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Some Kinematics of Inelastic High-Energy Electron Scattering

I. Introduction

Bernstein (CEA-31) has assembled useful formulas for the elastic scattering of high energy particles with zero rest energy. However, many collisions of such particles will be inelastic with the production of π-mesons, K-mesons and strange particles. This report is written to supplement that of Bernstein's. The assumption that the rest mass of the incident particle (electron) is zero is used here also; the error due to this approximation is negligible in the GeV region. The formulas are exact for gamma rays.

II. Maximum Energies of Particles Produced in Inelastic Collisions

Consider a light particle of energy E_2' striking a nucleon of energy $E_1' = m_1$ (see Table A for the details of the notation used.). From the collision more than two particles may emerge; however, a given particle produced in the collision will have a maximum energy only if a minimum number of particles is produced in the reaction. In the case of π-mesons production, this minimum number is two ($e + N \rightarrow N + \pi$), as it is also for strange particle production where no anti-particles are produced. Since an anti-particle can be produced only together with a particle, for anti-particle production the minimum number is three (example $e + N \rightarrow N + \Lambda + \bar{\Lambda}$).

Table A - Notation

E	=	Total energy of a particle, including rest energy
T	=	Kinetic energy of a particle
m	=	Rest energy of particle
p	=	Momentum of particle
$(\; ')$	=	Primes denote quantities in Laboratory system; unprimed quantities refer to center-of-momentum-system
$(\; _1)$	=	Sub-one denotes the target particle
$(\; _2)$	=	Sub-two denotes the incident particle (electron or γ)
$(\; _3)$	=	Sub-three denotes the particle of particular interest emerging from the collision
$(\; _4)$	=	Sub-four denotes the other particle(s) emerging from the collision
C	=	Velocity of light, taken as unity so that $E^2 = p^2 + m^2$
W	=	$E_1 + E_2$ = total energy before collision in center-of-momentum system
$\beta = v$	=	velocity of center-of-momentum system referred to the laboratory system
γ	=	$(1 - \beta^2)^{-1/2}$

It is easy to show that if three particles emerge from the collision, the condition that one of them have a maximum energy is that the other two particles move together. The proof follows. In the center-of-momentum system, calling the particles "3", "4" and "5", we have

$$E_3 = W - (E_4 - E_6) \qquad \text{Conservation of energy} \qquad (1)$$

$$p_3 = p_4 - p_5 \qquad \text{Conservation of momentum} \qquad (2)$$

It is obvious, that, for the maximum energy E_3, all particles must move along the same straight line, so that (2) can be taken as a scalar equation. Using the relativistic expression $E = \sqrt{p^2 + m^2}$, (1) becomes

$$E_3 = W - \left(\sqrt{p_4^2 + m_4^2} + \sqrt{p_5^2 + m_5^2}\right) = W - \left(\sqrt{p_4^2 + m_4^2} + \sqrt{(p_3 - p_4)^2 + m_5^2}\right) \qquad (3)$$

so
$$-\frac{dE_3}{dp_4} = \frac{p_4^2}{\sqrt{p_4^2 + m_4^2}} + \frac{p_3 - p_4}{\sqrt{(p_3 - p_4)^2 + m_5^2}} \left(\frac{dp_3}{dp_4} - 1\right) \qquad (4)$$

And for a given particle "3", when E_3 is a maximum, p_3 is also a maximu, so setting $\frac{dE_3}{dp_4} = \frac{dp_3}{dp_4} = 0$ in (4) and simplifying

$$p_4^2 m_5^2 = m_4^2 (p_3 - p_4)^2 = m_4^2 p_5^2$$

or
$$\frac{p_4}{p_5} = \frac{m_4}{m_5} \qquad (5)$$

which implies that $\beta_4 = \beta_5$ and the particles move together. Hence for the maximum energy calculation "4" and "5" may be considered as a single particle having the total mass $m_4 + m_5$. From now on, m_4 will represent this total mass.

The equations for transforming to the center-of-momentum system are identical with those in the elastic case and may be written

$$E_i = \gamma\,(E'_i - \beta p'_i); \qquad p_i = \gamma\,(p'_i - \beta E'_i)$$

and (6)

$$E_i = \gamma\,(E'_i - \beta p'_i); \qquad p'_i = \gamma\,(p_i + \beta E_i)$$

where

$$\beta = \frac{p_2}{m_1 + E'_2}\ ; \qquad \gamma = \frac{m_1 + E'_2}{W}\ ; \qquad W^2 = m_1^2 + m_2^2 + 2\,m_1 E'_2$$

From these

$$E_1 = \frac{m_1(m_1 + E'_2)}{W}\ ; \qquad E_2 = \frac{m_2^2 + m_1 E'_2}{W}\ ; \qquad p = \frac{m_1 p'_2}{W} \qquad (7)$$

where p is the common center-of-momentum momentum of "1" and"2" before collision. After collision, the momentum of 1 must equal that of 2, but in the inelastic case, this momentum, $\underline{p}$, need not be the same as p.

Using (7), writing conservation of energy and momentum in the center-of-momentum system, assuming equation (5) and solving for the maximum energy of particle "3" gives

$$E_3 = \frac{W^2 + m_3^2 - m_4^2}{2\,W}\ ; \qquad E_4 = \frac{W^2 + m_4^2 - m_3^2}{2\,W}$$

(8)

$$\underline{p}^2 = \left(\frac{W^2 + m_3^2 - m_4^2}{2\,W}\right) - m_3^2 = \left(\frac{W^2 + m_4^2 - m_3^2}{2\,W}\right) - m_4^2$$

and transforming with (6) back to the laboratory system

$$E'_3 = \gamma\left(\frac{W^2 + m_3^2 - m_4^2}{2\,W} + \beta\underline{p}\right)$$

$$= \frac{m_1 + E'_2}{W}\left(\frac{W^2 + m_3^2 - m_4^2}{2\,W} + \frac{p'_2}{m_1 + E'_2}\sqrt{\left(\frac{W^2 + m_3^2 - m_4^2}{2\,W} - m_3^2\right)}\right) \qquad (9)$$

For $E_4{}'$, one need simply interchange the subscripts 3 and 4.

The interastions of interest are those involving an electron or gamma ray striking a nucleon N with the production of some other particles. Because of the rule of associated production of hyperons with K-particles, not all energetically possible reactions can occur. Table B lists most of the reactions which are observed or which are believed possible. Following the example of Bethe and De Hoffmann, the cascade particle is written as Ω, rather than the more usual, but scarcely printable Ξ.

Threshold Production: When the incoming particle has just enough energy to produce a certain reaction, the center-of-momentum momentum after the collision is zero. This condition with equations (8) and (6) gives for the threshold of a reaction

$$E'_{2\,Th} = \frac{(m_3 + m_4)^2 - m_1^2 - m_2^2}{2 m_1} \tag{10}$$

It is interesting to note that in this case $E_3 = m_3$, $E_4 = m_4$, $p_3 = p_4 = 0$, so that, from (6)

$$E_3' = \gamma E_3$$
$$E_4' = \gamma E_4$$

so that in the laboratory system the particles produced move straight ahead with a kinetic energy

$$T_i = (\gamma - 1)\, m_i \tag{11}$$

where γ is given in equations (6). γ is about 2 for 6 GeV electrons on nucleons, so that even at threshold the produced particles will have a kinetic energy about equal to their rest energy.

The calculated values of the threshold energies for the reactions are also listed in Table B. The values of the masses of particles used are given in Table C. Table C also listes the maximum kinetic energy in the laboratory system at production for the various known particles, using for each particle the most favorable (kinematical) reaction. These values were computed from equation (9) for two separate energies,

Table B

Reaction	Threshold
$e + N \longrightarrow$	GeV
$N + \pi$	0,15
$\Lambda + K$	0,91
$\Sigma + K$	1,05
$N + K + \overline{K}$	1,51
$\Omega + 2K$	2,37
$N + N + \overline{N}$	3,75
$N + \Lambda + \overline{\Lambda}$	4,88
$N + \Lambda + \overline{\Sigma}$	5,14
$N + \overline{\Lambda} + \Sigma$	5,14
$N + \Sigma + \overline{\Sigma}$	5,41
$N + \Omega + \overline{\Omega}$	6,36

T a b l e C

Maximum Energy at Production

Particle	Rest Mass GeV	Reaction $e + N \longrightarrow$	Max.Kinetic Energy at 6 GeV	max.Kinetic Energy at 7,5 GeV
π	0,140	$N + \pi$	5,86	7,36
K	0,494	$\Lambda + K$	5,30	6,81
$\overline{K}$		$N + K + \overline{K}$	4,86	6,37
Λ	1,12	$\Lambda + K$	5,26	6,75
$\overline{\Lambda}$		$N + \Lambda + \overline{\Lambda}$	2,70	4,35
Σ	1,19	$\Sigma + K$	5,18	6,68
$\overline{\Sigma}$		$N + \Lambda + \overline{\Sigma}$	2,54	4,23
Ω	1,32	$\Omega + 2K$	4,60	6,16
$\overline{\Omega}$		$N + \Omega + \overline{\Omega}$	--	3,32
$\overline{N}$	0,938	$N + N + \overline{N}$	3,48	5,03

6 GeV and 7,5 GeV. The most important conclusion from this table, which could be surmised from equation (11) is that the kinetic energies of the particles at production will generally be very large. Even under conditions of production giving only a fraction of the maximum energy to the particles, in most cases they will still have energies of several hundreds MeV.

III. Meson Energies as a Function of Angle

When the particles produced in the reaction do not travel along the original line of the motion, as is usually the case, the particles do not have their maximum energy and this energy is a function of the angle made by the particle trajectory with that of the incoming particles. Herr Herrmann has verified that the following equations give the energy of particle 3 as a function of the angle φ' in the laborytory system, between the trajectory of 2 and that of 3:

$$E_3' = \frac{A + \beta \cos\varphi' \sqrt{A^2 - m_3^2 + m_3^2 \beta^2 \cos^2\varphi'}}{(1 - \beta^2 \cos^2\varphi')} \qquad (12)$$

where

$$A = \frac{m_3^2 + 2 m_1 E_2' + m_1^2 - m_4^2}{2(m_1 + E_2')} \qquad (13)$$

and all other quantities are as previously defined; this has been developed only for the case $m_2 = 0$, but negligible error results in using the results for very fast electrons.

Figures 1 and 2 snow as a function of the laboratory angle φ' the laboratory momentum

of the π-mesons produced by $\gamma + N \to N + \pi$
and of the K-mesons produced by $\gamma + N \to \Lambda + K$

for initial energies of 6 GeV and 7,5 GeV. The momentum is given in terms of c times the usual momentum and is expressed in the units GeV. For extreme relativistic particles, this is also the total energy of the particles. From the graphs, it is seen that the π-meson may be considered extreme relativistic at all angles (minimum cp' is about 3 times the rest mass) and the K-meson is extreme relativistic for angles less than 60^{o} (cp' about twice the rest energy) and has high energies at all angles.

These figures enable one to predict the initial momentum of the π- and K-meson beams from a target placed in the election beam of the synchrotron as a function of the angle between the electron beam and the meson beam. The π-meson beam will be complicated, however, by multiple production processes ($e + N \rightarrow N + n \cdot \pi$) in which several π-mesons (n = 2,3,.....etc.) may be produced in a single collision. For protons incident on the nucleon at 5 GeV energy, the multiple production processes become more probable than the single production. Although nothing is presently known about multiple production of mesons by electrons, presumably a similar effect will occur.

IV. Collisions with Moving Target

The target has so far been assumed to be at rest in the laboratory system before collision. In two cases of interest, the target may be moving fast enough to change the results appreciably. The first case is that of a nucleon in a heavy nucleus where it may have a kinetic energy up to about 25 MeV. Although this energy is very small compared to the bombarding energy, it represents sufficient energy in the center-of-momentum system to modify the problem. The second case is that in which two beams of particles are directed at each other. At present, this "colliding beam" experiment has only academic interest, for the cross-sections for interactions are so small that the beams must interseit many times or intermingle for an appreciable time in storage rings for a detectable number of interactions to occur. Such storage rings have not yet been built nor are any under construction at the present time.

Consider a target nucleon moving towards to incident electron with 25 MeV kinetic energy = T_1'. For the threshold production of particles it is convention to transform to a system in which the nucleon is at rest and calculate the energy of the electron in this system. Since 25 MeV is small compared to the nucleon rest energy, the velocity of the system ß' can be calculated from

$$\beta' = \sqrt{\frac{2T_1'}{m_1}} \qquad \text{and} \qquad \gamma' = \frac{E_1'}{m_1} \tag{14}$$

Then in the nucleon rest system, distinguished by double primes,

$$E_2'' = \gamma' \left(E_2' + \beta' p_2' \right) \cong \gamma' E_2' (1 + \beta') \tag{15}$$

the approximation following because $E_2' \cong p_2'$. Since γ' involves β^2 only, this is sufficiently close to

$$E_2'' = E_1' \left(1 + \beta' \right) \tag{16}$$

for small ß'. $\beta'^2 = \frac{2T_1'}{m_1} = \frac{2 \times 25}{938} = 0{,}26$,

independent of the incident particle energy and hence

$$E_2'' \cong 1{,}26\, E_2' \tag{17}$$

This means that any reaction possible on a stationary nucleus at energy E_2'' will also occur at the 25 % lower energy E_2' if nucleons in a heavy nucleus are used as the target. Hence the only reaction in Table B which is not possible with 6 GeV electrons on hydrogen, $e + N \rightarrow N + \Omega + \overline{\Omega}$ with a threshold of 6,36 GeV is possible with 6 GeV electrons on a heavy nucleus target. The yield, of course, would be very small, since not many of the nucleons would collide with the electrons with their maximum momentum directed just opposite to that of the electrons.

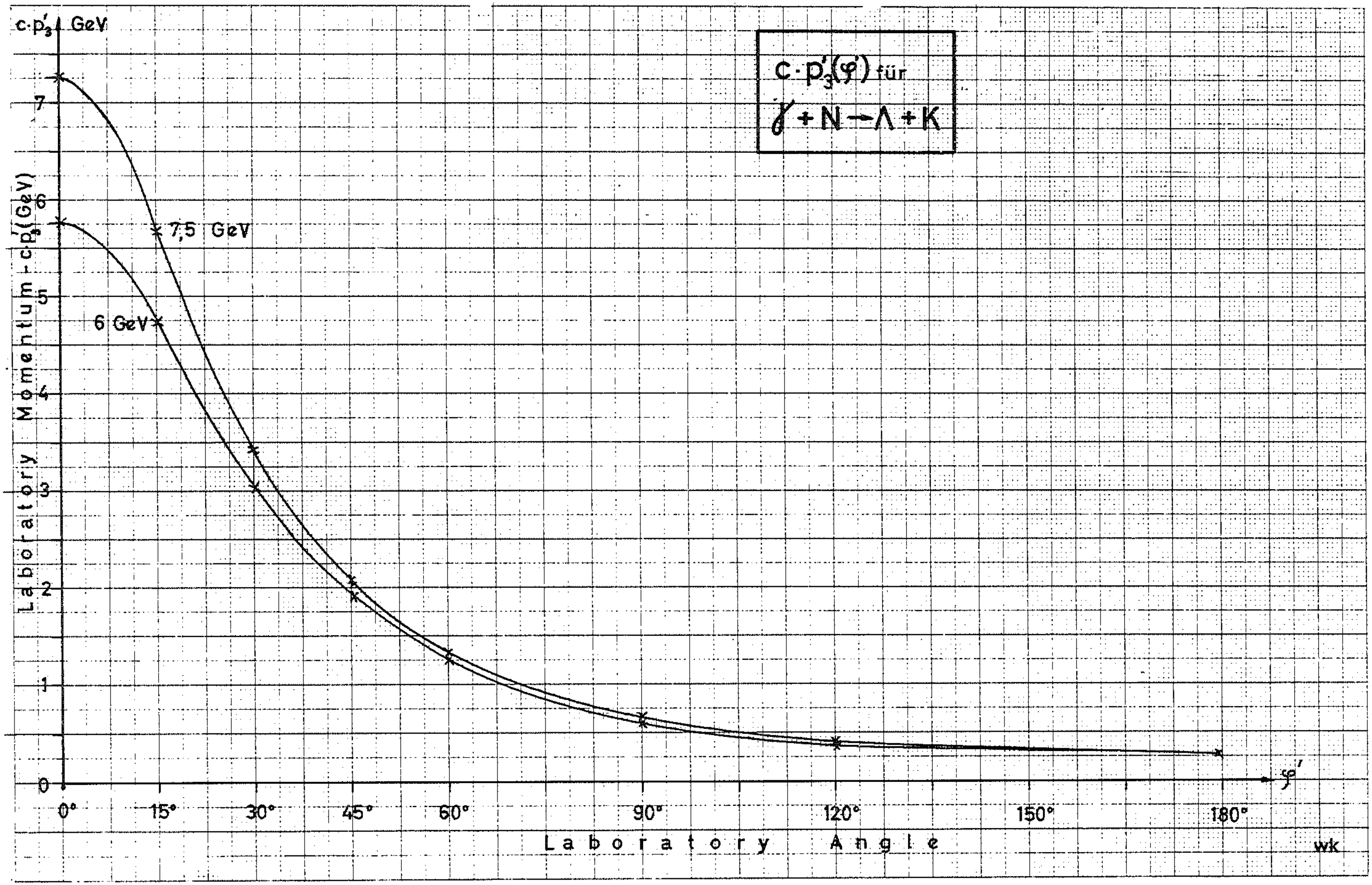
For threshold reaction, $E_2" = 1{,}26\ E_2' = 7{,}5$ GeV for 6 GeV electrons. Hence any reaction with a threshold less than 7,5 GeV can be produced with a heavy nucleus target. For reactions above threshold and for angular distributions, equation (17) does not give much information. One would have to consider the collision in detail in the (") system and then transform back to the laboratory system. It does not seem particularly interesting to work out the problem in detail because even in heavy nuclei the average nucleon momentum is zero and hence the momentum distribution will not very much affect the angular distribution of the energy of a produced particle; its primary effect will be to spread out the distribution in energy of the particles produced at a given angle. For experiments using such particles produced by heavy target bombardment this spread in momentum will have to be considered.

For the colliding beams experiment, the important parameter is the center-of-momentum energy W in (6);

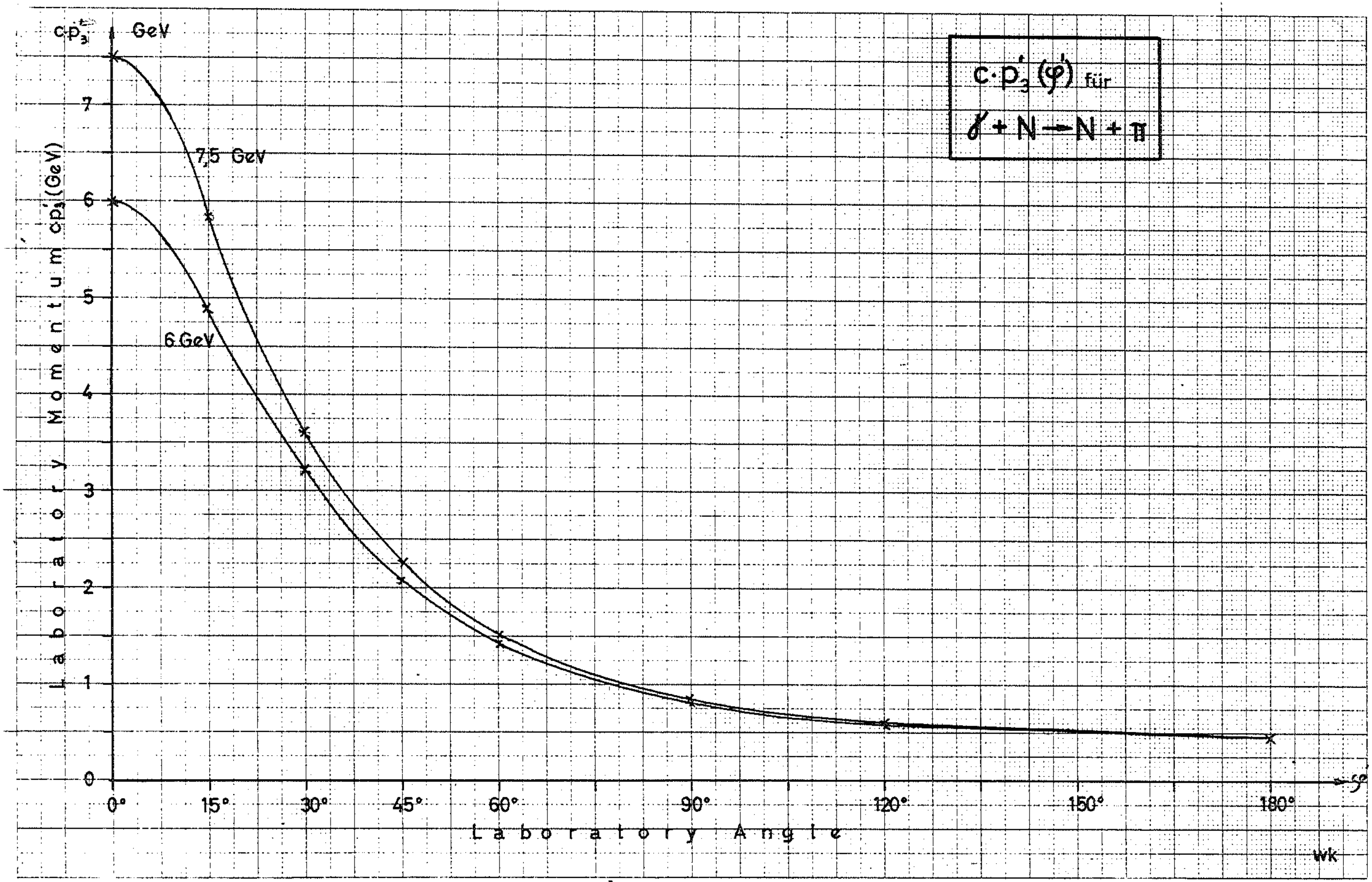
$$W = \sqrt{m_1^2 + m_2^2 + 2m_1 E_2'} \qquad \text{(stationary target)}$$

Hence for electron-electron collisions with a single beam of 6 GeV on stationary electrons, W is 78 MeV, a surprisingly low value. If two 6 GeV electrons collide head-on, however, W is obviously 12 GeV, since the laboratory and the center-of- momentum systems are the same. Thus the energy available for reactions is 154 times as great as for the single beam collision on stationary target. For a collision with a stationary electron to have W = 12 GeV would require the single electron beam to have an energy of $1{,}41 \times 10^5$ GeV, or $(154)^2 \times 6$ GeV. These examples show the enormous advantages to be gained if one could employ a colliding-beams machine.

E. Zimmermann



c·p'3 GeV
c · p'3(φ') für
γ + N → Λ + K
7,5 GeV
6 GeV
Laboratory Momentum - c·p'3 (GeV)
7
6
5
4
3
2
1
0
0°
15°
30°
45°
60°
90°
120°
150°
180°
φ'
Laboratory Angle
wk



c·p'₃ GeV
7,5 GeV
6 GeV
Laboratory Momentum cp'₃ (GeV)
7
6
5
4
3
2
1
0
0°
15°
30°
45°
60°
90°
120°
150°
180°
φ'
Laboratory Angle
c·p'₃ (φ') für
γ + N → N + π
wk