DESY-HERA 86-01
February 1986

Beam dynamics at low energy in DESY III

EMANUEL.A.KARANTZOULIS

Deutsches Electronen-Synchrotron DESY , Hamburg

and

JOHN.R.M.MAIDMENT

Rutherford Appleton Laboratories , England

DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY HAMBURG

1. *Introduction*

During injection and initial acceleration in DESY III, both longitudinal and transverse space charge forces play an important role in determining the beam dynamics. Section 2 introduces the longitudinal equations of motion in the absence of space charge which enable bunch length, momentum spread, etc. to be estimated without small-angle (linear) approximation.

In Section 3 these equations are modified to include the effect of longitudinal space charge arising from a stationary line-density particle distribution function and solved self-consistently.

A procedure for estimating the influence of a non-uniform transverse distribution on the transverse (betatron) oscillation frequencies is outlined in Section 4 and evaluated for the case of a parabolic density function.

Those interested in project related numbers may turn immediately to Section 5 which presents the results of evaluating the analytical model using DESY III parameters to yield rf voltage, momentum spread, transverse tune spreads, etc. versus time.

Section 6 discusses the validity of the assumptions made in the development of the analytical treatment and comments on future work initiated by this study.

2. *Longitudinal motion without space charge*

The synchrotron equations of motion in the absence of space charge may be derived from the Hamiltonian:

$$\boxed{H(W,\varphi) = \frac{h^2\Omega_0^2\eta}{2\beta^2 E_0}W^2 - \frac{e}{2\pi h}U(\varphi)} \tag{1}$$

where $W = \Delta E/h\Omega_0$, h is the harmonic number, Ω_0 is the angular revolution frequency, $\eta = 1/\gamma_t^2 - 1/\gamma^2$ and $U(\varphi)$ is the potential $U(\varphi) = \int_{\varphi_0}^{\varphi}\{V(\varphi) - V_s\}d\varphi$. $V(\varphi)$ is the applied voltage ($V_0 = V(\varphi_0)$) where the subscript 0 refers to the synchronous particle; then eV_0 is the synchronous energy gain per turn. The equations of motion can be deduced from the above Hamiltonian as:

$$\frac{d\varphi}{dt} \equiv \dot\varphi = \frac{\partial H}{\partial W} \quad \text{and} \quad \frac{dW}{dt} \equiv \dot W = -\frac{\partial H}{\partial \varphi} \tag{2}$$

substituting $J = h^2\Omega_0^2\eta/E_0\beta^2$ and $K = e/2\pi h$ equations (2) may be written:

$$\dot\varphi = JW \quad ; \quad \dot W = K\{V(\varphi) - V_0\} \tag{3}$$

and obtain the differential equation of φ alone

$$\ddot\varphi = JK\{V(\varphi) - V_0\} \tag{4}$$

which may be written as: $\frac{1}{2}\frac{d}{d\varphi}(\dot\varphi^2) = JK\{V(\varphi) - V_0\}$ hence $\dot\varphi^2 = 2JK\int_{\varphi_0}^{\varphi}\{V(\bar\varphi) - V_0\}d\bar\varphi + C$ (C being a constant to be specified) and finally

$$\dot\varphi = \{2JKU(\varphi) + C\}^{1/2} \tag{5}$$

1) Synchrotron period, τ
From (5) we have:

$$\tau = 2\int_{\varphi_1}^{\varphi_2}\{2JKU(\varphi) + C\}^{-1/2}d\varphi \tag{6}$$

where $\varphi_1 < \varphi_2$ are the limits of the considered oscillation. The extrema should satisfy the condition $\dot\varphi = 0$ i.e $2JKU(\varphi_1) + C = 0$, $2JKU(\varphi_2) + C = 0$ or

$$U(\varphi_1) = U(\varphi_2) \tag{7}$$

$$C = -2JKU(\varphi_2) \tag{8}$$

Equation (5) may then be written as:

$$\boxed{\dot\varphi = (2JK)^{1/2}\{U(\varphi) - U(\varphi_2)\}^{1/2}} \tag{9}$$

and equation (6) becomes:

$$\tau = 2\{2JK\}^{-1/2}\int_{\varphi_1}^{\varphi_2}\{U(\varphi) - U(\varphi_2)\}^{-1/2}d\varphi \tag{10}$$

2) Area of the phase space contour.
We may plot curves of constant H in (φ, W) phase space given by (1). One particular value, H_e , yields the closed curve enclosing maximum area, and which separates the inner, stable, region from the outer, unstable, one. Particles within this separatrix describe periodic motion around closed curves, whilst those outside follow open non periodic contours. The area contained within the separatrix is called the bucket area. The bunch area or emittance is that contained within the closed contour of limiting particle motion which may be coincident with the separatrix-'full bucket'. The bunch area A is given by $A = 2\int_{\varphi_1}^{\varphi_2} W\,d\varphi$ which using (3) and (9) may be written as,

$$A = 2\left(\frac{2K}{J}\right)^{1/2}\int_{\varphi_1}^{\varphi_2}\{U(\varphi) - U(\varphi_2)\}^{1/2}d\varphi \tag{11}$$

3) Maximum value of the canonical energy coordinate W
The maximum value of W, W_M, corresponds to $\varphi = \varphi_0$ as can be seen from the form of the potential and from eqs(1,2).(That is,for $\varphi = \varphi_0$ $\dot W = 0$ but $W \neq 0$).From equation (3) we get: $W_M = \frac{1}{J}\dot\varphi|_{\varphi=\varphi_0}$ or by (9)

$$W_M = \left(\frac{2K}{J}\right)^{1/2}\{U(\varphi_0) - U(\varphi_2)\}^{1/2} \tag{12}$$

4) Maximum momentum spread, bunch length, peak current.
From the definition of W and using $\frac{\Delta P}{P} = \frac{1}{\beta^2}\frac{\Delta E}{E}$ we get

$$\left.\frac{\Delta P}{P}\right|_M = \frac{hc}{\beta R}\left(\frac{2K}{J}\right)^{1/2}\{U(\varphi_0) - U(\varphi_2)\}^{1/2} \tag{13}$$

Transforming from the rf phase φ we obtain the bunch length , ℓ , as

$$\ell = \frac{R}{h}(\varphi_2 - \varphi_1) \tag{14}$$

Assuming the bunch is uniformly filled with N particles each of charge e the peak instantaneous current $\hat{I}$ may be written as

$$\hat{I} = \frac{2Ne\beta ch}{R}\frac{W_M}{A}$$

and using equations (11),(12)

$$\hat{I} = \frac{Ne\beta ch}{R}\frac{(U(\varphi_0) - U(\varphi_2))^{1/2}}{\int_{\varphi_1}^{\varphi_2}(U(\varphi) - U(\varphi_2))^{1/2}\,d\varphi} \tag{15}$$

In the appendix we apply the above formulae for the case of our interest i.e a sinusoidal voltage.

3. *Longitudinal motion including space charge*

Rather than assume a priori a given distribution we derive one in a self consistent fashion. A similar approach has been used in reference 1. Denoting $g(W,\varphi)(\equiv \frac{d^2N}{dW\,d\varphi})$ as the stationary phase space density we may write it as a function of the Hamiltonian (1) namely $g(W,\varphi) = g(H)$; then the line density is: $\lambda(\varphi) \equiv \int g(W,\varphi)dW$. The simplest choice for $g(H)$ is $g(H) = C(H_M - H)$ where H_M is the Hamiltonian value for which a particle describes the maximum contour of the bunch. Using (1,9,12) the line density may be written as: $\lambda(\varphi) = C(U(\varphi) + C_1)$ where C , C_1 are constants. Assuming that at the limits of the oscillation , $\varphi_1 < \varphi_2$, $\lambda(\varphi_1) = \lambda(\varphi_2) = 0$ we specify imediately $C_1 = -U(\varphi_2)$. The normalization constant C is obtained using $\int_{\varphi_1}^{\varphi_2}\lambda(\varphi) = N$. Then $C = N/f(\varphi_1,\varphi_2)$ where $f(\varphi_1,\varphi_2) = \int_{\varphi_1}^{\varphi_2}\{U(\varphi) - U(\varphi_2)\}d\varphi$ yielding

$$\boxed{\lambda(\varphi) = \frac{N(U(\varphi) - U(\varphi_2))}{f(\varphi_1,\varphi_2)}} \tag{16}$$

$\lambda(\varphi)$ has the same shape as the potential and $U(\varphi_2)$ is the potential of one end of the bunch. This distribution gives rise to a new potential $U_t(\varphi)$ defined as: $U_t(\varphi) = \int_{\varphi_0}^{\varphi}\{V_t(\varphi) - V_0\}d\varphi$ where $V_t(\varphi) = V(\varphi) - V_{sc}(\varphi)$. The subscript sc refers to quantities with space charge. Due to the self consistent treatment $U_t(\varphi_0) = 0$. Generally the same technique, as in the previous Section, can be applied in order to calculate beam characteristics starting from a new Hamiltonian namely

$$\boxed{H(W,\varphi) = \frac{1}{2}JW^2 - KU_t(\varphi)} \tag{17}$$

where $U_t(\varphi) = U(\varphi) - \Delta\lambda(\varphi)$ with Δ a constant in φ . Then equation (17) may be written as $H(W,\varphi) = H_0(W,\varphi) + H_{sc}(W,\varphi)$ and H_{sc} can be considered as a small perturbation. However it may be proved by straight forward application of the above

method that this distribution stabilises itself (that is the distribution does not feed back to the potential resulting in a different distribution and so on) . The space charge field may be written as:

$$E_{sc}(\varphi) = -\frac{eg_0}{4\pi\epsilon_0\gamma^2}\frac{h}{R}\frac{\partial\lambda(\varphi)}{\partial\varphi}$$

where g_0 is a geometric coupling factor given to first order by $g_0 = 1 + 2\ln\frac{b}{a}$ where b, a are chamber and beam radii respectively. Defining $Z_0 = \frac{1}{\epsilon_0 c} \equiv 377\ ohms$ (the free space impedance) we may write the voltage a particle sees per turn due to space charge as

$$V_{sc}(\varphi) = -2\pi h\frac{eg_0cZ_0}{4\pi\gamma^2}\frac{h}{R}\frac{\partial\lambda(\varphi)}{\partial\varphi}$$

or more generally,

$$V_{sc} = 2\pi h^2\, I_0\, Im(\frac{Z}{n})\frac{\partial\lambda_1(\varphi)}{\partial\varphi} \tag{18}$$

where $I_0 = Ne\Omega_0/2\pi$ and $(\frac{Z}{n})$ is the effective coupling impedence:

$$(\frac{Z}{n}) = j\left(\Omega_0 L - \frac{g_0 Z_0}{2\beta\gamma^2}\right)$$

$n = \Omega/\Omega_0$ and L is the chamber wall inductance. At low energies we may generally ignore the wall contribution and write: $(\frac{Z}{n}) = -j\left(\frac{g_0Z_0}{2\beta\gamma^2}\right)$. Thus the voltage becomes

$$V_t(\varphi) = \begin{cases} V(\varphi) - 2\pi h^2 I_0\, Im(\frac{Z}{n})\frac{\partial\lambda_1(\varphi)}{\partial\varphi} & \text{for } \varphi_1 < \varphi < \varphi_2 \\ V(\varphi) & \text{elsewhere} \end{cases}$$

where $\lambda(\varphi) = N\lambda_1(\varphi)$. Since $\frac{\partial\lambda_1(\varphi)}{\partial\varphi} = \frac{V(\varphi) - V_0}{f(\varphi_1,\varphi_2)}$ we see that the induced voltage has similar shape to the applied voltage. Writing again $U_t(\varphi) = U(\varphi) - \Delta\lambda(\varphi)$, Δ may be immediately obtained by differentiation as

$$\Delta = -\frac{Necg_0Z_0h^2}{2\gamma^2R} \tag{19}$$

thus the new potential reads:

$$\boxed{U_t(\varphi) = U(\varphi) + \frac{Necg_0Z_0h^2}{2\gamma^2R}\frac{U(\varphi) - U(\varphi_2)}{f(\varphi_1,\varphi_2)}} \tag{20}$$

consequently the instantaneous current, $I(\varphi)$, $\left(\equiv e\frac{dN}{d\varphi}\frac{d\varphi}{dt}\right)$ for $\varphi_1 < \varphi < \varphi_2$ may be written as:

$$I(\varphi) = 2\pi hI_0\frac{U(\varphi) - U(\varphi_2)}{f(\varphi_1,\varphi_2)} \tag{21}$$

Using equation (20) in the formulae of the previous Section we may derive all the beam characteristics under the influence of space charge. Just substitute U_t instead of U. Thus for example, the expression for the area, equation (11), will become:

$$A_t = 2\left(\frac{2K}{J}\right)^{1/2}\left(1 - \frac{\Delta}{f(\varphi_1,\varphi_2)}\right)^{1/2}\int_{\varphi_1}^{\varphi_2}\{U(\varphi) - U(\varphi_2)\}^{1/2}d\varphi$$

or

$$A_t = A\{1 - \frac{\Delta}{f(\varphi_1,\varphi_2)}\}^{1/2} \tag{22}$$

where A_t is the area modified by space charge and A as given by (11). Equation (12) becomes:

$$W_{t_M} = W_M\{1 - \frac{\Delta}{f(\varphi_1,\varphi_2)}\}^{1/2} \tag{23}$$

and the peak current (15) remains unaffected. Nevertheless if we demand that the bunch area should remain fixed as $A = \epsilon_p$, where ϵ_p is the longitudinal emittance, then

$$\hat{I}_t = \hat{I}\{1 - \frac{\Delta}{f(\varphi_1,\varphi_2)}\}^{1/2} \tag{24}$$

similarly the synchrotron period (equation (10)) will be modified:

$$\tau_t = \tau\{1 - \frac{\Delta}{f(\varphi_1,\varphi_2)}\}^{-1/2} \tag{25}$$

(See appendix for an application of the above formulae using a sinusoidal voltage)

4. *Transverse space charge effects*

At low proton velocity the electrostatic repulsion between particles is not compensated by the magnetic interaction and there remains a net defocussing self-force. This reduces the single particle incoherent betatron frequencies. In a bunched beam, with non uniform transverse density, and in practice also, this frequency (or tune) shift is both amplitude dependent and a function of the particle longitudinal position. There is then a transverse tune spread within which the individual particle tune varies at twice the synchrotron frequency as the particle oscillates from the low intensity head of the bunch via the high intensity middle to the low intensity tail and back. If the spread is too great particles may cross transverse non-linear betatron resonances leading to increased amplitudes and/or loss. We seek then to calculate the tune spread and via a knowledge of field imperfections obtain an estimate of the space charge limit.

The horizontal and vertical tune shifts $(\Delta Q_x\ ,\ \Delta Q_z)$ may be obtained from:

$$\Delta Q_{x,z} = \frac{2\pi\epsilon_0 r_p R^2}{e\beta^2\gamma^3 Q_{x,z}} < \frac{\partial E_{x,z}}{\partial x,z} > \tag{26}$$

where < > signifies an average over the ring and r_p is the classical proton radius. To evaluate E_x , E_z we assume a transversely elliptical beam with a parabolic distribution function. Thus if $h(x,z)$ is such a function, it has the form

$$h(x,z) = \frac{2\lambda}{ab\pi}(1 - u^2) \quad , u \le 1 \tag{27}$$

with $\lambda \equiv \lambda(\varphi)$ our previously defined linear density and

$$u^2 = \frac{x^2}{a^2} + \frac{z^2}{b^2} \tag{28}$$

where a , b are the semiaxes of the $u = 1$ ellipse. Using equation (27) and the potential theory for a solid ellipsoid [2] we may write the potential (reduced to 2-dimensions and neglecting image effects) as:

$$U(x,z) = -\frac{e\lambda}{2\pi\epsilon_0}\left\{\frac{2x^2}{a(a+b)} + \frac{2z^2}{b(a+b)} - \frac{1}{4}\left(\frac{1}{3}\frac{2a+b}{a^3(a+b)^2}x^4 + \frac{2}{ab(a+b)^2}x^2z^2 + \frac{1}{3}\frac{a+2b}{b^3(a+b)^2}z^4\right)\right\} \tag{29}$$

From (29) the electric fields E_x , E_z can be found:

$$E_x = -\frac{e\lambda}{\pi\epsilon_0}\left(\frac{2}{a(a+b)}x - \frac{2a+b}{6a^3(a+b)^2}x^3 - \frac{1}{2ab(a+b)^2}xz^2\right)$$
$$E_z = -\frac{e\lambda}{\pi\epsilon_0}\left(\frac{2}{b(a+b)}z - \frac{2b+a}{6b^3(a+b)^2}z^3 - \frac{1}{2ab(a+b)^2}zx^2\right) \tag{30}$$

and thus,

$$< \frac{dE_x}{dx} > = -\frac{e\lambda}{\pi\epsilon_0}\left\langle\frac{2}{a(a+b)} - \frac{2a+b}{2a^3(a+b)^2}x^2 - \frac{1}{2ab(a+b)^2}z^2\right\rangle$$
$$< \frac{dE_z}{dz} > = -\frac{e\lambda}{\pi\epsilon_0}\left\langle\frac{2}{b(a+b)} - \frac{2b+a}{2b^3(a+b)^2}z^2 - \frac{1}{2ab(a+b)^2}x^2\right\rangle \tag{31}$$

Finally using equation (26) we get:

$$\Delta Q_x(\xi,\chi) = -\frac{2r_p}{ec}\frac{\hat{I}_t}{\beta^2\gamma^2}\frac{R}{\beta\gamma}\frac{R}{Q_x}\left\langle\frac{2}{a(a+b)} - \frac{2a+b}{2a(a+b)^2}\xi^2 - \frac{b}{2a(a+b)^2}\chi^2\right\rangle$$
$$\Delta Q_z(\xi,\chi) = -\frac{2r_p}{ec}\frac{\hat{I}_t}{\beta^2\gamma^2}\frac{R}{\beta\gamma}\frac{R}{Q_z}\left\langle\frac{2}{b(a+b)} - \frac{2b+a}{2b(a+b)^2}\chi^2 - \frac{a}{2b(a+b)^2}\xi^2\right\rangle \tag{32}$$

where ξ , $\chi \le 1$ are the relative betatron amplitudes , $x = \xi a$, $z = \chi b$. In practice we know the transverse emittances corresponding to a given fraction δ of the total number of particles. The semiaxes , $(a'\ ,\ b')$ of the appropriate ellipse are given by

$$a' = (\beta_x\epsilon_x/\beta\gamma)^{1/2} \quad , \quad b' = (\beta_z\epsilon_z/\beta\gamma)^{1/2}$$

where ϵ_x , ϵ_z are normalized emittances containing δ . It can be shown that the value of u defining this ellipse is given by

$$\delta = u^2(2 - u^2)$$

and then a , b are related to the known a' , b' by $a' = ua$, $b' = ub$

5. *Application to DESY III*

A computer code has been written to solve the equations of Sections 3 and 4 for given input conditions. It is convenient to present the results obtained as a function of time during the acceleration cycle. We have used the specified magnet field rise, illustrated in Figure 1, to relate momentum to time. Injection is at a kinetic energy of 50 MeV and the peak field corresponds to a momentum of 7.5 GeV/c. Figure 1 also shows the consequent variation of the rf-frequency.

During the early part of acceleration we demand the minimum voltage sufficient to contain the bunch area thus yielding minimum peak momentum spread and hence smallest radial apperture. Once the region of linear field rise is reached ($t \geq 0.3\ s$) we choose to maintain constant rf-voltage. The required rf-voltage for the nominal bunch area of $7.2\ 10^{-2}$ eVs is shown in Figure 2 both for zero current and for the design intensity.

The variation of the synchronous phase angle, φ_0 , at design intensity is presented in Figure 3 whilst the change of small amplitude synchrotron frequency, f_{s0} , with time is shown in Figure 4. Figure 5 shows the time variation of bunch length and peak momentum spread at design intensity.

The spread of betatron frequencies within the beam, which we assume contains 95% of the particles within transverse normalised emittances of $8.\ 10^{-6}$ m, is shown in Figure 6. This is a plot of Q_x *vs* Q_z on which is indicated the position of transverse betatron resonances of order 3 and 4. The limits of the tune spread are given by taking (ξ, χ) in equation (32) to be (1,0),(0,0) and (0,1). The calculated spread is shown at three different times. Region A corresponds to injection energy immediately after trapping and the peak vertical tune shift , $\Delta Q_z \sim 0.25$. The spread is a maximum , $\Delta Q_z \sim 0.28$, some 20 ms after the start of acceleration ; this is shown as region B. When the momentum has increased to ~ 1.5 GeV/c (after approx. 0.5 s) the tune spread is contained within region C and thereafter the beam is an area notionally free of resonances of order 4 and lower. We note that the tune spreads due to natural chromaticity are much less, being at most $\pm.01$ in each plane.

6. *Discussion*

We have presented a selection of beam characteristics in DESY III calculated by including a self consistent model of longitudinal space charge forces. The time variation of parameters has been calculated by solving the equations at a given momentum and relating this to time via the specified field rise. This approach is only valid if the conditions are varied adiabatically. Quantitatively this is ensured by demanding the condition [3] $\epsilon = \frac{1}{\Omega_s^2}\frac{d\Omega_s}{dt} << 1$, where Ω_s is the angular synchrotron frequency. For the rf-voltage program given above ϵ, computed for the small angle frequency is of order 10^{-4} and we therefore feel justified in using this approach.

There is nevertheless a reduction of synchrotron frequency with phase amplitude. On the separatrix it drops to zero and the adiabaticity criterion is not satisfied. Using equation (25) to find the synchrotron frequency for a particle within 10^{-9} radians of the separatrix gives an effective value of $\epsilon \sim 10^{-2}$, confirming the validity of our assumption for large amplitudes also. If we demand a few percent more than the minimum rf-voltage this will reduce still further ϵ for the maximum phase amplitude.

The prediction of the transverse betatron frequency spreads may allow both an estimate of the non-linear transverse acceptance via tracking [4] and an assessment of possible multipole correction schemes. However, lacking detailed magnet field measurements, we may only judge the importance of acceptance reduction by using plausible data from the existing measurements [5].

APPENDIX

If we assume a sinusoidal voltage:

$$V(\varphi) = V \sin\varphi \qquad \text{(A1)}$$

the potential, according to our definition is:

$$U(\varphi) = -V(\cos\varphi - \cos\varphi_0 + (\varphi - \varphi_0)\sin\varphi_0) \qquad \text{(A2)}$$

Assuming no space charge the equations (3,4) may be written as:

$$\dot{\varphi} = JW \qquad \dot{W} = KV(\sin\varphi - \sin\varphi_0) \qquad \ddot{\varphi} = JKV(\sin\varphi - \sin\varphi_0) \qquad \text{(A3)}$$

For a full bucket the extrema of W are given from $\dot{W} = 0$ and correspond to $\varphi = \varphi_0$, $\varphi = \pi - \varphi_0$, and $\varphi = \varphi_1$ which may be obtained from equation (7) written as:

$$\cos\varphi_1 + \cos\varphi_0 + (\varphi_1 + \varphi_0 - \pi)\sin\varphi_0 = 0 \qquad \text{(A4)}$$

From the previous analysis it is known that for $\varphi = \varphi_0$ W is maximum whereas for $\varphi = \varphi_2 \equiv \pi - \varphi_0$ and $\varphi = \varphi_1$ W is zero. For synchronous acceleration at constant radius we have:

$$V \sin\varphi_0 = 20.958 R\dot{P} \qquad \text{(A5)}$$

with V in volts , P in GeV/c , R in meters. We see that $\dot{P} = 0$ (no acceleration,stationary bucket) for $\varphi_0 = 0$ or $\varphi_0 = \pi$. Below transition $(\eta < 0)$ $\varphi_0 = 0$ and above transition $(\eta > 0)$ $\varphi_0 = \pi$. From eqs (A3,A4) we get immediately that:

$$\begin{array}{llll} \text{for} & \varphi_0 = 0 & \varphi_1 = -\pi & \varphi_2 = \pi \\ \text{for} & \varphi_0 = \pi & \varphi_2 = 0 & \varphi_1 = 2\pi \end{array} \qquad \text{(A6)}$$

For $\varphi_0 = 0$ equation (9) becomes:

$$\dot{\varphi} = (-2KJV)^{1/2}\{\cos\varphi + 1\}^{1/2} \qquad \text{(A7)}$$

(N.B. Ignoring the limits, indefinite integration yields $\varphi = 4\tan^{-1}\exp(\sqrt{-KJV}\frac{s}{\beta c}) - \pi$ which is the typical form for a spatially localized'solution') . Integrating equation (10) from $-\pi$, π we get $\tau \to \infty$. Thus particles on the separatrix move very slowly and become stationary near the unstable fixed point π. However for $\varphi > -\pi$ and $\varphi < \pi$, τ exists and for very small oscillations it becomes independent of the amplitude (small angle approximation). From equation (11) we get:

$$A = 2\left(\frac{-2KV}{J}\right)^{1/2} \int_{-\pi}^{\pi} \{1 + \cos\varphi\}^{1/2} d\varphi = 8\left(\frac{-4KV}{J}\right)^{1/2}$$

substituting the values for K , J we obtain the formula for the area $\hat{A}$ of a full stationary bucket as:

$$\hat{A} = \frac{8R}{hc}\left(-\frac{2VE_0}{\pi\eta h}\right)^{1/2} \quad , \quad \eta < 0 \tag{A8}$$

and from equation (12) W_M becomes:

$$W_M = \left(-\frac{4KV}{J}\right) = \frac{\hat{A}}{8} \quad , \quad \eta < 0 \tag{A9}$$

The other equations may be obtained by simple substitution from (A8,A9). Above transition (second case of A6, $\eta > 0$) equation (A7) becomes:

$$\dot{\varphi} = (2KJV)^{1/2}\{1 - \cos\varphi\}^{1/2} \tag{A10}$$

and equation (11) yields:

$$A = 2\left(\frac{2KV}{J}\right)^{1/2} \int_{0}^{2\pi} \{1 - \cos\varphi\}^{1/2} = 8\left(\frac{4KV}{J}\right)^{1/2}$$

or

$$\hat{A} = \frac{8R}{hc}\left(\frac{2VE_0}{\pi\eta h}\right)^{1/2} \quad , \quad \eta > 0 \tag{A11}$$

and as before $W_M = \hat{A}/8$. Thus in general we can write

$$\hat{A} = \frac{8R}{hc}\left(\frac{2VE_0}{\pi|\eta|h}\right)^{1/2} \tag{A12}$$

The expression is more complicated for $\varphi_0 \neq 0$ even for a full bucket. Then the limits of stable oscillations are,

$$\varphi_1 \leq \varphi \leq \pi - \varphi_0 \quad , \quad \eta < 0$$

$$\pi - \varphi_0 \leq \varphi \leq \varphi_2 \quad , \quad \eta > 0$$

where φ_1 or φ_2 can be calculated from the transcedental equation (A4). The area may then be written as an elliptic integral:

$$A = \frac{\sqrt{2}}{8}\hat{A} \int_{\varphi_1}^{\pi-\varphi_0} \left|\cos\varphi + \cos\varphi_0 + (\varphi + \varphi_0 - \pi)\sin\varphi_0\right|^{1/2} d\varphi \tag{A13}$$

and similarly

$$W_M = \frac{\hat{A}}{8}\left|\cos\varphi_0 + (\varphi_0 - \frac{\pi}{2})\sin\varphi_0\right|^{1/2} \tag{A14}$$

For a given voltage and acceleration rate we may calculate φ_0 and evaluate (A13) numerically. For a partially filled moving bucket we may write the bunch area A_b as

$$A_b = \frac{\sqrt{2}}{8}\hat{A} \int_{\varphi_1}^{\varphi_2} \left|\cos\varphi - \cos\varphi_2 + (\varphi - \varphi_2)\sin\varphi_0\right|^{1/2} d\varphi \tag{A15}$$

with φ_1 , φ_2 given by

$$\cos\varphi_1 - \cos\varphi_2 + (\varphi_1 - \varphi_2)\sin\varphi_0 = 0 \tag{A16}$$

Simultaneous, numerical, solution of the above two equations results in the limits φ_1 , φ_2 for a particular φ_0 and bunch area. The expression for W_M becomes:

$$W_M = \frac{\sqrt{2}}{16}\hat{A}\left|\cos\varphi_0 - \cos\varphi_2 + (\varphi_0 - \varphi_2)\sin\varphi_0\right|^{1/2} \tag{A17}$$

The rest of the equations may be obtained from (A16,A17). The inclusion of space charge through our self-consistent model, leads to the modification of the above formulae in that the RHS are multiplied by the space charge factor m_t given by:

$$m_t = \left(1 - \frac{\Delta}{f(\varphi_1, \varphi_2)}\right)^{1/2} \tag{A18}$$

with Δ given by eqn(19), and the normalizing integral becomes:

$$f(\varphi_1, \varphi_2) = -V\left\{\sin\varphi_2 - \sin\varphi_1 - \frac{1}{2}(\varphi_2 - \varphi_1)(\cos\varphi_1 + \cos\varphi_2)\right\} \tag{A19}$$

The line density $\lambda(\varphi)$ using equation (16) is

$$\lambda(\varphi) = N\frac{\cos\varphi - \cos\varphi_2 + (\varphi - \varphi_2)\sin\varphi_0}{\sin\varphi_2 - \sin\varphi_1 - \frac{1}{2}(\varphi_2 - \varphi_1)(\cos\varphi_2 + \cos\varphi_1)} \tag{A20}$$

and is cos-squared shaped for long and parabolic for small bunches. Finally the multiplying factor m_t becomes:

$$m_t^2 = 1 - \frac{Neeg_0Z_0h^2}{2\gamma^2R}\,\frac{1}{V\left\{\sin\varphi_2 - \sin\varphi_1 - \frac{1}{2}(\varphi_2 - \varphi_1)(\cos\varphi_2 + \cos\varphi_1)\right\}} \tag{A21}$$

Thus for $\varphi_0 = 0$ (below transition) $m_t = \{1 + \frac{\Delta}{2\pi V}\}^{1/2}$ and the area is modified:

$$\hat{A}_t = \hat{A}\{1 + \frac{\Delta}{2\pi V}\}^{1/2}$$

Since $\Delta < 0$, the full bucket area for a given voltage is reduced by space charge. For $\varphi_0 = \pi$ (above transition) m_t becomes $m_t = \{1 - \frac{\Delta}{2\pi V}\}^{1/2}$ and the effect of space charge is to increase the bucket area. For $\sin\varphi_0 \neq 0$ (acceleration) we may solve the equations (A15) (modified by the space charge factor) and (A16). The numerical procedure becomes somewhat more complicated.

REFERENCES

1. A.Hofmann and F.Pedersen, IEEE Trans.Nucl.Sci. **NS-26** (1979), p. 3526.
2. O.D.Kellogg, "Foundations of Potential Theory", Springer-Verlag, 1967.
3. B.W.Montague, CERN 77-13 (1977), p. 68.
4. F.Schmidt, Private communication.
5. G.Hemmie, Private communication.

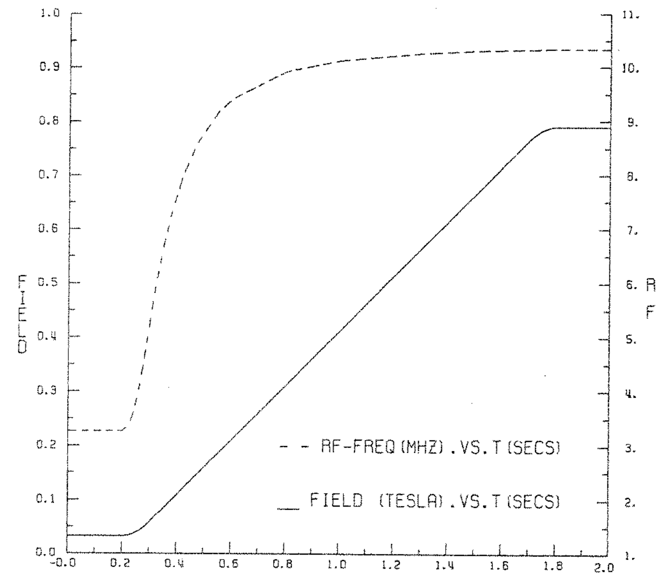

Fig. 1

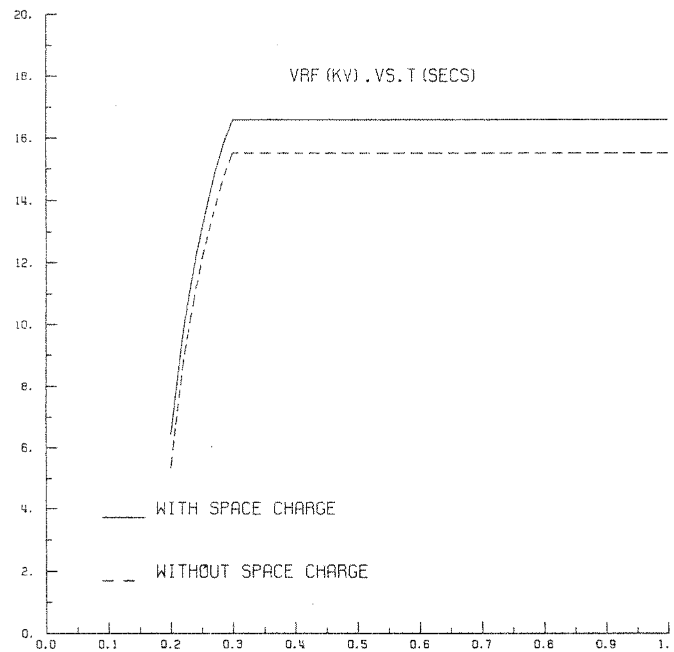

Fig. 2

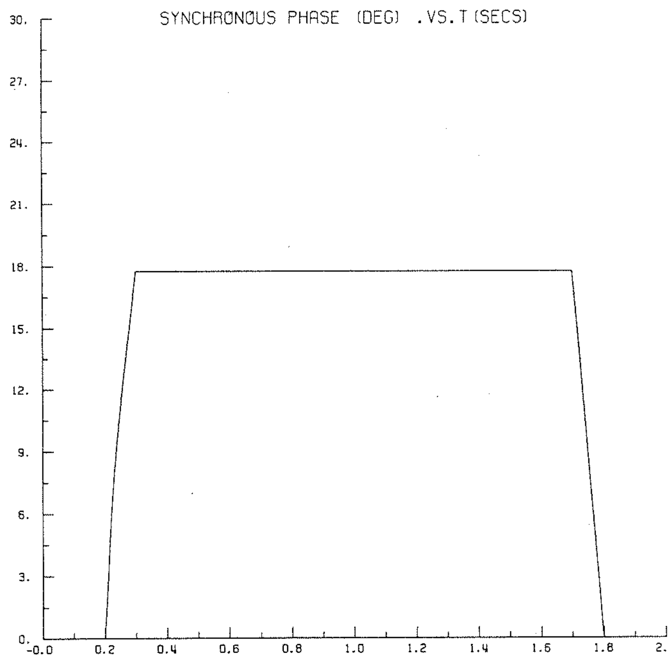

Fig. 3

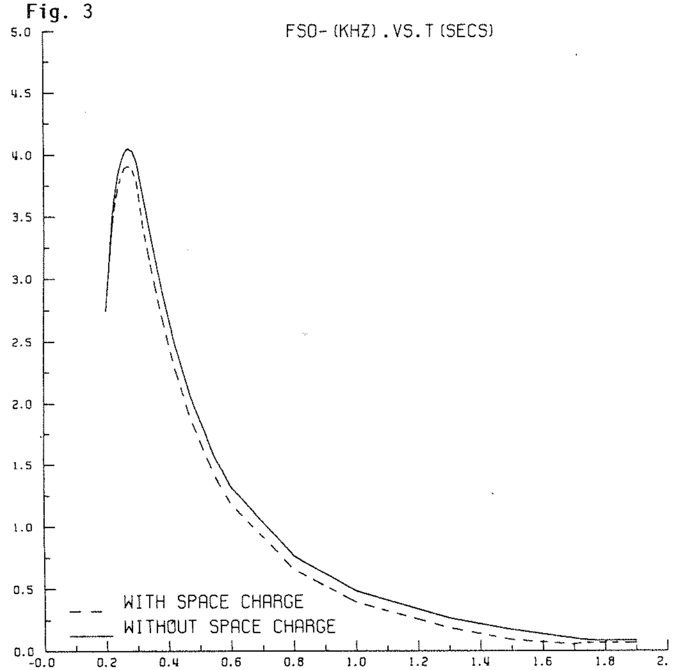

Fig. 4

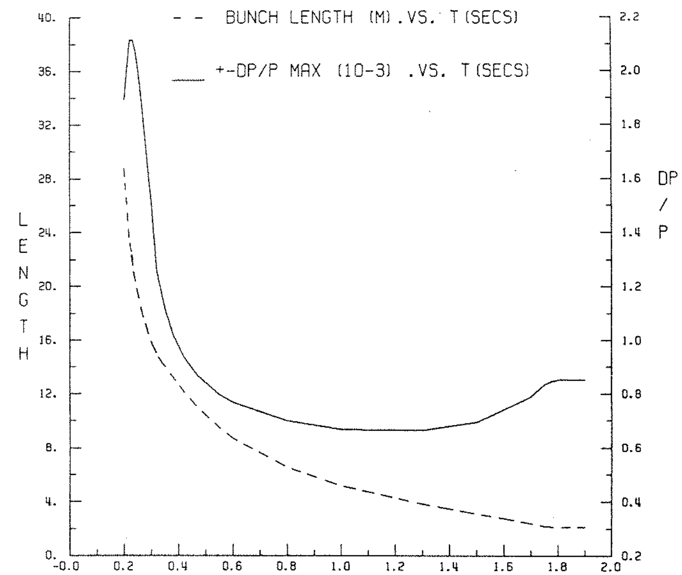

Fig. 5

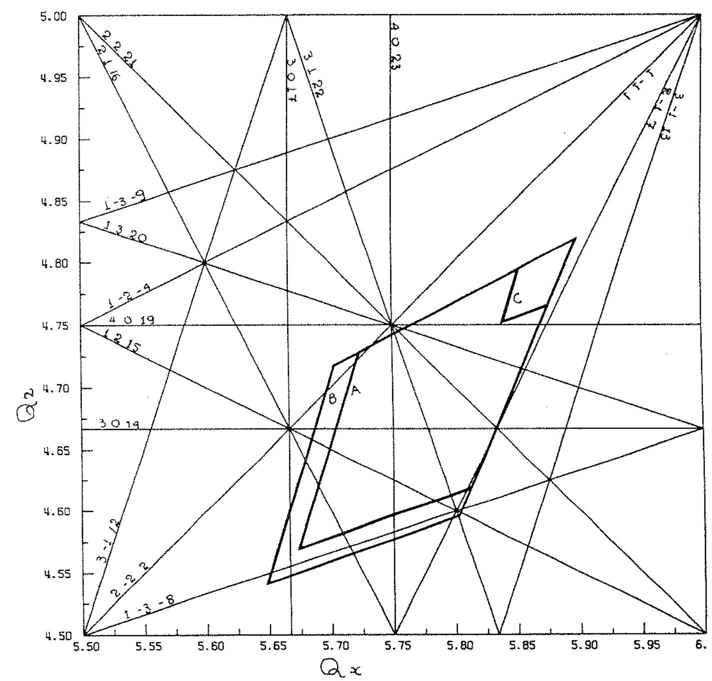

Fig. 6