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Current Commutation Relations in the Framework of General Quantum Field Theory

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Erratum

page 24: replace formula (51) by

$$\frac{2M}{p_o} \left[\frac{K\,(4p_o^{\,2})}{2M} \right]^2$$

and the following lines by:

therefore in the case of a free current where the vertex K is a constant, the sum of the one-particle and the partially connected three-particle contribution is covariant.[18] In the interacting case such an approximation to the commutator would violate causality and yield a non-covariant expression. In order to get a covariant answer in the one-particle approximation, one may attempt to construct a so-called "local one-particle approximation". Some aspects of this problem are discussed with the help of the Jost-Lehmann-Dyson representation in the next section.

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Abstract:

In this paper we give a rigorous formulation of Gell-Mann's equal time commutation relations in the framework of general quantum field theory. We show that sum rules of the Adler - Weisberger typ can be derived without making any additional assumptions on the high-energy behavior of amplitudes. We discuss two different methods of derivation, the first one is an improved version of the Fubini-Furlan method, the second one avoids the application of the Gauß-Theorem and uses the Jost-Lehmann-Dyson - representation directly for the commutator of currents. In neither case we get a dependence on the frame of reference.

I. Introductory remarks

The most convincing sucess of the equal time commutation relations between vector and axialvector currents originally proposed by M.Gell-Mann[1] is the derivation of sum rules of the Adler-Weisberger typ[2]. In the original presentation of Adler and Weisberger this derivation was very involved and assumptions on the interchange of certain limits had to be made. Fubini and Furlan[3] proposed subsequently a simpler and aestetically more appealing method, however, the connection of their results with the equal time commutation relation for the space integral of currents remained somewhat vague. Not only did the rough handling of Gauß's theorem induce certain ambiguities connected with the intermediate one particle contributions, but also was the impression given as if in addition to the equal time commutation relations assumptions on the high energy behavior (subtractions) had to be made.
We show that a careful formulation of the C.R. which takes into account the distribution theoretical aspects will take care of the high energy problem, whereas a more detailed discussion on the use of Gauß-theorem for field operators will resolve the ambiguities for low energies (i.e. the intermediate one particle ambiguity).

Finally we will discuss (in section 4) the application of the Jost-Lehmann-Dyson representation to the commutator. In this way one can give a treatment which does not rely on retarded functions and is in spirit very close to the original treatment of Adler and Weisberger.

We will discuss the mentioned statements in the framework of general quantum field theory[4]. However, the mathematical rigour of our presentation is modest and more on the level of the LSZ formulation than present day axiomatic field theory.

2. Definition of charges

The first problem we investigate is the question in what sense a "charge" operator Q can be connected with a conserved current $j_\mu(x)$:

$$\partial^\mu j_\mu(x) = 0 \tag{1}$$

We observe first, that irrespective of the conservation law (1) the matrix element

$$\langle \Phi | j_\mu(\vec{x},t) | \Psi \rangle \tag{2}$$

is a smooth and fast decreasing function in $\vec{x}$, whenever $|\Psi\rangle$ and $|\Phi\rangle$ are quasilocal states, i.e. states of the form

$$|\Phi\rangle = \mathcal{B}|0\rangle = \sum_{m=1}^{n} \int g_m(x_1 \cdots x_m)\, A_1(x_1) \cdots A_m(x_m)|0\rangle \tag{3}$$

where the A's are from the basic set of local fields (resp. currents) in terms of which the theory is defined, and g_m are fast decreasing smooth functions. Here we assumed $\langle j_\mu(x)\rangle_0 = 0$ and the restricted spectrum condition, i.e. the non occurrance of zero rest-mass states.

The smoothness property of (2) comes (due to translational invariance of the vacuum expectation values) directly from the smoothness of the g's, whereas the fall-off property for large $\vec{x}$ uses in addition locality and is a special case of the so-called linked cluster property [5]. Hence the spatial integral

$$\int \langle \Phi \,|\, j_\mu(\vec{x},t) \,|\, \Psi \rangle \, d^3x \tag{4}$$

always exists and defines a bilinear form.

In the conserved case we expect, however, to be able to define an operator Q called "charge", and we try for the connection with the charge density the following formula

$$Q = \lim_{R\to\infty} j_0(f_R, f_T) \tag{5}$$

i.e. we ask in what sense the sequence of unbounded operators $j_0\,(f_R, f_T)$ has an operator limit.

In choosing our space and time smearing functions f_R, f_T we followed the suggestion of Kastler, Robinson and Swieca [6]:

$$f_T \geq 0, \quad \mathrm{supp}\, f_T \subset [-T, T], \quad \int f_T(t)\, dt = 1 \tag{6a}$$

$$f_R(\vec{x}) = f_R(|\vec{x}|) = \begin{cases} 1 & |\vec{x}| < R \\ 0 & |\vec{x}| > R+L \end{cases} \tag{6b}$$

Hopefully the limes (5) turns out to be independent of T, so that $T \to o$ is superfluous.

We first want to show that (5) cannot exist in the sense of strong convergence:

<u>Statement I</u> $\quad \langle 0 | j_0(f_R, f_T)\, j_0(f_R, f_T) | 0 \rangle \underset{R \to \infty}{\longrightarrow} C R^2$

with $C \neq 0$ unless $j_\mu(x) \equiv 0$.

Proof: For a conserved current we have the following Källèn - Lehmann [7] representation:

$$\langle j_\mu(x)\, j_\nu(y) \rangle_0 = -i \int \left(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\varkappa^2} \right) \Delta^{(+)}_{\varkappa}(x-y)\, \rho(\varkappa^2)\, d\varkappa^2 \quad (7)$$

hence

$$\langle j_0(f_R, f_T)\, j_0(f_R, f_T) \rangle_0 = \int | p \tilde{f}_R(\vec{p}) |^2 g(p)\, d^3p$$

(8)

with $$g(p) = \frac{1}{2\pi} \int \frac{\rho(\varkappa^2)}{2\varkappa^2 \sqrt{p^2 + \varkappa^2}} \left| \tilde{f}_T\left(\sqrt{p^2 + \varkappa^2}\right) \right|^2$$

We have

$$\begin{aligned} p\, \tilde{f}_R(p) &= 2\pi \int \left(\frac{d}{dr} f_R(r) \right) \frac{\sin pr}{ip}\, r\, dr \\ &= \frac{2\pi}{ip} \int f'(r-R) \sin pr\, dr \end{aligned} \quad (9)$$

where $f'(r)$ is the derivative of (6b) for R = 0.

By change of variable:

$$p\tilde{f}_R(p) = \frac{2\pi}{ip}\int f'(\rho)\sin p(\rho+R)\,(\rho+R)\,d\rho \tag{10}$$

Inserting (10) into (8) using addition theorem for sin and cos and taking only the leading term in R we obtain:

$$\langle j_0(f_R,f_T)\, j_0(f_R,f_T)\rangle_0 = 8\pi^2 R^2\int\left(|\tilde{f}_1'(p)|^2+|\tilde{f}_2'(p)|^2\right)g(p)\,dp \tag{11}$$

with

$$\tilde{f}'_{\binom{1}{2}}(p) = \int f'(\rho)\,\rho\begin{pmatrix}\sin p\rho\\ \cos p\rho\end{pmatrix}d\rho \tag{12}$$

The coefficient of the leading term vanishes if and only if $\rho(x^2)\equiv 0$. According to a well known theorem [8] this is equivalent to $j_\mu(x)\equiv 0$.
It is easy to see that with our choice of infinity smooth test function in time, the leading term is approached faster than any inverse power in R.
Next we want to show that the limes (5) exists in the weak sense on a dense set of states. First we show for this purpose that the vacuum is annihilated weakly.

Statement 2

$$\lim_{R\to\infty}\langle\Phi\,|\,j_0(f_R,f_T)\,|\,0\rangle = 0 \tag{13}$$

for states Φ of the form $|\Phi\rangle = \int d^3x\, h(\vec{x})\, U(\vec{x}) B|0\rangle$ where B is quasilocal, i.e. of the form (3). $U(\vec{x})$ is the translational operator and $h(\vec{x})$ is a smooth function which decreases for large r such that

$$\lim_{r \to \infty} r^2 \, h(\vec{x}) = 0 \tag{14}$$

This statement is the transcription of a Lemma by Kastler, Robinson and Swieca [6] from their algebraic framework to the field theoretical framework.

Proof:

As in the paper of Kastler, Robinson and Swieca we "divide" $|\Phi\rangle$ by the energy operator. Here we use the fact that if $B|o\rangle$ is a quasilocal state of the form (3) with $\langle o|B|o\rangle = o$, then $|\psi\rangle = \frac{1}{H} B|o\rangle$ is again quasilocal. This is so since the smearing function $\tilde{g}_m(p_1 \ldots p_m)$ in (3) which according to the finite rest-mass spectrum condition can be chosen as

$$\tilde{g}_m(p_1 \ldots p_m) = 0 \text{ for } \sum_{i=1}^{m} p_{i0} < \frac{M}{2} \qquad \text{(M = smallest rest-mass)}$$

allow the division by $\sum p_{i0}$ and yield again smooth and fast decreasing test functions, and hence $|\Psi\rangle$ is again quasilocal. Therefore we have:

$$\langle \Phi | \frac{i}{H} \left[H, j_0(f_R, f_T) \right] | 0 \rangle = \int d^3x \; h(\vec{x}) \langle \Psi | U^{+}(\vec{x}) \, j_r(f_R', f_T) | 0 \rangle \tag{14}$$

where j_r = component along the radius vector $\vec{x}$.

Now we consider the left hand state as the sum of two states

$$\int h(\vec{x}) \, U(\vec{x}) |\Psi\rangle \, d^3x = \int_0^{R/2} h(\vec{x}) \, U(\vec{x}) \, |\Psi\rangle \, d^3x + \int_{R/2}^{\infty} h(\vec{x}) \, U(\vec{x}) |\Psi\rangle \, d^3x \tag{15}$$

The first state is effectively localized in the sphere with radius $^R/2$ and the second one behaves in norm as

$$\left\| \int_{R/2}^{\infty} h(\vec{x})\, U(\vec{x})\, |\Psi\rangle\, d^3x \right\| = \left\{ \int_{R/2}^{\infty} \int_{R/2}^{\infty} h(\vec{x})\, h(\vec{y})\, \langle \Psi |\, U(\vec{y}-\vec{x}) | \Psi \rangle\, d^3x\, d^3y \right\}^{1/2} \tag{16}$$

$$\sim O(R^{-1/2})$$

This estimate holds because of Ruelle's result [5)]

$$\lim_{(\vec{x}-\vec{y})^2 \to \infty} (\vec{x}-\vec{y})^{2N} \langle \Psi | U(\vec{y}-\vec{x}) | \Psi \rangle = 0$$

for all $N > 0$, and the assumed fall off properties of $h(\vec{x})$.

The contribution to (14) from the effectively localized first state is

$$\int_0^{R+L} d^3x \int d^3y\, h(\vec{x})\, f_R'(\vec{y})\, \langle \Psi |\, U(\vec{y}-\vec{x})\, j_r(0, f_T) | 0 \rangle \tag{17}$$

and hence because the matrix element vanishes again faster than any inverse power of $(\vec{x}-\vec{y})^2$, the integration (17) leads to a function of R which vanishes rapidly for $R \to \infty$. For the second state in (15) we use Schwarz inequality and obtain

$$\left| \int_{R/2}^{\infty} d^3x\, h(\vec{x})\, \langle \Psi |\, U^{+}(\vec{x})\, j_r(f_R', f_T) | 0 \rangle \right| \tag{18}$$

$$\leq \left\| \int_{R/2}^{\infty} d^3x\, h(\vec{x})\, U(\vec{x}) | \Psi \rangle \right\| \, \left\| j_r(f_R', f_T) | 0 \rangle \right\|$$

If we would use for f_R space smearing functions of the typ (6b) with L = constant, we obtain for the second norm

$$\lim_{R \to \infty} \| j_r(f_R', f_T) | 0 \rangle \| < C R$$

However, by using instead of (6b) a sequence of "stretched" functions

$$f_R(\vec{x}) = f\left(\frac{r}{R}\right)$$

where f(r) is a smooth function which is one inside a certain fixed radius and vanishes outside a larger radius, we obtain for the derivative

$$\frac{d}{dr} f\left(\frac{r}{R}\right) \leq \frac{1}{R} \max f'$$

and hence for the norm

$$\lim_{R\to\infty} \left\{ \int d^3x \int d^3y \; f_R'(\vec{x}) f_R'(\vec{y}) \left\langle j_r(\vec{x}, f_T)\, j_r(\vec{y}, f_T) \right\rangle_0 \right. \qquad (19)$$

$$< C R^{1/2}$$

Together with (16) we obtain a vanishing right hand side in (18) for $R \to \infty$.

We would like to mention that our estimates are optimal in any conserved current theory. This can easily be seen by taking a state $|\Phi\rangle = \int d^3x \, h(\vec{x}) \, j_0(\vec{x}, f_T)|0\rangle$ with $\lim\limits_{|\vec{x}|\to\infty} \vec{x}^2 h(\vec{x}) \neq 0$. Such a state is still normalizable, however, a consideration which is similar to the statement I shows that

$\lim\limits_{R\to\infty} \langle \Phi | j_0(f_R, f_T) | 0 \rangle$ vanishes if and only if $j_\mu \equiv 0$. Hence we have learned that in any theory the formula

$$Q = \underset{R\to\infty}{\mathrm{w.\,l.}} \; j_0(f_R, f_T) \qquad (20)$$

breaks down, if one of the states in which the weak limes is taken has a "long range".

If the connection between the $j_o\ (f_R, f_T)$ and a charge operator (2o) makes any sense, both operators should have a dense domain, which is independent of R. The "natural" domain of $j_o(f_R, f_T)$ are the quasilocal states and hence one would expect that Q has to have the vacuum in its domain. But then we can show that a nonconserved current can not give rise to an operator Q. This was first conjectured and made plausible by S. Coleman [9].

<u>Statement III</u> (Coleman) :

For a nonconserved current $\partial^\mu j_\mu(x) \equiv A(x) \neq 0$

the linear form

$$L(\Phi) = \lim_{R\to\infty} \langle \Phi | j_o(f_R, f_T) | 0 \rangle \tag{21}$$

is unbounded in $|\Phi\rangle$.

Here $|\Phi\rangle$ is again a quasilocal state $|\Phi\rangle = B|o\rangle$ with B as in the previous case.

<u>Proof</u>: Again dividing the state $|\Phi\rangle$ by the energy operator one obtains:

$$iL(\Phi) = \lim_{R\to\infty} \langle \Psi | \dot{j}_r (f_R', f_T) | 0 \rangle + \lim_{R\to\infty} \langle \Psi | A(f_R, f_T) | 0 \rangle \tag{22}$$

with $|\Psi\rangle = \frac{1}{H} |\Phi\rangle$

The first term vanishes according to the previous consideration. We want to show that the second term is unbounded in $|\Psi\rangle$.

For this purpose we choose a sequence of quasilocal states $|\Psi_\rho\rangle$ as

$$|\Psi_\rho\rangle = \| A(f_\rho, f_T)|0\rangle\|^{-1} A(f_\rho, f_T)|0\rangle \tag{23}$$

The norm behaves for large ρ like

$$\| A(f_\rho, f_T)|0\rangle\| = \left\{ \int d^3x \int d^3y \, f_\rho(\vec{x}) f_\rho(\vec{y}) \langle A(\vec{x}, f_T) A(\vec{y}, f_T)\rangle_0 \right\}^{1/2}$$

$$= \left\{ \frac{1}{2\pi} \int d^3p \, |\tilde{f}_\rho(p)|^2 \int dx^2 \frac{\sigma(x^2)}{2\sqrt{\vec{p}^2+x^2}} \left|\tilde{f}_T(\sqrt{\vec{p}^2+x^2})\right|^2 \right\}^{1/2} \quad (24)$$

$$\xrightarrow[\rho\to\infty]{} C \rho^{3/2} \quad \text{with} \quad C = 0 \Leftrightarrow A(x) = 0$$

Therefore

$$\lim_{R\to\infty} \langle \Psi_\rho | A(f_R, f_T)|0\rangle \xrightarrow[\rho\to\infty]{} \frac{1}{\rho^{3/2}} \lim_{R\to\infty} \int d^3p \, \tilde{f}^*_\rho(p) \tilde{f}_R(p) \times$$

$$\times \int dx^2 \frac{\sigma(x^2)}{2\sqrt{\vec{p}^2+x^2}} \left|\tilde{f}_T(\sqrt{\vec{p}^2+x^2})\right|^2 = \frac{1}{\rho^{3/2}} \int f_\rho(\vec{x}) d^3x \int dx^2 \frac{\sigma(x^2)}{2x} |\tilde{f}_T(x)|^2$$

$$\sim \rho^{3/2}$$

which can be made arbitrarily large by choice of $|\Psi_\rho\rangle$

Since $|\Phi_\rho\rangle = H|\Psi_\rho\rangle$

and hence $\| |\Phi_\rho\rangle\| = \frac{1}{C^2\rho^3} \rho^3 \int dx^2 \, \sigma(x^2) \, x |\tilde{f}_T(x)|^2 < \infty$,

$L(\Phi)$ is unbounded in $|\Phi\rangle$.

Let us now come back to the formula (2o) in the conserved case. We consider the action of $j_0(f_R, f_T)$ on the dense set of states $B|0\rangle$ as in formula (3) however, with compact support test functions $g_m(x, \ldots x_m)$

$$j_0(f_R, f_T) B|0\rangle = [j_0(f_R, f_T), B]|0\rangle + B \, j_0(f_R, f_T)|0\rangle \quad (25)$$

According to locality the first term is independent of R for large R and again has the form (3) with compact support test functions. The last term converges weakly to zero as $R \to \infty$. Hence the formula (25) defines an operator Q which has its domain all states (3) with compact support test functions and furthermore the operator Q can be applied repeatedly on this domain. It is just slightly more complicated to see that also quasilocal states; i.e. states with noncompact (but decreasing) test function and multiparticle in- (out) states with non - overlapping wave functions belong to the domain of Q.

Finally it is worthwhile to mention that all our considerations go through if the current has other tensorial indices in addition to the index in which the conservative law holds i.e. for currents

$$j_{\mu, \mu_1, \ldots \mu_n}(x)$$

In this case the decomposition of the two point function into standard covariants is more involved, however, due to the requirement that all relations hold for arbitrary $\mu_1, \ldots \mu_n$ we obtain the same result.

3. Formulation of equal time commutation relations

We want to consider at first the case of commutation relations between space integrals over time components of conserved or nonconserved current densities.

<u>Statement:</u> $$\langle \Phi | [j_0^{i}(f_R, f_T), j_0^{K}(f_R, f_T)] | \Phi \rangle = 0 \tag{26}$$

between TCP invariant states.

<u>Proof:</u> If θ is the TCP operator we have

$$\langle \Phi | \theta\theta\, j_0^{i}(f_R, f_T)\, j_0^{K}(f_R, f_T)\, \theta\theta | \Phi \rangle = \langle \Phi | j_0^{i}(f_R, f_T)\, j_0^{K}(f_R, f_T) | \Phi \rangle^{*}$$
$$= \langle \Phi | j_0^{K}(f_R, f_T)\, j_0^{i}(f_R, f_T) | \Phi \rangle$$

because of the choice of symmetric test functions f_R and f_T. Hence we have (26). Here the index i,K designates any vector or axial vector current. If the state $|\Phi\rangle$ is the vacuum we can due to the fact that $|0\rangle$ is rotational invariant omit the smearing in space.

In the literature one finds very often the statement that the vacuum expectation value of the equal time commutator vanishes. This is wrong because there is no equal time meaning to this quantity. Even for free field currents the two-point function although perfectly well defined as a Wightman distribution can not be given meaning for equal times. However, our symmetric time smearing process takes care of this problem, i.e. it truncates the matrix element automatically.

In order to avoid a lengthy discussion due to generalities we take as a model case the axial vector commutation relations of Adler and Weisberger. The currents $j_\mu(x)$ lead after smearing in time to one particle truncated expectation values

$$\langle\Psi| j_0^{(+)5}(\vec{x}, f_T)\, j_0^{(-)5}(\vec{y}, f_T) |\Phi\rangle - \langle\Psi|\Phi\rangle \langle j_0^{(+)5}(\vec{x}, f_T)\, j_0^{(-)5}(\vec{y}, f_T)\rangle_0 \qquad (27)$$

which are infinitly smooth functions in $\vec{x}$ and $\vec{y}$ and decrease in these variables faster than any inverse power. This statement is a direct consequence of Ruelle's results'[5] since the one particle (wave packet) states are quasilocal. Hence the integration with $f_R(x)\ f_R(y)$ and the limit $R \to \infty$ causes no difficulties. *

In our nonconserved current case the result will, however, depend on the time smearing function $f_T(t)$. The statement of equal time commutation relation in the case of our special expectation value now is the assertion that

$$\lim_{T\to 0} \lim_{R\to\infty} \langle\Psi| [j_0^{(+)5}(f_R, f_T), j_0^{(-)5}(f_R, f_T)] |\Phi\rangle = 2 \langle\Psi| I_3 |\Phi\rangle \qquad (28)$$

where I_3 is the 3rd component of the isospin operator. Such an assertion does not run into any obvious difficulties with the principles of quantum field theory. However, an explicit perturbation theoretical check in some renormalizable models would certainly add a lot to the credibilaty of relation (28). We will discuss this problem in a future paper.

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*) Here and in the following a smearing in space superfluous since the spatial integrals converge in the ordinary sense. We will, however, keep the f_R's because they serve as a convenient reminder that the spatial integration in general cannot be interchanged with other limits.

Symbollically we could write

$$\lim_{T\to 0}\lim_{R\to\infty}\left[\dot{j}_0^{(+)5}(f_R,f_T),\dot{j}_0^{(-)5}(f_R,f_T)\right]=2I_3 \tag{29}$$

If we only consider the left hand side between states which lead to fall off properties in $\vec{x}$ and $\vec{y}$ and hence to the existence of $R\to\infty$. All so-called quasilocal states certainly belong to the set of admissable states, but a more detailed investigation shows that (29) can also be taken between multi - particle in- (or out) states with nonoverlapping wave packets.

We will later see that the existence of $\lim\limits_{T\to 0}$ implies a high energy property which can be sharpened by saying how fast the right hand side is approached.

Consider now (ommitting the index 5)

$$\dot{j}_0^{(+)}(f_R,f_T)=-\int_0^t \frac{d}{d\tau}\,\dot{j}_0^{(+)}(f_R,f_T^{\tau})+\dot{j}_0^{(+)}(f_R,f_T^{t}) \tag{30a}$$

and

$$\dot{j}_0^{(-)}(f_R,f_T)=\int_{-t}^{0}\frac{d}{d\tau}\,\dot{j}_0^{(-)}(f_R,f_T^{\tau})+\dot{j}_0^{(-)}(f_R,f_T^{-t}) \tag{30b}$$

where $\quad \mathcal{U}(\tau)\,\dot{j}_0^{(\pm)}(f_R,f_T)\,\mathcal{U}^{\dagger}(\tau)=\dot{j}_0^{(\pm)}(f_R,f_T^{\tau})$

$$f_T^{\tau}(t)=f_T(t-\tau)$$

Lemma 1

$$\int_0^t \int_{-t}^0 d\tau\, d\tau' \lim_{R\to\infty} \langle\Psi| \Big\{ \Big[\frac{d}{d\tau} j_0^{(+)}(f_R, f_T^{\tau}),\ \frac{d}{d\tau'} j_0^{(-)}(f_R, f_T^{\tau'})\Big] \tag{31a}$$

$$- \big[\mathcal{D}^{(+)}(f_R, f_T^{\tau}),\ \mathcal{D}^{(-)}(f_R, f_T^{\tau'})\big]\Big\} |\Phi\rangle = 0$$

$$\int_{-t}^0 d\tau \lim_{R\to\infty} \langle\Psi| \Big[j_0^{(+)}(f_R, f_T^{t}),\ \frac{d}{d\tau} j_0^{(-)}(f_R, f_T^{\tau}) - \mathcal{D}^{(-)}(f_R, f_T^{\tau})\Big] |\Phi\rangle$$

$$- \int_0^t d\tau \lim_{R\to\infty} \langle\Psi| \Big[\frac{d}{d\tau} j_0^{(+)}(f_R, f_T^{\tau}) - \mathcal{D}^{(+)}(f_R, f_T^{\tau}),\ j_0^{(-)}(f_R, f_T^{-t})\Big] |\Phi\rangle \tag{31b}$$

$$= 0$$

Here $|\Phi\rangle$ and $|\Psi\rangle$ are quasilocal states and

$$\mathcal{D}^{(\pm)}(x) = \partial^{\mu} j_{\mu}^{(\pm)}(x)$$

Proof: Since $\frac{d}{d\tau} j_0^{(\pm)}(f_R, f_T^{\tau}) = j_r^{(\pm)}(f_R', f_T^{\tau}) + \mathcal{D}^{(\pm)}(f_R, f_T^{\tau})$

in order to prove (31a) we have to show that:

$$\langle\Psi| \big[j_r^{(+)}(f_R', f_T^{\tau}),\ \mathcal{D}^{(-)}(f_R, f_T^{\tau'})\big] |\Phi\rangle_{tr}$$

$$+ \langle\Psi| \big[\mathcal{D}^{(+)}(f_R, f_T^{\tau}),\ j_r^{(-)}(f_R', f_T^{\tau'})\big] |\Phi\rangle_{tr} \tag{32}$$

$$+ \langle\Psi| \big[j_r^{(+)}(f_R', f_T^{\tau}),\ j_r^{(-)}(f_R', f_T^{\tau'})\big] |\Phi\rangle_{tr} \xrightarrow[R\to\infty]{} 0$$

where tr (truncation) indicates subtraction of the vacuum expectation values*.

We prove that every single term in (32) goes to zero. Consider for example the first term explicitly

$$\int_R^{R+L} d^3x \int_0^{R+L} d^3y \; f_R'(x)\, f_R(y) \langle \Psi | [j_r^{(+)}(\vec{x}, f_T^{\tau}), \mathcal{D}^{(-)}(\vec{y}, f_T^{\tau'})] | \Phi \rangle_{tr}$$

According to Ruelle [5] the truncated matrix element

$$\langle \Psi | j_r^{(+)}(\vec{x}, f_T^{\tau})\, \mathcal{D}^{(-)}(\vec{y}, f_T^{\tau'}) | \Phi \rangle_{tr}, \quad \in \quad \mathcal{T}_{\vec{x}, \vec{y}} \tag{33}$$

i.e. is a smooth function which decreases rapidly in both variables $\vec{x}$ and $\vec{y}$. Hence after integration with $f_R(\vec{y})$ the remaining expression decreases rapidly in $\vec{x}$ and hence the integral over the ring $R \leq (\vec{x}) \leq R + L$ gives a decreasing function in R. Therefore the first commutator decreases rapidly as $R \to \infty$. The argument for the decrease of the other terms as well as for (31b) is the same.

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*) If the integration over the $\tilde{\iota}$'s is performed as in the Lemma the vacuum expectation value of the commutators vanishes, hence the truncation would be superfluous. However, working with the integrands only, the truncation is necessary for the existence of the limes $R \to \infty$.

Specializing $|\Phi\rangle = |\Psi\rangle = \int |p\rangle \psi(\vec{p}) \frac{d^3p}{2p_0}$ to a one particle state (proton state) with a smooth decreasing wave packet $\psi(\vec{p})$ (such states are quasilocal [5]) and using lemma I we obtain:

$$\lim_{R\to\infty} \langle\Psi|[j_0^{(+)}(f_R, f_T), j_0^{(-)}(f_R, f_T)]|\Psi\rangle$$

$$= -\int_0^t d\tau \int_{-t}^0 d\tau' \lim_{R\to\infty} \langle\Psi|[D^{(+)}(f_R, f_T^{\tau}), D^{(-)}(f_R, f_T^{\tau'})]|\Psi\rangle$$

$$+\int_{-t}^0 d\tau \lim_{R\to\infty} \langle\Psi|[j_0^{(+)}(f_R, f_T^{t}), D^{(-)}(f_R, f_T^{\tau})]|\Psi\rangle \qquad (34)$$

$$-\int_0^t d\tau \lim_{R\to\infty} \langle\Psi|[D^{(+)}(f_R, f_T^{\tau}), j_0^{(-)}(f_R, f_T^{-t})]|\Psi\rangle$$

$$+\lim_{R\to\infty} \langle\Psi|[j_0^{(+)}(f_R, f_T^{t}), j_0^{(-)}(f_R, f_T^{-t})]|\Psi\rangle$$

We now want to show, that the contribution for large t of the 2nd and 3rd term vanishes.

Since:

$$\langle p'|D^{(+)}(0)|p\rangle = (2\pi)^{-3} 2M\, \bar{u}(p')\gamma_5 u(p)\, K((p-p')^2) \xrightarrow[p\to p']{} 0$$

the one-particle intermediate state drops out and therefore the sum of the second and third term can be written as:

$$\int_{M+\mu}^{\infty} dW \int \frac{d^3p}{(2p_0)^2} |\Psi(p)|^2 \langle p|\{ j_0^{(+)}(0, f_T)\, E_{W,\vec{p}}\, D^{(-)}(0, f_T) \qquad (35)$$

$$- D^{(+)}(0, f_T)\, E_{W,\vec{p}}\, j^{(-)}(0, f_T)\}|p\rangle \frac{i}{p_0 - \sqrt{\vec{p}^2 + W^2}} \{ e^{i(p_0 - \sqrt{\vec{p}^2+W^2})t}$$

$$- e^{2i(p_0 - \sqrt{\vec{p}^2+W^2})t} \} - \text{term with } (+) \leftrightarrow (-) \text{ and } t \leftrightarrow -t$$

Here $E_{W,\vec{p}}$ is the prejector onto the (improper) subspace with momentum $\vec{p}$ and total mass $W \geq M + \mu$, where $M + \mu$ is the mass of the smallest two particle intermediate state. If the matrix element of this projector between states created by application of smeared out local fields would be a L_1 integrable function of W, then the time limit $t \to \infty$ in (35) would according to the Riemann-Lebesgue lemma oscillate to zero. This matrix element of the projector is closely related to the continuous contribution of the absorptive part in forward dispersion relations. If one could prove the local (in every finite W intervall) L - integrability of such absorptive parts, then the term (35) drops out in the limit $t \to \infty$. Such a property, although it is true in perturbation theory, has not been derived in general field theory and hence we are forced to make a technical assumption.

In the fourth term (34) one obtains the following one particle intermediate state contribution (neutron) for the case of equal masses of proton and neutron:

$$(2\pi)^6 \sum_i \int \frac{d^3p}{(2p_0)^2} \, |\Psi(p)|^2 \, \langle p| \, j_0^{(+)}(0) \, |p i\rangle \langle p i| \, j_0^{(-)}(0) \, |p\rangle \tag{36}$$

Here we already took $T \to 0$.

This term gives explicitly:

$$G_A^2 \int \frac{d^3p}{2p_0} \, |\Psi(p)|^2 \, \left(1 - \left(\frac{M}{p_0}\right)^2\right) \tag{37}$$

where G_A is defined by $K(0) = 2MG_A$

The first term on the right hand side can be written (integration over one τ) as:

$$-\int dx_0\, \theta(x_0)\, x_0 \int d^3x \,\langle p|[D^{(+)}(\tfrac{\vec{x}}{2}, f_T^{x_0/2}), D^{(-)}(-\tfrac{\vec{x}}{2}, f_T^{-x_0/2})]|p\rangle \quad (38)$$

where we have omitted the wave packet integration. This in turn one can write as

$$\lim_{\omega\to 0} \frac{\partial}{\partial\omega}\, i \int e^{i\omega x_0}\, \theta(x_0) \int d^3x \,\langle p|[D^{(+)}(\vec{x}/2, f_T^{x_0/2}), D^{(-)}(-\tfrac{\vec{x}}{2}, f_T^{-x_0/2})]|p\rangle \quad (39)$$

if the function is an analytic function in ω which approaches a continuous boundary value around $\omega = 0$.
That this is indeed the case will become evident later on.

Furthermore it is advantageous to write (39) as

$$\lim_{\omega\to 0} \frac{\partial}{\partial\omega} \lim_{\xi\to\omega^2} i \int e^{i\omega x_0 - i\sqrt{\omega^2-\xi}\,\vec{e}\cdot\vec{x}}\, \theta(x_0)\, \times$$
$$\times \langle p|[D^{(+)}(\vec{x}/2, f_T^{x_0/2}), D^{(-)}(-\vec{x}/2, f_T^{-x_0/2})]|p\rangle \quad (40)$$

This is allowed because if ξ is real and negative the integral is an analytic function of ω (since the commutator vanishes if $|\vec{x}| > |x_0| + 2T$) and due to the smearing in time there is no subtraction in the Hilbert-relation

$$R_{f_T}(\omega, 0, \xi) = \frac{1}{2\pi}\Big\{\int d\omega' \frac{M_-(\omega', 0, \xi)\,|\tilde{f}_T(\omega')|^2}{\omega' - \omega} \quad (41)$$
$$+ \int d\omega' \frac{M_+(\omega', 0, \xi)\,|\tilde{f}_T(\omega')|^2}{\omega' + \omega}$$

with [11])

$$M_{(\mp)}(\omega, 0, \xi) = \int d^4x\, e^{i\omega x_0 - i\sqrt{\omega^2-\xi}\,\vec{e}\cdot\vec{x}} \langle p|D^{(\pm)}(x/2)\, D^{(\mp)}(-x/2)|p\rangle \quad (42)$$

From now on the argument goes as in the case of derivation of dispersion relations. One uses the analyticity properties of $M_{\mp}(\omega, 0, \xi)$ in ξ in order to achieve the analytic continuation needed in (4o). Due to the presence of the time smearing function f_T there is, however, no high energy problem. As in the case of dispersion relations for scattering amplitudes, one separates the one particle contribution explicitly.

The only one particle contribution comes from M_- and is:

$$M_-^{(1)} = (2\pi)^4 \sum_i \int \langle p | \mathcal{D}^{(+)}(0) | q i \rangle \langle q i | \mathcal{D}^{(-)}(0) | p \rangle \, \delta_+(q^2 - M^2) \times \\ \times \delta(q_0 - \omega - p_0) \, \delta(\vec{p} - \vec{q} + \vec{e}\sqrt{\omega^2 - \xi}) \tag{43}$$

Because of

$$\sum_i \langle p | \mathcal{D}^{(+)}(0) | q i \rangle \langle q i | \mathcal{D}^{(-)}(0) | p \rangle = \\ (2\pi)^{-6}\, 2M\, \bar{u}(p)\, \gamma_5\, (-i\not q + M)\, \gamma_5\, u(p)\, K^2((p-q)^2) \\ = (2\pi)^{-6}\, 2M \quad \frac{-qp + M^2}{M} \quad K^2((p-q)^2) \tag{44}$$

we obtain

$$R_{f_T}^{(1)}(\omega, 0, \omega^2) = \frac{1}{(2\pi)^3} \, \frac{\omega^2 K^2(\omega^2)}{2p_0\omega + \omega^2} \left| \tilde{f}_T(-2p_0) \right|^2 \tag{45}$$

$$(2\pi)^3 \lim_{\omega \to 0} \frac{\partial}{\partial \omega} R_{f_T}^{(1)}(\omega, 0\, \omega^2) = \frac{1}{2p_0} K^2(0) \left| \tilde{f}_T(-2p_0) \right|^2 \\ \text{since} \quad K^2(0) = (2M)^2 G_A^2 \tag{46}$$

and $\tilde{f}_T \to 1$ for $T \to 0$ this term compensates exactly the term with the minus sign in (37).

The rest of the contribution in (4o) can be written with the help of the Goldberger-Treimann relation as [3]:

$$2p_0 \frac{G_A^2}{g_{NN\pi}^2 K_{NN\pi}^2(0)} \int_{\mu}^{\infty} \frac{d\omega}{\omega} \left[\sigma_{p\pi^-}^{tot}(\omega,0) - \sigma_{p\pi^+}^{tot}(\omega,0) \right] | \tilde{f}_T(\omega)|^2 \qquad (47)$$

where $\sigma_{p\pi^\pm}^{tot}(\omega,\xi)$ is analytic up to $\xi = \mu^2$ and $\sigma_{p\pi^\pm}^{tot}(\omega,\mu^2)$ is the physical total cross section of $p\pi^\pm$ scattering.

The wave packet $\psi(p)$ in (34) is now dropping out on both hand sides and we obtain the Adler-Weisberger relation:

$$1 = G_A^2 \left\{ 1 + \frac{2M^2}{g_{NN\pi}^2 K_{NN\pi}^2(0)} \frac{1}{\pi} \lim_{T \to 0} \times \right.$$

$$\times \int \frac{d\omega}{\omega} \left[\sigma_{p\pi^-}^{tot}(\omega,0) - \sigma_{p\pi^+}^{tot}(\omega,0) \right] | \tilde{f}_T(\omega)|^2 \qquad (48)$$

The existence of the limes $T \to 0$ gives the restriction on the high energy behavior of $\sigma_{p\pi^-}^{tot}(\omega,0) - \sigma_{p\pi^+}^{tot}(\omega,0)$ and hence leads to the absence of subtractions.
It is evident that the rate of convergence for $T \to 0$ in (30) is directly related to the rate of decrease of $\sigma_{p\pi^-}^{tot}(\omega,0) - \sigma_{p\pi^+}^{tot}(\omega,0)$
Therefore the high energy behavior of the extrapolated cross section has a direct space time significance in terms of equal time commutation relations.
The problem of the connection between $\sigma^{tot}(\omega,0)$ and $\sigma^{tot}(\omega,\mu^2)$ has been studied by Adler [2] in a model. If one could prove the analyticity of $\sigma^{tot}(\omega,\xi)$ in ξ not only in a strip around the real ξ-axis [17] but in a circle around $\xi = 0$ with radius μ, then by using the positiv definiteness of $\sigma^{tot}(\omega,\xi)$ on the real

axis and applying the maximum modulus principle one could derive

$$\sigma^{tot}(\omega, 0) \leq \sigma^{tot}(\omega, \mu)$$

as a model independent relation. Unfortunately this property does not allow to controll the difference appearing in (48) in a model independent way.

The independence of the sum rule on the frame of reference is in our language the independence on the wave packet Ψ . In order to see this independence it was very important to use locality of the currents (in the form of proven analytic properties). If we first insert a complete set of intermediate states in (38) before analytic continuation this independence on the frame of reference would (as in the original derivation of AW) not be manifest. The intermediate states after analytic continuation do not correspond in a simple way to the inter mediate state of the naive insertion.

If we insert in the commutator of

$$\lim_{\substack{R \to \infty \\ T \to 0}} \langle p | \left[j_0^{(+)}(f_R, f_T), j_0^{(-)}(f_R, f_T) \right] | p \rangle \tag{49}$$

the one particle intermediate state (neutron) we obtain:

$$2 p_0 \left(1 - \frac{M^2}{p_0^2}\right) G_A^2$$

whereas the "crossing symmetric" i.e. the partially connected contribution from the 3-particle intermediate state (one particle-antiparticle pair, one neutron)

$$\int \langle p| j_0^{(+)}(f_R, f_T)| q_1, q_2, q_3\rangle \langle q_1, q_2, q_3| j_0^{(-)}(f_R, f_T)| p\rangle \cdot$$

$$\times \prod_i \delta_+ (q_i^2 - m_i^2)\, d^4 q_i \tag{50}$$

$$= \int \langle 0| j_0^{(+)}(f_R, f_T)| q_2, q_3 = p\rangle \langle q_2, q_1 = p| j_0^{(-)}(f_R, f_T)|0\rangle\, \delta_+ (q_2^2 - m^2)\, d^4 q_2$$

leads to

$$\frac{2M}{p_0}\, G_A^2 \tag{51}$$

therefore their sum is covariant [18] and equal to the sum of the boundary term and the term from the one particle pole in the dispersion relation. The sum of these terms is part of a so - called "local" one-particle approximation. This point is discussed with the help of the Jost-Lehmann-Dyson representation in the next section

In this section we have studied the commutation relation between space integrals. Often one also formulates commutation relation between densities, for example

$$\lim_{T \to 0} \left[j_0^{(+)}(\vec{x}, f_T), j_0^{(-)}(\vec{y}, f_T)\right] = 2\, j_0^{(3)}(\vec{x}, 0)\, \delta(\vec{x} - \vec{y}) \tag{52}$$

or equivalently

$$\left[j_0^{(+)}(\vec{x}, t), j_0^{(-)}(\vec{y}, t)\right]_{tr} = 2\, j_0^{(3)}(\vec{x}, t)\, \delta(\vec{x} - \vec{y})$$

The validity of such commutation relations is more model - dependent, and therefore more doubtful. The derivation of sum rules is, however, much less complicated for that case. In the next section we will study such relations from the point of view of the Jost-Lehmann-Dyson representation.

4. Proof of sum rules by using the Jost-Lemann-Dyson representation

In this section we want to give a direct proof of sum rules by using the Jost-Lehmann-Dyson (JLD)-representation for the matrix element of current commutators.

Since we have previously understood the delicate points of equal time commutation relations with the use of testing functions, we feel free to take a more formal attitude in the following.

4.1 Definitions and kinematics

To avoid unnecessary complications with higher spins we consider the simple but nontrivial example of a truncated one-particle matrix element of the commutator between a vector current $j_\mu(x)$ and a scalar operator $t(y)$ for scalar particles of mass μ

$$F_\mu(x,y) \equiv \langle p_2 | [j_\mu(x), t(y)] | p_1 \rangle_{tr} \tag{53}$$

where $p_1^2 = p_2^2 = \mu^2$.

Then we define the Fourier transform of F_μ with respect to the relative coordinate x - y by

$$\tilde{F}_\mu(p, q, \Delta) \equiv \int d^4x \, e^{iqx} F_\mu(x/2, -x/2) \tag{54}$$

with $p \equiv \frac{p_1 + p_2}{2}$ and $\Delta \equiv p_2 - p_1$

We assume some selection rules which allow the following intermediate states in (53)*;

$$\langle p_2 | j_\mu | n \rangle \langle n | t | p_1 \rangle \neq 0 \quad \text{if} \quad p_n^2 = m^2 \quad \text{or} \quad p_n^2 \geq (m+\mu)^2 \tag{55a}$$

with $m \geq \mu$

and

$$\langle p_2 | t | n \rangle \langle n | j_\mu | p_1 \rangle \neq 0 \quad \text{if} \quad p_n^2 \geq (m+\mu)^2 \tag{55b}$$

According to eq. (55) $\tilde{\tilde{F}}_\mu$ has support for (in the special frame P = (a,o,o,o)):

$$\begin{aligned} q_0 &= -a + \sqrt{m^2 + \vec{q}^{\,2}} \\ q_0 &\geq -a + \sqrt{(m+\mu)^2 + \vec{q}^{\,2}} \\ q_0 &\leq a - \sqrt{(m+\mu)^2 + \vec{q}^{\,2}} \end{aligned} \tag{56}$$

Because of Lorentz covariance $\tilde{\tilde{F}}_\mu$ may be decomposed as follows

$$\tilde{\tilde{F}}_\mu (p, q, \Delta) = q_\mu B_1 + p_\mu B_2 + \Delta_\mu B_3 \tag{57}$$

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*) For these selection rules we think of a physical example: The one-particle states in (53) are π^+-mesons and j_μ resp. t transform with respect to SU_3 like the (+1) resp. (-1) component of a V-spin vector. Then the particles of mass m will be K-mesons.

where the B_i are invariant functions of the independent scalar products formed by the vectors P, q and Δ

$$\mathcal{B}_i = \mathcal{B}_i\left(\left(q-\frac{\Delta}{2}\right)^2, (q+p)^2, \Delta^2, (q-\Delta/2)\cdot\Delta\right) \tag{58}$$

The remaining invariants are determined by

$$p^2 + \frac{\Delta^2}{4} = \mu^2 \qquad \text{and} \qquad P\cdot\Delta = 0 \tag{59}$$

Furthermore we consider the expression

$$\mathcal{F}(x,y) \equiv \partial^\mu_{(x)}\, \mathcal{F}_\mu(x,y) \tag{60}$$

and the corresponding Fourier transform $\tilde{\mathcal{F}}$

$$\tilde{\mathcal{F}}(p,q,\Delta) \equiv -i\,(q-\Delta/2)^\mu\, \tilde{\mathcal{F}}_\mu(p,q,\Delta) \tag{61}$$

4.2 General consequences from locality

We assume, that $j_\mu(x)$ and $t(y)$ are local with respect to each other:

$$[j_\mu(x), t(y)] = 0 \quad \text{if} \quad |x-y|^2 < 0 \tag{62}$$

With (62) and Lorentz covariance we have JLD-representations [13] for the functions B_i resp. $\tilde{\mathcal{F}}$:

$$\mathcal{B}_i(p,q,\Delta) = \int d^4u \int ds\, \varepsilon(q_0-u_0)\, \delta((q-u)^2-s)\, \varphi_i(u,s,p,\Delta) \tag{63}$$

resp.

$$\tilde{F}(p,q,\Delta) = \int d^4u \int ds\, \varepsilon(q_0-u_0)\, \delta((q-u)^2-s)\, \psi(u,s,p,\Delta) \qquad (64)$$

(in the following we suppress the variables P and Δ in the JLD-spectral functions). The spectral functions φ_i and ψ have support in D(u,s)

$$D(u,s) = \left\{ (P \pm u) \in V_+,\ \sqrt{s} \geq \mathrm{Max}\left(0,\, m - \sqrt{(P+u)^2},\, m+\mu - \sqrt{(P-u)^2}\right) \right\}$$

Because $\tilde{F}$ is determined by the B_i through eq. (57) and (61), one possible spectral function ψ with support in D may be given in terms of the φ_i as follows

$$\begin{aligned} i\,\psi(u,s) = {} & \left[u^2 - \frac{(\Delta\cdot u)}{2} + s\right] \varphi_1(u,s) + (u\cdot p)\, \varphi_2(u,s) \\ & + \left[(u\cdot p) - \frac{\Delta^2}{2}\right] \varphi_3(u,s) + \frac{\partial}{\partial u^\mu} \int_0^s ds' \left[\varphi_1(u,s') \times \right. \\ & \left. \times \left(u^\mu - \frac{\Delta^\mu}{4}\right) + \frac{1}{2}\varphi_2(u,s')\, p^\mu + \frac{1}{2}\varphi_3(u,s')\, \Delta^\mu\right] \end{aligned} \qquad (65)$$

Now we make the additional assumption that $F_0(x,y)$ has an equal time meaning in the sense explained in the previous section. Necessary and sufficient for this would be the existence of the following integrals

$$\int d^4u \int ds\, \frac{\varphi_{2,3}(u,s)}{\sqrt{s}} \quad , \quad \int d^4u \int ds\, \varphi_1(u,s) \qquad (66)$$

With the conditions (66), F_0 is given at equal times as follows

$$\lim_{x_0 \to 0} F_0(x/2, -x/2) = \frac{\delta(\vec{x})}{2\pi} \int d^4u \int ds\, \varphi_1(u,s) \qquad (67)$$

An immediate consequence of eq. (67) is the Lorentz invariance of the matrix element of the equal time commutator between the charge

$$Q(x_0) \equiv \int d^3x\, j_0(x)$$

and the operator t

$$\langle p_2 | \left[Q(0), t(0) \right] | p_1 \rangle_{tr} \qquad (68)$$

From this fact an important conclusion for the approximate treatment of (68) may be drawn:
Every local approximation for $F_\mu(x,y)$ with JLD-spectral functions fulfilling (66) leads to a Lorentz invariant approximation for (68).

Now it is well known that an approximation which considers only a finite number of intermediate states in (53) is in nontrivial theories always a nonlocal approximation. Therefore, the covariance difficulties appearing in connection with older one-particle approximations of Fubini,Furlan and other authors [14] are a consequence of the violation of locality. In such cases it is necessary, to take into account a local completion [15] of the "primitive one - particle approximation".

4.3. Sum rules

Statement: If the $\varphi_i(u,s)$ decrease in s as fast as demanded by the existence of the integrals (66) we have the following equation for F_0 in the equal time limit

$$F_0\left(\vec{x}/2, -\vec{x}/2\right) = \delta(\vec{x}) \left[\mp \tilde{F}_{\substack{R\\A}} (0, \mu^2, \Delta^2, 0) + \frac{\varkappa_1(\Delta^2)}{2\pi} \right] \qquad (69)$$

where $\widetilde{F}_{\substack{R\\A}}$ is a retarded resp. advanced function (related to $\widetilde{\widetilde{F}}$) which is defined by its unsubtracted JLD-representation (according to eq. (58) we have $\widetilde{\widetilde{F}}_{\substack{R\\A}}(0,\mu^2,\Delta^2,0) = \widetilde{\widetilde{F}}_{\substack{R\\A}}\Big|_{q=\Delta/2}$):

$$\widetilde{F}_{\substack{R\\A}}(0,\mu^2,\Delta^2,\varepsilon) \equiv \pm \frac{i}{2\pi}\int d^4u \int ds \frac{\Psi(u,s)}{(\Delta/2 - u \pm i\varepsilon)^2 - s} \tag{70}$$

and

$$\mathcal{H}_1(\Delta^2) \equiv \begin{cases} 0 & \text{if } m > \mu \\ (2\pi)^{-2} g(0)\ G(\Delta^2) & \text{if } m = \mu \end{cases} \tag{71}$$

where g and G are vertex functions defined by the one-particle matrix elements of j_μ resp. t:

$$\langle p_2 | j_\mu(0) | p_2 + k_1 \rangle = (2\pi)^{-3} \left[(2p_2 + k_1)_\mu\ g(k_1^2) + k_{1\mu}\ h(k_1^2) \right] \tag{72a}$$

resp.

$$\langle p_1 + k_2 | t(0) | p_1 \rangle = (2\pi)^{-3}\ G(k_2^2) \tag{72b}$$

We have

$$h(k_1^2) \equiv 0 \quad \text{if} \quad m = \mu \tag{73}$$

by assuming time reversal- invariance and hermiticity of the neutral component of j_μ, i.e. in this case the one-particle matrix element of j_μ fulfills the continuity equation

$$k_1^\mu \langle p_2 | j_\mu(0) | p_2 + k_1 \rangle = 0 \quad \text{if} \quad m = \mu \tag{74}$$

Statement (69) contains not only the existence of the Fubini - Furlan-Rosetti relation [3]

$$\langle p_2 | [Q(0), t(0)] | p_1 \rangle - \langle p_2 | [Q(\pm\infty), t(0)] | p_1 \rangle$$
$$= \mp \int_{-\infty}^{+\infty} d^4x \, \theta(\pm x_0) \, \langle p_2 | [\partial^\mu j_\mu(x), t(0)] | p_1 \rangle \tag{75}$$

for our model, however, beyond this it gives a unique prescription for the definition of the different terms appearing in (75).

There are more advantages of the present method: It is known that in the case of axialvector currents and equal masses (i.e. $m = \mu$) the boundary terms and the one-particle contribution from the retarted commutator in (75) are not well defined seperately, only their sum exists. Such a problem does not arise in the derivation of sum rules with the aid of the JLD-representation.

Of course eq. (69) has not yet the form of a sum rule. In the case of zero momentum transfer, i.e. $\Delta = 0$, it is possible to extend the r.h. s. of eq. (69) to the r.h.s of a sum rule directly by using the JLD-representation, i.e. we have

$$-\int d^4u \int ds \, \frac{\Psi(u,s)}{u^2 - s} = \int_{-\infty}^{+\infty} ds' \, \frac{\tilde{\hat{F}}(0,s',0,0)}{s' - \mu^2} \tag{76}$$

The proof of eq.(76) follows immediately by insertion of the JLD-representation (64) for $\tilde{\hat{F}}$ into the r.h.s. of eq.(76) (we only have to use, that $\Psi(u,s)/_{\Delta=0} = 0$ for $u^2 = s$ as shown below).

According to eq. (56) and (74) $\tilde{\mathcal{F}}(0,s',0,0)$ has the following support

$$-\infty < s' \leq 2\mu^2-(m+\mu)^2 \tag{77a}$$

$$s'=m^2 \quad \text{if} \quad m>\mu \tag{77b}$$

$$(m+\mu)^2 \leq s' < \infty \tag{77c}$$

The physical interpretation of the r.h.s. of eq. (76) is the usual one, if the operators t(y) resp. $\partial^\mu j_\mu(x)$ may be identified with interpolating field operators describing asymptotically physical particles.

We only mention, that it is not possible to proof a relation analogous to eq. (76) for $\Delta \neq 0$ by only using the JLD-representation for $\tilde{\mathcal{F}}$ (this is connected with the possibility of proving dispersion relations for πN-scattering for instance in the forward case with the aid of the JLD-representation for the retarded commutator of the pion currents only [16], whereas for $\Delta \neq 0$ the proof is more complicated [17]).

Now we come to the proof of the statement eq. (69). With eq.(67) and (7o) our statement is equivalent to the following assertion.

<u>Assertion:</u>

$$-i\int d^4u \int ds \frac{\Psi(u,s)}{(\Delta/2-u\pm i\varepsilon)^2-s} = \int d^4u \int ds\, \varphi_1(u,s) - \chi_1(\Delta^2) \tag{78}$$

Proof:

Next we show that $\left((\Delta/2-u)^2=s\right) \notin$ supp $\psi(u,s)$:

As shown in appendix A we have the following decomposition for the φ_i :

$$\varphi_i(u,s) = \varphi_{i,0}(u,s) + \varphi_{i,1}(u,s) \tag{79}$$

with $\varphi_{i,1}(u,s) = 0$ if $s \leq (\Delta/2-u)^2$

(80)

and

$$\varphi_{i,0}(u,s) = \mathcal{H}_i(\Delta^2)\,\delta(s-\mu^2)\,\delta(u+p) \tag{81}$$

$$\mathcal{H}_i(\Delta^2) = 0 \quad \text{if} \quad m > \mu$$

(81) is unique up to terms which are different from zero only at the point $\{s=(\Delta/2-u)^2, u=-p\}$ and giving a vanishing contribution to the B_i. Of course the same decomposition is true for $\psi(u,s)$ with a function $\mathcal{H}_\psi(\Delta^2)$.

According to the kinematics we have

$$(q+p)^2 \xrightarrow[q\to\Delta/2]{} \mu^2$$

Therefore, with the decomposition (79) applied to ψ and the JLD-representation for $\tilde{F}$ eq. (64) we obtain:

$$\tilde{F}(p,q,\Delta) \xrightarrow[q\to\Delta/2]{} \mathcal{H}_\psi(\Delta^2)\,\delta\left((q+p)^2-\mu^2\right) \tag{82}$$

On the other hand the r.h.s. of eq. (82) vanishes for m = μ according to eq. (74). Therefore

$$\varkappa_{\psi}(\Delta^2) \equiv 0$$
$$\text{i.e.} \qquad \psi(u,s) = \psi_1(u,s) \tag{83}$$

Now by eq.(65) Ψ is a linear functional in the φ_i', therefore we have another decomposition for Ψ

$$\Psi[\varphi_i] = \Psi[\varphi_{i,0}] + \Psi[\varphi_{i,1}] \tag{84}$$

As can easily be seen with the aid of eq.(79)-(81) for the φ_i' and eq.(65), the decomposition (84) is identical with the decomposition (79) for Ψ , i.e.

$$\Psi[\varphi_{i,0}] = \Psi_0 \overset{\text{eq.(83)}}{=} 0$$
$$\Psi[\varphi_{i,1}] = \Psi_1 \tag{85}$$

With the prescription (85) for Ψ_1 , we may insert it into the l.h.s. of eq.(78) and get immediately by partial integration

$$-i \int d^4u \int ds \, \frac{\Psi(u,s)}{(\Delta/2-u)^2 - s} = \int d^4u \int ds \, \varphi_{1,1}(u,s)$$

Finally we have to show that the χ_1 defined by eq.(81) is identical with the χ_1 defined by eq. (71) for m = μ:

By the same arguments leading to eq. (82) we have

$$\mathcal{B}_1(p, q, \Delta) \xrightarrow[q \to \Delta/2]{} \mathcal{X}_1(\Delta^2)\, \delta((q+p)^2 - \mu^2) \tag{86}$$

On the other hand B_1 may be decomposed into the contributions from the one- and many-particle intermediate states

$$\mathcal{B}_1 = \mathcal{B}_1^{(1)} + \mathcal{B}_1^{(2)}$$

where we have according to the spectral conditions and eq. (72), (73) for $m = \mu$

$$\mathcal{B}_1^{(2)} \xrightarrow[q \to \Delta/2]{} 0$$

$$\mathcal{B}_1^{(1)} \xrightarrow[q \to \Delta/2]{} (2\pi)^{-2}\, g(0)\, G(\Delta^2)\, \delta((q+p)^2 - \mu^2)$$

4.4 Some remarks to "one-particle approximations"

In cases for which $\partial^\mu j_\mu(x)$ and $t(y)$ have no physical interpretation the sum rule is a formal one, i.e. not calculable. In such cases we would restrict ourselves to a one-particle approximation. As already discussed in subsection 4.3., the so-called "primitive one-particle approximation" $\overline{F}_\mu^{(1)}$ is a non-local one which leads in our model for $m > \mu$ to a non-Lorentz invariant expression for the matrix element of the equal-time commutator of the charge with $t(o)$. Therefore it is necessary to find a local one-particle approximation F_μ^I, which contains $\overline{F}_\mu^{(1)}$, i.e. we demand [15)]

$$\tilde{F}_\mu^I \xrightarrow[\substack{(q+p)^2 \to m^2 \\ (q+p) \in V_+}]{} \overline{F}_\mu^{(1)}$$

Statement:

Independent of its non-uniqueness $\mathcal{F}_\mu^{I}$ is different from the one-particle contribution in the sum rule.

Proof:

We consider the simplest case $\Delta = 0$ and m = μ. As shown in the previous subsection, the one-particle contribution to the sum rule is given by the contribution of the $\varphi_{i,\sigma}'$, i.e. we have in terms of $\tilde{\mathcal{F}}_\mu$:

$$\tilde{\mathcal{F}}_{\mu,0} = (2\pi)^{-2} \, (2p+q)_\mu \, \varepsilon(q_0+p_0) \, g(0) \, G(0) \, \delta((q+p)^2-\mu^2)$$

On the other hand $\tilde{\mathcal{F}}_\mu^{(1)}$ is given by

$$\tilde{\mathcal{F}}_\mu^{(1)} = (2\pi)^{-2} (2p+q)_\mu \, \Theta(q_0+p_0) \, g(q^2) \, G(q^2) \, \delta((q+p)^2-\mu^2)$$

i.e. the one-particle contribution to the sum rule only contains the coupling constants but not the full vertex structure of the matrix elements eq. (72).

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Appendix A

A theorem on the support of the JLD-spectral functions

The spectral functions $\varphi_i(u,s)$ have the well known support:

$$\mathcal{D}(u,s) = \left\{ (P \pm u) \in V_+ ,\ \sqrt{s} \geq \mathrm{Max}\left(0, m - \sqrt{(P+u)^2}, m+\mu - \sqrt{(P-u)^2}\right) \right\} \tag{A1}$$

We now decompose φ_i into two parts

$$\varphi_i = \varphi_{i,0} + \varphi_{i,1} \tag{A2}$$

where $\varphi_{i,\,{}^0_1}$ have support in $\mathcal{D}_{{}^0_1}$:

$$\mathcal{D}_0 = \mathcal{D} \cap \mathcal{U}_\varepsilon \quad \text{with} \quad \mathcal{U}_\varepsilon = \left\{ s = (\Delta/2 - u)^2 + \varepsilon,\ |\varepsilon| \to 0 \right\} \tag{A3}$$

$$\mathcal{D}_1 = \mathcal{D} - \mathcal{D}_0 \tag{A4}$$

Assertion: 1.) $\varphi_{i,0} = \chi_i(\Delta^2)\, \delta((\Delta/2 - u)^2 - s)\, \delta(u+p)$

with $\chi_i(\Delta^2) = 0$ if $m > \mu$

2.)

$$\varphi_{i,1}(u,s) = 0 \qquad \text{if} \qquad s \leq (\Delta/2 - u)^2$$

Proof:

The domain $\mathcal{D}_0$ consists of all points (u,s) for which the hyperboloids $(q-u)^2 - s = 0$ are admissable in the sense of Dyson [13] with the subsidary condition $(u,s) \in U_\varepsilon$, i.e. we have for $u \in \mathcal{D}_0$ in the Lorentz-frame P = (a,o,o,o) for fixed $\vec{\Delta}$ and a but arbitrary $\vec{q}$:

$$u_0 + f_\varepsilon(\vec{q}, \vec{\Delta}, a, u) \geq -a + \sqrt{m^2 + \vec{q}^{\,2}} \tag{A5}$$

$$u_0 - f_\varepsilon(\vec{q}, \vec{\Delta}, a, u) \leq a - \sqrt{(m+\mu)^2 + \vec{q}^{\,2}} \tag{A6}$$

with $f_\varepsilon(\vec{q}, \vec{\Delta}, a, u) \equiv \left[\vec{q}^{\,2} + \vec{u}(\vec{\Delta} - 2\vec{q}) + u_0^2 + \mu^2 - a^2 + \varepsilon\right]^{1/2}$

Up to this point we have not yet used the discrete nature of the one-particle hyperboloid

I.

Next we show that $U_\varepsilon \cap \mathcal{D} = 0$ for $m > \mu$:
Consider eq. (A5) and (A6) for one particular $\vec{q}$: $\vec{q} = \vec{\Delta}/2$
Then we obtain

$$u_0 + |u_0| + \frac{1}{2} \frac{\varepsilon}{|u_0|} \geq -a + \sqrt{m^2 + a^2 - \mu^2} > 0 \tag{A7}$$

$$u_0 - |u_0| - \frac{1}{2} \frac{\varepsilon}{|u_0|} \leq a - \sqrt{(m+\mu)^2 + a^2 - \mu^2} < 0 \tag{A8}$$

From (A8) we infer $u_0 < 0$. Therefore (A7) may be fulfilled only for m = μ and $\varepsilon \geq 0$ if $|\varepsilon| \to 0$.

II.

Now we show that for m = μ only the isolated point u = -p is contained in D_0.

For this purpose we must add to (A5), (A6) the discrete nature of the one-particle hyperboloid.

We must distinguish two different cases:

1. Non-overlapping hyperboloids, i.e. $\mathcal{B}_1 = 0$

   if $-a + \sqrt{\mu^2 + \vec{q}^{\,2}} < q_0 \leq -a + \sqrt{4\mu^2 + \vec{q}^{\,2}}$

   for arbitrary $\vec{q}$. According to the spectrum conditions this is the case for $a < 3/2\,\mu$

   Then we have:

   For fixed $\vec{\Delta}$ and a, u must fulfill (A6) and either (A9) or (A10) for every $\vec{q}$.

$$u_0 + f_\varepsilon(\vec{q}, \vec{\Delta}, a, u) = -a + \sqrt{\mu^2 + \vec{q}^{\,2}} \tag{A9}$$

$$u_0 + f_\varepsilon(\vec{q}, \vec{\Delta}, a, u) \geq -a + \sqrt{4\mu^2 + \vec{q}^{\,2}} \tag{A1o}$$

According to I. it is not possible to fulfill (A9) and (A1o) simultaneously. On the other hand (A6) + (A9) have the unique solution $u_0 = -a,\ \vec{u} = 0,\ \varepsilon = 0$.

2. Overlapping hyperboloids ($a > \frac{3}{2}\mu$), i,e. $\mathcal{B}_1 = 0$ if

   $-a + \sqrt{\mu^2 + \vec{q}^{\,2}} < q_0 < -a + \sqrt{4\mu^2 + \vec{q}^{\,2}}$

   only for $|\vec{q}| > |\vec{q}_s|$, where $|\vec{q}_s|$ is the intersection of the hyperboloids $q_0 = -a + \sqrt{\mu^2 + \vec{q}^{\,2}}$ and $q_0 = a - \sqrt{4\mu^2 + \vec{q}^{\,2}}$

   One gets

$$|\vec{q}_s|^2 = \frac{\vec{\Delta}^2}{4} - \frac{3\mu^2}{2}\left(1 - \frac{3\mu^2}{8a^2}\right) \tag{A11}$$

Then we have:

For fixed $\vec{\Delta}$ and a, u must fulfill (A6) for every $\vec{q}$ and

a) $u_0 + f_\varepsilon(\vec{q}, \vec{\Delta}, a, u) \in \mathcal{G} \quad \text{for } |\vec{q}| \leq |\vec{q}_s|$

with $\mathcal{G} = (\mathcal{G}_I \cap \overline{\mathcal{G}}_{II} \cap \mathcal{G}'_{II}) \cup \mathcal{G}_{II}$

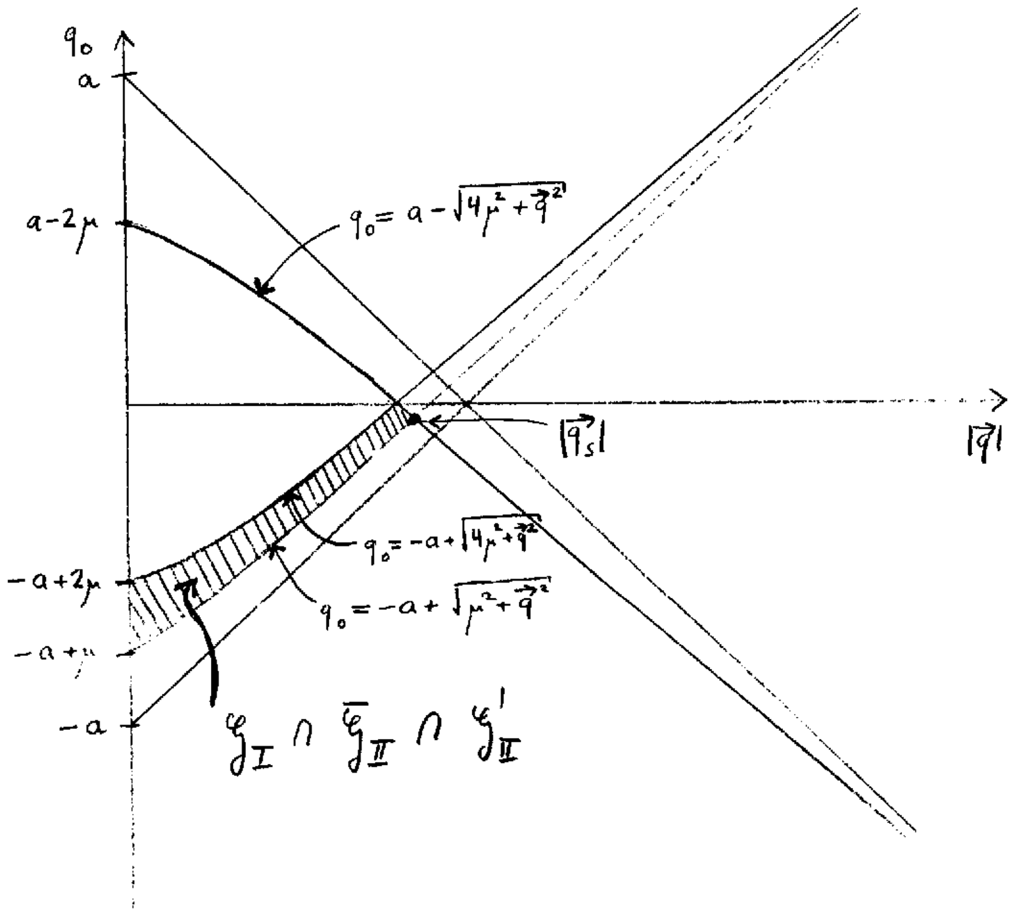
where

$$\mathcal{G}_I(q_0) = \{ q_0 \geq -a + \sqrt{\mu^2 + \vec{q}^2} \}$$

$$\mathcal{G}_{II}(q_0) = \{ q_0 \geq -a + \sqrt{4\mu^2 + \vec{q}^2} \}$$

$$\overline{\mathcal{G}}_{II}(q_0) = \{ q_0 \leq -a + \sqrt{4\mu^2 + \vec{q}^2} \}$$

$$\mathcal{G}'_{II}(q_0) = \{ q_0 \leq a - \sqrt{4\mu^2 + \vec{q}^2} \}$$



b) either (A9) or (A1o) for $|\vec{q}| > |\vec{q}_s|$

Now we have according to a) and b) by using explicitly the form of f_ε :
If for fixed u $(u_0 + f_\varepsilon) \in \mathcal{G} - \{-a + \sqrt{\mu^2 + \vec{q}^2}\}$ for $|\vec{q}| \leq |\vec{q}_s|$, then $(u_0 + f_\varepsilon) \in \mathcal{G}_{II}$ for $|\vec{q}| > |\vec{q}_s|$, in particular for $\vec{q} = \vec{\Delta}/2$ (remember: $|\vec{q}_s|^2 < \Delta^2/4$)!
But this is impossible as shown in I. Therefore, $u_0 + f_\varepsilon$ must fulfill in the overlapping case eq. (A9) too and we have again the unique solution $u_0 = -a$, $\vec{u} = 0$, $\varepsilon = 0$.

III. According to II. the point $s = (\Delta/2 - u)^2$ is an isolated point in D which is accompanied only by the isolated point $u = -p$. Therefore we have the following representation for $\varphi_{i,0}$:

$$\begin{aligned} \varphi_{i,0} &= \mathcal{H}_i(\Delta^2)\, \delta(s - (\Delta/2 - u)^2)\, \delta(u + p) \\ \mathcal{H}_i(\Delta^2) &= 0 \quad \text{if} \quad m > \mu \end{aligned} \tag{A12}$$

Derivatives of the δ-functions in (A12) cannot appear, because the B_i are proportional to $\delta((q+p)^2 - \mu^2)$ on the one-particle mass shell. This representation (A12) for $\varphi_{i,0}$ is unique up to terms, which are different from zero at the point $\{s = (\Delta/2 - u)^2;\ u = -p\}$ only but give a vanishing contribution to B_i.

<u>Corallary:</u>
The hyperboloid $(q-u)^2 - s = 0$ is not admissible for $s < (\Delta/2 - u)^2$ because it is a monoton function of s and the eq. (A7) and (A8) are inconsistent for $\varepsilon < 0$.
With that we have proved the second part of our assertion:

$$\varphi_{i,1}(u, s) = 0 \qquad \text{if} \quad s \leq (\Delta/2 - u)^2 .$$

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