DEUTSCHES ELEKTRONEN-SYNCHROTRON

DESY

DESY 64/2

Februar 1964

Theorie

ON A CLASSICAL STATISTICAL MODEL WITH THE NON-RELATIVISTIC INTEGRALS OF MOTION

by

H. Joos

H. Satz

Deutsches Elektronen-Synchrotron DESY

2 HAMBURG-GR. FLOTTBEK 1, NOTKESTIEG 1

ON A CLASSICAL STATISTICAL MODEL WITH THE NON-RELATIVISTIC INTEGRALS OF MOTION

by

H. Joos

H. Satz

Deutsches Elektronen-Synchrotron DESY

DESY 64/2

Abstract.

The classical non-relativistic phase space integral for fixed energy, momentum, angular momentum and center of mass is evaluated for large particle numbers by means of the central limit theorem of statistics. The problem is treated covariantly with respect to all transformations of the Galilei group. As result we get Ω_s as function of the invariants corresponding to the cms Energy E_o and angular momentum $\vec{L}_o$ in the form

$$\Omega_s(E_o, \vec{L}_o^2) = \Omega_s(E_o)\, F(\vec{L}_o^2, E_o)$$

is the well-known phase space at fixed Energy, momentum and center of mass, and $F(E_o, \vec{L}_o^2) = \left[\frac{3}{4\pi m R^2 E_o}\right]^{3/2} \exp\left(-\frac{3\vec{L}_o^2}{4 m R^2 E_o}\right)$ is a normalized probability density for the angular momentum $\vec{L}_o^2$.

§ 1. Introduction

Since FERMI's proposal [1] of a statistical model for elementary particle reactions, there have been several attempts to improve this approach in various directions [2]. We are particularly interested in a statistical theory which takes fully into account the ten fundamental conservation laws. While there has been some heuristic discussion on the rôle of conservation laws other than energy and momentum for the production cross-section in the statistical model [3], as yet little has been done in a systematic approach and a concise method of evaluation for this problem. In this paper we consider a classical phase-space integral invariant under the full Galilei group and evaluate it with the help of the central limit theorem of statistics [4]. We believe that the consequent treatment of this simplified model will give some insight into the relation between the space-time symmetry group and the statistical method.

As is well-known, a classical mechanical system invariant under the transformations of the Galilei group has the ten integrals of motion corresponding to the conservation of energy T, momentum $\vec{P}$, angular momentum $\vec{M}$ and to the linear time dependence of the center of mass coordinate $\vec{X}$. In the statistical treatment of an s-particle system with a certain set of r conserved quantities $F_i(x_1,\dots,x_s;p_1,\dots,p_s)$, $i=1,\dots,r$, the phase-space integral

$$\Omega_s(A_1,\dots,A_r) = \int\cdots\int \prod_{j=1}^{s} d^3p_j\, d^3x_j\ \delta^{(r)}(F_i - A_i) \tag{1.1}$$

plays as "structure function" a central rôle [5]. It is assumed in this expression that the measure of the hypersurface determined by $F_i(x_1,\ldots,x_s;p_1,\ldots,p_s)=A_i$ is finite; for the well-known case of an ideal gas in a box this condition is guaranteed by the positiveness of the energy and the spatial restrictions of the system. In a Galilei-invariant theory these spatial restrictions can only apply to the relative coordinates, while on the other hand the motion of the center of mass coordinate is in general unbounded. Therefore we have to restrict the integration to that part of the phase-space which corresponds to a fixed value of the center of mass coordinate. Since this coordinate is cyclic, i.e. is linearly time dependent and its conjugate momentum is a constant of motion, the thus restricted phase-space integral becomes time independent. In this sense the phase-space integral

$$\Omega_s(T,\vec{P},\vec{M},\vec{X}) = \int\cdots\int \prod_{j=1}^{s}\{d^3p_j\, d^3x_j\}\Big(e^{-\frac{1}{2sR^2}\sum_{i,j}(\vec{x}_i-\vec{x}_j)^2} \times$$
$$\times\, \delta\Big(\sum_i \frac{p_i^2}{2m}-T\Big)\delta^{(3)}\Big(\sum \vec{p}_i-\vec{P}\Big)\delta^{(3)}\Big(\sum_i \vec{x}_i\times\vec{p}_i-\vec{M}\Big)\delta^{(3)}\Big(\sum\vec{x}_i-s\vec{X}\Big)\Big) \qquad (1.2)$$

takes into account the ten integrals of motion connected with the Galilei group. The cut-off function $\Big(\exp\big[(-1/2sR^2)\sum_{i,j=1}^{s}(\vec{x}_i-\vec{x}_j)^2\big]\Big)$ provides an invariant restriction of the coordinate space, reflecting the idea of the interaction volume in the statistical model. We have chosen this particular form for calculational reasons [6]; one could as well choose any other invariant short range cut-off.

In order to discuss the transformation properties of the structure function $\Omega_s(T,\vec{P},\vec{M},\vec{X})$ we begin in the following section with some preliminaries on Galilei invariance. We evaluate in section III this phase-space integral for large particle numbers s by applying the central limit theorem in a covariant manner.

§ 2. Transformation Properties under the Galilei Group

The canonical coordinates of position and momentum of a single particle of mass m and the time coordinate transform as

$$\begin{aligned} \vec{x}\,' &= R\vec{x} + \vec{v}t + \vec{b} \\ \vec{p}\,' &= R\vec{p} + m\vec{v} \\ t' &= t + b_o \end{aligned} \tag{2.1}$$

upon transition from one inertial system to another. Here R denotes a three-dimensional space rotation, $\vec{v}$ the relative velocity of the two reference systems, $\vec{b}$ a spatial and b_o a time displacement. The transformations $g=(R,\vec{v},\vec{b},b_o)$ with the multiplication rule

$$\begin{aligned} (R,\vec{v},\vec{b},b_o)(R',\vec{v}\,',\vec{b}\,',b_o') &= (RR',\vec{v}+R\vec{v}\,',\vec{b}+b_o\vec{v}\,'+R\vec{b}\,',b_o+b_o') \\ (R,\vec{v},\vec{b},b_o)^{-1} &= (R^{-1},-R^{-1}\vec{v},-R^{-1}(\vec{b}-b_o\vec{v}),-b_o) \\ (1,0,0,0) &= \mathbb{1} \end{aligned} \tag{2.1a}$$

form the Galilei group. It is a ten-dimensional Lie-group which is generated by the infinitesimal rotations $\vec{M}$, accelerations $\vec{N}$, space translations $\vec{P}$ and time translations T, with the commutation relations

$$[M_i, M_j] = \varepsilon_{ijk} M_k \quad , [P_i, P_j] = 0 \, , [N_i, N_j] = 0$$
$$[M_i, N_j] = \varepsilon_{ijk} N_k \quad , [M_i, P_j] = \varepsilon_{ijk} P_k \, , \quad [N_i, T] = P_i \tag{2.2}$$
$$[M_i, T] = 0 \, , \quad [P_i, T] = 0$$

$$[N_i, P_j] = 0 \tag{2.2a}$$

if

$$(R, \vec{v}, \vec{b}, b_0) \rightarrow e^{-b_0 T + \vec{b}\vec{P}} e^{-\vec{v}\vec{N}} e(R) \tag{2.3}$$
$$e(R) = e^{\vec{z}\vec{M}} ,$$

Except for (2.2a) the commutation relations agree with the Poisson brackets of classical mechanics for energy T, linear momentum $\vec{P}$, angular momentum $\vec{M}$ and center of mass motion $\vec{N}=m\vec{X}-\vec{P}t$. These P.B. together with the equation of motion

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} - [T, F]_{PB} \tag{2.4}$$

immediately demonstrate that $A=(T,\vec{P},\vec{M},\vec{N})$ is a constant of motion.

The P.B. corresponding to (2.2a) is given by

$$[N_i, P_j]_{PB} = m\,\delta_{ij} \tag{2.5}$$

where m is the total mass of the system. It is therefore advantageous to extend the system of the C.R. (2.2) to a Lie algebra by introducing an additional infinitesimal generator Q which has the C.R.

$$[N_i, P_j] = Q\,\delta_{ij} \quad , \quad [M_i, Q] = [N_i, Q] = [P_i, Q] = [T, Q] = 0 \tag{2.6}$$

This corresponds to an extension of the Galilei group by phase transformations exp ($Q\delta$) which lie in the center of the extended group ("central extension")[7]. Hence in an irreducible representation of the extended Galilei group Q is a constant multiple of the unit matrix, and by choosing Q=m we get back a representation of the C.R. (2.2) and (2.5).

The adjoint representation of the extended Galilei group leads to transformations for the infinitesimal generators which are the same as those for the integrals of motion $(T,\vec{P},\vec{M},\vec{N},m)=(A_i)$. From the C.R. (2.2), (2.6) the adjoint representation

$$A_i' = g^{-1} A_i g = \sum_k D_{ik}(g) A_k$$

may be explicitely calculated:

$$\begin{aligned}
T' &= T + \vec{v} R \vec{P} + \tfrac{1}{2}\vec{v}^2 Q \\
\vec{P}' &= R\vec{P} + \vec{v} Q \\
\vec{M}' &= R\vec{M} + \vec{b}\times(R\vec{P}) - \vec{v}\times(R\vec{N}) - Q\,\vec{v}\times\vec{b} \\
\vec{N}' &= R\vec{N} - b_0 R\vec{P} + (\vec{b} - b_0\vec{v})Q \\
Q' &= Q
\end{aligned} \tag{2.7}$$

where $(A_i)=(T,\vec{P},\vec{M},\vec{N},Q)$ here denotes the infinitesimal generators. It is easily verified that the integrals of motion A_i, in the one-particle case

$$\begin{aligned}
T &= \frac{1}{2m}\vec{p}^2 & \vec{M} &= \vec{x}\times\vec{p} \\
\vec{P} &= \vec{p} & \vec{N} &= m\vec{x} - \vec{p}t
\end{aligned} \tag{2.8}$$

transform in the same way as the infinitesimal generators, provided the canonical coordinates and time transform as in

(2.1) and Q is put equal to m.

From the A_i one can construct three independent invariants[8] under the extended Galilei group:

$$E = QT - \frac{1}{2}\vec{P}^2 = QE_o \tag{2.9a}$$

$$\vec{L}^2 = (Q\vec{M} + \vec{P}\times\vec{N})^2 = Q^2\vec{L}_o^2 \tag{2.9b}$$

$$Q \tag{2.9c}$$

which have the physical meaning of CMS total energy E_o, total angular momentum $\vec{L}_o^2$, and total mass Q.

With these remarks we can establish the invariance of our phase-space integral (1.2). Consider for this purpose the integral

$$\begin{aligned}\bar{\Omega}_t(T,\vec{P},\vec{M},\vec{N}) = m^3\int\cdots\int\Big(\prod_{i=1}^{s} d^3p_i d^3x_i\Big)\, e^{-\frac{1}{2sR^2}\sum_{i,j}(x_i-x_j)^2}\,\delta\Big(\sum_i \frac{p_i^2}{2m} - T\Big)\times \\ \times\,\delta^{(3)}\Big(\sum_i \vec{p}_i - \vec{P}\Big)\,\delta^{(3)}\Big(\sum_i \vec{x}_i\times\vec{p}_i - \vec{M}\Big)\,\delta^{(3)}\Big(\sum(m\vec{x}_i - \vec{p}_i t) - \vec{N}\Big)\end{aligned} \tag{2.10}$$

For $\vec{N}=sm\vec{X}-\vec{P}t$ this integral is equal to (1.2). In the form (2.10) the spase-space integral can easily be shown to be Galilei-invariant. If in $\bar{\Omega}$ we transform A according to (2.7) with Q=sm, then this transformation can be compensated by a corresponding transformation in the integration variables, since the Jacobian is unity and the cut-off invariant. In particular, $\bar{\Omega}_t$ is time independent: $\bar{\Omega}_t=\bar{\Omega}_o$. As a Galilei invariant function $\bar{\Omega}_o(T,\vec{P},\vec{M},\vec{N})=\Omega(T,\vec{P},\vec{M},\vec{X})$ is a function only of the basic invariants: $\Omega(E,\vec{L}^2,Q)$.

In order to evaluate our phase-space integral in a covariant manner we have to include the transformations contragredient to the adjoint representation (2.7) in our discussion of invariance properties. Let $\xi = (\tau, \vec{\pi}, \vec{\mu}, \vec{\nu}, \omega)$ be a contravariant vector; then it transforms under the extended Galilei group as:

$$\begin{aligned}
\tau' &= \tau \\
\vec{\pi}' &= R\vec{\pi} - \vec{v}\tau + \vec{b}\times(R\vec{\mu}) + b_0 R\vec{\nu} \\
\vec{\mu}' &= R\vec{\mu} \\
\vec{\nu}' &= R\vec{\nu} + \vec{v}\times(R\vec{\mu}) \\
\omega' &= \omega - \vec{v}R\vec{\pi} + \frac{\vec{v}^2}{2}\tau - \vec{b}R\vec{\nu} - \vec{b}(\vec{v}\times R\vec{\mu})
\end{aligned} \tag{2.11}$$

By construction

$$(\xi, A) = \tau T + \vec{\pi}\vec{P} + \vec{\mu}\vec{M} + \vec{\nu}\vec{N} + \omega Q \tag{2.12}$$

is an invariant. In place of (2.9) we now have the invariants τ, $\vec{\mu}^2$ and $\vec{\mu}\vec{\nu}$.

Finally we shall need the invariants of the A and ξ under the "homogenous Galilei group" $(R,\vec{v},0,0)$ [9]. For the A these are

$$Q,\ E,\ \vec{L}^2,\ \vec{N}^2,\ \vec{M}\vec{N} \tag{2.13}$$

and for ξ

$$\tau,\ \vec{\mu}^2,\ \vec{\mu}\vec{\nu},\ (\vec{\pi}^2 - 2\omega\tau),\ (\tau\vec{\nu} + \vec{\mu}\times\vec{\pi})^2 \tag{2.14}$$

§ 3. The Application of the Central Limit Theorem.

In the following we want to evaluate the phase-space integral (1.2) resp. (2.10) in the limit of large particle numbers s. Due to the center of mass δ-function we may replace the exponent of the cut-off $\sum_{i,j}(x_i - x_j)^2$ by $2(s\sum x_i^2 - X^2)$ and thus obtain

$$\Omega_s(T,\vec{P},\vec{M},\vec{N},Q) = m^{-3(s-1)}\, e^{\vec{N}^2/sB^2}\, \bar{\Omega}_s(T,\vec{P},\vec{M},\vec{N},Q)$$

$$\bar{\Omega}_s(T,\vec{P},\vec{M},\vec{N},Q)\,\delta(Q - sm) = \tag{3.1}$$

$$= \int\!\!\cdots\!\!\int \prod_i (d^3p_i\, d^3n_i\, dq_i\, e^{-\vec{n}_i^2/B^2}\delta(q_i - m))\Big(\delta(\sum_i \frac{p_i^2}{2m} - T)\,\delta^{(3)}(\sum_i \vec{p}_i - \vec{P})\,\delta^{(3)}(\frac{1}{m}\sum_i \vec{n}_i\times\vec{p}_i - \vec{M})\times$$

$$\times\, \delta^{(3)}(\sum_i \vec{n}_i - \vec{N})\,\delta(\sum_i q_i - Q)\Big)$$

$$B = mR.$$

where we have added the mass δ-function to facilitate invariance considerations. As the cut-off function in the integrand now factorizes, the integral may be evaluated with help of the central limit theorem following the method of Khinchin[5] and its higher dimensional generalizations of Lurçat and Mazur[10].

In line with this method we introduce the generating function of the system defined as

$$\Phi_s(\xi) = \int \cdots \int d^{(11)}A \, e^{-(A,\xi)} \, \bar{\Omega}_s(A) \tag{3.2}$$

$$A = (T, \vec{P}, \vec{M}, \vec{N}, Q) ; \qquad \xi = (\tau, \vec{\pi}, \vec{\mu}, \vec{\nu}, \omega)$$

As an immediate consequence of the form of (3.1), $\Phi_s(\xi)$ factorizes into single particle generating functions

$$\Phi_s(\xi) = [\varphi(\xi)]^s \tag{3.3}$$

with

$$\varphi(\xi) = \int \cdots \int d^{(11)}a \, e^{-(a,\xi)} \, \varrho(a)$$

$$\varrho(a) = \delta^{(3)}(\vec{e} - \tfrac{1}{m}\vec{n} \times \vec{p}) \, \delta(\omega - \tfrac{\vec{p}^2}{2m}) \, \delta(q - m) \, e^{-\vec{n}^2/B^2} \tag{3.4}$$

$$a = (\omega, \vec{p}, \vec{e}, \vec{n}, q)$$

Now we can define the function

$$U_s^{\xi}(A) = \frac{1}{\Phi_s(\xi)} \, e^{-(A,\xi)} \, \bar{\Omega}_s(A) \, \delta(Q - sm) \tag{3.5}$$

From (3.1) and (3.3) we get

$$U_s^{\xi}(A) = \int \cdots \int \prod_{i=1}^{s} \left[d^{(11)}a_i \, \frac{e^{-(a_i,\xi)}}{\varphi(\xi)} \, \varrho(a_i) \right] \cdot \delta^{(11)}\left(\sum_{i=1}^{s} a_i - A\right) \tag{3.6}$$

which is positive and normalized to unity; hence it can be interpreted as probability density (frequency function) for the sum $A=\sum_{i=1}^{s} a_i$ of s independent eleven-dimensional random vector quantities with individual probability densities

$$u_{\xi}(a) = \frac{1}{\varphi(\xi)} e^{-(a\,\xi)} \varsigma(a) \tag{3.7}$$

Therefore we may evaluate (3.6) by applying the central limit theorem of statistics. Since from (3.3) and (3.5) follows

$$\bar{\Omega}_s(A)\,\delta(Q-sm) = e^{(A,\xi)} [\varphi(\xi)]^s\, \mathcal{U}_s^{\xi}(A) \tag{3.8}$$

we thus obtain an approximation for the phase-space integral (3.1) in the limit of large s.

For the leading term in s the central limit theorem[11] gives

$$\mathcal{U}_s^{\xi}(A) = \left[(2\pi)^5 \sqrt{\det \tilde{B}_{\mu\nu}}\right]^{-1} e^{-\frac{1}{2}\sum_{\mu,\nu=1}^{10} \tilde{B}_{\mu\nu}^{-1} [A_\mu - \bar{A}_\mu(\xi)][A_\nu - \bar{A}_\nu(\xi)]}\, \delta(Q-sm) \tag{3.9}$$

where the mean value is given by

$$\bar{A}_\mu(\xi) = \int\cdots\int d^{(10)}A \cdot A_\mu\, \mathcal{U}_s^{\xi}(A) = -\frac{\partial \log \bar{\Phi}_s(\xi)}{\partial \xi_\mu} = -s\,\frac{\partial \log \varphi(\xi)}{\partial \xi_\mu} \tag{3.10}$$

and the reduced dispersion matrix by

$$\tilde{B}_{\mu\nu}(\xi) = \int\cdots\int d^{(10)}A\, (A_\mu - \bar{A}_\mu(\xi))(A_\nu - \bar{A}_\nu(\xi))\, \mathcal{U}_s^{\xi}(A)$$

$$= \frac{\partial^2}{\partial \xi_\mu \partial \xi_\nu} \log \bar{\Phi}_s(\xi) = s\, \frac{\partial^2 \log \varphi(\xi)}{\partial \xi_\mu \partial \xi_\nu} \tag{3.11}$$

As the dispersion of q with the distribution $u_\xi(a)$ is zero, the distribution of Q becomes a δ-function and hence in the Gaussian there appears only a reduced ten-dimensional dispersion matrix ($\mu,\nu=1,\dots,10$).

From this short sketch of the "Khinchin Method" we see that the essential point is the replacement of the non-normalizable ϱ(a) by the normalizable probability density u_ξ(a), where ξ is arbitrary in the range of definition of the Laplace transform of ϱ(a). The expression (3.8) for the phase-space integral is independent of these provided $U_s^\xi(A)$ is the exact probability density of the A. With the approximative expression (3.9) this is only true up to terms of the neglected order in s.

The expression for $\overline{\Omega}_s(A)$ according to (3.8) - (3.11) is determined by $\varphi(\xi)$, which with (3.4) can be written as

$$\varphi(\xi) = e^{-\omega m}\int\!\dots\!\int d^3\vec{p}\, d^3\vec{n}\; e^{-\tau\frac{\vec{p}^2}{2m} - \vec{\pi}\vec{p} - \frac{\vec{\mu}}{m}(\vec{n}\times\vec{p}) - \vec{\nu}\vec{n} - \frac{\vec{n}^2}{B^2}} \qquad (3.12)$$

We evaluate this integral with help of the formula[12)]

$$\int\!\dots\!\int d^{(r)}z\; e^{-z^T\Gamma z - \eta^T z} = \frac{\pi^{r/2}}{(\det\Gamma)^{1/2}}\, e^{\frac{1}{4}\eta^T\Gamma^{-1}\eta} \qquad (3.13)$$

$$z^T = (z_1,\dots,z_r)\;;\quad \eta^T = (\eta_1,\dots,\eta_r)\;;\quad \Gamma = (\Gamma_{ik}),\; i,k=1,\dots,r$$

In our case we have $z^T=(\vec{p},\vec{n})$, $\eta^T=(\vec{\pi},\vec{\nu})$ and Γ can be written as a hyper-matrix

$$\Gamma = \begin{bmatrix} \tau/2m & 1/2m\cdot\vec{\mu}\,\vec{S} \\ -\frac{1}{2m}\vec{\mu}\,\vec{S} & \frac{1}{B^2} \end{bmatrix} \qquad (3.14)$$

with

$$S_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad S_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The following formulas simplify calculations:

$$(\vec{X}\vec{S})\vec{Y} = \vec{X} \times \vec{Y} \tag{3.15a}$$

$$(\vec{X}\vec{S})^3 = -\vec{X}^2(\vec{X}\vec{S}) \tag{3.15b}$$

$$\left[C + (\vec{S}\vec{X})^2\right]^{-1} = \frac{1}{C}\left(1 + \frac{1}{\vec{X}^2 - C}(\vec{S}\vec{X})^2\right) \tag{3.15c}$$

With these we get

$$\wp(\xi) = \tag{3.16}$$

$$\frac{(2m\pi)^3}{\sqrt{\frac{2m\tau}{B^2}\left(\frac{2m\tau}{B^2} - \vec{\mu}^2\right)}} \exp\left(\frac{m}{2\tau}(\vec{\pi}^2 - 2\tau\omega) - \frac{m(\tau\vec{\nu} + \vec{\mu}\times\vec{\pi})^2}{2\tau(\vec{\mu}^2 - 2m\tau/B^2)} + \frac{B^2(\vec{\mu}\vec{\nu})^2}{4(\vec{\mu}^2 - 2m\tau/B^2)}\right)$$

The Laplace transform (3.13) of $\rho(a)$ is defined for positive-definite Γ, which is the case if and only if

$$0 \leq \frac{B^2\vec{\mu}^2}{2m} < \tau \tag{3.17}$$

Therefore this inequality fixes the range of definition of $\wp(\xi)$.

Let us consider now the transformation properties of $\rho(a)$ and $\wp(\xi)$. By removing in (3.1) the factor $\exp(\vec{N}^2/sB^2)$ from the Galilei-invariant phase-space integral $\Omega_s(T,\vec{P},\vec{M},\vec{N},Q)$, we obtain an $\bar{\Omega}_s(T,\vec{P},\vec{M},\vec{N},Q)$ no longer invariant under the full group. As discussed in section 2, $\vec{N}^2$ is however invariant under the "homogeneous Galilei group"; therefore the same holds for $\bar{\Omega}_s(T,\vec{P},\vec{M},\vec{N},Q)$ and for $\rho(a)$. By treating ξ_μ as a vector transforming contravariant to a_μ we obtain for the Laplace transform $\varphi(\xi)$ of $\varsigma(a)$ the same transformation properties as those of $\varsigma(a)$. In fact we see from (2.14) and (3.16) that $\varphi(\xi)$ depends only on the invariants (2.14) of the homogeneous Galilei group.

With $\varphi(\xi)$ now explicitely given, we want to evaluate Ω_s in the limit of large s. For this we have to determine a particular ξ, which satisfies condition (3.17). We will not alter the transformation properties of the approximation for Ω_s if we choose as ξ_μ a function of A_μ transforming covariantly under the full Galilei group:

$$\xi_\mu = \alpha \frac{\partial E}{\partial A_\mu} + \beta \frac{\partial \vec{L}^2}{\partial A_\mu} + \gamma \frac{\partial Q}{\partial A_\mu} \qquad (3.18)$$

where α, β, γ depend only on the invariants $E, \vec{L}^2, Q$. Such a covariant choice which for all A_μ is in the region of definition (3.17) is given by

$$(\bar{\xi}_\mu) = \frac{3s}{2E}\left(\frac{\partial E}{\partial A_\mu}\right) = \frac{3s}{2E}(Q, -\vec{P}, 0, 0, T) \qquad (3.19)$$

With this ξ we get from (3.10) the first moments $\bar{A}_\mu(\bar{\xi})$:

$$(\bar{A}_\mu(\bar{\xi})) = \left(-s \frac{\partial \log \varphi}{\partial \xi_\mu}\Big|_{\xi_\mu = \bar{\xi}_\mu}\right) = (T, \vec{P}, 0, 0, Q) \qquad (3.20)$$

In the expression (3.19) for $\bar{\xi}$ we have chosen $\alpha = 3s/2E$ in order to get $\bar{A}_\mu(\bar{\xi}_\mu) = A_\mu$ for $\mu = 1, \dots, 4$. With this we partially follow a procedure of Khinchin[4] who in a simpler case can choose ξ such as to make $\bar{A}_\mu(\bar{\xi}) = A_\mu$ for all μ. For the reduced dispersion matrix (3.11) we get with (3.16) and (3.19)

$$\left[\tilde{B}_{\mu\nu}(\bar{\xi})\right] = \begin{bmatrix} \tilde{B}^{(1)}(\bar{\xi}) & \sigma \\ \sigma & \tilde{B}^{(2)}(\bar{\xi}) \end{bmatrix} \tag{3.21}$$

$$\tilde{B}^{(1)}(\bar{\xi}) = \frac{2E}{3sQ^2}\begin{bmatrix} \vec{P}^2 + E & , Q\vec{P} \\ Q\vec{P} & , Q^2 \cdot 1_3 \end{bmatrix} \; ; \; \tilde{B}^{(2)}(\bar{\xi}) = \frac{sB^2}{Q}\begin{bmatrix} \frac{2}{3}\frac{E}{Q} - \frac{(\vec{P}\vec{S})^2}{2Q} & , -\frac{1}{2}\vec{P}\vec{S} \\ \frac{1}{2}\vec{P}\vec{S} & , \frac{Q}{2} \end{bmatrix}$$

where we have used the same notation as in (3.14) and (3.15). With the help of the formula (3.15) it is a straight-forward calculation to determine

$$\det(\tilde{B}_{\mu\nu}) = \frac{2^4}{3^7}\,\frac{s^2 B^{12} E^{-8}}{Q^2} \tag{3.22}$$

and

$$\frac{1}{2}\sum_{\mu,\nu}^{10} \tilde{B}^{-1}_{\mu\nu}(A_\mu - \bar{A}_\mu(\bar{\xi}))(A_\nu - \bar{A}_\nu(\bar{\xi})) = \frac{3}{4s}\,\frac{\vec{L}^2}{EB^2} + \frac{\vec{N}^2}{sB^2} \tag{3.23}$$

Inserting these results in the expression (3.8) we get for Ω_s:

$$\Omega^a_s(E_o, \vec{L}_o^2) = \Omega^a_s(E_o)\,F(\vec{L}_o^2, E_o) \tag{3.24}$$

with

$$\Omega^a_s(E_o) = e\sqrt{\frac{\frac{3}{2}se}{2\pi}}\left(\frac{e}{\frac{3}{2}s}\right)^{\frac{3}{2}(s-1)} \frac{(2\pi^2 R^2 m E_o)^{\frac{3}{2}(s-1)}}{s^3 E_o} \tag{3.24a}$$

and

$$F(E_o, \vec{L}_o^2) = \left(\frac{3}{4\pi R^2 E_o}\right)^{3/2} e^{-\frac{3}{4mR^2E_o}\vec{L}_o^2} \tag{3.24b}$$

as the desired approximation of the Galilei invariant phase-space integral for large s.

§ 4. Conclusions

With formula (3.24) we have in our model obtained an expression for the phase-space integral which has quite the expected form:

(a) By the covariant choice of ξ our approximation $\Omega_s^a(E_0,\vec{L}_0)$ of $\Omega_s(E_0,\vec{L}_0^2)$ became explicitly covariant under the full Galilei group. (b) The expression for $\Omega_s^a(E_0,\vec{L}_0^2)$ factorizes[13] covariantly in $\Omega_s^a(E_0)$ and an angular momentum factor $F(E_0, \vec{L}_0^2)$. Here $\Omega_s^a(E_0)$ is the asymptotic form of the well-known non-relativistic phase-space integral at constant energy, momentum, center of mass, and with a Gaussian cut-off. This can easily be seen from the exact form[14]

$$\Omega_s(E_o) = \frac{1}{\Gamma(\frac{3}{2}(s-1))} \cdot \frac{\left[2\pi^2 R^2 m E_o\right]^{\frac{3}{2}(s-1)}}{s^3 E_o}$$

by applying Stirling's formula to the Γ-function.

(c) The function $F(E_0,\vec{L}_0^2)$ is normalized for all E_0:

$$\int d^3\vec{L}_o \, F(E_o,\vec{L}_o^2) = 1$$

and thus the probability density of the angular momentum $\vec{L}_0$. Its dispersion σ is

$$\sigma = \frac{2}{3} m R^2 E_o$$

and hence at a given energy the probability for an $\vec{L}_0^2 \gg \sigma$ is very small. For $\vec{L}_0^2$ fixed, $F(E_0,\vec{L}_0^2)$ as a function of E_0 has a maximum at

$$\hat{E}_o = \vec{L}_o^2 / 2mR^2$$

In our model the momentum mean square value of a particle is $\overline{\vec{p}^2}=2mE_0/s$, the mean square value of its distance from the center of mass $\overline{\vec{x}^2}=3R^2/2$ and that of the sine of the angle Θ between $\vec{p}$ and $\vec{x}$ is $\overline{\sin^2\Theta}=2/3$. Hence the probability that $\vec{L}_0^2$ takes on a particular value $\vec{\Lambda}_0^2$ is a maximum if the energy E_0 is such that $\vec{\Lambda}_0^2=2mR^2E_0=s\overline{(\vec{x}\times\vec{p})^2}$, i.e. if $\vec{\Lambda}_0^2$ is s times the single particle mean square angular momentum.

Keeping in mind that our model does not include relativistic and quantum effects, we nevertheless believe that it gives the essential features of phase-space integrals with fixed angular momentum in addition to energy and momentum conservation.[15)]

References:

1.) E. Fermi, Progr. theor. Phys. (Japan) 1, 510 (1950)

2.) For literature see: M. Kretzschmar, Ann. Rev. Nucl. Sc. 11, 1 (1961)

3.) Z. Koba, Nuov. 18, 608 (1961)
T. Ericson, Nuov. 21, 605 (1961)
F. Cerulus, Nuov. 22, 958 (1961)
H. Satz, Fortschr. 11, 445 (1963)
M. Neumann, An. acad. brasil. cienc. 31, 361, 487 (1959)

4.) A. I. Khinchin, Mathematical Foundations of Statistical Mechanics, Dover Publications (1949)
F. Lurçat, P. Mazur: Statistical Mechanical Evaluation of Phase Space Integrals (Preprint of Institut des Hautes Etudes Scientifique, Bures-sur-Yvette 1963)

5.) A. I. Khinchin, loc. cit.

6.) F. Cerulus, loc. cit.

7.) V. Bargmann, Ann. Math. 59, 1 (1954)
J. M. Levy-Leblond, Journ. of Math. Phys. 4, 776 (1963)

8.) M. Hamermesh, Ann. Phys. 9, 516 (1960)
Loinger, Ann. Phys. 20, 132 (1962)

9.) The homogeneous Galilei group is the non-relativistic limit of the homogeneous Zerenk group.

10) F. Lurçat, P. Mazur, loc. cit.

11) F. Lurçat, P. Mazur, loc. cit.

12) H. Cramér, Mathematical Methods of Statistics, p. 118, Princeton University Press, 9, printing 1961.

13) T. Ericson, loc. cit.
H. Satz, loc. cit.

14) Milburn, R. H. Revs. Mod. Phys. 27, 1, 1955

15) H. Satz, loc. cit.