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DESY - H 12

NOTE ON THE MEASUREMENT OF BEAM EMITTANCE

1. Measurement of beam envelope in a drift space

Using the notation of reference [1], the beam envelope $E(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}$ is given in terms of

$$\beta_o = \frac{1}{\varepsilon} E_o^2, \qquad \alpha_o = -\frac{1}{\varepsilon} E_o E_o' \quad \text{and} \quad \gamma_o = \frac{1+\alpha_o^2}{\beta_o}$$

by the equation

$$\beta(s) = \beta_o C^2(s) - 2\alpha_o C(s)S(s) + \gamma_o S^2(s) \qquad (1)$$

where C(s) and S(s) are the cosinelike and sinelike pricipal trajectories, respectively, and β_o, α_o and γ_o the beam parameters referring to the point s = 0.

In a drift space, we have

$$C(s) = 1 \qquad \text{and} \qquad S(s) = s$$

and therefore

$$\beta(s) = \beta_o - 2\alpha_o s + \gamma_o s^2 \qquad (2)$$

The amplitude function β(s) has a minimum, i.e. the beam

has a waist at the point

$$s_w = \frac{\alpha_o}{\gamma_o} \tag{3}$$

(see Fig. 1), and the beam width at the waist is given by

$$\beta(s_w) = \frac{1}{\gamma_o} \tag{4}$$

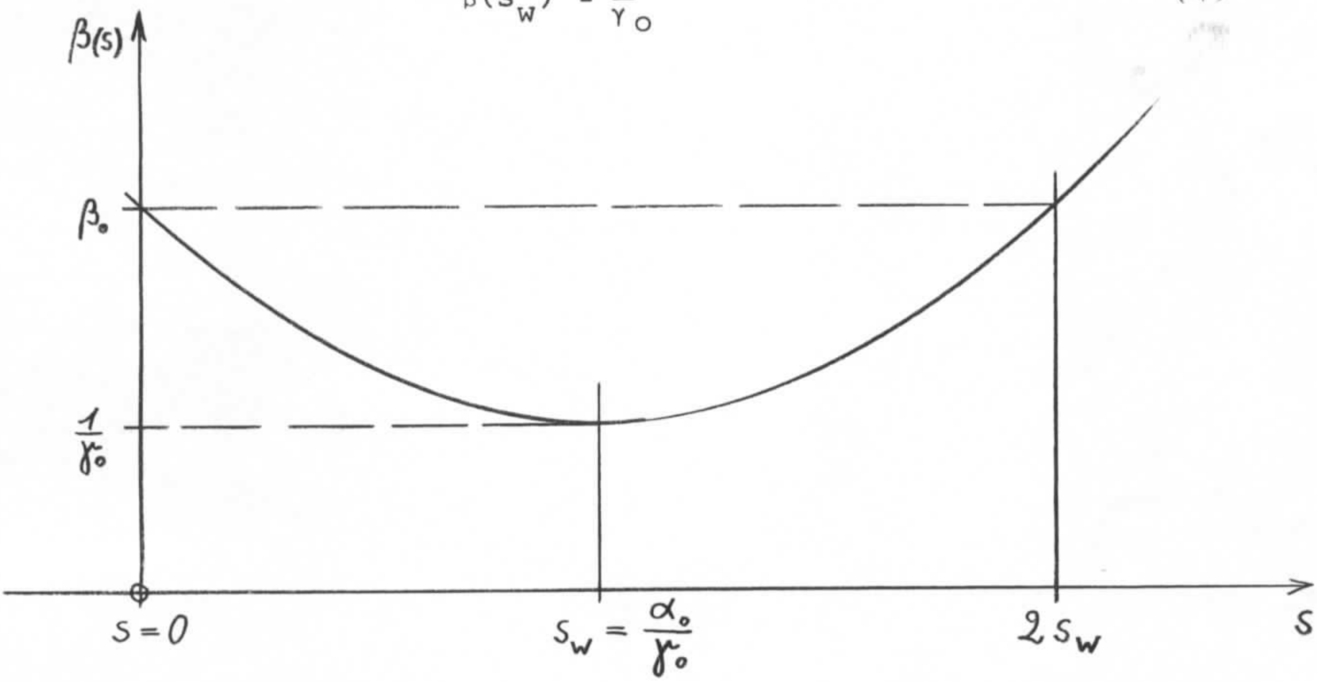


Figure 1: Amplitude function β(s) in a drift space

For a given $\beta_o = \frac{1}{\varepsilon} E_o^2$ at s = 0, the maximum distance of the waist from this point is obtained for $\alpha_o = 1$ and has the value

$$s_{w,max} = \frac{1}{2} \beta_o \tag{5}$$

In this case, the amplitude function at the waist is

$$\beta(s_{w,max}) = \frac{1}{2} \beta_o \tag{6}$$

Due to equation 2, the beam emittance at s = 0 as given by the beam parameters β_o, α_o and γ_o can be determined by measuring the beam envelope $E = \sqrt{\varepsilon}\sqrt{\beta}$ at different points s in the drift space and matching β(s) to these measured values [2]. This method calls for measuring

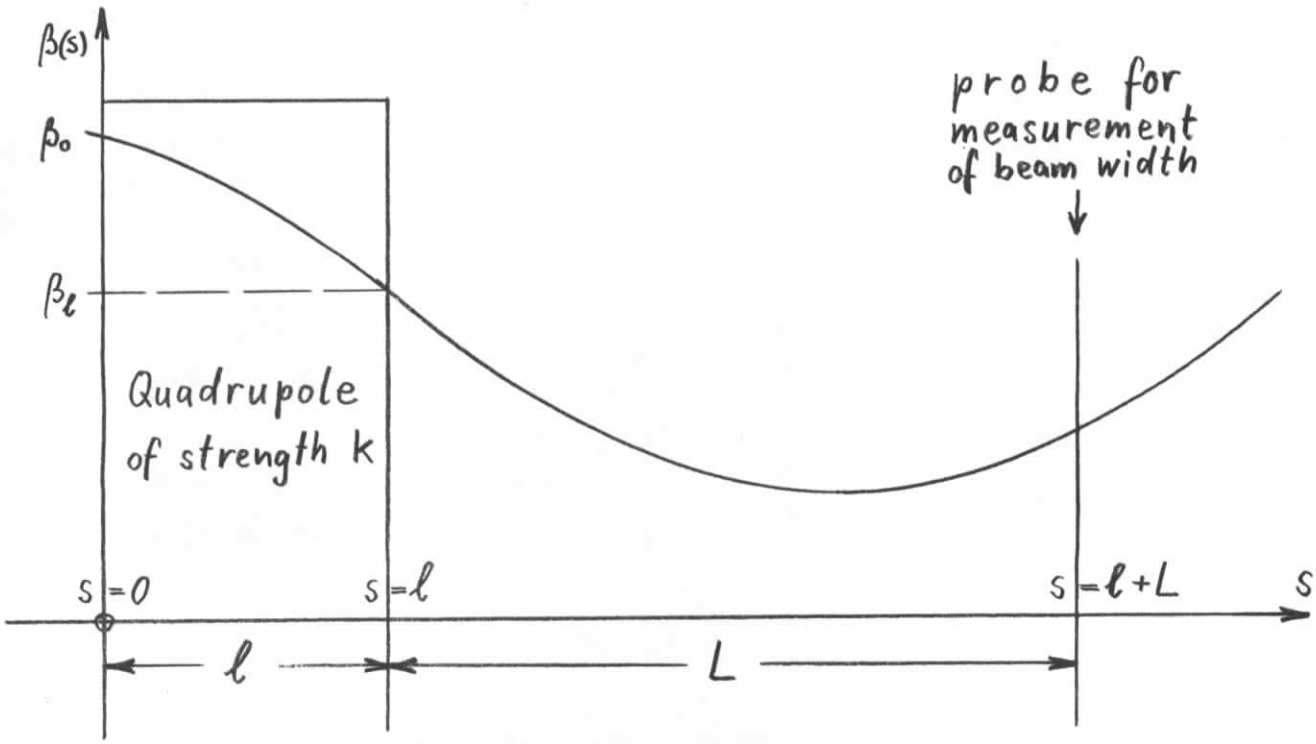
points in the vicinity of the waist as well as further away from it, since for $|s| \gg \beta_o$ the last term in equation 2 is dominating and determines $\varepsilon \cdot \gamma_o$, while the width and position of the waist determines $\frac{\varepsilon}{\gamma_o}$ and $\frac{\alpha_o}{\gamma_o}$, respectively.

2. Emittance measurement with quadrupole and fixed probe

In practice, instead of measuring at different points along the beam, it appears to be easier to move the beam waist across a fixed probe position by means of a quadrupole lens, as reported in reference [3], where the beam width for different lens strengths was determined photometrically from glass plates darkened by the beam. Since, according to equation 5, the maximum distance of the waist from the quadrupole is $\frac{1}{2}\beta_\ell$ (see figure 2), the probe distance L should be smaller than this value. Expressed in terms of the beam half width $E_\ell = \sqrt{\varepsilon}\sqrt{\beta_1}$ at the end of the quadrupole, this means

$$L < \frac{1}{2\varepsilon} E_\ell^2 \qquad (7)$$

Figure 2: Emittance measurement with quadrupole and fixed probe



for example, for an emittance $\varepsilon = 10^{-5}$ rad·m and a beam half width of 1 cm, the probe distance should be of the order of 2 - 3 meters only, and the quadrupole should be strong enough to produce a waist at 1 - 2 meters from its end.

In order to find the beam parameters at s = 0, we now have to use equation 1 for matching β to the measured beam half widths E at point $s = \ell + L$. Taking n different values E_i at different quadrupole strengths k_i, this yields the following system of linear equations for $\varepsilon\beta_o$, $\varepsilon\alpha_o$, $\varepsilon\gamma_o$,

$$\left\{\begin{array}{l} E_1^2 + \delta_1 = C_1^2\varepsilon\beta_o - 2C_1S_2\varepsilon\alpha_o + S_1^2\,\varepsilon\gamma_o \\ E_2^2 + \delta_2 = C_2^2\varepsilon\beta_o - 2C_2S_2\varepsilon\alpha_o + S_2^2\,\varepsilon\gamma_o \\ \quad\vdots \\ E_n^2 + \delta_n = C_n^2\varepsilon\beta_o - 2C_nS_n\varepsilon\alpha_o + S_n^2\,\varepsilon\gamma_o \end{array}\right. \qquad (8a)$$

For the n matching parameters δ_i one would, for instance, demand

$$\sum_{i=1}^{n} \left(\frac{\delta_i}{E_i}\right)^2 = \text{minimum}$$

which, by partial differentiation with respect to β_o, α_o and γ_o, yields the additional equations

$$\left\{\begin{array}{l} \sum_{i=1}^{n} \frac{1}{E_i^2}\, C_i^2\, \delta_i = 0 \\ \sum_{i=1}^{n} \frac{1}{E_i^2}\, C_iS_i\delta_i = 0 \\ \sum_{i=1}^{n} \frac{1}{E_i^2}\, S_i^2\, \delta_i = 0 \end{array}\right. \qquad (8b)$$

The principal trajectories C_i and S_i at the probe position depend on the quadrupole strength k_i as follows:

a) focusing case:

$$\begin{pmatrix} C_i & S_i \\ C_i' & S_i' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi_i & \frac{1}{\sqrt{k_i}} \sin\phi_i \\ -\sqrt{k_i}\sin\phi_i & \cos\phi_i \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi_i - L\sqrt{k_i}\sin\phi_i & \frac{1}{\sqrt{k_i}} \sin\phi_i + L\cos\phi_i \\ -\sqrt{k_i}\sin\phi_i & \cos\phi_i \end{pmatrix} \tag{9a}$$

with $\phi_i = \ell \sqrt{k_i}$

b) defocusing case

$$\begin{pmatrix} C_i & S_i \\ C_i' & S_i' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\phi_i & \frac{1}{\sqrt{k_i}} \sinh\phi_i \\ \sqrt{k_i}\sinh\phi_i & \cosh\phi_i \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\phi_i + L\sqrt{k_i}\sinh\phi_i & \frac{1}{\sqrt{k_i}} \sinh\phi_i + L\cosh\phi_i \\ \sqrt{k_i}\sinh\phi_i & \cosh\phi_i \end{pmatrix} \tag{9b}$$

with $\phi_i = \ell\sqrt{k_i}$

Thus, the system of linear equations 8 can easily be solved for $\varepsilon\beta_o$, $\varepsilon\alpha_o$ and $\varepsilon\gamma_o$ by a computer, and the beam emittance then follows from the relation

$$\varepsilon^2 = (\varepsilon\beta_o)(\varepsilon\gamma_o) - (\varepsilon\alpha_o)^2 \tag{10}$$

3. Emittance filter

The smallest praralellogram enclosing the beam ellipse touches the ellipse in 4 conjugated points (see reference [1], p. 175, and figure 3, where A and B are conjugated points). Therefore, if one wants to most efficiently restrict the beam emittance by means of two slit collimators positioned at s = 0 and $s = s_1$, respectively, the phase function $\emptyset(s)$ should increase by $\frac{\pi}{2}$ between them:

$$\emptyset(s_1) = \int_0^{s_1} \frac{1}{E^2(s)} \, ds = \frac{\pi}{2} \tag{11}$$

This can be achieved by placing a focusing lens between the collimators.

The setup, then, may also be used to measure the beam emittance by determining the beam half width E_o in the first collimator and, in addition, the half widths E_1 in the second collimator for various strengths of the quadrupole. Observing the product

$$\left(\frac{E_o}{S_1}\right) \cdot E_1 \tag{12}$$

as a function of quadrupole strength, it will assume a minimum for a certain pair of values E_1^* and S_1^*, and the beam emittance is then given by

$$\varepsilon = \frac{1}{S_1^*} E_o E_1^* \tag{13}$$

This may easily be seen as follows: The two factors in (12) are the "half widths" in y'- and y-direction of the phase plane parallelogram which is defined at point s_1 by the combined action of the two slit collimators (see

figure 3). The emittance of the inscribed maximum ellipse, therefore, is equal to the product (12), and whenever this product assumes a minimum, the maximum ellipse is equal to the beam ellipse.

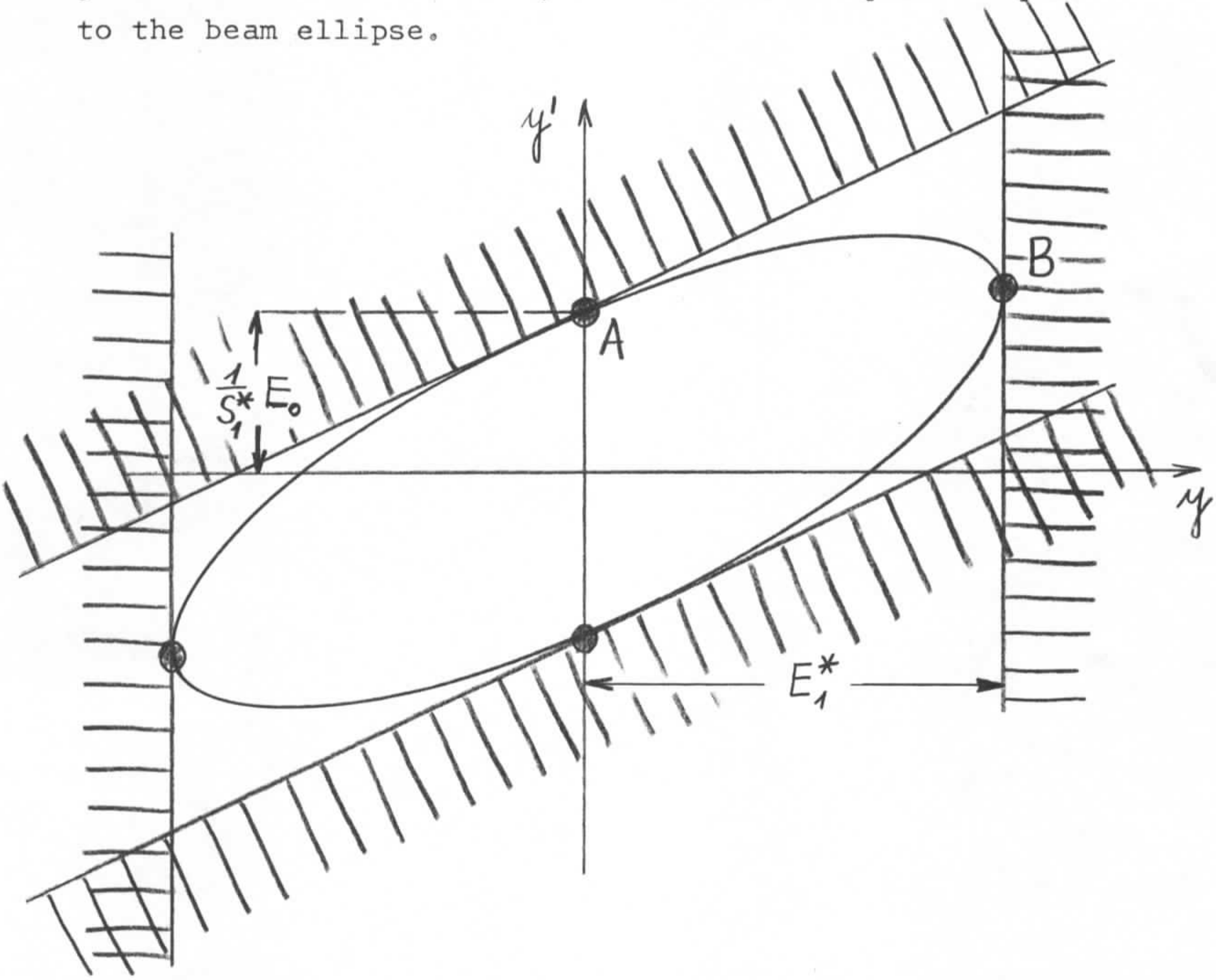


Figure 3: Phase plane parallelogram defined by two slit collimators.

References

1 K.G. Steffen, High Energy Beam Optics (Wiley 1965)

2 M. Placidi, private communication

3 F.W. Brasse, G. Hemmie and W. Schmidt, DESY 65/18 (1965)