \left[ -f(p_\sigma^i) + K_x \cdot x^i \right] \cdot \left\{ 2 + f(p_\sigma^i) + K_x \cdot x^i \right\} \ . \qquad (4.33)$$

Note that the transfer map described by (4.30) is symplectic for an arbitrary $\Delta s$ by construction and approximates the solution of the equations of motion in linear order of $\Delta s$. In the limit

$$\Delta s \longrightarrow 0$$

one obtains the exact solution of the canonical equations of motion corresponding to the starting Hamiltonian (2.1).

Remark:

Equations (4.32 b, c, d, e) have the same form as the relations (3.18 b, c, d, e) corresponding to the Hamiltonian $\mathcal{H}_{II}$. In (4.30 a, f) additional terms appear resulting from the Hamiltonian $\mathcal{H}_I$.

For example, eqn. (3.14) or (3.17a) is obtained from (4.27) (which is equivalent with (4.30a) ), if one replaces $\tilde{p}_x$ in (4.26) and (4.27) by $p_x$.

# 5 Summary

By extending Refs. [1, 2], we have shown how to solve the nonlinear canonical equations of motion for a thin - lens bending magnet and also for a thin - lens synchrotron magnet in the framework of the fully six–dimensional formalism, taking into account the exact Hamiltonian.

We achieve this with a technique different from Refs. [1, 2] by using a generating function in a way analogous to that suggested by E. Forest and K. Ohmi in Ref. [4].

Since the equations of motion (resulting from a Hamiltonian) are canonical, the transport maps obtained are automatically symplectic.

The equations derived are valid for arbitrary particle velocity, i.e. below and above transition energy, and shall be incorporated into the computer program SIXTRACK.

Following this thin - lens treatment for the conventional magnet types, a future task could be to try the construction of the symplectic thin - lens transfer map for the solenoid using the exact Hamiltonian (without expanding the square root).

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# Appendix A: A Symplectic Treatment of Quadrupoles, Skew Quadrupoles, Sextupoles, and Octupoles Taking into Account the Exact Hamiltonian

## A.1 The Hamiltonian

For a straight section containing quadrupoles, skew quadrupoles, sextupoles, and octupoles the Hamiltonian reads as [1, 2]:

$$\begin{aligned}\mathcal{H} &= p_\sigma - [1+f(p_\sigma)]\cdot\left\{1-\frac{p_x^2+p_z^2}{[1+f(p_\sigma)]^2}\right\}^{1/2} \\ &+\frac{1}{2}g\cdot x^2-\frac{1}{2}g\cdot z^2-N\cdot x\,z \\ &+\frac{\lambda}{6}\cdot(x^3-3\,xz^2)+\frac{\mu}{24}\cdot(z^4-6\,x^2\,z^2+x^4) \qquad \text{(A.1)}\end{aligned}$$

($g$, $N$, $\lambda$, and $\mu$ are defined in Ref. [1]).

In detail, one has:

| | | | |
|---|---|---|---|
| a) | $g \neq 0;$ | $N = \lambda = \mu = 0:$ | quadrupole; |
| b) | $N \neq 0;$ | $g = \lambda = \mu = 0:$ | skew quadrupole; |
| c) | $\lambda \neq 0;$ | $g = N = \mu = 0:$ | sextupole; |
| d) | $\mu \neq 0;$ | $g = N = \lambda = 0:$ | octupole. |

## A.2 Generating Function

Analogously to eqn. (3.9) we define:

$$
\begin{aligned}
& F(x,\, \bar{p}_x;\, z,\, \bar{p}_z;\, \sigma,\, \bar{p}_\sigma) \\
= \;& x \cdot \bar{p}_x + z \cdot \bar{p}_z + \sigma \cdot \bar{p}_\sigma + \Delta s \cdot \mathcal{H}(x,\, \bar{p}_x;\, z,\, \bar{p}_z;\, \sigma,\, \bar{p}_\sigma) \\
= \;& x \cdot \bar{p}_x + z \cdot \bar{p}_z + \sigma \cdot \bar{p}_\sigma \\
& - \Delta s \cdot [1 + f(\bar{p}_\sigma)] \cdot \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(\bar{p}_\sigma]^2} \right\}^{1/2} + \Delta s \cdot \bar{p}_\sigma \\
& + \Delta s \cdot \left\{ \frac{1}{2} g \cdot x^2 - \frac{1}{2} g \cdot z^2 - N \cdot x\, z \right. \\
& \qquad \left. + \frac{\lambda}{6} \cdot (x^3 - 3\, x z^2) + \frac{\mu}{24} \cdot (z^4 - 6\, x^2\, z^2 + x^4) \right\} .
\end{aligned}
\tag{A.2}
$$

## A.3 Transfer Map

With the generating function (A.2) the transfer map

$$
(x,\, p_x,\, z,\, p_z,\, \sigma,\, p_\sigma) \;\longrightarrow\; (\bar{x},\, \bar{p}_x,\, \bar{z},\, \bar{p}_z,\, \bar{\sigma},\, \bar{p}_\sigma)
$$

reads as:

$$
\begin{aligned}
\bar{x} \;=\; & \frac{\partial F}{\partial \bar{p}_x} \\
=\; & x - \Delta s \cdot [1 + f(\bar{p}_\sigma)] \cdot \frac{1}{2} \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(\bar{p}_\sigma]^2} \right\}^{-1/2} \cdot \frac{(-2\, \bar{p}_x)}{[1 + f(\bar{p}_\sigma)]^2} \\
=\; & x + \Delta s \cdot \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(\bar{p}_\sigma)]^2} \right\}^{-1/2} \cdot \frac{\bar{p}_x}{[1 + f(\bar{p}_\sigma)]} \;;
\end{aligned}
\tag{A.3a}
$$

$$
\begin{aligned}
p_x &= \frac{\partial F}{\partial x} \\
&= \bar{p}_x + \Delta s \cdot \left\{ +g \cdot x - N \cdot z + \frac{\lambda}{2} \cdot (x^2 - z^2) + \frac{\mu}{6} \cdot (x^3 - 3\,x\,z^2) \right\} ; \qquad \text{(A.3b)} \\
\bar{z} &= \frac{\partial F}{\partial \bar{p}_z} \\
&= z + \Delta s \cdot \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(\bar{p}_\sigma)]^2} \right\}^{-1/2} \cdot \frac{\bar{p}_z}{[1 + f(\bar{p}_\sigma)]} ; \qquad \text{(A.3c)} \\
p_z &= \frac{\partial F}{\partial z} \\
&= \bar{p}_z + \Delta s \cdot \left\{ -g \cdot z - N \cdot x - \lambda \cdot x\,z + \frac{\mu}{6} \cdot (z^3 - 3\,x^2\,z) \right\} ; \qquad \text{(A.3d)} \\
\bar{\sigma} &= \frac{\partial F}{\partial \bar{p}_\sigma} \\
&= \sigma + \Delta s \\
&\quad - \Delta s \cdot f'(\bar{p}_\sigma) \cdot \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(p_{\bar{\sigma}})]^2} \right\}^{1/2} \\
&\quad - \Delta s \cdot [1 + f(\bar{p}_\sigma)] \cdot \frac{1}{2} \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(\bar{p}_\sigma)]^2} \right\}^{-1/2} \cdot \frac{2\,[\bar{p}_x^2 + \bar{p}_z^2\,]}{[1 + f(\bar{p}_\sigma)]^3} \cdot f'(\bar{p}_\sigma) \\
&= \sigma + \Delta s \\
&\quad - \Delta s \cdot f'(\bar{p}_\sigma) \cdot \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(\bar{p}_\sigma)]^2} \right\}^{-1/2} \cdot \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(\bar{p}_\sigma)]^2} \right\} \\
&\quad - \Delta s \cdot f'(\bar{p}_\sigma) \cdot \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(\bar{p}_\sigma)]^2} \right\}^{-1/2} \cdot \frac{[p_x^2 + p_z^2]}{[1 + f(p_\sigma)]^2} \\
&= \sigma + \Delta s - \Delta s \cdot f'(\bar{p}_\sigma) \cdot \left\{ 1 - \frac{\bar{p}_x^2 + \bar{p}_z^2}{[1 + f(\bar{p}_\sigma)]^2} \right\}^{-1/2} ; \qquad \text{(A.3e)} \\
p_\sigma &= \frac{\partial F}{\partial \sigma} \\
&= \bar{p}_\sigma \, . \qquad \text{(A.3f)}
\end{aligned}
$$

From (A.3 b, d, e) we obtain:

$$
\bar{p}_x = p_x - \Delta s \cdot \left\{ +g \cdot x - N \cdot z + \frac{\lambda}{2} \cdot (x^2 - z^2) + \frac{\mu}{6} \cdot (x^3 - 3\,x\,z^2) \right\} ; \qquad \text{(A.4a)}
$$

$$\bar{p}_z = p_z - \Delta s \cdot \left\{ -g \cdot z - N \cdot x - \lambda \cdot x\, z + \frac{\mu}{6} \cdot (z^3 - 3\, x^2\, z) \right\} ; \tag{A.4b}$$

$$\bar{p}_\sigma = p_\sigma \, . \tag{A.4c}$$

The quantities $\bar{x}$, $\bar{z}$, and $\bar{\sigma}$ are then to be obtained from eqns. (A.3 a, c, e) by taking eqns. (A.4) into account.

Using the notation

$$\vec{y}^{\,i} \equiv \vec{y}(s_0) \, ; \tag{A.5}$$

$$\vec{y}^{\,f} \equiv \vec{y}(s_0 + \Delta s) \, , \tag{A.6}$$

we may finally write:

$$p_x^f = p_x^i - \Delta s \cdot \left\{ +g \cdot x^i - N \cdot z^i + \frac{\lambda}{2} \cdot [(x^i)^2 - (z^i)^2] + \frac{\mu}{6} \cdot [\,(x^i)^3 - 3\,(x^i)\,(z^i)^2\,] \right\} ; \tag{A.7a}$$

$$p_z^f = p_z^i - \Delta s \cdot \left\{ -g \cdot z^i - N \cdot x^i - \lambda \cdot (x^i)\,(z^i) + \frac{\mu}{6} \cdot [\,(z^i)^3 - 3\,(x^i)^2\,(z^i)\,] \right\} ; \tag{A.7b}$$

$$p_\sigma^f = p_\sigma^i \, ; \tag{A.7c}$$

$$x^f = x^i + \Delta s \cdot \left\{ 1 - \frac{(\,p_x^f\,)^2 + (\,p_z^f\,)^2}{[1 + f(p_\sigma^f)]^2} \right\}^{-1/2} \cdot \frac{p_x^f}{[1 + f(p_\sigma^f)]} \, ; \tag{A.7d}$$

$$z^f = z^i + \Delta s \cdot \left\{ 1 - \frac{(\,p_x^f\,)^2 + (\,p_z^f\,)^2}{[1 + f(p_\sigma^f)]^2} \right\}^{-1/2} \cdot \frac{p_z^f}{[1 + f(p_\sigma^f)]} \, ; \tag{A.7e}$$

$$\sigma^f = \sigma^i + \Delta s - \Delta s \cdot f'(p_\sigma^f) \cdot \left\{ 1 - \frac{(\,p_x^f\,)^2 + (\,p_z^f\,)^2}{[1 + f(p_\sigma^f)]^2} \right\}^{-1/2} . \tag{A.7f}$$

The relations (A.7 a - f) describe the thin - lens map of a lens consisting of a superposition of quadrupoles, skew quadrupoles, sextupoles, and octupoles. In particular, we get the map of a pure quadrupole, skew quadrupole, sextupole, and octupole.

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