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DERIVATION OF SUM RULES FOR ELECTROMAGNETIC ISOVECTOR FORM FACTORS

by

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Abstract

Starting with equal time commutation relations for the isospin current densities, sum rules for isovector form factors are derived in a manifestly covariant way. Locality is taken into account by means of dispersion relations. For scalar and pseudoscalar particles the most general sum rule is obtained. This sum rule is applied to electromagnetic form factors of mesons and nucleons.

§I. Introduction

Usually it is assumed that the well known commutation relations for the total isospin operator I_α

$$[I_\alpha, I_\beta] = i\,\varepsilon_{\alpha\beta\gamma}\, I_\gamma \tag{1,1}$$

imply that the time component of the isospin current density $j_{0,\alpha}(\vec{x},t)$ related to I_α by

$$I_\alpha = \int d^3x\; j_{0,\alpha}(\vec{x},t) \tag{1,2}$$

obeys equal time commutation relations of the following form [1]

$$[j_{0,\alpha}(\vec{x},t),\, j_{0,\beta}(\vec{x}',t)] = i\,\varepsilon_{\alpha\beta\gamma}\, j_{0,\gamma}(\vec{x},t)\,\delta^{(3)}(\vec{x}-\vec{x}') \tag{1,3}$$

In this paper we want to explore the consequences of these isospin current density commutation relations with respect to electromagnetic form factors of pions and nucleons. A first step in this direction has been made by Balachandran, Kummer and Pietschmann.[2] Taking equ (1,3) between one-nucleon states of momentum p' and p and inserting a complete set of intermediate states on the left hand side of equ (1,3) these authors obtained the isovector form factor of the nucleon as an infinite series of matrix elements of all possible electroproduction processes. This procedure has the disadvantage that the result is inconsistent with locality if the left hand side is approximated by a finite number of intermediate states. For the case p=p' one obtains only trivial results. A different method to study the content of the local commutation relations (1,3) has been pointed out by Adler [3] in connection with high energy neutrino reactions. He considered instead of $j_{0,\alpha}(\vec{x},t)$ the following momentum dependent operators:

$$I_\alpha(\vec{q},t) = \int d^3x\; e^{-i\vec{q}\vec{x}}\; j_{0,\alpha}(\vec{x},t) \tag{1,4}$$

For retarded products of such operators he derived equations in which equal time commutators of the $I_\alpha(\vec{q},t)$ appear. He started from noncovariant matrix elements and not all steps of this derivation can be followed with ease. It leads to nontrivial results also for p=p'. His procedure is similar to the technique used to derive a relation for the axial-vector coupling constant renormalization in β - decay.[4] Presumably the simplest way of deriving this relation is by means of dispersion relations devised by Fubini, Furlan and Rosetti.[5] These authors rederived Adler's relation in a manifestly covariant way. It is desirable to apply this method also in connection with the momentum dependent operators $I_\alpha(\vec{q},t)$ to obtain sum rules for electromagnetic or weak interaction form factors.[6] In this paper we derive sum rules with special reference to electroproduction in a completely covariant manner. We start with a matrix element which is related to the scattering amplitude of virtual photons of isospin one on nucleons. This matrix element is reduced further by applying Gauß theorem in section 2 and the most general sum rule for the case of scalar nucleons is established. Important steps are the application of dispersion relations for the retarded matrix element and the use of crossing relations. In section 3 this sum rule is further exploited for pion form factors. Section 4 contains the analogous derivations for nuleon form factors. Here we do not consider the most general case. We take p=p' and work only with the spin independent part of the matrix element.[7]

§2. Derivation of a General Sum Rule

We start with the equal time commutator of isospin currents in the following form:

$$\langle p'|[j_{0,\alpha}(x), j_{\gamma\beta}(0)]|p\rangle = \delta^{(3)}(\vec{x})\, i\,\varepsilon_{\alpha\beta\gamma}\,\langle p'|j_{0,\gamma}(0)|p\rangle \tag{2,1}$$

Here $j_{\mu,\alpha}(x)$ is the isospin current operator, so chosen that the operator $j_{\mu,\alpha}$ be hermitian $(\alpha=1,2,3)$. This commutator appears if we reduce the following matrix element

$$T_{\mu\nu,\alpha\beta} = -i(2\pi)^3\int d^4x\ \theta(x_0)\ e^{iq'x}\langle p'|[j_{\mu,\alpha}(x), j_{\nu\beta}(0)]|p\rangle \tag{2,2}$$

multiplied with q'_μ by applying Gauß theorem:

$$\begin{aligned}
q'^\mu\, T_{\mu\nu,\alpha\beta} &= -i(2\pi)^3\int d^4x\ \theta(x_0)\left(-i\frac{\partial}{\partial x_\mu}e^{iq'x}\right)\langle p'|[j_{\mu\alpha}(x), j_{\nu\beta}(0)]|p\rangle \\
&= -(2\pi)^3\int d^4x\ \theta(x_0)\frac{\partial}{\partial x_\mu}\left(e^{iq'x}\langle p'|[j_{\mu\alpha}(x), j_{\nu\beta}(0)]|p\rangle\right) \\
&\quad + (2\pi)^3\int d^4x\ \theta(x_0)\ e^{iq'x}\langle p'|[\partial^\mu j_{\mu\alpha}(x), j_{\nu\beta}(0)]|p\rangle \\
&= (2\pi)^3\int d^4x\ \delta(x_0)\ e^{iq'x}\langle p'|[j_{0,\alpha}(x), j_{\nu,\beta}(0)]|p\rangle \\
&\quad + (2\pi)^3\int d^4x\ \theta(x_0)\ e^{iq'x}\langle p'|[\partial^\mu j_{\mu,\alpha}(x), j_{\nu\beta}(0)]|p\rangle
\end{aligned} \tag{2,3}$$

The four vector q' has been chosen with an imaginary part in the forward light cone. Then the contribution at the boundary $x_0\to\infty$ vanishes. The electromagnetic interactions are taken into account in lowest perturbational order only. Then the isospin current can be considered to be conserved, so that

$$\partial^{\mu} j_{\mu,\alpha}(x) = 0 \tag{2,4}$$

and we have

$$q'^{\mu} T_{\mu\nu,\alpha\beta} = (2\pi)^3 \int d^4x\, \delta(x_0)\, e^{iq'x} \langle p' | [j_{0,\alpha}(x), j_{\nu\beta}(0)] | p \rangle \tag{2,5}$$

The right hand side of equ(2,5) can be evaluated with the help of the commutator equ (2,1) by specifying equ (2,5) to $\nu = 0$. This gives us the following sum rule:

$$q'^{\mu} T_{\mu 0,\alpha\beta} = i\, \varepsilon_{\alpha\beta\gamma} \langle p' | j_{0\gamma}(0) | p \rangle (2\pi)^3 \tag{2,6}$$

In the following two sections we shall reduce the left hand side of equ (2,6) for some special examples and shall try to express it by measurable quantities. First we shall consider the case of pions. Let us seperate the isospin part of the pion wave functions in the state vectors $|o\rangle$ and $|p'\rangle$. Then we consider $T_{\mu\nu,\alpha\beta}$ as s matrix in the isospin space of the pions. In this space $T_{\mu\nu,\alpha\beta}$ has the following decomposition:

$$T_{\mu\nu,\alpha\beta} = T^{(1)}_{\mu\nu}\, \delta_{\alpha\beta} + T^{(2)}_{\mu\nu}\, [t_\alpha, t_\beta] + T^{(3)}_{\mu\nu}\, [t_\alpha, t_\beta]_+ \tag{2,7}$$

where t_α $(\alpha = 1,2,3)$ is the isospin operator for the pion. The matrix elements $T^{(i)}_{\mu\nu}$ $(i = 1,2,3)$ may be decomposed into invariant functions. For $|p\rangle, |p'\rangle$ being state vectors of particles with spin zero, the most general form of the matrix elements $T^{(i)}_{\mu\nu}$ for all three $i = 1,2,3$ is(the index i will be omitted) :

$$\begin{aligned} T_{\mu\nu} = {} & T_1(\nu) P_\mu P_\nu + T_2(\nu) Q_\mu Q_\nu + T_3(\nu) K_\mu K_\nu + T_4(\nu) g_{\mu\nu} \\ & + T_5(\nu) P_\mu Q_\nu + T_6(\nu) Q_\mu P_\nu + T_7(\nu) P_\mu K_\nu + T_8(\nu) K_\mu P_\nu \\ & + T_9(\nu) Q_\mu K_\nu + T_{10}(\nu) K_\mu Q_\nu + T_{11}(\nu) N_\mu N_\nu \end{aligned} \tag{2,8}$$

Here we have introduced $q = q' + p' - p$ and used the conventional notation,

$$P = \tfrac{1}{2}(p' + p) \qquad \nu = PQ$$

$$Q = \tfrac{1}{2}(q' + q) \qquad N_\mu = \epsilon_\mu{}^{\nu\rho\sigma} P_\nu Q_\rho K_\sigma$$

$$K = q' - q = p - p'$$

Besides ν the invariant functions T_i depend on $P^2, Q^2, K^2, PK, Q\cdot K$. We have $P^2 + K^2/4 = \frac{1}{2}(p^2 + p'^2)$, $Q^2 + K^2/4 = \frac{1}{2}(q^2 + q'^2)$, $PK = \frac{1}{2}(p'^2 - p^2)$ and $QK = \frac{1}{2}(q^2 - q'^2)$. With this expansion we obtain the following expression for $q'^{\mu} T_{\mu\nu}$:

$$q'^{\mu} T_{\mu\nu} = P_\nu \left\{ (\nu + \tfrac{1}{2} PK) T_1 + (Q^2 + \tfrac{1}{2} QK) T_6 + (QK + \tfrac{1}{2} K^2) T_8 \right\}$$
$$+ Q_\nu \left\{ (Q^2 + \tfrac{1}{2} QK) T_2 + T_4 + (\nu + \tfrac{1}{2} PK) T_5 + (PK + \tfrac{1}{2} K^2) T_{10} \right\} \qquad (2,9)$$
$$+ K_\nu \left\{ (QK + \tfrac{1}{2} K^2) T_3 + \frac{T_4}{2} + (\nu + \tfrac{1}{2} PK) T_7 + (Q^2 + \tfrac{1}{2} QK) T_9 \right\}$$

It will be assumed that the invariant amplitudes $T_j^{(i)}(\nu)$ obey unsubtracted dispersion relations in ν with Q^2, K^2, and QK held constant

$$T_j^{(i)} = \frac{1}{\pi} \int_{-\infty}^{\infty} d\nu' \, \frac{A_j^{(i)}(\nu')}{\nu' - \nu} \qquad (2,10)$$

The absorptive parts $A_j^{(i)}(\nu)$ of $T_j^{(i)}(\nu)$ are determined by the decomposition of $A_{\mu\nu,\alpha\beta}$, where $A_{\mu\nu}$ is the absorptive part of $T_{\mu\nu}$:

$$T_{\mu\nu,\alpha\beta} = i A_{\mu\nu,\alpha\beta} + D_{\mu\nu,\alpha\beta} \qquad (2,11)$$

and $A_{\mu\nu,\alpha\beta}$ has the representation:

$$A_{\mu\nu,\alpha\beta} = - \tfrac{1}{2} (2\pi)^3 \int d^4x \, e^{iq'x} \langle p' | [J_{\mu\alpha}(x), J_{\nu\beta}(0)] | p \rangle \qquad (2,12)$$

Of course the isospin decomposition of $A_{\mu\nu,\alpha\beta}$ is as in equ(2,7) and the invariant functions $A_j^{(i)}(\nu)$ are defined by the expansion equ (2,8).Because of current conservation the amplitudes $A_j^{(i)}(\nu)$ are not all independent. From $q'^{\mu} A_{\mu\nu}^{(i)} = 0$ the following three equations are obtained :

$$(\nu + \tfrac{1}{2} PK) A_1^{(i)} + (Q^2 + \tfrac{1}{2} QK) A_6^{(i)} + (QK + \tfrac{1}{2} K^2) A_8^{(i)} = 0$$
$$(Q^2 + \tfrac{1}{2} QK) A_2^{(i)} + A_4^{(i)} + (\nu + \tfrac{1}{2} PK) A_5^{(i)} + (PK + \tfrac{1}{2} K^2) A_{10}^{(i)} = 0$$
$$(QK + \tfrac{1}{2} K^2) A_3^{(i)} + \tfrac{1}{2} A_4^{(i)} + (\nu + \tfrac{1}{2} PK) A_7^{(i)} + (Q^2 + \tfrac{1}{2} QK) A_9^{(i)} = 0 \qquad (2,13)$$

In the same way three more conditions result from $A_{\mu\nu}^{(i)} q^{\nu} = 0$ which we do not write down because they are not needed in the following discussion. The three conditions in equ (2,13) can be used to reduce $q'^{\mu} T_{\mu\nu}^{(i)}$. We obtain :

$$q'^{\mu} T_{\mu\nu}^{(i)} = - P_\nu \frac{1}{\pi} \int_{-\infty}^{\infty} d\nu' A_1^{(i)}(\nu') - Q_\nu \frac{1}{\pi} \int_{-\infty}^{\infty} d\nu' A_5^{(i)}(\nu')$$
$$- K_\nu \frac{1}{\pi} \int_{-\infty}^{\infty} d\nu' A_7^{(i)}(\nu') \qquad (2,14)$$

Now we show that the coefficients of Q_ν and K_ν in equ (2,14) vanish for $i = 2$ as a consequence of crossing symmetry and PT-invariance.

For the full matrix element crossing symmetry tells us that

$$\left(T_{\mu\nu,\alpha\beta}(p', q'; p, q) \right)^* = T_{\mu\nu,\alpha\beta}(p, -q; p', -q') \qquad (2,15)$$

From this relation the crossing symmetry relations of the invariant amplitudes can be read off:

$$\left(T_j^{(i)}(\nu)\right)^* = T_j^{(i)}(-\nu) \quad \text{for} \quad \begin{array}{ll} i=1,3\,; & j=1,2,3,4,9,10,11 \\ i=2\,; & j=5,6,7,8 \end{array}$$

$$\left(T_j^{(i)}(\nu)\right)^* = -T_j^{(i)}(-\nu) \quad \text{for} \quad \begin{array}{ll} i=1,3\,; & j=5,6,7,8 \\ i=2 & j=1,2,3,4,9,10,11 \end{array} \qquad (2,16)$$

From PT- invariance we deduce that the absorptive part $A_{\mu\nu,\alpha\beta}$ (equ (2,12)) is real. Then it is evident from the crossing relations equ(2.16) that the integrals over $A_5^{(2)}$ and $A_7^{(2)}$ vanish. In the following we shall explore only that part of the sum rule equ (2,6) where the right hand side does not vanish that means on the left hand side we consider only the term which is antisymmetric in α and β. In that part only the amplitudes $A_j^{(2)}(\nu)$ appear in the sum rule. With the result of equ (2,14) the sum rule then has the simple form:

$$i\,\varepsilon_{\alpha\beta\gamma}\,(2\pi)^3\langle p'|\,j_{0,\gamma}(0)\,|p\rangle = -[t_\alpha, t_\beta]\,P_0\,\frac{1}{\pi}\int_{-\infty}^{\infty} d\nu'\,A_1^{(2)}(\nu') \qquad (2,17)$$

We remark that in this derivation only the equal time commutation relation equ (1,3) between the zero components of the current densities was needed and no extension to other components of the current density was necessary.

3. Sum Rule for Electromagnetic Form Factors of Mesons.

The amplitude $A_1^{(2)}(\nu)$ is connected with the antisymmetric isovector part of the amplitude for scattering of virtual photons by pions. In the following we shall exploit the sum rule equ (2,17) by keeping in the integral only terms comming from the pion intermediate state and from the resonant production of vector mesons. Besides the pion the main contribution comes from $\gamma+\pi^0\rightarrow\omega$. The $\gamma+\pi^0\rightarrow\varphi$ contribution can also be calculated but from the experimental limit on $\varphi\rightarrow\pi^0+\gamma$ we see that the φ contributes to equ (2,17) at most one tenth of the ω contribution. The one-particle terms are calculated from

$$\begin{aligned} A_{\mu\nu,\alpha\beta} = -\pi \int d^4n\; \theta(n_0)\; \delta(n^2-\kappa^2) \\ \times\{ (2\pi)^3\langle p'|\,j_{\mu,\alpha}(0)\,|n\rangle (2\pi)^3\langle n|\,j_{\nu,\beta}(0)\,|p\rangle\; \delta(n-p'-q') \\ -(2\pi)^3\langle p'|\,j_{\nu\beta}(0)\,|n\rangle (2\pi)^3\langle n|\,j_{\nu,\beta}(0)\,|p\rangle\; \delta(n-(p-q'))\} \end{aligned} \qquad (3,1)$$

κ^2 is the mass of the intermediate state. The current matrix element of the pion has the following form:

$$(2\pi)^3\langle p'|\,j_{\mu,\alpha}(0)\,|p\rangle = G_\pi\left((p'-p)^2\right)(p'+p)_\mu\, t_\alpha \qquad (3,2)$$

G_π is the electromagnetic form factor of the pion normalized to $G_\pi(0) = 1$ The left hand side of equ (2,17) is

$$i\,\varepsilon_{\alpha\beta\gamma}\,(2\pi)^3 \langle p'|\,j_{\mu,\gamma}(0)\,|p\rangle = G_\pi(k^2)\; 2P_0\,[t_\alpha, t_\beta] \tag{3,4}$$

The single-particle terms are easily calculated from equ (3,1). Since we need only the amplitude $A_1^{(2)}(\nu)$ for the sum rule the other amplitudes $A_j^{(i)}$ are not written down.

$$A_1^{(2)}(\nu) = -2\pi\; G_\pi(q'^2)\; G_\pi(q^2) \left\{ \delta((P+Q)^2 - m_\pi^2)\,\Theta(P_0+Q_0) + \delta((P-Q)^2 - m_\pi^2)\,\Theta(P_0-Q_0) \right\} \tag{3 5}$$

For the ω contribution we need the $\omega\pi\gamma$ vertex for virtual photons. We write it in the following form:

$$(2\pi)^3 \langle p' r'|\,j_{\mu,\alpha}(0)\,|p,\gamma\rangle = \delta_{\alpha\gamma}\; G_{\pi\omega}((p'-p)^2)\; \epsilon_\mu{}^{\nu\rho\sigma} p_\nu\, p'_\rho\, e_\sigma^{r'} \tag{3,6}$$

p' and $e^{r'}$ are momentum and polarization vector of the ω, p is the momentum of the pion. The function $G_{\pi\omega}$ is the electromagnetic $\pi-\omega$ transition form factor. The contribution of the ω to $A_1^{(2)}$ is :

$$A_1^{(2)}(\nu) = -\frac{\pi}{2}\; G_{\pi\omega}(q'^2)\; G_{\pi\omega}(q^2)\left(\tfrac{1}{4}k^2 - Q^2\right)$$
$$\times\left\{ \delta((P+Q)^2 - m_\omega^2)\,\Theta(P_0+Q_0) + \delta((P-Q)^2 - m_\omega^2)\,\Theta(P_0-Q_0)\right\} \tag{3,7}$$

Now we include also the φ meson as intermediate state. For the φ particle contribution we have equ (3,7) with $G_{\pi\omega}$ replaced by $G_{\pi\varphi}$ etc. Under the hypothesis that these three mesons already saturate the sum rule we obtain the following equation between the electromagnetic form factors of π, ω and φ :

$$G_\pi(k^2) = G_\pi(q'^2)\,G_\pi(q^2) + \frac{1}{4}\left(\frac{1}{4}k^2 - Q^2\right)\left\{ G_{\pi\omega}(q'^2)\,G_{\pi\omega}(q^2) + G_{\pi\varphi}(q'^2)\,G_{\pi\varphi}(q^2)\right\} \tag{3,8}$$

A special case of this relation has been derived by Cabibbo and Radicati [8] starting from commutation relations for the electric dipole moment operators. We set q =q' and have from equ (3,8)

$$1 = G_\pi^2(q^2) - \frac{q^2}{4}\left(G_{\pi\omega}^2(q^2) + G_{\pi\varphi}^2(q^2)\right) \tag{3,9}$$

If we take the first derivative of this last equation and set $q^2=0$ we arrive at :

$$\overline{r_\pi^2} = \frac{3}{4}\left(G_{\pi\omega}^2(0) + G_{\pi\varphi}^2(0)\right) \tag{3,10}$$

In the following we neglect the contribution of the φ and replace $G_{\pi\omega}^2$ by the partial width $\Gamma_{\omega\pi\gamma}$ of the decay $\omega \to \pi^0 + \gamma$:

$$\Gamma_{\omega\pi\gamma} = \alpha\,\frac{G_{\pi\omega}^2}{24}\left(\frac{m_\omega^2 - m_\pi^2}{m_\omega}\right)^3 \tag{3,11}$$

the formula of Cabibbo and Radicati is rederived.[8] We insert the experimental number for $\Gamma_{\omega\pi\gamma} = (1.3 + 0.3)$ Mev. Then the result for the pion radius is $(\overline{r_\pi^2})^{1/2} = (0.35 \pm 0.05)\frac{\hbar}{m_\pi c}$ not to far away from a recent experimentally determined value $(\overline{r_\pi^2})^{1/2} = (0.50 \pm 0.14)\frac{\hbar}{m_\pi c}$ [9]

The special case q=q' is advantageous also in other respects. In this case the contribution of the continuum to the integral in equ (2,17) can be expressed by electroproduction cross sections on pions:

$$e + \pi^\alpha \longrightarrow e' + \text{(state of isospin I)}$$

The relation between the continuum part of $A_1^{(2)}(\nu)$ and the longitudinal and transversal electroproduction cross section will be derived in connection with sum rules for nucleons in the next section. We denote by $\sigma_T(q^2, s)$ and $\sigma_L(q^2, s)$ the transversal and longitudinal photoproduction cross section of virtual photons for excitation energy $s = (p+q)^2$. In terms of these cross sections the sum rule equ (2,17) with the one-pion contribution separated is :

$$1 = G_\pi^2(q^2) + \frac{q^2}{8\pi^2\alpha}\int_{9m_\pi^2}^{\infty} ds \frac{1}{\sqrt{(pq)^2 - m_\pi^2 q^2}}$$

$$\times \left\{ \sigma^V(q^2, s)_{\gamma+\pi^0 \to I=0} + \sigma^V(q^2, s)_{\gamma+\pi^+ \to I=1} - \frac{5}{4}\sigma^V(q^2, s)_{\gamma+\pi^0 \to I=2} \right\} \quad (3,12)$$

where $\sigma^V(q^2, s)$ is the isovector part of the sum $\sigma_L(q^2, s) + \sigma_T(q^2, s)$

§4. Sum Rules for Nucleon Electromagnetic Form Factors.

In this section we shall derive a sum rule for the isovector part of the electromagnetic form factor of the nucleon. Again we separate the isospin part of the nucleon wave functions in the state vectors $|p, r\rangle$ and $|p', r'\rangle$ which now decribe protons or neutrons of momentum p and p' and polarization r and r' respectively. The matrix element $T_{\mu\nu,\alpha\beta}$ defined in equ (2,2) is decomposed in the isospin space of nucleons as follows:

$$T_{\mu\nu,\alpha\beta} = T_{\mu\nu}^{(1)} \delta_{\alpha\beta} + T_{\mu\nu}^{(2)} \tfrac{1}{2}[\tau_\alpha, \tau_\beta] \quad (4,1)$$

The two isospin independent amplitutes $T_{\mu\nu}^{(1)}$ and $T_{\mu\nu}^{(2)}$ obey the crossing relations:

$$\left(T_{\mu\nu}^{(1)}(p', r', q'; p, r, q)\right)^* = T_{\mu\nu}^{(1)}(p, r, -q'; p', r', -q)$$

$$\left(T_{\mu\nu}^{(2)}(p', r', q'; p, r, q)\right)^* = -T_{\mu\nu}^{(2)}(p, r, -q'; p', r', -q) \quad (4,2)$$

In this paper we shall not explore the full content of the equal time current commutation relations applied to nucleons. We consider the matrix element $T_{\mu\nu,\alpha\beta}(p'r'q';\, p r q)$ only for the special case $r = r'$. Furthermore we take the average over the polarization r. This way we use only the spin independent part of the scattering amplitude $T_{\mu\nu,\alpha\beta}$. We call this polarization average again $T_{\mu\nu,\alpha\beta}(p'q';\, p q)$. The decomposition of $T^{(a)}_{\mu\nu}(p'q';\, p q)(\alpha \to \nu)$ is given by equ (2,8) written down for matrix elements of a spin zero particle. Then all derivations of section 2 are also valid for this special nucleon matrix element. The sum rule can be read off immediately from equ (2,17):

$$i\,\varepsilon_{\alpha\beta\gamma}\,(2\pi)^3\,\frac{1}{2}\sum_{r}\langle p'r|\,j_{0,\gamma}(0)\,|p r\rangle = -\frac{1}{2}[\tau_\alpha,\tau_\beta]\,P_0\,\frac{1}{\pi}\int_{-\infty}^{\infty} d\omega\, A_1^{(a)}(\omega) \qquad (4,3)$$

Now we further specialize to p=p'. Then the integral on the right hand side of equ (4,3) can be expressed by form factors for elastic and inelastic scattering of electrons on unpolarized nucleons. It is clear that only the isovector part of these form factors contributes to equ (4,3). The term dependent directly on the elastic nucleon form factor is easily established by evaluating the one-nucleon contribution to the absorptive part $A_{\mu\nu}$ (equ (18). The nuleon form factors

$$F_1 = \frac{G_E - {}^{q^2}\!/_{4m^2}\, G_M}{1 - {}^{q^2}\!/_{4m^2}} \quad \text{and} \quad F_2 = \frac{G_M - G_E}{1 - {}^{q^2}\!/_{4m^2}} \qquad (4,4)$$

are defined by (m is the mass of the nucleon) :

$$(2\pi)^3 \langle p'|\,j_{\mu,\gamma}(0)\,|p\rangle = \bar{u}(p')\,\frac{\tau_\gamma}{2}\left\{\gamma_\mu F_1((p-p')^2) + i\sigma_{\mu\nu}(p'-p)^\nu \frac{1}{2m} F_2((p'-p)^2)\right\} u(p) \qquad (4,5)$$

For p = p' the left hand side of equ(4,3) is :

$$i\,\varepsilon_{\alpha\beta\gamma}\,(2\pi)^3 \langle p|\,j_{0,\gamma}(0)\,|p\rangle = i\,\varepsilon_{\alpha\beta\gamma}\,\frac{\tau_\gamma}{2}\,\bar{u}(p)\gamma_0 u(p) = \frac{1}{2}[\tau_\alpha,\tau_\beta]\,p_0 \qquad (4,6)$$

and the one-nucleon contribution to $A_1^{(a)}(\omega)$ is :

$$A_1^{(a)}(\omega) = -\pi\,\frac{G_E - {}^{q^2}\!/_{4m^2}\, G_M}{1 - {}^{q^2}\!/_{4m^2}} \times \left\{\delta((p+q)^2 - m^2)\,\Theta(p_0+q_0) + \delta((p-q)^2 - m^2)\,\Theta(p_0-q_0)\right\} \qquad (4,7)$$

The continuum part in equ(4,3) will be expressed by the inelastic form factors $\sigma_L(q^2, s)$ and $\sigma_T(q^2, s)$ as introduced by Gourdin.[10)] s is the total excitation energy $s = (p+q)^2$.

Although the dependence of the cross section for scattering of electrons by unpolarized nucleons has been derived by many people we repeat it here for completeness and we derive shortly the relation between $A_1^{(a)}$ and σ_L and σ_T respectively.

The cross section for the scattering of electrons by unpolarized protons is calculated from

$$d\sigma = \frac{\alpha^2}{q^4} \frac{1}{\sqrt{(pk)^2 - m_e^2 m^2}} \left(-t_{\mu\nu} A_{\mu\nu}\right) \frac{1}{\pi} \frac{d^3k'}{2k_0'} \qquad (4,8)$$

where k and k' are the momenta of the initial and final electron. m_e stands for the mass of the electron. $t_{\mu\nu}$ is the current tensor of the electron vertex :

$$\begin{aligned} t_{\mu\nu} &= \frac{1}{2} \sum_{r,r'} \tilde{u}(k',r')\gamma_\mu u(k,r)\, \tilde{u}(k,r)\gamma_\nu u(k',r') \\ &= 2\left(k_\mu k_\nu' + k_\mu' k_\nu\right) + 2 g_{\mu\nu}\left(m_e^2 - kk'\right) \end{aligned} \qquad (4,9)$$

and $A_{\mu\nu}$ is the current tensor of the proton vertex:

$$\begin{aligned} A_{\mu\nu} = &-\pi \int ds \int d^4p_n\, \Theta(p_{n0})\, \delta(p+q-p_n)\, \delta(p_n^2 - s) \\ &\times (2\pi)^3 \langle p | j_\mu | p_n \rangle (2\pi)^3 \langle p_n | j_\nu | p \rangle \end{aligned} \qquad (4,10)$$

With the expansion

$$A_{\mu\nu} = a\, p_\mu p_\nu + b\left(p_\mu q_\nu + q_\mu p_\nu\right) + c\, q_\mu q_\nu + d\, g_{\mu\nu} \qquad (4,11)$$

the cross section equ (4.8) is transformed in:

$$d\sigma = \frac{\alpha^2}{q^4} \frac{1}{|p\cdot k|} \frac{1}{\pi}\, 4m^2 k_0 k_0' \cos^2\theta/_2 \left\{ a - \frac{2d}{m^2}\, tg^2\theta/_2 \right\} \frac{d^3k'}{2k_0'} \qquad (4,12)$$

We have set $m_e = 0$ and have evaluated the tensor product in the laboratory frame as usual. Then θ is the electron scattering angle in that frame. The corresponding photoproduction cross section $(q^2 = 0)$ is:

$$\sigma_T = \frac{e^2}{|p\cdot q|} \frac{1}{2}(-d) \qquad (4,13)$$

The transversal cross section σ_T for virtual photons $(q^2 \neq 0)$ is defined accordingly:

$$\sigma_T = \frac{e^2}{\sqrt{(pq)^2 - m^2q^2}} \frac{1}{2}(-d) \qquad (4,14)$$

and is then substituted for d in the electroproduction cross section:

$$\begin{aligned} d\sigma = &\frac{\alpha^2}{(-q^2)} \frac{1}{|p\cdot k|} \frac{1}{2\pi^2} \frac{4m^2 k_0 k_0' \cos^2\theta/_2}{\sqrt{(pq)^2 - m^2 q^2}} \\ &\times \left\{ \sigma_L + \sigma_T + \frac{(pq)^2 - m^2q^2}{-q^2m^2}\, \sigma_T\, 2tg^2\theta/_2 \right\} \end{aligned} \qquad (4,15)$$

The amplitude a has been replaced by

$$\sigma_L + \sigma_T = -\frac{e^2}{2q^2} \sqrt{(pq)^2 - m^2q^2}\; a \qquad (4.16)$$

in accordance with the usual conventions. With these definitions the continuum contribution to $A_1^{(2)}(\nu)$ is:

$$A_1^{(2)}(\nu) = \pi \frac{1}{2\pi^2\alpha} \int_{(m+\sqrt{q^2})^2}^{\infty} ds \frac{q^2}{\sqrt{(pq)^2 - m^2q^2}} \left(\sigma_T^{(2)}(q^2, s) + \sigma_L^{(2)}(q^2, s) \right)$$
$$\times \left\{ \delta((p+q)^2 - s)\, \Theta(p_0+q_0) + \delta((p-q)^2 - s)\, \Theta(p_0 - q_0) \right\} \qquad (4,17)$$

In equ(4,17) the cross sections are also defined with respect to the isospin factor $\frac{1}{2}[\tau_\alpha, \tau_\beta]$. It is convenient however to introduce the cross sections for total isospin I. The conversion is easily done since

$$\sigma^{(2)} = (\sigma)_{1/2} - \frac{1}{2}(\sigma)_{3/2} \qquad (4,18)$$

The vector part of the electroproduction cross section is: $(\sigma)_{1/2} + (\sigma)_{3/2}$

With these definitions the sum rule derived from equ (4,3),(4,7) and(4,17) goes over in its final form:

$$1 = \frac{G_E^{V2} - \frac{q^2}{4m^2} G_M^{V2}}{1 - \frac{q^2}{4m^2}} - \frac{q^2}{2\pi^2\alpha} \int ds \frac{1}{\sqrt{(pq)^2 - m^2q^2}}$$
$$\times \left\{ (\sigma_L^V + \sigma_T^V)_{1/2} - \frac{1}{2}(\sigma_L^V + \sigma_T^V)_{3/2} \right\} \qquad (4,19)$$

This equation agrees with the sum rule as derived by Adler for neutrino reactions [3] specified to electroproduction by Gourdin [7] and Bucella, Veneziano and Gatto.[7]

Let us compute the first q^2 derivative of equ(4,19) and put $q^2=0$ Using

$$G_E^{V\prime}(0) = \frac{1}{6}\left\{ \overline{r_{EP}^2} - \overline{r_{EN}^2} \right\}$$
$$G_M^V(0) = \mu_P - \mu_N \qquad (4,20)$$
$$\sigma_L(0, s) = 0$$

we immediately obtain a sum rule derived in ref. 8 from commutation relations of electric dipole moment operaors:

$$\left(\frac{\mu_P - \mu_N}{2m}\right)^2 - \frac{1}{4m^2} = \frac{1}{3}\left\{ \overline{r_{EP}^2} - \overline{r_{EN}^2} \right\} + \frac{1}{2\pi^2\alpha} \int_{\omega_0}^{\infty} \frac{d\omega}{\omega} \left((\sigma^V(\omega))_{3/2} - \sigma^V(\omega)_{1/2} \right) \qquad (4,21)$$

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