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A SIMPLE MODEL OF AN RF - FEEDBACK SYSTEM AND ITS BEHAVIOUR IN THE TIME DOMAIN

by

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1 INTRODUCTION

Beam loading [1] [2] [3] may cause severe problems in accelerators. To counteract transient beam loading various methods can be applied, for example an rf - feedback system.[4]
An rf - feedback system will be necessary to reduce transient beam loading effects in the proton rings of the HERA - project.[5] In the following a simple model of an rf - feedback system is examined in the time domain. This simplified approach is advantageous because it offers an insight into the physical properties of such a system. After describing the rf - feedback system the impedance of the system is given and the voltage induced by the bunches is derived. It is assumed that the distance between bunches is short as compared to the filling time of the cavity feedback system, so that the bunch current is quasi-sinusoidal. It is shown that the rf - feedback system may be represented by an infinite number of simple resonators, however only one or two resonators of this set determine the behaviour of the system. Problems which may arise as a result of rf - feedback voltage overshoot or rf frequency variation during acceleration are investigated.

2 DESCRIPTION OF THE RF - FEEDBACK SYSTEM

The rf - feedback system consists of a simple resonator and a feedback circuit, which is an amplifier with gain constant k (Fig. 1).
Without the feedback loop the fourier components of the bunch current $\tilde{I}_B$, which is identical

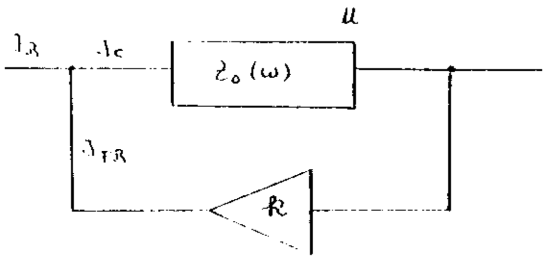

Figure 1: The RF - FEEDBACK SYSTEM

with the resonator current $\tilde{I}_r$ in this case, and the resonator voltage $\hat{U}$ are related by

$$\hat{U} = Z_0(\omega)\,\tilde{I}_B \tag{1}$$

with the resonator impedance

$$Z_0(\omega) = \frac{R_S}{1 + iQ(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega})} \tag{2}$$

R_S shunt impedance
Q Q - factor
ω_0 resonance frequency

Including the feedback loop the resonator current is given by

$$\tilde{I}_r = \tilde{I}_B - \tilde{I}_{FB}, \tag{3}$$

where the feedback current is obtained from

$$\tilde{I}_{FB} = k \cdot \hat{U} \tag{4}$$

Letting τ denote the time delay due to the finite length of cables and delay within the amplifier, the fourier components of the feedback current at time t and time $t-\tau$ are related in the following way

$$\begin{aligned}\tilde{I}_{FB_\tau} = \mathcal{F}\{I_{FB}(t-\tau)\} &= \int_{-\infty}^{\infty} dt\, e^{-i\omega t} I_{FB}(t-\tau) \\ &= e^{-i\omega\tau}\mathcal{F}\{I_{FB}(t)\}\end{aligned} \tag{5}$$

and therefore the resonator current can be rewritten in the form

$$\tilde{I}_c = \tilde{I}_B(t) - e^{i\omega t}\tilde{I}_{FB} \tag{6}$$

Using eq. 4 the resonator current may be written as

$$\tilde{I}_c = \tilde{I}_B - k\,e^{-i\omega\tau}\hat{U} \tag{7}$$

The resonator voltage with feedback is expressed by

$$\hat{U} = Z_0(\omega) \cdot \tilde{I}_c \tag{8}$$

Inserting eq. 7 and eq. 2 into eq. 8 yields

$$\hat{U} = \frac{R_S}{1 + iQ(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}) + \bar{k}e^{-i\omega\tau}} \cdot \tilde{I}_B \tag{9}$$

with

$$\bar{k} = R_S \cdot k$$

Thus the rf - feedback system is described by an impedance of the following form

$$Z(\omega) = \frac{R_S}{1 + iQ(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}) + \bar{k}e^{-i\omega\tau}} \tag{10}$$

The impedance of the simple resonator is purely resistive on resonance, that is

$$Z(\omega_0) > 0$$

The rf - feedback system should have the same property and this requires

$$\omega_0 \cdot \tau = 2\pi \cdot m \tag{11}$$

where m is any integer.

3 CALCULATION OF THE RF - FEEDBACK SYSTEM VOLTAGE

3.1 RESPONSE OF THE RF - FEEDBACK SYSTEM TO ANY GIVEN EXCITATION

The rf - feedback system voltage $G(t)$ is given by

$$G(t) = \int_{-\infty}^{\infty} d\omega\, e^{i\omega t} \tilde{F}(\omega) Z(\omega) \tag{12}$$

with

$$\tilde{F}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau} F(\tau) \tag{13}$$

$F(t)$ is any exciting function satisfying

$$F(t) = 0 \quad , t \leq 0 \tag{14}$$

The impedance of the rf - feedback system $Z(\omega)$ (10) is analytical in the lower half of the complex plane. Therefore $G(t)$ has the following properties

$$G(t) = 0 \quad , t < 0 \tag{15}$$

and $G(t)$ is finite for positive t. Inserting eq. 13 into eq. 12 and using eq. 14 as well as the analytical properties of $Z(\omega)$ leads to

$$G(t) = \int_0^t d\tau\, F(\tau) \int_{-\infty}^{\infty} d\omega\, e^{i\omega(t-\tau)} Z(\omega) \tag{16}$$

The integral

$$\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t-\tau)} Z(\omega) \tag{17}$$

is evaluated with the calculus of residues, the result being

$$\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t-\tau)} Z(\omega) = 2\pi i \sum_{\omega_n} e^{i\omega_n(t-\tau)} a_n \tag{18}$$

with

$$a_n = Res\,(Z(\omega))_{\omega=\omega_n}$$

The sum must be taken over all poles ω_n of the impedance in the upper half plane. It follows that the rf - feedback system voltage is given by

$$G(t) = i \sum_{\omega_n} a_n e^{i\omega_n t} \int_0^t d\tau\, F(\tau) e^{-i\omega_n \tau} \tag{19}$$

The impedance of the rf - feedback system without the feedback loop is given by eq. 2. This function has two poles ω_1, ω_2 and two related residues. The frequencies ω_1, ω_2 have the property

$$\omega_1 = -\omega_2^* \tag{20}$$

where the $*$ denotes complex conjugate. Relation (20) is responsible for the reality of the simple resonator voltage. The impedance of the rf - feedback system with feedback loop possesses an infinite number of poles $\{\omega_n\}$, as is shown in Appendix A1. The frequencies $\{\omega_n\}$ of the rf - feedback system have the property, that for any frequency ω_i there exists just one frequency ω_j with

$$\omega_i = -\omega_j^* \tag{21}$$

so that the reality of the rf - feedback system voltage is guaranteed. The rf - feedback system voltage (21) consist of an infinite number of real terms each describing a simple resonator voltage. Thus the rf - feedback system can be described by an infinite series of simple resonators with different resonance frequencies and Q - factors but the same shunt impedance.

3.2 RESPONSE TO SINUSOIDAL EXCITATION

The exciting function $F(t)$ is chosen to be

$$F(\tau) = \theta(\tau) \sin\omega\tau \tag{22}$$

where $\theta(\tau)$ is the Heaviside step function defined by

$$\theta(\tau) = \begin{cases} 0 & if\ \tau \leq 0 \\ 1 & if\ \tau \geq 0 \end{cases} \tag{23}$$

Using

$$\int_0^t d\tau\, e^{-i\omega_n\tau} \sin\omega\tau = e^{-i\omega_n t} \frac{\omega}{\omega_n^2 - \omega^2} \{\cos\omega t + i\frac{\omega_n}{\omega} \sin\omega t\} - \frac{\omega}{\omega_n^2 - \omega^2}$$

eq. 19 can be rewritten in the following form

$$G(t) = -i \sum_{\omega_n} a_n \frac{\omega}{\omega_n^2 - \omega^2} \{e^{i\omega_n t} - \cos\omega t - i\frac{\omega_n}{\omega} \sin\omega t\} \tag{24}$$

3.3 THE RF - FEEDBACK SYSTEM VOLTAGE

Poles of the rf - feedback system impedance

The poles of the impedance of the rf - feedback system are given by the zero points of the denominator of $Z(\omega)$ given in eq. 10. Thus the ω_n are the solutions of

$$1 + iQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) + \tilde{k}\, e^{-i\omega\tau} = 0 \tag{25}$$

To simplify the subsequent equations the following quantities are introduced

$$\begin{aligned} \omega &= \omega_0 (x_r + i\, x_i) \\ \varphi &= \omega \cdot \tau \end{aligned} \tag{26}$$

Multiplying eq 25 with ω / ω_0 and using the relations 26 yields

$$
\begin{aligned}
x_r - 2Qx_i x_r + \bar{k}e^{x_i\varphi}(x_r \cos x_r\varphi + x_i \sin x_r\varphi) &= 0 \\
x_i + Q(x_r^2 - x_i^2 - 1) + \bar{k}e^{x_i\varphi}(x_i \cos x_r\varphi - x_r \sin x_r\varphi) &= 0
\end{aligned}
\tag{27}
$$

For $\bar{k} = 0$ the solution of eq. 27 is given by the well known poles of a simple resonator

$$
\begin{aligned}
x_i &= \frac{1}{2Q} \\
x_r &= \sqrt{1 - x_i^2}
\end{aligned}
\tag{28}
$$

Eq. 27 reveals the following property of the solution ω: If

$$\omega = \omega_0(x_r + ix_i) \tag{29}$$

is a solution of eq. 27 then another solution is

$$\tilde{\omega} = -\omega^* \tag{30}$$

This guarantees the reality of the rf - feedback system voltage. The exact solutions of eq. 27 can only be obtained numerically, but it is possible to find solutions given in Appendix A.1 which solve eq. 27 approximately.

Residues of the rf - feedback system impedance

The singularities of the rf - feedback system impedance given by the solutions of eq. 27 are poles of order one and the residues are therefore obtained by the following formula

$$a_n = \frac{R_S}{\frac{d}{d\omega}\left(1 + iQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} + \bar{k}e^{-i\omega\tau}\right)\right)_{\omega = \omega_n}} \tag{31}$$

Substituting eq. 27 and 26 into 31 leads to

$$a_n = \frac{\omega_n (A_n + iB_n)}{A_n^2 + B_n^2} \tag{32}$$

with

$$A_n = 1 - 2Qx_i + \bar{k}e^{x_i\varphi}((1 + \varphi x_i)\cos x_r\varphi - \varphi x_r \sin x_r\varphi) \tag{33}$$

$$B_n = 2Qx_r + \bar{k}e^{x_i\varphi}(\varphi x_r)\cos x_r\varphi + (1 + \varphi x_i)\sin x_r\varphi) \tag{34}$$

where

$$\omega_n = \omega_0(x_r + ix_i)$$

denotes the poles of the impedance (10)

Calculation of the rf - feedback system voltage

The voltage is given by eq. (20). Inserting the values of the residues (32) into eq. (20) yields

$$G(t) = i\sum_{\omega_n} \frac{A_n + iB_n}{A_n^2 + B_n^2} \frac{\omega\omega_n}{\omega_n^2 - \omega^2} \{e^{i\omega_n t} - \cos\omega t - i\frac{\omega_n}{\omega}\sin\omega t\}$$

or

$$G(t) = i\sum_{\omega_n} \frac{A_n + iB_n}{A_n^2 + B_n^2} \frac{z \cdot r}{z^2 - r^2} \{e^{i\omega_n t} - \cos\omega t - i\frac{z}{r}\sin\omega t\} \tag{35}$$

with

$$z = \frac{\omega_n}{\omega_0} \; , \; r = \frac{\omega}{\omega_0}$$

The term

$$\mathcal{A} = \frac{A_n + iB_n}{A_n^2 + B_n^2} \frac{z \cdot r}{z^2 - r^2}$$

will be called amplitude in the following.
In Appendix A.2 an alternative form of (35) is given so that further evaluation may be preformed using a computer.

4 INVESTIGATION OF THE TIME DEPENDENCE OF THE RF - FEEDBACK SYSTEM VOLTAGE

4.1 GENERAL BEHAVIOUR

As mentioned on page 6 the rf - feedback system is composed of a series of resonators with different Q - factors and resonance frequencies. As a rule the behaviour of the rf - feedback system is usually determined by one or two of these resonators only (see Fig. 2 and 3). The most important resonators are those with the smallest damping time t_d

$$t_d = \frac{1}{x_i \omega_0} \tag{36}$$

and therefore those with the smallest imaginary part x_i. These resonators can be found by inspection of the formulae for the zeroes in Appendix A.1. One of the important resonances is given by

$$x_r = \frac{4n + 1}{2\varphi}\pi \approx 1 \tag{37}$$

because this solution has the smallest imaginary part given by (65) or (66) respectively and the greatest amplitude among the resonators given by (63). The solution with frequency zero (56) and (57) is less important as it usually has a small amplitude and a large damping time. The other important solution is given by (70) and (73) namely

$$x_r = 1 - \epsilon \; , \; \epsilon \ll 1 \tag{38}$$

Fig. 2 and 3 reveal a problem. Fig. 2 shows the desired time dependence of the rf - feedback system voltage. The voltage increases to its final value and never exceeds it. The problem is

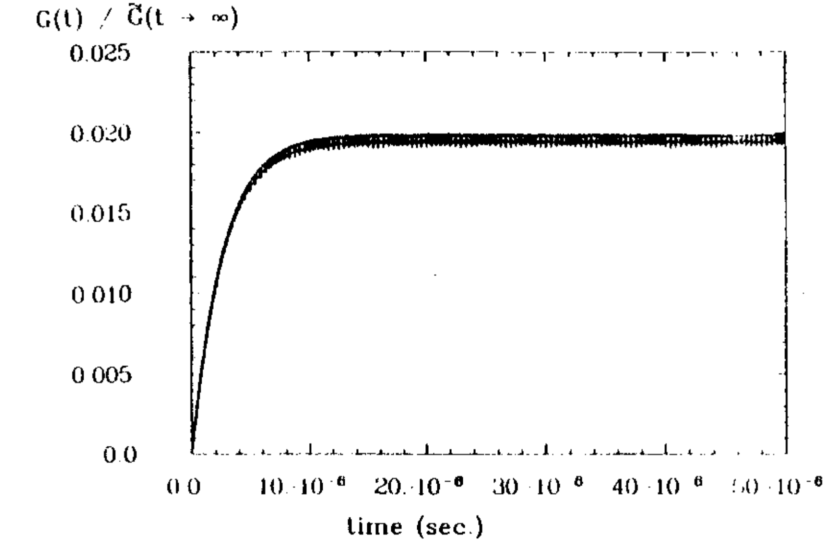

Figure 2: feedback voltage versus time ($Q = 4000$ $\omega_0 = 50$ MHz $\tau \approx 503$ ns, $k = 50$) ($\tilde{G}(t \to \infty)$ is the simple resonator voltage for large times)

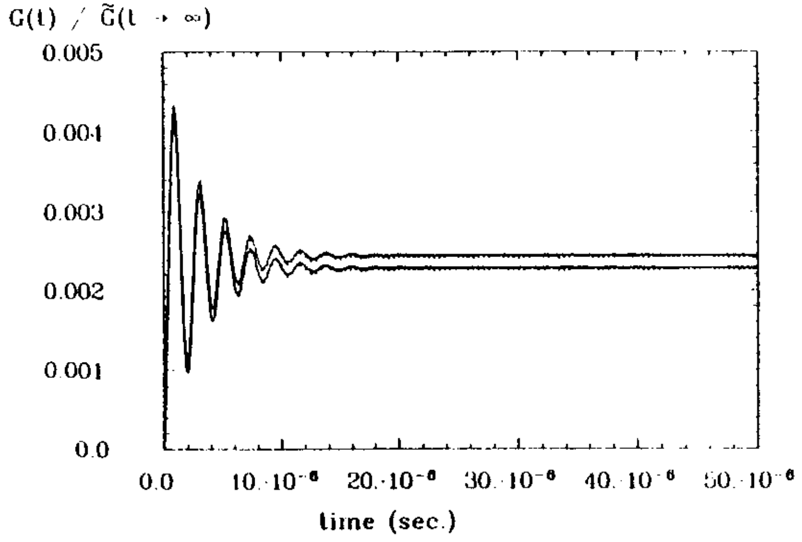

Figure 3: feedback voltage versus time ($Q = 4000$ $\omega_0 = 50$ MHz $\tau \approx 503$ ns, $k = 400$) ($\tilde{G}(t \to \infty)$ is the simple resonator voltage for large times)

that in some cases the voltage rings (Fig. 3) and overswings occur which may be dangerous. In the following section the conditions governing the appearence of voltage overshoot are investigated. Considering only the most important resonances which means those with

$$\frac{x_r}{r} \approx 1 \quad \frac{x_i}{r} \ll 1 \tag{39}$$

the time dependence of the rf - feedback system voltage is given by (A.2 77 and 78)

$$x_1(t) \sim \cos r\omega_0 t - e^{-ix_i\omega_0 t} \cos x_r\omega_0 t \tag{40}$$

or

$$x_2(t) \sim \sin r\omega_0 t - e^{-ix_i\omega_0 t} \sin x_r\omega_0 t \tag{41}$$

These relations may be rewritten in the following form

$$x_1 \sim \cos r\omega_0 t \,(1 - e^{-ix_i\omega_0 t}) - A(t)\cos(r\omega_0 t + \alpha(t)) \tag{42}$$

or

$$x_2 \sim \sin r\omega_0 t \,(1 - e^{-ix_i\omega_0 t}) - A(t)\sin(r\omega_0 t + \alpha(t)) \tag{43}$$

with

$$A(t) = 2e^{-ix_i\omega_0 t} \sin\frac{r - x_r}{2}\omega_0 t$$

$$\alpha(t) = \frac{x_r - r}{2}\omega_0 t - \frac{\pi}{2}$$

Thus the time dependence of x_1 consists of an oscillation

$$(1 - e^{-ix_i\omega t}) \cos r\omega t$$

and a part which is called beat

$$A(t)\cos(r\omega t + \alpha(t))$$

with an exponentially decreasing amplitude and a time dependent phase. If $t = 0$ both parts are 180° degrees out of phase and more or less cancel each other. But as the time approaches

$$t \to \frac{\pi}{(r - x_r)\omega_0}$$

both parts come into phase, sum up and the voltage may ring. If

$$\frac{x_i}{|r - x_r|} < \frac{1}{4} \tag{44}$$

overswings occur. The condition is derived in Appendix A.3. Even when the inequalilty 44 is not fulfilled some degree of voltage overswing occurs (Fig. 4). This is due to 44 being a somewhat pessimistic criterion, however the smaller the ratio defined in 44 the greater is the voltage overshoot (Fig. 5).

Amplitude of oscillation and beat versus time for different values of the ratio [illegible]

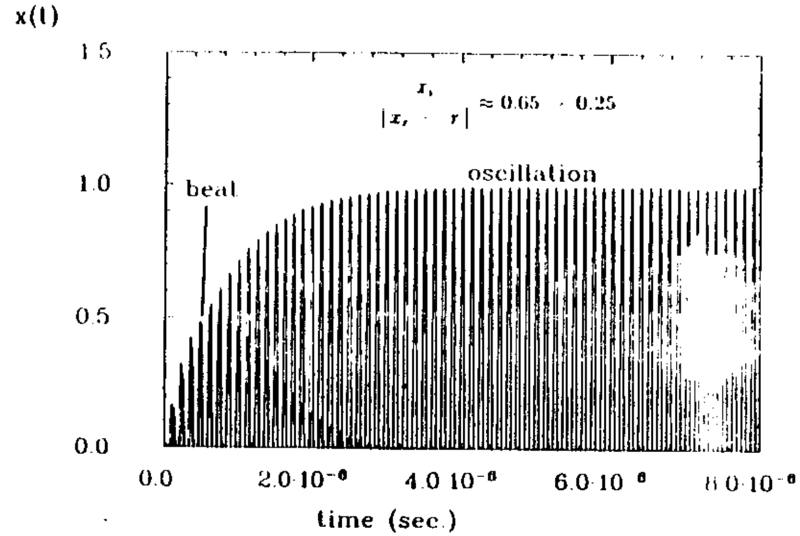

Figure 4:

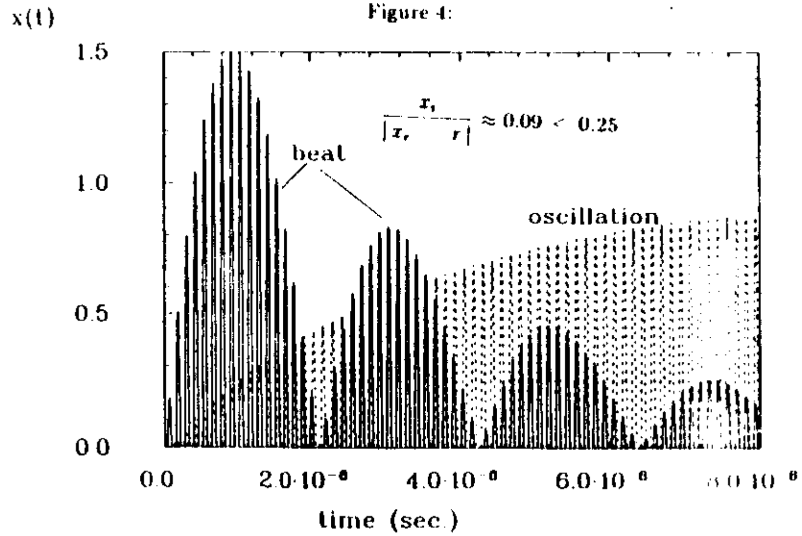

Figure 5:

4.2 BEHAVIOUR FOR SPECIAL CASES

Fig. 6 and 7 show the time dependence of the simple resonator voltage with

$$Q = 4000 \qquad \omega_0 = 50\ \mathrm{MHz} \tag{45}$$

and of the rf - feedback system voltage with

$$\tau \approx 503\ \mathrm{ns} \qquad \Rightarrow \quad \omega_0 \cdot \tau = 8\pi \tag{46}$$

for different values of the gain constant $\hat{k}$. The gain constant $\hat{k}$ must not exceed a maximum value

$$\hat{k}_{max} = 485 \tag{47}$$

in order that the voltage remains finite (see appendix A.1 eq. (69)). As long as

$$\hat{k} \leq 0.2\ \hat{k}_{max} \tag{48}$$

the voltage does not ring (Fig. 7). If $\hat{k}$ approaches $\hat{k}_{max}$ the voltage rings. The overswings can have amplitudes twice as big as the voltage for large times. For small values of $\hat{k}$ ($\hat{k} < 50$) the voltage is determined by the frequency given in (70). For increasing $\hat{k}$ the imaginary part of (63) with

$$x_r \approx 1$$

decreases while the imaginary part of (70) increases. Thus the time dependence is at first determined by solution (70) then by both solutions (70 and 63) and finally mainly by (63).
The acceleration of protons causes a special problem. During the acceleration cycle the frequency of the resonator ω_0 must be increased with increasing velocity v of the protons. The ratio between the frequency at injection and ejection is given by

$$\frac{\omega_{ej}}{\omega_{in}} = \frac{\beta_{ej}}{\beta_{in}} \tag{49}$$

with

$$\beta = \frac{v}{c}$$

Now the following problem arises. Condition 11

$$\omega_0 \tau = 2\pi m$$

can only be satisfied for one special value ω lying in the interval $\{\omega_{in}, \omega_{ej}\}$ for a given time delay τ. The time delay could be tuned to each value ω_0 between ω_{in} and ω_{ej} but there might exist an optimum choice. During the acceleration cycle the frequency of the most important resonator might be changed to a value which lies in the interval $\{\omega_{in}, \omega_{ej}\}$ so that this resonance is crossed during acceleration and the voltage increases.
In the following part two cases are studied
1. τ is chosen so that

$$\omega_{in} \cdot \tau_1 = 2\pi \cdot n \tag{50}$$

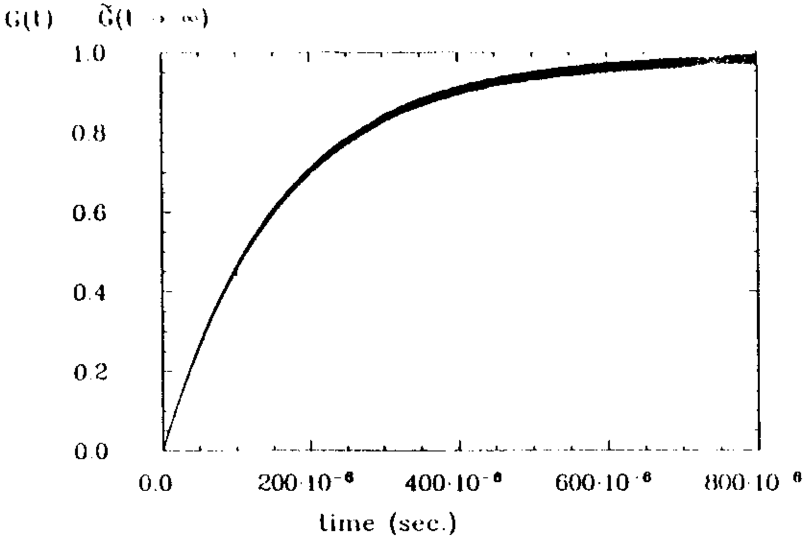

Figure 6: Voltage of the simple resonator ($Q = 4000$ $\omega_0 = 50$ MHz) { $\tilde{G}(t \to \infty)$ is the simple resonator voltage for large times }

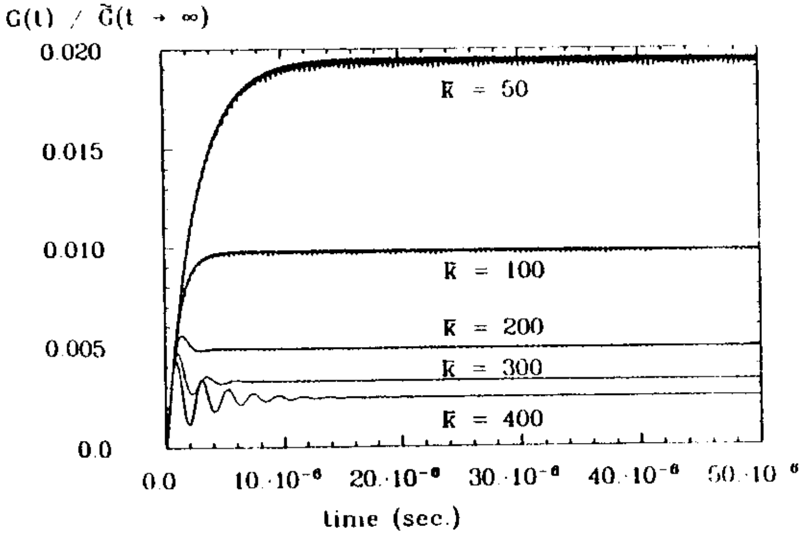

Figure 7: Voltage of the rf - feedback system for different values of the gain constant k ($Q = 4000$, $\omega_0 = 50$ MHz , $\tau \approx 503$ ns) { $\tilde{G}(t \to \infty)$ is the simple resonator voltage for large times }

2. τ is chosen so that

$$\omega_{ej} \cdot \tau_2 = 2\pi \cdot n \tag{51}$$

with $n = 4$.
The frequency at injection is

$$\omega_{in} = 50\,\mathrm{MHz} \;\Rightarrow\; \tau_1 \approx 503\ \mathrm{ns} \tag{52}$$

and is increased by 1% hence

$$\omega_{ej} = 50.5\,\mathrm{MHz} \;\Rightarrow\; \tau_2 \approx 497\ \mathrm{ns} \tag{53}$$

Fig. 8 and 9 show the time dependence of the rf - feedback system voltage for both cases. The voltage is plotted in both cases at injection and ejection and an intermediate value. As can be seen the small change of ω_0 during acceleration does not affect the voltage very much. This behaviour can be understood by looking at the formulae for the zeroes. If the delay is chosen to fulfil 1. or 2. respectively, the most important resonance describing the rf - feedback system is given by equation (70)

$$x_r \approx 1 - \epsilon \quad , \epsilon \ll 1$$

in both cases. If

$$\varphi = 2\pi \cdot n + \delta$$

then

$$x_r \approx 1 - \epsilon - \delta'$$

as is shown in Appendix A.1. If τ is tuned to ω_{in} (ω_{ej}) then δ and δ' are positive (negative) and the frequency of the most important resonatot discribing the rf - feedback system is shifted to lower (higher) values during acceleration. Thus in both cases 1. 2. it is impossible to cross this important resonance during acceleration.

5 CONCLUSION

In this paper a simple model of an rf - feedback system is presented. The investigation of the system is performed in the time domain. It is shown that the rf - feedback system is composed of an infinite number of resonators, but only one or two resonances determine the general behaviour.
The suppression of the bunch induced voltage is limited by the maximum allowed gain constant of the amplifier. The maximum gain constant, defined by the requirement that the system should be stable, has been calculated. As long as the gain constant is smaller than 20% of the maximum value an overswing does not appear. If the gain constant exceeds 20% of the maximum gain the voltage rings and the overswings might be twice as big as the final voltage for large times.
Changing the rf-frequency during acceleration by 1% does not cause any problem.

6 ACKNOWLEDGEMENT

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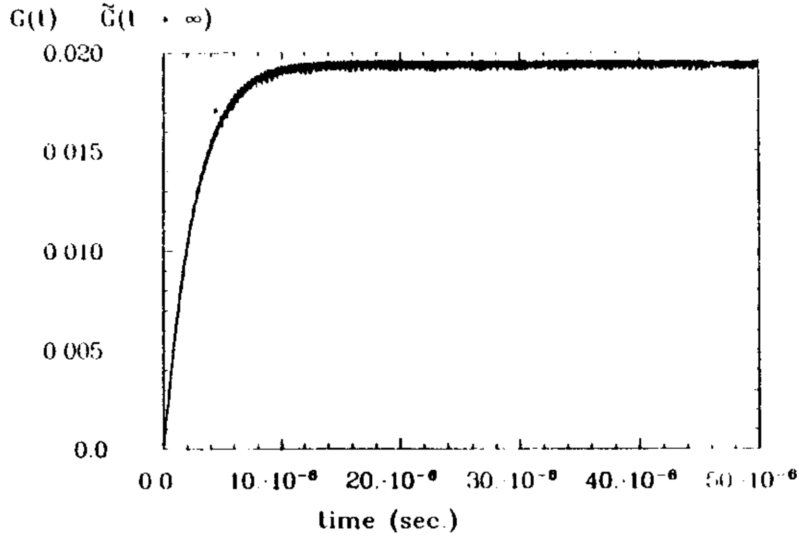

Figure 8: Change of the rf - frequency; case 1 { $\tilde{G}(t \to \infty)$ is the simple resonator voltage for large times }

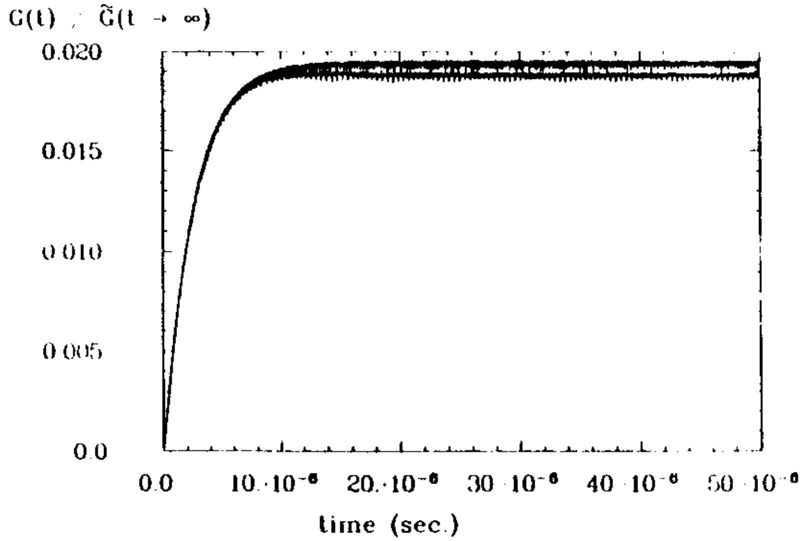

Figure 9: Change of the rf - frequency; case 2 { $\tilde{G}(t \to \infty)$ is the simple resonator voltage for large times }

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A APPENDIX

A.1

It is convenient to rewrite 27 in the following form

$$e^{x_i\varphi}\cos x_r\varphi = \frac{q}{k}x_i + \frac{Q}{k}\frac{x_i}{x_i^2 + x_r^2} - \frac{1}{k} \tag{54}$$

$$e^{x_i\varphi}\sin x_r\varphi = \frac{q}{k}x_r - \frac{Q}{k}\frac{x_r}{x_i^2 + x_r^2} \tag{55}$$

Eq. 55 is solved by $x_r = 0$ thus one of the zeroes is given by

$$x_r = 0 \tag{56}$$

$$e^{x_i\varphi} = \frac{Q}{k}(x_i + \frac{1}{x_i}) - \frac{1}{k} \tag{57}$$

Assuming that $x_r^2 \gg x_i^2$ and that the term $\frac{1}{k}$ can be neglected eq. (54) and (55) are transformed to

$$e^{x_i\varphi}\cos x_r\varphi = \frac{q}{k}x_i(1 + \frac{1}{x_r^2}) \tag{58}$$

$$e^{x_i\varphi}\sin x_r\varphi = \frac{q}{k}x_r(1 - \frac{1}{x_r^2}) \tag{59}$$

With

$$x_r\varphi = x_r^{(0)}\varphi - \delta \quad x_r^{(0)}\varphi = \frac{4n+1}{2}\pi \tag{60}$$

where n is any integer and considering

$$\cos x_r\varphi = \sin\delta$$
$$\sin x_r\varphi = \cos\delta$$

yields

$$\tan\delta = \frac{x_i\, 1 + 1/x_r^2}{x_r\, 1 - 1/x_r^2} \tag{61}$$

$$e^{x_i\varphi} = \frac{Q}{k}\sqrt{x_i^2(1 + \frac{1}{x_r^2})^2 + x_r^2(1 - \frac{1}{x_r^2})^2} \tag{62}$$

The last two equations can be solved approximately if $\delta \ll 1$ and $x_r \neq 1$ and the solutions are given by

$$x_r = x_r^{(0)} - \delta \tag{63}$$

$$\delta = \frac{x_i\, 1 + 1/x_r^2}{x_r\, 1 - 1/x_r^2} \tag{64}$$

$$x_i = \frac{1}{\varphi}\ln\{\frac{Q}{k}x_r^{(0)}(1 - \frac{1}{x_r^{(0)2}})\} \quad if\ x_r^{(0)} > 1 \tag{65}$$

$$x_i = \frac{1}{\varphi}\ln\{\frac{Q}{k}\frac{1}{x_r^{(0)}}\} \quad if\ x_r^{(0)} < 1 \tag{66}$$

where $x_r^{(0)}$ is given by (60).
In order that the rf - feedback system voltage remains finite it is neccesary that the imaginary part of the zeroes is positive. This requires

$$\begin{array}{ll} \frac{Q}{k}x_r^{(0)}(1 - \frac{1}{x_r^{(0)2}}) \geq 1 & n > m \\ \frac{Q}{k}\frac{1}{x_r^{(0)}} > 1 & n \leq m \end{array} \tag{67}$$

where n is defined by (60) and m by

$$\varphi = 2\pi \cdot m \tag{68}$$

Both inequalities (67) are satisfied if $\bar{k}$ does not exceed a maximum value given by

$$\bar{k}_{max} = Q(1 + \frac{1}{4m})(1 - \frac{1}{(1 + \frac{1}{4m})^2}) \tag{69}$$

Another zero is given by

$$x_r = 1 - \epsilon \quad 2\pi m\epsilon \ll 1 \tag{70}$$

if

$$\varphi = 2\pi \cdot m \tag{71}$$

$$\Rightarrow \sin x_r\varphi = 2\pi m\epsilon \tag{72}$$

and

$$x_r^2 \gg x_i^2$$

Then eq. (55) is fulfilled and eq. (54) gives the value of x_i

$$e^{x_i\varphi} \approx 2\frac{Q}{k}x_i - \frac{1}{k} \tag{73}$$

If φ is not determined by (71) but is given by

$$\varphi = 2\pi m + \delta \tag{74}$$

with

$$\delta \ll 2\pi m$$

the real part of the zero (70) is changed by

$$x_r = 1 - \epsilon - \delta' \quad \delta' = \frac{\delta}{2\pi m} \tag{75}$$

so that eq. (72) is still satisfied. Thus if φ is changed by a small value δ then x_r is changed by $-\frac{\delta}{2\pi m}$. To summarize:
The rf - feedback system impedance has an infinite number of poles. Though the solutions given above are only approximations they give an insight into the behaviour of the voltage. As a rule the approximate solutions are in good agreement with the exact solutions but for the most important zeroes the exact values are taken to calculate the voltage.

A.2

In the following eq. (35) is transformed into a convenient form which is useful for further evaluation. The voltage is given by

$$G(t) = -i \sum_{\omega_n} \frac{A_n + iB_n}{A_n^2 + B_n^2} \frac{z \cdot r}{z^2 - r^2} \{e^{i\omega_n t} - \cos \omega t - i \frac{z}{r} \sin \omega t\}$$

The term

$$\frac{z \cdot r}{z^2 - r^2}$$

can be changed to

$$\frac{z \cdot r}{z^2 - r^2} = \frac{W_r - iW_i}{C_n} \tag{76}$$

with

$$C_n = (x_r^2 - x_i^2 - r^2) + 4x_r^2 x_i^2$$
$$W_r = r\, x_r (x_r^2 + x_i^2 - r^2)$$
$$W_i = r\, x_i (x_r^2 + x_i^2 + r^2)$$

The time dependence of eq (35) can be rewritten in the following form

$$e^{i\omega_n t} - \cos \omega t - \frac{z}{r} \sin \omega t = \alpha(t) + i\beta(t)$$

with

$$\alpha(t) = e^{-x_i \omega t} \cos x_r \omega t - \cos r\omega_0 t + \frac{x_i}{r} \sin r\omega_0 t \tag{77}$$

$$\beta(t) = e^{-x_i \omega t} \sin x_r \omega t - \frac{x_r}{r} \sin r\omega_0 t \tag{78}$$

Thus the voltage can be expressed in the form

$$G(t) = -i \sum_{\omega_n} \frac{A_n + iB_n}{A_n^2 + B_n^2} \frac{W_r - iW_i}{C_n} \{\alpha(t) + i\beta(t)\} \tag{79}$$

This form has the advantage that the quantities A_n B_n C_n W_i W_r $\alpha(t)$ $\beta(t)$ are all real-valued numbers or functions respectively. The poles of the rf - feedback system impedance have the property

$$\omega_n = -\omega_n^*$$

(compare eq. (29) and (30))
Thus to every solution there exists another with the same imaginary part but with negative real part. Now the quantities A_n B_n C_n W_i W_r $\alpha(t)$ $\beta(t)$ are even or odd functions of x_r respectively

$$A_n(x_r) = A_n(x_r)$$
$$B_n(x_r) = -B_n(x_r)$$
$$C_n(x_r) = C_n(x_r)$$
$$W_i(x_r) = W_i(x_r)$$
$$W_r(x_r) = -W_r(x_r)$$
$$\alpha(x_r) = \alpha(x_r)$$
$$\beta(x_r) = -\beta(x_r)$$

Using these relations the voltage can be obtained by the following formula

$$G(t) = -i \sum_{x_r > 0}^{\omega_n} \frac{A_n + iB_n}{A_n^2 + B_n^2} \frac{W_r - iW_i}{C_n} \{\alpha(t) + i\beta(t)\}$$
$$+ \frac{A_n - iB_n}{A_n^2 + B_n^2} \frac{W_r + iW_i}{C_n} \{\alpha(t) - i\beta(t)\}$$
$$- i \sum_{x_r = 0}^{\omega_n} \frac{A_n + iB_n}{A_n^2 + B_n^2} \frac{W_r - iW_i}{C_n} \{\alpha(t) + i\beta(t)\}$$

hence

$$G(t) = 2 \cdot \sum_{x_r > 0}^{\omega_n} \frac{B_n W_r - A_n W_i}{(A_n^2 + B_n^2) C_n} \alpha(t) + \frac{A_n W_r + B_n W_i}{(A_n^2 + B_n^2) C_n} \beta(t)$$
$$- \frac{W_i}{A_n \cdot C_n} \alpha(t) \Big|_{x_r = 0} \tag{80}$$

A.3

In the following the derivation of eq. (44) is given. The extreme values of the amplitude of the beat occur at

$$t_{max} = \frac{2}{|r - x_r| \omega_0} \arctan \frac{|r - x_r|}{2x_i}$$

Certainly overswings will occur if the maximum amplitude of the beat is greater than 1:

$$|A(t_{max})| > 1$$

$$\Rightarrow \frac{e^{-\frac{\arctan x}{x}}}{\sin(\arctan x)} < 2 \quad x = \frac{|r - x_r|}{2x_i}$$

This inequality may be solved graphically (see Fig. 10) and the result is

$$x > 2$$
$$\Rightarrow \frac{x_i}{|r - x_r|} < \frac{1}{4}$$

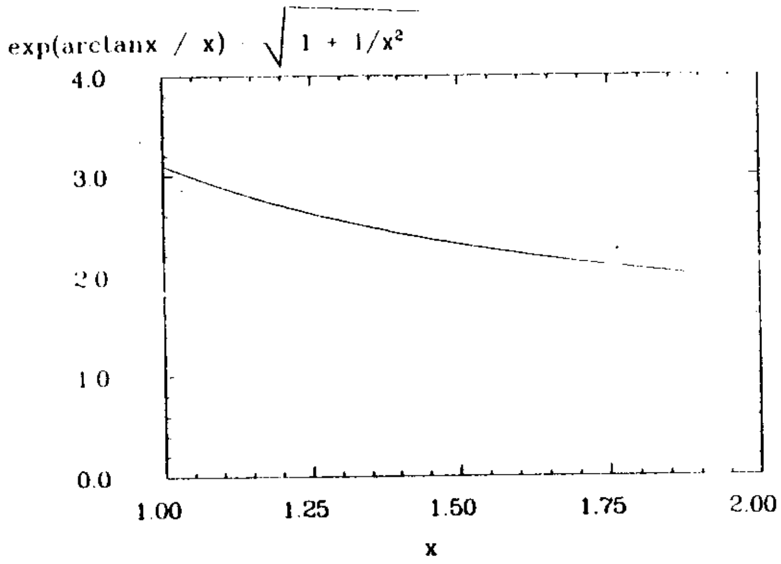

Figure 10: