DESY HERA 81/19
December 1981

Periodic dispersion suppressors

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Abstract

In large storage rings, the dispersion is usually matched to vanish in the straight section. This matching is usually done in the last section of the arc, called "dispersion suppressor". It is shown that dispersion suppressors can be designed with strictly periodic amplitude functions that are fully matched to the periodic amplitude functions in the normal cells of the arc. This, however, requires in the arc and in the dispersion suppressor special phase advances per cell and a particular ratio of bending angles, which may not be practical for other reasons. Nevertheless, the given special solutions and the underlying analytic expressions for the dispersion at the matching point provide some basic insight into the mechanism of dispersion suppression and are therefore presented here.

Dispersion in a periodic focusing structure

The dispersion generated in an optical system starting at s=0 can be generally written[1] as

$$D(s) = -S(s)\int_0^s \frac{1}{\rho} C(\sigma)d\sigma + C(s)\int_0^s \frac{1}{\rho} S(\sigma)d\sigma$$

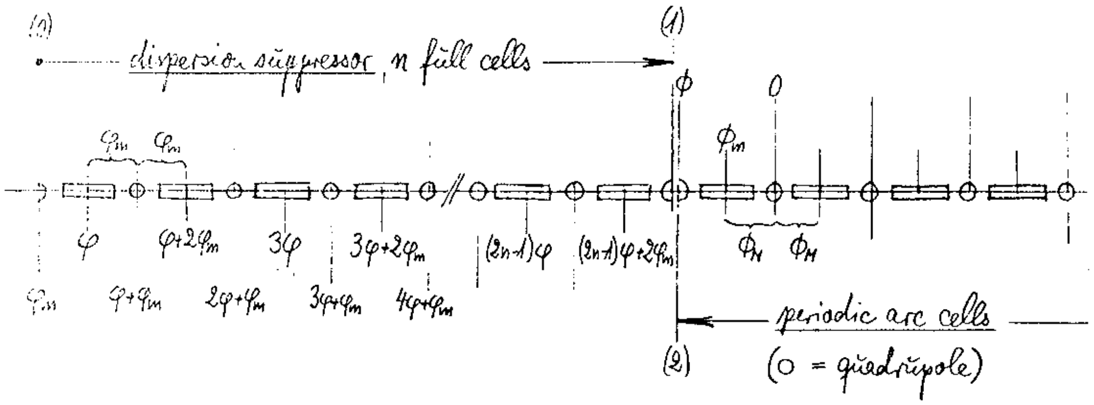
$$= -S(s)\cdot\sum_i \frac{\ell_i}{\rho_i} C_{m,i} + C(s)\cdot\sum_i \frac{\ell_i}{\rho_i} S_{m,i}$$

where S and C are the sinelike and cosinelike trajectories and $S_{m,i}$ and $C_{m,i}$ their values at the midplane of the i-th magnet of length ℓ_i and bending strength $\frac{1}{\rho_i}$. Differentiation with respect to s yields the slope of the dispersion

$$D'(s) = -S'(s)\int_0^s \frac{1}{\rho} C(\sigma)d\sigma + C'(s)\int_0^s \frac{1}{\rho} S(\sigma)d\sigma \;.$$

$$= -S'(s)\cdot\sum_i \frac{\ell_i}{\rho_i} C_{m,i} + C'(s)\cdot\sum_i \frac{\ell_i}{\rho_i} S_{m,i}$$

Let us assume the following periodic dispersion suppressor and arc structure, with betatron phases as indicated:



The dispersion suppressor consists of n identical full cells and has a periodic horizontal amplitude function with a betatron phase advance φ per half cell.

Then, with

$$\begin{cases} C = \dfrac{\sqrt{\beta}}{\sqrt{\beta_o}}\cos\varphi \\ C' = \dfrac{1}{\sqrt{\beta_o}\sqrt{\beta}}(\sin\varphi + \alpha\cos\varphi) \end{cases} ; \quad \begin{cases} S = \sqrt{\beta_o}\sqrt{\beta}\sin\varphi \\ S' = \dfrac{\sqrt{\beta_o}}{\sqrt{\beta}}(\cos\varphi - \alpha\sin\varphi) \end{cases}$$

we have at point (1), with $\alpha = 0$, the dispersion and slope

$$D_1 = \varepsilon_m\sqrt{\beta_1}\sqrt{\beta_m}\,[-\Sigma_c\cdot\sin(2n\varphi+\varphi_m) + \Sigma_s\cdot\cos(2n\varphi+\varphi_m)]$$

$$D'_1 = \varepsilon_m\frac{\sqrt{\beta_m}}{\sqrt{\beta_1}}[-\Sigma_c\cdot\cos(2n\varphi+\varphi_m) + \Sigma_s\cdot\sin(2n\varphi+\varphi_m)]$$

where

$$\Sigma_c = \sum_{i=1}^{n}[\cos(2i-1)\varphi + \cos\{(2i-1)\varphi + 2\varphi_m\}]$$

$$\Sigma_s = \sum_{i=1}^{n}[\sin(2i-1)\varphi + \sin\{(2i-1)\varphi + 2\varphi_m\}]$$

and $\varepsilon_m = \frac{\ell_m}{\rho_m}$ the bending angle of the dispersion suppressor magnets.

After some arithmetic, using the relations

$$\sum_{i=1}^{n} \sin(2i-1)\varphi = \frac{\sin n\varphi}{\sin\varphi} \cdot \sin n\varphi$$

$$\sum_{i=1}^{n} \cos(2i-1)\varphi = \frac{\sin n\varphi}{\sin\varphi} \cdot \cos n\varphi$$

this yields for the dispersion and its derivative at the end (1) of the periodic dispersion suppressor the simple expressions (see also ref. 3)

$$D_1 = -\sqrt{\beta_1}\sqrt{\beta_m}\ \varepsilon_m \cos\varphi_m \cdot \frac{2\sin^2 n\varphi}{\sin\varphi}$$

$$D_1' = -\frac{\sqrt{\beta_m}}{\sqrt{\beta_1}}\ \varepsilon_m \cos\varphi_m \cdot \frac{\sin 2n\varphi}{\sin\varphi} \qquad (1)$$

For the periodic dispersion in the arc, we similarly obtain at the end (2) of the arc cell

$$D_2 = -\sqrt{\beta_2}\sqrt{\beta_M}\ \varepsilon_M \cos\Phi_M \cdot \frac{1}{\sin\Phi}$$

$$D_2' = 0 \qquad (2)$$

Matching between dispersion suppressor and periodic arc cell

In order to match dispersion and horizontal amplitude function at the end of the dispersion suppressor into the periodic arc cell, we must satisfy the matching conditions

$$D_1 = D_2 \qquad (3)$$

$$D_1' = D_2' = 0 \qquad (4)$$

$$\beta_1 = \beta_2 \qquad (5)$$

Eq. (4) requires that the total phase advance in the dispersion suppressor must be a multiple of π:

$$2n\varphi = k \cdot \pi$$

For $k = 2$, D_1 would vanish also, and eq. (3) could not be satisfied. This leaves us with $k = 1$ or $K = 3$, and the phase advances 2φ per full dispersion suppressor cell must be as follows

k \ n	1	2	3	4	5
1	-	90°	60°	45°	36°
3	-	-	-	135°	108°

Eq. (3) then writes, with $\beta_1 = \beta_2$ according to eq. (5),

$$2 \cdot \sqrt{\beta_m}\ \varepsilon_m \frac{\cos\varphi_m}{\sin\varphi} = \sqrt{\beta_M}\ \varepsilon_M \frac{\cos\Phi_M}{\sin\Phi} \qquad (6)$$

In the simple case where all cells, including the arc, are identical except for the bending, we thus find that the bending angle in the dispersion suppressor magnets must be half of the bending angle in the arc magnets:

$$\varepsilon_m = \frac{1}{2} \cdot \varepsilon_M$$

Such an example is shown in Fig. 4 (1), where 4 dispersion suppressor cells of $2\varphi = 45°$ phase advance are matched to the arc cells which have the same phase advance. As shown in the figure, the matching is retained when the focusing and defocusing quadrupoles ar interchanged (a → b). The example shown has equal strength in both types of quadrupoles. The system parameters are given in Tab. 1.

In a more complex case we can use the fact that a given maximum amplitude function at the end of the cell can be obtained with two different phase advances, which allows to match the periodic amplitude function between cells of same length but different phase advance. In the approximation of ref. 2), with a half cell length ℓ and equal strength in focusing and defocusing quadrupoles, the normalized maximum amplitude function is given by

$$\frac{\hat{\beta}}{\ell} = \frac{1}{\sin\Phi}\left(\frac{1+\sin\Phi}{1-\sin\Phi}\right)^{1/2}$$

This is shown in Fig. 1 and in a different form in Fig. 2, where the sines of the two phase advances Φ_1, Φ_2 belonging to the same $\hat{\beta}$ are shown as a function of the ratio $\sin\Phi_2/\sin\Phi_1$.

In order to fullfill condition (4) we choose, from the table on p. 4, for the dispersion suppressor a cell phase advance of

$$2\varphi = 2\Phi_1 = 90^\circ,\ 135^\circ,\ \text{or}\ 108^\circ;$$

the corresponding phase advance $2\Phi = 2\,\Phi_2$ in the arc cell is then taken from Fig. 2. In addition, for dispersion matching (eq. (3)), the ratio $\varepsilon_m/\varepsilon_M$ of magnet bending angles is chosen according to eq. (6).

With[2])

$$\frac{\beta_m}{\ell} \approx \frac{\overline{\beta}}{\ell} = \frac{1}{2\ell}(\hat{\beta} + \check{\beta}) = \frac{2}{\sin 2\varphi}$$

this can be written

$$2 \cdot \frac{\varepsilon_m}{\varepsilon_M} = R(\varphi) = \sqrt{\frac{\ell}{L}} \cdot \frac{\dfrac{\sin\varphi\sqrt{\sin 2\varphi}}{\cos\varphi_m}}{\dfrac{\sin\Phi\sqrt{\sin 2\Phi}}{\cos\Phi_M}}$$

Using the further approximation

$$\varphi_m \approx \overline{\varphi} = \frac{1}{2}\varphi - \frac{\ell}{12}\left(\frac{1}{\check{\beta}} - \frac{1}{\hat{\beta}}\right) = \frac{1}{2}\varphi - \frac{1}{6}\sin\varphi\,\tan\varphi,$$

the ratio $R(\varphi)$ was evaluated for $\ell = L$ and graphically displayed in Fig. 3.

The dispersion suppressors thus constructed have a periodic beta function in the horizontal plane only. Maintaining equal cell lengths $\ell = L$, a periodic beta function also in the vertical plane can only be obtained by making the foc. and defoc. cell quadrupoles differently strong, in the dispersion suppressor as well as in the arc. Such an example, rather exotic, with 4 cells of $2\varphi = 135^\circ$ horizontal phase advance each, and matched to an arc cell of $2\varphi = 44.4^\circ$ horizontal phase advance, is shown in Fig. 4 II. The vertical phase advances are 57° and 70.7°, respectively, and the ration of bending angles is $\varepsilon_m/\varepsilon_M = 3$ (see Tab. 1).

Let us finally consider the case where the length of the dispersion suppressor cell is treated as an additional variable. Then, the phase advances can be chosen independently, e.g. $2\varphi = 90^\circ$ in the dispersion suppressor and $2\Phi = 45^\circ$ in the arc cell, and the horizontal beta functions in case of equal foc. and defoc. quadrupole strengths can be matched by increasing the length ℓ according to Fig. 1. In our example, this will also provide a fairly good match of the vertical amplitude function since, in the range of small phase advances, $\hat{\beta}$ and $\check{\beta}$ vary rather proportionally. The ratio of bending angles is again determined by eq. (8), with $\ell/L > 1$ in this case.

It might also be interesting to look for periodic dispersion suppressors consisting of (2n - 1) half cells, but this cannot be pursued at the moment.

Acknowledgement

I wish to thank F. Willeke, who first used the concept of the periodic dispersion suppresor and expressed its dispersion in a simple closed form, for discussions and for checking the results.

References

1) K. Steffen: High Energy Beam Optics; p. 113 ff., Wiley, New York (1965)

2) K. Steffen: Simplified formulae for properties and scaling of large electron storage rings; Internal Report DESY PET-79/03 (March 1979)

3) F. Willeke: "Verbotene" Q-Werte bei PETRA; Internal Report DESY PET-81/28 (Nov. 1981)

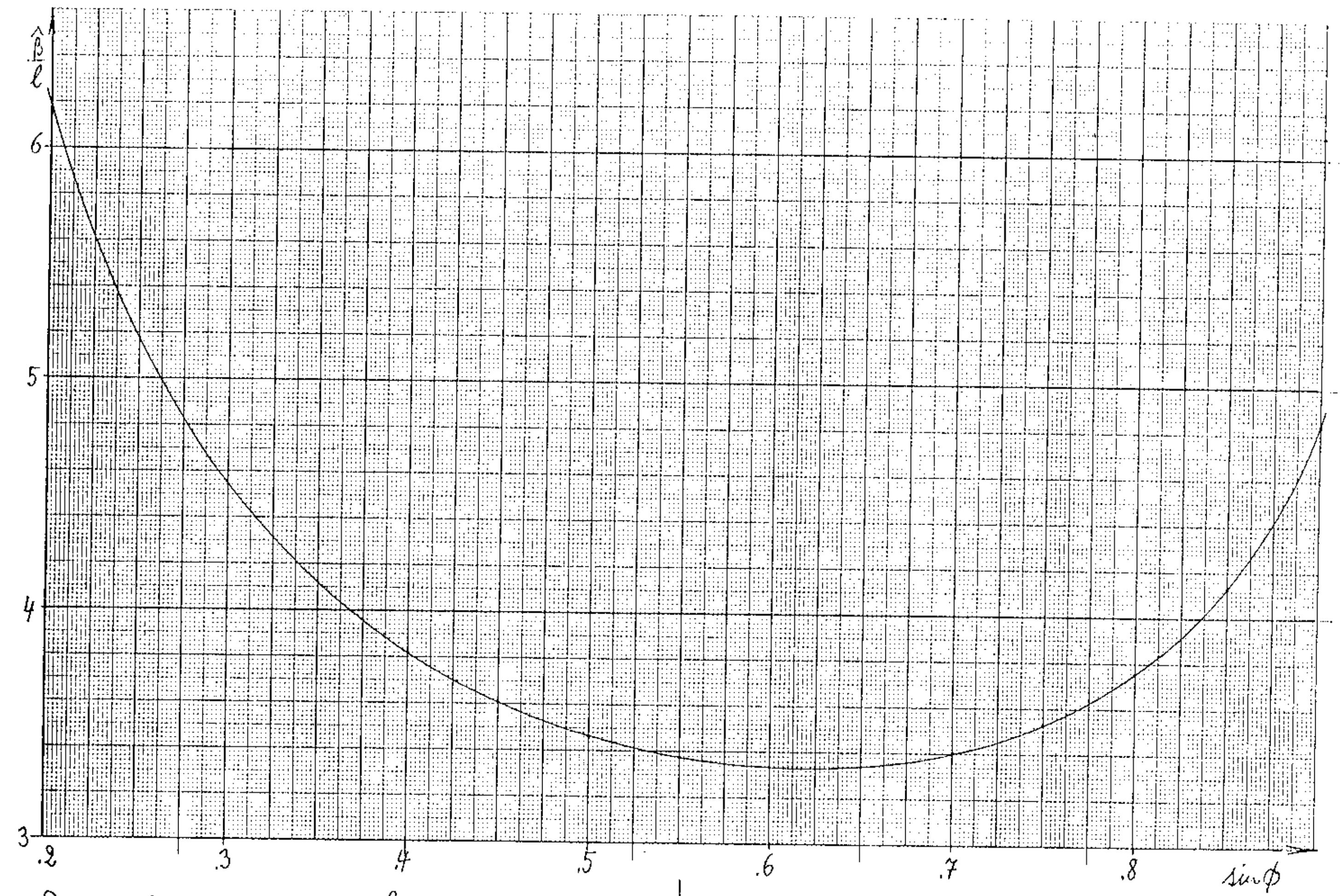


Fig. 1: Max. rel. amplitude function in the idealized cell as a function of phase advance per half cell.

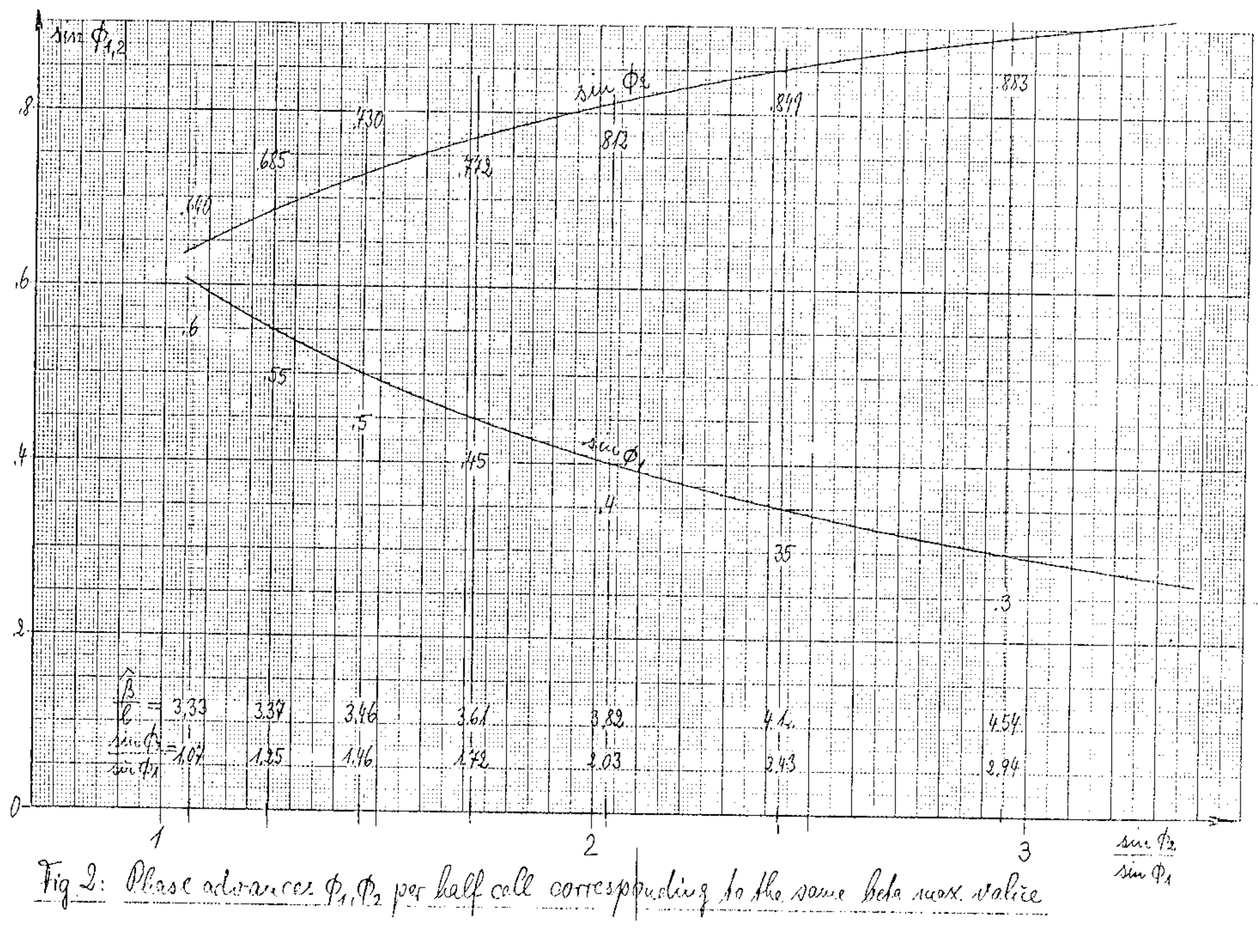


Fig. 2: Phase advances Φ_1, Φ_2 per half cell corresponding to the same beta max. value

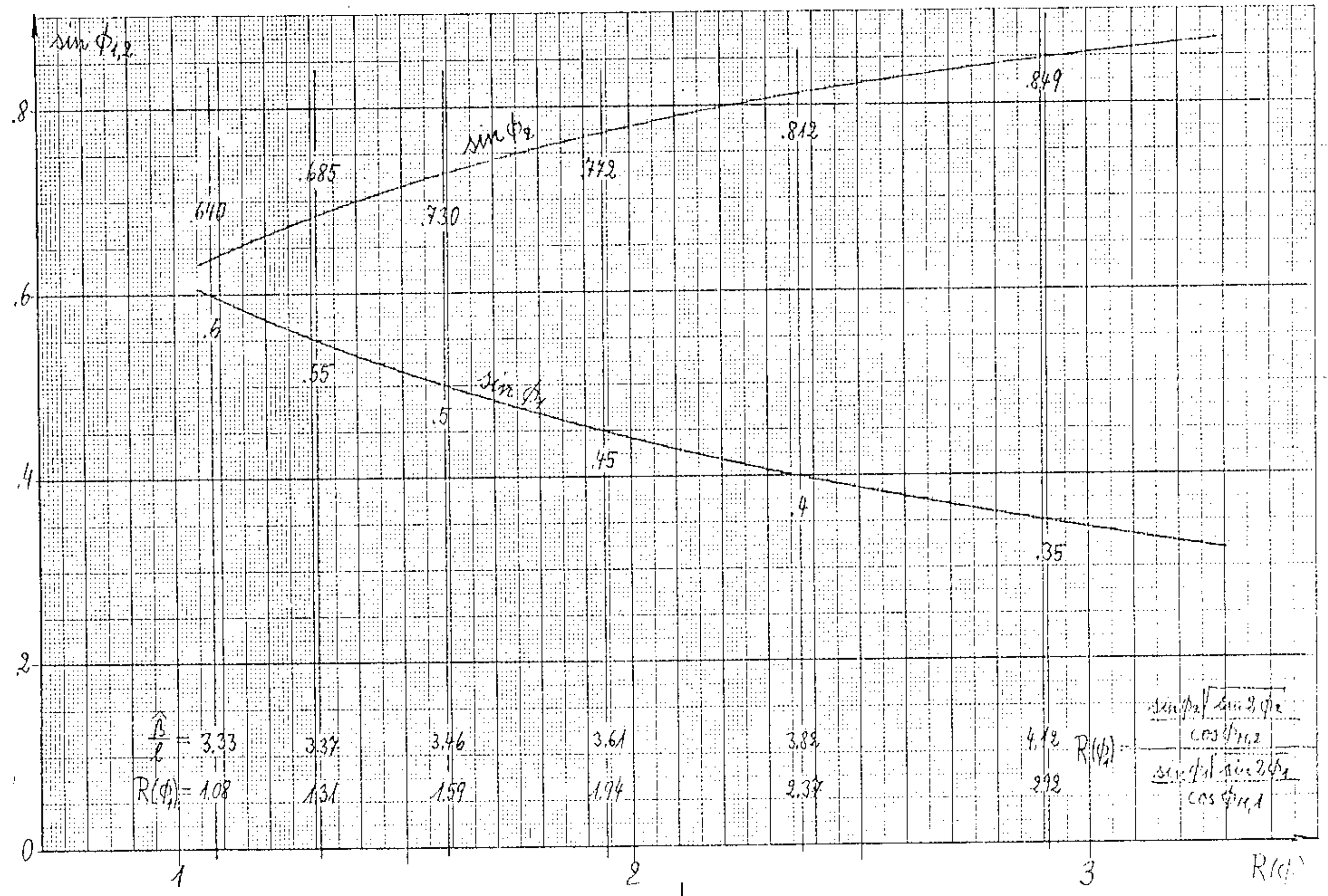


Fig. 3: Phase advances Φ_1, Φ_2 per half cell as function of the ratio R.

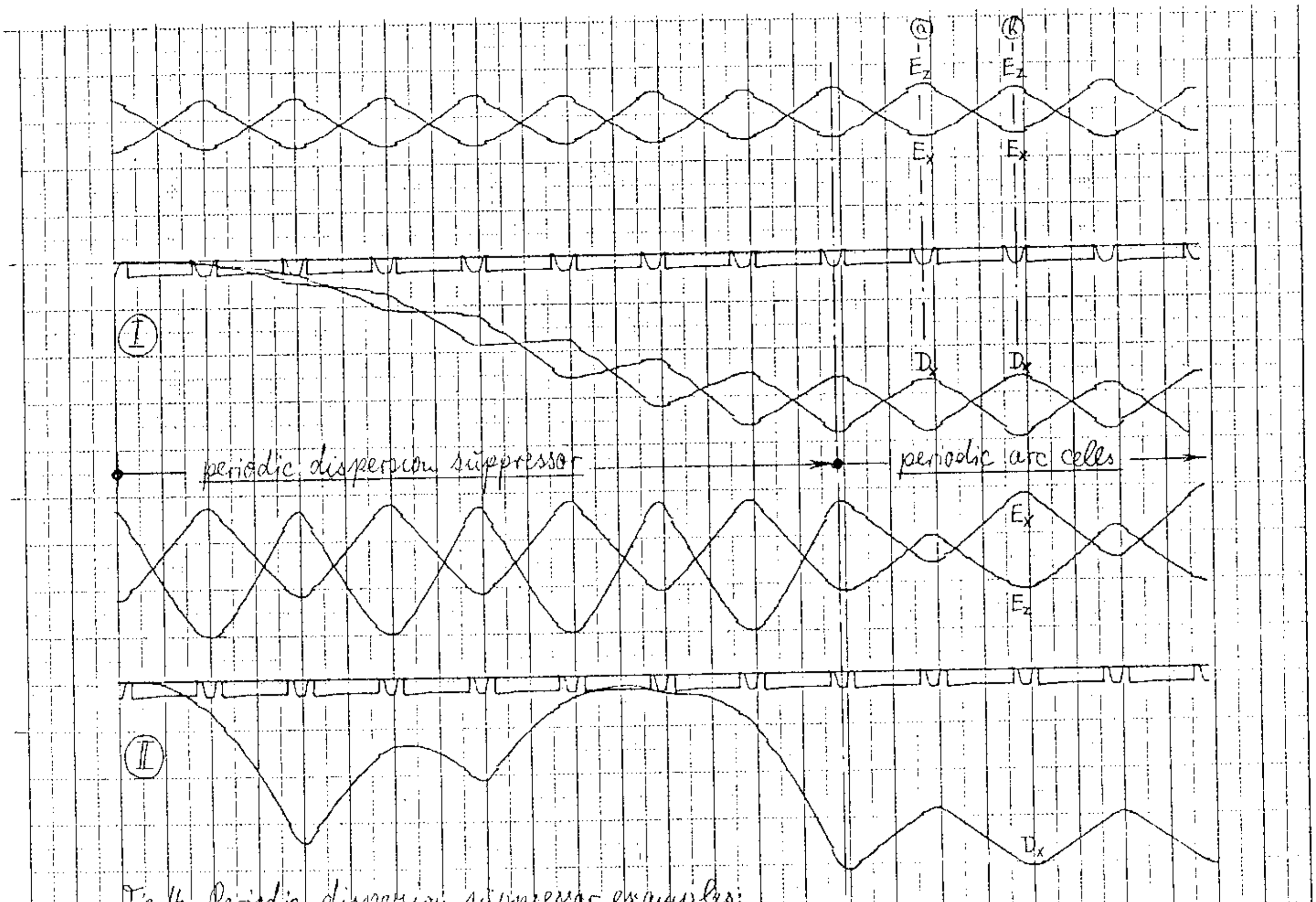


Fig. 4: Periodic dispersion suppressor examples: beam envelopes E and dispersion D

half cell length $l = L = 7.846$ m
quadrupole length $l_q = 1$ m
bending magnet length $l_m = 5.446$ m

example #		quadrupole strengths k_f	k_d	hor. — amplitude $\hat{\beta}_x$	$\check{\beta}_x$	functions — vert $\hat{\beta}_z$	$\check{\beta}_z$	phase advance per cell $2\Phi_x$	$2\Phi_z$	dispersion $\hat{D}$	$\check{D}$	magnet bend. angle ε
		[m^{-2}]		[m]		[m]		[deg]		[m]		[mrad]
	disp. suppr.											5.04
I		.102	-.102	30.2	13.8	30.4	13.9	45°	45°			
	arc cell									.64	.44	10.08
II	disp. suppr.	.2416	-.16	34.3	2.13	35.2	7.72	134.9°	57.0°			29.95
II	arc cell	.1185	-.1432	34.3	12.80	23.5	7.72	44.4°	70.7°	.71	.44	10.08

Tab. 1: Parameters of the dispersion suppressor systems of Fig. 4